

Immersion and Invariance-based Disturbance Observer and Its Application to Safe Control

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Abstract—When the disturbance input matrix is nonlinear, existing disturbance observer design methods rely on the solvability of a partial differential equation or the existence of an output function with a uniformly well-defined disturbance relative degree, which can pose significant limitations. This note introduces a systematic approach for designing an Immersion and Invariance-based Disturbance Observer (IIDOB) that circumvents these strong assumptions. The proposed IIDOB ensures the disturbance estimation error is globally uniformly ultimately bounded by approximately solving a partial differential equation while compensating for the approximation error. Furthermore, by integrating IIDOB into the framework of control barrier functions, a filter-based safe control design method for control-affine systems with disturbances is established where the filter is used to generate an alternative disturbance estimation signal with a known derivative. Sufficient conditions are established to guarantee the safety of the disturbed systems. Simulation results demonstrate the effectiveness of the proposed method.

Index Terms—Safe control, control barrier functions, immersion and invariance, disturbance observer.

I. INTRODUCTION

Designing feedback controllers that guarantee the safety specification of a system has attracted significant attention in the past decades [1], [2], [3], [4]. Inspired by automotive safety applications, [5], [6], [7] proposed reciprocal and zeroing Control Barrier Functions (CBFs) that generalize previous barrier conditions to only require a single sub-level set to be controlled invariant. By including the CBF condition in a convex Quadratic Program (QP), a CBF-QP-based controller is generated in real time and acts as a safety filter that modifies potentially unsafe control inputs in a minimally invasive fashion. Various robust CBF approaches have been proposed for systems with model uncertainties and external disturbances [8], [9], [10], [11]; however, most of these robust CBF methods consider the worst-case of disturbances, resulting in overly conservative control behaviors.

To reduce the adverse effects of disturbances/uncertainties on system performance, several works integrating disturbance/uncertainty estimation and compensation techniques into the CBF-QP framework have been proposed recently [12],

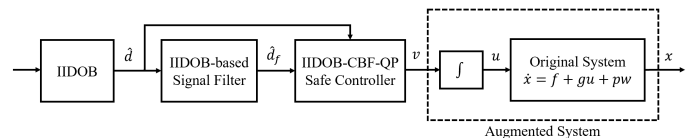


Figure 1: Configuration of the proposed IIDOB-CBF-QP method that consists of three components: (i) an IIDOB used for disturbance estimation, (ii) a filter that can generate an alternative disturbance estimation signal with a known derivative, and (iii) an IIDOB-CBF-QP-based safe controller that can ensure safety of the closed-loop system.

[13], [14], [15]. In our previous work [12], the Disturbance Observer (DOB) presented in [16] was incorporated into the CBF-QP framework for the first time. Compared with other robust control schemes, DOB-based control has two main advantages: (i) the DOB can be designed independently and added to a baseline controller to improve its robustness and disturbance attenuation capability; (ii) in the presence of disturbances/uncertainties, the nominal performance of the baseline controller can be recovered by the DOB-based controller [17], [18], [19].

Nevertheless, the design of DOBs is non-trivial and highly problem-specific. Specifically, designing a DOB requires the existence of two functions that can ensure the asymptotic stability of the error dynamics and the satisfaction of a partial differential equation (PDE) simultaneously (more details will be given in the next section) [17]. Fulfilling these two requirements is challenging, and existing methods rely on relatively strong assumptions, e.g., the disturbance relative degree is uniformly well-defined [16], [20]. A systematic and computationally feasible method for constructing DOBs for generic nonlinear control-affine systems is still lacking.

The contribution of this note is twofold: (i) Inspired by the Immersion and Invariance (I&I) technique [21], [22], [23], we propose a systematic approach for designing I&I-based Disturbance Observer (IIDOB) for general nonlinear control-affine systems without imposing the strong assumptions adopted by existing DOB design methods, such as the solvability of a PDE or the existence of an output function with a uniformly well-defined disturbance relative degree. By approximately solving the PDE and compensating for the approximation error, the proposed IIDOB ensures that the disturbance estimation error is globally Uniformly Ultimately Bounded (UUB). (ii) We propose a filter-based IIDOB-CBF-QP safe control design approach for control-affine systems with disturbances (see Fig. 1). We design a filter to obtain an alternative disturbance esti-

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mation signal with a known derivative and provide sufficient conditions that ensure the safety of the disturbed system. The remainder of this note is organized as follows: the background and the problem statement are provided in Section II; the proposed IIDOB is presented in Section III; the IIDOB-CBF-QP-based safe control strategy is provided in Section IV; numerical simulation results are provided in Section V; and finally, the conclusion is drawn in Section VI.

Notation: For a given positive integer n , denote $[n] = \{1, 2, \dots, n\}$. For a column vector $x \in \mathbb{R}^n$ or a row vector $x \in \mathbb{R}^{1 \times n}$, denote x_i as the i -th entry of x and $\|x\|$ as its 2-norm. Denote I_n as an identity matrix with dimension $n \times n$. For a given matrix $A \in \mathbb{R}^{n \times m}$, A_{ij} is the (i, j) -th entry of A , A_j is the j -th column of A , and $\|A\|$ is its Frobenius norm. Denote $\text{diag}[a_1, a_2, \dots, a_n] \in \mathbb{R}^{n \times n}$ as a diagonal matrix with diagonal entries $a_1, a_2, \dots, a_n \in \mathbb{R}$. The gradient $\frac{\partial h}{\partial x} \in \mathbb{R}^{n \times 1}$ is considered as a row vector, where $x \in \mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a function with respect to x . For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with respect to $x \in \mathbb{R}^n$, $\frac{\partial f}{\partial x}$ denotes the Jacobian matrix whose (i, j) -th entry is $\frac{\partial f_i}{\partial x_j}$.

II. BACKGROUND AND PROBLEM STATEMENT

A. Background

Consider a control-affine system $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are known and locally Lipschitz continuous functions. Define a safe set $\mathcal{C}$ as

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}, \quad (1)$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is a sufficiently smooth function. The function h is called a CBF of (input) relative degree 1 if $\sup_{u \in \mathbb{R}^m} [L_f h + L_g h u + \gamma h] \geq 0$ holds for all $x \in \mathbb{R}^n$, where $\gamma > 0$ is a given positive constant, and $L_f h = \frac{\partial h}{\partial x} f$ and $L_g h = \frac{\partial h}{\partial x} g$ are Lie derivatives [7]. It was proven in [7] that any Lipschitz continuous controller $u \in \{u : L_f h + L_g h u + \gamma h \geq 0\}$ will ensure the safety of the closed-loop system, i.e., the forward invariance of $\mathcal{C}$.

Now consider the following control-affine system with disturbances:

$$\dot{x} = f(x) + g(x)u + p(x)w(t) \quad (2)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^l$ is the disturbance, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, and $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l}$ are known functions. Provided that the disturbance w is bounded, robust CBF-based methods can be adopted to ensure the safety of system (2) [8], [9]. In existing robust CBF-based methods, safety is achieved by sacrificing the nominal performance, as the worst-case of the disturbances is considered in the safe controller design. Therefore, trajectories of the closed-loop system will stay in a shrunk subset of the original safe set $\mathcal{C}$, implying that the performance of these controllers is conservative.

DOB is one of the most effective tools for estimating and compensating disturbances/uncertainties in nonlinear control design, and has been extensively applied to numerous systems [17], [18], [19]. Our previous work [12] integrated DOBs into the CBF-QP framework, and proposed a DOB-CBF-QP controller with safety guarantees. However, designing DOBs for control-affine system (2) is non-trivial.

Suppose that w is slowly time-varying, that is, $\dot{w} \approx 0$ for any $t \geq 0$. As shown in [17], [20], the DOB for system (2) has the following structure:

$$\hat{w} = z + q(x), \quad (3a)$$

$$\dot{z} = -l(x)p(x)z - l(x)[f(x) + g(x)u + p(x)q(x)], \quad (3b)$$

where $\hat{w}$ is the disturbance estimation, $z \in \mathbb{R}^l$ is the internal state of the DOB. The function $l(x)$, known as the DOB gain, and the function $q(x)$ should be designed such that [17]

$$\frac{\partial q(x)}{\partial x} = l(x), \quad (4)$$

and the error dynamics is globally asymptotically stable for any $x \in \mathbb{R}^n$:

$$\dot{e}_w + l(x)p(x)e_w = 0, \quad (5)$$

where $e_w = w - \hat{w}$ is the disturbance estimation error.

Designing l and q is a challenging and highly case-specific task in general [17]. Several methods have been proposed based on relatively strong assumptions. If p has full column rank for any $x \in \mathbb{R}^n$, then one can select q by solving the PDE $\frac{\partial q}{\partial x} = p^\dagger$, where $p^\dagger$ denotes the left inverse of p [24]; however, when $n > 1$, this PDE is generally unsolvable and even when solvable, its closed-form solution is hard to obtain. If the disturbance relative degree is uniformly well-defined with respect to an output function $s(x)$, an approach for designing q is proposed in [16], [20]; however selecting such a function s is challenging and its existence is not guaranteed (e.g., there may exist $x^* \in \mathbb{R}^n$ such that $p(x^*) = 0$). A practically useful approach involves treating $p w$ as the total disturbance and assuming that $\frac{d}{dt}(p w)$ is bounded [25], [26]; however, this assumption is rather restrictive because $\frac{d}{dt}(p w)$ explicitly relies on u and x . As will be shown in Section III, we will provide a systematic approach for designing DOBs that avoids the issues of the aforementioned methods.

B. Problem Statement

Consider system (2) and the safe set defined in (1), where h is a sufficiently smooth function. Recall that g_j denotes the j -th column of g for $j \in [m]$, and p_i denotes the i -th column of p for $i \in [l]$. System (2) is said to have a vector Input Relative Degree (IRD) $\mathcal{I} = (\sigma_1, \sigma_2, \dots, \sigma_m)$ at a given point $x_0 \in \mathbb{R}^n$ if $L_{g_j} L_f^{k_j} h(x) = 0$ for any $k \in [\sigma_j - 2]$, $j \in [m]$, and for all x in a neighborhood of x_0 , and $L_{g_j} L_f^{\sigma_j - 1} h(x_0) \neq 0$ holds for any $j \in [m]$ [27, Remark 5.1.1]. Similarly, system (2) is said to have a vector Disturbance Relative Degree (DRD) $\mathcal{D} = (\nu_1, \nu_2, \dots, \nu_l)$ at a given point $x_0 \in \mathbb{R}^n$ if $L_{p_j} L_f^{k_j} h(x) = 0$ for any $k \in [\nu_j - 2]$, $j \in [l]$, and for all x in a neighborhood of x_0 , and $L_{p_j} L_f^{\nu_j - 1} h(x_0) \neq 0$ holds for any $j \in [l]$ [28]. Note that because system (2) is multiple-input-single-output with h as the output, the definitions of vector IRD and vector DRD above are slight modifications of those given in [27], [28].

In this note, with a slight abuse of notation, we will call $r_I = \min \mathcal{I}$ and $r_D = \min \mathcal{D}$ as the minimum IRD and the minimum DRD of system (2) with respect to function h at a given point $x_0 \in \mathbb{R}^n$, respectively; that is, r_I (or r_D) denotes the number of times h has to be differentiated to have at least one component of u (or w) explicitly appearing.

Next, a standard assumption for DOB design is given.

Assumption 1: The disturbance w and its derivative are bounded as $\|w\| \leq \omega_0$ and $\|\dot{w}\| \leq \omega_1$, where ω_0 and ω_1 are positive constants not necessarily known in DOB design.

The first problem investigated in this note is to design a disturbance estimation law to estimate the total disturbance

$$d(x, t) = p(x)w(t). \quad (6)$$

Problem 1: Consider system (2) with $f, g \in C^1$ and $p \in C^2$ and suppose Assumption 1 holds. Design a DOB-based estimation law to estimate the total disturbance d online.

Using the DOB-based estimation of the total disturbance, the second problem investigated in this note is to design a feedback control law such that system (2) is safe.

Problem 2: Consider system (2) with $f, g \in C^1$ and $p \in C^2$ and the safe set $\mathcal{C}$ defined in (1). Suppose that Assumption 1 holds and $r_I = r_D$ for system (2) with respect to h . Given the DOB developed via solving Problem 1, design a feedback control law such that system (2) is safe, i.e., $h(x(t)) \geq 0$ for any $t > 0$ provided $h(x(0)) > 0$.

Remark 1: In Problem 2, if $r_I < r_D$, the disturbance can be directly decoupled from the system via state feedback control [29]. The case $r_I > r_D$ will be explored in our future work. Note that we don't assume the minimum DRD of system (2) is uniformly well-defined as in [16], [20], i.e., there may exist $x_0 \in \mathbb{R}^n$ such that $L_{p_j} L_f^{\nu_j-1} h(x_0) = 0$ for any $j \in [l]$.

Remark 2: In this note, we aim to estimate the total disturbance d rather than the disturbance w for two main reasons: first, since no assumption is imposed on p except for $p \in C^2$, the disturbance w may not be uniquely determined in general; second, Problem 2 can be solved by using the information of d only.

III. IIDOB DESIGN

Inspired by the I&I technique [21], [22], [23], we propose an IIDOB design approach to solve Problem 1 in this section.

First, we augment system (2) with an additional integrator:

$$\dot{x} = f(x) + g(x)u + p(x)w(t), \quad (7a)$$

$$\dot{u} = v, \quad (7b)$$

where v denotes the auxiliary control input to be designed, and u is considered as a state variable of the augmented system. The relationship between system (2) and the augmented system (7) is illustrated in Fig. 1. As will be shown in this section and Section IV, the auxiliary control input v will be used in the design of IIDOB, and it will be generated from solving the IIDOB-CBF-QP. The control input u for the original system (2) will be obtained through integrating v .

Define a time-varying set $\mathcal{M}(t) = \{(x, \hat{x}, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m : \xi(t) + \beta(\hat{x}, x, u) - p(x)w(t) = 0\}$, where $\hat{x}$ denotes the state estimation and ξ, β are known functions that will be all specified later. Define

$$\hat{d} = \xi + \beta \quad (8)$$

as the estimated total disturbance, and the disturbance estimation error as

$$e_d = \hat{d} - d. \quad (9)$$

It is clear that if the system trajectories are restricted to $\mathcal{M}(t)$, the disturbance estimation is accurate. We also define

$$z = \frac{\xi + \beta - d}{r}, \quad (10)$$

where r is the scaling factor governed by an adaptive law yet to be designed. It is clear that $e_d = rz$. Our IIDOB design will render e_d globally UUB [30, Definition 4.6] by guaranteeing that z is globally UUB and r remains bounded. Note that $\dot{z}$, the time derivative of z , can be expressed as

$$\begin{aligned} \dot{z} = & -\frac{\dot{r}}{r}z + \frac{1}{r} \left(\dot{\xi} + \frac{\partial \beta}{\partial x}(f + gu + pw) + \frac{\partial \beta}{\partial u}v + \frac{\partial \beta}{\partial \hat{x}}\dot{\hat{x}} \right. \\ & \left. - p\dot{w} - \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu + pw)w_i \right), \end{aligned} \quad (11)$$

where p_i denotes the i -th column of p , $i \in [l]$. Define

$$\psi(x, u) = \frac{\eta}{2} \left[\|p\|^2 + \sum_{i=1}^l \left(\left\| \frac{\partial p_i}{\partial x}(f + gu) \right\|^2 + \left\| \frac{\partial p_i}{\partial x}p \right\|^2 \right) \right] + \gamma \quad (12)$$

where $\gamma, \eta > 0$ are tuning parameters and γ denotes the observer gain.

If $\delta(x, u) \in \mathbb{R}^n$ is a solution to the following PDE:

$$\frac{\partial \delta}{\partial x} = \psi I_n, \quad (13)$$

then the DOB design becomes straightforward by following [21]. Specifically, one can design the total disturbance estimation as $\hat{d} = \xi + \delta$, where ξ is governed by $\dot{\xi} = -\frac{\partial \xi}{\partial x}(f + gu + \xi + \delta) - \frac{\partial \xi}{\partial u}v$. Invoking (7) and (9), one can see $\dot{e}_d = -\psi e_d - p\dot{w} - \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu + pw)w_i$. By selecting a candidate Lyapunov function $V = \frac{1}{2}e_d^\top e_d$, one can easily verify that $\dot{V} = e_d^\top \left(-\psi e_d - p\dot{w} - \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu + pw)w_i \right) \leq -\psi \|e_d\|^2 + \omega_1 \|p\| \|e_d\| + \sum_{i=1}^l \omega_0 \left\| \frac{\partial p_i}{\partial x}(f + gu) \right\| \|e_d\| + \sum_{i=1}^l \omega_0^2 \left\| \frac{\partial p_i}{\partial x}p \right\| \|e_d\| \leq -\gamma \|e_d\|^2 + \frac{1}{2\eta}(\omega_1^2 + l\omega_0^2 + l\omega_0^4)$, which indicates that e_d is globally UUB. However, when $n > 1$, solving (13) is extremely challenging in principle, and even a solution to (13) may not exist [23]. To tackle this issue, we will follow [23] to first “approximately solve” (13) and then use $\dot{r}$ to compensate for the approximation error.

Recall that $f, g \in C^1$ and $p \in C^2$. Then, $\psi \in C^1$, and it is easy to verify that there exist continuous functions $\delta_{ij} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $i, j \in [n]$, such that [31], [32]:

$$\begin{aligned} & \psi(\hat{x}_1, \dots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \dots, \hat{x}_n, u) - \psi(x, u) \\ & = - \sum_{j=1}^n \delta_{ij}(x, \hat{x}, u) e_j, \end{aligned} \quad (14)$$

where

$$e = \hat{x} - x \quad (15)$$

and e_j denotes the j -th entry of e . The following theorem shows that our IIDOB design ensures the disturbance estimation error e_d is globally UUB.

Theorem 1: Consider system (7) where $f, g \in C^1$ and $p \in C^2$, and suppose Assumption 1 holds. If the disturbance estimation law $\hat{d}$ is designed as:

$$\hat{d} = \xi + \beta, \quad (16a)$$

$$\Lambda = (k_1 + k_2 r^2) I_n + \frac{cr^2}{2} \text{diag}[\|\Delta_1\|^2, \dots, \|\Delta_n\|^2], \quad (16b)$$

$$\beta = \begin{bmatrix} \int_0^{x_1} \psi(\tau, \hat{x}_2, \hat{x}_3, \dots, \hat{x}_n, u) d\tau \\ \int_0^{x_2} \psi(\hat{x}_1, \tau, \hat{x}_3, \dots, \hat{x}_n, u) d\tau \\ \vdots \\ \int_0^{x_n} \psi(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n-1}, \tau, u) d\tau \end{bmatrix}, \quad (16c)$$

$$\dot{\hat{x}} = f + gu + \hat{d} - \Lambda e, \quad (16d)$$

$$\dot{\xi} = -\frac{\partial \beta}{\partial x}(f + gu + \hat{d}) - \frac{\partial \beta}{\partial u}v - \frac{\partial \beta}{\partial \hat{x}}\dot{\hat{x}}, \quad (16e)$$

$$\dot{r} = -\theta(r-1) + \frac{cr}{2} \sum_{j=1}^n e_j^2 \|\Delta_j\|^2, \quad r(0) > 1, \quad (16f)$$

where ψ is defined in (12), e is defined in (15), $\gamma, c, \theta > 0$ are positive constants satisfying $\gamma > \frac{n}{2c} + \theta$, $\Delta_j = \text{diag}[\delta_{1j}, \delta_{2j}, \dots, \delta_{nj}] \in \mathbb{R}^{n \times n}$ with δ_{ij} defined in (14) for $i, j \in [n]$, and $k_1, k_2 > 0$ are positive constants satisfying $k_2 > \frac{1}{4\gamma - 2n/c - 4\theta}$, then e_d is globally UUB.

Proof: Recall that $e_d = rz$. To prove that e_d is globally UUB, we will first show z is globally UUB, and then r is bounded. Substituting (16e) into (11) yields

$$\dot{z} = -\frac{\dot{r}}{r}z - \frac{\partial \beta}{\partial x}z - \frac{1}{r} \left(p\dot{w} + \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu + pw)w_i \right). \quad (17)$$

Recall that $\psi \in C^1$. According to the fundamental theorem of calculus, one can see that

$$\frac{\partial \beta}{\partial x} = \text{diag}[\psi(x_1, \hat{x}_2, \dots, \hat{x}_n, u), \psi(\hat{x}_1, x_2, \hat{x}_3, \dots, \hat{x}_n, u), \dots, \psi(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n-1}, x_n, u)]. \quad (18)$$

Define $e_\psi = \left\| \psi(x, u)I_n - \frac{\partial \beta}{\partial x} \right\|$ as the ‘‘approximation error’’ induced by approximately solving (13) using β designed in (16c). Intuitively, from (18) one can see that if $\hat{x}$ is very close to x , e_ψ would be negligible. Note that the influence of e_ψ will be eliminated by $\dot{r}$ as shown in the following analysis.

Then, substituting (14) into (18) yields

$$\frac{\partial \beta}{\partial x} = \psi(x, u)I_n - \sum_{j=1}^n \Delta_j e_j, \quad (19)$$

and substituting (12) and (19) into (17) yields

$$\begin{aligned} \dot{z} = & -\frac{\dot{r}}{r}z - \gamma z - \frac{\eta}{2} \left[\sum_{i=1}^l \left(\left\| \frac{\partial p_i}{\partial x}(f + gu) \right\|^2 + \left\| \frac{\partial p_i}{\partial x}p \right\|^2 \right) + \|p\|^2 \right] z \\ & + \sum_{j=1}^n \Delta_j e_j z + \frac{1}{r} \left(-p\dot{w} - \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu + pw)w_i \right). \end{aligned} \quad (20)$$

From (16f) one can easily verify that $r \geq 1$ for any $t > 0$ because the set $\{r : r \geq 1\}$ is invariant. Substituting (16f) into (20) gives

$$\begin{aligned} \dot{z} = & \theta \frac{r-1}{r}z - \frac{c}{2} \sum_{j=1}^n e_j^2 \|\Delta_j\|^2 z - \left[\frac{\eta}{2} \sum_{i=1}^l \left(\left\| \frac{\partial p_i}{\partial x}(f + gu) \right\|^2 + \left\| \frac{\partial p_i}{\partial x}p \right\|^2 \right) + \frac{\eta}{2} \|p\|^2 - \sum_{j=1}^n \Delta_j e_j \right] z + \frac{1}{r} \left(-p\dot{w} \right. \\ & \left. - \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu)w_i - \sum_{i=1}^l \frac{\partial p_i}{\partial x}pw_i \right) - \gamma z. \end{aligned} \quad (21)$$

Meanwhile, subtracting (7) from (16d) yields

$$\dot{e} = \hat{d} - d - \Lambda e = rz - \Lambda e. \quad (22)$$

Next, we prove that z is globally UUB. Define a candidate Lyapunov function as $V = \frac{1}{2}z^\top z$, whose time derivative is

$$\begin{aligned} \dot{V} \stackrel{(21)}{=} & \theta \frac{r-1}{r} \|z\|^2 - \frac{c}{2} \sum_{j=1}^n e_j^2 \|\Delta_j\|^2 \|z\|^2 + z^\top \sum_{j=1}^n \Delta_j e_j z \\ & - \frac{\eta}{2} \left[\sum_{i=1}^l \left(\left\| \frac{\partial p_i}{\partial x}(f + gu) \right\|^2 + \left\| \frac{\partial p_i}{\partial x}p \right\|^2 \right) + \|p\|^2 \right] \|z\|^2 \end{aligned}$$

$$\begin{aligned} & + \frac{z^\top}{r} \left(-p\dot{w} - \sum_{i=1}^l \frac{\partial p_i}{\partial x}(f + gu + pw)w_i \right) - \gamma \|z\|^2 \\ \leq & \theta \frac{r-1}{r} \|z\|^2 - \frac{c}{2} \sum_{j=1}^n e_j^2 \|\Delta_j\|^2 \|z\|^2 + \sum_{j=1}^n \|\Delta_j\| \|e_j\| \|z\|^2 \\ & - \frac{\eta}{2} \left[\sum_{i=1}^l \left(\left\| \frac{\partial p_i}{\partial x}(f + gu) \right\|^2 + \left\| \frac{\partial p_i}{\partial x}p \right\|^2 \right) + \|p\|^2 \right] \|z\|^2 \\ & - \gamma \|z\|^2 + \frac{\|z\|}{r} \left(\|p\| \omega_1 + \sum_{i=1}^l \left\| \frac{\partial p_i}{\partial x}(f + gu) \right\| \omega_0 \right. \\ & \left. + \sum_{i=1}^l \left\| \frac{\partial p_i}{\partial x}p \right\| \omega_0^2 \right) \\ \leq & - \left(\gamma - \theta - \frac{n}{2c} \right) \|z\|^2 + \frac{1}{2\eta r^2} (\omega_1^2 + l\omega_0^2 + l\omega_0^4) \\ \leq & -\kappa \|z\|^2 + \omega, \end{aligned} \quad (23)$$

where

$$\kappa = \gamma - \frac{n}{2c} - \theta > 0, \quad (24a)$$

$$\omega = \frac{1}{2\eta} (\omega_1^2 + l\omega_0^2 + l\omega_0^4) > 0, \quad (24b)$$

the first and second inequality arise from Cauchy-Schwarz inequality, and the last inequality comes from the fact that $r \geq 1$. Therefore, one can see that

$$\|z\| \leq \sqrt{\|z(0)\|^2 e^{-2\kappa t} + \frac{\omega}{\kappa}} = \varrho_z(t), \quad (25)$$

which indicates that z is UUB. Note that selecting a larger κ will result in a smaller final bound of $\|z\|$. However, the convergence of $\|z\|$ does not imply the convergence of e_d unless r is bounded. To show the boundedness of r , we construct an augmented candidate Lyapunov function W as $W = V + \frac{1}{2}e^\top e + \frac{1}{2}r^2$, whose time derivative satisfies

$$\begin{aligned} \dot{W} \stackrel{(22)}{\leq} & -\kappa \|z\|^2 + \omega - e^\top \Lambda e + e^\top rz - \theta r(r-1) \\ & + \frac{cr^2}{2} \sum_{j=1}^n e_j^2 \|\Delta_j\|^2 \\ \stackrel{(16b)}{=} & -\kappa \|z\|^2 + \omega - k_1 \|e\|^2 - k_2 r^2 \|e\|^2 - \theta r(r-1) + e^\top rz \\ \leq & -\kappa \|z\|^2 + \omega - k_1 \|e\|^2 - k_2 r^2 \|e\|^2 + k_2 r^2 \|e\|^2 \\ & + \frac{1}{4k_2} \|z\|^2 - \frac{\theta}{2} r^2 + \frac{\theta}{2} \\ = & - \left(\kappa - \frac{1}{4k_2} \right) \|z\|^2 - k_1 \|e\|^2 - \frac{\theta}{2} r^2 + \left(\frac{\theta}{2} + \omega \right) \\ \leq & -\chi W + \left(\frac{\theta}{2} + \omega \right), \end{aligned} \quad (26)$$

where $\chi = \min \left\{ 2\kappa - \frac{1}{2k_2}, 2k_1, \theta \right\}$. From (26) we have

$$r \leq \sqrt{2W(0)e^{-\chi t} + \frac{\theta + 2\omega}{\chi}} = \varrho_r(t). \quad (27)$$

Recall that $e_d = rz$. From (25) and (27), it is easy to conclude that e_d is globally UUB. This completes the proof.

Remark 3: From (25) and (27), one can see that $\lim_{t \rightarrow \infty} \|e_d(t)\| \leq \sqrt{\frac{\omega(\theta+2\omega)}{\kappa\chi}}$, implying that the ultimate bound of e_d can be made arbitrarily small by appropriately choosing the tuning parameters $\gamma, \theta, c, k_1, k_2$ in the IIDOB design. In practice, the selection of these parameters should

reflect a trade-off between reducing the ultimate disturbance estimation error and achieving a desired transient performance.

Remark 4: From (16c) one can see that β is obtained via calculating an (indefinite) integral, whose explicit form is hard to obtain in general. In practice, a numerical integration can be adopted to compute β . Moreover, since $\psi \in C^1$, $\frac{\partial \beta}{\partial \hat{x}}$ can be computed using the Leibniz integral rule [33] as $\left(\frac{\partial \beta}{\partial \hat{x}}\right)_{ij} = \int_0^{x_i} \frac{\partial}{\partial \hat{x}_j} \psi(\hat{x}_1, \dots, \hat{x}_{i-1}, \tau, \hat{x}_{i+1}, \dots, \hat{x}_n) d\tau$, where $\left(\frac{\partial \beta}{\partial \hat{x}}\right)_{ij}$ denotes the ij -th entry of $\frac{\partial \beta}{\partial \hat{x}}$.

Remark 5: The IIDOB design method proposed in this work can be seen as an extension of the I&I estimator design approach given in [31], [32] since unknown time-varying functions, instead of unknown constant parameters, are considered in this work. Note that the definition of the scaled estimation error z and the design of β, ξ are different from their counterparts in [31], [32].

Before the end of this section, we show the design of an IIDOB-based tracking controller, which could be used as a nominal controller in the IIDOB-CBF-QP in Section IV. Note that $\hat{d}$ can be expressed as

$$\dot{\hat{d}} = \dot{\xi} + \frac{\partial \beta}{\partial x} \dot{x} + \frac{\partial \beta}{\partial \hat{x}} \dot{\hat{x}} + \frac{\partial \beta}{\partial u} v \stackrel{(16)}{=} -r \frac{\partial \beta}{\partial x} z. \quad (28)$$

The following proposition presents an IIDOB-based tracking control law provided the right inverse of g exists.

Proposition 1: Consider system (7) and suppose that all conditions of Theorem 1 hold such that the IIDOB shown in (16) exists. Suppose that κ defined in (24a) is greater than 1, and the right inverse of g exists for any $x \in \mathbb{R}^n$. Given a reference trajectory $x_d(t)$ where $x_d(t)$ and $\dot{x}_d(t)$ are bounded, $\forall t \geq 0$, if the control law is designed as

$$u_d = -g^\dagger \left(f + \left(\alpha_1 + \frac{1}{2} r^2 \right) e_x + \hat{d} - \dot{x}_d \right), \quad (29a)$$

$$v = -\alpha_2 e_u + \mathcal{G}_1 - g^\top e_x - \frac{\|\mathcal{G}_2\|^2}{2} e_u, \quad (29b)$$

where $e_x = x - x_d$, $e_u = u - u_d$, $g^\dagger$ is the right inverse of g , $\mathcal{G}_1 = \frac{\partial u_d}{\partial t} + \frac{\partial u_d}{\partial x} (f + gu + \hat{d}) + \frac{\partial u_d}{\partial r} \dot{r}$, $\alpha_1, \alpha_2 > 0$ are positive constants, and $\mathcal{G}_2 = r \left(\frac{\partial u_d}{\partial d} \frac{\partial \beta}{\partial x} + \frac{\partial u_d}{\partial x} \right)$, then the tracking error e_x is globally UUB.

Proof: Define $V_1 = \frac{1}{2} e_x^\top e_x + \frac{1}{2} z^\top z$ as a candidate Lyapunov function where z is defined in (10). Note that $\dot{V}_1 \stackrel{(23)}{\leq} e_x^\top (f + gu + pw - \dot{x}_d) - \kappa \|z\|^2 + \omega \stackrel{(29a)}{\leq} -(\alpha_1 + \frac{1}{2} r^2) \|e_x\|^2 - r e_x^\top z + e_x^\top g e_u - \kappa \|z\|^2 + \omega \leq -\alpha_1 \|e_x\|^2 - (\kappa - \frac{1}{2}) \|z\|^2 + e_x^\top g e_u + \omega$, where ω is defined in (24b). Since u_d is a function of $x, r, \hat{d}$ and t , its derivative is $\dot{u}_d = \frac{\partial u_d}{\partial t} + \frac{\partial u_d}{\partial x} (f + gu + \hat{d}) + \frac{\partial u_d}{\partial r} \dot{r} - r \left(\frac{\partial u_d}{\partial d} \frac{\partial \beta}{\partial x} + \frac{\partial u_d}{\partial x} \right) z = \mathcal{G}_1 - \mathcal{G}_2 z$. Then, we define $V_2 = V_1 + \frac{1}{2} e_u^\top e_u$ as an augmented Lyapunov function candidate, whose derivative satisfies $\dot{V}_2 \stackrel{(28)}{\leq} \dot{V}_1 + e_u^\top (v - \mathcal{G}_1 + \mathcal{G}_2 z) \leq -\alpha_1 \|e_x\|^2 - (\kappa - 1) \|z\|^2 + e_u^\top g^\top e_x + e_u^\top (v - \mathcal{G}_1) + \frac{\|e_u\|^2 \|\mathcal{G}_2\|^2}{2} + \omega \stackrel{(29b)}{\leq} -\alpha_1 \|e_x\|^2 - (\kappa - 1) \|z\|^2 - \alpha_2 \|e_u\|^2 + \omega$. Thus, $\dot{V}_2 \leq -\vartheta V_2 + \omega$, where $\vartheta = \min\{2\alpha_1, 2\kappa - 2, 2\alpha_2\}$. Hence, one can see that $\|e_x\| \leq \sqrt{2V_2(0)e^{-\vartheta t} + \frac{2\omega}{\vartheta}}$, indicating e_x is globally UUB. This completes the proof.

When g has no full row rank, an IIDOB-based tracking controller similar to Proposition 1 can still be designed by

following the backstepping technique [34], provided some control Lyapunov function conditions hold. The details are omitted due to space limitation.

Remark 6: The dynamic surface control [35] or command filter [36] technique can be adopted to bypass the tedious calculation of partial derivatives of u_d . For example, the idea of the dynamic surface control is to let u_d defined in (29a) pass a low-pass filter $\epsilon \dot{u}_d^f = -u_d^f + u_d$, where u_d^f is the filtered signal and ϵ is a small time constant. Then, one can replace u_d with u_d^f and use $\dot{u}_d^f$ directly in the design of v , instead of computing partial derivatives of u_d .

IV. IIDOB-CBF-QP-BASED SAFE CONTROLLER

In this section, we will present an IIDOB-CBF-QP-based safe control design method to solve Problem 2.

We will design a safe controller v based on the augmented system shown in (7) that is used for the IIDOB design in the preceding section. Two issues need to be addressed in this design: (i) The time derivative of $\hat{d}$ is indispensable in control design and it depends on z , which is unknown since z relies on w , as shown in (28); however, considering the worst-case of $\hat{d}$ may lead to unnecessary conservatism. (ii) The minimum DRD of system (7) is lower than its minimum IRD (i.e., d appears prior to v when one differentiates h), which makes the direct decoupling of the disturbance from the system difficult even if the disturbance is precisely estimated [29].

We address the first challenge by designing a filter to obtain an alternative disturbance estimation signal whose derivative is known. Specifically, given an IIDOB shown in (16), we design the following filter:

$$\dot{\hat{d}}_f = - \left(T_1 + T_2 r^2 \left\| \frac{\partial \beta}{\partial x} \right\|^2 \right) (\hat{d}_f - \hat{d}), \quad (30)$$

where $\hat{d}_f$ denotes the filtered disturbance estimation with $\hat{d}_f(0) = \hat{d}(0)$, r is governed by (16f), β is given in (16c), and $T_1, T_2 > 0$ are tuning parameters. The filter shown in (30) is a modified low-pass filter by adding an additional term $-T_2 r^2 \left\| \frac{\partial \beta}{\partial x} \right\|^2 (\hat{d}_f - \hat{d})$ whose usefulness will be clear from the proof of Lemma 1. From (30) one can see that the derivative of $\hat{d}_f$ is completely known. Define the filtering error e_f as

$$e_f = \hat{d}_f - \hat{d}. \quad (31)$$

The following result shows that $\hat{d}_f$ is close to $\hat{d}$ in the sense that e_f is bounded by a known time-varying function whose ultimate bound can be arbitrarily small by choosing appropriate parameters.

Lemma 1: Consider the augmented system (7), the IIDOB as shown in (16), and the filter given in (30). If Assumption 1 holds and $T_2 > \frac{1}{4\kappa}$, where κ is defined in (24a), then the filtering error e_f satisfies

$$\|e_f(t)\| \leq \sqrt{\left(\|z(0)\|^2 - \frac{2\omega}{\zeta} \right) e^{-\zeta t} + \frac{2\omega}{\zeta}} = \varrho_f(t) \quad (32)$$

for any $t \geq 0$, where $\zeta = \min\{2T_1, 2\kappa - \frac{1}{2T_2}\}$ and ω is defined in (24b).

Proof: Substituting (28) into (30) gives

$$\dot{e}_f = -T_1 e_f - T_2 r^2 \left\| \frac{\partial \beta}{\partial x} \right\|^2 e_f + r \frac{\partial \beta}{\partial x} z. \quad (33)$$

Construct a candidate Lyapunov function V_f as

$$V_f = \frac{1}{2}e_f^\top e_f + \frac{1}{2}z^\top z, \quad (34)$$

whose derivative satisfies

$$\begin{aligned} \dot{V}_f &\stackrel{(23)}{\leq} -T_1\|e_f\|^2 - T_2r^2\left\|\frac{\partial\beta}{\partial x}\right\|^2\|e_f\|^2 + re_f^\top \frac{\partial\beta}{\partial x}z - \kappa\|z\|^2 + \omega \\ &\leq -T_1\|e_f\|^2 - T_2r^2\left\|\frac{\partial\beta}{\partial x}\right\|^2\|e_f\|^2 + r\left\|\frac{\partial\beta}{\partial x}\right\|\|e_f\|\|z\| - \kappa\|z\|^2 + \omega \\ &\leq -T_1\|e_f\|^2 - \left(\kappa - \frac{1}{4T_2}\right)\|z\|^2 + \omega \\ &\leq -\zeta V_f + \omega, \end{aligned} \quad (35)$$

where the second inequality comes from Cauchy-Schwarz inequality. Note that $V_f(0) = \frac{1}{2}\|z(0)\|^2$ because $\hat{d}_f(0) = \hat{d}(0)$. It is standard to obtain (32) from (35).

Next, we will present the safe controller design. Consider system (2) and the safe set $\mathcal{C}$ defined in (1). Suppose that $r_I = r_D = \iota \geq 1$ for system (2) with respect to h ; clearly, because of (7b), $r_D < r_I$ for the augmented system (7). To address the second issue above, we define a set of functions $h_0, h_1, \dots, h_\iota$ as

$$h_0(x) = h, \quad (36a)$$

$$h_i(x) = \left(\frac{d}{dt} + \lambda_{i-1}\right) \circ h_{i-1}, \quad i \in [\iota - 1], \quad (36b)$$

$$\begin{aligned} h_\iota(x, u, r, \hat{d}_f) &= \frac{\partial h_{\iota-1}}{\partial x}(f + gu + \hat{d}_f) - \frac{(1+r^2)\left\|\frac{\partial h_{\iota-1}}{\partial x}\right\|^2}{2\tilde{\rho}(\zeta - \lambda_{\iota-1})} \\ &\quad - \tilde{\rho}\omega + \lambda_{\iota-1}h_{\iota-1}, \end{aligned} \quad (36c)$$

where $\tilde{\rho} > 0, \lambda_i > 0 (i = 0, 1, \dots, \iota - 1)$ are tuning parameters, and $\omega > 0$ is the constant defined in (24b). Based on the IIDOB shown in (16), the filter given in (30), and the notations above, the following result provides a safe controller v that ensures the forward invariance of $\mathcal{C}$ for system (2).

Theorem 2: Consider system (2) and the safe set $\mathcal{C}$ defined in (1). Suppose that all conditions of Theorem 1 hold such that the IIDOB shown in (16) exists. Suppose that $r_I = r_D = \iota \geq 1$ for system (2) with respect to h , and there exist positive constants $\rho, \tilde{\rho}$, and $\lambda_i (i = 0, 1, \dots, \iota)$ such that $\lambda_\iota < 2\kappa, \lambda_{\iota-1} < \zeta, h_i(x(0)) > 0$ for $i = 0, \dots, \iota - 2, h_{\iota-1}(x(0)) - \tilde{\rho}V_f(e_f(0), z(0)) > 0$, and $h_\iota(x(0), u(0), r(0), \hat{d}_f(0)) - \frac{\rho}{2}\|z(0)\|^2 > 0$, where κ, ζ , and V_f are defined in (24a), (32), and (34), respectively. If $\sup_{v \in \mathbb{R}^m} [\psi_0 + \psi_1 v] \geq 0$ holds for any $u \in \mathbb{R}^m, r \in [1, \varrho_r], \|\hat{d}\| \leq \|p\|\omega_0 + \varrho_d, \|\hat{d}_f\| \leq \|p\|\omega_0 + \varrho_d + \varrho_f, \|e_f\| \leq \varrho_f, x \in \mathcal{C}$, and $t \geq 0$, where $\varrho_d = \varrho_z \varrho_r$ with ϱ_z and ϱ_r defined in (25) and (27), respectively, ϱ_f is defined in (32), and

$$\begin{aligned} \psi_0 &= \frac{\partial h_\iota}{\partial x}(f + gu + \hat{d}) + \frac{\partial h_\iota}{\partial r}\dot{r} + \frac{\partial h_\iota}{\partial \hat{d}_f}\dot{\hat{d}}_f - \frac{r^2\left\|\frac{\partial h_\iota}{\partial x}\right\|^2}{\rho(4\kappa - 2\lambda_\iota)} \\ &\quad - \rho\omega + \lambda_\iota h_\iota, \end{aligned} \quad (37a)$$

$$\psi_1 = \frac{\partial h_\iota}{\partial u}, \quad (37b)$$

with $\dot{r}, \dot{\hat{d}}_f$, and h_ι defined in (16f), (30), and (36c), respectively, then any Lipschitz controller $v \in K_{BF} = \{v \in \mathbb{R}^m : \psi_0 + \psi_1 v \geq 0\}$ will ensure $h \geq 0$ for all $t \geq 0$.

Proof: Define $H_1(x, u, r, \hat{d}_f, z) = h_\iota - \frac{\rho}{2}z^\top z$, where h_ι is given in (36c). Since

$$\begin{aligned} \dot{H}_1 &\stackrel{(23)}{\geq} \frac{\partial h_\iota}{\partial x}(f + gu + pw) + \frac{\partial h_\iota}{\partial u}v + \frac{\partial h_\iota}{\partial r}\dot{r} + \frac{\partial h_\iota}{\partial \hat{d}_f}\dot{\hat{d}}_f \\ &\quad + \rho\kappa\|z\|^2 - \rho\omega \\ &= \frac{\partial h_\iota}{\partial x}(f + gu + \hat{d}) + \frac{\partial h_\iota}{\partial u}v + \frac{\partial h_\iota}{\partial r}\dot{r} + \frac{1}{2}\rho\lambda_\iota\|z\|^2 - \rho\omega \\ &\quad + \frac{\partial h_\iota}{\partial \hat{d}_f}\dot{\hat{d}}_f - r\frac{\partial h_\iota}{\partial x}z + \rho\left(\kappa - \frac{\lambda_\iota}{2}\right)\|z\|^2 \\ &\geq \frac{\partial h_\iota}{\partial x}(f + gu + \hat{d}) + \frac{\partial h_\iota}{\partial r}\dot{r} + \frac{\partial h_\iota}{\partial \hat{d}_f}\dot{\hat{d}}_f - \frac{r^2\left\|\frac{\partial h_\iota}{\partial x}\right\|^2}{\rho(4\kappa - 2\lambda_\iota)} \\ &\quad + \frac{1}{2}\rho\lambda_\iota\|z\|^2 + \frac{\partial h_\iota}{\partial u}v - \rho\omega \\ &= \psi_0 + \psi_1 v - \lambda_\iota H_1, \end{aligned} \quad (38)$$

any $v \in K_{BF}$ will result in $\dot{H}_1 \geq -\lambda_\iota \bar{H}_1$. Noting that $h_\iota(x(0), u(0), r(0), \hat{d}_f(0)) - \frac{\rho}{2}\|z(0)\|^2 > 0 \implies H_1(x(0), u(0), r(0), \hat{d}_f(0), z(0)) > 0$, we can conclude that $H_1 \geq 0$, which implies that $h_\iota \geq 0$, for any $t \geq 0$.

Define another function as $H_2(x, z, e_f) = h_{\iota-1} - \tilde{\rho}V_f$ where V_f is given in (34). Note that

$$\begin{aligned} \dot{H}_2 + \lambda_{\iota-1}H_2 &\stackrel{(35)}{\geq} \frac{\partial h_{\iota-1}}{\partial x}(f + gu + \hat{d}_f) - r\frac{\partial h_{\iota-1}}{\partial x}z - \frac{\partial h_{\iota-1}}{\partial x}e_f - \tilde{\rho}\omega \\ &\quad + \lambda_{\iota-1}h_{\iota-1} + \frac{(\zeta - \lambda_{\iota-1})\tilde{\rho}}{2}(\|z\|^2 + \|e_f\|^2) \\ &\geq \frac{\partial h_{\iota-1}}{\partial x}(f + gu + \hat{d}_f) - \frac{(1+r^2)\left\|\frac{\partial h_{\iota-1}}{\partial x}\right\|^2}{2\tilde{\rho}(\zeta - \lambda_{\iota-1})} - \tilde{\rho}\omega + \lambda_{\iota-1}h_{\iota-1} \\ &= h_\iota \geq 0. \end{aligned} \quad (39)$$

Since $h_{\iota-1}(x(0)) - \tilde{\rho}V_f(e_f(0), z(0)) > 0 \implies H_2(x(0), z(0), e_f(0)) > 0$, we have $H_2 \geq 0$, which implies that $h_{\iota-1} \geq 0$ for all $t \geq 0$. According to (36b), one can conclude that $h_i \geq 0 \implies h_{i-1} \geq 0$ for any $i \in [\iota - 1]$ because $h_i(x(0)) > 0, i = 0, 1, \dots, \iota - 2$ [37]. Therefore, one can see $h \geq 0$ for all $t \geq 0$. This completes the proof.

The safe controller v proposed in Theorem 2 can be obtained by solving the following convex IIDOB-CBF-QP:

$$\begin{aligned} \min_v \quad &\|v - v_{nom}\|^2 \\ \text{s.t.} \quad &\psi_0 + \psi_1 v \geq 0, \end{aligned} \quad (40)$$

where ψ_0, ψ_1 are given in (37) and v_{nom} is any given nominal control law that is potentially unsafe (e.g., the IIDOB-based tracking controller given in Proposition 1). Note that the IIDOB-CBF-QP (40) is convex and can be efficiently solved online. In fact, (40) has a closed-form solution that can be expressed as [6]:

$$v = \begin{cases} v_{nom}, & \text{if } \psi_0 + \psi_1 v_{nom} \geq 0, \\ v_{nom} - \frac{\psi_0 + \psi_1 v_{nom}}{\psi_1^\top \psi_1} \psi_1^\top, & \text{otherwise.} \end{cases}$$

V. SIMULATION

In this section, two simulation examples are presented to demonstrate the effectiveness of the proposed method.

Example 1: Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ x_1 x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 + \sin^2(x_1) \end{bmatrix}}_{g(x)} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{p(x)} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} w(t), \quad (41)$$

where $x = [x_1 \ x_2]^\top$ is the state, $u = [u_1 \ u_2]^\top$ is the control input, and w is the disturbance. It is easy to verify that f, g, p are smooth functions. In the simulation, we choose the disturbance as $w = 5 \sin(t) + 2 \cos(2t) + 4 \sin(3t) + 3 \cos(4t)$, which implies that Assumption 1 holds with $\omega_0 = 8, \omega_1 = 26$. We select the initial condition as $x_1(0) = x_2(0) = -0.5$, and the reference trajectory of x is $x_d(t) = [x_{1d}(t) \ x_{2d}(t)]^\top$, where $x_{1d}(t) = 2 \sin(t)$ and $x_{2d}(t) = 2 \cos(t)$.

We choose parameters $\gamma = 100, k_1 = k_2 = 10, \theta = 10, c = 0.5$, and $\alpha_1 = \alpha_2 = 50$ in Theorem 1 and Proposition 1. It is easy to verify that all conditions of Theorem 1 hold so that the IIDOB is designed as shown in (16). We design a tracking controller for (41) using the dynamic surface technique in [35] with the filter parameter $\epsilon = 0.001$ (see Remark 6). As shown in Fig. 2, the disturbance estimation of the proposed IIDOB is accurate and the tracking performance is satisfactory.

Next, we consider two safety sets $\mathcal{C}_1 = \{x \in \mathbb{R}^2 : x_1 + 1 \geq 0\}$ and $\mathcal{C}_2 = \{x \in \mathbb{R}^2 : 1 - x_2 \geq 0\}$, which aim to keep $x_1 \geq -1$ and $x_2 \leq 1$. Define $h_1 = x_1 + 1$ and $h_2 = 1 - x_2$. One can easily verify that the minimum IRD and the minimum DRD of system (41) with respect to h_1, h_2 are both 1, i.e., $r_I = r_D = 1$. We choose parameters $\rho = \tilde{\rho} = 1, \lambda_0 = \lambda_1 = 50, T_1 = 50, T_2 = 1$ in Theorem 2, and let other parameters the same as above. We also choose the nominal controller as the tracking controller designed above. One can verify that all conditions of Theorem 2 are satisfied, so that the IIDOB-CBF-QP-based controller (40) can ensure $h_1 \geq 0$ and $h_2 \geq 0$ for all $t \geq 0$. Indeed, as shown in Fig. 3, trajectories of x_1 (or x_2) always remain within the safety set $\mathcal{C}_1$ (or $\mathcal{C}_2$).

We also compare the tracking performance of the proposed controller (40) with the robust CBF approach proposed in [8]. As shown in Fig. 3, although the robust CBF controller can always ensure safety, its tracking performance of the reference trajectories inside the safe region is not as good as our proposed controller (40).

Example 2: Consider a two-linked planar robot manipulator:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau + J^\top(q)F_d(t), \quad (42)$$

where $q = [q_1 \ q_2]^\top$ is the joint angle, $\dot{q} = [\dot{q}_1 \ \dot{q}_2]^\top$ is the joint angular velocity, $\tau \in \mathbb{R}^2$ is the control input, F_d is the external disturbance satisfying Assumption 1, and $M \in \mathbb{R}^{2 \times 2}, C \in \mathbb{R}^{2 \times 2}, G \in \mathbb{R}^2$, and $J \in \mathbb{R}^{2 \times 2}$ denote the inertia matrix, the Coriolis/centrifugal matrix, the gravity term, and the Jacobian, respectively.

It can be seen (42) can be expressed in the form of (2) with $x = [q_1 \ q_2 \ \dot{q}_1 \ \dot{q}_2]^\top$, $f = [\dot{q}^\top - (C + G)^\top M^{-1}]^\top$, $g = [0_{2 \times 2} \ M^{-1}]^\top$, and $p = [0_{2 \times 2} \ JM^{-1}]^\top$. It is obvious that f, g, p are smooth functions. The expression of M, C, G and physical parameters are chosen the same as those in [38]. Note that the Jacobian J is singular when $q_2 = 0$ such that it is impossible to uniquely recover F_d even if $\ddot{q}$ is available. The reference trajectory of q is $q_d = [q_{1d}(t) \ q_{2d}(t)]^\top$, where $q_{1d}(t) = q_{2d}(t) = 2 \sin(t)$; the nominal controller is designed by following Proposition 1; the disturbance is selected as $F_d =$

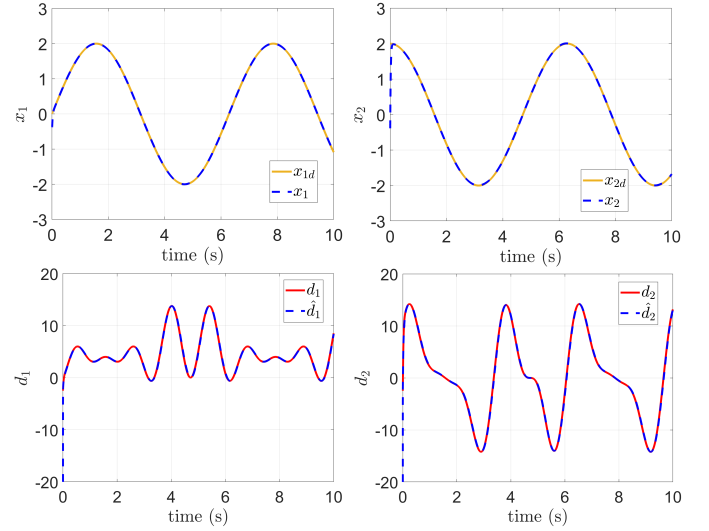


Figure 2: Simulation results of the proposed IIDOB and IIDOB-based tracking controller for system (41). (top) Trajectories of the states x_1, x_2 and the reference signals x_{1d}, x_{2d} ; (bottom) trajectories of the total disturbances d_1, d_2 and the estimated total disturbances $\hat{d}_1, \hat{d}_2$.

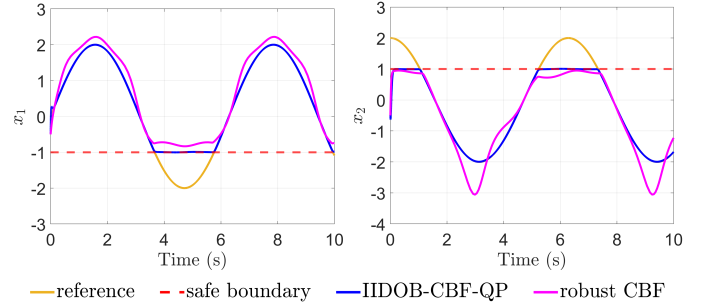


Figure 3: Simulation results of the IIDOB-CBF-QP-based controller (40) and the robust CBF-based controller proposed in [8]. Both controllers can ensure safety, but controller (40) has better tracking performance inside the safe region.

$[d_1 \ d_2]^\top$ with $d_1 = d_2 = 5 \sin(t) + 2 \cos(2t) + 4 \sin(3t) + 3 \cos(4t)$ such that Assumption 1 holds with $\omega_0 = 11, \omega_1 = 37$; four CBFs are selected as $h_1 = q_1 + 1, h_2 = -q_1 + 1.5, h_3 = q_2 + 1.2$, and $h_4 = -q_2 + 1$, which aim to ensure $-1 \leq q_1 \leq 1.5$ and $-1.2 \leq q_2 \leq 1$. It can be verified that the minimum IRD and the minimum DRD of system (42) with respect to $h_i, i \in [4]$, are 2, i.e., $r_I = r_D = 2$. We select the control parameters as $\gamma = 250, c = 5, \theta = 50, k_1 = k_2 = 20, \alpha_1 = \alpha_2 = 50, \rho = \tilde{\rho} = 1, \lambda_0 = 25, \lambda_1 = 30, \lambda_2 = 50, T_1 = 250, T_2 = 1$ in Theorem 1 and 2.

The simulation results are presented in Fig. 4. One can observe that the disturbance is precisely estimated by the proposed IIDOB (16), and the IIDOB-CBF-QP-based controller (40) can ensure the safety because the trajectories of q_1, q_2 remain within the safety sets.

VI. CONCLUSION

This note introduced a systematic approach for designing IIDOB for general nonlinear control-affine systems without imposing restrictive assumptions employed by existing DOB

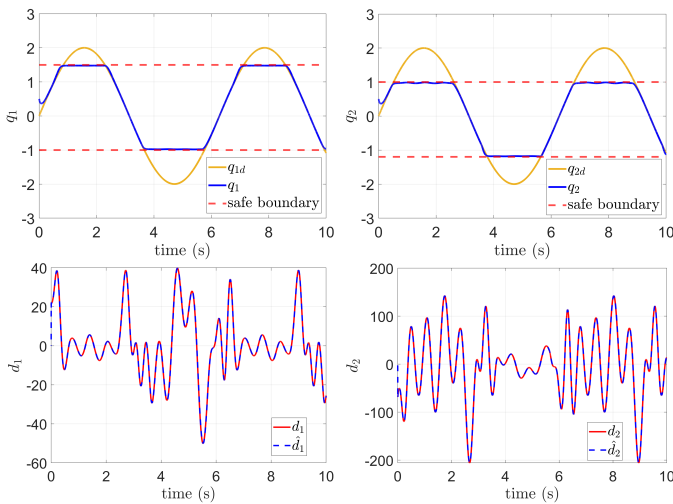


Figure 4: Simulation results of the proposed IIDOB-CBF-QP-based safe control for the two-linked planar robot (42). One can see that the disturbance estimation is accurate and the safety is ensured. Moreover, the tracking performance of the nominal controller is well preserved inside the safe region.

design strategies. Based on that, a filter-based IIDOB-CBF-QP safe control design method was proposed. The numerical simulation results demonstrated the estimation accuracy achieved by the IIDOB and the superior performance of the proposed safe controller compared to the robust CBF-QP-based methods. Future studies include extending the proposed method to hybrid systems and applying it to robotic systems such as bipedal robots subject to rigid impacts with the ground.

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