

Quiver DT Invariants and Log Gromov–Witten Theory of Toric Varieties

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ABSTRACT. We review how log Gromov–Witten invariants of toric varieties can be used to express quiver Donaldson–Thomas invariants in terms of the simpler attractor Donaldson–Thomas invariants. This is an exposition of joint work with Pierrick Bousseau [6, 8].

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0. Introduction

Donaldson–Thomas (DT) invariants of a quiver with potential can be expressed in terms of simpler attractor DT invariants by a universal formula. The coefficients in this formula are calculated combinatorially using attractor flow trees. In this paper, we give an introduction to the main result of [8] showing that these coefficients are genus 0 log Gromov–Witten invariants of d -dimensional toric varieties, where d is the number of vertices of the quiver.

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1. Quiver DT invariants

1.1. Quiver representations and moduli spaces. A **quiver** is a finite oriented graph Q . We denote by

- Q_0 : the finite set of vertices of Q ,
- Q_1 : the finite set of arrows (oriented edges) of Q ,
- $s : Q_1 \rightarrow Q_0$ the source map which maps an arrow to its source, and;
- $t : Q_1 \rightarrow Q_0$ the target map, which maps an arrow to its target.

We illustrate a quiver in Figure 1.1.

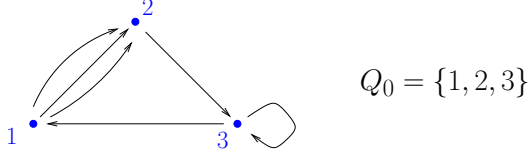


FIGURE 1.1. A quiver

DEFINITION 1.1. A **representation** of a quiver Q is an assignment of

- (i) a finite-dimensional vector space V_i over $\mathbb{C}$ for each vertex $i \in Q_0$, and
- (ii) a $\mathbb{C}$ -linear transformation $f_\alpha \in \text{Hom}(V_{s(\alpha)}, V_{t(\alpha)})$ for each arrow $\alpha \in Q_1$.

Representations of Q naturally form an abelian category: a morphism between (V_i, f_α) and (V'_i, f'_α) is the data of linear maps $V_i \rightarrow V'_i$ intertwining the maps f_α and f'_α . In what follows, we fix

$$N = \mathbb{Z}^{Q_0} = \bigoplus_{i \in Q_0} \mathbb{Z}e_i,$$

and $N^\oplus = \mathbb{N}^{Q_0} \subset N$. The **dimension** of a quiver representation (V_i, f_α) is defined as the vector

$$\gamma = (\gamma_i)_{i \in Q_0} \in N^\oplus \subset N,$$

where $\gamma_i := \dim V_i$ – see Figure 1.2 for an illustration.

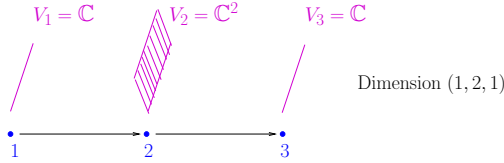


FIGURE 1.2. A quiver with dimension vector $\gamma = (1, 2, 1)$

There is a notion of stability due to King in the context of quiver representations, defined as follows [33]. Denote $M := \text{Hom}(N, \mathbb{Z})$ the dual lattice to N and set

$$M_{\mathbb{R}} = \text{Hom}(N, \mathbb{R}) = M \otimes \mathbb{R}.$$

DEFINITION 1.2 (King's stability). A **stability parameter** for $\gamma \in N$ is a point $\theta \in \gamma^\perp := \{\theta \in M_{\mathbb{R}}, \theta(\gamma) = 0\} \subset M_{\mathbb{R}}$. A quiver representation V of dimension γ is **θ -stable** (resp. **θ -semistable**) if for all non-zero strict subrepresentation V' of V we have $\theta(\dim(V')) < 0$ (resp. $\theta(\dim(V')) \leq 0$).

Given a dimension vector $\gamma \in N^\oplus$ and a stability parameter $\theta \in M_{\mathbb{R}}$, one obtains by geometric representation theory [33, Proposition 4.2] a moduli space $\mathcal{M}_\gamma^\theta$ of S-equivalence classes of θ -semistable quiver representations of Q of dimension γ . It is a quasi-projective variety over $\mathbb{C}$, which is actually projective if Q is acyclic.

When working with a non-acyclic quiver Q , we consider the additional data of a **potential** W , that is, a formal linear combination of oriented cycles of Q . A choice

of potential W defines a **trace function** $\mathrm{Tr}(W)_\gamma^\theta : \mathcal{M}_\gamma^\theta \rightarrow \mathbb{C}$ on each of the moduli spaces $\mathcal{M}_\gamma^\theta$: if c is an oriented cycle of arrows $\alpha_1, \dots, \alpha_n$ of Q , $\mathrm{Tr}(c)_\gamma^\theta : \mathcal{M}_\gamma^\theta \rightarrow \mathbb{C}$ is defined by

$$V = (V_i, f_\alpha) \mapsto \mathrm{Tr}(f_{\alpha_n} \circ \dots \circ f_{\alpha_1})$$

and we set

$$\mathrm{Tr}(W)_\gamma^\theta = \sum_c \lambda_c \mathrm{Tr}(c)_\gamma^\theta$$

if W is the linear combination $\sum_c \lambda_c c$ of oriented cycles.

1.2. DT invariants. Let (Q, W) be a quiver with potential, $\gamma \in N^\oplus$ a dimension vector, and θ a generic stability parameter for γ . The Donaldson-Thomas (DT) invariant $\Omega_\gamma^{+, \theta}$ is an integer which is a virtual count of the critical points of the trace function $\mathrm{Tr}(W)_\gamma^\theta$ on the moduli space $\mathcal{M}_\gamma^\theta$ of θ -semistable representations of dimension γ . We provide a more concrete description of this count in what follows.

In the situation when the θ -stable locus in $\mathcal{M}_\gamma^\theta$ is empty, we have $\Omega_\gamma^{+, \theta} = 0$, and else, $\Omega_\gamma^{+, \theta}$ is the Euler characteristic of $\mathcal{M}_\gamma^\theta$ valued in the perverse sheaf obtained by applying the vanishing-cycle functor for $\mathrm{Tr}(W)_\gamma^\theta$ to the intersection cohomology sheaf [18, 19, 39]:

$$\Omega_\gamma^{+, \theta} := e(M_\gamma^\theta, \phi_{\mathrm{Tr}(W)_\gamma^\theta}(IC[-\dim \mathcal{M}_\gamma^\theta])).$$

For example, if γ is primitive and $W = 0$, then $\Omega_\gamma^{+, \theta}$ is simply the topological Euler characteristic¹ of $\mathcal{M}_\gamma^\theta$. The DT invariants $\Omega_\gamma^{+, \theta}$ can be repackaged into the **rational DT invariants**

$$\bar{\Omega}_\gamma^{+, \theta} := \sum_{\substack{\gamma' \in N \\ \gamma = k\gamma', k \in \mathbb{Z}_{\geq 1}}} \frac{(-1)^{k-1}}{k^2} \Omega_{\gamma'}^{+, \theta},$$

which can also be defined using the motivic Hall algebra [31, 34, 45, 46].

The quiver DT invariants $\Omega_\gamma^{\theta, +}$ have a piecewise constant dependence on $\theta \in \gamma^\perp$: there is a natural wall and chamber structure on $M_\mathbb{R}$ as illustrated in Figure 1.3, and while the quiver DT invariants remain constant as long as θ remains inside a chamber, they change by a universal **wall-crossing formula** [31, 34] once we cross a wall and pass to another chamber – see 1.3.

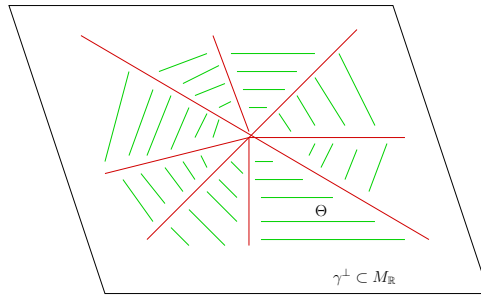


FIGURE 1.3. A wall and chamber structure

¹Here, we are considering the “unsigned” DT invariants $\Omega_\gamma^{+, \theta}$, and not the signed DT invariants Ω_γ^θ , which are equal to $(-1)^{\dim \mathcal{M}_\gamma^\theta} e(\mathcal{M}_\gamma^\theta)$ when γ is primitive and $W = 0$.

EXAMPLE 1.3. Let Q be the n -Kronecker quiver, consisting of two vertices connected by n arrows as illustrated in Figure 1.4. Fix the dimension vector $\gamma := \dim(V) = (1, 1) \in N$ and choose a stability parameter $\theta = (\theta_1, -\theta_1) \in \gamma^\perp \subset M_{\mathbb{R}}$. The data of n linear maps $\mathbb{C} \rightarrow \mathbb{C}$ is given by a tuple of n complex numbers $(\xi_1, \dots, \xi_n)$, and a global rescaling of $(\xi_1, \dots, \xi_n)$ by a non-zero constant produces isomorphic representations. For $\theta_1 > 0$, θ -semistability requires $(\xi_1, \dots, \xi_n) \neq (0, \dots, 0)$, and in this case, the moduli space of θ -semistable quiver representations $\mathcal{M}_\gamma^\theta$ is $\mathbb{CP}^{n-1}$, and so

$$\Omega_\gamma^{\theta,+} = e(\mathbb{CP}^{n-1}) = n.$$

On the other hand, for $\theta_1 < 0$ we have $\mathcal{M}_\gamma^\theta = \emptyset$, and so $\Omega_\gamma^{\theta,+} = 0$.

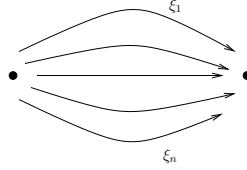


FIGURE 1.4. The n -Kronecker quiver

Quiver DT invariants are extensively studied both in the physics as well as in the mathematics literature. In physics they appear in the context of supersymmetric quantum mechanics, and of BPS states in $\mathcal{N} = 2$ four-dimensional quantum field theories and string theory compactifications [5, 10, 38], whereas in mathematics they are closely related to geometric DT theory, which concerns counts of coherent sheaves or special Lagrangians in Calabi–Yau threefolds [31, 34, 50] – see Table 1.

EXAMPLE 1.4. The 3-Kronecker quiver appears in the description of the BPS spectrum of $\mathcal{N} = 2$, four-dimensional $SU(3)$ super Yang-Mills theory [23]. In this context, explicit values of quiver DT invariants are calculated –see [23, A.2] and Figure 1.5. Here, for every $\gamma = (\gamma_1, \gamma_2) \in N^\oplus$, the stability parameter $\theta = (\theta_1, \theta_2) \in \gamma^\perp$, and so with $n_1\theta_1 + n_2\theta_2 = 0$, satisfies $\theta_1 > 0$ (and so $\theta_2 < 0$). If $\theta_1 < 0$, the DT invariant is zero, as in Example 1.3.

1.3. The flow tree formula. Quiver DT invariants can be expressed in terms of a simpler set of quiver DT invariants, known as attractor DT invariants defined as follows. Let $\langle -, - \rangle$ be the skew-symmetrized Euler form on

$$N = \mathbb{Z}^{Q_0} = \bigoplus_{i \in Q_0} \mathbb{Z}e_i.$$

It is the skew-symmetric form on N defined by $\langle e_i, e_j \rangle = a_{ij} - a_{ji}$, where a_{ij} is the number of arrows from i to j in Q . Given $\gamma \in N$, $\langle \gamma, - \rangle$ is a point in the hyperplane $\gamma^\perp$ of stability parameters for γ , referred to as the **attractor point**.

DEFINITION 1.5. [4, 35, 42] Fix a dimension vector $\gamma \in N^\oplus$. The **attractor DT invariant** for γ is defined by $\Omega_\gamma^* := \Omega_\gamma^\theta$, where θ is a small generic perturbation in $\gamma^\perp$ of the attractor point $\langle \gamma, - \rangle$ ².

²One can show that the attractor DT invariant is independent of the choice of perturbation, see [42, Thm 3.7].

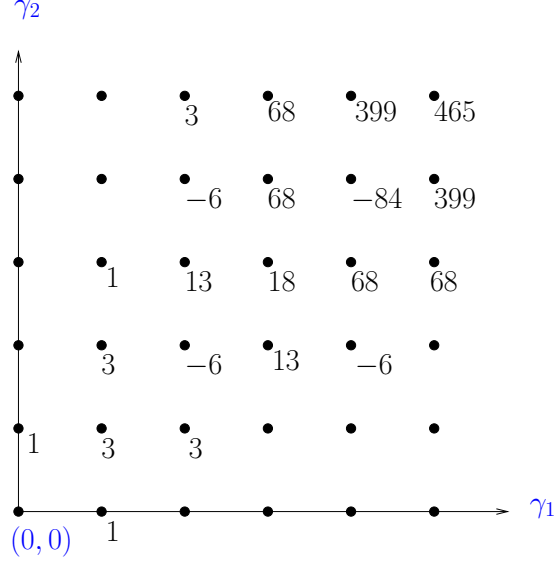
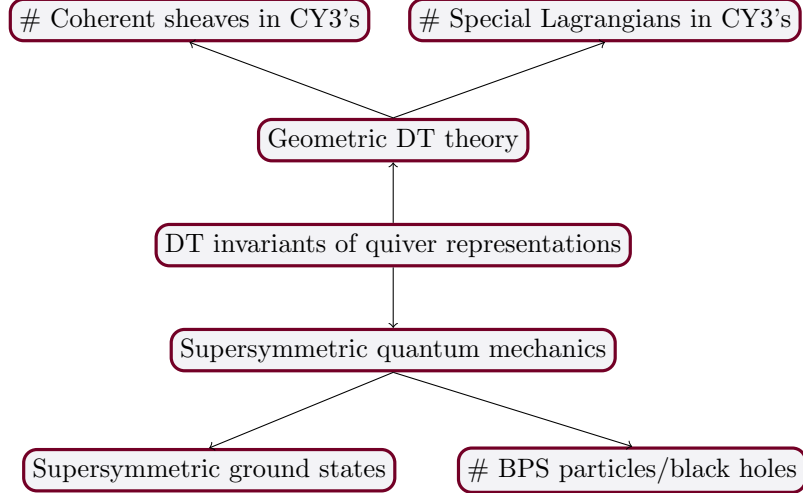

 FIGURE 1.5. Values of Ω_γ^θ for the 3-Kronecker quiver [23, A.2]


TABLE 1. DT invariants of quivers in maths and physics

The attractor DT invariants turn out to be particularly simple in many situations.

EXAMPLE 1.6. *It is shown by Bridgeland [16] that if Q is acyclic then*

$$\Omega_\gamma^\star = \begin{cases} 1 & \text{if } \gamma = e_i \text{ for some } i \in Q_0 \\ 0 & \text{otherwise} \end{cases}$$

This theorem has been generalized by Lang Mou to some situations when Q is not acyclic, but admits a so called red to green sequence [40].

EXAMPLE 1.7. *Given a toric Calabi-Yau 3-fold, one can construct a quiver with a potential (Q, W) such that*

$$D^b \text{Rep}(Q, W) \cong D^b \text{Coh}(X),$$

where $D^b \text{Rep}(Q, W)$ is the bounded derived category of representations of (Q, W) and $D^b \text{Coh}(X)$ is the bounded derived category of coherent sheaves on X [41].

Beaujard–Manschot–Pioline [11] and Mozgovoy–Pioline [42] conjectured in this situation that one gets $\Omega_\gamma^\star = 0$ unless either $\gamma = e_i$ for some $i \in Q_0$ (in which case one has $\Omega_\gamma^\star = 1$) or γ is in the kernel of $\langle -, - \rangle$. A more precise conjecture including the values of all non-trivial attractor invariants has been formulated by Descombes [20]. This conjecture is proven for local $\mathbb{P}^2$ by Bousseau–Descombes–Le Floch–Pioline [14].

More generally, attractor DT invariants are expected to take a simple form in physics set-ups without dynamical gravity, such as string compactifications on non-compact Calabi-Yau 3-folds. For theories with dynamical gravity, such as string compactifications on compact Calabi-Yau 3-folds, one typically expects attractor invariants to be sufficiently non-trivial to account for the entropy of black holes [38].

Fix a dimension vector $\gamma \in N^\oplus$, and a generic stability parameter $\theta \in \gamma^\perp$. By an iterative application of the wall-crossing formula [35], quiver DT invariants can be obtained in terms of attractor DT invariants by a formula of the form

$$(1.1) \quad \bar{\Omega}_\gamma^\theta = \sum_{\gamma = \gamma_1 + \dots + \gamma_r} \frac{1}{|\text{Aut}((\gamma_i)_i)|} F_r^\theta(\gamma_1, \dots, \gamma_r) \prod_{i=1}^r \bar{\Omega}_{\gamma_i}^\star,$$

where the sum is over all possible decompositions of γ into $\gamma_1 + \dots + \gamma_r$ such that $\gamma_i \in N^\oplus$ for $1 \leq i \leq r$ and $r \in \mathbb{N}$, the term $|\text{Aut}((\gamma_i)_i)|$ is the order of the group of permutation symmetries of the decomposition $\gamma = \gamma_1 + \dots + \gamma_r$, and the coefficients $F_r^\theta(\gamma_1, \dots, \gamma_r)$ are sums of contributions from attractor trees with leaves decorated by $\gamma_1, \dots, \gamma_r$ and with root at θ :

$$F_r^\theta(\gamma_1, \dots, \gamma_r) = \sum_{T \in \mathcal{T}_{\gamma_1, \dots, \gamma_r}^\theta} F_{r,T}^\theta(\gamma_1, \dots, \gamma_r).$$

These trees, as illustrated in Figure 1.6, are oriented trees with one root vertex corresponding to the point $\theta \in M_\mathbb{R}$ and edges have direction vectors determined by the decompositions of the dimension vector γ .

By construction, attractor trees satisfy the “tropical balancing condition” – that is, weighted directions around edges add up to zero. Hence, they can be viewed as tropical trees. As illustrated in Figure 1.6, these trees have in general vertices of valency bigger than 3, and calculating their contributions in this case is technically challenging, as it requires the application of the wall-crossing formula.

In joint work with Bousseau [6], we proved the **flow tree formula** conjectured by Alexandrov–Pioline [4] computing the coefficients $F_r^\theta(\gamma_1, \dots, \gamma_t)$ in terms of binary trees as illustrated in Figure 1.7, counted with simple tropical multiplicities attached to the trivalent vertices. These binary trees are obtained as generic trivalent perturbations of the non-trivalent attractor trees.

In the following section we explain how to interpret the coefficients $F_r^\theta(\gamma_1, \dots, \gamma_r)$ in (1.1) geometrically in terms of counts of rational curves in toric varieties, by relating contributions of binary trees to such curve counts.

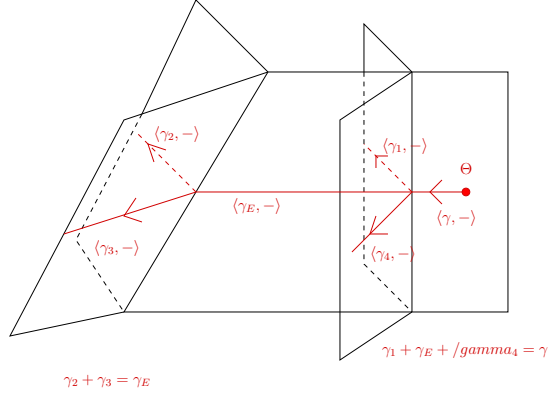
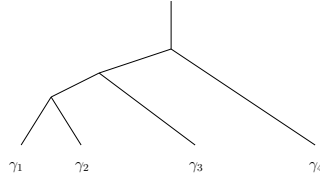


FIGURE 1.6. An attractor tree with higher valent vertices


 FIGURE 1.7. A binary tree with leaves labelled by $\gamma_1, \dots, \gamma_n$, for $n = 4$

2. Log Gromov–Witten invariants of toric varieties

The main result of [6] relates DT invariants of quivers to log Gromov–Witten invariants given by counts of log curves toric varieties. After briefly explaining the toric varieties we consider and the enumerative problem of curves, we provide further details of log geometry and log Gromov–Witten theory. In the final section we state the relationship between counts of log curves in toric varieties and quiver DT invariants, and provide a brief summary of the proof.

2.1. From quivers to enumerative geometry in toric varieties. Fix a quiver Q , a dimension vector $\gamma \in N^\oplus$ and a decomposition

$$\gamma = \gamma_1 + \dots + \gamma_r.$$

Recall that we denote by $\langle -, - \rangle$ the skew-symmetrized Euler form on N . We assume that $\langle \gamma_i, - \rangle \neq 0$ for all $1 \leq i \leq r$, and $\langle \gamma, - \rangle \neq 0$.

Let Σ be a fan in $M_\mathbb{R}$ of a smooth projective toric variety X_Σ containing the rays $\mathbb{R}_{\geq 0}\langle \gamma_i, - \rangle$ for all $1 \leq i \leq r$. We consider hypersurfaces $H_i \subset D_i$ defined by equations

$$(z^{\frac{\gamma_i}{|\gamma_i|}} + c_i)|_{D_i} = 0,$$

where $|\gamma_i|$ is the divisibility of γ_i in N , and c_i is a general constant.

The count of log curves which we show are related to quiver DT invariants in [6] are given by appropriately defined counts of marked genus 0 stable maps

$$f : (C, \{p_1, \dots, p_{r+1}\}) \rightarrow X_\Sigma$$

satisfying

- (i) $f(p_i) \in H_i$ for all $1 \leq i \leq r$
- (ii) The contact order of f with D_i at p_i is the divisibility of $\langle \gamma_i, - \rangle$ in M .

See Figure 2.1 for an illustration.

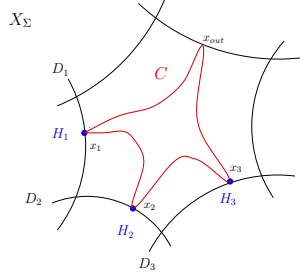


FIGURE 2.1. A log curve in a toric variety with prescribed tangencies along the toric boundary

To count such curves with prescribed tangency conditions, we use log Gromov–Witten theory developed by Abramovich–Chen [1] and Gross–Siebert [29]. In the following sections we briefly review log geometry and log Gromov–Witten theory before stating a precise relationship between counts of log curves and quiver DT invariants.

2.2. Log schemes. We start reviewing a couple of definitions from log geometry to fix our notation. For familiarity with log geometry we refer to [32, 44]. All the monoids considered below are assumed to be commutative.

DEFINITION 2.1. *Let X be a scheme. A **prelog structure** on X is a sheaf of monoids $\mathcal{M}$ on X together with a homomorphism of monoids $\alpha: \mathcal{M} \rightarrow \mathcal{O}_X$ where we consider the structure sheaf $\mathcal{O}_X$ as a monoid with respect to multiplication. A pre log structure on X is called a **log structure** if α induces an isomorphism*

$$\alpha|_{\alpha^{-1}(\mathcal{O}_X^\times)}: \alpha^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_X^\times.$$

*We call a scheme X endowed with a log structure a **log scheme**, and denote the log structure on X by $(\mathcal{M}_X, \alpha_X)$, or sometimes by omitting the structure homomorphism from the notation, just by $\mathcal{M}_X$. We denote a scheme X , endowed with a log structure by $(X, \mathcal{M}_X)$.*

All log schemes below are assumed to be fine and saturated (fs) as in [44, I, §1.3]. The combinatorial information of the log structure is encoded via tropical geometry using the ghost sheaf as defined below.

DEFINITION 2.2. *Let X be a log scheme. The **ghost sheaf** of X is the sheaf on X defined by*

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / \alpha^{-1}(\mathcal{O}_X^\times).$$

*The **tropicalization** $\Sigma(X)$ of X is an abstract polyhedral cone complex: $\Sigma(X)$ is the collection of cones $\sigma_{\bar{x}} := \text{Hom}(\mathcal{M}_{X, \bar{x}}, \mathbb{R}_{\geq 0})$ for every geometric point $\bar{x} \rightarrow X$, glued together using the natural generization and specialization maps, see [2, §2.1] for details.*

EXAMPLE 2.3. Let $D \subset X$ be a divisor. Let $j: X \setminus D \hookrightarrow X$ denote the inclusion map. The divisorial log structure on X is the pair $(\mathcal{M}_{(X,D)}, \alpha_X)$, where $\mathcal{M}_{(X,D)}$ is the sheaf of regular functions on X , that restrict to units on $X \setminus D$. That is,

$$\mathcal{M}_{(X,D)} = j_*(\mathcal{O}_{X \setminus D}^\times) \cap \mathcal{O}_X$$

The structure homomorphism α_X is given by the inclusion $\alpha_X: \mathcal{M}_{(X,D)} \hookrightarrow \mathcal{O}_X$.

REMARK 2.4. The log structure we consider on the toric variety X_Σ constructed from the data of a quiver, as discussed in 2.1, is the divisorial log structure defined by the toric boundary divisor D_Σ in X_Σ , that is, the natural anti-canonical divisor formed by the union of divisors fixed under the torus action on X_Σ .

Note that, for any toric variety X_Σ endowed with the divisorial log structure defined by the toric boundary divisor, the tropicalization of the resulting log scheme is naturally identified with the toric fan Σ associated to X .

Analogous to assigning a sheaf to a presheaf, we can assign a log structure to a prelog structure, using a fibered coproduct, as follows.

DEFINITION 2.5. Let $\alpha: \mathcal{P} \rightarrow \mathcal{O}_X$ be a prelog structure on X . We define the log structure associated to the prelog structure $(\mathcal{P}, \alpha)$ on X as follows. Set

$$\mathcal{P}^a = \mathcal{P} \oplus \mathcal{O}_X^\times / \{(p, \alpha(p)^{-1}) \mid p \in \alpha^{-1}(\mathcal{O}_X^\times)\}$$

and define the structure homomorphism $\alpha^a: \mathcal{P}^a \rightarrow \mathcal{O}_X$ by

$$\alpha^a(p, h) = h \cdot \alpha(p)$$

One can easily check that $(\mathcal{P}^a, \alpha^a)$ is a log structure on X .

Let Y be a log scheme, and let $f: X \rightarrow Y$ be a scheme theoretic morphism. Then, the log structure on Y , given by $\alpha_Y: \mathcal{M}_Y \rightarrow \mathcal{O}_Y$, induces a log structure on X defined as follows. First define a prelog structure, by considering the composition

$$f^{-1}\mathcal{M}_Y \rightarrow f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X.$$

Then, we endow X with the log structure associated to this prelog structure, as in Definition 2.5. We refer to this log structure as the *pull back log structure* or the *induced log structure* on X , and denote it by $\mathcal{M}_X = f^*\mathcal{M}_Y$.

EXAMPLE 2.6. For every scheme X , $\mathcal{M}_X = \mathcal{O}_X^\times$ define a log structure on X , called the *trivial log structure*. In particular, taking $X = \text{Spec } \mathbb{C}$, we call $(\text{Spec } \mathbb{C}, \mathbb{C}^\times)$ the *trivial log point*.

EXAMPLE 2.7. Let $X = \text{Spec } \mathbb{C}$. Define $\mathcal{M}_X = \mathbb{C}^\times \oplus \mathbb{N}$, and $\alpha_X: \mathcal{M}_X \rightarrow \mathbb{C}$ as follows.

$$\alpha_X(x, n) = \begin{cases} x & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

The corresponding log scheme $O_0 = (\text{Spec } \mathbb{C}, \mathbb{C}^\times \oplus \mathbb{N})$ is called the *standard log point*. One can check that the log structure on the standard log point is the same as the pull-back of the divisorial log structure on $\mathbb{A}^1$ with the divisor $D = \{0\} \subset \mathbb{A}^1$. One also obtains a more general log point $(\text{Spec } \mathbb{C}, \mathbb{C}^\times \oplus Q)$ by replacing $\mathbb{N}$ by a more general monoid Q such that $Q^\times = \{0\}$. It is the pullback to the torus fixed point of the divisorial log structure on the affine toric variety $\text{Spec } \mathbb{C}[Q]$. Its tropicalization is the cone $\text{Hom}(Q, \mathbb{R}_{\geq 0})$.

DEFINITION 2.8. A **morphism of log schemes** $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ is a morphism of schemes $f: X \rightarrow Y$ along with a homomorphism of sheaves of monoids $f^\sharp: f^{-1}\mathcal{M}_Y \rightarrow \mathcal{M}_X$ such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^\sharp} & \mathcal{M}_X \\ \alpha_Y \downarrow & & \downarrow \alpha_X \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^*} & \mathcal{O}_X. \end{array}$$

is commutative. Here, f^* is the usual pull-back of regular functions defined by the morphism f . Given a morphism of log spaces $f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$, we denote by $\underline{f}: X \rightarrow Y$ the underlying morphism of schemes. By abuse of notation, the underlying morphism on topological spaces is also denoted by $\underline{f}$.

We refer to [25, Defn. 3.23] for the notion of *log smooth morphism*. We recall that a toric variety is log smooth over the trivial log point. If $\pi: \mathcal{X} \rightarrow \mathbb{A}^1$ is a toric degeneration of toric varieties, then π can be naturally viewed as a log smooth morphism. In restriction to $t = 0$, we get a log smooth morphism $X_0 \rightarrow O_0$, that is, X_0 is log smooth over the standard log point O_0 . We also refer to [25, Remark 3.25] for the notion of **integral** morphism of log schemes.

2.3. Stable log maps. We recall that a ℓ -marked stable map with target a scheme X is a map $f: C \rightarrow X$, where C is a proper nodal curve with ℓ marked smooth points $\mathbf{x} = (x_1, \dots, x_\ell)$, such that the group of automorphisms of C fixing $\mathbf{x}$ and commuting with f is finite.

The notion of stable map has a natural generalization to the setting of logarithmic geometry [1, 29].

DEFINITION 2.9. Let $X \rightarrow B$ be a log morphism between log schemes. A **ℓ -marked stable log map with target $X \rightarrow B$** is a commutative diagram of log schemes

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \\ W & \longrightarrow & B, \end{array}$$

together with a tuple of sections $\mathbf{x} = (x_1, \dots, x_\ell)$ of π , where π is a proper log smooth and integral morphism of log schemes, such that, for every geometric point w of W , the restriction of $\underline{f}$ to w with the marked points $\underline{\mathbf{x}}(w)$ is an ordinary stable map, and such that, if $U \subset C$ is the non-critical locus of $\underline{\pi}$, we have $\overline{\mathcal{M}}_C|_U \simeq \pi^* \overline{\mathcal{M}}_W \oplus \bigoplus_{j=1}^\ell (x_j)_* \mathbb{N}_W$.

When the scheme underlying W in Definition 2.9 is a point, then W is a log point $(\text{Spec } \mathbb{C}, \mathbb{C}^* \oplus Q)$ as in Example 2.7. The general enumerative theory of stable log maps, called log Gromov–Witten theory and developed in [29, 1], is based on the notion of **basic stable log map**. For a basic stable log map over a log point $(\text{Spec } \mathbb{C}, \mathbb{C}^* \oplus Q)$, the monoid Q is the so-called **basic monoid**. The basic monoid is uniquely determined by the combinatorial type of the stable log map and has a natural tropical geometric interpretation: the dual cone $\text{Hom}(Q, \mathbb{R}_{\geq 0})$ is the base of a universal family of tropical curves with given combinatorial type. We provide a more precise description of the combinatorial type of a stable log map, and of basic monoids below.

The **combinatorial type** τ of a stable log map $f: C/W \rightarrow X/S$ consists of:

- (i) The dual intersection graph $G = G_C$ of C , with set of vertices $V(G)$, set of edges $E(G)$, and set of legs $L(G)$.
- (ii) The map $\sigma: V(G) \cup E(G) \cup L(G) \rightarrow \Sigma(X)$ mapping $x \in C$ to $(\overline{\mathcal{M}}_{X,f(x)})_{\mathbb{R}}^{\vee}$.
- (iii) The contact data $u_p \in \overline{\mathcal{M}}_{X,f(p)}^{\vee} = \text{Hom}(\overline{\mathcal{M}}_{X,f(p)}, \mathbb{N})$ and $u_q \in \text{Hom}(\overline{\mathcal{M}}_{X,f(q)}, \mathbb{Z})$ at marked points p and nodes q of C .

DEFINITION 2.10. *Given a combinatorial type τ of a stable log map $f: C/W \rightarrow X/S$, we define the associated **basic monoid** Q by first defining its dual*

$$Q_{\tau}^{\vee} = \left\{ ((V_{\eta})_{\eta}, (e_q)_q) \in \bigoplus_{\eta} \overline{\mathcal{M}}_{X,f(\eta)}^{\vee} \oplus \bigoplus_q \mathbb{N} \mid \forall q: V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}$$

where the sum is over generic points η of C and nodes q of C . We then set

$$Q_{\tau} := \text{Hom}(Q_{\tau}^{\vee}, \mathbb{N}).$$

Note that, by definition Q_{τ} indeed only depends on the combinatorial type of $f: C/W \rightarrow X/S$. Geometrically, one can interpret $Q_{\tau, \mathbb{R}}^{\vee} := \text{Hom}(Q_{\tau}, \mathbb{R}_{\geq 0})$ as the moduli cone of tropical curves of fixed combinatorial type.

Given a stable log map $f: C/W \rightarrow X/S$, one can show that there is a canonical map $Q \rightarrow \overline{\mathcal{M}}_W$, where Q is the basic monoid defined by the combinatorial type of f .

DEFINITION 2.11. *A stable log map $f: C/W \rightarrow X/S$ is said to be **basic** if the natural map of monoids $Q \rightarrow \overline{\mathcal{M}}_W$ is an isomorphism.*

For every $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$ and $u = (u_1, \dots, u_k)$ with $u_i \in |\Sigma(X)|$, we denote by $\mathcal{M}_{g,u}(X/S, \beta)$ the moduli space of genus g basic stable log maps to X/S of class β and with k marked points of contact data

$$u = u_1, \dots, u_k.$$

THEOREM 2.12 (Abramovich–Chen [1], Gross–Siebert [29], 2011). *If X/S is projective, then the moduli space $\mathcal{M}_{g,u}(X/S, \beta)$ is a proper Deligne–Mumford stack. Furthermore, If X/S is log smooth, then the moduli space $\mathcal{M}_{g,u}(X/S, \beta)$ admits a natural virtual fundamental class $[\mathcal{M}_{g,u}(X/S, \beta)]^{\text{virt}}$.*

2.4. Flow trees to tropical and log curves. Recall from §1.3 that quiver DT invariants can be calculated using a universal formula from simpler attractor DT invariants. Further, the coefficients $F_{r,T}^{\theta}(\gamma_1, \dots, \gamma_r)$ are indexed by attractor tropical trees T in $M_{\mathbb{R}}$. In this section we explain how to interpret these coefficients in terms of counts of log curves.

In [43], Nishinou–Siebert used toric degenerations and log deformation theory to prove a correspondence theorem between counts of genus 0 log curves in n -dimensional toric varieties and counts of tropical curves in $\mathbb{R}^n$. The counts of genus zero log curves considered in [43] are shown to correspond to genus zero log Gromov–Witten invariants by Mandel–Ruddat [37]. This tropical-log correspondence theorem is established in the context when the basic monoid is trivial. In our paper with Bousseau [8], we generalize this tropical-log correspondence to the situation of non-trivial basic monoids. In other words, we work with “families” of tropical trees corresponding to log curves over basic monoids of rank equal to the dimension of the family.

Fix a quiver Q , a dimension vector $\gamma \in N^\oplus$ and a decomposition $\gamma = \gamma_1 + \dots + \gamma_r$, such that $\langle \gamma_i, - \rangle \neq 0$ for all $1 \leq i \leq r$ and $\langle \gamma, - \rangle \neq 0$. We denote by d the number of vertices of Q , so that $N \simeq \mathbb{Z}^d$ and $M_{\mathbb{R}} \simeq \mathbb{R}^d$. Let X_Σ be a toric variety as in §2.1, with toric divisors D_i containing hypersurfaces H_i . Writing $\mathbf{H} = (H_1, \dots, H_r)$, let $\mathcal{M}_{\mathbf{H}}^{\log}(X_\Sigma)$ be the moduli space of genus 0 basic stable log maps $f : (C, \{p_1, \dots, p_{r+1}\}) \rightarrow X_\Sigma$ satisfying

- (i) $f(p_i) \in H_i$ for all $1 \leq i \leq r$
- (ii) The contact order of f with D_i at p_i is $\langle \gamma_i, - \rangle \in M$.
- (iii) The contact order of f with the toric boundary divisor of X_Σ at p_{r+1} is $-\langle \gamma, - \rangle$.

For general $\mathbf{H}$, one shows in [8, Theorem 2.8] that the moduli space $\mathcal{M}_{\mathbf{H}}^{\log}(X_\Sigma)$ is $(d-2)$ -dimensional and naturally stratified by the combinatorial types of stable log curves. For every combinatorial type ρ , there is a stratum $\mathcal{M}_{\rho, \mathbf{H}}^{\log}(X_\Sigma)$ consisting of stable log maps of type ρ , and which is of codimension $\text{rk } Q_\rho^{\text{gp}}$ in $\mathcal{M}_{\mathbf{H}}^{\log}(X_\Sigma)$, where Q_ρ is the basic monoid of ρ . In particular, types ρ with $\text{rk } Q_\rho^{\text{gp}} = (d-2)$ correspond to 0-dimensional strata of $\mathcal{M}_{\mathbf{H}}^{\log}(X_\Sigma)$, and we denote by $N_{\rho, \mathbf{H}}^{\text{toric}}(X_\Sigma)$ the log Gromov–Witten counts of the stable log maps of type ρ . The tropicalizations of these stable log maps are $(d-2)$ -dimensional families of tropical curves parametrized by the cones $\text{Hom}(Q_\rho, \mathbb{R}_{\geq 0})$.

We construct such $(d-2)$ -dimensional families ρ_T of tropical curves in $M_{\mathbb{R}}$ from an attractor tree T , contributing in the formula (1.1) for quiver DT invariants. To do this, we extend the root of T to infinity to obtain a tropical curve with leaves constrained to lie in the hyperplanes $\gamma_i^\perp$. We then consider a deformation of this tropical curve while preserving the combinatorial type and the constraints on the leaves – see Figure 2.2.

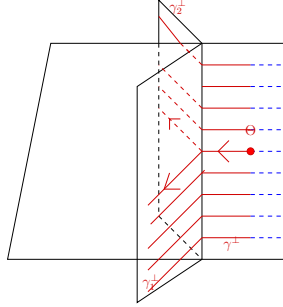


FIGURE 2.2. A family of tropical curves associated to an attractor tree

We denote by $N_{\rho_T, \mathbf{H}}^{\text{toric}}(X_\Sigma)$ the corresponding log Gromov–Witten invariant counting the finitely many stable log maps in $\mathcal{M}_{\mathbf{H}}^{\log}(X_\Sigma)$ of type ρ_T – see [8, Defn. 1.20]. Our main result is [8, Theorem B]:

THEOREM 2.13. *The coefficients $F_{r, T}^\theta(\gamma_1, \dots, \gamma_r)$ expressing the contribution to $F_r^\theta(\gamma_1, \dots, \gamma_r)$ of an attractor tree T in 1.1 satisfy*

$$F_{r, T}^\theta(\gamma_1, \dots, \gamma_r) = N_{\rho_T, \mathbf{H}}^{\text{toric}}(X_\Sigma).$$

We provide a brief summary of the main steps of the proof here. The first step is to construct a toric degeneration $\mathcal{X} \rightarrow \mathbb{A}^1$ of X_Σ and of the constraints $\mathbf{H}$. We then use a degeneration formula expressing the invariants $N_{\rho_T, H}^{\text{toric}}(X_\Sigma)$ of the general fibers X_Σ as a sum of invariants $N_{\rho_S}^{\text{toric}}(\mathcal{X}_0)$ of the special fiber $\mathcal{X}_0$, where S are binary trees in $M_{\mathbb{R}}$ deforming T . We then prove an explicit formula

$$N_{\rho_S}^{\text{toric}}(\mathcal{X}_0) = \prod_v |\langle \gamma_{v'}, \gamma_{v''} \rangle|$$

computing $N_{\rho_S}^{\text{toric}}(\mathcal{X}_0)$ as a product over local contributions of the trivalent vertices v of S , where v' and v'' denote the children of v , that is, each of the vertices v' and v'' are connected to v by a single edge which is not contained in the unique path lying between v and the root. This formula is proved using the theory of punctured log maps due to Abramovich–Chen–Gross–Siebert [3] to count log maps by gluing punctured log maps. Finally, by the flow tree formula proved in [6] and briefly reviewed at the end of §1.3, one also calculates,

$$F_{r, T}^\theta(\gamma_1, \dots, \gamma_r) = \sum_S \prod_v |\langle \gamma_{v'}, \gamma_{v''} \rangle|.$$

We then conclude that $F_{r, T}^\theta(\gamma_1, \dots, \gamma_r) = N_{\rho_T, H}^{\text{toric}}(X_\Sigma)$.

Another way to construct a geometrical object out of a quiver is based on the theory of cluster algebras [22] and cluster varieties [21]. Starting with a quiver Q with d vertices, one can construct a d -dimensional cluster variety using cluster transformations in a way prescribed by the combinatorics of Q .

In a paper in preparation with Bousseau, we provide a correspondence between quiver DT invariants, in the situation when attractor DT invariants are trivial, and counts of log curves in cluster varieties. The proof will use a comparison between the stability scattering diagram of Bridgeland [17] for quiver DT invariants and the canonical scattering diagram of Gross–Siebert [30] for log Gromov–Witten invariants of log Calabi–Yau varieties, using as intermediate steps the cluster scattering diagram of Gross–Hacking–Keel–Kontsevich [27] and the comparison results obtained in [9, 7]. This will generalize the Kronecker–Gromov–Witten correspondence previously shown in the $d = 2$ case [12, 24, 28, 47, 48]. For variants of the Kronecker–Gromov–Witten correspondence in dimension two see also [13, 15, 49]. Such a correspondence between quiver DT invariants and log curve counts in cluster varieties is compatible with Theorem 2.13 using that cluster varieties are obtained by non-toric blow-ups of toric varieties [26], and so admit degenerations into a toric variety as in [9, 28, 36]. In particular, log curve counts in cluster varieties are naturally related to log curve counts in toric varieties appearing in Theorem 2.13.

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