

THREE-DIMENSIONAL RANDOM WAVE COUPLING ALONG A
BOUNDARY AND AN ASSOCIATED INVERSE PROBLEM*MAARTEN V. DE HOOP[†], JOSSELIN GARNIER[‡], AND KNUT SØLNA[§]

Abstract. We consider random wave coupling along a flat boundary in dimension three, where the coupling is between surface and body modes and is induced by scattering by a randomly heterogeneous medium. In an appropriate scaling regime we obtain a system of radiative transfer equations which are satisfied by the mean Wigner transform of the mode amplitudes. We provide a rigorous probabilistic framework for describing solutions to this system using that it has the form of a Kolmogorov equation for some Markov process. We then prove statistical stability of the smoothed Wigner transform under the Gaussian approximation. We conclude with analyzing the nonlinear inverse problem for the radiative transfer equations and establish the unique recovery of phase and group velocities as well as power spectral information for the medium fluctuations from the observed smoothed Wigner transform. The mentioned statistical stability is essential in monitoring applications where the realization of the random medium may change.

Key words. waves in random media, waveguide, radiative transfer, paraxial equation, asymptotic analysis

MSC codes. 76B15, 35Q99, 60F05

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1. Introduction. Radiative transfer [10] has been used for a long time to model waves in heterogeneous media like Earth's crust [17, 24, 27, 28], biological tissue [2], the atmosphere, and the ocean [1, 14]. The mathematical theory of radiative transfer in open random media, which involves only body waves, is well established [3, 4, 7, 6, 13, 22]. However, in a half-space, the coupling between surface waves propagating along a boundary and body waves propagating in the bulk medium has remained a challenging problem [15, 18, 26, 29]. In dimension two a preliminary approach had been applied in the context of coupled mode theory in [8] and an analysis of the mean Wigner transform (which satisfies a form of radiative transfer equation) was presented in [11]. Here we deal with the full three-dimensional problem which includes diffractive effects and we analyze the statistical stability of the Wigner transform, which makes it possible to study the associated inverse problem.

The coupling between surface and body waves is essential in understanding, for example, seismic coda (formed by scattered waves from numerous heterogeneities).

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Here, we analyze the coupling in dimension three. We consider a novel anisotropic scaling in the random medium fluctuations and we study an associated inverse problem. Interestingly, seismograms recently acquired with SEIS on Mars show a behavior that fits the hypotheses of our analysis about the properties of its crust [20]. The results reported in [20] suggest that (i) attenuation is much weaker than scattering, (ii) the scattering properties are stratified, and (iii) smooth models of heterogeneity seem appropriate and forward scattering is dominant. This is the regime we address in our paper, and this justifies the relevance of the inversion procedure based on the radiative transfer model for the coda waves because multiple scattering hampers the identification and analysis of ballistic waves in a low-attenuation, strongly scattering medium [24].

To describe the energy transport in the seismic coda, Margerin, Bajadas, and Campillo [18] introduced a system of radiative transfer equations for coupled surface and body waves in a scalar approximation for a half-space with a special Robin boundary condition. They identified cross sections for surface-to-body and body-to-surface waves scattering. They followed a phenomenological approach to obtain the specific energy density of surface and body waves in a medium containing a homogeneous distribution of point scatterers. Zeng [29] and Sato [23] have shown the importance of each mode of wave propagation and their possible conversions, in the formation of the seismic coda, suggesting that the scattered wave energy at different arrival times may be dominated by different types of waves. Maeda, Sato, and Nishimura [15] made important contributions in this regard, modeling the coda envelopes in an inhomogeneous elastic half-space including P , SV , and Rayleigh waves in the single scattering approximation. Messaoudi, Cottreau, and Gomez [21] developed a scaling limit theory for the radiative transfer equations for scalar waves in the presence of a boundary via the use of a method of images. The results give a modification of the intensity in a domain within one wavelength of the boundary and where the modification depends on the boundary condition.

Analytical solutions of the radiative transfer equations are known only in some special cases. In [9] Celorio et al. used a spectral element model to numerically validate predictions of the radiative transfer equations in a context of scalar waves, while Shearer and Earle [25] introduced a heuristic Monte Carlo seismic phonon method. A seismic phonon can be identified with a packet of energy. Barajas, Margerin, and Campillo [5] analyzed phenomenologically the phonon propagation taking a probabilistic approach similar to the one presented by Zhang et al. [30]. They studied the typical time a seismic phonon passes through each part of the medium, and the typical time it spends in each mode of propagation (here, as a body or a surface wave). Accounting for the directionality of seismic phonons was earlier introduced by Margerin, Sens-Schönfelder, and Margerin [19]. In our paper, we provide a rigorous probabilistic framework for describing solutions to our system of radiative transfer equations using that this system has the form of a Kolmogorov equation for some Markov process.

We conclude with analyzing the nonlinear inverse problem for radiative transfer accounting for the coupling between surface and body modes and prove the unique recovery of phase and group velocities as well as the power spectrum of the medium fluctuations from the observed smoothed Wigner transform. The proof makes use of an expansion of the associated albedo operator. In a follow-up paper we will give a proof of unique recovery of the deterministic background wave speed from the phase velocities assuming that it is piecewise constant in the boundary normal coordinate or depth. Barajas, Margerin, and Campillo [5] considered the linearized version of this inverse problem. We prove statistical stability of the smoothed Wigner transform,

which implies that the coda is particularly amenable to time-lapse detection of small changes in the background wave speed.

The paper is organized as follows. In section 2 we describe the configuration of open random waveguides and wave propagation in these. In section 3, we introduce the random medium fluctuations and their anisotropic scaling and give a description of the stochastic evolution of the Fourier coefficients of the wavefield taking the form of an Itô–Schrödinger equation. In section 4, we introduce the Wigner transform and its mean and obtain the system of radiative transfer equations it satisfies. We also present a probabilistic representation of the mean Wigner transform. In section 5, we establish statistical stability of the smoothed Wigner transform. In section 6, we introduce and analyze the transport albedo kernel associated with the system of radiative transfer equations and its expansion; this expansion is used in section 7, where we study the inverse problem for the system of radiative transfer equations.

2. Wave propagation in open, random three-dimensional waveguides. We consider the three-dimensional scalar wave equation in the half-space $\mathbb{R}^2 \times (0, +\infty)$:

$$(2.1) \quad \left[\frac{n^2(x, y, z)}{c_o^2} \partial_t^2 - \Delta \right] p(t, x, y, z) = f(t, y, z) \delta(x),$$

for $(x, y) \in \mathbb{R}^2$, $z \in (0, +\infty)$, $t \in \mathbb{R}$, with $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. We assume a Dirichlet boundary condition at the surface $z = 0$: $p(t, x, y, z = 0) = 0$. The excitation is due to a source located in the plane $x = 0$. The medium is quiescent before the source excitation: $p(t, x, y, z) = 0$ for $t \ll 0$.

2.1. Wave propagation in the ideal waveguide. We refer to the deterministic case without random perturbations in the waveguide as the ideal waveguide case. The pressure field in ideal waveguides is given by

$$(2.2) \quad p_o(t, x, y, z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{p}_o(\omega, x, y, z) e^{-i\omega t},$$

with Fourier coefficients satisfying the Helmholtz equation

$$(2.3) \quad \left[\partial_x^2 + \partial_y^2 + \partial_z^2 + \frac{\omega^2 n_o^2(z)}{c_o^2} \right] \hat{p}_o(\omega, x, y, z) = -\hat{f}(\omega, y, z) \delta(x),$$

for $(x, y) \in \mathbb{R}^2$, $z \in (0, \infty)$. In the ideal waveguide the index of refraction is (x, y) -independent and equal to $n_o(z)$.

Assumption 2.1. The function $n_o(z)$ is such that $n_o(0) = n_0$, $n_o(z)$ is nonincreasing on $[0, d]$ from n_0 to $n_1 < n_0$, and $n_o(z) = n_1$ for $z \geq d$.

We denote $k = \omega/c_o$. The spectral problem associated to the one-dimensional Schrödinger operator

$$(\partial_z^2 + k^2 n_o^2(z))\phi(z) = \gamma\phi(z)$$

with Dirichlet boundary condition at $z = 0$ has been well studied [16, 11]:

- The spectrum is of the form $(-\infty, n_1^2 k^2) \cup \{\beta_N^2, \dots, \beta_1^2\}$.
- The N modal wavenumbers β_j are positive and $n_1^2 k^2 < \beta_N^2 < \dots < \beta_1^2 < n_0^2 k^2$. We have $N \geq 1$ when ω is large enough.
- The functions ϕ_j , $j = 1, \dots, N$, are the modes corresponding to the discrete spectrum. They decay exponentially in z for $z > d$.

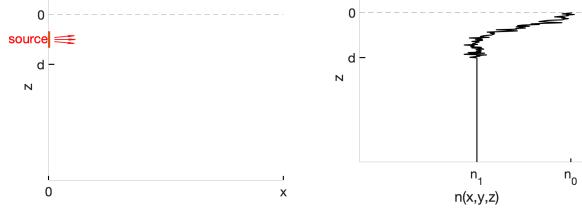


FIG. 2.1. *Left:* Set-up in the plane $y = 0$. *Right:* profile of the perturbed index of refraction $n(x, y, z) = \sqrt{n_o^2(z) + \mu_n(x, y, z)}$ as a function of z for fixed x and y . The perturbation is localized in the region $z \in (0, d)$.

- The functions ϕ_γ , $\gamma \in (-\infty, n_1^2 k^2)$, are the modes corresponding to the continuous spectrum. They are oscillatory and bounded at infinity.
- The set of modes is complete in $L^2(0, +\infty)$.

The pressure field in the ideal waveguide can be expanded as

$$(2.4) \quad \hat{p}_o(\omega, x, y, z) = \sum_{j=1}^{N(\omega)} \hat{p}_{o,j}(\omega, x, y) \phi_j(\omega, z) + \int_{-\infty}^{n_1^2 k^2} \hat{p}_{o,\gamma}(\omega, x, y) \phi_\gamma(\omega, z) d\gamma,$$

where the mode amplitudes satisfy the uncoupled system:

$$(2.5) \quad [\partial_x^2 + \partial_y^2 + \beta_j^2(\omega)] \hat{p}_{o,j} = -\hat{f}_j(\omega, y) \delta(x),$$

$$(2.6) \quad [\partial_x^2 + \partial_y^2 + \gamma] \hat{p}_{o,\gamma} = -\hat{f}_\gamma(\omega, y) \delta(x),$$

with $\hat{f}_j(\omega, y) = \int_0^\infty \hat{f}(\omega, y, z) \phi_j(\omega, z) dz$ and $\hat{f}_\gamma(\omega, y) = \int_0^\infty \hat{f}(\omega, y, z) \phi_\gamma(\omega, z) dz$.

2.2. Wave propagation in perturbed waveguides. The pressure field in the randomly perturbed waveguide satisfies the perturbed Helmholtz equation

$$(2.7) \quad \left[\partial_x^2 + \partial_y^2 + \partial_z^2 + \frac{\omega^2}{c_o^2} (n_o^2(z) + \mu_n(x, y, z)) \right] \hat{p}(\omega, x, y, z) = -\hat{f}(\omega, y, z) \delta(x)$$

for $(x, y) \in \mathbb{R}^2$, $z \in (0, \infty)$. Here μ_n models the perturbation of the index of refraction. It is a random and zero-mean process. It is stationary in (x, y) and mixing in x . It is supported in $\mathbb{R}^2 \times [0, d]$ (see Figure 2.1).

Any function can be expanded on the complete set of the eigenfunctions of the Schrödinger operator. In particular, the solution of the perturbed Helmholtz equation (2.7) can be expanded as the superposition of modes:

$$(2.8) \quad \hat{p}(\omega, x, y, z) = \sum_{j=1}^{N(\omega)} \hat{p}_j(\omega, x, y) \phi_j(\omega, z) + \int_{-\infty}^{n_1^2 k^2} \hat{p}_\gamma(\omega, x, y) \phi_\gamma(\omega, z) d\gamma.$$

3. Medium with anisotropic fluctuations. We assume that the source in (2.7) is of the form

$$(3.1) \quad f(t, y, z) = G^\varepsilon(t) F(\varepsilon y, z),$$

where ε is a small dimensionless parameter defined as the ratio of the central wavelength λ_o and the transverse width r_o of the source. The function $G^\varepsilon(t)$ is supposed to have carrier frequency ω_o , associated to the central wavelength $\lambda_o = 2\pi c_o / \omega_o$. We

will give the hypotheses that G^ε should satisfy in section 4. The separable form (3.1) of the source is assumed for simplicity but is not essential for the analysis. Standard diffraction theory states that this source generates a paraxial beam and that the Rayleigh length for this beam is of the order of $r_o^2/\lambda_o = \lambda_o/\varepsilon^2$. The Rayleigh length can here be defined as the distance along the x axis from the beam waist (located in the source plane in our case) to the place where the beam radius (in y) is doubled by diffraction. Therefore, we look at the wavefield p solution of (2.7) at a cross-range scale (in y) of order $O(\varepsilon^{-1})$, similar to r_o , with λ_o being $O(1)$ in our scaling. Moreover, we consider a range scale (in x) of order $O(\varepsilon^{-2})$, similar to the Rayleigh length (see Figure 3.1). We rename the field in this scaling as

$$(3.2) \quad p^\varepsilon(t, X, Y, z) = p\left(t, \frac{X}{\varepsilon^2}, \frac{Y}{\varepsilon}, z\right).$$

We also assume that the medium perturbation in (2.7) is of the form

$$(3.3) \quad \mu_n(x, y, z) = \varepsilon \mu(x, \varepsilon y, z).$$

The process μ_n is anisotropic with a vertical correlation length (in z) of the order of the wavelength, a horizontal correlation length in x of the order of the wavelength, and a horizontal correlation length in y of the order of the beam radius. The standard deviation (of the order of ε) of the process μ_n is such that the cumulative scattering effects are of order one as $\varepsilon \rightarrow 0$. In Appendix A we address the case when the medium perturbation is isotropic in the two horizontal directions and of the form (A.1). We derive there the radiative transfer equation (A.8) that is satisfied by the mean Wigner transform of the field (A.9) and that does not contain all the terms that are included in the forthcoming radiative transfer equation (4.4). Indeed, the anisotropy introduced in (3.3) is exactly the one that produces a radiative transfer equation in which all conversion mechanisms between the different types of modes are of the same order, hence, it is the right model to produce the most comprehensive and general radiative transfer equation. We remark that the radiative transfer equation

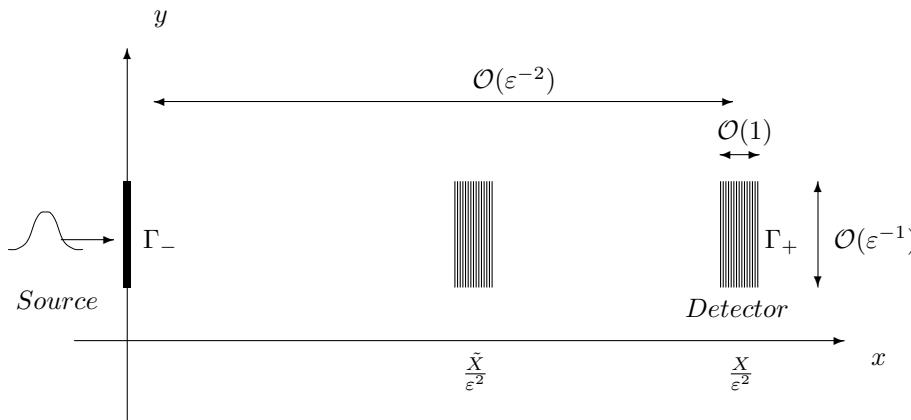


FIG. 3.1. The source is located to the left and its support projected on the horizontal spatial (surface) plane is the strip Γ_- , while the transmitted field is observed as a function of time on the set Γ_+ in the horizontal plane and centered at range X/ε^2 and then subsequently processed to form the smoothed Wigner transform. The figure also illustrates the situation with an additional measurement centered at \tilde{X}/ε^2 .

(A.9) can be formally derived from the radiative transfer equation (4.4) by neglecting the terms that induce conversion between different surface modes (i.e., the first term in the right-hand side of (4.4)).

The Fourier transform of (3.2) is given by the scaled version of (2.8):

$$(3.4) \quad \hat{p}^\varepsilon(\omega, X, Y, z) = \sum_{j=1}^{N(\omega)} \hat{p}_j^\varepsilon(\omega, X, Y) \phi_j(\omega, z) + \int_{-\infty}^{n_1^2 k^2} \hat{p}_\gamma^\varepsilon(\omega, X, Y) \phi_\gamma(\omega, z) d\gamma.$$

As the wave field evolves in the range direction (x) the Fourier transformed field changes and incorporates diffraction effects as well as mode coupling due to the random medium fluctuations. After taking out a rapidly oscillating phase to get the modal amplitudes the following proposition gives the description of this stochastic evolution of the Fourier transform \hat{p}^ε .

PROPOSITION 3.1. *Let*

$$(3.5) \quad \hat{\alpha}_j^\varepsilon(\omega, X, Y) = \frac{\sqrt{\beta_j(\omega)}}{\hat{G}^\varepsilon(\omega)} \hat{p}_j^\varepsilon(\omega, X, Y) \exp(-i\beta_j(\omega)X/\varepsilon^2).$$

As $\varepsilon \rightarrow 0$, $\hat{\alpha}^\varepsilon(\omega, X, Y) = (\hat{\alpha}_j^\varepsilon(\omega, X, Y))_{j=1}^N$ converges weakly and in distribution to the diffusion Markov process $\hat{\alpha}(\omega, X, Y) = (\hat{\alpha}_j(\omega, X, Y))_{j=1}^N$. The limit processes $\hat{\alpha}_j(\omega, X, Y)$ solve the Itô–Schrödinger equations

$$(3.6) \quad \begin{aligned} d\hat{\alpha}_j(\omega, X, Y) &= \frac{i}{2\beta_j(\omega)} \partial_Y^2 \hat{\alpha}_j(\omega, X, Y) dX + i\hat{\alpha}_j(\omega, X, Y) \circ dB_j(\omega, X, Y) \\ &+ \sum_{l=1, l \neq j}^{N(\omega)} \frac{1}{\sqrt{2}} \hat{\alpha}_l(\omega, X, Y) \circ (idB_{j,l}(\omega, X, Y) - d\tilde{B}_{j,l}(\omega, X, Y)) \\ &+ \frac{1}{2} (-\Lambda_j^r(\omega) - i\Lambda_j^s(\omega) + i\kappa_j(\omega)) \hat{\alpha}_j(\omega, X, Y) dX, \end{aligned}$$

for $X > 0$, starting from

$$(3.7) \quad \hat{\alpha}_j(\omega, X=0, Y) = \frac{F_j(\omega, Y)}{2i\sqrt{\beta_j(\omega)}}, \quad F_j(\omega, Y) = \int_0^{+\infty} F(Y, z) \phi_j(\omega, z) dz.$$

The correlated Brownian fields $B_j(\omega, X, Y)$ are Gaussian processes that satisfy

$$(3.8) \quad \mathbb{E}[B_j(\omega, X, Y)] = 0,$$

$$(3.9) \quad \mathbb{E}[B_j(\omega, X, Y) B_l(\omega, X', Y')] = \min\{X, X'\} R_{jl}(\omega, Y - Y'),$$

with R_{jl} defined by

$$(3.10) \quad \begin{aligned} R_{jl}(\omega, Y) &= \frac{k^4(\omega)}{4\beta_j(\omega)\beta_l(\omega)} \int_0^\infty \int_0^d \int_0^d \mathbb{E}[\mu(0, 0, z)\mu(x, Y, z')] \phi_j(\omega, z)^2 \phi_l(\omega, z')^2 dz dz' dx \\ &+ \frac{k^4(\omega)}{4\beta_j(\omega)\beta_l(\omega)} \int_0^\infty \int_0^d \int_0^d \mathbb{E}[\mu(0, 0, z)\mu(x, -Y, z')] \phi_l(\omega, z)^2 \phi_j(\omega, z')^2 dz dz' dx. \end{aligned}$$

The Brownian fields $B_{j,l}(\omega, X, Y)$, $\tilde{B}_{j,l}(\omega, X, Y)$ are independent and identically distributed for $j < l$ and satisfy $B_{l,j} = B_{j,l}$, $\tilde{B}_{l,j} = -\tilde{B}_{j,l}$,

$$(3.11) \quad \mathbb{E}[B_{j,l}(\omega, X, Y)] = 0,$$

$$(3.12) \quad \mathbb{E}[B_{j,l}(\omega, X, Y)B_{j,l}(\omega, X', Y')] = \min\{X, X'\}\Gamma_{jl}^c(\omega, Y - Y'),$$

where

$$(3.13) \quad \Gamma_{jl}^c(\omega, Y) = \frac{k^4(\omega)}{4\beta_j\beta_l(\omega)} \int_0^\infty \mathcal{R}_{jl}(\omega, x, Y) \cos((\beta_l(\omega) - \beta_j(\omega))x) dx, \quad j \neq l,$$

$$(3.14) \quad \Gamma_{jl}^s(\omega, Y) = \frac{k^4(\omega)}{4\beta_j\beta_l(\omega)} \int_0^\infty \mathcal{R}_{jl}(\omega, x, Y) \sin((\beta_l(\omega) - \beta_j(\omega))x) dx, \quad j \neq l,$$

$$(3.15) \quad \begin{aligned} \mathcal{R}_{jl}(\omega, x, Y) = & \int_0^d \int_0^d \phi_j \phi_l(\omega, z) \mathbb{E}[\mu(0, 0, z) \mu(x, Y, z')] \phi_j \phi_l(\omega, z') dz dz' \\ & + \int_0^d \int_0^d \phi_l \phi_j(\omega, z) \mathbb{E}[\mu(0, 0, z) \mu(x, -Y, z')] \phi_l \phi_j(\omega, z') dz dz', \end{aligned}$$

$$(3.16) \quad \Lambda_j^c(\omega) = \sum_{l=1, l \neq j}^{N(\omega)} \Gamma_{jl}^c(\omega, 0),$$

$$(3.17) \quad \Lambda_j^r(\omega) = \int_0^{n_1^2 k^2(\omega)} \frac{k^4(\omega)}{4\sqrt{\gamma}\beta_j(\omega)} \int_0^\infty \mathcal{R}_{j\gamma}(\omega, x) \cos((\sqrt{\gamma} - \beta_j(\omega))x) dx d\gamma,$$

$$(3.18) \quad \Lambda_j^s(\omega) = \sum_{l=1, l \neq j}^{N(\omega)} \Gamma_{jl}^s(\omega, 0) + \int_0^{n_1^2 k^2(\omega)} \frac{k^4(\omega)}{4\sqrt{\gamma}\beta_j(\omega)} \int_0^\infty \mathcal{R}_{j\gamma}(\omega, x) \sin((\sqrt{\gamma} - \beta_j(\omega))x) dx d\gamma,$$

$$(3.19) \quad \kappa_j(\omega) = \int_{-\infty}^0 \frac{k^4(\omega)}{4\sqrt{|\gamma|}\beta_j(\omega)} \int_0^\infty \mathcal{R}_{j\gamma}(\omega, x) \cos(\beta_j(\omega)x) e^{-\sqrt{|\gamma|}x} dx d\gamma,$$

$$(3.20) \quad \mathcal{R}_{j\gamma}(\omega, x) = 2 \int_0^d \int_0^d \phi_j \phi_\gamma(\omega, z) \mathbb{E}[\mu(0, 0, z) \mu(x, 0, z')] \phi_j \phi_\gamma(\omega, z') dz dz'.$$

The Itô–Schrödinger equations (3.6) allow us to describe moments of the wave field and the statistics of the transmitted wave. In the case of the unperturbed waveguide there is no coupling between the modal amplitudes and only the first term on the right-hand side of the Itô–Schrödinger equation, the lateral Laplacian term, is present and describes diffraction in the lateral direction. In the case of the perturbed waveguide the terms involving the driving Brownian fields, $B_j, B_{j,l}, \tilde{B}_{j,l}$, reflect the effects of the random medium fluctuations. Note that the driving Brownian fields are correlated, as the modes sense the same medium, with the correlation reflecting the correlations of the medium fluctuations when projected on products of eigenfunctions of the one-dimensional unperturbed Schrödinger operator. The term involving B_j is the one also seen in the scalar, open medium case [13] and gives a randomization of the mode due to random forward and lateral scattering. The terms involving $B_{j,l}, \tilde{B}_{j,l}$ produce a coupling between different modes due to the random perturbations in the waveguide and are of the type seen in the two-dimensional case [11].

Proof. For a fixed frequency ω , we expand the wave field as in (2.8). Substituting the result into (2.7) and taking into account the form (3.3) of the random medium fluctuations, we find that the complex mode amplitudes satisfy the coupled equations for $x \neq 0$:

$$(3.21) \quad \partial_x^2 \hat{p}_j + \partial_y^2 \hat{p}_j + \beta_j^2 \hat{p}_j = -\varepsilon k^2 \sum_{l=1}^N C_{j,l}(x, \varepsilon y) \hat{p}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{j,\gamma'}(x, \varepsilon y) \hat{p}_{\gamma'} d\gamma',$$

for $j = 1, \dots, N$,

$$(3.22) \quad \partial_x^2 \hat{p}_\gamma + \partial_y^2 \hat{p}_\gamma + \gamma \hat{p}_\gamma = -\varepsilon k^2 \sum_{l=1}^N C_{\gamma,l}(x, \varepsilon y) \hat{p}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{\gamma,\gamma'}(x, \varepsilon y) \hat{p}_{\gamma'} d\gamma',$$

for $\gamma \in (-\infty, n_1^2 k^2)$, where

$$\begin{aligned} C_{j,l}(x, y) &= (\phi_j, \phi_l \mu(x, y, \cdot))_{L^2}, & C_{j,\gamma'}(x, y) &= (\phi_j, \phi_{\gamma'} \mu(x, y, \cdot))_{L^2}, \\ C_{\gamma,l}(x, y) &= (\phi_\gamma, \phi_l \mu(x, y, \cdot))_{L^2}, & C_{\gamma,\gamma'}(x, y) &= (\phi_\gamma, \phi_{\gamma'} \mu(x, y, \cdot))_{L^2}, \end{aligned}$$

and $(\cdot, \cdot)_{L^2}$ stands for the standard scalar product in $L^2(0, +\infty)$. We introduce the generalized forward-going and backward-going mode amplitudes,

$$(3.23) \quad \{\hat{a}_j(x, y), \hat{b}_j(x, y), j = 1, \dots, N\} \text{ and } \{\hat{a}_\gamma(x, y), \hat{b}_\gamma(x, y), \gamma \in (0, n_1^2 k^2)\},$$

which are defined such that

$$(3.24) \quad \begin{aligned} \hat{p}_j(x, y) &= \frac{1}{\sqrt{\beta_j}} (\hat{a}_j(x, y) e^{i\beta_j x} + \hat{b}_j(x, y) e^{-i\beta_j x}), \\ \partial_x \hat{p}_j(x, y) &= i\sqrt{\beta_j} (\hat{a}_j(x, y) e^{i\beta_j x} - \hat{b}_j(x, y) e^{-i\beta_j x}), \quad j = 1, \dots, N, \end{aligned}$$

and

$$(3.25) \quad \begin{aligned} \hat{p}_\gamma(x, y) &= \frac{1}{\gamma^{1/4}} (\hat{a}_\gamma(x, y) e^{i\sqrt{\gamma} x} + \hat{b}_\gamma(x, y) e^{-i\sqrt{\gamma} x}), \\ \partial_x \hat{p}_\gamma(x, y) &= i\gamma^{1/4} (\hat{a}_\gamma(x, y) e^{i\sqrt{\gamma} x} - \hat{b}_\gamma(x, y) e^{-i\sqrt{\gamma} x}), \quad \gamma \in (0, n_1^2 k^2). \end{aligned}$$

We then substitute (3.24)–(3.25) into (3.21)–(3.22) in order to obtain the coupled system of random differential equations satisfied by the mode amplitudes in (3.23),

$$(3.26) \quad \begin{aligned} \partial_x \hat{a}_j &= \frac{i}{2\beta_j} \partial_y^2 \hat{a}_j + \frac{i\varepsilon k^2}{2} \sum_{l'=1}^N \frac{C_{j,l'}(x, \varepsilon y)}{\sqrt{\beta_{l'} \beta_j}} [\hat{a}_{l'} e^{i(\beta_{l'} - \beta_j)x} + \hat{b}_{l'} e^{i(-\beta_{l'} - \beta_j)x}] \\ &+ \frac{i\varepsilon k^2}{2} \int_0^{n_1^2 k^2} \frac{C_{j,\gamma'}(x, \varepsilon y)}{\sqrt[4]{\gamma'} \sqrt{\beta_j}} [\hat{a}_{\gamma'} e^{i(\sqrt{\gamma'} - \beta_j)x} + \hat{b}_{\gamma'} e^{i(-\sqrt{\gamma'} - \beta_j)x}] d\gamma' \\ &+ \frac{i\varepsilon k^2}{2} \int_{-\infty}^0 \frac{C_{j,\gamma'}(x, \varepsilon y)}{\sqrt{\beta_j}} \hat{p}_{\gamma'} e^{-i\beta_j x} d\gamma', \end{aligned}$$

$$(3.27) \quad \begin{aligned} \partial_x \hat{a}_\gamma &= \frac{i}{2\sqrt{\gamma}} \partial_y^2 \hat{a}_\gamma + \frac{i\varepsilon k^2}{2} \sum_{l'=1}^N \frac{C_{\gamma,l'}(x, \varepsilon y)}{\sqrt[4]{\gamma} \sqrt{\beta_{l'}}} [\hat{a}_{l'} e^{i(\beta_{l'} - \sqrt{\gamma})x} + \hat{b}_{l'} e^{i(-\beta_{l'} - \sqrt{\gamma})x}] \\ &+ \frac{i\varepsilon k^2}{2} \int_0^{n_1^2 k^2} \frac{C_{\gamma,\gamma'}(x, \varepsilon y)}{\sqrt[4]{\gamma'} \gamma} [\hat{a}_{\gamma'} e^{i(\sqrt{\gamma'} - \sqrt{\gamma})x} + \hat{b}_{\gamma'} e^{i(-\sqrt{\gamma'} - \sqrt{\gamma})x}] d\gamma' \\ &+ \frac{i\varepsilon k^2}{2} \int_{-\infty}^0 \frac{C_{\gamma,\gamma'}(x, \varepsilon y)}{\sqrt[4]{\gamma}} \hat{p}_{\gamma'} e^{-i\sqrt{\gamma} x} d\gamma'. \end{aligned}$$

We have similar equations for \hat{b}_j and \hat{b}_γ . The evanescent mode amplitudes \hat{p}_γ , $\gamma \in (-\infty, 0)$, satisfy (3.22). We then take the scaled coordinates $y = Y/\varepsilon$ and $x = X/\varepsilon^2$ to get a system for $\hat{a}_j^\varepsilon(\omega, X, Y) = \hat{a}_j(\omega, X/\varepsilon^2, Y/\varepsilon)$ and $\hat{b}_j^\varepsilon(\omega, X, Y) = \hat{b}_j(\omega, X/\varepsilon^2, Y/\varepsilon)$, where we can use diffusion approximation results [12, Chapter 20] in a case with rapidly oscillating phase terms. As shown in [12, section 20.2.6], the forward scattering approximation is valid in our scaling regime, that is to say, we can make the approximation $\hat{b}_j^\varepsilon \simeq 0$ and $\hat{b}_\gamma^\varepsilon \simeq 0$. Indeed, for the type of medium fluctuations we consider the coupling terms between the forward and backward propagating modes are small. The coupling with the evanescent modes can be integrated out and only gives rise to an effective deterministic phase modulation (this is the term in $\kappa_j(\omega)$ in (3.6); see [12, section 20.2.5]). The evolution equations for the forward mode amplitudes \hat{a}_j^ε then take the form

$$(3.28) \quad \begin{aligned} \partial_X \hat{a}_j^\varepsilon &= \frac{i}{2\beta_j} \partial_Y^2 \hat{a}_j^\varepsilon + \frac{ik^2}{2\varepsilon} \sum_{l'=1}^N \frac{C_{j,l'}(\frac{X}{\varepsilon^2}, Y)}{\sqrt{\beta_{l'}\beta_j}} \hat{a}_{l'}^\varepsilon e^{i(\beta_{l'} - \beta_j)\frac{X}{\varepsilon^2}} \\ &\quad + \frac{ik^2}{2\varepsilon} \int_0^{n_1^2 k^2} \frac{C_{j,\gamma'}(\frac{X}{\varepsilon^2}, Y)}{\sqrt[4]{\gamma'} \sqrt{\beta_j}} \hat{a}_{\gamma'}^\varepsilon e^{i(\sqrt{\gamma'} - \beta_j)\frac{X}{\varepsilon^2}} d\gamma' + \frac{i\kappa_j}{2} \hat{a}_j^\varepsilon \end{aligned}$$

with the initial condition

$$\hat{a}_j^\varepsilon(\omega, X=0, Y) = \frac{F_j(\omega, Y)\hat{G}^\varepsilon(\omega)}{2i\sqrt{\beta_j(\omega)}}.$$

This system is in a form which allows for the application of the diffusion approximation theory set forth in [12, section 20.3.1]. This establishes the diffusion limit for $\hat{a}_j^\varepsilon(\omega, X, Y) = \hat{a}_j^\varepsilon(\omega, X, Y)/\hat{G}^\varepsilon(\omega)$. The Itô–Schrödinger equation (3.6) is the Itô diffusion corresponding to this limit. \square

4. The mean Wigner transform.

4.1. The radiative transfer equation satisfied by the mean Wigner transform. We first formulate the radiative transfer equation satisfied by the mean Wigner transform of the mode amplitudes. We assume from now on that the Wigner transform of the source function $G^\varepsilon(t)$ satisfies

$$(4.1) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} d\omega' e^{-i\omega' t} \hat{G}^\varepsilon \left(\omega + \varepsilon^2 \frac{\omega'}{2} \right) \overline{\hat{G}^\varepsilon \left(\omega - \varepsilon^2 \frac{\omega'}{2} \right)} = W_G(t, \omega).$$

The typical situation we have in mind is a partially coherent source, for instance, a zero-mean process with covariance function

$$\left\langle G^\varepsilon \left(t + \frac{\tau}{2} \right) \overline{G^\varepsilon \left(t - \frac{\tau}{2} \right)} \right\rangle = \varepsilon^2 \exp \left(-\frac{\varepsilon^4 t^2}{4\sigma_t^2} - \frac{\tau^2}{2\tau_c^2} - i\omega_o \tau \right),$$

where $\langle \cdot \rangle$ is the statistical average, ω_o is the carrier frequency, τ_c is the coherence time, and σ_t/ε^2 is the duration of the source envelope. We then have

$$W_G(t, \omega) = \sqrt{2\pi} \tau_c \exp \left(-\frac{(\omega - \omega_o)^2 \tau_c^2}{2} - \frac{t^2}{4\sigma_t^2} \right).$$

PROPOSITION 4.1. *The mean Wigner transform of the j th surface mode defined by*

(4.2)

$$\begin{aligned} \mathcal{W}_j(t, \omega, X, \kappa_x, Y, \kappa_y) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}^3} d\omega' dx' dY' e^{-i\omega' t - i\kappa_x x' - i\kappa_y Y'} \\ &\times \mathbb{E} \left[\widehat{p}_j^\varepsilon \left(\omega + \varepsilon^2 \frac{\omega'}{2}, X + \varepsilon^2 \frac{x'}{2}, Y + \frac{Y'}{2} \right) \overline{\widehat{p}_j^\varepsilon \left(\omega - \varepsilon^2 \frac{\omega'}{2}, X - \varepsilon^2 \frac{x'}{2}, Y - \frac{Y'}{2} \right)} \right] \end{aligned}$$

has the form

$$(4.3) \quad \mathcal{W}_j(t, \omega, X, \kappa_x, Y, \kappa_y) = W_j(t, \omega, X, Y, \kappa_y) \delta(\kappa_x - \beta_j(\omega)),$$

where the W_j 's satisfy the system of radiative transfer equations

$$\begin{aligned} &\partial_X W_j + \frac{\kappa_y}{\beta_j(\omega)} \partial_Y W_j + \frac{1}{v_j(\omega)} \partial_t W_j \\ &= \frac{1}{2\pi} \sum_{l=1, l \neq j}^{N(\omega)} \int_{\mathbb{R}} \check{\Gamma}_{jl}^c(\omega, \kappa_y') [W_l(\kappa_y - \kappa_y') - W_j(\kappa_y)] d\kappa_y' \\ (4.4) \quad &+ \frac{1}{2\pi} \int_{\mathbb{R}} \check{R}_{jj}(\omega, \kappa_y') [W_j(\kappa_y - \kappa_y') - W_j(\kappa_y)] d\kappa_y' - \Lambda_j^r(\omega) W_j, \end{aligned}$$

starting from

(4.5)

$$W_j(t, \omega, X = 0, Y, \kappa_y) = \frac{\pi W_G(t, \omega)}{2\beta_j(\omega)^2} \int_{\mathbb{R}} F_j \left(\omega, Y + \frac{Y'}{2} \right) \overline{F_j} \left(\omega, Y - \frac{Y'}{2} \right) e^{-i\kappa_y Y'} dY'.$$

Here

$$(4.6) \quad v_j(\omega) = 1/\beta_j'(\omega),$$

$$(4.7) \quad \check{\Gamma}_{jl}^c(\omega, \kappa) = \int_{\mathbb{R}} \Gamma_{jl}^c(\omega, Y) e^{-i\kappa Y} dY,$$

$$(4.8) \quad \check{R}_{jj}(\omega, \kappa) = \int_{\mathbb{R}} R_{jj}(\omega, Y) e^{-i\kappa Y} dY.$$

We note that $\check{\Gamma}_{jl}^c(\omega, \kappa) \geq 0$ and $\check{R}_{jj}(\omega, \kappa) \geq 0$ because they are proportional to the power spectral densities of stationary processes in (x, Y) ($\int \phi_j \phi_l(z) \mu(x, Y, z) dz$ and $\int \phi_j^2(z) \mu(x, Y, z) dz$, respectively).

In radiative transfer equation (4.4), v_j is the group velocity of the j th surface mode, $\check{\Gamma}_{jl}^c$ and \check{R}_{jj} are the scattering cross-coefficients ($\check{\Gamma}_{jl}$ is the conversion rate from the l th surface mode to the j th surface mode and \check{R}_{jj} is the conversion rate between different κ_y components of the j th surface mode), and Λ_j^r is the extinction coefficient that takes into account leakage from the j th surface mode toward the body modes. The scattering and extinction coefficients depend on the two-point statistics of the fluctuations of the random medium.

Proof. We denote

$$U_{j,l}^\varepsilon(\omega, \omega', X, Y, Y') = \widehat{a}_j \left(\omega + \frac{\varepsilon^2 \omega'}{2}, \frac{X}{\varepsilon^2}, \frac{Y}{\varepsilon} + \frac{Y'}{2\varepsilon} \right) \overline{\widehat{a}_l \left(\omega - \frac{\varepsilon^2 \omega'}{2}, \frac{X}{\varepsilon^2}, \frac{Y}{\varepsilon} - \frac{Y'}{2\varepsilon} \right)},$$

and similarly for $U_{j,\gamma}^\varepsilon$, etc. By expanding $\beta_j(\omega \pm \varepsilon^2 \omega'/2)$ at ω up to terms of order ε^2 so as to keep all nonnegligible terms in the phases, we get that $U_{j,l}^\varepsilon$ satisfies

$$\begin{aligned} \partial_X U_{j,l}^\varepsilon &= \frac{i}{\beta_j} \partial_Y \partial_{Y'} U_{j,l}^\varepsilon \\ &+ i(\kappa_j - \kappa_l) U_{j,l}^\varepsilon + \frac{ik^2}{2\varepsilon} \sum_{l'=1}^N \frac{C_{j,l'}(\frac{X}{\varepsilon^2}, Y + \frac{Y'}{2})}{\sqrt{\beta_{l'} \beta_j}} U_{l',l}^\varepsilon e^{i(\beta_{l'} - \beta_j) \frac{X}{\varepsilon^2}} e^{i(\beta'_{l'} - \beta'_j) \frac{X \omega'}{2}} \\ &- \frac{ik^2}{2\varepsilon} \sum_{l'=1}^N \frac{C_{l,l'}(\frac{X}{\varepsilon^2}, Y - \frac{Y'}{2})}{\sqrt{\beta_{l'} \beta_l}} U_{j,l'}^\varepsilon e^{i(\beta_l - \beta_{l'}) \frac{X}{\varepsilon^2}} e^{i(\beta'_{l'} - \beta'_l) \frac{X \omega'}{2}} \\ &+ \frac{ik^2}{2\varepsilon} \int_0^{n_1^2 k^2} \frac{C_{j,\gamma'}(\frac{X}{\varepsilon^2}, Y + \frac{Y'}{2})}{\sqrt[4]{\gamma'} \sqrt{\beta_j}} U_{\gamma',l}^\varepsilon e^{i(\sqrt{\gamma'} - \beta_j) \frac{X}{\varepsilon^2}} d\gamma' e^{-i\beta'_j \frac{X \omega'}{2}} \\ &- \frac{ik^2}{2\varepsilon} \int_0^{n_1^2 k^2} \frac{C_{l,\gamma'}(\frac{X}{\varepsilon^2}, Y - \frac{Y'}{2})}{\sqrt[4]{\gamma'} \sqrt{\beta_l}} U_{j,\gamma'}^\varepsilon e^{-i(\sqrt{\gamma'} - \beta_l) \frac{X}{\varepsilon^2}} d\gamma' e^{-i\beta'_l \frac{X \omega'}{2}}, \end{aligned}$$

and we get similar equations for $U_{j,\gamma}^\varepsilon$. We introduce

$$V_{j,l}^\varepsilon(t, \omega, X, Y, Y') = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega'(t - (\beta'_j(\omega) + \beta'_l(\omega)) \frac{X}{2})} U_{j,l}^\varepsilon(\omega, \omega', X, Y, Y') d\omega',$$

and similarly for $V_{j,\gamma}^\varepsilon$. These quantities satisfy

$$\begin{aligned} \partial_X V_{j,l}^\varepsilon + \frac{\beta'_j(\omega) + \beta'_l(\omega)}{2} \partial_t V_{j,l}^\varepsilon &= \frac{i}{2\beta_j} \partial_Y \partial_{Y'} V_{j,l}^\varepsilon + i(\kappa_j - \kappa_l) V_{j,l}^\varepsilon \\ &+ \frac{ik^2}{2\varepsilon} \sum_{l'=1}^N \frac{C_{j,l'}(\frac{X}{\varepsilon^2}, Y + \frac{Y'}{2})}{\sqrt{\beta_{l'} \beta_j}} V_{l',l}^\varepsilon e^{i(\beta_{l'} - \beta_j) \frac{X}{\varepsilon^2}} \\ &- \frac{ik^2}{2\varepsilon} \sum_{l'=1}^N \frac{C_{l,l'}(\frac{X}{\varepsilon^2}, Y - \frac{Y'}{2})}{\sqrt{\beta_{l'} \beta_l}} V_{j,l'}^\varepsilon e^{i(\beta_l - \beta_{l'}) \frac{X}{\varepsilon^2}} \\ &+ \frac{ik^2}{2\varepsilon} \int_0^{n_1^2 k^2} \frac{C_{j,\gamma'}(\frac{X}{\varepsilon^2}, Y + \frac{Y'}{2})}{\sqrt[4]{\gamma'} \sqrt{\beta_j}} V_{\gamma',l}^\varepsilon e^{i(\sqrt{\gamma'} - \beta_j) \frac{X}{\varepsilon^2}} d\gamma' \\ &- \frac{ik^2}{2\varepsilon} \int_0^{n_1^2 k^2} \frac{C_{l,\gamma'}(\frac{X}{\varepsilon^2}, Y - \frac{Y'}{2})}{\sqrt[4]{\gamma'} \sqrt{\beta_l}} V_{j,\gamma'}^\varepsilon e^{-i(\sqrt{\gamma'} - \beta_l) \frac{X}{\varepsilon^2}} d\gamma'. \end{aligned}$$

The completion of the proof is as follows: We first establish a diffusion-approximation result for $V_{j,l}^\varepsilon$, then establish the equation satisfied by the expectation, and finally take a Fourier transform in Y' to get the desired result. \square

From the previous proposition we can formulate the radiative transfer equation satisfied by the normal derivative of the wave field at the surface, which is the measured quantity in seismology, for instance.

PROPOSITION 4.2. *The mean Wigner transform defined by*

$$\begin{aligned}
 \mathcal{W}(t, \omega, X, \kappa_x, Y, \kappa_y) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}^3} d\omega' dx' dY' e^{-i\omega' t - i\kappa_x x' - i\kappa_y Y'} \\
 &\quad \times \mathbb{E} \left[\partial_z \widehat{p}^\varepsilon \left(\omega + \varepsilon^2 \frac{\omega'}{2}, X + \varepsilon^2 \frac{x'}{2}, Y + \frac{Y'}{2}, z=0 \right) \right. \\
 &\quad \times \left. \partial_z \widehat{p}^\varepsilon \left(\omega - \varepsilon^2 \frac{\omega'}{2}, X - \varepsilon^2 \frac{x'}{2}, Y - \frac{Y'}{2}, z=0 \right) \right] \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} d\tau dx' dY' e^{i\omega\tau - i\kappa_x x' - i\kappa_y Y'} \\
 &\quad \times \mathbb{E} \left[\partial_z p^\varepsilon \left(\frac{t}{\varepsilon^2} + \frac{\tau}{2}, X + \varepsilon^2 \frac{x'}{2}, Y + \frac{Y'}{2}, z=0 \right) \right. \\
 &\quad \times \left. \partial_z p^\varepsilon \left(\frac{t}{\varepsilon^2} - \frac{\tau}{2}, X - \varepsilon^2 \frac{x'}{2}, Y - \frac{Y'}{2}, z=0 \right) \right] \\
 (4.9) \quad &
 \end{aligned}$$

has the form

$$(4.10) \quad \mathcal{W}(t, \omega, X, \kappa_x, Y, \kappa_y) = \sum_{j=1}^{N(\omega)} \partial_z \phi_j(\omega, 0)^2 W_j(t, \omega, X, Y, \kappa_y) \delta(\kappa_x - \beta_j(\omega)),$$

where the W_j 's satisfy (4.4).

4.2. Probabilistic representation of the mean Wigner transform. It is possible to give a probabilistic representation of the mean Wigner transforms $W_j(t, \omega, X, Y, \kappa_y)$ because (4.4) has the form of a Kolmogorov equation for a Markov process that we describe in this subsection. This gives a mathematical background to the phenomenological approach to the radiative transfer theory based on seismic phonons discussed in [5, 25]. We first introduce the Markov process for a fixed ω .

(i) Let $(J_n, K_n)_{n \in \mathbb{N}}$ be a jump Markov process with state space $\{1, \dots, N(\omega)\} \times \mathbb{R}$ and transition kernel

$$\begin{aligned}
 \mathbb{E}[f(J_1, K_1) | J_0 = j, K_0 = \kappa_y] &= \frac{1}{\Lambda_j^c(\omega) + R_{jj}(\omega, 0)} \\
 &\times \frac{1}{2\pi} \left[\sum_{l \neq j} \int_{\mathbb{R}} \check{\Gamma}_{jl}^c(\omega, \kappa_y') f(l, \kappa_y - \kappa_y') d\kappa_y' + \int_{\mathbb{R}} \check{R}_{jj}(\omega, \kappa_y') f(j, \kappa_y - \kappa_y') d\kappa_y' \right],
 \end{aligned}$$

where $\Lambda_j^c(\omega) = \sum_{l \neq j} \Gamma_{jl}^c(\omega)$. This transition kernel describes the type of mode conversion that can happen from a mode (j, κ_y) . When a mode conversion happens from (j, κ_y) , with probability $\Gamma_{jl}^c(\omega)/[\Lambda_j^c(\omega) + R_{jj}(\omega, 0)]$ it is a mode conversion to (l, κ_y') with $l \neq j$ and with some $\kappa_y' \neq \kappa_y$ (this conversion affects both j and κ_y) and with probability $R_{jj}(\omega, 0)/[\Lambda_j^c(\omega) + R_{jj}(\omega, 0)]$ it is a mode conversion to (j, κ_y') with $\kappa_y' \neq \kappa_y$ (this conversion only affects κ_y). In the first case, the new κ_y' is chosen randomly according to the distribution with the probability density function $\kappa_y' \mapsto \check{\Gamma}_{jl}^c(\omega, \kappa_y' - \kappa_y)/[2\pi \Gamma_{jl}^c(\omega, 0)]$. In the second case, the new κ_y' is chosen randomly according to the distribution with the probability density function $\kappa_y' \mapsto \check{R}_{jj}(\omega, \kappa_y' - \kappa_y)/[2\pi R_{jj}(\omega, 0)]$.

(ii) Conditioned on $(J_n, K_n)_{n \in \mathbb{N}}$, let $(D_n)_{n \in \mathbb{N}}$ be a sequence of independent exponential random variables with parameters $\Lambda_{J_n}^c(\omega) + R_{J_n J_n}(\omega, 0)$. These parameters govern the intensity rates of the mode conversions. The probability that there is a mode conversion from (j, κ_y) between X and $X + \delta X$ is $[\Lambda_j^c(\omega) + R_{jj}(\omega, 0)]\delta X$ for small δX .

(iii) For any $X \in [0, +\infty)$, set $(J_X, K_X) = (J_n, K_n)$ if $\sum_{m=0}^{n-1} D_m \leq X < \sum_{m=0}^n D_m$ (with the convention $\sum_{m=0}^{-1} D_m = 0$) and introduce $(Y_X, T_X)_{X \in (0, +\infty)}$ as the solution of

$$\partial_X Y_X = -\frac{K_X}{\beta_{J_X}(\omega)}, \quad \partial_X T_X = -\frac{1}{v_{J_X}(\omega)}.$$

Then $(J_X, K_X, Y_X, T_X)_{X \in [0, +\infty)}$ is a Markov process with infinitesimal generator

$$\begin{aligned} \mathcal{L}f(j, \kappa_y, Y, t) = & \frac{1}{2\pi} \left[\sum_{l \neq j} \int_{\mathbb{R}} \check{\Gamma}_{jl}^c(\omega, \kappa'_y) [f(l, \kappa_y - \kappa'_y) - f(j, \kappa_y)] d\kappa'_y \right. \\ & \left. + \int_{\mathbb{R}} \check{R}_{jj}(\omega, \kappa'_y) [f(j, \kappa_y - \kappa'_y) - f(j, \kappa_y)] d\kappa'_y \right] - \frac{\kappa_y}{\beta_j(\omega)} \partial_Y f - \frac{1}{v_j(\omega)} \partial_t f. \end{aligned}$$

This implies that the solution $u(X, j, \kappa_y, y, t)$ of the Kolmogorov equation $\partial_X u = \mathcal{L}u$, $u(X=0, j, \kappa_y, y, t) = u_0(j, \kappa_y, y, t)$, has the representation

$$u(X, j, \kappa_y, Y, t) = \mathbb{E}[u_0(J_X, K_X, Y_X, T_X) | J_0 = j, K_0 = \kappa_y, Y_0 = Y, T_0 = t].$$

The solution of the damped Kolmogorov equation $\partial_X u = \mathcal{L}u + \mathcal{V}u$, with $\mathcal{V}f(j) = -\Lambda_j^r(\omega)f(j)$, has the representation (by the Feynman–Kac formula):

$$\begin{aligned} u(X, j, \kappa_y, Y, t) = & \mathbb{E} \left[u_0(J_X, K_X, Y_X, T_X) \right. \\ & \left. \times \exp \left(- \int_0^X \Lambda_{J_{X'}}^r(\omega) dX' \right) \middle| J_0 = j, K_0 = \kappa_y, Y_0 = Y, T_0 = t \right]. \end{aligned}$$

Accordingly, the Wigner transform solution of (4.4) can be expressed as

$$\begin{aligned} W_j(t, \omega, X, Y, \kappa_y) = & \mathbb{E} \left[W_{J_X}(T_X, \omega, 0, Y_X, K_X) \right. \\ (4.11) \quad & \left. \times \exp \left(- \int_0^X \Lambda_{J_{X'}}^r(\omega) dX' \right) \middle| J_0 = j, K_0 = \kappa_y, Y_0 = Y, T_0 = t \right], \end{aligned}$$

where $W_j(t, \omega, 0, Y, \kappa_y)$ is the initial Wigner transform at $X = 0$.

An alternative representation in which the evolution variable is t rather than X is as follows. Replace step (ii) by the following: Conditionally to $(J_n, K_n)_{n \in \mathbb{N}}$, let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of independent exponential random variables with parameters $v_{J_n}(\omega)[\Lambda_{J_n}^c(\omega) + R_{J_n J_n}(\omega, 0)]$. Replace step (iii) by the following: For any $t \in [0, +\infty)$, set $(J_t, K_t) = (J_n, K_n)$ if $\sum_{m=0}^{n-1} \tau_m \leq t < \sum_{m=0}^n \tau_m$ and introduce $(Y_t, X_t)_{t \in (0, +\infty)}$ as the solution of

$$\partial_t Y_t = -\frac{K_t v_{J_t}(\omega)}{\beta_{J_t}(\omega)}, \quad \partial_t X_t = -v_{J_t}(\omega).$$

Then $(J_t, K_t, Y_t, X_t)_{t \in [0, +\infty)}$ is a Markov process with infinitesimal generator

$$\begin{aligned} \mathcal{L}f(j, \kappa_y, Y, x) = & \frac{v_j(\omega)}{2\pi} \left[\sum_{l \neq j} \int_{\mathbb{R}} \check{\Gamma}_{jl}^c(\omega, \kappa'_y) [f(l, \kappa_y - \kappa'_y) - f(j, \kappa_y)] d\kappa'_y \right. \\ & \left. + \int_{\mathbb{R}} \check{R}_{jj}(\omega, \kappa'_y) [f(j, \kappa_y - \kappa'_y) - f(j, \kappa_y)] d\kappa'_y \right] - \frac{\kappa_y v_j(\omega)}{\beta_j(\omega)} \partial_Y f - v_j(\omega) \partial_x f. \end{aligned}$$

The Wigner transforms can then be expressed as

$$W_j(t, \omega, X, Y, \kappa_y) = \mathbb{E} \left[W_{J_t}(0, \omega, X_t, Y_t, K_t) \right. \\ \left. \times \exp \left(- \int_0^t v_{J_{t'}}(\omega) \Lambda_{J_{t'}}^r(\omega) dt' \right) \middle| J_0 = j, K_0 = \kappa_y, Y_0 = Y, X_0 = X \right],$$

where $W_j(0, \omega, X, Y, \kappa_y)$ is the initial Wigner transform at $t = 0$.

5. Statistical stability of the Wigner transform. The previous section gives a complete characterization of the mean Wigner transform (4.9) of the normal derivative of the wave field at the surface. The normal derivative of the wave field is indeed the quantity that can be measured in the experimental configuration in which receivers can only be deployed at the surface. From these measurements it is possible to compute the Wigner transform defined by

$$W^\varepsilon(t, \omega, X, \kappa_x, Y, \kappa_y) = \frac{1}{2\pi} \int_{\mathbb{R}^3} d\omega' dx' dY' e^{-i\omega' t - i\kappa_x x' - i\kappa_y Y'} \\ \times \frac{\partial_z \widehat{p}^\varepsilon \left(\omega + \varepsilon^2 \frac{\omega'}{2}, X + \varepsilon^2 \frac{x'}{2}, Y + \frac{Y'}{2}, z = 0 \right)}{\partial_z \widehat{p}^\varepsilon \left(\omega - \varepsilon^2 \frac{\omega'}{2}, X - \varepsilon^2 \frac{x'}{2}, Y - \frac{Y'}{2}, z = 0 \right)}. \quad (5.1)$$

From the measurements we cannot, however, compute the *mean* Wigner transform, which is an average over the possible realizations of the random medium, because only one realization of the random medium is available. In this section, we show that a smoothed version of the Wigner transform (see (5.2)) is statistically stable, in the sense that its typical value for one realization is close to its expectation. This means that the smoothed Wigner transform can be measured and it can be related to quantities that are characteristic of the medium by the radiative transfer equation. This gives the right framework to solve a well-posed inverse problem in order to estimate the medium.

We remark that the mean Wigner transform (4.9) is the limit expectation of the Wigner transform (5.1). However, this quantity is practically difficult to compute because of the three improper integrals over ω' , x' , and Y' , and it is not statistically stable in the sense that its standard deviation is large compared to its expectation (which is shown below). Let us define $\psi(s) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{s^2}{2})$ and choose positive smoothing parameters $\Omega_s, X_s, Y_s, T_s, K_{xs}, K_{ys}$. We can then introduce the smoothed Wigner transform

$$W_s(t, \omega, X, \kappa_x, Y, \kappa_y) = \lim_{\varepsilon \rightarrow 0} \sqrt{2\pi} \int_{\mathbb{R}^6} d\omega' dx' dY' d\omega'' dx'' dY'' e^{-i\omega'' t - i\kappa_x x'' - i\kappa_y Y''} \\ \times \frac{\partial_z \widehat{p}^\varepsilon \left(\omega + \varepsilon^2 \omega' + \varepsilon^2 \frac{\omega''}{2}, X + \varepsilon^2 x' + \varepsilon^2 \frac{x''}{2}, Y + Y' + \frac{Y''}{2}, z = 0 \right)}{\partial_z \widehat{p}^\varepsilon \left(\omega + \varepsilon^2 \omega' - \varepsilon^2 \frac{\omega''}{2}, X + \varepsilon^2 x' - \varepsilon^2 \frac{x''}{2}, Y + Y' - \frac{Y''}{2}, z = 0 \right)} \\ \times \psi(\omega'' T_s) \frac{1}{\Omega_s} \psi \left(\frac{\omega'}{\Omega_s} \right) \frac{1}{X_s} \psi \left(\frac{x'}{X_s} \right) \psi(x'' K_{xs}) \frac{1}{Y_s} \psi \left(\frac{Y'}{Y_s} \right) \psi(Y'' K_{ys}). \quad (5.2)$$

We note that the domains of the integrals are essentially bounded by the cut-off function ψ so the overall integral becomes tractable. Using the general relation

$$\int_{\mathbb{R}} d\tau' e^{-i\eta\tau'} \frac{1}{\tau_s} \psi\left(\frac{\tau - \tau'}{\tau_s}\right) = e^{-i\eta\tau} \sqrt{2\pi} \psi(\eta\tau_s),$$

we get the equivalent representation

(5.3)

$$\begin{aligned} W_s(t, \omega, X, \kappa_x, Y, \kappa_y) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^6} dt' d\omega' dX' d\kappa'_x dY' d\kappa'_y W^\varepsilon(t', \omega', X', \kappa'_x, Y', \kappa'_y) \\ &\quad \times \frac{1}{T_s} \psi\left(\frac{t - t'}{T_s}\right) \frac{1}{\varepsilon^2 \Omega_s} \psi\left(\frac{\omega - \omega'}{\varepsilon^2 \Omega_s}\right) \\ &\quad \times \frac{1}{\varepsilon^2 X_s} \psi\left(\frac{X - X'}{\varepsilon^2 X_s}\right) \frac{1}{K_{xs}} \psi\left(\frac{\kappa_x - \kappa'_x}{K_{xs}}\right) \frac{1}{Y_s} \psi\left(\frac{Y - Y'}{Y_s}\right) \frac{1}{K_{ys}} \psi\left(\frac{\kappa_y - \kappa'_y}{K_{ys}}\right), \end{aligned}$$

which shows that (5.2) is indeed a smoothed version of the Wigner transform (5.1), with smoothing in all variables $(t, \omega, X, \kappa_x, Y, \kappa_y)$. The smoothing is carried out by a convolution with a Gaussian kernel with width $(T_s, \varepsilon^2 \Omega_s, \varepsilon^2 X_s, K_{xs}, Y_s, K_{ys})$.

PROPOSITION 5.1. *We have*

(5.4)

$$\begin{aligned} \mathbb{E}[W_s(t, \omega, X, \kappa_x, Y, \kappa_y)] &= \sum_{j=1}^{N(\omega)} \int_{\mathbb{R}^3} W_j(t', \omega, X, Y', \kappa'_y) \frac{1}{T_s} \psi\left(\frac{t - t'}{T_s}\right) \frac{1}{Y_s} \psi\left(\frac{Y - Y'}{Y_s}\right) \frac{1}{K_{ys}} \psi\left(\frac{\kappa_y - \kappa'_y}{K_{ys}}\right) dt' d\kappa'_y dY' \\ &\quad \times \frac{1}{K_{xs}} \psi\left(\frac{\kappa_x - \beta_j(\omega)}{K_{xs}}\right) \partial_z \phi_j(\omega, 0)^2. \end{aligned}$$

Using the Gaussian approximation (more exactly, using Isserlis' formula to compute the fourth-order moments of the wave field in terms of the second-order moments) and considering that X_s is large enough so that $|\beta_j(\omega) - \beta_l(\omega)|X_s \gg 1$ for all $j \neq l$, the variance can be approximated by

(5.5)

$$\begin{aligned} \text{Var}(W_s(t, \omega, X, \kappa_x, Y, \kappa_y)) &= \sum_{j=1}^{N(\omega)} \int_{\mathbb{R}^6} W_j\left(t' + \frac{t''}{2}, \omega, X, Y' + \frac{Y''}{2}, \kappa'_y + \frac{\kappa''_y}{2}\right) W_j\left(t' - \frac{t''}{2}, \omega, X, Y' - \frac{Y''}{2}, \kappa'_y - \frac{\kappa''_y}{2}\right) \\ &\quad \times \frac{1}{T_s^2} \psi\left(\frac{t - t'}{T_s}\right)^2 \frac{1}{Y_s} \psi\left(\frac{Y - Y'}{Y_s}\right)^2 \frac{1}{K_{ys}} \psi\left(\frac{\kappa_y - \kappa'_y}{K_{ys}}\right)^2 \psi(\Omega_s t'')^2 \psi(Y_s \kappa''_y)^2 \psi(K_{ys} Y'')^2 \\ &\quad \times dt' dt'' dY' dY'' d\kappa'_y d\kappa''_y \frac{2(2\pi)^3}{K_{xs}^2} \psi\left(\frac{\kappa_x - \beta_j(\omega)}{K_{xs}}\right)^2 \partial_z \phi_j(\omega, 0)^4. \end{aligned}$$

A complete expression of the variance is given in the proof. The first result (5.4) shows that there is a smoothing of the mean Wigner transform in t of order T_s , a smoothing in Y of order Y_s , a smoothing in κ_y of order K_{ys} , and a smoothing in κ_x of order K_{xs} . The effect of the smoothing in ω and X is negligible, as long as the

smoothing parameters are of order ε^2 . The smoothing in ω is, however, important because it reduces the variance. Indeed, the second result (5.5) shows that the relative variance (also called coefficient of variation) is of order

$$(5.6) \quad \frac{\text{Var}(W_s)}{\mathbb{E}[W_s]^2} = O\left(\frac{1}{T_s} \times \min(\Delta t, \Omega_s^{-1}) \times \min(\Delta \kappa_y, Y_s^{-1}) \times \min(\Delta Y, K_{ys}^{-1})\right),$$

where $\Delta \kappa_y$, respectively Δt , ΔY , is the width of the mean Wigner transform in κ_y , respectively in t , Y .

Proof. The first result is obtained by substituting (4.10) into the expectation of (5.3). Using the Gaussian approximation makes it possible to express the expectation of the square of W_s in terms of a double sum (over j and l) and multiple integrals involving the product of two terms W_j and W_l (the solutions of (4.4)):

$$\begin{aligned} & \text{Var}(W_s(t, \omega, X, \kappa_x, Y, \kappa_y)) \\ &= \sum_{j,l=1}^{N(\omega)} \int_{\mathbb{R}^6} dt_1 d\kappa_{y1} dY_1 dt_2 d\kappa_{y2} dY_2 W_j(t_1, \omega, X, Y_1, \kappa_{y1}) W_l(t_2, \omega, X, Y_2, \kappa_{y2}) \\ & \quad \times \frac{2(2\pi)^3}{T_s^2 K_{xs}^2 Y_s K_{ys}} \psi\left(\frac{\frac{t_1+t_2}{2} - t}{T_s}\right)^2 \psi\left(\frac{\frac{\kappa_{y1}+\kappa_{y2}}{2} - \kappa_y}{K_{ys}}\right)^2 \psi\left(\frac{\frac{Y_1+Y_2}{2} - Y}{Y_s}\right)^2 \\ & \quad \times \psi\left(\frac{\frac{\beta_j(\omega) + \beta_l(\omega)}{2} - \kappa_x}{K_{xs}}\right)^2 \psi(\Omega_s(t_1 - t_2))^2 \psi(Y_s(\kappa_{y1} - \kappa_{y2}))^2 \psi(K_{ys}(Y_1 - Y_2))^2 \\ & \quad \times \psi((\beta_j(\omega) - \beta_l(\omega))X_s)^2 \partial_z \phi_j(\omega, 0)^2 \partial_z \phi_l(\omega, 0)^2. \end{aligned}$$

The condition that X_s is large then reduces the double sum over j, l to a single sum and gives the desired result. \square

We study the relative variance (5.6). We first remark that ΔY is in fact the radius of the transverse envelope of the wave field and $\Delta \kappa_y$ is the reciprocal of the transverse correlation radius of the wave field. For a coherent beam we have $\Delta Y \Delta \kappa_y \sim 1$, but for a partially coherent or incoherent wave as we deal with here, we may have $\Delta Y \Delta \kappa_y \gtrsim 1$ or $\Delta Y \Delta \kappa_y \gg 1$. If we want to resolve (by a factor $R_s > 1$) the dependence of the Wigner transform with respect to the variables Y and κ_y , we need to choose $Y_s = \Delta Y / R_s$ and $K_{ys} = \Delta \kappa_y / R_s$.

If Ω_s^{-1} is larger than Δt , then the smoothing in ω in (5.2) plays no role and the relative variance is of order

$$(5.7) \quad \frac{\text{Var}(W_s)}{\mathbb{E}[W_s]^2} = O\left(\frac{\Delta t}{T_s} \times \min\left(\Delta Y \Delta \kappa_y, \frac{R_s^2}{\Delta Y \Delta \kappa_y}\right)\right).$$

If Ω_s^{-1} is smaller than Δt , then the smoothing in ω in (5.2) plays an important role and the relative variance is of order

$$(5.8) \quad \frac{\text{Var}(W_s)}{\mathbb{E}[W_s]^2} = O\left(\frac{1}{\Omega_s T_s} \times \min\left(\Delta Y \Delta \kappa_y, \frac{R_s^2}{\Delta Y \Delta \kappa_y}\right)\right),$$

which can be made smaller than one if Ω_s^{-1} is small enough. This self-averaging effect is due to the small coherence frequency of the field, and this is the main mechanism that can ensure the statistical stability of the smoothed Wigner transform. Note that the partial spatial coherence of the wave field can also give some stability. Indeed, if

$\Delta Y \Delta \kappa_y \gg 1$ and we choose R_s such that $1 < R_s^2 \ll \Delta Y \Delta \kappa_y$ (and therefore $Y_s K_{ys} \gg 1$), then we get an extra reduction of the relative variance by the factor $Y_s K_{ys}$:

$$(5.9) \quad \frac{\text{Var}(W_s)}{\mathbb{E}[W_s]^2} = O\left(\frac{1}{\Omega_s T_s} \times \frac{1}{Y_s K_{ys}}\right).$$

To summarize, the smoothing parameters are chosen as follows to ensure both resolution and stability: We take $T_s \ll \Delta t$, $K_{ys} \ll \Delta \kappa_y$, $Y_s \ll \Delta Y$ to get good resolution. We take $\Omega_s^{-1} \ll T_s$ to get good stability. The coefficient of variation is then given by

$$(5.10) \quad \frac{\text{Var}(W_s)}{\mathbb{E}[W_s]^2} = O\left(\frac{1}{\Omega_s T_s} \times \min(\Delta \kappa_y, Y_s^{-1}) \times \min(\Delta Y, K_{ys}^{-1})\right),$$

which is always smaller than $O(\frac{\Delta Y \Delta \kappa_y}{\Omega_s T_s})$ whatever the values of Y_s and K_{ys} . If the coefficient of variation is not small enough for the considered application, it is possible to reduce it by taking $Y_s = \Delta Y / (\Delta Y \Delta \kappa_y)^\alpha$ and $K_{ys} = \Delta \kappa_y / (\Delta Y \Delta \kappa_y)^\alpha$ for some $\alpha \in (0, 1)$, so that $\frac{\text{Var}(W_s)}{\mathbb{E}[W_s]^2} = O(\frac{(\Delta Y \Delta \kappa_y)^{2\alpha-1}}{\Omega_s T_s})$. As α becomes close to zero, we lose resolution but we gain stability.

The stability results of this section will be important in the robust estimation of the (background) medium and source parameters, which we discuss in section 7.

6. Transport Albedo kernel. We develop an expression for what we refer to as the Albedo kernel which describes the mapping between the source and the measurements in our framework. This description is useful in order to analyze the announced inverse problem of inferring model parameters from the observed data. We discuss this inverse problem in more detail in the next section.

We consider the mean Wigner transforms W_j , $j = 1, \dots, N$, solving the radiative transfer equations (4.4). For a fixed frequency ω we define the associated effective Albedo kernel $\mathcal{A}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0})$ that is the $N(\omega) \times N(\omega)$ matrix solution of

$$(6.1) \quad \begin{aligned} \partial_X \mathcal{A} + \text{diag}_j \left\{ \frac{\kappa_y}{\beta_j(\omega)} \right\} \partial_Y \mathcal{A} + \text{diag}_j \left\{ \frac{1}{v_j(\omega)} \right\} \partial_t \mathcal{A} = -\text{diag}_j \left\{ \Lambda_j^c(\omega) + \Lambda_j^r(\omega) \right\} \mathcal{A} \\ + \frac{1}{2\pi} \int_{\mathbb{R}} \text{diag}_j \left\{ \check{R}_{jj}(\omega, \kappa_y') \right\} [\mathcal{A}(\kappa_y - \kappa_y') - \mathcal{A}(\kappa_y)] d\kappa_y' \\ + \frac{1}{2\pi} \int_{\mathbb{R}} \check{\Gamma}(\omega, \kappa_y') \mathcal{A}(\kappa_y - \kappa_y') d\kappa_y', \end{aligned}$$

with initial condition

$$(6.2) \quad \mathcal{A}|_{X=0} = \delta(Y - Y_0) \delta(\kappa_y - \kappa_{y,0}) \delta(t - t_0) \mathcal{I}_N,$$

where \mathcal{I}_N is the $N \times N$ identity matrix and

$$(6.3) \quad \check{\Gamma}(\omega, \kappa_y) = \{\check{\Gamma}_{j,l}^c(\omega, \kappa_y) \mathbb{1}_{j \neq l}\}_{j,l=1}^{N(\omega)}.$$

The Albedo kernel relates the initial Wigner transform $(W_l^{(0)})_{l=1}^N$ at $x = 0$ to the mean transmitted Wigner transform $(W_j)_{j=1}^N$ at the range $x = X/\varepsilon^2$:

$$(6.4) \quad \begin{aligned} W_j(t, \omega, X, Y, \kappa_y) = \sum_{l=1}^N \int_{\mathbb{R}^3} \mathcal{A}_{j,l}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) \\ \times W_l^{(0)}(\omega, t_0, Y_0, \kappa_{y,0}) dt_0 dY_0 d\kappa_{y,0}. \end{aligned}$$

We next develop a closed-form expression for \mathcal{A} . For $n \geq 0$, let $\mathcal{A}^{(n)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0})$ solve

$$\begin{aligned}
 & \partial_X \mathcal{A}^{(n)} + \text{diag}_j \left\{ \frac{\kappa_y}{\beta_j(\omega)} \right\} \partial_Y \mathcal{A}^{(n)} + \text{diag}_j \left\{ \frac{1}{v_j(\omega)} \right\} \partial_t \mathcal{A}^{(n)} \\
 &= -\text{diag}_j \left\{ \Lambda_j^c(\omega) + \Lambda_j^r(\omega) \right\} \mathcal{A}^{(n)} \\
 &+ \frac{1}{2\pi} \int_{\mathbb{R}} \text{diag}_j \left\{ \check{R}_{jj}(\omega, \kappa'_y) \right\} [\mathcal{A}^{(n)}(\kappa_y - \kappa'_y) - \mathcal{A}^{(n)}(\kappa_y)] d\kappa'_y \\
 (6.5) \quad &+ \frac{1}{2\pi} \int_{\mathbb{R}} \check{\Gamma}(\omega, \kappa'_y) \mathcal{A}^{(n-1)}(\kappa_y - \kappa'_y) d\kappa'_y,
 \end{aligned}$$

starting from

$$(6.6) \quad \mathcal{A}^{(n)}|_{X=0} = \delta(Y - Y_0) \delta(\kappa_y - \kappa_{y,0}) \delta(t - t_0) \mathcal{I}_N \mathbb{1}_0(n), \quad n = 0, 1, \dots$$

Here we use the convention $\mathcal{A}^{(-1)} \equiv 0$. We then find that

$$\begin{aligned}
 (6.7) \quad & \mathcal{A}^{(0)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) = (2\pi)^{-2} \text{diag}_j \left\{ \delta \left(t - \frac{X}{v_j(\omega)} - t_0 \right) e^{-(\Lambda_j^c(\omega) + \Lambda_j^r(\omega))X} \right. \\
 & \times \int_{\mathbb{R}^2} e^{\int_0^X (R_{jj}(\omega, \eta + X' \xi / \beta_j(\omega)) - R_{jj}(\omega, 0)) dX'} e^{i((Y_0 - Y + \kappa_{y,0} X / \beta_j(\omega)) \xi + (\kappa_{y,0} - \kappa_y) \eta)} d\eta d\xi \left. \right\},
 \end{aligned}$$

and for $n \geq 1$

$$\begin{aligned}
 & \mathcal{A}^{(n)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) \\
 &= \int_0^X dX' \int_{\mathbb{R}^3} dt' dY' d\kappa'_y \mathcal{A}^{(0)}(t, \omega, X - X', Y, \kappa_y; t', Y', \kappa'_y) \\
 (6.8) \quad & \times \frac{1}{2\pi} \int_{\mathbb{R}} \check{\Gamma}(\omega, \tilde{\kappa}_y) \mathcal{A}^{(n-1)}(t', \omega, X', Y', \kappa'_y - \tilde{\kappa}_y; t_0, Y_0, \kappa_{y,0}) d\tilde{\kappa}_y.
 \end{aligned}$$

We introduce the operator $\mathcal{L}_X^{(0)}$ defined by

$$\begin{aligned}
 (6.9) \quad & \mathcal{L}_X^{(0)} \Phi(t, Y, \kappa_y) := \int_{\mathbb{R}^3} dt' dY' d\kappa'_y \frac{1}{2\pi} \int_{\mathbb{R}} d\tilde{\kappa}_y \mathcal{A}^{(0)}(t, \omega, X, Y, \kappa_y; t', Y', \tilde{\kappa}_y) \check{\Gamma}(\omega, \tilde{\kappa}_y - \kappa'_y) \\
 & \times \Phi(t', Y', \kappa'_y)
 \end{aligned}$$

for any square integrable test function $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^{N(\omega) \times N(\omega)}$. The operator $\mathcal{L}_X^{(0)}$ is such that

$$(6.10) \quad \|\mathcal{L}_X^{(0)}\|_2 \leq \vartheta(\omega) := N(\omega) \max_{j \neq l \in \{1, \dots, N(\omega)\}} \Gamma_{j,l}^c(\omega, 0)$$

with

$$(6.11) \quad \|\Phi\|_2^2 := \sum_{j,l=1}^{N(\omega)} \int_{\mathbb{R}^3} dt dY d\kappa_y \Phi_{j,l}(t, Y, \kappa_y)^2.$$

We then have convergence of $\sum_{j=0}^n \mathcal{A}^{(j)}$ to \mathcal{A} in the sense we describe next. With $\Phi \in L^2$ it follows that we can write $\mathcal{A}\Phi = \sum_{j=0}^{\infty} \mathcal{A}^{(j)}\Phi$, satisfying the precision bound at order n :

$$(6.12) \quad \left\| \left(\mathcal{A} - \sum_{j=0}^n \mathcal{A}^{(j)} \right) \Phi \right\|_2 (X) \leq \frac{(\vartheta(\omega)X)^{n+1}}{(n+1)!} e^{X\vartheta(\omega)} \|\Phi\|_2,$$

where we used that $\|\mathcal{A}^{(0)}\|_2(X) \leq 1$. Note that we have a rapid decay of the higher order terms (large n) in the case of small medium fluctuations so that $\vartheta(\omega)$ is small. We discuss in more detail in the next section how a particular source condition corresponds to a particular form of Φ so that the approximations $\sum_{j=0}^n \mathcal{A}^{(j)} \Phi$ converge to the full response function $\mathcal{A}\Phi$ as n goes to infinity. For the second-order term, we have

$$(6.13) \quad \begin{aligned} \mathcal{A}_{j,l}^{(1)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) &= (2\pi)^{-3} \int_{\mathbb{R}^3} d\eta d\xi d\tilde{\kappa}_y \check{\Gamma}_{j,l}^c(\omega, \tilde{\kappa}_y - \kappa_{y,0}) \\ &\times e^{\mathcal{H}_j(\omega, X - s_{j,l}^*, \eta, \xi) + \mathcal{H}_l(\omega, s_{j,l}^*, \eta + \xi(X - s_{j,l}^*)/\beta_j, \xi)} \\ &\times e^{i(\xi(Y_0 - Y + (X - s_{j,l}^*)\tilde{\kappa}_y/\beta_j + s_{j,l}^*\kappa_{y,0}/\beta_l) + \eta(\tilde{\kappa}_y - \kappa_y))} \times \mathbb{1}_{(0,X)}(s_{j,l}^*), \end{aligned}$$

in which

$$(6.14)$$

$$\mathcal{H}_j(\omega, X, \eta, \xi) = \int_0^X [R_{jj}(\omega, \eta + X'\xi/\beta_j(\omega)) - R_{jj}(\omega, 0)] dX' - [\Lambda_j^c(\omega) + \Lambda_j^r(\omega)] X$$

and $s_{j,l}^*$ is the solution of

$$(6.15) \quad t - t_0 = \frac{X - s_{j,l}^*}{v_j} + \frac{s_{j,l}^*}{v_l},$$

corresponding to ‘‘effective scattering’’ from mode l to mode j at propagation distance $s_{j,l}^*$ and an ‘‘effective travel time’’ in mode l being $s_{j,l}^*/v_l$ and in mode j being $(X - s_{j,l}^*)/v_j$.

The coefficient \check{R}_{jj} determines the conversion rates between different κ_y components of the j th surface mode and it depends on low-frequency components of the fluctuations of the random medium. The coefficients $\check{\Gamma}_{jl}^c$ determine the conversion rates between different surface modes ($j \neq l$) and they depend on high-frequency components of the fluctuations of the random medium because of the presence of the term $\cos(\beta_l(\omega) - \beta_j(\omega))x$ in (3.13). We will consider a regime in which the coefficients \check{R}_{jj} are larger than the coefficients $\check{\Gamma}_{jl}^c$. More exactly, we will consider propagation distances X such that

$$(6.16) \quad X R_{jj}(\omega, 0) \gg 1, \quad X \Lambda_j^c(\omega) \ll 1,$$

and we will call it the strongly heterogeneous regime.

The second hypothesis in (6.16) implies that the term $\mathcal{A}^{(1)}$ is much smaller than $\mathcal{A}^{(0)}$ but much larger than the remainder $\sum_{n=2}^{\infty} \mathcal{A}^{(n)}$. The first hypothesis in (6.16) makes it possible to obtain simplified expressions for $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(0)}$. Indeed, assuming that the first hypothesis in (6.16) holds and that, additionally, the covariance function of the medium fluctuations is smooth enough so that we can expand

$$(6.17) \quad R_{jj}(\omega, Y) = R_{jj}(\omega, 0) \left[1 - \frac{1}{2} \frac{Y^2}{\ell_j(\omega)^2} + o\left(\frac{Y^2}{\ell_j(\omega)^2}\right) \right],$$

then we obtain

$$(6.18) \quad \begin{aligned} \mathcal{A}^{(0)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) &= \text{diag}_j \left\{ \delta \left(t - \frac{X}{v_j(\omega)} - t_0 \right) \right. \\ &\times \left. \frac{e^{-(\Lambda_j^c(\omega) + \Lambda_j^r(\omega))X}}{\sqrt{r_j X}} \psi \left(\frac{\kappa_{y,0} - \kappa_y}{\sqrt{r_j X}} \right) \frac{1}{\sqrt{\frac{r_j X^3}{12\beta_j^2}}} \psi \left(\frac{Y_0 - Y + \frac{X(\kappa_{y,0} + \kappa_y)}{2\beta_j}}{\sqrt{\frac{r_j X^3}{12\beta_j^2}}} \right) \right\} \end{aligned}$$

with

$$(6.19) \quad r_j(\omega) = \frac{R_{jj}(\omega, 0)}{\ell_j(\omega)^2}.$$

Moreover, we find for $j \neq l$ that

$$(6.20) \quad \begin{aligned} \mathcal{A}_{j,l}^{(1)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) &= \frac{1}{2\pi} \int_{\mathbb{R}} d\tilde{\kappa}_y \tilde{\Gamma}_{j,l}^c(\omega, \tilde{\kappa}_y - \kappa_{y,0}) \\ &\times \frac{1}{\sqrt{h_{j,l}}} \psi\left(\frac{\tilde{\kappa}_y - \kappa_y}{\sqrt{h_{j,l}}}\right) \frac{1}{\sqrt{H_{j,l}}} \psi\left(\frac{Y_0 - Y + K_{j,l}(\tilde{\kappa}_y)}{\sqrt{H_{j,l}}}\right) \\ &\times \exp\left(-(\Lambda_j^c(\omega) + \Lambda_j^r(\omega))(X - s_{j,l}^*) - (\Lambda_l^c(\omega) + \Lambda_l^r(\omega))s_{j,l}^*\right) \times \mathbb{1}_{(0, X)}(s_{j,l}^*) \end{aligned}$$

with $s_{j,l}^*$ defined by (6.15) and

$$\begin{aligned} H_{j,l} &= \frac{r_j^2 \frac{(X - s_{j,l}^*)^4}{3\beta_j^2} + r_l^2 \frac{(s_{j,l}^*)^4}{3\beta_l^2} + 2r_j r_l s_{j,l}^* (X - s_{j,l}^*) \left(\frac{s_{j,l}^* (X - s_{j,l}^*)}{\beta_j \beta_l} + \frac{2(X - s_{j,l}^*)^2}{3\beta_j^2} + \frac{2(s_{j,l}^*)^2}{3\beta_l^2} \right)}{4h_{j,l}}, \\ K_{j,l}(\tilde{\kappa}_y) &= \frac{\frac{(X - s_{j,l}^*)^2}{\beta_j} r_j(\kappa_y + \tilde{\kappa}_y) + 2s_{j,l}^* (X - s_{j,l}^*) \left(\frac{r_j \kappa_{y,0}}{\beta_l} + \frac{r_l \kappa_y}{\beta_j} \right) + \frac{(s_{j,l}^*)^2}{\beta_l} r_l(\kappa_y - \tilde{\kappa}_y + 2\kappa_{y,0})}{2h_{j,l}}, \\ h_{j,l} &= r_j(X - s_{j,l}^*) + r_l s_{j,l}^*. \end{aligned}$$

In the next section, we analyze how this simplified form of the Albedo kernel allows us to identify how model parameters can be inferred from the observed data.

7. Source response, data separation, and parameter identification. In this section, we consider the situation with a Gaussian source and measuring the transmitted field in the form of the smoothed Wigner transform in (5.2) in the strongly heterogeneous random medium case (6.16). We show that in this case, we can stably identify the parameters of the RTE in (4.4).

From Proposition 5.1 and the previous section, we find that the mean smoothed Wigner transform $\mathcal{W}_s = \mathbb{E}[W_s]$ can be expanded up to second order as

$$(7.1) \quad \mathcal{W}_s(t, \omega, X, \kappa_x, Y, \kappa_y) = \mathcal{W}_s^{(0)}(t, \omega, X, \kappa_x, Y, \kappa_y) + \mathcal{W}_s^{(1)}(t, \omega, X, \kappa_x, Y, \kappa_y),$$

$$(7.2) \quad \begin{aligned} \mathcal{W}_s^{(n)}(t, \omega, X, \kappa_x, Y, \kappa_y) &= \sum_{j,l=1}^{N(\omega)} \partial_z \phi_j(\omega, 0)^2 \int_{\mathbb{R}^3} \mathcal{T}_{j,l}^{(n)}(t', \omega, X, Y', \kappa_y') \frac{1}{T_s} \psi\left(\frac{t - t'}{T_s}\right) \\ &\times \frac{1}{Y_s} \psi\left(\frac{Y - Y'}{Y_s}\right) \frac{1}{K_{ys}} \psi\left(\frac{\kappa_y - \kappa_y'}{K_{ys}}\right) dt' dY' d\kappa_y' \\ &\times \frac{1}{K_{xs}} \psi\left(\frac{\kappa_x - \beta_j(\omega)}{K_{xs}}\right), \quad n = 0, 1, \end{aligned}$$

in terms of the first ($n = 0$) and second ($n = 1$) order response functions

$$(7.3) \quad \begin{aligned} \mathcal{T}_{j,l}^{(n)}(t, \omega, X, Y, \kappa_y) &:= \int_{\mathbb{R}^3} dt_0 dY_0 d\kappa_{y,0} \mathcal{A}_{j,l}^{(n)}(t, \omega, X, Y, \kappa_y; t_0, Y_0, \kappa_{y,0}) W_l^{(0)}(t_0, \omega, Y_0, \kappa_{y,0}). \end{aligned}$$

We assume that the source terms $G^\varepsilon(t)$ and $F(Y, z)$ in (3.1) are such that

$$(7.4) \quad W_G(t, \omega) = \frac{\sqrt{\pi} w_G(\omega)}{\sigma_t} \exp\left(-\frac{t^2}{4\sigma_t^2}\right), \quad F_j(\omega, Y) = \frac{f_j(\omega)}{\pi} \exp\left(-\frac{Y^2}{2\sigma_y^2}\right),$$

for some positive constants σ_t, σ_y and smooth functions $w_G(\omega), f_j(\omega)$. We then find by (4.5) that

$$(7.5) \quad W_j^{(0)}(t, \omega, Y, \kappa_y) = \frac{|w_j(\omega)|^2 \sigma_y}{\beta_j^2(\omega) \sigma_t} \exp\left(-\frac{t^2}{4\sigma_t^2} - \frac{Y^2}{\sigma_y^2} - \sigma_y^2 \kappa_y^2\right)$$

with $w_j(\omega) = \sqrt{w_G(\omega)} f_j(\omega)$. Figure 3.1 illustrates the wave propagation configuration that we consider. Note first that via the projections in (2.4) we reduce the problem to an evolution problem in two space dimensions with x being the propagation (range) direction and y the lateral space dimension. A source emits in the plane $x = 0$ which then produces a boundary condition on the incoming boundary Γ_- (a strip on the y axis), and the field is measured on the thin set Γ_+ in the $x-y$ plane. Here, ε is a small parameter which characterizes the transverse radius and the propagation distance in relation to the wavelength which is order one in our scaling. The propagated field is measured on the receiver array at $x = X/\varepsilon^2$, denoted Γ_+ , for a time duration $\mathcal{O}(1)$ and subsequently processed to form the smoothed Wigner transform in (5.2). Note that in the figure (x, y) denote the unscaled coordinates while the scaled coordinates are $(X, Y) = (\varepsilon^2 x, \varepsilon y)$. We remark that the Albedo kernel developed in the previous section gives the mapping from the initial data for the Wigner transform to the left, at $x = 0$, in Figure 3.1 to the mean transmitted Wigner transform to the right, at the range X/ε^2 .

We assume that $\Omega_s^{-1} \ll T_s$, so that the coefficient of variation in (5.6) is much smaller than one. We remark that we do not consider additive measurement noise, although this could easily have been included. We emphasize that the medium perturbations are responsible for generating the mode coupling and producing the particular form for the Wigner transform in (7.2). As we show below, this form can be used for parameter estimation. What is important to note here is that to identify the medium information carried by the particular form in (7.2) we need to perform an a priori smoothing step. This signal-to-noise enhancement step is indeed not so often discussed in the transport literature, but nevertheless is important in the processing of the measurements.

We discuss next how parameters in the radiative transport equations can be identified based on these measurements. We have

(7.6)

$$\begin{aligned} \mathcal{T}_{j,l}^{(0)}(t, \omega, X, Y, \kappa_y) &= \left(\frac{|w_j(\omega)|^2 \sigma_y 4\pi^{3/2}}{\beta_j X} \right) \frac{1}{\sqrt{2\sigma_t}} \psi\left(\frac{t - X/v_j}{\sqrt{2\sigma_t}}\right) \frac{1}{\sqrt{\Sigma_j}} \psi\left(\frac{\kappa_y}{\sqrt{\Sigma_j}}\right) \\ &\times e^{-(\Lambda_j^c(\omega) + \Lambda_j^r(\omega))X} \frac{1}{\sqrt{r_j X (1/3 + 1/(2\sigma_y^2 \Sigma_j)) + 2\beta_j^2 \sigma_y^2/X^2}} \\ &\times \psi\left(\frac{\kappa_y (1 + 1/(2\sigma_y^2 \Sigma_j)) - 2Y\beta_j/X}{\sqrt{r_j X (1/3 + 1/(2\sigma_y^2 \Sigma_j)) + 2\beta_j^2 \sigma_y^2/X^2}}\right) \mathbb{1}_l(j) \end{aligned}$$

with $\Sigma_j(\omega) = r_j(\omega)X + \frac{1}{2\sigma_y^2}$, and

$$\begin{aligned}
 \mathcal{T}_{j,l}^{(1)}(t, \omega, X, Y, \kappa_y) &= \int_{\mathbb{R}^4} dY_0 d\kappa_{y,0} d\tilde{\kappa}_y dt_0 \left(\frac{\tilde{\Gamma}_{j,l}^c(\omega, \tilde{\kappa}_y - \kappa_{y,0}) |w_l(\omega)|^2 \sigma_y}{2\pi \beta_l^2(\omega) \sigma_t} \right) \\
 &\times \exp \left(-\frac{t_0^2}{4\sigma_t^2} - \frac{Y_0^2}{\sigma_y^2} - \sigma_y^2 \kappa_{y,0}^2 \right) \frac{1}{\sqrt{h_{j,l}}} \psi \left(\frac{\tilde{\kappa}_y - \kappa_y}{\sqrt{h_{j,l}}} \right) \\
 &\times \frac{1}{\sqrt{H_{j,l}}} \psi \left(\frac{Y_0 - Y + K_{j,l}(\tilde{\kappa}_y)}{\sqrt{H_{j,l}}} \right) \\
 (7.7) \quad &\times \exp \left(-(\Lambda_j^c(\omega) + \Lambda_j^r(\omega))(X - s_{j,l}^*) - (\Lambda_l^c(\omega) + \Lambda_l^r(\omega))s_{j,l}^* \right) \times \mathbb{1}_{(0,X)}(s_{j,l}^*).
 \end{aligned}$$

We make some simplifying assumptions to get more explicit expressions to articulate the parameter estimation context. In the regime (6.16) $\mathcal{W}_s^{(0)}$ is the leading contribution; moreover, we assume narrow temporal source and smoothing, that is,

$$(7.8) \quad (\Lambda_j^c(\omega) + \Lambda_j^r(\omega)) X \ll 1,$$

$$(7.9) \quad T_s \ll \sigma_t \ll \min_{j \neq l} |v_j^{-1}(\omega) - v_l^{-1}(\omega)|X, \quad K_{xs} \ll \min_{2 \leq j \leq N(\omega)} |\beta_j - \beta_{j-1}| \wedge \beta_{N(\omega)},$$

$$(7.10) \quad Y_s \ll \sigma_y \wedge \sqrt{r_j(\omega)X}(X/\beta_j(\omega)), \quad K_{ys} \ll \sigma_y^{-1} \wedge \sqrt{r_j(\omega)X} \wedge \sigma_y \beta_j(\omega)/X,$$

for all $j = 1, \dots, N(\omega)$. Note then that the term $\psi((\kappa_x - \beta_j)/K_{xs})$ in (7.2) makes it possible to use the support in the variable κ_x and the dispersion in phase velocities to separate the contributions to \mathcal{W}_s associated with different j values (corresponding to observing transmitted surface mode j). Moreover, note that $\mathcal{T}_{j,l}^{(0)} = 0, j \neq l$ and $\mathcal{T}_{j,j}^{(1)} = 0$ and that the term $\mathbb{1}_{(0,X)}(s_{j,l}^*)$ is supported for $t - t_0 \in X(v_j \wedge v_l, v_j \vee v_l)$ in view of (6.15). This then allows us to additionally separate the contributions to \mathcal{W}_s associated with different source modes l due to the dispersion in group velocities and in view of the fact that the observations have a high signal-to-noise ratio due to the smoothing. We then find the following lemma.

LEMMA 7.1. *Assume that the assumptions in Proposition 5.1 are satisfied and that $\Omega_s^{-1} \ll T_s$ so that the coefficient of variation is much smaller than one. Moreover, assume that the source assumptions in (6.16) and (7.8)–(7.10) are satisfied. Then the components of the second-order smoothed Wigner transform*

$$\begin{aligned}
 \mathcal{W}_s^{(n,j,l)}(t, \omega, X, \kappa_x, Y, \kappa_y) &= \partial_z \phi_j(\omega, 0)^2 \int_{\mathbb{R}^3} \mathcal{T}_{j,l}^{(n)}(t', \omega, X, Y', \kappa_y') \frac{1}{T_s} \psi \left(\frac{t - t'}{T_s} \right) \\
 &\times \frac{1}{Y_s} \psi \left(\frac{Y - Y'}{Y_s} \right) \frac{1}{K_{ys}} \psi \left(\frac{\kappa_y - \kappa_y'}{K_{ys}} \right) dt' dY' d\kappa_y' \\
 &\times \frac{1}{K_{xs}} \psi \left(\frac{\kappa_x - \beta_j(\omega)}{K_{xs}} \right), \quad n = 0, 1, \quad j, l = 1, \dots, N(\omega),
 \end{aligned}$$

can be identified.

Next, we discuss the identification of the parameters of the radiative transfer equation assuming additionally that the range X is known. We note first that the group and phase velocities $v_j, \omega/\beta_j$, and the parameters

$$(7.11) \quad \sigma_t, \quad \sigma_y, \quad r_j, \quad \partial_z \phi_j^2(\omega, 0) |w_j(\omega)|^2,$$

can be identified from the leading-order smoothed Wigner response $\mathcal{W}_s^{(0)}$.

Now, we consider the second-order smoothed Wigner function contributions

$$(7.12) \quad \begin{aligned} \mathcal{W}_s^{(1)}(t, \omega, X, \kappa_x, Y, \kappa_y) := & \sum_{j,l=1}^{N(\omega)} \partial_z \phi_j^2(\omega, 0) \int_{\mathbb{R}^3} \mathcal{T}_{j,l}^{(1)}(t', \omega, X, Y', \kappa_y') \\ & \times \frac{1}{T_s} \psi\left(\frac{t-t'}{T_s}\right) \frac{1}{Y_s} \psi\left(\frac{Y-Y'}{Y_s}\right) \frac{1}{K_{ys}} \psi\left(\frac{\kappa_y - \kappa_y'}{K_{ys}}\right) dt' dY' d\kappa_y' \frac{1}{K_{xs}} \psi\left(\frac{\kappa_x - \beta_j(\omega)}{K_{xs}}\right). \end{aligned}$$

We assume, finally, a fast decorrelation on the scale ℓ_y of the medium fluctuations in the lateral direction, so that

$$(7.13) \quad \ell_y \ll \sigma_y \wedge \frac{1}{\sqrt{r_j(\omega)X}}.$$

This then additionally allows for estimation of

$$(7.14) \quad \partial_z \phi_j^2(\omega, 0) \check{\Gamma}_{j,l}^c(\omega, 0) |w_l(\omega)|^2, \quad j \neq l = 1, \dots, N(\omega).$$

In summary, using also that $\check{\Gamma}$ is symmetric, we get the following identification result.

PROPOSITION 7.2. *Assume that the conditions in Lemma 7.1 are satisfied, that the identification conditions (7.13) are satisfied, and additionally that X is known; then with the observations being the smoothed Wigner transform, this allows for identification of the group and phase velocities*

$$(7.15) \quad v_j(\omega), \quad \omega/\beta_j(\omega), \quad j = 1, \dots, N(\omega),$$

as well as the source parameters

$$(7.16) \quad \sigma_t, \quad \sigma_y, \quad |w_j(\omega)|, \quad j = 1, \dots, N(\omega),$$

and the (coupling) medium parameters

$$(7.17) \quad |\partial_z \phi_j(\omega, 0)|, \quad r_j, \quad \check{\Gamma}_{j,l}^c(\omega, 0), \quad j \neq l = 1, \dots, N(\omega).$$

In Proposition 7.2 we assumed that the distance to the source is known. However, from the form of the smoothed Wigner transform \mathcal{W}_s observations at arrays centered at two values for the range coordinate x allow us to identify the distance from the source. We have the following corollary.

COROLLARY 7.3. *Assume that we measure on two arrays separated in range by a distance of order $\mathcal{O}(\varepsilon^{-2})$ compared to the wavelength; then the parameters in Proposition 7.2 can be estimated without a priori knowledge of range.*

We illustrate the situation with two measurement arrays in Figure 3.1. Here, we have focused on wave propagation in the transport regime and can conclude that in the high-frequency source and strongly heterogeneous random medium regime, we can identify the parameters of the radiative transfer equation (4.4).

8. Discussion. We have developed a general framework for studying coupling between body and surface modes along a boundary with random medium layers beneath it, and an associated inverse problem. We shed light on various aspects motivated by recent descriptions and observations of coupling published in the seismology literature. While the inverse problem addressed the recovery of surface-wave phase and group velocities, in a forthcoming paper we will present the unique recovery of the background wave speed from the phase velocities if this is piecewise constant in the boundary normal coordinate. The statistical stability of the “data” makes this framework well suited for sensitive background monitoring applications when the realizations of the random medium fluctuations may change in time. We expect that the results obtained here carry over to the Rayleigh system.

Appendix A. The sedimentary paraxial regime. We assume again that the source is of the form (3.1). As above we rescale the field as (3.2). This scaling corresponds to the one discussed in the previous sections and illustrated in Figure 3.1.

However, here we assume that the medium perturbation is of the form

$$(A.1) \quad \mu_n(x, y, z) = \varepsilon^{3/2} \mu(\varepsilon x, \varepsilon y, z).$$

The process μ_n is anisotropic with a vertical correlation length (in z) of the order of the wavelength and horizontal correlation lengths (in x and y) of the order of the beam radius. It differs from the scaling discussed above in that the correlation length in the x coordinate is of the order of the beam radius, rather than small compared to this radius. The standard deviation (of the order of $\varepsilon^{3/2}$) of the process μ_n is such that the cumulative scattering effects are of order one as $\varepsilon \rightarrow 0$.

The Fourier transform of (3.2) is given by the scaled version of (2.8),

$$(A.2) \quad \widehat{p}^\varepsilon(\omega, X, Y, z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j^\varepsilon(\omega, X, Y) \phi_j(\omega, z) + \int_{-\infty}^{n_1^2 k^2} \widehat{p}_\gamma^\varepsilon(\omega, X, Y) \phi_\gamma(\omega, z) d\gamma,$$

and the modal amplitudes satisfy the following proposition.

PROPOSITION A.1. *Let $\widehat{\alpha}_j^\varepsilon(\omega, X, Y)$ be defined by (3.5). As $\varepsilon \rightarrow 0$, $\widehat{\alpha}^\varepsilon(\omega, X, Y) = (\widehat{\alpha}_j^\varepsilon(\omega, X, Y))_{j=1}^N$ converges weakly and in distribution to the diffusion Markov process $\widehat{\alpha}(\omega, X, Y) = (\widehat{\alpha}_j(\omega, X, Y))_{j=1}^N$. The limit processes $\widehat{\alpha}_j(\omega, X, Y)$ solve the Itô-Schrödinger equations*

$$(A.3) \quad d\widehat{\alpha}_j(\omega, X, Y) = \frac{i}{2\beta_j(\omega)} \partial_Y^2 \widehat{\alpha}_j(\omega, X, Y) dX + i\widehat{\alpha}_j(\omega, X, Y) \circ dB_j(\omega, X, Y),$$

for $X > 0$, starting from

$$(A.4) \quad \widehat{\alpha}_j(\omega, X=0, Y) = \frac{F_j(\omega, Y)}{2i\sqrt{\beta_j(\omega)}}, \quad F_j(\omega, Y) = \int_0^{+\infty} F(Y, z) \phi_j(\omega, z) dz.$$

Equations (A.3) are uncoupled for different j , but they are driven by correlated Brownian fields $B_j(\omega, X, Y)$, satisfying

$$(A.5) \quad \mathbb{E}[B_j(\omega, X, Y)] = 0, \quad \mathbb{E}[B_j(\omega, X, Y) B_l(\omega, X', Y')] = \min\{X, X'\} R_{jl}(\omega, Y - Y'),$$

with R_{jl} defined by (3.10).

The situation that the medium fluctuations in (A.1) have an isotropic scaling in the horizontal variables rather than the anisotropic scaling in (3.3) gives a qualitatively different description of the modal amplitudes. By comparing Propositions A.1 and 3.1 we find that in the horizontally isotropic case the modal amplitudes are dynamically uncoupled, while they are dynamically coupled in the anisotropic case. Note, however, that in both cases the correlations of the Brownian fields B_j give statistical coupling of the mode amplitudes. We consider next the radiative transfer equations for the Wigner transform modes and from the modal amplitude equations we find the following proposition.

PROPOSITION A.2. *If G^ε satisfies (4.1), then the mean Wigner transform*

$$(A.6) \quad \mathcal{W}_j(t, \omega, X, \kappa_x, Y, \kappa_y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}^3} d\omega' dx' dY' e^{-i\omega' t - i\kappa_x x' - i\kappa_y Y'} \\ \times \mathbb{E} \left[\widehat{p}_j^\varepsilon \left(\omega + \varepsilon^2 \frac{\omega'}{2}, X + \varepsilon^2 \frac{x'}{2}, Y + \frac{Y'}{2} \right) \overline{\widehat{p}_j^\varepsilon \left(\omega - \varepsilon^2 \frac{\omega'}{2}, X - \varepsilon^2 \frac{x'}{2}, Y - \frac{Y'}{2} \right)} \right]$$

has the form

$$(A.7) \quad \mathcal{W}_j(t, \omega, X, \kappa_x, Y, \kappa_y) = W_j(t, \omega, X, Y, \kappa_y) \delta(\kappa_x - \beta_j(\omega)),$$

where W_j satisfies the radiative transfer equation

$$(A.8) \quad \partial_X W_j + \frac{\kappa_y}{\beta_j(\omega)} \partial_Y W_j + \frac{1}{v_j(\omega)} \partial_t W_j = \frac{1}{2\pi} \int_{\mathbb{R}} \check{R}_{jj}(\omega, \kappa_y') [W_j(\kappa_y - \kappa_y') - W_j(\kappa_y)] d\kappa_y',$$

starting from (4.5). Here $v_j(\omega) = 1/\beta_j'(\omega)$ and \check{R}_{jj} is defined by (4.7).

Note that (A.8) is (4.4) in the special case $\check{\Gamma}^c \equiv 0$. Furthermore, note that $\check{\Gamma}_{jl}^c$ is small if the relative phase velocity contrast $|\beta_j - \beta_l|$ is large relative to the coherence length of the fluctuations the range direction (see (3.13)), in which case the dynamic mode amplitude coupling becomes small.

We can then conclude that in the case of isotropic fluctuations in the horizontal directions we have the following proposition.

PROPOSITION A.3. *The mean Wigner transform (4.9) has the form*

$$(A.9) \quad \mathcal{W}(t, \omega, X, \kappa_x, Y, \kappa_y) = \sum_{j=1}^{N(\omega)} \partial_z \phi_j(\omega, 0)^2 W_j(t, \omega, X, Y, \kappa_y) \delta(\kappa_x - \beta_j(\omega)),$$

where the W_j 's satisfy (A.8).

Finally, we remark that in this scaling regime, via a smoothing step as in section 5, we can obtain stable estimates of the group and phase velocities at the central frequency ω , $v_j(\omega), \omega/\beta_j(\omega)$, as well as information about the scattering kernels $\kappa \mapsto \check{R}_{jj}(\omega, \kappa)$ for $j = 1, \dots, N(\omega)$.

REFERENCES

- [1] L. A. APRESYAN AND Y. A. KRAVTSOV, *Radiation Transfer: Statistical and Wave Aspects*, Gordon and Breach, Amsterdam, 1995.
- [2] S. R. ARRIDGE AND J. C. SCHOTLAND, *Optical tomography: Forward and inverse problems*, Inverse Problems, 25 (2009), 123010.

- [3] G. BAL, T. KOMOROWSKI, AND L. RYZHIK, *Kinetic limits for waves in a random medium*, Kinet. Relat. Models, 3 (2010), pp. 529–644.
- [4] G. BAL AND O. PINAUD, *Imaging using transport models for wave-wave correlations*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1071–1093.
- [5] A. BARAJAS, L. MARGERIN, AND M. CAMPILLO, *Coupled body and surface wave sensitivity kernels for coda-wave interferometry in a three-dimensional scalar scattering medium*, Geophys. J. Int., 230 (2022), pp. 1013–1029.
- [6] L. BORCEA AND J. GARNIER, *Derivation of a one-way radiative transfer equation in random media*, Phys. Rev. E, 93 (2016), 022115.
- [7] L. BORCEA AND J. GARNIER, *Polarization effects for electromagnetic wave propagation in random media*, Wave Motion, 63 (2016), pp. 179–208.
- [8] L. BORCEA, J. GARNIER, AND K. SØLNA, *Onset of energy equipartition among surface and body waves*, Proc. A, 477 (2021).
- [9] M. CELORIO, E. CHALJUB, L. MARGERIN, AND L. STEHLY, *Propagation of 2-D SH waves in random media: Insights from ab initio numerical simulations and transport theory*, Front. Earth Sci., 10 (2022), 1033109.
- [10] S. CHANDRASEKAR, *Radiative Transfer*, Dover, New York, 1960.
- [11] M. V. DE HOOP, J. GARNIER, AND K. SØLNA, *System of radiative transfer equations for coupled surface and body waves*, Z. Angew. Math. Phys., 73 (2022), pp. 1–31.
- [12] J.-P. FOUCQUE, J. GARNIER, G. PAPANICOLAOU, AND K. SØLNA, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer, New York, 2007.
- [13] J. GARNIER AND K. SØLNA, *Coupled paraxial wave equations in random media in the white-noise regime*, Ann. Appl. Probab., 19 (2009), pp. 318–346.
- [14] A. ISHIMARU, *Wave Propagation and Scattering in Random Media*, Academic Press, New York, 1978.
- [15] T. MAEDA, H. SATO, AND T. NISHIMURA, *Synthesis of coda wave envelopes in randomly inhomogeneous elastic media in a half-space: Single scattering model including Rayleigh waves*, Geophys. J. Int., 172 (2008), pp. 130–154.
- [16] R. MAGNANINI AND F. SANTOSA, *Wave propagation in a 2-d optical waveguide*, SIAM J. Appl. Math., 61 (2000), pp. 1237–1252.
- [17] L. MARGERIN, *Introduction to radiative transfer of seismic waves*, in *Seismic Earth: Array Analysis of Broadband Seismograms*, Geophys. Monogr. 157, A. Levander and G. Nolet, eds., American Geophysical Union, Washington, DC, 2005, pp. 229–252.
- [18] L. MARGERIN, A. BAJARAS, AND M. CAMPILLO, *A scalar radiative transfer model including the coupling between surface and body waves*, Geophys. J. Int., 219 (2019), pp. 1092–1108.
- [19] L. MARGERIN, T. PLANÈS, J. MAYOR, AND M. CALVET, *Sensitivity kernels for coda-wave interferometry and scattering tomography: Theory and numerical evaluation in two-dimensional anisotropically scattering media*, Geophys. J. Int., 204 (2016), pp. 650–666.
- [20] S. MENINA, L. MARGERIN, T. KAWAMURA, P. LOGNONNÉ, J. MARTI, M. DRILLEAU, M. CALVET, N. COMPAIRE, R. GARCIA, F. KARAKOSTAS, N. SCHMERR, M. VAN DRIEL, S. C. STÄHLER, M. PLASMAN, D. GIARDINI, S. CARRASCO, B. KNAPMEYER-ENDRUN, G. SAINTON, AND W. B. BANERDT, *Energy envelope and attenuation characteristics of high-frequency (HF) and very-high-frequency (VF) Martian events*, Bull. Seismol. Soc. Amer., 111 (2021), pp. 3016–3034, <https://doi.org/10.1785/0120210127>.
- [21] A. MESSAOUDI, R. COTTEREAU, AND C. GOMEZ, *Boundary Effects in Radiative Transfer of Acoustic Waves in a Randomly Fluctuating Half-Space*, arXiv:2211.17007v1, 2022.
- [22] L. RYZHIK, G. PAPANICOLAOU, AND J. B. KELLER, *Transport equations for elastic and other waves in random media*, Wave Motion, 24 (1996), pp. 327–370.
- [23] H. SATO, *Multiple isotropic scattering model including ps conversions for the seismogram envelope formation*, Geophys. J. Int., 117 (1994), pp. 487–494.
- [24] H. SATO, M. C. FEHLER, AND T. MAEDA, *Seismic Wave Propagation and Scattering in the Heterogeneous Earth*, Springer, Berlin, 2012.
- [25] P. M. SHEARER AND P. S. EARLE, *The global short-period wavefield modelled with a Monte Carlo seismic phonon method*, Geophys. J. Int., 158 (2004), pp. 1103–1117.
- [26] R. SNIEDER, *Scattering of surface waves*, in *Scattering and Inverse Scattering in Pure and Applied Sciences*, R. Pike and P. Sabatier, eds., Academic Press, New York, 2002, pp. 562–577.
- [27] N. P. TRÉGOURÈS AND B. A. VAN TIGGELEN, *Quasi-two-dimensional transfer of elastic waves*, Phys. Rev. E, 66 (2002), 036601.
- [28] R.-S. WU, *Multiple scattering and energy transfer of seismic waves – separation of scattering effect from intrinsic attenuation. I. Theoretical modelling*, Geophys. J. Int., 82 (1985), pp. 57–80.

- [29] Y. ZENG, *Scattered surface wave energy in the seismic coda*, Pure Appl. Geophys., 163 (2006), pp. 533–548.
- [30] T. ZHANG, C. SENS-SCHÖNFELDER, AND L. MARGERIN, *Sensitivity kernels for static and dynamic tomography of scattering and absorbing media with elastic waves: A probabilistic approach*, Geophys. J. Int., 225 (2021), pp. 1824–1853.