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Article

A Parabolic Analog of a Theorem of Beilinson and Schechtman

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For a simple, simply connected, complex group G, we prove an explicit formula to compute the Atiyah class of parabolic determinant of cohomology line bundle on the moduli space of parabolic G-bundles. This generalizes an earlier result of Beilinson-Schechtman.

1 Introduction

Lie algebroids play an important role in the geometry of sheaves on manifolds. For the case of Atiyah algebras associated to principal bundles, the Atiyah exact sequence packages the information of connections and, more generally, twisted differential operators on the bundle. Of particular interest are Atiyah algebras associated to line bundles on moduli spaces of bundles on curves. These are almost always constructed via descent from a bigger parameter space. A fundamental question is therefore how to relate the behavior of Atiyah algebras of these natural line bundles to infinitesimal joint deformations of the moduli spaces under this correspondence.

This question was addressed in the fundamental work of Beilinson-Schechtman [4]. In the context of relative moduli stacks of vector bundles over families of smooth projective curves, the main result of [4] describes the Atiyah algebra of the determinant of cohomology in terms of a direct image of a *trace complex* constructed from the Atiyah algebra of the universal bundle. This construction is closely related to the "localization functor" [3, 5].

In [13], Ginzburg gave an alternative construction that is more amenable to the case of principal bundles. This is based on a general correspondence between quasi-Lie algebras and certain differential graded Lie algebras. Applied to the moduli problem, this time for principal bundles, it can be seen from work of Bloch-Esnault [10] that the direct image of the dgla constructed by Ginzburg also computes the Atiyah algebra of the determinant of cohomology.

The main goal of this paper is to extend these constructions to the case of moduli stacks of principal bundles with parabolic structures. In order to state the result, let us introduce some notation. Let $\mathcal{C} \longrightarrow S$ be a versal family of smooth projective curves with n marked points $p_1, \cdots, p_n$. Fix a simple, simply connected complex algebraic group G with Lie algebra $\mathfrak{g}$. Choose parabolic subgroups $P_1, \cdots, P_n$ of G and associated weights $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$. Let $M_G^{\boldsymbol{\alpha},rs} = M_G^{\boldsymbol{\alpha},rs}(\mathcal{C}/S) \to S$ be the relative moduli space (over S) of regularly stable parabolic G-bundles. On $\mathcal{C} \times_S M_G^{\boldsymbol{\alpha},rs}$ there exists local universal bundles $\mathcal{P}$.

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Let ${}^{par}\mathfrak{S}^{\bullet}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs},\pi,S}(\mathcal{P})$ denote the relative Ginzburg complex associated to $\mathcal{P}$. These local complexes glue together to give a global complex on $\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}$ (even though the global $\mathcal{P}$ does not exist).

On the other hand, we consider the sheaf of strongly parabolic relative Atiyah algebras ${}^{spar}\mathbf{At}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P})$ defined via the parabolic orbifold correspondence satisfying the short exact sequence

$$0\to \mathbf{SPar}(\mathcal{P})\to {}^{spar}\mathbf{At}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P})\to \mathcal{T}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(-D)\to 0,$$

where $\mathbf{SPar}(\mathcal{P})$ is the sheaf of strongly parabolic endomorphisms and D is the divisor of marked points. Now consider the sheaf of quasi Lie algebras ${}^{spar}\widetilde{\mathbf{At}}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P})$ obtained as the pull back of the following sequence:

$$0\to \Omega_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}\to ({}^{spar}\mathbf{At}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P})(D))^{\vee}\to (\mathbf{SPar}(\mathcal{P})(D))^{\vee}\to 0,$$

via the isomorphism of sheaf of parabolic endomorphisms $\mathbf{Par}(\mathcal{P})\cong(\mathbf{SPar}(\mathcal{P})(D))^{\vee}$. Moreover, by construction $\mathrm{R}^1\pi_*{}^{spar}\widetilde{\mathbf{At}}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P})$ is a sheaf of relative Atiyah algebras on $M_G^{\boldsymbol{\alpha},rs}$, where $\pi:\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}\to M_G^{\boldsymbol{\alpha},rs}$ is the projection. We first show that

$$\mathrm{R}^1\pi_*({}^{par}\mathfrak{S}^{\bullet}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs},\pi,S}(\mathcal{P}))\simeq \mathrm{R}^1\pi_*({}^{par}\mathfrak{S}^{-1}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P}))\simeq \mathrm{R}^1\pi_*({}^{spar}\widetilde{\mathbf{At}}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P}))$$

We refer the reader to Sections 3 and 4 for more details. Finally, we related it to relative Atiyah algebras of parabolic determinant of cohomologies.

Given a nontrivial holomorphic embedding $\phi: G\longrightarrow \mathrm{SL}_r$, there is an associated determinant of cohomology line bundle $\mathcal{L}_\phi\longrightarrow M_G^{\boldsymbol{\alpha},rs}$. Let $\mathrm{At}_{M_G^{\boldsymbol{\alpha},rs}/S}(\mathcal{L}_\phi)$ denote the relative Atiyah algebra of $\mathcal{L}_\phi$. Then the main result of this paper is the following.

Theorem 1.1. On $M_G^{\boldsymbol{\alpha},rs}$ there is a natural isomorphism of Atiyah algebras

$$\frac{1}{m_\phi}\mathrm{At}_{M_G^{\boldsymbol{\alpha},rs}/S}(\mathcal{L}_\phi)\simeq \mathrm{R}^1\pi_*({}^{par}\mathfrak{S}^{-1}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P}))\simeq \mathrm{R}^1\pi_*({}^{spar}\widetilde{\mathbf{At}}_{\mathcal{C}\times_S M_G^{\boldsymbol{\alpha},rs}/M_G^{\boldsymbol{\alpha},rs}}(\mathcal{P})),$$

where m_ϕ is the Dynkin index of the associated homomorphism $\phi_*:\mathfrak{g}\to\mathfrak{sl}_r$ given by the ratio of the normalized Killing forms.

In a recent paper [16] the result of Beilinson-Schechtman was used in an integral way to give an algebraic proof of the existence of a flat projective connection (a *Hitchin connection*) on the bundle of generalized theta functions for vector bundles on families of curves. One of the main motivations of the present paper was to apply Theorem 1.1 in the same way to obtain a Hitchin connection for theta functions associated to parabolic G-bundles. This is carried out in [9].

2 Quasi-Lie Algebras and Extensions of Atiyah Algebras

2.1 Basic definitions

In this section, we recall a correspondence stated in Ginzburg [13] between quasi-Lie algebras and their associated differential graded Lie algebras. We also recall from Beilinson-Schechtman [4] a natural classes of Atiyah algebras associated to a family of curves.

2.1.1 *Quasi-Lie algebras*

First we recall the definition of a quasi-Lie algebra. Let $\widetilde{\mathfrak{g}}$ be a vector space equipped with a skew-symmetric bilinear map

$$[\ ]:\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}\longrightarrow\widetilde{\mathfrak{g}}.$$

Let $Z\subset\widetilde{\mathfrak{g}}$ be a linear subspace.

Definition 2.1. A triple $(\widetilde{\mathfrak{g}}, Z, [\ ])$ as above is called a quasi-Lie algebra if the following holds:

1) the subspace Z is central with respect to $[\ ]$, and
2) the bracket $[\ ]$ descends to give a Lie algebra structure on $\widetilde{\mathfrak{g}}/Z$.

We will see that in the setting of Atiyah algebras, these quasi-Lie algebras arise naturally. Now recall the notion of a differential graded Lie algebra (dgla).

Definition 2.2. A *differential graded Lie algebra* (dgla) is a vector space $\mathfrak{S} := \bigoplus_i \mathfrak{S}^i$ together with a bilinear map $\{\mathfrak{S}^i, \mathfrak{S}^j\} \subset \mathfrak{S}^{i+j}$ and a differential $d : \mathfrak{S}^i \longrightarrow \mathfrak{S}^{i+1}$ satisfying the following:

- $\{x, y\} = (-1)^{|x||y|+1}\{y, x\}$, where $|z| = i$ for $z \in \mathfrak{S}^i$,
- $(-1)^{|x||z|}\{x, \{y, z\}\} + (-1)^{|y||x|}\{y, \{z, x\}\} + (-1)^{|y||z|}\{z, \{y, x\}\} = 0$, and
- $d\{x, y\} = \{dx, y\} + (-1)^{|x|}\{x, dy\}$.

A morphism of dglas is a graded linear map $\mathfrak{S} \longrightarrow \mathfrak{S}'$ that preserves the Lie bracket and commutes with the differentials.

The following lemma of Ginzburg [13, Lemma 7.7] gives a correspondence between quasi-Lie algebras and a certain class of dglas.

Lemma 2.3. Let $(\widetilde{\mathfrak{g}}, Z, [\ ])$ be a quasi-Lie algebra equipped with a symmetric Z-valued, Z-invariant bilinear form $\langle\ \rangle : \mathbf{Sym}^2\,\widetilde{\mathfrak{g}} \longrightarrow Z$ such that the following hold:

- $\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0$, and
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = d(\langle [x, y], z\rangle)$.

Then there exists a dgla $\mathfrak{S} = \mathfrak{S}^{-2} \oplus \mathfrak{S}^{-1} \oplus \mathfrak{S}^0$, where $\mathfrak{S}^0 := \mathfrak{g}$, $\mathfrak{S}^{-1} := \widetilde{\mathfrak{g}}$, $\mathfrak{S}^{-2} := Z$, the differential is given by inclusion and quotient, and with the bracket given by the formula $\{x, y\} = \langle x, y\rangle$; and $\{x, dy\} = [x, y] + \langle x, y\rangle$ for $x, y \in \widetilde{\mathfrak{g}}$. Conversely given a dgla as above satisfying $\{\mathfrak{S}^{-2}, d\mathfrak{S}^{-1}\} \subset \ker d$, there exists a quasi-Lie algebra along with a symmetric Z-valued, Z-invariant bilinear form.

2.1.2 *Dglas associated to families of curves*

Let $\pi : X \to T$ be a smooth morphism of relative dimension one parametrized by T; the curves in this family are not assumed to be proper. The relative holomorphic tangent bundle $\mathcal{T}_{X/T}$ fits in the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{T}_{X/T} \longrightarrow \mathcal{T}_X \xrightarrow{d\pi} \pi^*\mathcal{T}_T \longrightarrow 0.$$

Let $\mathcal{T}_{X,\pi} \subset \mathcal{T}_X$ denote the subsheaf $d\pi^{-1}(\pi^{-1}\mathcal{T}_T)$. Clearly the sheaf $\mathcal{T}_{X,\pi}$ has the structure of Lie algebra with Lie bracket coming from that on $\mathcal{T}_X$ and there is an exact sequence of Lie algebras

$$0 \longrightarrow \mathcal{T}_{X/T} \longrightarrow \mathcal{T}_{X,\pi} \xrightarrow{d\pi} \pi^{-1}\mathcal{T}_T \longrightarrow 0.$$

Consider the dgla given by $\mathcal{T}_\pi^\bullet = \bigoplus_i \mathcal{T}^i$, where $\mathcal{T}^i$ is zero for $i \neq \{0, -1\}$, $\mathcal{T}^{-1} := \mathcal{T}_{X/T}$ and $\mathcal{T}^0 := \mathcal{T}_{X,\pi}$. This dgla $\mathcal{T}_\pi^\bullet$ carries a natural action of $\pi^{-1}\mathcal{O}_T$ and a map

$$\overline{\epsilon} : \mathcal{T}_\pi^\bullet \longrightarrow H^0(\mathcal{T}_\pi^\bullet) = \pi^{-1}(\mathcal{T}_T) \tag{1}$$

given by $d\pi$. The relative de Rham complex $\Omega^\bullet_{X/T} = (\mathcal{O}_X \to \omega_{X/T})$ with $V^0 := \mathcal{O}_X$ and $V^1 := \Omega_{X/T}$ is naturally a *dg*-module $V^\bullet := V^0 \oplus V^1$ for $\mathcal{T}_\pi^\bullet$, which is compatible with the $\pi^{-1}\mathcal{O}_T$ action on both sides.

2.1.3 *Atiyah algebras as $R^0\pi_*$ of dglas*

Let $\pi : X \longrightarrow T$ be a family of curves as before. We discuss the notion of π-algebras following Beilinson-Scechtman [4], which are quasi-isomorphic to extensions of a complex of Atiyah algebras by the de Rham complex $\Omega^\bullet_{X/T}$.

Definition 2.4 ([4, § 1.2.1]). An $\mathcal{O}_S$-Lie algebra $\mathcal{A}^\bullet$ on X is a dgla together with a $\pi^{-1}\mathcal{O}_S$-module structure and a morphism $\epsilon_{\mathcal{A}} : \mathcal{A}^\bullet \longrightarrow \mathcal{T}^\bullet_\pi$ that satisfies the condition $[a, fb] = \overline{\epsilon}_{\mathcal{A}}(a)(f)b + f[a, b]$, where $\overline{\epsilon}_{\mathcal{A}} = \overline{\epsilon} \circ \epsilon_{\mathcal{A}}$ (see (1)). A π-algebra $\mathcal{A}^\bullet$ is an $\mathcal{O}_S$-Lie algebra together with a three term filtration

$$0 = \mathcal{A}^\bullet_{-3} \subset \mathcal{A}^\bullet_{-2} \subset \mathcal{A}^\bullet_{-1} \subset \mathcal{A}^\bullet_0 = \mathcal{A}^\bullet$$

such that the following hold:

1) $[\mathcal{A}^\bullet_i, \mathcal{A}^\bullet_j] \subset \mathcal{A}^\bullet_{i+j}$, $\mathcal{O}_S \cdot \mathcal{A}^\bullet_i \subset \mathcal{A}^\bullet_i$,
2) $\mathcal{A}^\bullet_{-2} \simeq \Omega^\bullet_{X/S}[2]$ as $\mathcal{O}_S$-modules,
3) $\mathcal{A}^\bullet_{-1}/\mathcal{A}^\bullet_{-2}$ is acyclic,
4) $\epsilon_{\mathcal{A}} : \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet/\mathcal{A}^\bullet_{-1} \simeq \mathcal{T}^\bullet_\pi$, and
5) the ad action of $\mathcal{A}^\bullet/\mathcal{A}^\bullet_{-1}$ on $\mathcal{A}^\bullet_{-2}$ coincides with the $\mathcal{T}^\bullet_\pi$-action on $\Omega^\bullet_{X/T}[2]$.

By the above definition of a π-algebra, we get an exact sequence

$$0 \longrightarrow \mathcal{A}^\bullet_{-2} \longrightarrow \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet/\mathcal{A}^\bullet_{-2} \longrightarrow 0. \tag{2}$$

This will be called an Ω-extension of $\mathcal{A}^\bullet/\mathcal{A}^\bullet_{-2}$. Now assume that the map π is proper. Suppose we are given any π-algebra $\mathcal{A}^\bullet$ fitting in an exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet_{X/T}[2] \longrightarrow \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet/\mathcal{A}^\bullet_{-2} \longrightarrow 0. \tag{3}$$

Proposition 2.5 ([4, § 1.2.3]). The short exact sequence

$$0 \longrightarrow \mathcal{O}_T \longrightarrow R^0\pi_*\mathcal{A}^\bullet \longrightarrow \mathcal{T}_T \longrightarrow 0$$

defines an Atiyah algebra on T.

2.1.4 *π-algebras associated to Atiyah algebras*

Let $\pi : X \longrightarrow T$ be a family of curves that are not necessarily projective, and let $\mathcal{A}$ be an R-Atiyah algebra on X. There is a natural π-algebra associated to $\mathcal{A}$. Consider the $\mathcal{O}_T$-Lie algebra $\mathcal{A}^\bullet_\pi$ defined by: $\mathcal{A}^{-1}_\pi := \epsilon^{-1}_{\mathcal{A}}\mathcal{T}_{X/T}$ and $\mathcal{A}^0_\pi := \epsilon^{-1}_{\mathcal{A}}\mathcal{T}_{X,\pi}$. There is a canonical surjective map $\epsilon_{\mathcal{A}} : \mathcal{A}^\bullet_\pi \longrightarrow \mathcal{T}^\bullet_\pi$ whose kernel is $\mathbf{Cone}\,\mathrm{id}_R$.

Definition 2.6. A Ω-extension ${}^\#\mathcal{A}^\bullet$ of $(\mathcal{A}, R)$ is a Ω extension of $\mathcal{A}^\bullet_\pi$ together with an $\mathcal{O}_X$-module structure on ${}^\#\mathcal{A}^{-1}$ such that

- the $\mathcal{O}_X$-action is compatible with the action on $\mathcal{A}^{-1}_\pi$, and
- the component $[\,]_{-1,-1} : {}^\#\mathcal{A}^{-1} \otimes {}^\#\mathcal{A}^{-1} \longrightarrow {}^\#\mathcal{A}^{-2} = \mathcal{O}_X$ is a differential operator along the fibers.

There is a commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_X & = & {}^{\#}\mathcal{A}^{-2} & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{\#}\mathcal{A}^{-1} & \longrightarrow & \mathcal{A}_\pi^{-1} := \mathcal{A}_{X/T} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & {}^{\#}\mathcal{A}^{0} & = & \mathcal{A}_\pi^{0} := \mathcal{A}_{X,\pi} & \longrightarrow & 0. \end{array} \tag{4}$$

Observe that the $\Omega_{X/T}$-extension is, by definition, a π-algebra, where the filtration ${}^{\#}\mathcal{A}^{\bullet}_{-2}$ is given by $\Omega^{\bullet}_{X/T}[2]$.

Later will also need to vary T with respect to S and consider $\mathcal{T}^{\bullet}_{\pi,S}$ where degree -1 term is same as $\mathcal{T}^{\bullet}_{\pi}$ and the degree zero term is $\mathcal{T}_{X,\pi,S} := d\pi^{-1}(\pi^{-1}(\mathcal{T}_{T/S}))$. In the relative set-up one similarly defines $\mathcal{A}^{\bullet}_{X,\pi,S}$ by modifying the zero-th term to be $\mathcal{A}^{-0}_{X,\pi,S} := \epsilon_{\mathcal{A}}^{-1}\mathcal{T}_{X,\pi,S}$. The resulting pushforward $R^0\pi_*\mathcal{A}^{\bullet}_{X,\pi,S}$ is a relative Atiyah algebra satisfying the fundamental exact sequence

$$0 \to \mathcal{O}_T \to R^0\pi_*(\mathcal{A}^{\bullet}_{X,\pi,S}) \to \mathcal{T}_{T/S} \to 0$$

2.2 Principal bundles

2.2.1 *The Ginzburg complex*

We continue with the earlier notation. Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. We will denote by $\kappa_{\mathfrak{g}}$ the normalized Cartan-Killing form on $\mathfrak{g}$ and consider the corresponding isomorphism

$$\nu_{\mathfrak{g}} : \mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}^{\vee} \tag{5}$$

Let $\Pi : P \longrightarrow X$ be a holomorphic principal G bundle; we use the convention that G acts on the right of P. Automorphisms $\mathbf{Aut}(P)|_U$ of P over $U \subset X$ are by definition G-equivariant automorphisms of $\Pi^{-1}(U)$, that is, $F : P|_U \longrightarrow P|_U$ satisfying $F(pg) = F(p)g$ for all $g \in G$; we do not assume that $\Pi \circ F = \Pi$. The group of automorphisms $\mathbf{Aut}(P)|_U$ is generated by the invariant vector fields

$$\mathbf{aut}_U(P) := \left\{ Y \in \Gamma(\Pi^{-1}(U), TP) \mid (R_g)_* Y = Y, \ \ \forall\ g \in G \right\}.$$

Then $\mathbf{aut}(P)$ defines a coherent sheaf of $\mathcal{O}_X$-modules. We refer to the subsheaf $\mathrm{At}_{X/T}(P) \subset \mathbf{aut}(P)$ that projects by $d\Pi$ to $\mathcal{T}_{X/T} \subset TX$ as the *relative Atiyah algebra of* P. We have an exact sequence

$$0 \longrightarrow \mathbf{ad}(P) \longrightarrow \mathrm{At}_{X/T}(P) \longrightarrow \mathcal{T}_{X/T} \longrightarrow 0. \tag{6}$$

We will explain the inclusion map on the left. Recall that a section of $\mathbf{ad}(P)$ is identified with a function $f : P \longrightarrow \mathfrak{g}$ satisfying $f(pg) = \mathbf{Ad}_{g^{-1}} f(p)$. For $Y \in \mathfrak{g}$, let $Y^{\sharp}$ denote the fundamental vector field on P generated by Y. Then $Y^{\sharp}(pg) = (R_g)_*(\mathbf{Ad}_g Y)^{\sharp}(p)$. The map $\mathbf{ad}(P) \longrightarrow \mathbf{aut}(P)$ in (6) is $f \longmapsto Y$, where $Y(p) = f(p)^{\sharp}$. With this definition,

$$Y(pg) = (\mathbf{Ad}_{g^{-1}} f(p))^{\sharp}(pg) = (R_g)_*(f(p))^{\sharp} = Y(p),$$

so Y is invariant and lies in $\mathbf{aut}(P)$ as the kernel of $d\Pi$. The following will be important when we investigate universal bundles.

Remark 2.7. Let $Z(G)$ denote the (finite) center of G, and let $\overline{P} := P/Z(G)$ the associated principal bundle for the adjoint group $\overline{G} := G/Z(G)$. Then there are canonical isomorphisms $\mathbf{ad}(\overline{P}) \simeq \mathbf{ad}(P)$ and $\mathrm{At}_{X/T}(\overline{P}) \simeq \mathrm{At}_{X/T}(P)$.

Dualizing (6) gives a quasi-Lie algebra structure on $\mathrm{At}_{X/T}(P)^\vee$:

$$0 \longrightarrow \Omega_{X/T} \longrightarrow \mathrm{At}_{X/T}(P)^\vee \xrightarrow{j} \mathbf{ad}(P)^\vee \longrightarrow 0. \tag{7}$$

Identify $\mathbf{ad}(P)$ with $\mathbf{ad}(P)^\vee$ using $\nu_{\mathfrak{g}}$ in (5), and denote $\mathrm{At}_{X/T}(P)^\vee$ by $\widetilde{\mathfrak{g}}_P$. We have the following quasi-Lie algebra:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathrm{At}_{X/T}(P)^\vee & \longrightarrow & \mathbf{ad}(P)^\vee & \longrightarrow & 0 \\ & & \| & & \uparrow & & \cong\uparrow{\scriptstyle \nu_{\mathfrak{g}}} & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \widetilde{\mathfrak{g}}_P & \longrightarrow & \mathbf{ad}(P) & \longrightarrow & 0. \end{array} \tag{8}$$

As in Lemma 2.3, associated to $\widetilde{\mathfrak{g}}_P$ is a dgla, $\mathfrak{S}^{\mathrm{i}}_{X/T}(P)$, which we call the *Ginzburg complex* for P. Explicitly,

$$\mathfrak{S}^{\mathrm{i}}_{X/T}(P) = \begin{cases} 0 & \mathrm{i} \neq -2, -1, 0 \\ \mathcal{O}_X & \mathrm{i} = -2 \\ \widetilde{\mathfrak{g}}_P & \mathrm{i} = -1 \\ \mathfrak{g}_P & \mathrm{i} = 0. \end{cases}$$

We will later consider a relative version of it where T varies.

2.2.2 *The Bloch-Esnault complex*

Let $\mathcal{E} \longrightarrow X$ be a holomorphic vector bundle. Like the Ginzburg complex, the Bloch-Esnault complex $\mathcal{B}^\bullet_{X/T}(\mathcal{E})$, [10], is nonzero in degrees $-2, -1$, and 0, with $\mathcal{B}^{-2}_{X/T}(\mathcal{E}) = \mathcal{O}_X$. To define the other terms, let $X \simeq \Delta \subset X \times_T X$ be the relative diagonal, and let $\mathrm{At}_{X/T}(\mathcal{E})$ denote the relative Atiyah algebra of $X \longrightarrow T$. Then $\mathcal{B}^0_{X/T}(\mathcal{E}) = \mathbf{End}(\mathcal{E})$. Set $\mathcal{E}' := \mathcal{E} \otimes \Omega_{X/T}$, and define

$$\widetilde{\mathcal{B}}^{-1}_{X/T}(\mathcal{E}) = \frac{\mathcal{E} \boxtimes \mathcal{E}'(\Delta)}{\mathcal{E} \boxtimes \mathcal{E}'(-\Delta)}$$

on $X \times_T X$. Note that $\mathcal{O}_{X\times_T X}(\Delta)\big|_\Delta = \mathcal{T}_{X/T}$. Then $\mathcal{B}^{-1}_{X/T}(\mathcal{E})$ is defined by pushing out with the trace:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\mathcal{E} \boxtimes \mathcal{E}'}{\mathcal{E} \boxtimes \mathcal{E}'(-\Delta)} & \longrightarrow & \dfrac{\mathcal{E} \boxtimes \mathcal{E}'(\Delta)}{\mathcal{E} \boxtimes \mathcal{E}'(-\Delta)} & \longrightarrow & \dfrac{\mathcal{E} \boxtimes \mathcal{E}'(\Delta)}{\mathcal{E} \boxtimes \mathcal{E}'} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathbf{End}(\mathcal{E}) \otimes \Omega_{X/T} & \longrightarrow & \widetilde{\mathcal{B}}^{-1}_{X/T}(\mathcal{E}) & \longrightarrow & \mathbf{End}(\mathcal{E}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbf{tr}} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathcal{B}^{-1}_{X/T}(\mathcal{E}) & \longrightarrow & \mathbf{End}(\mathcal{E}) & \longrightarrow & 0. \end{array}$$

Here $\mathbf{tr}$ denote the trace $\mathbf{End}(\mathcal{E}) \longrightarrow \mathcal{O}_X$ of endomorphisms.

We will actually need the traceless version $\mathcal{B}^\bullet_{0,X/T}(\mathcal{E})$, where $\mathcal{B}^{-2}_{0,X/T}(\mathcal{E}) = \mathcal{B}^{-2}_{X/T}(\mathcal{E})$ while $\mathcal{B}^0_{0,X/T}(\mathcal{E})$ is defined by the inclusion map into the traceless relative Atiyah algebra and $\mathcal{B}^{-1}_0(\mathcal{E})$ is defined by pulling back the extension $\mathcal{B}^{-1}_{X/T}(\mathcal{E})$ over the sheaf $\mathbf{End}_0(\mathcal{E})$ of traceless endomorphisms.

2.2.3 *Associated bundles*

For any simple Lie algebra $\mathfrak{g}$, recall $\nu_{\mathfrak{g}}$ in (5).

Lemma 2.8. Let $\phi_* : \mathfrak{g} \longrightarrow \mathfrak{s}$ be a nonzero homomorphism of simple Lie algebras. Consider the linear map $\underline{\psi} : \mathfrak{s} \longrightarrow \mathfrak{g}$ given by the following diagram:

$$\begin{array}{ccccc} & & \mathfrak{s}^\vee & \xrightarrow{\phi_*^t} & \mathfrak{g}^\vee \\ & & \cong\uparrow \nu_{\mathfrak{s}} & & \nu_{\mathfrak{g}}^{-1}\downarrow\cong \\ \mathfrak{g} & \xrightarrow{\phi_*} & \mathfrak{s} & \xrightarrow{\underline{\psi}} & \mathfrak{g}. \end{array} \tag{9}$$

Then $\underline{\psi} \circ \phi_* = m_\phi \, \mathrm{Id}_{\mathfrak{g}}$, where as mentioned in the Introduction, m_ϕ is the Dynkin index of ϕ_*. Moreover, replacing $\underline{\psi}$ by $\psi := (m_\phi)^{-1}\underline{\psi}$,

$$\psi \circ \phi_* = \mathrm{Id}_{\mathfrak{g}}\,.$$

Proof. This follows form a direct calculation and the definition of Dynkin index. ■

The following lemma is straightforward.

Lemma 2.9. Let $\psi = \frac{1}{m_\phi}(\nu_{\mathfrak{g}}^{-1} \circ \phi_*^t \circ \nu_{\mathfrak{s}})$ be as in Lemma 2.8, and consider the map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{s}$ defined by $\nu_{\mathfrak{s}}^{-1} \circ \psi^t \circ \nu_{\mathfrak{g}}$:

$$\begin{array}{ccccccc} \mathfrak{g} & \xrightarrow{\nu_{\mathfrak{g}}} & \mathfrak{g}^\vee & \xrightarrow{\psi^t} & \mathfrak{s}^\vee & \xrightarrow{\nu_{\mathfrak{s}}^{-1}} & \mathfrak{s}. \\ & \searrow & & & & \nearrow & \\ & & & \varphi & & & \end{array} \tag{10}$$

Then $\varphi = \frac{1}{m_\phi}\phi_*$.

Let $\phi : G \longrightarrow \mathrm{SL}(V)$ be a nontrivial holomorphic representation and $\mathcal{E}_\phi = P \times_G V$ the corresponding vector bundle associated to a principal G-bundle $\Pi : P \longrightarrow X$. Sections of $\mathcal{E}_\phi$ are functions $\sigma : P \longrightarrow V$ satisfying the condition that $\sigma(pg) = \phi(g^{-1})\sigma(p)$ for all $g \in G$. The adjoint bundle $\mathbf{ad}(P) = P \times_G \mathfrak{g}$ maps to the traceless endomorphism bundle $\mathbf{End}_0(\mathcal{E}_\phi)$ using $\phi_* := d\phi : \mathfrak{g} \longrightarrow \mathfrak{sl}(V)$, and we shall use the same notation ϕ_* for this map $\mathbf{ad}(P) \longrightarrow \mathbf{End}_0(\mathcal{E}_\phi)$. Notice that in this case, the map ψ defined in Lemma 2.8 is G-equivariant, and hence defines a homomorphism $\mathbf{End}_0(\mathcal{E}_\phi) \longrightarrow \mathbf{ad}(P)$. A G-invariant vector field Y on P defines a differential operator on sections by $Y(\sigma)$ $(= d\sigma(Y))$. Invariance of Y guarantees that $Y(\sigma)$ is again equivariant with respect to ϕ, and so defines a section of $\mathcal{E}_\phi$. It is clear that the symbol of this operator is $\Pi_* Y$. Therefore, denoting the relative Atiyah algebra of $\mathcal{E}_\phi \longrightarrow X$ by $\mathrm{At}_{X/T}(\mathcal{E}_\phi)$, we have constructed a map (cf. Atiyah [1, p. 188])

$$\widetilde{\phi} : \mathrm{At}_{X/T}(P) \longrightarrow \mathrm{At}_{X/T}(\mathcal{E}_\phi). \tag{11}$$

The following is a consequence of the above.

Proposition 2.10. For a principal G-bundle $P \longrightarrow X$, a representation $\phi : G \longrightarrow \mathrm{SL}(V)$, and the associated vector bundle $\mathcal{E}_\phi \longrightarrow X$, there is a natural map $\widetilde{\psi} : \mathrm{At}_{X/T}(\mathcal{E}_\phi) \longrightarrow \mathrm{At}_{X/T}(P)$ that makes the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{ad}(P) & \longrightarrow & \mathrm{At}_{X/T}(P) & \longrightarrow & \mathcal{T}_{X/T} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle \widetilde{\phi}} & \searrow^{\mathrm{Id}} & \| & & \\ 0 & \longrightarrow & \mathbf{End}_0(\mathcal{E}_\phi) & \longrightarrow & \mathrm{At}_{X/T}(\mathcal{E}_\phi) & \longrightarrow & \mathcal{T}_{X/T} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & \swarrow & \downarrow{\scriptstyle \widetilde{\psi}} & \swarrow & \| & & \\ 0 & \longrightarrow & \mathbf{ad}(P) & \longrightarrow & \mathrm{At}_{X/T}(P) & \longrightarrow & \mathcal{T}_{X/T} & \longrightarrow & 0. \end{array} \tag{12}$$

Here $\psi : \mathbf{End}_0(\mathcal{E}_\phi) \longrightarrow \mathbf{ad}(P)$ is induced from the map ψ constructed in Lemma 2.8 and $\widetilde{\psi}$ is the map obtained by pushing out the exact sequence of $\mathrm{At}_{X/T}(\mathcal{E}_\phi)$ via ψ.

2.2.4 *Relating the Ginzburg and Bloch-Esnault complexes*

We now compare the Ginzburg complex for a principal bundle P with the Bloch-Esnault complex for the bundle $\mathcal{E}_\phi$ associated to P via a nontrivial representation $\phi : G \longrightarrow \mathrm{SL}(V)$. Identifying the endomorphism bundle $\mathbf{End}_0(\mathcal{E}_\phi)$ with its dual using the trace homomorphism, consider the map

$$\phi^{\mathrm{t}}_* : \mathbf{End}_0(\mathcal{E}_\phi) \cong (\mathbf{End}_0(\mathcal{E}_\phi))^\vee \longrightarrow \mathbf{ad}(P)^\vee.$$

Let $\mathcal{K}_\phi = \ker \phi^{\mathrm{t}}_*$. The following is important for us.

Proposition 2.11. There is a lift of the inclusion $\mathcal{K}_\phi \longrightarrow \mathbf{End}_0(\mathcal{E}_\phi)$:

$$\begin{array}{ccc} & & \mathcal{B}^{-1}_{0,X/T}(\mathcal{E}_\phi) \\ & \nearrow & \downarrow \\ \mathcal{K}_\phi & \longrightarrow & \mathbf{End}_0(\mathcal{E}_\phi) \end{array} \tag{13}$$

Proof. We begin by describing a general situation. Namely, for bundles $\mathcal{E}, \mathcal{F}$, we find a local lifting Φ_α of the map

$$\frac{\mathcal{E} \boxtimes \mathcal{F}'(\Delta)}{\mathcal{E} \boxtimes \mathcal{F}'(-\Delta)} \longrightarrow \frac{\mathcal{E} \boxtimes \mathcal{F}'(\Delta)}{\mathcal{E} \boxtimes \mathcal{F}'} \simeq \mathbf{Hom}(\mathcal{F}, \mathcal{E}) \longrightarrow 0.$$

Choose local trivializations of $X \longrightarrow T$ and coordinate neighborhoods U_α, z_α on a fixed fiber. We set $\varphi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}$, so $\varphi'_{\alpha\beta} = dz_\alpha/dz_\beta$. The lift is defined by choosing a (holomorphic) connection ∇_α on $\mathcal{F}^*$.

Given sections u and v of $\mathcal{E}$ and $\mathcal{F}^*$ respectively on U_α, define on $U_\alpha \times U_\alpha$

$$\Phi_\alpha(u(q) \otimes v(q)) = \frac{u(p) \boxtimes v(q) dz_\alpha(q)}{z_\alpha(p) - z_\alpha(q)} + u(p) \boxtimes \nabla_\alpha v(q) \mod \mathcal{I}_\Delta. \tag{14}$$

To show that this is well defined, let $f(p, q) \in \mathcal{I}_\Delta$. Multiplying on the right-hand side, we have

$$\left(\frac{f(p,q) dz_\alpha(q)}{z_\alpha(p) - z_\alpha(q)} + \frac{\partial f}{\partial q}(p, q)\right) u(p) \boxtimes v(q) \mod \mathcal{I}_\Delta. \tag{15}$$

Since

$$f(p, q) = \frac{\partial f}{\partial q}(p, p)(z_\alpha(q) - z_\alpha(p)) \mod \mathcal{I}^2_\Delta,$$

we see that (15) vanishes, and so (14) gives a well-defined lift. Set $\mathscr{A}_{\alpha\beta} = \nabla_\alpha - \nabla_\beta$ and $\Theta_{\alpha\beta} = (\varphi''_{\alpha\beta}/\varphi'_{\alpha\beta}) dz_\beta$. Then $\mathscr{A}_{\alpha\beta} \in \mathbf{End}(\mathcal{F}^*) \otimes \Omega_{X/T}$ is a 1-cocycle representing the Atiyah class of $\mathcal{F}^*$, and $\Theta_{\alpha\beta} \in \Omega_{X/T}$ is a cocycle for the affine structure (cf. [14, p. 164]).

Notice that

$$\begin{aligned} z_\alpha(p) - z_\alpha(q) &= \varphi_{\alpha\beta}(z_\beta(p)) - \varphi_{\alpha\beta}(z_\beta(q)) \\ &= \varphi'_{\alpha\beta}(z_\beta(q))(z_\beta(p) - z_\beta(q)) + \frac{1}{2}\varphi''_{\alpha\beta}(z_\beta(q))(z_\beta(p) - z_\beta(q))^2 + \cdots \end{aligned}$$

from which we have

$$\frac{dz_\alpha(q)}{z_\alpha(p) - z_\alpha(q)} = \frac{dz_\beta(q)}{z_\beta(p) - z_\beta(q)} - \frac{1}{2}\Theta_{\alpha\beta} \mod \mathcal{I}_\Delta.$$

The cocycle $\Phi_{\alpha\beta} = \Phi_\alpha - \Phi_\beta \in \mathbf{Hom}\left(\mathbf{Hom}(\mathcal{F}, \mathcal{E}), \mathbf{Hom}(\mathcal{F}, \mathcal{E}) \otimes \Omega_{X/T}\right)$ is then given by

$$\Phi_{\alpha\beta}(u \otimes v) = u \otimes \mathscr{A}_{\alpha\beta} v - \frac{1}{2} u \otimes v \otimes \Theta_{\alpha\beta}.$$

In the case of $\mathcal{F} = \mathcal{E}$, we may write simply

$$\Phi_{\alpha\beta} : \mathbf{End}(\mathcal{E}) \longrightarrow \mathbf{End}(\mathcal{E}) \otimes \Omega_{X/T} : \beta \longmapsto \beta\left(\mathscr{A}_{\alpha\beta} - \frac{1}{2}\Theta_{\alpha\beta} \cdot \mathrm{I}\right).$$

The extension class for $\mathcal{B}_0^{-1}(\mathcal{E})$ is then given by the trace of endomorphisms

$$\mathbf{tr}\,\Phi_{\alpha\beta} : \mathbf{End}_0(\mathcal{E}) \longrightarrow \Omega_{X/T}, \quad \beta \longmapsto \mathbf{tr}(\beta \mathscr{A}_{\alpha\beta}). \tag{16}$$

since $\mathbf{tr}\,\beta = 0$.

Finally, to complete the proof we must show that if $\mathcal{E} = \mathcal{E}_\phi$, then $\mathbf{tr}\,\Phi_{\alpha\beta}$ vanishes on $\mathcal{K}_\phi$. When $\mathcal{E} = \mathcal{E}_\phi$, we may choose local holomorphic connections on P, so that $\mathscr{A}_{\alpha\beta}$ is in the image of ϕ_*. But $\mathcal{K}_\phi$ consists precisely of endomorphisms that are orthogonal to these under the trace pairing. Thus, the proposition follows from the expression in (16). ∎

It is shown in [16, Thm. B.2.6] that the exact sequence for $\mathcal{B}^{-1}_{0,X/T}(\mathcal{E})$ is dual to the (traceless) Atiyah algebra sequence for $\mathcal{E}$ using the trace map. In the case of $\mathcal{E}_\phi$, we have the natural map $\widetilde{\phi}_*^t : \mathrm{At}_{X/T}(\mathcal{E}_\phi)^\vee \longrightarrow \mathrm{At}_{X/T}(P)^\vee$. Then, by Proposition 2.11 we have the following:

Corollary 2.12. Let $P \longrightarrow X$ be a principal G-bundle, $\phi : G \longrightarrow \mathrm{SL}(V)$ a holomorphic representation, and $\mathcal{E}_\phi \longrightarrow X$ the associated vector bundle. Then the degree -1 part of the Bloch-Esnault complex $\mathcal{B}^{-1}_{X/T}(\mathcal{E}_\phi)$ is the pullback of the -1 part of the Ginzburg complex in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathcal{B}^{-1}_{0,X/T}(\mathcal{E}_0) & \longrightarrow & \mathbf{End}_0(\mathcal{E}_\phi) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \widetilde{\phi}_*^t} & & \downarrow{\scriptstyle \phi_*^t} & & \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathfrak{S}^{-1}_{X/T}(P) & \longrightarrow & \mathbf{ad}(P) & \longrightarrow & 0.
\end{array}$$

Consider the map $\widetilde{\psi} : \mathrm{At}_{X/T}(\mathcal{E}_\phi) \longrightarrow \mathrm{At}_{X/T}(P)$ obtained in Proposition 2.10 along with its dual $\widetilde{\psi}^* : \mathrm{At}_{X/T}(P)^\vee \longrightarrow \mathrm{At}_{X/T}(\mathcal{E}_\phi)$. We summarize the above discussions in the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathfrak{S}^{-1}_{X/T}(P) & \longrightarrow & \mathbf{ad}(P) & \longrightarrow & 0 \\
 & & & \searrow & \cong\downarrow & \nearrow{\scriptstyle \nu_{\mathfrak{g}}} & & & \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathrm{At}_{X/T}(P)^\vee & \longrightarrow & \mathbf{ad}(P)^\vee & \longrightarrow & 0 \\
 & & {\scriptstyle m_\phi \mathrm{Id}}\downarrow & & {\scriptstyle m_\phi \mathrm{Id}}\downarrow & {\scriptstyle \widehat{\phi}}\searrow & {\scriptstyle m_\phi \widetilde{\psi}^*}\downarrow & {\scriptstyle -\phi}\swarrow & {\scriptstyle m_\phi \psi^t}\downarrow \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathrm{At}_{X/T}(\mathcal{E}_\phi)^\vee & \longrightarrow & \mathbf{End}_0(\mathcal{E}_\phi)^\vee & \longrightarrow & 0 \\
 & & \| & \searrow & \cong\downarrow & & \downarrow{\scriptstyle -\nu^{-1}_{\mathfrak{sl}(r)}} & & \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathcal{B}^{-1}_{0,X/T}(\mathcal{E}_\phi) & \longrightarrow & \mathbf{End}_0(\mathcal{E}_\phi) & \longrightarrow & 0.
\end{array} \tag{17}$$

In (17), $\widehat{\phi}$ is just the map obtained by composition in the middle column of (17). Observe that

$$\nu^{-1}_{\mathfrak{sl}(r)} \circ m_\phi \psi^* \circ \nu_{\mathfrak{g}} = m_\phi(\nu^{-1}_{\mathfrak{sl}(r)} \circ \psi^* \circ \nu_{\mathfrak{g}}) = m_\phi(\frac{1}{m_\phi}\phi) = \phi. \tag{18}$$

Composing the maps that appear in the above diagram we get the following.

Proposition 2.13. The map $\phi : \mathbf{ad}(P) \longrightarrow \mathbf{End}_0(\mathcal{E}_\phi)$ extends to $\widehat{\phi} : \mathfrak{S}^{-1}_{X/T}(P) \longrightarrow \mathcal{B}^{-1}_{0,X/T}(\mathcal{E}_\phi)$ which restricts to multiplication by the Dynkin index m_ϕ on $\Omega_{X/T}$. Taking push-forward by $R^1\pi$ yields the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_T & \longrightarrow & R^1\pi_*(\mathfrak{S}^{-1}_{X/T}(P)) & \longrightarrow & R^1\pi_*(\mathbf{ad}(P)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m_\phi \,\mathrm{Id}} & & \downarrow{\scriptstyle \widehat{\phi}} & & \downarrow{\scriptstyle -\phi} & & \\ 0 & \longrightarrow & \mathcal{O}_T & \longrightarrow & R^1\pi_*(\mathcal{B}^{-1}_{0,X/T}(\mathcal{E}_\phi)) & \longrightarrow & R^1\pi_*(\mathbf{End}_0(\mathcal{E}_\phi)) & \longrightarrow & 0. \end{array} \tag{19}$$

2.2.5 *The relative extension class*

In this section we consider the special case where $X \longrightarrow T$ is simply a product $X = C \times T$. Let $P \longrightarrow X$ be a principal G-bundle such that the restriction of it to $C \times \{t\}$ is regularly stable for every t. We wish to compute the extension class of the top (and hence also the bottom) row of (19). Since we assume the curve is fixed, for future reference we call this the *relative extension class*. To state the result, let M^{rs}_G denote the moduli space of regularly stable bundles on C. Since this a coarse moduli space, there is a morphism $\varphi : T \longrightarrow M^{rs}_G$. By the deformation theory of principal G-bundles there is a homomorphism $\mathcal{T}T \longrightarrow R^1\pi_*(\mathbf{ad}(P))$. Via the Dolbeault isomorphism, we have

$$H^1(T, (R^1\pi_*(\mathbf{ad}(P))^\vee) \longrightarrow H^1(T, (\mathcal{T}T)^\vee) \simeq H^{1,1}_{\overline{\partial}}(T)$$

and so the extension class of (19) defines a class in $H^{1,1}_{\overline{\partial}}(T)$. On the other hand, there is another natural class $\varphi^*[\Theta] \in H^{1,1}_{\overline{\partial}}(T)$ due to Atiyah-Bott, which we define below. The result is then:

Theorem 2.14. Let $X = C \times T$ as above. The image of the extension class of (19) in $H^{1,1}_{\overline{\partial}}(T)$ is exactly the pullback $\varphi^*[\Theta]$ of the Atiyah-Bott-Narasimhan-Goldman form.

We first define Θ. Under the assumptions, the direct image $R^1\pi_*(\mathbf{ad}(P)) \longrightarrow T$ is locally free and its fiber at $t \in T$ isomorphic to $H^1(C, \mathbf{ad}(P_t))$, where $P_t = P|_{C\times\{t\}}$. By Dolbeault isomorphism, this is isomorphic to $H^{0,1}_{\overline{\partial}}(C, \mathbf{ad}(P_t))$.

Fix a maximal compact subgroup $K \subset G$. Let ρ_K be the Cartan involution of $\mathfrak{g}$ that fixes Lie(K) and acts on Lie$(K)^\perp$ as multiplication by -1. The Narasimhan-Seshadri-Ramanathan theorem [15, 17] asserts the existence of a C^∞ reduction of structure group $P^t_K \subset P_t$ of P_t to K, satisfying the condition that the associated Chern connection on P_t is flat. The Chern connection defines harmonic representatives $\mathcal{H}^{0,1}(C, \mathbf{ad}(P_t))$ of the Dolbeault group $H^{0,1}_{\overline{\partial}}(C, \mathbf{ad}(P_t))$. The Cartan involution ρ_K produces a conjugate linear involution ρ' of $\mathbf{ad}(P_t) \otimes T^*C \otimes \mathbb{C}$ that exchanges $\Omega^{1,0}_C(\mathbf{ad}(P_t))$ and $\Omega^{0,1}_C(\mathbf{ad}(P_t))$ and preserves the harmonic forms; this ρ' is also called the Hodge $*$ operator. Then a hermitian inner product on $\mathcal{H}^{0,1}(C, \mathbf{ad}(P_t))$ is given by

$$\langle \alpha, \beta \rangle = \sqrt{-1}\int_C (\alpha \wedge \rho'(\beta))_{\mathfrak{g}},$$

where α and β are harmonic representatives. The almost complex structure on M^{rs}_G at $\varphi(t)$ is given by ρ' on harmonic 1-forms. The tangent space to the space of flat K-connections at the point corresponding to the Narasimhan-Seshadri-Ramanathan connection on P^t_K coincides with $H^1(C, \underline{\mathbf{ad}}(P^t_K))$, where $\underline{\mathbf{ad}}(P^t_K)$ is the local system. We note that $H^1(C, \underline{\mathbf{ad}}(P^t_K))$ is identified with $H^{0,1}_{\overline{\partial}}(C, \mathbf{ad}(P))$ by the map $\alpha \longmapsto \alpha - \sqrt{-1}\rho(\alpha)$. The almost complex structure on $H^1(C, \underline{\mathbf{ad}}(P^t_K))$ is given by ρ'. For $a, b \in H^1(C, \underline{\mathbf{ad}}(P^t_K))$, the Riemannian metric is given by

$$\langle a, b \rangle = -\int_C (a \wedge *b)_{\mathfrak{g}} = 2\mathrm{Re}\,\langle \alpha, \beta \rangle,$$

where $\alpha, \beta \in H^{0,1}_{\overline{\partial}}(C, \mathbf{ad}(P))$ correspond to a, b respectively. Finally, the *Atiyah-Bott-Narasimhan-Goldman* symplectic form is given by $\Theta(a, b) = 2\mathrm{Im}\,\langle \alpha, \beta \rangle$. It is closed and of type $(1, 1)$, and so defines a class in $H^{1,1}_{\overline{\partial}}(T)$.

Proof of Theorem 2.14. First, we note the following:

1) $\mathrm{At}(P)^\vee$ is the sheaf of invariant 1-forms on P, that is, $\varphi \in \Gamma(\pi^{-1}(U), T^*P)$ such that $R_g^*\varphi = \varphi$ for all $g \in G$.
2) $\mathbf{ad}(P)^\vee$ is the sheaf of maps $P \longrightarrow \mathfrak{g}^*$ that are equivariant with respect to the co-adjoint action, that is, $f(pg) = f(p) \circ \mathbf{Ad}_g$.
3) The map j is defined explicitly by: $j(\varphi)(p)(X) = \varphi(X_p^\sharp)$, for $X \in \mathfrak{g}$. We have

$$\begin{aligned} j(\varphi)(pg)(X) &= \varphi(pg)(X_{pg}^\sharp) = \varphi(pg)\left((R_g)_*(\mathbf{Ad}_g X)_p^\sharp\right) \\ &= R_g^*\varphi((\mathbf{Ad}_g X)_p^\sharp) = j(\varphi) \circ \mathbf{Ad}_g(X) \end{aligned}$$

consequently $j(\varphi)$ satisfies the correct invariance.

A relative holomorphic connection ω on P gives a holomorphic splitting of (7). Explicitly, if φ is a local section of $\mathbf{ad}(P)^\vee$, then define $\omega(\varphi)$ in $\mathrm{At}(P)^\vee$, by $\omega(\varphi)(Y) = \varphi(\omega(Y))$. Notice that

$$j(\omega(\varphi))(p)(Y) = \omega(\varphi)(Y_p^\sharp) = \varphi(\omega(Y_p^\sharp)) = \varphi(Y),$$

so this map is a splitting of the sequence in (7). Let M_G^{dR} denote the moduli space of holomorphic G-connections on a fixed curve C. Let $M_G^{dR,rs}$ denote the open subset where the underlying bundle P is regularly stable. Hence, there is the forgetful map $p : M_G^{dR,rs} \longrightarrow M_G^{rs}$, and this is a submersion. Therefore, we can find local holomorphic sections. With this understood, let $\{\alpha_j\}$ be a local holomorphic frame for the bundle $R^1\pi_*(\mathbf{ad}(P)^\vee)$ over an open set $U \subset T$, and let $\{\alpha_j^*\}$ be the dual frame. From the previous paragraph, after shrinking U, we may find a holomorphic family of relatively flat connections ω_U^{hol} for the restriction of P to $C \times U$. Then a lift of the identity endomorphism of $R^1\pi_*(\mathbf{ad}(P)^\vee)$ to $R^1\pi_*(\mathrm{At}(P)^\vee) \otimes (R^1\pi_*(\mathbf{ad}(P)^\vee))^\vee$ is given by

$$s_U = \sum_j \omega_U^{hol}(\alpha_j) \otimes \alpha_j^*.$$

For open subsets U, V, the difference $\sigma_{UV} = s_U - s_V$ is valued in $R^1\pi_*(\Omega_C) \otimes (R^1\pi_*(\mathbf{ad}(P)^*))^*$, and the 1-cocycle $\{\sigma_{UV}\}$ represents the extension class.

We now shift to the Dolbeault picture. First, the identification $R^1\pi_*(\Omega_C) \simeq \mathbb{C}$ is given by integration along the fiber C. Next, using the Killing form we identify $\mathbf{ad}\,P \simeq (\mathbf{ad}\,P)^\vee$. Consider the map $\mathcal{T}T \longrightarrow R^1\pi_*(adP)$; then as discussed above the extension class defines via the Dolbeault isomorphism a $\overline{\partial}$-closed (1, 1)-form on T. Let $P_K \subset P$ be the reduction of structure group given by the Narasimhan-Seshadri-Ramanathan theorem. The Chern connection on P_K extends to a connection ω_A on P that restricts to a flat connection on each P_t, although ω_A does not vary holomorphically in $t \in T$. We can write $\omega_U^{hol} = \omega_A + B_U$, where B_U is an invariant $\mathfrak{g}$-valued relative 1-form on P that vanishes on vertical vector fields. Let $b_U = \sum_j B_U^{hol}(\alpha_j) \otimes \alpha_j^*$. Then since ω_A is globally defined, we have $s_U - s_V = b_U - b_V$. Hence, $\{b_U\}$ gives a C^∞ trivialization of the 1-cocycle $\{\sigma_{UV}\}$. By definition, the Dolbeault representative of the extension class is therefore given by the global (0, 1)-form $\{\overline{\partial} b_U\}$. Now, since ω_U^{hol} is holomorphic, the extension class is represented by

$$\Omega_A = -\sum_j \overline{\partial}\omega_A(\alpha_j) \otimes \alpha_j^*.$$

We calculate this form at a given point $t \in U$. Choose local holomorphic coordinates $s_1, \cdots, s_N$ centered at the point $[P_t]$ in M_G^{rs} corresponding to t. We may so arrange that the holomorphic sections $\alpha_j(s_1, \cdots, s_N)$ of $\mathcal{T}M_G^{rs}$ in a neighborhood $[P_t]$ satisfy the condition $\alpha_j(0) = \partial/\partial s_j$. Thus, we have also $\alpha_j^*(0) = ds_j$. Each $\alpha_j(s)$ defines a Dolbeault class in $H^{0,1}(C, \mathbf{ad}\,\mathcal{P}_s)$, for which we use the same notation. Let $\rho_s(\alpha_j(s))$ be the hermitian conjugate with respect to ρ_s. With this notation, the Chern connection is

given by

$$\omega_A(s) = \omega_A(0) + \sum t_j(s)\alpha_j(s) + \sum \bar{t}_j(s)\rho_s(\alpha_j(s)) \mod \mathcal{T}^\vee M_G^{rs},$$

where $t_j(0) = 0, \partial t_j/\partial s_i(0) = \delta_{ij}$ and $\partial t_j/\partial \bar{s}_i(0) = 0$. We therefore have

$$\omega_A(\alpha_k) = \omega_A(0)(\alpha_k) + \sum \bar{t}_j(\rho(\alpha_j), \alpha_k)$$
$$\bar{\partial}\omega_A(\alpha_k)(0) = \sum (\rho(\alpha_j), \alpha_k) d\bar{s}_j.$$

Recalling that $H^1(C, K_C) \simeq \mathbb{C}$ is gotten by integration over C we have that Ω_A is the pullback of the form

$$\sum \int_C (\rho(\alpha_j), \alpha_k) ds_k \wedge d\bar{s}_j = i \sum \langle \alpha_k, \alpha_j \rangle ds_k \wedge d\bar{s}_j$$

and this is precisely the Atiyah-Bott-Narasimhan-Goldman symplectic form. ■

3 Determinant of Cohomology and Beilinson-Schechtman Classes

In this section we reinterpret Theorem 2.14 in terms of the quasi-isomorphisms between the Bloch-Esnault complex, the trace complex of Beilinson-Schechtman, and the Ginzburg complex. Finally, we related all of these to the Atiyah class of the determinant of coholomology line bundle.

3.1 Definitions

Recall the notation $\mathcal{E}' := \mathcal{E} \otimes \Omega_{X/T}$. The sheaf $\mathcal{D}^{\leq 1}(\mathcal{E})$ of first order differential operators on $\mathcal{E}$ can be identified with $\frac{\mathcal{E}\boxtimes\mathcal{E}'(2\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'}$. There is an exact sequence of sheaves

$$0 \longrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'(\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'} \longrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'(2\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'} \longrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'(2\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'(\Delta)} \longrightarrow 0. \tag{20}$$

Similarly, we use another short exact sequence from [4]:

$$0 \longrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'}{\mathcal{E}\boxtimes\mathcal{E}'(-\Delta)} \longrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'(2\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'(-\Delta)} \longrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'(2\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'} \longrightarrow 0. \tag{21}$$

Since we have $\mathrm{At}_{X/T}(\mathcal{E}) \subset \mathcal{D}^{\leq 1}_{X/T}(\mathcal{E})$, pulling back (21) we get a quasi-Lie algebra that fits into the following short exact sequence:

$$0 \longrightarrow \mathbf{End}(\mathcal{E}) \otimes \Omega_{X/T} \longrightarrow {}^{\mathrm{tr}}\widetilde{\mathcal{A}}_{X/T}(\mathcal{E})^{-1} \longrightarrow \mathrm{At}_{X/T}(\mathcal{E}) \longrightarrow 0. \tag{22}$$

Pushing it forward via the trace map $\mathbf{tr} : \mathbf{End}(\mathcal{E}) \longrightarrow \mathcal{O}_X$ we get that

$$0 \longrightarrow \Omega_{X/T} \longrightarrow {}^{\mathrm{tr}}\mathcal{A}_{X/T}(\mathcal{E})^{-1} \longrightarrow \mathrm{At}_{X/T}(\mathcal{E}) \longrightarrow 0. \tag{23}$$

Recall that $\widetilde{\mathcal{B}}^{-1}_{X/T}(\mathcal{E}) := \frac{\mathcal{E}\boxtimes\mathcal{E}'(\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'(-\Delta)}$, and define $\widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E})$ to be the trace free version. Now consider the pushforwards of $\widetilde{\mathcal{B}}^{-1}_{X/T}(\mathcal{E})$ and $\widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E})$ via $\mathbf{tr}$, and denote them by $\mathcal{B}^{-1}_{X/T}(\mathcal{E})$ and $\mathcal{B}^{-1}_{0,X/T}(\mathcal{E})$ respectively. The natural inclusion map

$$\mathbf{End}(\mathcal{E}) \cong \frac{\mathcal{E}\boxtimes\mathcal{E}'(\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'} \hookrightarrow \frac{\mathcal{E}\boxtimes\mathcal{E}'(2\Delta)}{\mathcal{E}\boxtimes\mathcal{E}'}$$

fits the above objects into the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}(\mathcal{E})\otimes\Omega_{X/T} & \longrightarrow & {}^{\mathrm{tr}}\widetilde{\mathcal{A}}_{X/T}(\mathcal{E})^{-1} & \longrightarrow & \mathrm{At}_{X/T}(\mathcal{E}) & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{End}(\mathcal{E})\otimes\Omega_{X/T} & \longrightarrow & \widetilde{\mathcal{B}}^{-1}_{X/T}(\mathcal{E}) & \longrightarrow & \mathrm{End}(\mathcal{E}) & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{End}(\mathcal{E})\otimes\Omega_{X/T} & \longrightarrow & \widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E}) & \longrightarrow & \mathrm{End}_0(\mathcal{E}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{tr}} & \searrow^{\mathrm{tr}} & \downarrow{\scriptstyle \mathrm{tr}_*} & \searrow^{\mathrm{tr}_*} & \| & & \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathcal{B}^{-1}_{0,X/T}(\mathcal{E}) & \longrightarrow & \mathrm{End}_0(\mathcal{E}) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathcal{B}^{-1}_{X/T}(\mathcal{E}) & \longrightarrow & \mathrm{End}(\mathcal{E}) & \longrightarrow & 0.
\end{array} \tag{24}$$

3.2 Relative set-up

Now, consider the relative set-up. Therefore, we have a smooth scheme T over S with connected fibers, and $\pi : X \longrightarrow T$ is a family of connected smooth curves of genus g. We have short the exact sequence:

$$0 \longrightarrow \mathcal{T}_{X/T} \longrightarrow \mathcal{T}_{X/S} \longrightarrow \pi^*\mathcal{T}_{T/S} \longrightarrow 0.$$

Now as $\mathcal{O}_T$ modules, $\pi^{-1}(\mathcal{T}_{T/S}) \subset \pi^*(\mathcal{T}_{T/S})$. Define $\mathcal{T}_{X,\pi,S} := d\pi^{-1}(\pi^{-1}(\mathcal{T}_{T/S}))$. Then we have

$$0 \longrightarrow \mathcal{T}_{X/T} \longrightarrow \mathcal{T}_{X,\pi,S} \xrightarrow{d\pi} \pi^{-1}\mathcal{T}_{T/S} \longrightarrow 0. \tag{25}$$

Let $\mathbf{At}_{X,\pi,S}(\mathcal{E})$ denote the Atiyah algebra satisfying the fundamental exact sequence

$$0 \longrightarrow \mathrm{End}(\mathcal{E}) \to \mathbf{At}_{X,\pi,S}(\mathcal{E}) \longrightarrow \mathcal{T}_{X,\pi,S} \longrightarrow 0$$

Observe that $\mathbf{At}_{X/T}(\mathcal{E}) \subset \mathbf{At}_{X,\pi,S}(\mathcal{E}) \subset \mathbf{At}_{X/S}(\mathcal{E})$.

Definition 3.1. The *Beilinson-Schechtman trace complex* ${}^{\mathrm{tr}}\mathcal{A}_{X,\pi,S}(\mathcal{E})^\bullet$ is the $\Omega_{X/T}$-extension of the dgla associated the quasi-Lie algebra ${}^{\mathrm{tr}}\mathcal{A}_{X/T}(\mathcal{E})^{-1}$, that is, ${}^{\mathrm{tr}}\mathcal{A}_{X,\pi,S}(\mathcal{E})^{(0)} = \mathbf{At}_{X,\pi,S}(\mathcal{E})$, ${}^{\mathrm{tr}}\mathcal{A}_{X,\pi,S}(\mathcal{E})^{-1} = {}^{\mathrm{tr}}\mathcal{A}_{X/T}(\mathcal{E})^{-1}$ and ${}^{\mathrm{tr}}\mathcal{A}_{X,\pi,S}(\mathcal{E})^{(-2)} = \mathcal{O}_X$.

Throughout the rest of this section we will have the assumption that there is a splitting of the short exact sequence in (25).

This condition holds for example in the case of fiber products. We then use the splitting of (25) to pull-back the Atiyah algebra $\mathbf{At}_{X,\pi,S}(\mathcal{E})$ further via $\pi^{-1}\mathcal{T}_{T/S}$ to obtain $\mathcal{B}_{X,\pi,S}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}\,\mathcal{E} & \longrightarrow & \mathbf{At}_{X,\pi,S}(\mathcal{E}) & \longrightarrow & \mathcal{T}_{X,\pi,S} & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{End}\,\mathcal{E} & \longrightarrow & \mathcal{B}^0_{X,\pi,S}(\mathcal{E}) & \longrightarrow & \pi^{-1}\mathcal{T}_{T/S} & \longrightarrow & 0.
\end{array}$$

Let $\mathcal{B}^0_{0,X,\pi,S}(\mathcal{E})$ be the pushout of the exact sequence defining $\mathcal{B}^0_{X,\pi,S}(\mathcal{E})$ via the quotient homomorphism $\mathbf{End}(\mathcal{E}) \longrightarrow \mathbf{End}_0(\mathcal{E})$.

Definition 3.2. The *Bloch-Esnault complex* $\mathcal{B}^\bullet_{X,\pi,S}(\mathcal{E})$ is the three-term complex consisting of the locally free sheaves $\mathcal{B}^0_{X,\pi,S}(\mathcal{E})$, $\mathcal{B}^{-1}_{X/T}(\mathcal{E})$ and $\mathcal{B}^{-2}_{X,\pi,S}(\mathcal{E}) \cong \mathcal{O}_X$ in degrees 0, −1 and −2 respectively, and zero otherwise.

We will denote by $\mathcal{B}^\bullet_{0,X/T,S}$ the traceless Bloch-Esnault complex as considered in [19] and [16].

3.3 Determinant of cohomology

Let $\pi : X \to T$ be a family of smooth projective curves and let $\mathcal{E}$ be a vector bundle X. Then the object $R\pi_*\mathcal{E}$ in the bounded derived category of T is represented by a two term complex $\mathcal{E}_0 \to \mathcal{E}_1$. The determinant of cohomology $\mathcal{L}$ up to isomorphism is defined by

$$\mathcal{L} := \bigwedge^{top} \mathcal{E}_1 \otimes \bigwedge^{top} \mathcal{E}_0^{\vee}$$

We refer the reader to [9, § 6.3] for more details on determinant of cohomology $\mathcal{L}$. We recall the following result of Beilinson-Schechtman [4] that connects the Atiyah algebra of $\mathbf{At}_{X/T}(\mathcal{L})$ and the trace complex of $\mathcal{E}$ and also a result of Bloch-Esnault [10] connecting the trace complex with the Bloch-Esnault complex.

Proposition 3.3. The Atiyah algebra $\mathbf{At}_{T/S}(\mathcal{L}^{-1})$ is isomorphic to the relative Atiyah algebra $R^0\pi_*({}^{tr}\mathcal{A}_{X,\pi,S}(\mathcal{E})^\bullet)$. If $\mathcal{B}^\bullet_{X,\pi,S}(\mathcal{E})$ is the relative Bloch-Esnault complex associated to a family of curves $X \longrightarrow T$ parametrized by $T \to S$ such that the exact sequence in Equation (25) splits, then $\mathcal{B}^\bullet_{X,\pi,S}(\mathcal{E})$ is quasi-isomorphic to the trace complex ${}^{tr}\mathcal{A}_{X,\pi,S}(\mathcal{E})^\bullet$.

3.4 Relative Ginzburg complex

Let $\mathcal{P}$ be a principal G-bundle on $X \longrightarrow T \longrightarrow S$, and consider the Atiyah algebra $\mathbf{At}_{X,\pi}(\mathcal{P})$ with the fundamental exact sequence

$$0 \longrightarrow \mathfrak{g}_{\mathcal{P}} \longrightarrow \mathbf{At}_{X,\pi,S}(\mathcal{P}) \longrightarrow \mathcal{T}_{X,\pi,S} \longrightarrow 0.$$

We then use the splitting of the exact sequence in (25) to pull-back further to define $\mathfrak{S}^0(\mathcal{P})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g}_{\mathcal{P}} & \longrightarrow & \mathbf{At}_{X,\pi,S}(\mathcal{P}) & \longrightarrow & \mathcal{T}_{X,\pi,S} & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathfrak{g}_{\mathcal{P}} & \longrightarrow & \mathfrak{S}^0_{X,\pi,S}(\mathcal{P}) & \longrightarrow & \pi^{-1}\mathcal{T}_{T/S} & \longrightarrow & 0. \end{array}$$

We consider the following three-term complex $\mathfrak{S}^\bullet_X, \pi, S(\mathcal{P})$ that will be referred to as the *relative Ginzburg complex*:

$$\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P}) := \begin{cases} \mathcal{O}_X \text{ if } i = -2, \\ \mathfrak{S}^{-1}_{X/T}(\mathcal{P}) \text{ if } i = -1, \\ \mathfrak{S}^0_{X,\pi,S}(\mathcal{P}) \text{ ifi } = 0, \\ 0, \text{ otherwise.} \end{cases}$$

Let $\phi : G \longrightarrow SL(V)$ be a holomorphic representation of dimension r, and let $\mathcal{E}_\phi$ be the associated vector bundle. Then one recovers the Bloch-Esnault complex $\mathcal{B}^\bullet_{0,X,\pi,S}(\mathcal{E}_\phi)$.

We have the following commutative diagram in which the horizontal map is the relative Ginzburg quasi-Lie algebra associated to a relative principal bundle $\mathcal{P}$

$$\begin{array}{ccccccccc} & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \mathfrak{S}^{-1}_{X/T}(\mathcal{P}) & \longrightarrow & \mathfrak{g}_{\mathcal{P}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathfrak{S}^0_{X,\pi,S}(\mathcal{P}) & = & \mathfrak{S}^0_{X,\pi,S}(\mathcal{P}). & & \end{array} \tag{26}$$

The above diagram can be written as a short exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet_{X/T}[2] \longrightarrow \mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P}) \longrightarrow \mathfrak{R}^\bullet(\mathcal{P}) := (\mathfrak{g}_{\mathcal{P}} \longrightarrow \mathfrak{S}^0_{X,\pi,S}(\mathcal{P})) \longrightarrow 0. \tag{27}$$

The complex $\mathfrak{R}^\bullet(\mathcal{P})$ is quasi-isomorphic to $\pi^{-1}\mathcal{T}_{T/S}$. Hence, there is a natural map of complexes $\pi^{-1}\mathcal{T}_{T/S} \cong \mathfrak{R}^\bullet(\mathcal{P}) \longrightarrow \mathfrak{g}_{\mathcal{P}}[1]$. Taking pushforward with π, we get a map $\mathcal{T}_{T/S} \cong \pi_*(\pi^{-1}\mathcal{T}_{T/S}) \longrightarrow R^1\pi_{n*}\mathfrak{g}_{\mathcal{P}}$. We wish to compute $R^0\pi_*\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P})$.

To compute $R^0\pi_*\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P})$, we need to compute the zero-th hypercohomology. Choose fine resolutions (local on $\mathcal{M}$),

$$\begin{array}{ccccc}
\mathfrak{G}^{-2}(\mathcal{P}) & \xrightarrow{f} & \mathfrak{G}^{-1}(\mathcal{P}) & \xrightarrow{g} & \mathfrak{G}^{0}(\mathcal{P}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}_0(\mathcal{P}) & \xrightarrow{f} & \mathcal{M}_0(\mathcal{P}) & \xrightarrow{g} & \mathcal{N}_0(\mathcal{P}) \\
\downarrow d & & \downarrow d & & \downarrow d \\
\mathcal{L}_1(\mathcal{P}) & \xrightarrow{f} & \mathcal{M}_1(\mathcal{P}) & \xrightarrow{g} & \mathcal{N}_1(\mathcal{P}) \\
\downarrow d & & \downarrow d & & \downarrow d \\
\vdots & & \vdots & & \vdots
\end{array}$$

and compute the zero-th cohomology of the total complex $\mathcal{C}_{-1} \xrightarrow{D_{-1}} \mathcal{C}_0 \xrightarrow{D_0} \mathcal{C}_1$

$$\mathcal{C}_{-1} = \mathcal{L}_1 \oplus \mathcal{M}_0 \; \mathcal{C}_0 = \mathcal{L}_2 \oplus \mathcal{M}_1 \oplus \mathcal{N}_0 \; \mathcal{C}_1 = \mathcal{L}_3 \oplus \mathcal{M}_2 \oplus \mathcal{N}_1, \text{ and the differentials are}$$

$$D_{-1} : \mathcal{C}_{-1} \longrightarrow \mathcal{C}_0, \;\; (\ell_1, m_0) \longmapsto (d\ell_1, dm_0 - f\ell_1, gm_0)$$
$$D_0 : \mathcal{C}_0 \longrightarrow \mathcal{C}_1, \;\; (\ell_2, m_1, n_0) \longmapsto (d\ell_2, dm_1 + f\ell_2, dn_0 - gm_1).$$

We note three facts in the next lemma the proofs of which are immediate in view of the assumptions.

Lemma 3.4. Let $\pi : X \longrightarrow T$ be as above. Then the following hold:

1) $R^2\pi_*\mathcal{O}_X = \{0\}$ (since the relative dimension is 1).
2) $R^0\pi_*\mathfrak{g}_{\mathcal{P}} = \{0\}$ (assuming the stable locus is non-empty).
3) Since the fibers are connected, assume that the natural map

$$\pi_*(\pi^{-1}\mathcal{T}_{T/S}) \cong \mathcal{T}_{T/S} \to R^1\pi_*\mathfrak{g}_{\mathrm{P}} \tag{28}$$

is an isomorphism. Then the map $R^1\pi_*(\mathfrak{S}^{-1}_{X/T}(\mathcal{P})) \longrightarrow R^1\pi_*(\mathfrak{S}^0_{X,\pi,S}(\mathcal{P}))$ is zero as it factors through $R^1\pi_*\mathfrak{g}_{\mathcal{P}}$.

We prove the next proposition under the assumption of Statement (iii) in Lemma 3.4.

Proposition 3.5. There is an isomorphism $R^0\pi_*\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P}) \simeq R^1\pi_*\mathfrak{S}^{-1}_{X/T}(\mathcal{P})$ that is the identity on $\mathcal{O}_T$ and $\mathcal{T}_{T/S}$.

Proof. Let $(\ell_2, m_1, n_0) \in \ker D_0$. By Lemma 3.4(i) we may take $\ell_2 = 0$. Hence, $(0, m_1, n_0) \mapsto (0, dm_1, dn_0 - gm_1) = (0, 0, 0)$. So m_1 defines a class in $R^1\pi_*(\mathfrak{S}^{-1}_{X/T}(\mathcal{P}))$. The second condition says that gm_1 defines the zero class in $R^1\pi_*(\pi^{-1}\mathcal{T}_{T/S})$. But by Part (iii) of Lemma 3.4 this is automatic. Now n_0 is such that $dn_0 = gm_1$. By (ii) and (iii) of Lemma 3.4, we have $R^0\pi_*\mathfrak{S}^0_{X,\pi,S}(\mathcal{P}) = \{0\}$. This means that n_0 is uniquely determined. Hence, the hypercohomology gives $R^1\pi_*\mathfrak{S}^{-1}_{X/T}(\mathcal{P})$. ∎

Pick a representation $\phi : \mathfrak{g} \to \mathfrak{sl}_r$, and consider the adjoint $\mathbf{ad} : \mathfrak{sl}_r \to \mathfrak{sl}(\mathfrak{sl}_r)$. This gives a map of the corresponding simply connected groups, and now the corresponding associated construction first via ϕ gives a vector bundle $\mathcal{E}_\phi$ and then taking the adjoint we have the bundle $\mathbf{End}(\mathcal{E}_\phi)$. Assuming the condition stated in (iii) of Lemma 3.4, we have the following proposition, which is a generalization of results in [16, Prop. 5.0.2] and [19]:

Proposition 3.6. There is a natural isomorphism between $R^0\pi_*\mathcal{B}^\bullet_{X,\pi,S}(End_0(\mathcal{E}_\phi))$ of the Bloch-Esnault complex and $R^0\pi_*\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P})$ of the relative Ginzburg complex, and the isomorphism fits in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^0\pi_*\Omega^\bullet_{X/T}[2] \cong \mathcal{O}_T & \longrightarrow & R^0\pi_*\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P}) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m_\phi\cdot 2r} & & \downarrow\cong & & \downarrow{\scriptstyle -\mathrm{id}} & & \\ 0 & \longrightarrow & R^0\pi_*\Omega^\bullet_{X/T}[2] \cong \mathcal{O}_T & \longrightarrow & R^0\pi_*\mathcal{B}^\bullet_{X,\pi,S}(\mathbf{End}_0(\mathcal{E}_\phi)) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0. \end{array} \tag{29}$$

Proof. The composition of maps $\mathbf{ad}\circ\phi : \mathfrak{g} \longrightarrow \mathfrak{sl}(\mathfrak{sl}_r)$ gives a map $\mathrm{At}_{X/T}(\mathcal{P}) \longrightarrow \mathrm{At}_{X/T}(\mathbf{End}_0(\mathcal{E}_\phi))$ (see (11) and [1]). Moreover, the representation also gives the map $\mathfrak{S}^{-1}_{X/T}(\mathcal{P}) \longrightarrow \mathcal{B}^{-1}_{0,X/T}(\mathbf{End}(\mathcal{E}_\phi))$ (see Proposition 2.13). These two together give a map between the complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_{X/T}[2] & \longrightarrow & \mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P}) & \longrightarrow & \pi^{-1}\mathcal{T}_{T/S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m_\phi\cdot 2r} & & \downarrow & & \downarrow{\scriptstyle -\mathrm{id}} & & \\ 0 & \longrightarrow & \Omega^\bullet_{X/T}[2] & \longrightarrow & \mathcal{B}^\bullet_{X,\pi,S}(\mathbf{End}(\mathcal{E}_\phi)) & \longrightarrow & \pi^{-1}\mathcal{T}_{T/S} & \longrightarrow & 0. \end{array} \tag{30}$$

Taking pushforward $R^1\pi_*$ of the above, we obtain the desired result. ■

Now assume that $R^1\pi_*\mathfrak{g}_\mathcal{P}$ is isomorphic to $\mathcal{T}_{T/S}$ under the natural map in (28). Consequently, combining Propositions 3.5 and 3.6, we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1\Omega_{X/T} \cong \mathcal{O}_T & \longrightarrow & R^1\pi_*\mathfrak{S}^{-1}_{X/T}(\mathcal{P}) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0 \\ & & \| & & \downarrow\cong & & \| & & \\ 0 & \longrightarrow & R^0\pi_*\Omega^\bullet_{X/T}[2] \cong \mathcal{O}_T & \longrightarrow & R^0\pi_*\mathfrak{S}^\bullet_{X,\pi,S}(\mathcal{P}) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m_\phi\cdot 2r} & & \downarrow\cong & & \downarrow{\scriptstyle -\mathrm{id}} & & \\ 0 & \longrightarrow & R^0\pi_*\Omega^\bullet_{X/T}[2] \cong \mathcal{O}_T & \longrightarrow & R^0\pi_*\mathcal{B}^\bullet_{X,\pi,S}(\mathbf{End}_0(\mathcal{E}_\phi)) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0 \\ & & \| & & \cong\uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_T & \longrightarrow & R^0\pi_*({}^{\mathrm{tr}}\mathcal{A}_{X,\pi,S}(\mathbf{End}_0(\mathcal{E}_\phi))^\bullet) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0 \\ & & \| & & \cong\uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_T & \longrightarrow & \mathrm{At}_{T/S}(\mathcal{L}_\phi^{-\otimes 2r}) & \longrightarrow & \mathcal{T}_{T/S} & \longrightarrow & 0. \end{array} \tag{31}$$

The isomorphism between the third and fourth rows is due to Bloch-Esnault [10], and the isomorphism between the fourth and fifth rows is due to Beilinson-Schechtman [4]. Here, $\mathcal{L}_\phi$ is the determinant of cohomology associated to the family $\mathcal{E}_\phi$.

We have the following theorem under the assumption that $\mathcal{T}_{T/S} \cong R^1\pi_*\mathfrak{g}_\mathcal{P}$ for the map in (28).

Theorem 3.7. The relative Atiyah sequence for $\frac{1}{m_\phi}\mathrm{At}_{T/S}(\mathcal{L}_\phi)$ is isomorphic to

$$0 \longrightarrow \mathcal{O}_{M_G^{rs}} \longrightarrow R^1\pi_{n,*}(\mathfrak{S}^{-1}_{X/T}(\mathcal{P})) \longrightarrow \mathcal{T}_{T/S} \longrightarrow 0. \tag{32}$$

This justifies the computation of the relative extension class of the short exact sequence via Dolbeault methods in Theorem 2.14.

3.5 Associated bundles and pullback

In this section we discuss the relation between the relative Ginzburg complex for the moduli space M_G and the pull-back of Bloch-Esnault complex of M_{SL_r} associated to a representation $\phi : G \hookrightarrow SL_r$. As

previously, let M_G^{rs} denote the moduli space parametrizing the regularly stable principal G-bundles, and denote by $M^s_{\mathrm{SL}_r}$ the moduli space of stable rank r vector bundles of trivial determinant. Let $\phi : G \hookrightarrow \mathrm{SL}_r$ be such that G is not contained in any proper parabolic subgroup of SL_r. With this condition, we know from [8] that a stable principal G-bundle P produces a stable SL_r bundle.

Consider the map $\phi : M_G^{rs} \longrightarrow M^s_{\mathrm{SL}_r}$ taking $[P]$ to $[\mathcal{E}_\phi]$, where $\mathcal{E}_\phi = P \times_\phi \mathbb{C}^r$ is the associated vector bundle. Let $\mathscr{E} \to M^s_{\mathrm{SL}_r} \times_S \mathcal{C}$ and $\mathscr{P} \to M_G^{rs} \times_S \mathcal{C}$ be the universal bundles (since they exist on a covering by étale open subsets, we can treat as if they exist). Then the associated vector bundle $\mathscr{E}_\phi = \mathscr{P} \times_\phi \mathbb{C}^r$ is the pull-back of $\mathscr{E}$ via ϕ. We have the following diagram of maps:

$$\begin{array}{ccccc}
\mathscr{E}_\phi & & \longrightarrow & & \mathscr{E} \\
\downarrow & & & & \downarrow \\
\mathfrak{X}_G := M_G^{rs} \times_S \mathcal{C} & & \xrightarrow{f} & & \mathfrak{X}_{\mathrm{SL}_r} := M^s_{\mathrm{SL}_r} \times_S \mathcal{C} \\
\downarrow \pi_{w,G} & \searrow^{\pi_{n,G}} & & \swarrow_{\pi_{w,\mathrm{SL}_r}} & \downarrow \pi_{n,\mathrm{SL}_r} \\
M_G^{rs} & & \xrightarrow{\phi} & & M^s_{\mathrm{SL}_r} \\
\| & & & & \downarrow \pi_{e,\mathrm{SL}_r} \\
\mathcal{C} & \| & & \| & \mathcal{C} \\
 & \searrow_{\pi_{s,G}} & \downarrow \pi_{e,G} & \swarrow_{\pi_{s,\mathrm{SL}_r}} & \\
 & & S & = & S.
\end{array} \tag{33}$$

Recall that we have the following commutative diagram on $M^s_{\mathrm{SL}_r}$ that connects the Atiyah algebra of the determinant bundle with the Atiyah algebra defined by Bloch-Esnault [10]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}} & \longrightarrow & \mathrm{At}_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E})^\vee & \longrightarrow & \mathrm{End}_0(\mathscr{E})^\vee & \longrightarrow & 0 \\
 & & \| & & \cong\uparrow & & -\nu_{\mathfrak{sl}(r)}\uparrow\cong & & \\
0 & \longrightarrow & \Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}} & \longrightarrow & \mathcal{B}^{-1}_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E}) & \longrightarrow & \mathrm{End}_0(\mathscr{E}) & \longrightarrow & 0.
\end{array} \tag{34}$$

Taking pushforward and combining the results of Beilinson-Schechtman [4], Bloch-Esnault, [10], and Baier-Bolognesi-Martens-Pauly, [16], we get the following commutative diagram of maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{R}^1\pi_{n,\mathrm{SL}_r*}\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}} & \longrightarrow & \mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathrm{At}_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E})^\vee) & \longrightarrow & \mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathrm{End}_0(\mathscr{E})^\vee) & \longrightarrow & 0 \\
 & & \| & & \cong\uparrow & & -\nu_{\mathfrak{sl}(r)}\uparrow\cong & & \\
0 & \longrightarrow & \mathrm{R}^1\pi_{n,\mathrm{SL}_r*}\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}} & \longrightarrow & \mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E})) & \longrightarrow & \mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathrm{End}_0(\mathscr{E})) & \longrightarrow & 0 \\
 & & \cong\uparrow & & \cong\uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{M^s_{\mathrm{SL}_r}} & \longrightarrow & \mathrm{At}_{M^s_{\mathrm{SL}_r}/S}(\mathcal{L}^{-1}) & \longrightarrow & \mathcal{T}_{M^s_{\mathrm{SL}_r}/S} & \longrightarrow & 0
\end{array}$$

where $\mathcal{L} = \det(R\pi_{n*}\mathscr{E})$ is the determinant of cohomology of the family $\mathscr{E}_\phi$. Pulling back the exact sequence at the bottom of the diagram by the map $\phi : M_G^{rs} \longrightarrow M^s_{\mathrm{SL}_r}$, we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{M_G^{rs}} \longrightarrow \mathrm{At}_{M_G^{rs}/S}(\phi^*(\mathcal{L})) \longrightarrow \mathcal{T}_{M_G^{rs}/S} \longrightarrow 0.$$

We wish to connect the following two exact sequences:

$$0 \longrightarrow \phi^*(\mathrm{R}^1\pi_{n,\mathrm{SL}_r*}\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M_{\mathrm{SL}_r}}) \longrightarrow \phi^*(\mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E}))) \longrightarrow \phi^*\mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathrm{End}_0(\mathscr{E})) \longrightarrow 0, \tag{35}$$

$$0 \longrightarrow \mathcal{O}_{M_G^{rs}} \cong \mathrm{R}^1\pi_{n*}\Omega_{\mathfrak{X}_G/M_G^{rs}} \longrightarrow \mathrm{R}^1\pi_{n*}(\mathfrak{S}^{-1}_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{P})) \longrightarrow \mathcal{T}_{M_G^{rs}/S} \longrightarrow 0.$$

By Proposition 1.13, it is enough to relate (35) with the following (see (19)):

$$0 \longrightarrow \mathcal{O}_{M_G^{rs}} \longrightarrow \mathrm{R}^1\pi_{n,G*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E}_\phi)) \longrightarrow \mathrm{R}^1\pi_{n*}(\mathbf{End}_0(\mathscr{E}_\phi)) \longrightarrow 0.$$

Proposition 3.8. There is an isomorphism

$$\mathrm{R}^1\pi_{n,G*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E}_\phi)) \simeq \phi^*(\mathrm{R}^1\pi_{n,\mathrm{SL}_r*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E}))).$$

Proof. Consider the following diagram:

$$\begin{array}{ccc}
\mathscr{E}_\phi \simeq f^*(\mathscr{E}) & \longrightarrow & \mathscr{E} \\
\downarrow & & \downarrow \\
M_G^{rs} \times_S C =: \mathfrak{X}_G & \xrightarrow{f} & \mathfrak{X}_{\mathrm{SL}_r} := M^{s}_{\mathrm{SL}_r} \times_S C \\
\downarrow{\scriptstyle \pi_{n,G}} & & \downarrow{\scriptstyle \pi_{n,\mathrm{SL}_r}} \\
M_G^{rs} & \xrightarrow{\phi} & M^{s}_{\mathrm{SL}_r}.
\end{array}$$

This induces a map of the following exact sequences of Atiyah algebras as in [1]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{End}_0(\mathscr{E}_\phi) & \longrightarrow & \mathrm{At}_{\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}_\phi) & \longrightarrow & \mathcal{T}_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & f^*\,\mathbf{End}_0(\mathscr{E}) & \longrightarrow & f^*\,\mathrm{At}_{\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E}) & \longrightarrow & f^*\mathcal{T}_{\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}} & \longrightarrow & 0.
\end{array}$$

Dualizing, we obtain

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f^*\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}} & \longrightarrow & f^*(\mathrm{At}_{\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E})^\vee) & \longrightarrow & f^*((\mathbf{End}_0(\mathscr{E}))^\vee) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\cong & & \downarrow & & \\
0 & \longrightarrow & \Omega_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & \mathrm{At}_{\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}_\phi)^\vee & \longrightarrow & (\mathbf{End}_0(\mathscr{E}_\phi))^\vee & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle q^{-1}_{\mathscr{E}_\phi}}\downarrow\cong & {\scriptstyle q_{\mathscr{E}}} & {\scriptstyle -\nu^{-1}_{\mathfrak{sl}(r)}}\downarrow\cong & {\scriptstyle -\nu_{\mathfrak{sl}(r)}} & \\
0 & \longrightarrow & \Omega_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & \mathcal{B}^{-1}_{0,\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}_\phi) & \longrightarrow & \mathbf{End}_0(\mathscr{E}_\phi)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & f^*\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}} & \longrightarrow & f^*(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E})) & \longrightarrow & f^*\,\mathbf{End}_0(\mathscr{E}) & \longrightarrow & 0.
\end{array}$$

Hence, by composing we get the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & \mathcal{B}^{-1}_{0,\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}_\phi) & \longrightarrow & \mathbf{End}_0(\mathscr{E}_\phi) & \longrightarrow & 0 \\
 & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \\
0 & \longrightarrow & f^*\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}} & \longrightarrow & f^*(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^{s}_{\mathrm{SL}_r}}(\mathscr{E})) & \longrightarrow & f^*\,\mathbf{End}_0(\mathscr{E}) & \longrightarrow & 0.
\end{array}$$

Applying $R^1\pi_{n,G*}$, we conclude that the following extensions are isomorphic:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^1\pi_{n,G*}\Omega_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & R^1\pi_{n,G*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}_\phi)) & \longrightarrow & R^1\pi_{n,G*}(\mathbf{End}_0(\mathscr{E}_\phi)) & \longrightarrow & 0 \\
 & & \cong\uparrow & & \uparrow & & \cong\uparrow & & \\
0 & \longrightarrow & R^1\pi_{n,G*}f^*\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}} & \longrightarrow & R^1\pi_{n,G*}(f^*(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E}))) & \longrightarrow & R^1\pi_{n,G*}f^*(\mathbf{End}_0(\mathscr{E})) & \longrightarrow & 0.
\end{array}$$

Thus, to finish the proof of the proposition we need to show that

$$R^1\pi_{n,G*}(f^*(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E}))) \cong \phi^*(R^1\pi_{n,\mathrm{SL}_r*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}))).$$

But since the fibers of the morphism π_{n,SL_r} are smooth projective curves, we conclude that $R^2\pi_{n,\mathrm{SL}_r*}\mathcal{F}$ is zero for any coherent sheaf $\mathcal{F}$ which is flat over M_{SL_r}. Moreover, $R^1\pi_{n,\mathrm{SL}_r*}(\mathcal{B}_0^{-1}(\mathscr{E}))$ is locally free. Hence by base change of cohomologies for flat morphisms, we get the required isomorphism. This completes the proof. ∎

Consider the following diagram where m_ϕ be the Dynkin index of the embedding $\mathfrak{g} \longrightarrow \mathfrak{sl}(V)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{M_G} & \longrightarrow & \phi^*\,\mathbf{At}_{M^s_{\mathrm{SL}_r}/S}(\mathcal{L}^{-1}) & \longrightarrow & \phi^*\mathcal{T}_{M^s_{\mathrm{SL}_r}/S} & \longrightarrow & 0 \\
 & & \| & & \cong\downarrow & & \downarrow{\scriptstyle KS_{M^s_{\mathrm{SL}_r}/S}} & & \\
0 & \to & \phi^*\left(R^1\pi_{n,\mathrm{SL}_r*}\Omega_{\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}\right) & \to & \phi^*\left(R^1\pi_{n,\mathrm{SL}_r*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_{\mathrm{SL}_r}/M^s_{\mathrm{SL}_r}}(\mathscr{E})\right) & \to & \phi^*\left(R^1\pi_{n,\mathrm{SL}_r*}(\mathbf{End}_0(\mathscr{E})\right) & \to & 0 \\
 & & \| & & \cong\uparrow & & \| & & \\
0 & \longrightarrow & R^1\pi_{n,G*}\Omega_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & R^1\pi_{n,G*}(\mathcal{B}^{-1}_{0,\mathfrak{X}_G/M_G^{rs}}(\mathscr{E}_\phi)) & \longrightarrow & R^1\pi_{n,G*}(\mathbf{End}_0(\mathscr{E}_\phi)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle m_\phi} & & \uparrow & & \uparrow{\scriptstyle -\phi_*} & & \\
0 & \longrightarrow & R^1\pi_{n,G*}\Omega_{\mathfrak{X}_G/M_G^{rs}} & \longrightarrow & R^1\pi_{n,G*}(\mathfrak{S}^{-1}_{\mathfrak{X}_G/M_G^{rs}}(\mathscr{P})) & \longrightarrow & \mathcal{T}_{M_G^{rs}/S} & \longrightarrow & 0.
\end{array}$$

In this diagram, by Proposition 3.8 we get the isomorphism of the first two rows. Finally, the map between the second and the third row follows from Proposition 1.13 and Proposition 19. Thus, we get that $R^1\pi_{n,G*}(\mathfrak{S}^{-1}_{\mathfrak{X}_G/M_G^{rs}}(\mathscr{P}))$ is isomorphic to the Atiyah algebra $\frac{1}{m_\phi}\phi^\bullet\,\mathbf{At}_{M^s_{\mathrm{SL}_r}/S}(\mathcal{L})$, where $\phi^\bullet$ denote the pull-back in the category of Atiyah algebras.

4 Parabolic Analog of Beilinson-Schechtman Construction

We now extend the previous considerations to the case of parabolic bundles. In order to analyze parabolic Atiyah algebras for families of parabolic bundles on a curve C, we adopt notion of Γ-linearized bundles on a Galois cover $\widehat{C} \longrightarrow C$ with Galois group Γ. See [9,§ 6] for more details.

4.1 Parabolic vector bundles

Let $\widetilde{\mathcal{E}}$ be a vector bundle of rank r on a family of ramified Γ-cover of curves $\widetilde{\pi} : \widetilde{X} \longrightarrow T$ ramified along $\widehat{D}$. In other words, there is a natural projection $p : \widetilde{X} \longrightarrow X$, which is a ramified Γ-covering such that $\widetilde{\pi} = \pi \circ p$. Let $D := p(\widehat{D}) \subset X$ be the divisor or marked points. Let $\widetilde{\mathcal{E}}$ be a family of vector bundles on $\widetilde{X}$, which is Γ-linearized. Let $\mathcal{E}$ be the vector bundle on X defined by the invariant pushforward of the Γ-bundle $\widetilde{\mathcal{E}}$. By the discussion in [9, § 7], the vector bundle $\mathcal{E}$ comes equipped with a parabolic structure supported on D. Recall the trace-zero relative Atiyah sequence

$$0 \longrightarrow \mathbf{End}_0(\widetilde{\mathcal{E}}) \longrightarrow \mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}}) \longrightarrow \mathcal{T}_{\widetilde{X}/T} \longrightarrow 0. \tag{36}$$

By Seshadri [18], we can identify the sheaf of parabolic endomorphism $\mathbf{Par}(\widetilde{\mathcal{E}})$ with $p_*^{\Gamma}\,\mathbf{End}_0(\widetilde{\mathcal{E}})$. With this set-up we can consider the *parabolic Atiyah algebra* with the following fundamental exact sequence:

$$0 \longrightarrow \begin{array}{c}\mathbf{Par}_0(\widetilde{\mathcal{E}})\\ \cong p_*^{\Gamma}(\mathbf{End}_0(\widetilde{\mathcal{E}})\end{array} \longrightarrow \begin{array}{c}{}^{par}\mathrm{At}_{X/T}(\mathcal{E})\\ := p_*^{\Gamma}\mathrm{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})\end{array} \longrightarrow \mathcal{T}_{X/T}(-D) \longrightarrow 0.$$

As before, $\widehat{D}$ is the relative ramification divisor in $\widetilde{X}$; consider the log-relative traceless Atiyah sequence obtained from (36)

$$0 \longrightarrow \mathbf{End}_0(\widetilde{\mathcal{E}})(-\widehat{D}) \longrightarrow (\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}}))(-\widehat{D}) \longrightarrow \mathcal{T}_{\widetilde{X}/T}(-\widehat{D}) \longrightarrow 0.$$

Since all the objects are naturally Γ-linearized, we can apply the invariant push-forward functor p_*^{Γ} to get the *strongly parabolic Atiyah algebra* ${}^{spar}\mathrm{At}_{X/T}(\mathcal{E})$ with the fundamental exact sequence

$$0 \longrightarrow \begin{array}{c}\mathbf{SPar}_0(\mathcal{E})\\ \cong p_*^{\Gamma}\left(\mathbf{End}_0(\widetilde{\mathcal{E}})(-\widehat{D})\right)\end{array} \longrightarrow p_*^{\Gamma}\left(\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})(-\widehat{D})\right) \longrightarrow \mathcal{T}_{X/T}(-D) \longrightarrow 0$$

Tensoring with $\mathcal{O}_X(D)$ we get,

$$0 \longrightarrow \mathbf{SPar}_0(\mathcal{E})(D) \longrightarrow p_*^{\Gamma}\left(\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})(-\widehat{D})\right)(D) \longrightarrow \mathcal{T}_{X/T} \longrightarrow 0.$$

As in Section 2.1, consider the dual exact sequence:

$$0 \longrightarrow \Omega_{X/T} \longrightarrow \left(p_*^{\Gamma}\left(\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})(-\widehat{D})\right)(D)\right)^{\vee} \longrightarrow (\mathbf{SPar}_0(\mathcal{E})(D))^{\vee} \longrightarrow 0. \tag{37}$$

The trace pairing $\kappa_{\mathfrak{sl}(r)} : \mathbf{End}_0(\widetilde{\mathcal{E}}) \otimes \mathbf{End}_0(\widetilde{\mathcal{E}}) \to \mathcal{O}_{\widetilde{X}}$ gives an $\mathcal{O}_{\widetilde{X}}(-\widehat{D})$ valued pairing

$$\kappa_{\mathfrak{sl}(r)} : \mathbf{End}_0(\widetilde{\mathcal{E}}) \otimes \mathbf{End}_0(\widetilde{\mathcal{E}})(-\widehat{D}) \longrightarrow \mathcal{O}_{\widetilde{X}}(-\widehat{D}).$$

Taking Γ-invariant push forward p_*^{Γ} of the above exact sequence we get a map

$$\kappa_{\mathfrak{sl}(r)} : \mathbf{Par}_0(\mathcal{E}) \otimes \mathbf{SPar}_0(\mathcal{E}) \longrightarrow \mathcal{O}_X(-D).$$

Now by multiplying by $\mathcal{O}_X(D)$ on both sides we get the following:

Proposition 4.1. The trace induces a nondegenerate pairing

$$\kappa_{\mathfrak{sl}(r)} : \mathbf{Par}_0(\mathcal{E}) \otimes \mathbf{SPar}_0(\mathcal{E})(D) \longrightarrow \mathcal{O}_X,$$

which identifies $(\mathbf{SPar}_0(\mathcal{E})(D))^{\vee} \cong \mathbf{Par}_0(\mathcal{E})$.

We pull back the sequence in (37) via the map $\nu_{\mathfrak{sl}(r)}$ to get the following quasi-Lie algebra which we denote by ${}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{E})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & \left(p_*^{\Gamma}\left(\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})(-\widehat{D})\right)(D)\right)^{\vee} & \longrightarrow & (\mathbf{SPar}_0(\mathcal{E})(D))^{\vee} & \longrightarrow & 0\\ & & \| & & \cong\uparrow & & \nu_{\mathfrak{sl}(r)}\uparrow & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{E}) & \longrightarrow & \mathbf{Par}_0(\mathcal{E}) & \longrightarrow & 0. \end{array} \tag{38}$$

4.1.1 *Parabolic Bloch-Esnault complex*

By the construction of Beilinson-Schechtman [4], we get an exact sequence of sheaves:

$$0 \longrightarrow \mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T} \longrightarrow {}^{tr}\widetilde{\mathcal{A}}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})^{-1} \longrightarrow \mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}}) \longrightarrow 0.$$

Recall that since $\widetilde{X} \to T$ is a Γ-cover of curves, and now assume that $\widetilde{\mathcal{E}}$ is Γ-equivariant. This implies that all terms in the above exact sequence have a Γ-action. Taking Γ-invariant pushforward we get the following:

$$0 \longrightarrow p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) \longrightarrow p_*^{\Gamma}({}^{tr}\widetilde{\mathcal{A}}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})^{-1}) \longrightarrow p_*^{\Gamma}(\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})) \longrightarrow 0. \tag{39}$$

Here $\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})$ is the relative Atiyah algebra of the bundle of the vector bundle $\widetilde{\mathcal{E}}$. The last term of the above is the *parabolic Atiyah algebra* ${}^{par}\mathbf{At}_{X/T}(\mathcal{E})$. Pulling back the exact sequence in (39) via the natural inclusion $\mathbf{Par}_0(\mathcal{E}) \hookrightarrow {}^{par}\mathbf{At}(\mathcal{E})$, we get an exact sequence:

$$0 \longrightarrow p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) \longrightarrow {}^{par}\widetilde{\mathcal{B}}_{0,X/T}^{-1}(\mathcal{E}) \longrightarrow \mathbf{Par}_0(\mathcal{E}) \longrightarrow 0. \tag{40}$$

Taking invariant pushforward with respect to the trace of an endomorphism $\mathbf{tr} : \mathcal{E}nd(\widetilde{\mathcal{E}}) \to \mathcal{O}_{\widetilde{X}}$, we get another map $\mathbf{tr} : p_*^{\Gamma}\,\mathbf{End}(\widetilde{\mathcal{E}} \otimes \Omega_{\widetilde{X}/T}) \longrightarrow p_*^{\Gamma}\Omega_{\widetilde{X}/T}$.

Since $\Omega_{\widetilde{X}/T} \cong p^*\Omega_{X/T} \otimes \mathcal{O}(\widehat{D})$, where $\widehat{D}$ is the relative ramification $p : \widetilde{X} \to X$. We can identify $p_*^{\Gamma}(\Omega_{\widetilde{X}/T}) \cong \Omega_{X/T}$. This in turn gives a map

$$\mathbf{tr} : p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) \longrightarrow \Omega_{X/T}. \tag{41}$$

Taking pushforward of the exact sequence in (39) via the parabolic trace in (41), we get the following exact sequence:

$$0 \longrightarrow \Omega_{X/T} \longrightarrow {}^{par}\mathcal{B}_{0,X/T}^{-1}(\mathcal{E}) \longrightarrow \mathbf{Par}_0(\mathcal{E}) \longrightarrow 0. \tag{42}$$

We can summarize the above discussion in the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_*\left(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}\right) & \longrightarrow & p_*({}^{tr}\widetilde{\mathcal{A}}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})^{-1}) & \longrightarrow & p_*\left(\mathbf{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})\right) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) & \longrightarrow & p_*^{\Gamma}({}^{tr}\widetilde{\mathcal{A}}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})^{-1}) & \longrightarrow & {}^{par}\mathbf{At}_{\widetilde{X}/T}(\mathcal{E}) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) & \longrightarrow & {}^{par}\widetilde{\mathcal{B}}_{\widetilde{X}/T}^{-1}(\mathcal{E}) & \longrightarrow & \mathbf{Par}_0(\mathcal{E}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathbf{tr}} & & \downarrow{\scriptstyle \mathbf{tr}_*} & & \| & & \\
0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{par}\mathcal{B}_{0,X/T}^{-1}(\mathcal{E}) & \longrightarrow & \mathbf{Par}_0(\mathcal{E}) & \longrightarrow & 0.
\end{array}$$

Recall from Section 2.1.1.1 in Beilinson-Schechtman [4, Lemma (a)], that there is a residue pairing

$$\widetilde{\mathbf{Res}} : \Omega_{\widetilde{X}/T} \boxtimes \Omega_{\widetilde{X}/T}(3\Delta) \longrightarrow \mathcal{O}_{\widetilde{X}}.$$

The following theorem connects the parabolic Ginzburg dgla defined above to the quasi Lie algebra ${}^{spar}\widetilde{\mathbf{At}}_{X/T}(\mathcal{E})$ defined by (38).

Theorem 4.2. There is an isomorphism induced by invariant push-forward of the residue pairing between the quasi-Lie algebras ${}^{spar}\widetilde{\mathbf{At}}_{X/T}(\mathcal{E})$ and ${}^{par}\mathcal{B}_{0,X/T}^{-1}(\mathcal{E})$, which induces an isomorphism of

exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{E}) & \longrightarrow & \mathrm{Par}_0(\mathcal{E}) & \longrightarrow & 0 \\ & & \| & & \downarrow\cong & & \cong\downarrow -\mathrm{id} & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{par}\mathcal{B}^{-1}_{0,X/T}(\mathcal{E}) & \longrightarrow & \mathrm{Par}_0(\mathcal{E}) & \longrightarrow & 0. \end{array}$$

Proof. Let $\widetilde{\mathcal{E}}' := \widetilde{\mathcal{E}} \otimes \Omega_{\widetilde{X}/T}$, as in Beilinson-Schechtman, the sheaf of first order differential operators $\mathcal{D}^{\leq 1}(\widetilde{\mathcal{E}})$ can be identified with $\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}$. There is a natural exact sequence of sheaves

$$0 \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)} \longrightarrow 0.$$

Moreover, all the objects of the above exact sequence are Γ-linearized. In particular, apply the invariant pushforward functor we get the following commutative diagram:

$$0 \longrightarrow p^{\Gamma}_*\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow p^{\Gamma}_*\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow p^{\Gamma}_*\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)} \longrightarrow 0.$$

Similarly we have another short exact sequence from [4] that is used in constructing the trace complex.

$$0 \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow 0.$$

Pulling back the exact sequence above by the inclusion $\mathbf{End}(\widetilde{\mathcal{E}}) \cong \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \hookrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}$. we get a short exact sequence

$$0 \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \longrightarrow \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow 0.$$

Taking invariant pushforward functor we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^{\Gamma}_*\left(\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)}\right) & \longrightarrow & p^{\Gamma}_*\left(\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)}\right) & \longrightarrow & p^{\Gamma}_*\left(\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}\right) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & p^{\Gamma}_*\left(\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)}\right) & \longrightarrow & p^{\Gamma}_*\left(\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)}\right) & \longrightarrow & p^{\Gamma}_*\left(\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}\right) & \longrightarrow & 0. \end{array}$$

Now there is a natural nondegenerate bilinear form:

$$\langle , \rangle : \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \otimes \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow \mathcal{O}_{\widetilde{X}}. \tag{43}$$

The above form $\langle , \rangle$ vanishes identically restricted to $\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \otimes \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}$ and hence it descends to a form

$$\langle , \rangle : \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \otimes \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)} \longrightarrow \mathcal{O}_{\widetilde{X}} \text{ and } \langle , \rangle : \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \otimes \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \longrightarrow \mathcal{O}_{\widetilde{X}}. \tag{44}$$

Under the canonical identifications of $\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'} \cong \mathbf{End}(\widetilde{\mathcal{E}})$ we get

$$\frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(\Delta)} \cong \mathbf{End}(\widetilde{\mathcal{E}}) \otimes \mathcal{T}_{\widetilde{X}/T} \text{ and } \frac{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}}\boxtimes\widetilde{\mathcal{E}}'(-\Delta)} \cong \mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}. \tag{45}$$

Lemma B.2.8 in [16] shows that the bilinear form $\langle\,,\,\rangle$ in (43) and (44) can be identified as

$$-\kappa_{\mathfrak{gl}(r)} : \mathbf{End}(\widetilde{\mathcal{E}}) \otimes \mathbf{End}(\widetilde{\mathcal{E}}) \longrightarrow \mathcal{O}_{\widetilde{X}} \text{ and } \kappa_{\mathfrak{gl}(r)} : \mathbf{End}(\widetilde{\mathcal{E}}) \otimes \mathcal{T}_{\widetilde{X}/T} \otimes \mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T} \longrightarrow \mathcal{O}_{\widetilde{X}}, \tag{46}$$

where $\kappa_{\mathfrak{gl}(r)}$ is the trace of the product of the two endomorphisms. Now the invariant pushforward of $\langle\,,\,\rangle$ induces a nondegenerate $\mathcal{O}_X(-D)$ valued form

$$\langle\,,\,\rangle : p_*^{\Gamma} \frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(-\Delta)} \otimes p_*^{\Gamma}\left(\frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'}(-\widehat{D})\right) \longrightarrow p_*^{\Gamma}\mathcal{O}_{\widetilde{X}}(-\widehat{D}) \cong \mathcal{O}_X(-D),$$

which gives the following nondegenerate form

$$\langle\,,\,\rangle : p_*^{\Gamma} \frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(-\Delta)} \otimes \left(p_*^{\Gamma}\left(\frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(2\Delta)}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'}(-\widehat{D})\right)\right)(D) \longrightarrow \mathcal{O}_X$$

that restricts to

$$\langle\,,\,\rangle : p_*^{\Gamma}\left(\frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'}\right) \otimes \left(p_*^{\Gamma}\left(\frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(\Delta)}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'}(-\widehat{D})\right)\right)(D) \longrightarrow \mathcal{O}_X. \tag{12}$$

$$\langle\,,\,\rangle : p_*^{\Gamma}\left(\frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(-\Delta)}\right) \otimes \left(p_*^{\Gamma}\left(\frac{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'}{\widetilde{\mathcal{E}} \boxtimes \widetilde{\mathcal{E}}'(\Delta)}(-\widehat{D})\right)\right)(D) \longrightarrow \mathcal{O}_X. \tag{13}$$

The identification of parabolic and strongly parabolic endomorphism as invariant pushforward and invariants pushforward of the identification in (46) tell us that the form $\langle\,,\,\rangle$ in (47) can be identified with

$$-\kappa_{\mathfrak{gl}(r)} : \mathbf{Par}(\widetilde{\mathcal{E}}) \otimes \mathbf{SPar}(\widetilde{\mathcal{E}})(D) \longrightarrow \mathcal{O}_X.$$

Similarly, the form in (48) can be identified with

$$\langle\,,\,\rangle : p_*^{\Gamma}\left(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}\right) \otimes \left(p_*^{\Gamma}\left(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \mathcal{T}_{\widetilde{X}/T}(-\widehat{D})\right)\right)(D) \longrightarrow \mathcal{O}_X.$$

Now the identification of $\left(p_*^{\Gamma}\left(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \mathcal{T}_{\widetilde{X}/T}(-\widehat{D})\right)\right)(D) \cong \mathcal{T}_{X/T}$ gives the following commutative:

$$\begin{array}{ccc} \langle\,,\,\rangle : \left(p_*^{\Gamma}\left(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}\right)\right) \otimes \left(p_*^{\Gamma}\left(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \mathcal{T}_{\widetilde{X}/T}(-\widehat{D})\right)\right)(D) & \longrightarrow & \mathcal{O}_X \\ \downarrow \cong & & \| \\ \Omega_{X/T} \otimes \mathcal{T}_{X/T} & \longrightarrow & \mathcal{O}_X. \end{array}$$

This induces an isomorphism of ${}^{par}\mathcal{B}^{-1}_{0,X/T}(\mathcal{E})$ with ${}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{E})$ that restricts to $\mathbf{id}$ on $\Omega_{X/T}$ and $-\mathbf{id}$ on $\mathbf{Par}_0(\mathcal{E})$. ■

The following proposition connects $R^1\pi_*$ of the parabolic Ginzburg dgla with $R^1\widetilde{\pi}_*$ of the dgla constructed by Bloch-Esnault.

Proposition 4.3. There are inclusion maps $\Omega_{X/T} \to p_*\Omega_{\widetilde{X}/T}$ and $\mathbf{Par}_0(\mathcal{E}) \cong p_*^{\Gamma}\,\mathbf{End}_0(\mathcal{E}) \hookrightarrow p_*\,\mathbf{End}_0(\widetilde{\mathcal{E}})$ that extend to a map of the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_*\Omega_{\widetilde{X}/T} & \longrightarrow & p_*\mathcal{B}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}}) & \longrightarrow & p_*\,\mathbf{End}_0(\widetilde{\mathcal{E}}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{par}\mathcal{B}^{-1}_{0,X/T}(\mathcal{E}) & \longrightarrow & \mathbf{Par}_0(\mathcal{E}) & \longrightarrow & 0. \end{array}$$

Proof. First we prove that $p_*^{\Gamma}(\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}})) \cong {}^{par}\widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E})$. Consider the short exact sequence

$$0 \longrightarrow p_*(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) \longrightarrow p_*(\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}})) \longrightarrow p_*(\mathbf{End}_0(\widetilde{\mathcal{E}})) \longrightarrow 0.$$

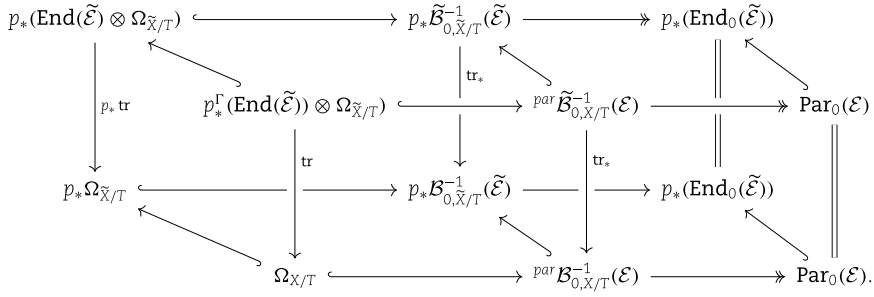
Taking invariants with respect to Γ, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_*(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) & \longrightarrow & p_*\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}}) & \longrightarrow & p_*(\mathbf{End}_0(\widetilde{\mathcal{E}})) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) & \longrightarrow & p_*^{\Gamma}(\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}})) & \longrightarrow & p_*^{\Gamma}(\mathbf{End}_0(\widetilde{\mathcal{E}})) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) & \longrightarrow & p_*^{\Gamma}({}^{\mathrm{tr}}\widetilde{\mathcal{A}}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})^{-1}) & \longrightarrow & p_*^{\Gamma}(\mathrm{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}})) & \longrightarrow & 0.
\end{array}$$

The inclusion of the second row into the third follows from the invariant pushforward of the first two rows of the diagram (24). Since both ${}^{par}\widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E})$ and $p_*^{\Gamma}(\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}}))$ are extensions of $p_*^{\Gamma}(\mathbf{End}_0(\widetilde{\mathcal{E}}))$ by $p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T})$ obtained as a sub extension of $p_*^{\Gamma}(\mathrm{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}}))$ by $p_*^{\Gamma}(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T})$ via the inclusion $p_*^{\Gamma}(\mathbf{End}_0(\widetilde{\mathcal{E}})) \hookrightarrow p_*^{\Gamma}(\mathrm{At}_{\widetilde{X}/T}(\widetilde{\mathcal{E}}))$, it follows that $p_*^{\Gamma}(\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}})) \cong {}^{par}\widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E})$. Now the above gives the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p_*(\mathbf{End}(\widetilde{\mathcal{E}}) \otimes \Omega_{\widetilde{X}/T}) & \longrightarrow & p_*\widetilde{\mathcal{B}}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}}) & \longrightarrow & p_*(\mathbf{End}_0(\widetilde{\mathcal{E}})) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p_*^{\Gamma}\left(\mathbf{End}(\widetilde{\mathcal{E}})) \otimes \Omega_{\widetilde{X}/T}\right) & \longrightarrow & {}^{par}\widetilde{\mathcal{B}}^{-1}_{0,X/T}(\mathcal{E}) & \longrightarrow & \mathbf{Par}_0(\mathcal{E}) & \longrightarrow & 0.
\end{array}$$

Pushing forward with respect to the trace of endomorphism $\mathbf{tr}$, the above commutative diagram implies the existence of the following commutative diagram all of whose rows are short exact sequences:



The bottom level of the above diagram gives the required result. ■

Recall that we have a diagram relating the families of curves parametrized by T:

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{p} & X \\
 & \searrow^{\widetilde{\pi}} & \downarrow{\scriptstyle \pi} \\
 & & T
\end{array}$$

Taking $R^1\pi_*$ of all the terms of the commutative diagram in Proposition 4.3, and using the fact that $R^1\widetilde{\pi} = R^1\pi \circ p_*$, we get the following proposition:

Proposition 4.4. The following diagram is commutative:

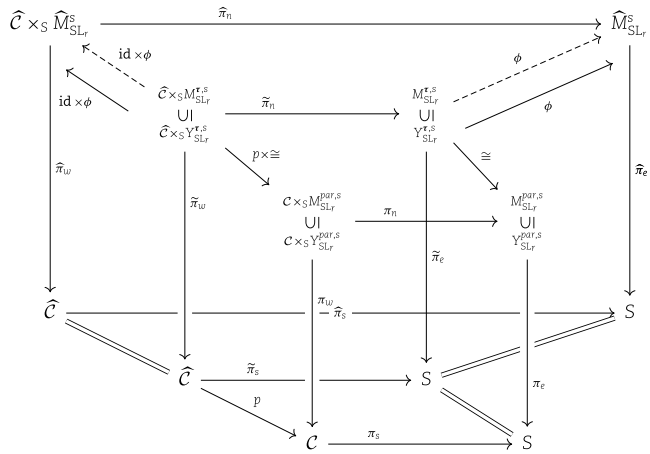
$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1\widetilde{\pi}_*\Omega_{\widetilde{X}/T} \simeq \mathcal{O}_T & \longrightarrow & R^1\widetilde{\pi}_*(\mathcal{B}^{-1}_{0,\widetilde{X}/T}(\widetilde{\mathcal{E}})) & \longrightarrow & R^1\widetilde{\pi}_*\,\mathbf{End}_0(\widetilde{\mathcal{E}}) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R^1\pi_*\Omega_{X/T} \simeq \mathcal{O}_T & \longrightarrow & R^1\pi_*({}^{par}\mathcal{B}^{-1}_{0,X/T}(\mathcal{E})) & \longrightarrow & R^1\pi_*\,\mathbf{Par}_0(\mathcal{E}) & \longrightarrow & 0. \end{array}$$

4.2 Parabolic Atiyah algebras and moduli of parabolic bundles

Let $\mathcal{C} \to S$ be a family of curves, and let $\widehat{\mathcal{C}} \to S$ be a family of Γ-covers. Consider the relative family $\widehat{M}^{s}_{\mathrm{SL}_r}$ parametrizing the moduli space of stable SL_r bundles on $\widehat{\mathcal{C}} \to S$. Let $M^{\boldsymbol{\tau},s}_{\mathrm{SL}_r,S}$ and $M^{par,s}_{\mathrm{SL}_r}$ be the relative moduli spaces of stable (Γ, SL_r) and stable parabolic bundles on $\widehat{\mathcal{C}} \to S$ and $\mathcal{C} \to S$, respectively. Without loss of generality assume that the interior of $\widehat{M}^{ss}_{\mathrm{SL}_r}$ is non-empty, otherwise our main theorem is trivially true.

Now by the discussion in [9, §7.2], the invariant pushforward functor induces an isomorphism between $M^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$ (respectively, the semi-stable moduli space $M^{\boldsymbol{\tau},ss}_{\mathrm{SL}_r}$) and $M^{par,s}_{\mathrm{SL}_r}$ (respectively, $M^{par,ss}_{\mathrm{SL}_r}$). The data $\boldsymbol{\tau}$ and the covering family $\widehat{\mathcal{C}}$ depend on the data of the parabolic weights that defines the parabolic semistability. By [7], we get a map $\phi : M^{\boldsymbol{\tau},ss}_{\mathrm{SL}_r} \to \widehat{M}^{ss}_{\mathrm{SL}_r}$, which extends to a map $\widehat{\mathcal{C}} \times_S M^{\boldsymbol{\tau},ss}_{\mathrm{SL}_r} \to \widehat{\mathcal{C}} \times_S \widehat{M}^{ss}_{\mathrm{SL}_r}$. The map ϕ may not preserve stability, however ϕ being finite, the complement of the inverse image $Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r} := \phi^{-1}\widehat{M}^{s}_{\mathrm{SL}_r}$ of the stable locus has codimension at least two in $M^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$ provided genus of the orbifold curve $\mathscr{C} = [\widehat{C}/\Gamma]$ is at least 2 if $r \neq 2$ and at least 3 if $r = 2$. We refer the reader to [9, Lemma 8.3] for more details.

We have the following diagrams that connect all the objects described above:



The rational maps are regular over $Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$, which will be also denoted by the same notation. The image of $Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$ under the invariant pushforward isomorphism of $M^{\boldsymbol{\tau},ss}_{\mathrm{SL}_r} \cong M^{par,ss}_{\mathrm{SL}_r}$ will be denoted by $Y^{par,s}_{\mathrm{SL}_r}$. By definition $Y^{par,s}_{\mathrm{SL}_r} \hookrightarrow M^{par,s}_{\mathrm{SL}_r}$.

Let $\widehat{\mathcal{E}}$ be the universal bundle (which exist in the étale topology) on $\widehat{\mathcal{C}} \times_S \widehat{M}^{ss}_{\mathrm{SL}_r}$ and $\widetilde{\mathcal{E}}$ be its pull-back to $\widehat{\mathcal{C}} \times_S M^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$. We denote by $\mathcal{E}$ the universal parabolic bundle, which we can assume to exist without loss of generality (see Remark 1). As in the diagram let $\widetilde{\pi}_n : \widehat{C} \times_S Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r} \longrightarrow Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$ denote the projection and similarly consider the projection $\widehat{\pi}_n : \widehat{C} \times_S \widehat{M}^{s}_{\mathrm{SL}_r} \longrightarrow \widehat{M}^{s}_{\mathrm{SL}_r}$

Let $\mathcal{L}$ be the determinant of cohomology line bundle on $\widehat{M}^{s}_{\mathrm{SL}_r}$. Now, as before, combining the results of Baier-Bolognesi-Martens-Pauly [16], Beilinson-Schechtman [4], Bloch-Esnault [10], and Sun-Tsai [19], we get an isomorphism of the Atiyah algebras $\mathrm{At}_{\widehat{M}^{s}_{\mathrm{SL}_r}/S}(\mathcal{L}^{-1})$ with $R^1\widehat{\pi}_{n*}(\mathcal{B}^{-1}_{0,\widehat{\mathcal{C}}\times_S\widehat{M}^{s}_{\mathrm{SL}_r}/\widehat{M}^{s}_{\mathrm{SL}_r}}(\widehat{\mathcal{E}}))$, which makes

the diagram of fundamental sequences of Atiyah algebras commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\widehat{M}^s_{\mathrm{SL}_r}} & \longrightarrow & \mathrm{At}_{\widehat{M}^s_{\mathrm{SL}_r}/S}(\mathcal{L}^{-1}) & \longrightarrow & \mathcal{T}_{\widehat{M}^s_{\mathrm{SL}_r}/S} & \longrightarrow & 0 \\ & & \| & & \downarrow\cong & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\widehat{M}^s_{\mathrm{SL}_r}} & \longrightarrow & R^1\widehat{\pi}_{n*}(\mathcal{B}^{-1}_{0,\widehat{\mathcal{C}}\times_S\widehat{M}^s_{\mathrm{SL}_r}/\widehat{M}^s_{\mathrm{SL}_r}}(\widehat{\mathcal{E}})) & \longrightarrow & \mathcal{T}_{\widehat{M}^s_{\mathrm{SL}_r}/S} & \longrightarrow & 0. \end{array}$$

Pulling back by ϕ we get an isomorphism of $\phi^*\mathrm{At}_{\widehat{M}^s_{\mathrm{SL}_r}/S}(\mathcal{L})$ and $\phi^*R^1\widehat{\pi}_{n*}(\mathcal{B}^{-1}_{0,\widehat{\mathcal{C}}\times_S\widehat{M}^s_{\mathrm{SL}_r}/\widehat{M}^s_{\mathrm{SL}_r}}(\widehat{\mathcal{E}}))$. Moreover the base change theorems implies that the later is isomorphic to $R^1\widetilde{\pi}_{n*}(\mathcal{B}^{-1}_{0,\widehat{\mathcal{C}}\times_S Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}/Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}}(\widehat{\mathcal{E}}))$. (Observe that ϕ^* of an Atiyah algebra may not be an Atiyah algebra.) We have the following result.

Theorem 4.5. There is an isomorphism of the relative Atiyah algebras $\mathrm{At}_{M^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}/S}(\phi^*\mathcal{L}^{-1})$ with $R^1\pi_{n*}({}^{par}\mathcal{B}^{-1}_{0,\mathcal{C}\times_S M^{par,s}_{\mathrm{SL}_r}/M^{par,s}_{\mathrm{SL}_r}}(\mathcal{E}))$ that restricts to the identity map on $\Omega_{M^{par,s}_{\mathrm{SL}_r}/S}$.

Proof. Applying Proposition 4.4 with $\widetilde{X} = \widehat{\mathcal{C}}\times_S Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$, $X = \mathcal{C}\times_S Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$, and $T = Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}$, we get an isomorphism between $\mathbf{At}_{Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}/S}(\phi^*\mathcal{L})$ and $R^1\pi_{n*}{}^{par}\mathcal{B}^{-1}_{0,\mathcal{C}\times_S Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}/Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}}(\mathcal{E})$ over $Y^{par,s}_{\mathrm{SL}_r}$. Now both these sheaves $\mathbf{At}_{Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}/S}(\phi^*\mathcal{L}^{-1})$ and $R^1\pi_{n*}({}^{par}\mathcal{B}^{-1}_{0,\mathcal{C}\times_S Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}/Y^{\boldsymbol{\tau},s}_{\mathrm{SL}_r}}(\mathcal{E}))$ are locally free (hence reflexive) and extend over $M^{par,s}_{\mathrm{SL}_r}$. Since they are isomorphic on an open subset whose complement has codimension at least two, the isomorphism actually extends to all of $M^{par,s}_{\mathrm{SL}_r}$. ■

Remark 4.6. The proof of Theorem 2.14 applies in this parabolic setting as well. The fact that the Atiyah-Bott-Narasimhan-Goldman symplectic form is in the class of $\phi^{*\mathcal{L}}$ is one of the results of [11] (see also [6]).

4.3 Parabolic G-bundles

4.3.1 *Parabolic bundles*

Let $p : \widehat{C} \to C$ be a ramified covering with Galois group Γ, so $C = \widehat{C}/\Gamma$. Let $D \subset C$ and $\widehat{D} \subset \widehat{C}$ denote the branching loci in C and $\widehat{C}$ respectively. Let $\widehat{\pi} : \widehat{P} \to \widehat{C}$ be a Γ-principal G-bundle, that is, $\widehat{P}$ is a principal G-bundle, and there is a representation $\Gamma \to \mathbf{Aut}(\widehat{P}) : \gamma \mapsto F_\gamma$ such that the actions of Γ and G on $\widehat{P}$ commute. For an open set $U \subset C$, define

$$\mathbf{aut}_\Gamma(\widehat{P})(U) = \left\{X \in \mathbf{aut}(\widehat{P})(p^{-1}(U)) \mid X(F_\gamma(p)) = (F_\gamma)_*(X(p)) \ \forall\ \gamma \in \Gamma\right\}.$$

This presheaf defines an $\mathcal{O}_C$-coherent sheaf. Notice that for $X \in \mathbf{aut}_\pi(\widehat{P})$, the vector field $\widehat{v}$ on $\widehat{C}$ is Γ-invariant, and so descends to a vector field on C that vanishes along the ramification divisor $D \subset C$. Hence, we have a map $\mathbf{aut}_\Gamma(\widehat{P}) \to TC(-D)$.

Let P be a parabolic G-bundle on C and let $\widehat{P}$ be a (Γ, G)-bundle on $\widehat{C}$ such that P and $\widehat{P}$ are related by invariant pushforward.

Definition 4.7. We define the parabolic Atiyah algebra ${}^{par}\mathrm{At}(P)$ of P to be $\mathbf{aut}_\Gamma(\widehat{P})$.

Suppose $\widehat{P}$ is the frame bundle of a vector bundle $\widehat{\mathcal{E}} \to \widehat{C}$, and let $\mathcal{E} \to C$ denote the sheaf of invariant sections of $\widehat{\mathcal{E}}$. The parabolic Atiyah algebra ${}^{par}\mathbf{At}(\mathcal{E})$ is $\mathbf{aut}_\Gamma(\widehat{P})$. We wish to describe the kernel $\mathbf{ad}_\Gamma(\widehat{P})$ as a subsheaf of $\mathbf{aut}_\Gamma(\widehat{P}) \to TC(-D)$. Let $\widehat{U} \subset \widehat{C}$ be a neighborhood of $w_0 \in \widehat{D}$, and let $\Gamma_0 \subset \Gamma$ be the isotropy group of w_0. We assume that Γ_0 stabilizes $\widehat{U}$ and only $w_0 \in \widehat{U}$ has nontrivial isotropy. Choose a section s of $\widehat{P}$ over $\widehat{U}$. For each $\gamma \in \Gamma_0$ there is $\rho_\gamma : \widehat{U} \to G$ defined by: $F_\gamma(s(w)) = s(\gamma w)\rho_\gamma(w)$.

Definition 4.8. Define $\mathbf{ad}_\Gamma(\widehat{P})(U) = \left\{f \in \mathbf{ad}(\widehat{P})(\widehat{U}) \mid f(s(\gamma w)) = \mathrm{Ad}_{\rho_\gamma(w)} f(s(w))\right\}$.

The following is straightforward.

Proposition 4.9. The above definition is independent of the choice of section s.

Extending the definition for neighborhoods at each of the branch points defines $\mathbf{ad}_\Gamma(\widehat{P})$ globally. We also note that the inclusion map $\mathbf{ad}(\widehat{P}) \to \mathbf{aut}(\widehat{P})$ restricts to an inclusion map $\mathbf{ad}_\Gamma(\widehat{P}) \to \mathbf{aut}_\Gamma(\widehat{P})$. Indeed, for $f \in \mathbf{ad}_\Gamma(\widehat{P})$, it suffices to check the condition on the vector field $X = f^\sharp$ along the section s, and this follows from the equivariance of F_γ:

$$f(F_\gamma(s(w))) = f(s(\gamma w)\rho_\gamma(w)) = \mathbf{Ad}_{\rho_\gamma(w)^{-1}} f(s(\gamma w)) = f(s(w)).$$

Hence we have the following short exact sequence of sheaves of Lie algebras on C:

$$0 \longrightarrow \mathbf{ad}_\Gamma(\widehat{P}) \longrightarrow \mathbf{aut}_\Gamma(\widehat{P}) \longrightarrow TC(-D) \longrightarrow 0.$$

Suppose that $\widehat{P}$ is the frame bundle of a vector bundle $\widehat{\mathcal{E}} \to \widehat{C}$, and let $\mathcal{E} \to C$ denote the sheaf of invariant sections of $\widehat{\mathcal{E}}$. The parabolic endomorphism bundle of $\mathcal{E}$, namely $\mathbf{Par}(\mathcal{E})$, is $\mathbf{ad}_\Gamma(\widehat{P})$. We refer to the above sequence as the *fundamental sequence* for parabolic Atiyah algebras. From the discussion above, we have an exact sequence

$$0 \longrightarrow \mathbf{Par}(\mathcal{E}) \longrightarrow {}^{par}\mathbf{At}(\mathcal{E}) \longrightarrow TC(-D) \longrightarrow 0.$$

We note that this is nothing but the Γ-invariant push-forward of the Atiyah algebra exact sequence on $\widehat{C}$.

4.3.2 *Strongly parabolic Atiyah algebras*

Next, we define a "strongly parabolic" version of this construction. Set

$$\mathbf{aut}(\widehat{P})(-\widehat{D}) := \{X \in \mathbf{aut}(\widehat{P}) \mid X(p) = 0 \text{ for } \widehat{\pi}(p) \in \widehat{D}\}.$$
$$\mathbf{ad}(\widehat{P})(-\widehat{D}) := \{f \in \mathbf{ad}(\widehat{P}) \mid f(p) = 0 \text{ for } \widehat{\pi}(p) \in \widehat{D}\}.$$

Then we have the restricted short exact sequence

$$0 \longrightarrow \mathbf{ad}(\widehat{P})(-\widehat{D}) \longrightarrow \mathbf{aut}(\widehat{P})(-\widehat{D}) \longrightarrow T\widehat{C}(-\widehat{D}) \longrightarrow 0.$$

Now if $\widehat{P}$ is the frame bundle of $\widetilde{E}$, then by Seshadri's correspondence (see Appendix B), the Γ-invariant part $\mathbf{ad}_\Gamma(\widehat{P}(-\widehat{D}))$ is identified with the *strongly* parabolic endomorphisms $\mathbf{SPar}(P)$. Motivated by this observation, we have:

Definition 4.10. The strongly parabolic Atiyah algebra ${}^{spar}\mathrm{At}(P)$ of a Γ-linearized principal bundle $\widehat{P}$ on $\widehat{C}$ is defined to be the Γ-invariant part of $\mathbf{aut}(\widehat{P})(-\widehat{D})$.

If $\widehat{P}$ is the frame bundle of a vector bundle $\widehat{\mathcal{E}}$, then we denote it by ${}^{spar}\mathbf{At}(\mathcal{E}) := \mathbf{aut}_\Gamma(\widehat{P})$. Noting that $p_*(T\widehat{C}(-\widehat{D}))^\Gamma = TC(-D)$, we have the following short exact sequence on C

$$0 \longrightarrow \mathbf{SPar}(\mathcal{E}) \longrightarrow {}^{spar}\mathbf{At}(\mathcal{E}) \longrightarrow TC(-D) \longrightarrow 0.$$

4.3.3 *Determinant line bundle for parabolic G-bundles*

Let $\widehat{\mathcal{P}}$ be a family of (Γ, G)-bundles parametrized by T as in the previous section, and let $\mathcal{P}$ be the family of parabolic G-bundles obtained by applying the invariant pushforward functor.

Consider the relative parabolic Atiyah algebra ${}^{par}\mathbf{At}_{X/T}(\mathcal{P}) := p_*^\Gamma(\mathrm{At}_{\widehat{X}/T}(\widehat{\mathcal{P}})$ and the strongly parabolic Atiyah algebra ${}^{spar}\mathbf{At}_{X/T}(\mathcal{P}) = p_*^\Gamma(\mathrm{At}_{\widehat{X}/T}(\widehat{\mathcal{P}})(-\widehat{D}))$. As in the case of parabolic vector bundles, they fit in the following fundamental exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathbf{Par}(\mathcal{P}) \longrightarrow {}^{par}\mathbf{At}_{X/T}(\mathcal{P}) \longrightarrow \mathcal{T}_{X/T}(-D) \longrightarrow 0, \\ 0 &\longrightarrow \mathbf{SPar}(\mathcal{P}) \longrightarrow {}^{spar}\mathbf{At}_{X/T}(\mathcal{P}) \longrightarrow \mathcal{T}_{X/T}(-D) \longrightarrow 0, \end{aligned} \tag{49}$$

where $\mathbf{Par}(\mathcal{P})$ (respectively, $\mathbf{SPar}(\mathcal{P})$) denote the parabolic (respectively, strongly parabolic) endomorphism bundle of $\mathcal{P}$. As in the case of parabolic vector bundles we get the following quasi-Lie algebra:

$$0 \longrightarrow \Omega_{X/T} \longrightarrow ({}^{par}\mathbf{At}_{\widetilde{X}/T}(\mathcal{P})(D))^{\vee} \longrightarrow (\mathbf{SPar}(\mathcal{P})(D))^{\vee} \longrightarrow 0. \tag{50}$$

The Cartan-Killing form $\kappa_{\mathfrak{g}}$ gives an identification

$$\nu_{\mathfrak{g}}^{-1} : (\mathbf{SPar}(\mathcal{P})(D))^{\vee} \xrightarrow{\sim} \mathbf{Par}(\mathcal{P}). \tag{51}$$

Pulling back the exact sequence in (50) by the above isomorphism, we get a quasi-Lie algebra ${}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{P})$ fitting into the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & (p_*^{\Gamma}(\mathrm{At}_{\widetilde{X}/T}(\widehat{\mathcal{P}})(-\widehat{D}))(D))^{\vee} & \longrightarrow & (\mathbf{SPar}(\mathcal{P})(D))^{\vee} \cong (p_*^{\Gamma}(\mathbf{ad}(\widehat{\mathcal{P}})(-\widehat{D}))(D))^{\vee} & \longrightarrow & 0 \\ & & \| & & \cong\uparrow & & \cong\uparrow & & \\ 0 & \longrightarrow & \Omega_{X/T} & \longrightarrow & {}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{P}) & \longrightarrow & \mathbf{Par}(\mathcal{P}) & \longrightarrow & 0 \end{array}$$

which should be considered as a parabolic G-bundle analog of Ginzburg's dgla considered in [13]. Recall that we have the quasi-Lie algebra $\mathfrak{S}^{-1}(\widehat{\mathcal{P}})$ associated to a family $\widehat{\mathcal{P}}$ of principal G-bundles satisfying:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{\widetilde{X}/T} & \longrightarrow & \mathrm{At}_{\widetilde{X}/T}(\widehat{\mathcal{P}})^{\vee} & \longrightarrow & \mathbf{ad}(\widehat{\mathcal{P}})^{\vee} & \longrightarrow & 0 \\ & & \| & & \cong\uparrow & & \cong\uparrow & & \\ 0 & \longrightarrow & \Omega_{\widetilde{X}/T} & \longrightarrow & \mathfrak{S}^{-1}_{\widetilde{X}/T}(\widehat{\mathcal{P}}) & \longrightarrow & \mathbf{ad}(\widehat{\mathcal{P}}) & \longrightarrow & 0. \end{array}$$

Now since $\widehat{\mathcal{P}}$ is Γ-linearized, we conclude that all objects in the above exact sequence are Γ-linearized. We define ${}^{par}\mathfrak{S}^{-1}_{X/T}(P) := p_*^{\Gamma}\mathfrak{S}^{-1}_{\widetilde{X}/T}(\widehat{\mathcal{P}})$.

Taking Γ-invariant pushforward of the bottom row, we get the following extension:

$$0 \longrightarrow \Omega_{X/T} \longrightarrow {}^{par}\mathfrak{S}^{-1}_{X/T}(\mathcal{P}) \longrightarrow \mathbf{Par}(\mathcal{P}) \longrightarrow 0.$$

We now have the following proposition:

Proposition 4.11. There is an isomorphism ${}^{par}\mathfrak{S}^{-1}_{X/T}(\mathcal{P}) \xrightarrow{\sim} {}^{spar}\widetilde{\mathrm{At}}_{X/T}(\mathcal{P})$, which induces identity maps on $\Omega_{X/T}$ and $\mathbf{Par}(\mathcal{P})$.

Proof. Recall that there is a natural nondegenerate pairing $\langle , \rangle : \mathfrak{S}^{-1}_{\widetilde{X}/T}(\widehat{\mathcal{P}}) \times \mathrm{At}_{\widetilde{X}/T}(\widehat{\mathcal{P}}) \to \mathcal{O}_{\widetilde{X}}$. Tensoring it with $\mathcal{O}_{\widetilde{X}}(-\widehat{D})$, we get a $\mathcal{O}_{\widetilde{X}}(-\widehat{D})$-valued pairing

$$\langle , \rangle : \mathfrak{S}^{-1}_{\widetilde{X}/T}(\widehat{\mathcal{P}}) \times \mathrm{At}_{\widetilde{X}/T}(\widehat{\mathcal{P}})(-\widehat{D}) \longrightarrow \mathcal{O}_{\widetilde{X}}(-\widehat{D}).$$

Taking invariant pushforward, we get the following nondegenerate pairing:

$$p_*^{\Gamma}\langle , \rangle : p_*^{\Gamma}\mathfrak{S}^{-1}_{\widetilde{X}/T}(\widehat{\mathcal{P}}) \times p_*^{\Gamma}(\mathrm{At}_{\widetilde{X}/T}(\widehat{\mathcal{P}})(-\widehat{D})) \longrightarrow p_*^{\Gamma}\mathcal{O}_{\widetilde{X}}(-\widehat{D}).$$

This produces a duality between ${}^{par}\mathfrak{S}^{-1}_{X/T}(\mathcal{P})$ and $p_*^{\Gamma}(\mathrm{At}_{\widetilde{X}/T}(\widehat{\mathcal{P}})(-\widehat{D}))(D)$. This completes the proof of the proposition. ∎

4.4 The general set of parabolic G-bundles

Let $M_G^{\mathbf{r},ss}$ (respectively, $M_G^{\mathbf{r},rs}$) be the moduli space of semistable (respectively, regularly stable) (Γ, G) bundles on a curve $\widehat{C}$, and let $\phi : G \to \mathrm{SL}_r$ be a representation. We assume without loss of generality that $\widehat{M}_G^{rs}$ is non-empty. Note that for a semistable (Γ, G), the underlying G-bundle is semistable [2, 7]. We also use the same notation for a relative family of Γ covers $\widehat{\mathcal{C}} \to S$. Consider the induced maps

$M_G^{\boldsymbol{\tau},ss} \to \widehat{M}_G^{ss} \xrightarrow{\phi} \widehat{M}_{\mathrm{SL}_r}^{ss}$. Let $\widehat{M}_G^{rs}$ be the locus of regularly stable bundles on $\widehat{C}$. Then by [12], we get that the complement of the regularly stable locus is at least two provided $g \geq 2$ and G is any simple group different from SL_2 and $g \geq 3$ if the Lie algebra of G has a $\mathfrak{sl}(2)$ factor. Now as in the SL_r case, let $Y_G^{\boldsymbol{\tau},rs}$ be the inverse image of $\widehat{M}_G^{rs}$ in $M_G^{\boldsymbol{\tau},ss}$. Moreover, the complement of $Y_G^{\boldsymbol{\tau},rs}$ has codimension at least two provided the genus $g(\mathscr{C})$ of the orbifold curve $\mathscr{C} = [\widehat{C}/\Gamma]$ (cf. [9, Lemma 8.3]) determined by $\boldsymbol{\tau}$ is at least three, or if the Lie algebra of G has no $\mathfrak{sl}(2)$ or $\mathfrak{sp}(4)$ factor, $g(\mathscr{C}) \geq 2$.

As before let $\mathcal{L}_\phi$ be the pull-back of the determinant of cohomology $\mathcal{L}$ to $M_G^{\boldsymbol{\tau},ss}$. Then by applying Proposition 4.11, and Theorem 3.7, we get the following.

Corollary 4.12. There is a natural isomorphism of Atiyah algebras over the regularly stable locus $Y_G^{\boldsymbol{\tau},rs}$:

$$\frac{1}{m_\phi}\mathrm{At}_{M_G^{\boldsymbol{\tau},rs}/S}(\mathcal{L}_\phi) \xrightarrow{\sim} \mathrm{R}^1\pi_{n*}\left({}^{par}\mathfrak{S}^{-1}_{\mathcal{C}\times_S M_G^{\boldsymbol{\tau},rs}/M_G^{\boldsymbol{\tau},rs}}(\mathcal{P})\right).$$

Since by assumption, the complement of $Y_G^{\boldsymbol{\tau},rs}$ in $M_G^{\boldsymbol{\tau},rs}$ is at least two, the above isomorphism extends over the entire space $M_G^{\boldsymbol{\tau},rs}$.

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References

1. Atiyah, M. F. "Complex analytic connections in fibre bundles." *Trans. Amer. Math. Soc.* **85** (1957, ISSN 0002-9947): 181–207. https://doi.org/10.1090/S0002-9947-1957-0086359-5. https://doi.org/10.2307/1992969.
2. Balaji, V., I. Biswas, and D. S. Nagaraj. "Principal bundles over projective manifolds with parabolic structure over a divisor." *Tohoku Math. J. (2)* **53**, no. 3 (2001, ISSN 0040-8735. doi: 10.2748/tmj/1178207416): 337–67. https://doi.org/10.2748/tmj/1178207416.
3. Beĭlinson, A., and J. Bernstein. "A proof of Jantzen conjectures." *Seminar*, Vol. 16 of *Adv. Soviet Math*, edited by Gelfand I. M. 1–50. Providence, RI: Amer. Math. Soc, 1993.
4. Beĭlinson, A. A., and V. V. Schechtman. "Determinant bundles and Virasoro algebras." *Comm. Math. Phys.* **118** (1988): 651. https://doi.org/10.1007/BF01221114. http://projecteuclid.org/euclid.cmp/1104162170.
5. Beĭlinson, A., and V. Drinfeld. *Quantization of Hitchin's Integrable System and Hecke Eigensheaves*. Preprint, 1988.
6. Biswas, I., and N. Raghavendra. "Determinants of parabolic bundles on Riemann surfaces." *Proc. Indian Acad. Sci. Math. Sci.* **103**, no. 1 (1993, ISSN 0253-4142): 41–71. https://doi.org/10.1007/BF02837895. https://doi.org/10.1007/BF02837895.
7. Biswas, I. "Parabolic bundles as orbifold bundles." *Duke Math. J.* **88**, no. 2 (1997, ISSN 0012-7094. doi: 10.1215/S0012-7094-97-08812-8): 305–25. https://doi.org/10.1215/S0012-7094-97-08812-8.
8. Biswas, I. Stable bundles and extension of structure group. *Differential Geom. Appl.* **23** (2005): 67. doi: 10.1016/j.difgeo.2005.03.006. URL https://doi.org/10.1016/j.difgeo.2005.03.006.
9. Biswas, I., S. Mukhopadhyay, and R. Wentworth. "A Hitchin connection on nonabelian theta functions for parabolic G-bundles." *J. Reine Angew. Math.* **803** (2023, ISSN 0075-4102,1435-5345. doi: 10.1515/crelle-2023-0049): –137, 181. https://doi.org/10.1515/crelle-2023-0049.
10. Bloch, S., and H. Esnault. "Relative algebraic differential characters." *Motives, Polylogarithms and Hodge Theory, Part I (Irvine, CA, 1998)*, Vol. **3** of *Int. Press Lect. Ser.* 47–73. Somerville, MA: Int. Press, 2002.
11. Daskalopoulos, G. D., and R. A. Wentworth. "Geometric quantization for the moduli space of vector bundles with parabolic structure." *Geometry, Topology and Physics (Campinas, 1996)*, 119–55. Berlin: de Gruyter, 1997.
12. Faltings, G. "Stable G-bundles and projective connections." *J. Algebraic Geom.* **2**, no. 3 (1993): 507–68 ISSN 1056-3911.

13. Ginzburg, V. Resolution of diagonals and moduli spaces. In *The Moduli Space of Curves (Texel Island, 1994)*, Vol. **129** of *Progr. Math.*, pp. 231–266. Boston, MA: Birkhäuser Boston, 1995. doi: 10.1007/978-1-4612-4264-2_9.
14. Gunning, R. C. *Lectures on Riemann Surfaces*. Princeton Mathematical Notes. Princeton, N.J.: Princeton University Press, 1966.
15. Narasimhan, M. S. and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, **82** (1965. ISSN 0003-486X. doi: 10.2307/1970710): 540–567
16. Pauly, C., J. Martens, M. Bolognesi, and T. Baier. "The Hitchin connection in arbitrary characteristic." *J. Inst. Math. Jussieu* **22**, no. 1 (2023, ISSN 1474-7480,1475-3030. doi: 10.1017/S1474748022000196): 449–92. https://doi.org/10.1017/S1474748022000196.
17. Ramanathan, A. "Stable principal bundles on a compact Riemann surface." *Math. Ann.* **213** (1975, ISSN 0025-5831): 129–52. https://doi.org/10.1007/BF01343949. https://doi-org.proxy-um.researchport.umd.edu/10.1007/BF01343949.
18. Seshadri, C. S. "Moduli of π-vector bundles over an algebraic curve." *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, 139–260. Rome: Edizioni Cremonese, 1970.
19. Sun, X., and I.-H. Tsai. "Hitchin's connection and differential operators with values in the determinant bundle." *J. Differential Geom.* **66** (2004, ISSN 0022-040X). https://doi.org/10.4310/jdg/1102538613. http://projecteuclid.org/euclid.jdg/1102538613.