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# THE GLOBAL EXISTENCE OF STRONG SOLUTIONS FOR A NON-ISOTHERMAL IDEAL GAS SYSTEM

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**Abstract** We investigate the global existence of strong solutions to a non-isothermal ideal gas model derived from an energy variational approach. We first show the global well-posedness in the Sobolev space $\boldsymbol{H}^{\mathbf{2}}(\mathbb{R}^{\mathbf{3}})$ for solutions near equilibrium through iterated energy-type bounds and a continuity argument. We then prove the global well-posedness in the critical Besov space $\dot{\boldsymbol{B}}^{\mathbf{3/2}}_{\mathbf{2,1}}$ by showing that the linearized operator is a contraction mapping under the right circumstances.

**Key words** thermal fluid equations; energy-variational method; well-posedness theory for PDE; paraproduct calculus

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## 1 Introduction

Starting from a given free energy, Lai-Liu-Tarfulea [20] established a general framework for deriving non-isothermal fluid models by combining classical thermodynamic laws and the energetic variational approach (see [15, 18]). As an application, three full non-isothermal systems (the non-isothermal ideal gas, non-isothermal porous media, and non-isothermal generalized porous media equations) are established based on three specific free energies. What is more, under some appropriate assumptions on the conductivity coefficient $\kappa_3$, a maximum/minimum principle is developed for the first two models by adapting an idea originally from the work

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[25]. These maximum/minimum principles establish the positivity of the absolute temperature, which implies the thermodynamic consistency of the corresponding models.

However, [20] does not address the long time behavior (the existence and uniqueness) of the solution to the non-isothermal models mentioned, which is the core theory for the partial differential system. At present, there are many results on the existence and behavior of weak solutions to various non-isothermal fluid models; see [9, 10, 12, 13, 24] for the Navier-Stokes-Fourier system, which is a powerful generalization of the classical Navier-Stokes equations and is used to model thermodynamic fluid flow, [7] for the non-isothermal general Ericksen-Leslie system, [8] for the non-isothermal Cahn-Hilliard equation, [17] for the non-isothermal Poisson-Nernst-Planck-Fourier system, and [21] for the Brinkman-Fourier system with ideal gas equilibrium.

This paper aims to study the global well-posedness of the following non-isothermal ideal gas system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t \rho = \kappa_1 \Delta(\rho\theta), \\ \kappa_2(\rho\theta)_t - \kappa_1(\kappa_1+\kappa_2)\nabla\cdot(\theta\nabla(\rho\theta)) = \nabla\cdot(\kappa_3\nabla\theta)\,. \end{cases} \tag{1.1}$$

For the reader's convenience, we briefly sketch the construction of (1.1). As can be seen in the model, the main unknown variables are:

1. a non-negative measurable function $\rho = \rho(t,x)$ which denotes the mass density;
2. a positive measurable function $\theta = \theta(t,x)$ representing the absolute temperature.

In addition, a vector field $u = u(t,x)$, denoting the velocity field of the fluid, will be used as an intermediate variable.

For an ideal gas, we have the following definition of free energy:

$$\Psi(\rho,\theta) = \kappa_1\theta\rho\ln\rho - \kappa_2\rho\theta\ln\theta. \tag{1.2}$$

Then the (specific) entropy of the system, denoted by $\eta$, and the (specific) internal energy, denoted by $e$, are connected to the free energy $\Psi$ by the standard Helmholtz relation (see formula (2.5.26) in the classical book [5])

$$\begin{cases} \eta(\rho,\theta) := -\partial_\theta\Psi, \\ e(\rho,\theta) := \Psi - \partial_\theta\Psi\theta = \Psi + \eta\theta, \\ \eta_\theta = \dfrac{\kappa_2\rho}{\theta}. \end{cases} \tag{1.3}$$

The total energy and total dissipation are then chosen to be

$$E^{\text{total}} = \int_{\Omega_t^x} \Psi(\rho,\theta)\mathrm{d}x \qquad \text{and} \qquad \mathcal{D}^{\text{total}} = \frac{1}{2}\int_{\Omega_t^x} \rho u^2 \mathrm{d}x.$$

Employing the energetic variational approach then establishes the following Darcy type diffusion law:

$$\begin{cases} p = \Psi_\rho\rho - \rho = \kappa_1\rho\theta, \\ \nabla p = -\rho u, \\ \partial_\theta p = \kappa_1\rho. \end{cases} \tag{1.4}$$

We remark that, according to [2, 23], the internal energy and pressure are both linearly proportional to the product of temperature and density. It is easy to verify this fact by combining (1.2), (1.3) and (1.4).

Now, we rewrite the internal energy function in terms of the new state variables $\rho$ and $\eta$, giving that

$$e_1(\rho,\eta) = e\left(\rho, \theta(\rho,\eta)\right), \tag{1.5}$$

which then implies that

$$\begin{cases} e_{1\eta} = \theta, \quad e_{1\rho} = \Psi_\rho, \\ \nabla p = \rho \nabla e_{1\rho} + \eta \nabla e_{1\eta}. \end{cases} \tag{1.6}$$

We recall the continuity equation for a closed system

$$\rho_t + \nabla\cdot(\rho u) = 0, \tag{1.7}$$

and combine this with the two classical thermodynamic laws, the first of which relates to the rate of change of the internal energy with dissipation and heat:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \nabla\cdot W + \nabla\cdot q. \tag{1.8}$$

Here $W$ denotes the amount of thermodynamic work done by the system on its surroundings and $q$ denotes the quantity of energy supplied to the system as heat. The second thermodynamic law describes the evolution of the entropy

$$\partial_t \eta + \nabla\cdot(\eta u) = \nabla\cdot\left(\frac{q}{\theta}\right) + \Delta, \tag{1.9}$$

where $\Delta \geq 0$ denotes the rate of entropy production. Fourier's law then yields that

$$q = \kappa_3 \nabla\theta, \tag{1.10}$$

where $\kappa_3$ denotes the material conductivity (which may depend on $\rho$ and $\theta$). Combining (1.6), (1.7), (1.8), (1.9) and (1.10), we obtain that

$$\begin{aligned} \frac{\mathrm{d}e_1(\rho,\eta)}{\mathrm{d}t} &= e_{1\rho}\rho_t + e_{1\eta}\eta_t \\ &= e_{1\rho}\left(-\nabla\cdot(\rho u)\right) + e_{1\eta}\left(-\nabla\cdot(\eta u) + \nabla\cdot\left(\frac{q}{\theta}\right) + \Delta\right) \\ &= -\nabla\cdot\left(e_{1\rho}\rho u + e_{1\eta}\eta u\right) + \left(\rho\nabla e_{1\rho} + \eta \nabla e_{1\eta}\right)\cdot u + \theta\nabla\cdot\left(\frac{q}{\theta}\right) + \theta\Delta \\ &= \nabla\cdot W + \nabla p\cdot u + \nabla\cdot q - \frac{q}{\theta}\cdot\nabla\theta + \theta\Delta \\ &= \nabla\cdot W - \rho u^2 + \nabla\cdot q - \frac{\kappa_3|\nabla\theta|^2}{\theta} + \theta\Delta. \end{aligned} \tag{1.11}$$

Therefore,

$$\begin{cases} W = -\left(e_{1\rho}\rho + e_{1\eta}\eta\right)u, \\ \Delta = \dfrac{1}{\theta}\left(\rho|u|^2 + \dfrac{\kappa_3|\nabla\theta|^2}{\theta}\right), \end{cases} \tag{1.12}$$

which in turn gives that

$$\begin{aligned} \eta_t + \nabla\cdot(\eta u) &= \eta_\theta(\theta_t + u\cdot\nabla\theta) + \eta_\rho(\rho_t + u\cdot\nabla\rho) + \eta\nabla\cdot u \\ &= \eta_\theta(\theta_t + u\cdot\nabla\theta) + \eta_\rho\left(-\rho\nabla\cdot u\right) + \eta\nabla\cdot u \\ &= \eta_\theta(\theta_t + u\cdot\nabla\theta) + \left(\eta - \eta_\rho\rho\right)\nabla\cdot u \\ &= \eta_\theta(\theta_t + u\cdot\nabla\theta) + \partial_\theta p \nabla\cdot u \\ &= \nabla\cdot\left(\frac{q}{\theta}\right) + \Delta = \nabla\cdot\left(\frac{q}{\theta}\right) + \frac{1}{\theta}\left(\rho|u|^2 + \frac{q\cdot\nabla\theta}{\theta}\right), \end{aligned} \tag{1.13}$$

which finally yields that

$$\eta_\theta(\theta_t + u\cdot\nabla\theta) + \partial_\theta p \nabla\cdot u = \nabla\cdot\left(\frac{q}{\theta}\right) + \frac{1}{\theta}\left(\rho|u|^2 + \frac{q\cdot\nabla\theta}{\theta}\right). \tag{1.14}$$

Combining (1.14), (1.3) and (1.4) alows us to conclude that

$$\frac{\kappa_2\rho}{\theta}(\theta_t + u\cdot\nabla\theta) + \kappa_1\rho\nabla\cdot u = \nabla\cdot\left(\frac{\kappa_3\nabla\theta}{\theta}\right) + \frac{1}{\theta}\left(-\kappa_1\nabla(\rho\theta)\cdot u + \frac{\kappa_3|\nabla\theta|^2}{\theta}\right), \tag{1.15}$$

so that

$$\kappa_2(\rho\theta)_t - \kappa_1(\kappa_1+\kappa_2)\nabla\cdot(\theta\nabla(\rho\theta)) = \nabla\cdot(\kappa_3\nabla\theta)\,, \tag{1.16}$$

which completes the derivation of the non-isothermal ideal gas model (1.1).

Our main goal in the present work is to establish the well-posedness for system (1.1). Motivated by similar works on the classical Navier-Stokes equations ([6, 14]), we first discuss our choice of working spaces. We observe that (1.1) is invariant under the transformation

$$\begin{aligned}&(\rho(t,x),\theta(t,x)) \longrightarrow (\rho(\lambda^2 t,\lambda x),\theta(\lambda^2 t,\lambda x)),\\ &(\rho_0(x),\theta_0(x)) \longrightarrow (\rho_0(\lambda x),\theta_0(\lambda x)).\end{aligned} \tag{1.17}$$

**Definition 1.1** A function space $E\subset\mathcal{S}'(\mathbb{R}^3)\times\mathcal{S}'(\mathbb{R}^3)$ is called a critical space if the associated norm is invariant under the transformation (1.17).

Obviously $\dot{H}^{3/2}\times\dot{H}^{3/2}$ is a critical space for the initial data, but $\dot{H}^{3/2}$ is not included in $L^\infty$. We cannot expect to get $L^\infty$ control on the density and the temperature by taking that $(\rho_0-1,\theta_0-1)\in\dot{H}^{3/2}\times\dot{H}^{3/2}$. Moreover, the product between functions does not extend continuously from $\dot{H}^{3/2}\times\dot{H}^{3/2}$ to $\dot{H}^{3/2}$, so we will run into difficulties when estimating the nonlinear terms. Similarly to the Navier-Stokes system studied in [6], we could use homogeneous Besov spaces $\dot{B}^s_{2,1}(\mathbb{R}^3)$ (defined in [1, Chapter 2]). $\dot{B}^{3/2}_{2,1}$ is an algebra embedded in $L^\infty$ which allows us to control the density and temperature from above without requiring more regularity on the derivatives of $\rho_0$ and $\theta_0$.

Our first result proves the global well-posedness for (1.1) when the initial data is close to a stable equilibrium $(\underline{\rho},\underline{\theta})$ in the subcritical space $H^2\times H^2$. The working space $X(T)$ is defined by the norm

$$\|u\|^2_{X(T)} := \sup_{0\le t\le T}\|u(t)\|^2_{H^2} + \int_0^T\left(\|\nabla u\|^2_{H^2} + \|\partial_t u\|^2_{H^1}\right)\mathrm{d}t$$

for any distribution $u$ and for $T>0$.

**Theorem 1.2** Let $\underline{\rho}$, $\underline{\theta}>0$ be fixed constants. There exist two positive constants, $\alpha$ and $M$, such that, for all $\rho_0$ and $\theta_0$ where $(\rho_0-\underline{\rho},\theta_0-\underline{\theta})\in H^2\times H^2$ and

$$\|\rho_0-\underline{\rho}\|_{H^2} + \|\theta_0-\underline{\theta}\|_{H^2} \le \alpha, \tag{1.18}$$

system (1.1) has a unique global solution $(\rho,\theta)$ with $(\rho-\underline{\rho},\theta-\underline{\theta})\in X(T)$ for all $T>0$. Moreover, if we define that $c:=\rho-\underline{\rho}$ and $\tau:=\theta-\underline{\theta}$, then

$$\|(c,\tau)\|_{X(T)} \le M\alpha. \tag{1.19}$$

Our second main result then establishes the existence and uniqueness of a solution to system (1.1) for initial data close to a stable equilibrium $(\underline{\rho},\underline{\theta})$ in the critical space $\dot{B}^{3/2}_{2,1}\times\dot{B}^{3/2}_{2,1}$. For convenience, we assume that $\underline{\rho}=\underline{\theta}=1$. The working space $E(T)$ is then defined by

$$E(T) := \left\{u\in\mathcal{C}\left([0,T],\dot{B}^{3/2}_{2,1}\right),\quad \nabla^2 u\in L^1\left(0,T;\dot{B}^{3/2}_{2,1}\right)\right\},\quad T>0.$$

**Theorem 1.3** There exist two positive constants, $\alpha$ and $M$, such that, for all $(a_0,\tau_0)\in\dot{B}^{3/2}_{2,1}\times\dot{B}^{3/2}_{2,1}$ with

$$\|a_0\|_{\dot{B}^{3/2}_{2,1}}+\|\tau_0\|_{\dot{B}^{3/2}_{2,1}}\le\alpha, \tag{1.20}$$

system (1.1) has a unique global solution $(\rho,\theta)$ with initial data $\theta_0=\tau_0+1$ and $\rho_0=1/(1+a_0)$. Moreover, if we define that $\rho=1/(1+a)$ and $\tau=\theta-1$, then, for all $T>0$,

$$\|(a,\tau)\|_{E(T)}\le M\alpha. \tag{1.21}$$

The rest of the paper unfolds as follows: Section 2 will present some basic tools in Fourier analysis: the Littlewood-Paley decomposition and the paraproduct calculus in Besov spaces. Section 3 will prove the global existence and uniqueness result in Soblolev spaces (Theorem 1.2). Section 4 will prove the global well-posedness result in the critical Besov space by using Banach's fixed point Theorem.

## 2 Notation and Preliminaries

For any $1\le p\le\infty$ and measurable $f:\mathbb{R}^n\to\mathbb{R}$, we will use $\|f\|_{L^p(\mathbb{R}^n)}$, or simply $\|f\|_p$, to denote the usual $L^p$ norm. For a vector valued function $f=(f^1,\cdots,f^m)$, we still denote that $\|f\|_p:=\sum\limits_{j=1}^{m}\|f^j\|_p$.

For any $0<T<\infty$ and any Banach space $\mathbb{B}$ with a norm $\|\cdot\|_{\mathbb{B}}$, we will use the notation $C([0,T],\mathbb{B})$ or $C^0_t\mathbb{B}$ to denote the space of continuous $\mathbb{B}$-valued functions endowed with the norm

$$\|f\|_{C([0,T],\mathbb{B})}:=\max_{0\le t\le T}\|f(t)\|_{\mathbb{B}}.$$

Also, for $1\le p\le\infty$, we define that $\|f\|_{L^p_t\mathbb{B}([0,T])}:=\|\|f(t)\|_{\mathbb{B}}\|_{L^p_t([0,T])}$.

We shall adopt the following convention for the Fourier transform:

$$\hat{f}(\xi)=\int_{\mathbb{R}^n}f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\mathrm{d}x;\qquad f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

For $s\in\mathbb{R}$, the fractional Laplacian $|\nabla|^s$ corresponds to the Fourier multiplier $|\xi|^s$ defined as

$$\widehat{|\nabla|^s f}(\xi)=|\xi|^s\hat{f}(\xi)$$

whenever it is well-defined. For $s\ge0$ and $1\le p<\infty$, we define the semi-norm and norms as

$$\|f\|_{\dot{W}^{s,p}}=\||\nabla|^sf\|_p,\qquad \|f\|_{W^{s,p}}=\||\nabla|^sf\|_p+\|f\|_p.$$

When $p=2$, we denote that $\dot{H}^s=\dot{W}^{s,2}$ and that $H^s=W^{s,2}$, in accordance with the usual notation.

For any two quantities $X$ and $Y$, we denote that $X\lesssim Y$ if $X\le CY$ for some constant $C>0$. Similarly, $X\gtrsim Y$ if $X\ge CY$ for some $C>0$. We denote that $X\sim Y$ if $X\lesssim Y$ and $Y\lesssim X$. The dependence of the constant $C$ on other parameters or constants are usually clear from the context, so we will often suppress this dependence. We shall denote that $X\lesssim_{Z_1,Z_2,\cdots,Z_k}Y$ if $X\le CY$ and that the constant $C$ depends on the quantities $Z_1,\cdots,Z_k$.

For any two quantities $X$ and $Y$, we shall denote that $X\ll Y$ if $X\le cY$ for some sufficiently small constant $c$. The smallness of the constant $c$ is usually clear from the context.

The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is different from the usual Vinogradov notation in number theory or asymptotic analysis.

We will need to use the Littlewood-Paley (LP) frequency projection operators. To fix the notation, let $\phi_0$ be a radial function in $C_c^\infty(\mathbb{R}^n)$ satisfying

$$0 \le \phi_0 \le 1, \quad \phi_0(\xi) = 1 \ \text{ for } |\xi| \le 1, \quad \phi_0(\xi) = 0 \ \text{ for } |\xi| \ge 7/6.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$, which is supported in $\frac{1}{2} \le |\xi| \le \frac{7}{6}$. For any $f \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define that

$$\widehat{S_j f}(\xi) = \phi_0(2^{-j}\xi)\hat{f}(\xi), \qquad \widehat{\Delta_j f}(\xi) = \phi(2^{-j}\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}^n.$$

We will denote that $P_{>j} = I - S_j$ ($I$ is the identity operator) and, for any $-\infty < a < b < \infty$, that $P_{[a,b]} = \sum\limits_{a \le j \le b} \Delta_j$. Sometimes, for simplicity of notation (and when there is no obvious confusion), we will write $f_j = \Delta_j f$ and $f_{a \le \cdot \le b} = \sum\limits_{j=a}^{b} f_j$. By using the support property of $\phi$, we have that $\Delta_j \Delta_{j'} = 0$ whenever $|j - j'| > 1$.

Thanks to the above Littlewood-Paley decomposition, a number of functional spaces can be characterized. Let us give the definition of homogeneous Besov spaces first.

**Definition 2.1** For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^3)$, we set that

$$\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} = \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j u\|^r_{L^p} \right)^{\frac{1}{r}},$$

with the usual modification if $r = \infty$.

We then define the Besov space by $\dot{B}^s_{p,r} = \{u \in \mathcal{S}'(\mathbb{R}^3), \ \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} < \infty\}$. In the what follows, for the convenience of notation, we always use $\dot{B}^s_{p,r}$ instead of $\dot{B}^s_{p,r}(\mathbb{R}^3)$, and similar notations for other norms. Let us now state some classical properties for the Besov spaces without giving the proofs.

**Proposition 2.2** The following properties hold:

1) Derivatives: we have that $\|\nabla u\|_{\dot{B}^{s-1}_{p,r}} \le C\|u\|_{\dot{B}^s_{p,r}}$.

2) Sobolev embedding: if $p_1 \le p_2$ and $r_1 \le r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-\frac{3}{p_1}+\frac{3}{p_2}}_{p_2,r_2}$.
If $s_1 > s_2$ and $1 \le p, r_1, r_2 \le +\infty$, then $\dot{B}^{s_1}_{p,r_1} \hookrightarrow \dot{B}^{s_2}_{p,r_2}$.

3) Algebraic property: for $s > 0$, $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra.

4) Real interpolation: $\left(\dot{B}^{s_1}_{p,r}, \dot{B}^{s_2}_{p,r}\right)_{\theta,r'} = \dot{B}^{\theta s_1 + (1-\theta)s_2}_{p,r'}$.

We recall some product laws in Besov spaces coming directly from the paradifferential calculus of Bony (see [4]).

**Proposition 2.3** We have the following product laws:

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} + \|v\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}} \quad \text{if} \quad s > 0,$$

$$\|uv\|_{\dot{B}^{s_1}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p,r}}\|v\|_{\dot{B}^{s_2}_{p,r}} \quad \text{if} \quad s_1 \le \frac{3}{2}, s_2 > \frac{3}{2} \quad \text{and} \quad s_1 + s_2 > 0,$$

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{3}{2}}_{p,r}} \lesssim \|u\|_{\dot{B}^{s_1}_{p,r}}\|v\|_{\dot{B}^{s_2}_{p,r}} \quad \text{if} \quad s_1, s_2 < \frac{3}{2} \quad \text{and} \quad s_1 + s_2 > 0,$$

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p,r}}\|v\|_{\dot{B}^{3/2}_{p,r} \cap L^\infty} \quad \text{if} \quad |s| < \frac{3}{2}.$$

Moreover, if $r=1$, the third inequality also holds for $s_1, s_2 \le \frac{3}{2}$ and $s_1+s_2>0$.

## 3 Global Well-Posedness in Sobolev Spaces

The present section is dedicated to proving Theorem 1.2. Before starting, we assume that $c := \rho - \underline{\rho}, \tau := \theta - \underline{\theta}$. We first rewrite (1.1) as

$$\begin{cases} \partial_t c - \kappa_1 \underline{\theta} \Delta c - \kappa_1 \underline{\rho} \Delta \tau = \kappa_1 \Delta(c\tau), \\ \rho \kappa_2 \partial_t \tau - \kappa_1 \kappa_2 \nabla \tau \cdot \nabla(\rho\theta) - \kappa_1^2 \nabla \cdot (\theta \nabla(\rho\theta)) = \nabla \cdot (\kappa_3(\theta) \nabla \tau), \\ (c, \tau) \mid_{t=0} = (c_0, \tau_0), \end{cases} \tag{3.1}$$

with

$$c_0 = \rho_0 - \underline{\rho}, \tau_0 = \theta_0 - \underline{\theta}.$$

For simplicity, here we assume that $\underline{\rho} = \underline{\theta} = 1$. Furthermore, we decompose the coefficients $\kappa_3(\theta) = \bar{\kappa}_3 + \widetilde{\kappa}_3(\tau)$, which satisfies that $\widetilde{\kappa}_3(0) = 0$. We also assume that $\widetilde{\kappa}_3'$ and $\widetilde{\kappa}_3''$ exist and are bounded. Then (3.1) can be written as

$$\begin{cases} \partial_t c - \kappa_1 \Delta c - \kappa_1 \Delta \tau = \kappa_1 \Delta(c\tau), \\ \kappa_2 \partial_t \tau - (\kappa_1^2 + \bar{\kappa}_3) \Delta \tau - \kappa_1^2 \Delta c = \kappa_1(\kappa_1+\kappa_2)\Big(\nabla\tau \cdot \nabla c + \nabla \tau \cdot \nabla \tau + \nabla \tau \cdot \nabla(\tau c)\Big) \\ \qquad + \kappa_1^2 \Delta(c\tau) + \nabla \cdot (\widetilde{\kappa}_3(\tau) \nabla \tau) - \kappa_2 c \partial_t \tau, \\ (c, \tau) \mid_{t=0} = (c_0, \tau_0). \end{cases} \tag{3.2}$$

The principle of the proof of Theorem 1.2 is a very classical one. We use an iterative method to establish the approximate solutions to the perturbed system (3.2). Define the first term in the sequence as $(c^0(t,x), \tau^0(t,x)) = (0,0)$ everywhere on $\mathbb{R}^+ \times \mathbb{R}^3$. We then define $(c^{n+1}(t,x), \tau^{n+1}(t,x))$ by induction, as the solution to the linear approximate system

$$\begin{cases} \partial_t c^{n+1} - \kappa_1 \Delta c^{n+1} - \kappa_1 \Delta \tau^{n+1} = H^n, \\ \kappa_2 \partial_t \tau^{n+1} - (\kappa_1^2 + \bar{\kappa}_3) \Delta \tau^{n+1} - \kappa_1^2 \Delta c^{n+1} = I^n, \\ (c^{n+1}, \tau^{n+1}) \mid_{t=0} = (c_n, \tau_n) \end{cases} \tag{3.3}$$

with

$$(c_n, \tau_n) = (S_n c_0, S_n \tau_0), \tag{3.4}$$

$$H^n = \kappa_1 \Delta(c^n \tau^n), \tag{3.5}$$

$$\begin{aligned} I^n = {} & \kappa_1(\kappa_1+\kappa_2)\Big(\nabla \tau^n \cdot \nabla c^n + \nabla \tau^n \cdot \nabla \tau^n + \nabla \tau^n \cdot \nabla(\tau^n c^n)\Big) \\ & + \kappa_1^2 \Delta(c^n \tau^n) + \nabla \cdot (\widetilde{\kappa}_3(\tau^n) \nabla \tau^n) - \kappa_2 c^n \partial_t \tau^n, \end{aligned} \tag{3.6}$$

where the low frequency cut-off operator $S_n$ is as defined in Section 2.

In the next two subsections, we will show that the sequence of approximate solutions $\{(c^n(t,x), \tau^n(t,x))\}_{n\in\mathbb{N}}$ is uniformly bounded (and moreover Cauchy) in $X(T)$ for all $T>0$. As mentioned in Section 1, the working space $X(T)$ is defined by the norm

$$\|u\|^2_{X(T)} := \sup_{0\le t\le T} \|u(t)\|^2_{H^2} + \int_0^T \Big(\|\nabla u\|^2_{H^2} + \|\partial_t u\|^2_{H^1}\Big) \mathrm{d}t$$

for any distribution $u$ and $T>0$.

### 3.1 Uniform Bound in the Critical Regularity Case

In this part, we prove a uniform estimate in $X(T)$ for $(c^n,\tau^n)$. Denote that

$$\alpha=\|c_0\|_{H^2}+\|\tau_0\|_{H^2}.$$

We are going to prove the existence of a positive $M$ such that, if $\alpha$ is small enough, then the solution belongs to the space $L^\infty_T(H^2)\times L^\infty_T(H^2)$ and also satisfies that

$$\|(c^n,\tau^n)\|^2_{X(T)}\le M\alpha^2. \tag{3.7}$$

Clearly, (3.7) holds for $(c^0,\tau^0)$. Assuming that (3.7) holds for $(c^n,\tau^n)$, we will show that it also holds for $(c^{n+1},\tau^{n+1})$.

**Step 1** $L^2$ energy estimate

Taking the $L^2$ inner product with $c^{n+1}$ and $\tau^{n+1}$ with respect to the first and second equations, one has that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c^{n+1}\|^2_{L^2}+\kappa_1\|\nabla c^{n+1}\|^2_{L^2}&=-\kappa_1\int_{\mathbb{R}^3}\nabla\tau^{n+1}\cdot\nabla c^{n+1}\,\mathrm{d}x-\kappa_1\int_{\mathbb{R}^3}\nabla(c^n\tau^n)\cdot\nabla c^{n+1}\mathrm{d}x\\&:=I_1+I_2,\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}\frac{1}{2}\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\tau^{n+1}\|^2_{L^2}+(\kappa_1^2+\bar\kappa_3)\|\nabla\tau^{n+1}\|^2_{L^2}&=-\kappa_1^2\int_{\mathbb{R}^3}\nabla\tau^{n+1}\cdot\nabla c^{n+1}\,\mathrm{d}x+\int_{\mathbb{R}^3}I^n\cdot\tau^{n+1}\mathrm{d}x\\&:=I_3+I_4+I_5+I_6+I_7,\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}&I_1=-\kappa_1\int_{\mathbb{R}^3}\nabla\tau^{n+1}\cdot\nabla c^{n+1}\,\mathrm{d}x,\quad I_2=-\kappa_1\int_{\mathbb{R}^3}\nabla(c^n\tau^n)\cdot\nabla c^{n+1}\,\mathrm{d}x,\\&I_3=-\kappa_1^2\int_{\mathbb{R}^3}\nabla\tau^{n+1}\cdot\nabla c^{n+1}\,\mathrm{d}x,\quad I_4=-\kappa_1^2\int_{\mathbb{R}^3}\nabla(c^n\tau^n)\cdot\nabla\tau^{n+1}\,\mathrm{d}x,\\&I_5=\kappa_1(\kappa_1+\kappa_2)\int_{\mathbb{R}^3}\Big(\nabla c^n\cdot\nabla\tau^n+\nabla\tau^n\cdot\nabla\tau^n+\nabla\tau^n\cdot\nabla(\tau^nc^n)\Big)\tau^{n+1}\,\mathrm{d}x,\\&I_6=-\int_{\mathbb{R}^3}\widetilde\kappa_3(\tau^n)\nabla\tau^n\cdot\nabla\tau^{n+1}\,\mathrm{d}x,\quad I_7=-\kappa_2\int_{\mathbb{R}^3}c^n\partial_t\tau^n\ \tau^{n+1}\,\mathrm{d}x.\end{aligned}$$

First, by the Hölder and Cauchy inequalities, we have that

$$I_1\le\frac{1}{2}\kappa_1\|\nabla\tau^{n+1}\|^2_{L^2}+\frac{1}{2}\kappa_1\|\nabla c^{n+1}\|^2_{L^2},\quad I_3\le\frac{1}{2}\kappa_1^2\|\nabla\tau^{n+1}\|^2_{L^2}+\frac{1}{2}\kappa_1^2\|\nabla c^{n+1}\|^2_{L^2}.$$

Then, by the linear combination of (3.8) and (3.9), one can get that

$$\begin{aligned}&\kappa_1(1+\delta)\frac{\mathrm{d}}{\mathrm{d}t}\|c^{n+1}\|^2_{L^2}+\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\tau^{n+1}\|^2_{L^2}+\delta\kappa_1^2\|\nabla c^{n+1}\|^2_{L^2}+(2\bar\kappa_3-\delta\kappa_1^2)\|\nabla\tau^{n+1}\|^2_{L^2}\\&\le2\left(\kappa_1(1+\delta)I_2+I_4+I_5+I_6+I_7\right),\end{aligned}\tag{3.10}$$

where $\delta$ is chosen to satisfy that $\delta\kappa_1^2=\bar\kappa_3$. To bound $I_2$, we use Hölder's inequality to obtain that

$$\begin{aligned}I_2&=-\kappa_1\int_{\mathbb{R}^3}\tau^n\nabla c^n\cdot\nabla c^{n+1}\,\mathrm{d}x-\kappa_1\int_{\mathbb{R}^3}c^n\nabla\tau^n\cdot\nabla c^{n+1}\,\mathrm{d}x\\&\le\kappa_1\|\tau^n\|_{L^\infty}\|\nabla c^n\|_{L^2}\|\nabla c^{n+1}\|_{L^2}+\kappa_1\|c^n\|_{L^\infty}\|\nabla c^{n+1}\|_{L^2}\|\nabla\tau^n\|_{L^2}\\&\le\frac{C}{\delta}\Big(\|\tau^n\|^2_{L^\infty}\|\nabla c^n\|^2_{L^2}+\|\nabla\tau^n\|^2_{L^2}\|c^n\|^2_{L^\infty}\Big)+\frac{1}{32}\delta\kappa_1^2\|\nabla c^{n+1}\|^2_{L^2}.\end{aligned}\tag{3.11}$$

Similarly, $I_4$ and $I_5$ can be bounded by

$$\begin{aligned} I_4 &\le \frac{C\kappa_1^2}{\delta}\Big(\|\tau^n\|_{L^\infty}^2\|\nabla c^n\|_{L^2}^2 + \|\nabla\tau^n\|_{L^2}^2\|c^n\|_{L^\infty}^2\Big) + \frac{1}{32}\delta\kappa_1^2\|\nabla\tau^{n+1}\|_{L^2}^2, \\ I_5 &\le \kappa_1(\kappa_1+\kappa_2)\|\nabla\tau^n\|_{L^2}\|\tau^{n+1}\|_{L^\infty} \\ &\quad \times\Big(\|\nabla c^n\|_{L^2} + \|\nabla\tau^n\|_{L^2} + \|\nabla\tau^n\|_{L^2}\|c^n\|_{L^\infty} + \|\nabla c^n\|_{L^2}\|\tau^n\|_{L^\infty}\Big). \end{aligned} \tag{3.12}$$

For $I_6$, notice that $\widetilde{\kappa}_3(0)=0$. Then we use Taylor's formula and Hölder's inequality to get that

$$\begin{aligned} I_6 &\le C\|\tau^n\|_{L^\infty}\|\nabla\tau^n\|_{L^2}\|\nabla\tau^{n+1}\|_{L^2} \\ &\le \frac{C}{\delta\kappa_1^2}\|\tau^n\|_{L^\infty}^2\|\nabla\tau^n\|_{L^2}^2 + \frac{1}{32}\delta\kappa_1^2\|\nabla\tau^{n+1}\|_{L^2}^2. \end{aligned} \tag{3.13}$$

For the last term $I_7$, we use the Hölder's inequality and Sobolev embedding to obtain that

$$\begin{aligned} I_7 &= -\kappa_2\int_{\mathbb{R}^3} c^n\ \partial_t\tau^n\ \tau^{n+1}\,\mathrm{d}x \le \kappa_2\|c^n\|_{L^3}\|\partial_t\tau^n\|_{L^2}\|\tau^{n+1}\|_{L^6} \\ &\le \frac{C\kappa_2^2}{\delta\kappa_1^2}\|c^n\|_{H^1}^2\|\partial_t\tau^n\|_{L^2}^2 + \frac{1}{32}\delta\kappa_1^2\|\nabla\tau^{n+1}\|_{L^2}^2. \end{aligned} \tag{3.14}$$

Using the bounds (3.11)–(3.14) in (3.10) and integrating over $[0,T]$, we get that

$$\begin{aligned} &\|c^{n+1}\|_{L^2}^2 + \|\tau^{n+1}\|_{L^2}^2 + \int_0^T\Big(\|\nabla c^{n+1}\|_{L^2}^2 + \|\nabla\tau^{n+1}\|_{L^2}^2\Big)\mathrm{d}t \\ \le\ & C\Big(\|c_0\|_{L^2}^2 + \|\tau_0\|_{L^2}^2\Big) + C\int_0^T\Big(\|\tau^n\|_{L^\infty}^2\|\nabla c^n\|_{L^2}^2 + \|\nabla\tau^n\|_{L^2}^2\|c^n\|_{L^\infty}^2\Big)\mathrm{d}t \\ &+ C\int_0^T\|\nabla\tau^n\|_{L^2}\|\tau^{n+1}\|_{L^\infty}\Big(\|\nabla c^n\|_{L^2} + \|\nabla\tau^n\|_{L^2} + \|\nabla\tau^n\|_{L^2}\|c\|_{L^\infty} + \|\nabla c^n\|_{L^2}\|\tau^n\|_{L^\infty}\Big)\mathrm{d}t \\ &+ C\int_0^T\Big(\|\tau^n\|_{L^\infty}^2\|\nabla\tau^n\|_{L^2}^2 + \|c^n\|_{H^1}^2\|\partial_t\tau^n\|_{L^2}^2\Big)\mathrm{d}t, \end{aligned} \tag{3.15}$$

where the constant $C$ only depends on $\kappa_1,\kappa_2$ and $\bar{\kappa}_3$. By the Sobolev embedding $H^2\hookrightarrow L^\infty$ in $\mathbb{R}^3$, and using assumption (3.7), we obtain that

$$\begin{aligned} &\|c^{n+1}\|_{L^2}^2 + \|\tau^{n+1}\|_{L^2}^2 + \int_0^T\Big(\|\nabla c^{n+1}\|_{L^2}^2 + \|\nabla\tau^{n+1}\|_{L^2}^2\Big)\mathrm{d}t \\ \le\ & C\alpha^2 + CM^2\alpha^4 + CM\alpha^2\|\tau^{n+1}\|_{X(T)}^{\frac12} + CM^{\frac32}\alpha^3\|\tau^{n+1}\|_{X(T)}^{\frac12}. \end{aligned} \tag{3.16}$$

**Step 2** $\dot{H}^2$ energy estimate

Due to the equivalence of $\|(c^{n+1},\tau^{n+1})\|_{H^2}$ with $\|(c^{n+1},\tau^{n+1})\|_{L^2}+\|(c^{n+1},\tau^{n+1})\|_{\dot{H}^2}$, it is sufficient to bound the homogeneous $\dot{H}^2$ energy of $(c^{n+1},\tau^{n+1})$. Applying $\partial_i^2$ for $i=1,2,3$ to (3.2) and then taking the $L^2$ inner product with $(\partial_i^2c^{n+1},\partial_i^2\tau^{n+1})$, respectively, we find that

$$\begin{aligned} &\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2c^{n+1}\|_{L^2}^2 + \kappa_1\|\nabla\partial_i^2c^{n+1}\|_{L^2}^2 \\ =\ & -\kappa_1\int_{\mathbb{R}^3}\nabla\partial_i^2\tau^{n+1}\cdot\nabla\partial_i^2c^{n+1}\,\mathrm{d}x - \kappa_1\int_{\mathbb{R}^3}\nabla\partial_i^2(c^n\tau^n)\cdot\nabla\partial_i^2c^{n+1}\,\mathrm{d}x \\ \le\ & \frac12\kappa_1\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2 + \frac12\kappa_1\|\nabla\partial_i^2c^{n+1}\|_{L^2}^2 + J_1, \end{aligned} \tag{3.17}$$

and

$$\frac12\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2\tau^{n+1}\|_{L^2}^2 + (\kappa_1^2+\bar{\kappa}_3)\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2 \le \frac12\kappa_1^2\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2 + \sum_2^5 J_i. \tag{3.18}$$

Here, we denote $J_1$ to $J_5$ by

$$J_1 = -\kappa_1 \int_{\mathbb{R}^3} \nabla\partial_i^2(c^n\tau^n)\cdot\nabla\partial_i^2 c^{n+1}\,\mathrm{d}x, \quad J_2 = -\kappa_1^2 \int_{\mathbb{R}^3} \nabla\partial_i^2(c^n\tau^n)\cdot\nabla\partial_i^2 \tau^{n+1}\,\mathrm{d}x,$$
$$J_3 = \kappa_1(\kappa_1+\kappa_2)\int_{\mathbb{R}^3} \partial_i^2\Big(\nabla c^n\cdot\nabla\tau^n + \nabla\tau^n\cdot\nabla\tau^n + \nabla\tau^n\cdot\nabla(\tau^n c^n)\Big)\,\partial_i^2\tau^{n+1}\,\mathrm{d}x,$$
$$J_4 = -\int_{\mathbb{R}^3} \partial_i^2(\widetilde{\kappa}_3(\tau^n)\nabla\tau^n)\cdot\nabla\partial_i^2\tau^{n+1}\,\mathrm{d}x, \quad J_5 = \kappa_2\int_{\mathbb{R}^3} \partial_i^2(c^n\partial_t\tau^n)\ \partial_i^2\tau^{n+1}\,\mathrm{d}x.$$

By choosing a suitable $\delta$ as in (3.10), a linear combination of (3.17) and (3.18) implies that

$$\begin{aligned}&\kappa_1(1+\delta)\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2 c^{n+1}\|_{L^2}^2 + \kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2\tau^{n+1}\|_{L^2}^2\\ &\quad + \delta\kappa_1^2\|\nabla\partial_i^2 c^{n+1}\|_{L^2}^2 + (2\bar{\kappa}_3 - \delta\kappa_1^2)\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2\\ &\le 2\left(\kappa_1(1+\delta)J_1 + J_2 + J_3 + J_4 + J_5\right).\end{aligned}\tag{3.19}$$

By Hölder's inequality and Sobolev embedding, one has that

$$\begin{aligned}J_1 &\le C\kappa_1\Big(\|\tau\|_{L^\infty}\|\nabla\partial_i^2 c^n\|_{L^2} + \|\nabla\partial_i c^n\|_{L^6}\|\partial_i\tau^n\|_{L^3}\\ &\quad + \|\nabla c^n\|_{L^3}\|\partial_i^2\tau^n\|_{L^6} + \|c^n\|_{L^\infty}\|\nabla\partial_i^2\tau^n\|_{L^2}\\ &\quad + \|\nabla\partial_i\tau^n\|_{L^6}\|\partial_i c^n\|_{L^3} + \|\nabla\tau^n\|_{L^3}\|\partial_i^2 c^n\|_{L^6}\Big)\|\nabla\partial_i^2 c^{n+1}\|_{L^2}\\ &\le C\kappa_1\Big(\|\tau^n\|_{H^2}\|\nabla c^n\|_{H^2} + \|c^n\|_{H^2}\|\nabla\tau^n\|_{H^2}\Big)\|\nabla\partial_i^2 c^{n+1}\|_{L^2}\\ &\le \frac{C}{\delta}\Big(\|\tau^n\|_{H^2}^2\|\nabla c^n\|_{H^2}^2 + \|c^n\|_{H^2}^2\|\nabla\tau^n\|_{H^2}^2\Big) + \frac{1}{32}\delta\kappa_1^2\|\nabla\partial_i^2 c^{n+1}\|_{L^2}^2.\end{aligned}\tag{3.20}$$

Similarly, we have that

$$J_2 \le \frac{C\kappa_1^2}{\delta}\Big(\|\tau^n\|_{H^2}^2\|\nabla c^n\|_{H^2}^2 + \|c^n\|_{H^2}^2\|\nabla\tau^n\|_{H^2}^2\Big) + \frac{1}{32}\delta\kappa_1^2\|\nabla\partial_i^2 c^{n+1}\|_{L^2}^2.\tag{3.21}$$

and

$$\begin{aligned}J_3 &\le C(\kappa_1^2+\kappa_2^2)\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}\Big(\|\nabla\tau^n\|_{L^3}\|\nabla\partial_i c^n\|_{L^6} + \|\nabla c^n\|_{L^3}\|\nabla\partial_i\tau^n\|_{L^6}\\ &\quad + \|\nabla\tau^n\|_{L^3}\|\nabla\partial_i\tau^n\|_{L^6} + \|c^n\|_{L^\infty}\|\nabla\tau^n\|_{L^3}\|\nabla\partial_i\tau^n\|_{L^6}\\ &\quad + \|\tau^n\|_{L^\infty}\|\nabla c^n\|_{L^3}\|\nabla\partial_i\tau^n\|_{L^6} + \|\tau^n\|_{L^\infty}\|\nabla\tau^n\|_{L^3}\|\nabla\partial_i c^n\|_{L^6}\\ &\quad + \|\nabla c^n\|_{L^\infty}\|\nabla\tau^n\|_{L^3}\|\nabla\tau^n\|_{L^6}\Big)\\ &\le C(\kappa_1^2+\kappa_2^2)\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}\Big(\|\tau^n\|_{H^2} + \|c^n\|_{H^2} + \|\tau^n\|_{H^2}^2 + \|c^n\|_{H^2}^2\Big)\\ &\quad \times\Big(\|\nabla\partial_i^2 c^n\|_{L^2} + \|\nabla\partial_i^2\tau^n\|_{L^2} + \|\nabla^2\tau^n\|_{L^2}\Big)\\ &\le \frac{C(\kappa_1^2+\kappa_2^2)^2}{\delta\kappa_1^2}\Big(\|\tau^n\|_{H^2}^2 + \|c^n\|_{H^2}^2 + \|\tau^n\|_{H^2}^4 + \|c^n\|_{H^2}^4\Big)\\ &\quad \times\Big(\|\nabla\partial_i^2 c^n\|_{L^2}^2 + \|\nabla\partial_i^2\tau^n\|_{L^2}^2 + \|\nabla^2\tau^n\|_{L^2}^2\Big) + \frac{1}{2}\delta\kappa_1^2\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2.\end{aligned}\tag{3.22}$$

Similarly, for $J_4$, we have that

$$\begin{aligned}J_4 &\le C\|\tau^n\|_{L^\infty}\|\nabla\partial_i^2\tau^n\|_{L^2}\|\nabla\partial_i^2\tau^{n+1}\|_{L^2} + \|\partial_i\tau^n\|_{L^3}\|\nabla\partial_i\tau^n\|_{L^6}^2\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}\\ &\quad + \|\partial_i\tau^n\|_{L^\infty}\|\partial_i\tau^n\|_{L^3}^2\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}\\ &\le \frac{C}{\delta\kappa_1^2}\|\tau^n\|_{H^2}^2\|\nabla\partial_i\tau^n\|_{L^2}^2 + \frac{1}{2}\delta\kappa_1^2\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2.\end{aligned}\tag{3.23}$$

As for the last term $J_5$, we split it into two pieces and roughly estimate these as

$$\begin{aligned} J_5 &= \kappa_2 \int_{\mathbb{R}^3} \partial_i^2(c^n \partial_t \tau^n)\ \partial_i^2 \tau^{n+1}\, \mathrm{d}x \\ &= -\kappa_2 \int_{\mathbb{R}^3} \partial_i c^n\ \partial_t \tau^n\ \partial_i^3 \tau^{n+1}\, \mathrm{d}x - \kappa_2 \int_{\mathbb{R}^3} c^n\ \partial_t \partial_i \tau^n\ \partial_i^3 \tau^{n+1}\, \mathrm{d}x \\ &\le C(\kappa_1,\kappa_2)\Big(\|c^n\|_{H^2}^2 \|\partial_t \tau^n\|_{H^1}^2\Big) + \frac{1}{2}\delta \kappa_1^2 \|\nabla \partial_i^2 \tau^{n+1}\|_{L^2}^2. \end{aligned} \tag{3.24}$$

To close the above $\dot{H}^2$ energy estimate we will need additional bounds on $\partial_t \tau^{n+1}$ and $\partial_t \partial_i \tau^{n+1}$. For that, applying $\partial_i^k$ with $k=0,1$ and $i=1,2,3$ to the equation for $\tau^{n+1}$, then taking the $L^2$ inner product with $\partial_t \partial_i^k \tau^{n+1}$, we obtain that

$$\begin{aligned} &\kappa_2 \|\partial_t \partial_i^k \tau^{n+1}\|_{L^2}^2 + \frac{1}{2}(\kappa_1^2 + \bar{\kappa}_3)\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \partial_i^k \tau^{n+1}\|_{L^2}^2 \\ \le{}& C\frac{\kappa_1^4}{\kappa_2}\|\Delta \partial_i^k c^{n+1}\|_{L^2}^2 + C\frac{\kappa_1^4}{\kappa_2}\|\Delta \partial_i^k (c^n \tau^n)\|_{L^2} + C\|\partial_i^k\big(\nabla c^n \cdot \nabla \tau^n\big)\|_{L^2}^2 \\ &+ C\|\partial_i^k\big(\nabla \tau^n \cdot \nabla \tau^n\big)\|_{L^2}^2 + C\|\partial_i^k\big(\nabla \tau^n \cdot \nabla(\tau^n c^n)\big)\|_{L^2}^2 \\ &+ C\frac{1}{\kappa_2}\|\partial_i^k \nabla\big(\widetilde{\kappa}_3(\tau^n)\nabla \tau^n\big)\|_{L^2}^2 + \kappa_2 \|\partial_i^k\big(c^n \partial_t \tau^n\big)\|_{L^2}^2 + \frac{1}{2}\kappa_2 \|\partial_t \partial_i^k \tau^{n+1}\|_{L^2}^2, \end{aligned} \tag{3.25}$$

which implies that

$$\begin{aligned} &\frac{1}{2}\kappa_2 \|\partial_t \partial_i^k \tau^{n+1}\|_{L^2}^2 + \frac{1}{2}(\kappa_1^2 + \bar{\kappa}_3)\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \partial_i^k \tau^{n+1}\|_{L^2}^2 \\ \le{}& C\frac{\kappa_1^4}{\kappa_2}\|\Delta \partial_i^k c^{n+1}\|_{L^2}^2 + C\frac{\kappa_1^4}{\kappa_2}\|\Delta \partial_i^k (c^n \tau^n)\|_{L^2}^2 + C\|\partial_i^k\big(\nabla c^n \cdot \nabla \tau^n\big)\|_{L^2}^2 \\ &+ C\|\partial_i^k\big(\nabla \tau^n \cdot \nabla \tau^n\big)\|_{L^2}^2 + C\|\partial_i^k\big(\nabla \tau^n \cdot \nabla(\tau^n c^n)\big)\|_{L^2}^2 \\ &+ C\frac{1}{\kappa_2}\|\partial_i^k \nabla\big(\widetilde{\kappa}_3(\tau^n)\nabla \tau^n\big)\|_{L^2}^2 + \kappa_2 \|\partial_i^k\big(c^n \partial_t \tau^n\big)\|_{L^2}^2. \end{aligned} \tag{3.26}$$

As in the $\dot{H}^2$ estimate, for the right hand side of (3.26), we have that

$$\begin{aligned} &\|\Delta \partial_i^k (c^n \tau^n)\|_{L^2}^2 + \|\partial_i^k (\nabla c^n \nabla \tau^n)\|_{L^2}^2 + \|\partial_i^k (\nabla \tau^n \nabla \tau^n)\|_{L^2}^2 \\ \le{}& C\Big(\|c^n\|_{H^2}^2 + \|\tau^n\|_{H^2}^2\Big)\Big(\|\nabla c^n\|_{H^2}^2 + \|\nabla \tau^n\|_{H^2}^2\Big), \end{aligned} \tag{3.27}$$

$$\|\partial_i^k \nabla\big(\widetilde{\kappa}_3(\tau^n)\nabla \tau^n\big)\|_{L^2}^2 \le C\|\tau^n\|_{H^2}^2 \|\nabla \tau^n\|_{H^2}^2, \tag{3.28}$$

$$\|\partial_i^k\big(c^n \partial_t \tau^n\big)\|_{L^2}^2 \le C\|c^n\|_{H^2}^2 \|\partial_t \tau^n\|_{H^1}^2, \tag{3.29}$$

and

$$\|\partial_i^k\big(\nabla \tau^n \cdot \nabla(\tau^n c^n)\big)\|_{L^2}^2 \le C\Big(||c^n\|_{H^2}^2 \|\tau^n\|_{H^2}^2 + \|\tau^n\|_{H^2}^4\Big)\|\nabla c^n\|_{H^2}^2. \tag{3.30}$$

Multiplying by $\frac{\kappa_2}{4C\kappa_1^2}\delta$ on both sides of (3.26) and combining the resulting inequality with (3.27)–(3.30) and (3.19)–(3.24), we get that

$$\begin{aligned} &\|\partial_i^2 c^{n+1}\|_{L^2}^2 + \|\partial_i^2 \tau^{n+1}\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \partial_i^k \tau^{n+1}\|_{L^2}^2 \\ &+ \|\nabla \partial_i^2 c^{n+1}\|_{L^2}^2 + \|\nabla \partial_i^2 \tau^{n+1}\|_{L^2}^2 + \|\partial_t \partial_i^k \tau^{n+1}\|_{L^2}^2 \\ \le{}& C(\kappa_1,\kappa_2,\bar{\kappa}_3)\Big(\|\tau^n\|_{H^2}^2 + \|c^n\|_{H^2}^2 + \|\tau^n\|_{H^2}^4 + \|c^n\|_{H^2}^4\Big) \\ &\times \Big(\|\nabla c\|_{H^2}^2 + \|\nabla \tau\|_{H^2}^2 + \|\partial_t \tau^n\|_{H^1}^2\Big). \end{aligned} \tag{3.31}$$

Integrating (3.31) over $[0,T]$ in time, using assumption (3.7), and combining these with the $L^2$ estimate and Young's inequality, we finally have that

$$\|(c^{n+1},\tau^{n+1})\|_{X(T)}^2 \le C\alpha^2 + CM^2\alpha^4 + CM^3\alpha^6.$$

For $M$ sufficiently large and $\alpha$ sufficiently small (compared to $C$), we see that (3.7) holds for $(c^{n+1},\tau^{n+1})$, and therefore for all $n\in\mathbb{N}$.

### 3.2 Cauchy Sequence in $X(T)$

In this part, we shall show that the sequence $\{c^n,\tau^n\}_{n\in\mathbb{N}}$ is Cauchy in $X(T)$. For this, we consider the difference between two solutions $(\delta c^{n+1},\delta\tau^{n+1})$ with

$$\delta c^{n+1} = c^{n+2} - c^{n+1}, \quad \delta\tau^{n+1} = \tau^{n+2} - \tau^{n+1}.$$

Then $(\delta c^{n+1},\delta\tau^{n+1})$ satisfies that

$$\begin{cases} \partial_t\delta c^{n+1} - \kappa_1\Delta\delta c^{n+1} - \kappa_1\Delta\delta\tau^{n+1} = \delta H^n, \\ \kappa_2\partial_t\delta\tau^{n+1} - (\kappa_1^2+\bar{\kappa}_3)\Delta\delta\tau^{n+1} - \kappa_1^2\Delta\delta c^{n+1} = \delta I^n, \\ (\delta c^{n+1},\delta\tau^{n+1})\mid_{t=0} = (\delta c_{n+1},\delta\tau_{n+1}) \end{cases} \tag{3.32}$$

with

$$(\delta c_n,\delta\tau_n) = ((S_{n+1}-S_n)c_0, (S_{n+1}-S_n)\tau_0), \tag{3.33}$$

$$\delta H^n = \kappa_1\Delta(c^{n+1}\tau^{n+1}) - \kappa_1\Delta(c^n\tau^n), \tag{3.34}$$

$$\begin{aligned} \delta I^n = {} & \kappa_1(\kappa_1+\kappa_2)\Big(\nabla\tau^{n+1}\cdot\nabla c^{n+1} + \nabla\tau^{n+1}\cdot\nabla\tau^{n+1} + \nabla\tau^{n+1}\nabla(\tau^{n+1}c^{n+1})\Big) \\ & - \kappa_1(\kappa_1+\kappa_2)\Big(\nabla\tau^n\cdot\nabla c^n + \nabla\tau^n\cdot\nabla\tau^n + \nabla\tau^n\cdot\nabla(\tau^n c^n)\Big) \\ & + \kappa_1^2\Delta(c^{n+1}\tau^{n+1}) - \kappa_1^2\Delta(c^n\tau^n) + \nabla\cdot(\widetilde{\kappa}_3(\tau^{n+1})\nabla\tau^{n+1}) \\ & - \nabla\cdot(\widetilde{\kappa}_3(\tau^n)\nabla\tau^n) - \Big(\kappa_2 c^{n+1}\partial_t\tau^{n+1} - \kappa_2 c^n\partial_t\tau^n\Big). \end{aligned} \tag{3.35}$$

Observe that each term in $H^n$ and $I^n$ from (3.3) is either quadratic or cubic in $(c^n,\tau^n)$. After taking the differences and computing the $X(T)$-norm, we will essentially find that

$$\|(\delta c^{n+1},\delta\tau^{n+1})\|_{X(T)}^2 \le C\times\sup_{k\in\mathbb{N}}\big(\|(c^k,\tau^k)\|_{X(T)}\big)\times\|(\delta c^n,\delta\tau^n)\|_{X(T)}^2.$$

Since the sequence $\{(c^n,\tau^n)\}_{n\in\mathbb{N}}$ is bounded by (3.7) uniformly in $n$, we can choose $\alpha$ sufficiently small such that the above inequality produces a geometrically convergent Cauchy sequence.

In the remainder of this section, we will perform this lengthy calculation. We follow the same two steps as those used in Section 3.1 to show (3.7).

**Step 1** $L^2$ energy estimate

Taking the $L^2$ inner product with $\delta c^{n+1}$ and $\delta\tau^{n+1}$ with respect to the first and second equations, we arrive at

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c^{n+1}\|_{L^2}^2 + \kappa_1\|\nabla\delta c^{n+1}\|_{L^2}^2 \\ = & -\kappa_1\int_{\mathbb{R}^3}\nabla\delta\tau^{n+1}\cdot\nabla\delta c^{n+1}\,\mathrm{d}x + \kappa_1\int_{\mathbb{R}^3}\delta H^n\cdot\delta c^{n+1}\,\mathrm{d}x \end{aligned} \tag{3.36}$$

and

$$\frac{1}{2}\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\tau^{n+1}\|_{L^2}^2 + (\kappa_1^2+\bar{\kappa}_3)\|\nabla\delta\tau^{n+1}\|_{L^2}^2$$

$$= -\kappa_1^2 \int_{\mathbb{R}^3} \nabla\delta\tau^{n+1} \cdot \delta\nabla c^{n+1}\,\mathrm{d}x + \int_{\mathbb{R}^3} \delta I^n \cdot \delta\tau^{n+1}\,\mathrm{d}x. \tag{3.37}$$

By using the Hölder and Cauchy inequalities, a linear combination of (3.36) and (3.37) yields that

$$\begin{aligned}
&\frac{1}{2}\kappa_1(1+\delta)\frac{\mathrm{d}}{\mathrm{d}t}\|\delta c^{n+1}\|_{L^2}^2 + \frac{1}{2}\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\tau^{n+1}\|_{L^2}^2 \\
&+ \frac{1}{2}\delta\kappa_1^2\|\nabla\delta c^{n+1}\|_{L^2}^2 + (\bar{\kappa}_3 - \frac{1}{2}\delta\kappa_1^2)\|\nabla\delta\tau^{n+1}\|_{L^2}^2 \\
\le{}& \kappa_1(1+\delta)\int_{\mathbb{R}^3} \delta H^n \cdot \delta c^{n+1}\,\mathrm{d}x + \int_{\mathbb{R}^3} \delta I^n \cdot \delta\tau^{n+1}\,\mathrm{d}x.
\end{aligned} \tag{3.38}$$

To bound $\int_{\mathbb{R}^3} \delta H^n \cdot \delta c^{n+1}\,\mathrm{d}x$, we first rewrite this as

$$\begin{aligned}
\int_{\mathbb{R}^3} \delta H^n \cdot \delta c^{n+1}\,\mathrm{d}x &= -\kappa_1 \int_{\mathbb{R}^3} \nabla(c^{n+1}\tau^{n+1}) - \nabla(c^n\tau^n) \cdot \nabla\delta c^{n+1}\,\mathrm{d}x \\
&= -\kappa_1 \int_{\mathbb{R}^3} \Big((\nabla c^{n+1} - \nabla c^n)\tau^{n+1} + \nabla c^n(\tau^{n+1} - \tau^n)\Big) \cdot \nabla\delta c^{n+1}\,\mathrm{d}x \\
&\quad - \kappa_1 \int_{\mathbb{R}^3} \Big((c^{n+1} - c^n)\nabla\tau^{n+1} + c^n(\nabla\tau^{n+1} - \nabla\tau^n)\Big) \cdot \nabla\delta c^{n+1}\,\mathrm{d}x \\
&:= K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

We estimate the $K_i$ terms as follows:

$$\begin{aligned}
K_1 &:= -\kappa_1 \int_{\mathbb{R}^3} (\nabla c^{n+1} - \nabla c^n)\tau^{n+1} \cdot \nabla\delta c^{n+1}\,\mathrm{d}x \\
&\le C\|\tau^{n+1}\|_{L^\infty}\|\nabla\delta c^n\|_{L^2}\|\nabla\delta c^{n+1}\|_{L^2} \\
&\le C\|\tau^{n+1}\|_{H^2}^2\|\nabla\delta c^n\|_{L^2}^2 + \frac{1}{32}\delta\kappa_1^2\|\nabla\delta c^{n+1}\|_{L^2}.
\end{aligned} \tag{3.39}$$

Similarly, $K_2$, $K_3$, and $K_4$ can be bounded by

$$\begin{aligned}
K_2 + K_3 + K_4 \le{}& C\Big(\|\nabla c^n\|_{L^2}^2\|\delta\tau^n\|_{H^2}^2 + \|\delta c^n\|_{H^2}^2\|\nabla\tau^{n+1}\|_{L^2}^2 \\
&+ \|c^n\|_{H^2}^2\|\nabla\delta\tau^n\|_{L^2}^2\Big) + \frac{1}{32}\delta\kappa_1^2\|\nabla\delta c^{n+1}\|_{L^2}.
\end{aligned} \tag{3.40}$$

For $\int_{\mathbb{R}^3} \delta I^n \cdot \delta\tau^{n+1}\,\mathrm{d}x$, we first rewrite $\delta I^n$ as

$$\begin{aligned}
\delta I^n ={}& \kappa_1(\kappa_1+\kappa_2)\Big(\nabla\delta\tau^n \cdot \nabla c^{n+1} + \nabla\tau^n \cdot \nabla\delta c^n + \nabla\delta\tau^n \cdot (\nabla\tau^{n+1} + \nabla\tau^n)\Big) \\
&+ \kappa_1(\kappa_1+\kappa_2)\Big(\nabla\delta\tau^n \cdot \nabla\tau^{n+1}c^{n+1} + \nabla\tau^n \cdot \nabla\delta\tau^n c^{n+1} + \nabla\tau^n \cdot \nabla\tau^n\delta c^n\Big) \\
&+ \kappa_1(\kappa_1+\kappa_2)\Big(\nabla\delta\tau^n \cdot \nabla c^{n+1}\tau^{n+1} + \nabla\tau^n \cdot \nabla c^{n+1}\delta\tau^n + \nabla\tau^n \cdot \nabla\delta c^n\tau^n\Big) \\
&+ \kappa_1^2\nabla \cdot \Big(\nabla\delta c^n\tau^{n+1} + \nabla c^n\delta\tau^n + \delta c^n\nabla\tau^{n+1} + c^n\nabla\delta\tau^n\Big) \\
&+ \nabla \cdot \Big(\big(\widetilde{\kappa}_3(\tau^{n+1}) - \widetilde{\kappa}_3(\tau^n)\big)\nabla\tau^{n+1} + \widetilde{\kappa}_3(\tau^n)\nabla\delta\tau^n\Big) \\
&- \kappa_2\delta c^n\partial_t\tau^{n+1} - \kappa_2 c^n\partial_t\delta\tau^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^3} \delta I^n \cdot \delta\tau^{n+1}\,\mathrm{d}x \le{}& C\Big(\|\nabla\delta\tau^n\|_{L^2}\|\nabla\tau^{n+1}\|_{L^3}\|c^{n+1}\|_{L^\infty} + \|\nabla\delta\tau^n\|_{L^2}\|\nabla\tau^n\|_{L^3}\|c^{n+1}\|_{L^\infty} \\
&+ \|\nabla\tau^n\|_{L^2}\|\nabla\tau^n\|_{L^3}\|\delta c^n\|_{L^\infty} + \|\nabla\delta\tau^n\|_{L^2}\|\nabla c^{n+1}\|_{L^3}\|\tau^{n+1}\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&+\|\delta\tau^n\|_{L^\infty}\|\nabla\tau^n\|_{L^3}\|\nabla c^{n+1}\|_{L^2}+\|\nabla\tau^n\|_{L^3}\|\nabla\delta c^n\|_{L^2}\|\tau^n\|_{L^\infty}\\
&+\|\nabla\delta\tau^n\|_{L^2}\|\nabla c^{n+1}\|_{L^3}+\|\nabla\delta c^n\|_{L^2}\|\nabla\tau^n\|_{L^3}\\
&+\|\nabla\delta\tau^n\|_{L^2}\|\nabla\tau^{n+1}\|_{L^3}+\|\nabla\delta\tau^n\|_{L^2}\|\nabla\tau^n\|_{L^3}\\
&+\|\delta c^n\|_{L^3}\|\partial_t\tau^{n+1}\|_{L^2}+\|c^n\|_{L^3}\|\partial_t\delta\tau^n\|_{L^2}\Big)\|\delta\tau^{n+1}\|_{L^6}\\
&+C\Big(\|\nabla\delta c^n\|_{L^2}\|\tau^{n+1}\|_{L^\infty}+\|\nabla c^n\|_{L^2}\|\delta\tau^n\|_{L^\infty}\\
&+\|\delta c^n\|_{L^\infty}\|\nabla\tau^{n+1}\|_{L^2}+\|c^n\|_{L^\infty}\|\nabla\delta\tau^n\|_{L^2}\\
&+\|\delta\tau^n\|_{L^\infty}\|\nabla\tau^{n+1}\|_{L^2}+\|\tau^n\|_{L^\infty}\|\nabla\delta\tau^n\|_{L^2}\Big)\|\nabla\delta\tau^{n+1}\|_{L^2},
\end{aligned}\tag{3.41}
$$

where in the estimation of terms involving $\widetilde{\kappa}_3(\tau)$ we have used Taylor's formula and Hölder's inequality. Plugging estimates (3.39)–(3.41) into (3.38), integrating over $[0,T]$, and using the uniform bound (3.7) found in the previous subsection, we get that

$$
\begin{aligned}
&\|\delta c^{n+1}\|_{L^2}^2+\|\delta\tau^{n+1}\|_{L^2}^2+\int_0^T\big(\|\nabla\delta c^{n+1}\|_{L^2}^2+\|\nabla\delta\tau^{n+1}\|_{L^2}^2\big)\mathrm{d}t\\
\le{}& C\|(\delta c_{n+1},\delta\tau_{n+1})\|_{L^2}^2+CM\alpha^2\|(\delta c^n,\delta\tau^n)\|_{X(T)}\\
&+CM^2\alpha^4\|(\delta c^n,\delta\tau^n)\|_{X(T)}+\frac14\|(\delta c^{n+1},\delta\tau^{n+1})\|_{X(T)}.
\end{aligned}\tag{3.42}
$$

**Step 2** $\dot H^2$ energy estimate

As in the proof of the uniform bound, applying $\partial_i^2$ for $i=1,2,3$ to (3.32) and then taking the $L^2$ inner product with $(\partial_i^2\delta c^{n+1},\partial_i^2\delta\tau^{n+1})$, we find that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2\delta c^{n+1}\|_{L^2}^2+\kappa_1\|\nabla\partial_i^2\delta c^{n+1}\|_{L^2}^2\\
={}&-\kappa_1\int_{\mathbb{R}^3}\nabla\partial_i^2\delta\tau^{n+1}\cdot\nabla\partial_i^2\delta c^{n+1}\,\mathrm{d}x+\kappa_1\int_{\mathbb{R}^3}\partial_i^2\delta H^n\cdot\partial_i^2\delta c^{n+1}\,\mathrm{d}x\\
\le{}&\frac12\kappa_1\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2+\frac12\kappa_1\|\nabla\partial_i^2c^{n+1}\|_{L^2}^2+\int_{\mathbb{R}^3}\partial_i^2\delta H^n\cdot\partial_i^2\delta c^{n+1}\,\mathrm{d}x.
\end{aligned}\tag{3.43}
$$

and

$$
\begin{aligned}
&\frac12\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2\delta\tau^{n+1}\|_{L^2}^2+(\kappa_1^2+\bar\kappa_3)\|\nabla\partial_i^2\delta\tau^{n+1}\|_{L^2}^2\\
={}&-k_1^2\int_{\mathbb{R}^3}\nabla\partial_i^2\delta\tau^{n+1}\cdot\partial_i^2\nabla\delta c^{n+1}\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_i^2I^n\cdot\nabla\partial_i^2\tau^{n+1}\,\mathrm{d}x\\
\le{}&\frac12\kappa_1^2\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2+\frac12\kappa_1^2\|\nabla\partial_i^2c^{n+1}\|_{L^2}^2+\int_{\mathbb{R}^3}\partial_i^2I^n\cdot\nabla\partial_i^2\tau^{n+1}\,\mathrm{d}x.
\end{aligned}\tag{3.44}
$$

Choosing a suitable $\delta$, as was done in Section 3.1, yields, through a linear combination of (3.17) and (3.18), that

$$
\begin{aligned}
&\frac12\kappa_1(1+\delta)\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2c^{n+1}\|_{L^2}^2+\frac12\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2\tau^{n+1}\|_{L^2}^2\\
&+\frac12\delta\kappa_1^2\|\nabla\partial_i^2c^{n+1}\|_{L^2}^2+(\bar\kappa_3-\frac12\delta\kappa_1^2)\|\nabla\partial_i^2\tau^{n+1}\|_{L^2}^2\\
\le{}&C\int_{\mathbb{R}^3}\partial_i^2\delta H^n\cdot\partial_i^2\delta c^{n+1}\,\mathrm{d}x+C\int_{\mathbb{R}^3}\partial_i^2\delta I^n\cdot\partial_i^2\tau^{n+1}\,\mathrm{d}x.
\end{aligned}\tag{3.45}
$$

Recalling the expressions of $\delta H^n$ and $\delta I^n$, by Hölder's inequality, the Sobolev embedding, and

Cauchy's inequality, we get that

$$
\begin{aligned}
\int_{\mathbb{R}^3} \partial_i^2 \delta H^n \cdot \partial_i^2 \delta c^{n+1}\,\mathrm{d}x \le{}& \|\partial_i^2 \nabla \delta c^n\|_{L^2}^2 \|\tau^{n+1}\|_{L^\infty}^2 + \|\partial_i \nabla \delta c^n\|_{L^6}^2 \|\partial_i \tau^{n+1}\|_{L^3}^2 \\
&+ \|\nabla \delta c^n\|_{L^6}^2 \|\partial_i^2 \tau^{n+1}\|_{L^3}^2 + \|\partial_i^2 \nabla c^n\|_{L^2}^2 \|\delta \tau^n\|_{L^\infty}^2 \\
&+ \|\partial_i \nabla \delta c^n\|_{L^6}^2 \|\partial_i \tau^n\|_{L^3}^2 + \|\nabla c^n\|_{L^6}^2 \|\partial_i^2 \delta \tau^n\|_{L^3}^2 \\
&+ \|\partial_i^2 \delta c^n\|_{L^6}^2 \|\nabla \tau^{n+1}\|_{L^3}^2 + \|\partial_i \delta c^n\|_{L^6}^2 \|\partial_i \nabla \tau^{n+1}\|_{L^3}^2 \\
&+ \|\delta c^n\|_{L^\infty}^2 \|\partial_i^2 \nabla \tau^{n+1}\|_{L^2}^2 + \|\partial_i^2 c^n\|_{L^3}^2 \|\nabla \delta \tau^n\|_{L^6}^2 \\
&+ \|\partial_i \delta c^n\|_{L^6}^2 \|\partial_i \nabla \tau^n\|_{L^3}^2 + \|c^n\|_{L^\infty}^2 \|\partial_i^2 \nabla \delta \tau^n\|_{L^2}^2 \\
&+ \frac{1}{4}\|\partial_i^2 \nabla \delta c^{n+1}\|_{L^2}. \qquad (3.46)
\end{aligned}
$$

The estimate on the term involving $\delta I^n$ is similar to (3.46). For the sake of brevity, we omit the details. Plugging the resulting inequality and (3.46) into (3.45) and performing the time integration over $[0,T]$, we then have that

$$
\begin{aligned}
&\|\partial_i^2 \delta c^{n+1}\|_{L^2}^2 + \|\partial_i^2 \delta \tau^{n+1}\|_{L^2}^2 + \int_0^T \Big(\|\nabla \partial_i^2 c^{n+1}\|_{L^2}^2 + \|\nabla \partial_i^2 \tau^{n+1}\|_{L^2}^2\Big)\mathrm{d}t \\
\le{}& C\|(\delta c_{n+1}, \delta \tau_{n+1})\|_{H^2}^2 + CM\alpha^2 \|(\delta c^n, \delta \tau^n)\|_{X(T)}^2 \\
&+ CM^2\alpha^4 \|(\delta c^n, \delta \tau^n)\|_{X(T)}^2 + \frac{1}{4}\|(\delta c^{n+1}, \delta \tau^{n+1})\|_{X(T)}^2. \qquad (3.47)
\end{aligned}
$$

As before, we still require additional estimates on $\partial_t \delta \tau^{n+1}$ and $\partial_t \partial_i \delta \tau^{n+1}$. Applying $\partial_i^k$ with $k = 0, 1$ and $i = 1, 2, 3$ to the equation for $\delta \tau^{n+1}$, taking the $L^2$ inner product with $\partial_t \partial_i^k \delta \tau^{n+1}$, one has that

$$
\begin{aligned}
&\kappa_2 \|\partial_t \partial_i^k \delta \tau^{n+1}\|_{L^2}^2 + \frac{1}{2}(\kappa_1^2 + \bar{\kappa}_3)\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \delta \partial_i^k \tau^{n+1}\|_{L^2}^2 \\
={}& \kappa_1^2 \int_{\mathbb{R}^3} \Delta \partial_i^k \delta c^{n+1} \cdot \partial_t \partial_i^k \delta \tau^{n+1}\,\mathrm{d}x + \int_{\mathbb{R}^3} \partial_i^k \delta I^n \cdot \partial_t \partial_i^k \delta \tau^{n+1}\,\mathrm{d}x. \qquad (3.48)
\end{aligned}
$$

which implies that

$$
\|\partial_t \partial_i^k \tau^{n+1}\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \partial_i^k \tau^{n+1}\|_{L^2}^2 \le C\|\Delta \partial_i^k c^{n+1}\|_{L^2}^2 + C\int_{\mathbb{R}^3} \partial_i^k \delta I^n \cdot \partial_t \partial_i^k \delta \tau^{n+1}\,\mathrm{d}x. \qquad (3.49)
$$

As in the $\dot{H}^2$ estimate, for the right hand side of (3.49), we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \partial_i \delta I^n \cdot \partial_t \partial_i \delta \tau^{n+1}\,\mathrm{d}x \\
\le{}& C\Big(\|\nabla \partial_i \delta \tau^n\|_{L^6} \|\nabla \tau^{n+1}\|_{L^3} \|c^{n+1}\|_{L^\infty} + \|\nabla \delta \tau^n\|_{L^6} \|\nabla \partial_i \tau^{n+1}\|_{L^3} \|c^{n+1}\|_{L^\infty} \\
&+ \|\nabla \delta \tau^n\|_{L^6} \|\nabla \tau^{n+1}\|_{L^6} \|\partial_i c^{n+1}\|_{L^6} + \|\nabla \partial_i \delta \tau^n\|_{L^6} \|\nabla \tau^n\|_{L^3} \|c^{n+1}\|_{L^\infty} \\
&+ \|\nabla \delta \tau^n\|_{L^6} \|\nabla \partial_i \tau^n\|_{L^3} \|c^{n+1}\|_{L^\infty} + \|\nabla \delta \tau^n\|_{L^6} \|\nabla \tau^n\|_{L^6} \|\partial_i c^{n+1}\|_{L^6} \\
&+ \|\nabla \partial_i \tau^n\|_{L^6} \|\nabla \tau^n\|_{L^3} \|\delta c^n\|_{L^\infty} + \|\nabla \tau^n\|_{L^6} \|\nabla \tau^n\|_{L^6} \|\partial_i \delta c^n\|_{L^6} \\
&+ \|\nabla \partial_i \delta \tau^n\|_{L^6} \|\nabla c^{n+1}\|_{L^3} \|\tau^{n+1}\|_{L^\infty} + \|\nabla \delta \tau^n\|_{L^2} \|\nabla \partial_i c^{n+1}\|_{L^3} \|\tau^{n+1}\|_{L^\infty} \\
&+ \|\nabla \delta \tau^n\|_{L^6} \|\nabla c^{n+1}\|_{L^6} \|\partial_i \tau^{n+1}\|_{L^6} + \|\partial_i \delta \tau^n\|_{L^6} \|\nabla \tau^n\|_{L^6} \|\nabla c^{n+1}\|_{L^6} \\
&+ \|\delta \tau^n\|_{L^6} \|\nabla \partial_i \tau^n\|_{L^6} \|\nabla c^{n+1}\|_{L^6} + \|\delta \tau^n\|_{L^6} \|\nabla \tau^n\|_{L^6} \|\nabla \partial_i c^{n+1}\|_{L^6} \\
&+ \|\nabla \partial_i \tau^n\|_{L^3} \|\nabla \delta c^n\|_{L^6} \|\tau^n\|_{L^\infty} + \|\nabla \tau^n\|_{L^6} \|\nabla \partial_i \delta c^n\|_{L^3} \|\tau^n\|_{L^\infty} \\
&+ \|\nabla \tau^n\|_{L^6} \|\nabla \delta c^n\|_{L^6} \|\partial_i \tau^n\|_{L^6} + \|\nabla \partial_i \delta \tau^n\|_{L^6} \|\nabla c^{n+1}\|_{L^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla\delta\tau^n\|_{L^6}\|\nabla\partial_i c^{n+1}\|_{L^3}+\|\nabla\partial_i\delta c^n\|_{L^6}\|\nabla\tau^n\|_{L^3}+\|\nabla\delta c^n\|_{L^6}\|\nabla\partial_i\tau^n\|_{L^3}\\
&+\|\nabla\partial_i\delta\tau^n\|_{L^6}\|\nabla\tau^{n+1}\|_{L^3}+\|\nabla\delta\tau^n\|_{L^6}\|\nabla\partial_i\tau^{n+1}\|_{L^3}+\|\nabla\partial_i\delta\tau^n\|_{L^6}\|\nabla\tau^n\|_{L^3}\\
&+\|\nabla\delta\tau^n\|_{L^6}\|\nabla\partial_i\tau^n\|_{L^3}+\|\partial_i\delta c^n\|_{L^3}\|\partial_t\tau^{n+1}\|_{L^6}+\|\delta c^n\|_{L^\infty}\|\partial_t\partial_i\tau^{n+1}\|_{L^2}\\
&+\|\partial_i c^n\|_{L^3}\|\partial_t\delta\tau^n\|_{L^6}+\|c^n\|_{L^\infty}\|\partial_t\partial_i\delta\tau^n\|_{L^2}+\|\nabla\partial_i\delta c^n\|_{L^2}\|\tau^{n+1}\|_{L^\infty}\\
&+\|\nabla\delta c^n\|_{L^3}\|\partial_i\tau^{n+1}\|_{L^6}+\|\nabla\partial_i c^n\|_{L^2}\|\delta\tau^n\|_{L^\infty}+\|\nabla c^n\|_{L^3}\|\partial_i\delta\tau^n\|_{L^6}\\
&+\|\partial_i\delta c^n\|_{L^6}\|\nabla\tau^{n+1}\|_{L^3}+\|\delta c^n\|_{L^\infty}\|\nabla\partial_i\tau^{n+1}\|_{L^2}+\|\partial_i c^n\|_{L^6}\|\nabla\delta\tau^n\|_{L^3}\\
&+\|c^n\|_{L^6}\|\nabla\partial_i\delta\tau^n\|_{L^3}+\|\partial_i\delta\tau^n\|_{L^6}\|\nabla\tau^{n+1}\|_{L^3}+\|\delta\tau^n\|_{L^\infty}\|\nabla\partial_i\tau^{n+1}\|_{L^2}\\
&+\|\partial_i\tau^n\|_{L^6}\|\nabla\delta\tau^n\|_{L^3}+\|\tau^n\|_{L^\infty}\|\nabla\partial_i\delta\tau^n\|_{L^2}\Big)\|\partial_t\partial_i\delta\tau^{n+1}\|_{L^2}.
\end{aligned}
$$

Inserting the above estimate into (3.39), combining it with (3.47), and integrating the resulting inequality in time, we then use the uniform bound (3.7), combined with the $L^2$ estimate used in Step 1 and Young's inequality to give that

$$
\begin{aligned}
\|(\delta c^{n+1},\delta\tau^{n+1})\|^2_{X(T)}\le{}& C\|(\delta c_{n+1},\delta\tau_{n+1})\|^2_{H^2}+CM\alpha^2\|(\delta c^n,\delta\tau^n)\|^2_{X(T)}\\
&+CM^2\alpha^4\|(\delta c^n,\delta\tau^n)\|^2_{X(T)}.
\end{aligned}\tag{3.50}
$$

From (3.33) and standard Littlewood-Paley theory, we have that

$$
\|(c_0,\tau_0)\|_{H^2}\approx\sum_{j\in\mathbb{Z}}\|(\Delta_j c_0,\Delta_j\tau_0)\|_{H^2}\gtrsim\sum_{n=0}^{\infty}\|(\delta c_n,\delta\tau_n)\|_{H^2}.
$$

Then, if we fix the initial data $(c_0,\tau_0)$ with $\alpha$ being sufficiently small, (3.50) implies that

$$
\|(\delta c^{n+1},\delta\tau^{n+1})\|_{X(T)}\le r_n+\frac{1}{2}\|(\delta c^n,\delta\tau^n)\|_{X(T)},\tag{3.51}
$$

where $r_n=\|(\delta c_n,\delta\tau_n)\|_{H^2}$ is summable. For any $l\ge m\ge n$, we can then conclude (by a triangle inequality and repeated application of (3.51), and using the uniform bound (3.7) at the end) that

$$
\|(c^l,\tau^l)-(c^m,\tau^m)\|_{X(T)}\le 2\sum_{j=n}^{l-1}r_j+\frac{M\alpha^2}{2^{n-m}},
$$

which shows that the sequence $\{(c^n,\tau^n)\}_{n\in\mathbb{N}}$ constructed above is Cauchy.

**Proof of Theorem 1.2** From the uniform estimates obtained in Section 3.1 and the Cauchy sequence in Section 3.2, we can send $n$ to $\infty$ and obtain a limit in $X(T)$, called $(c,\tau)$, with

$$
\begin{aligned}
&c^n\to c\quad\text{in } L^\infty_T(H^2),\quad \nabla c^n\to c\quad\text{in } L^2_T(H^2),\\
&\tau^n\to\tau\quad\text{in } L^\infty_T(H^2),\quad (\Delta\tau^n,\partial_t\tau^n)\to(\Delta\tau,\partial_t\tau)\quad\text{in } L^2_T(H^2).
\end{aligned}\tag{3.52}
$$

Therefore $(\rho,\theta):=(c+1,\tau+1)$ is a classical solution to system (1.1).

The uniqueness follows by examining the system of equations for the difference between two solutions. This system is nearly identical to (3.32), except that the initial data is $(0,0)$. The same strategy as to that above yields a control on the size of the difference in $X(T)$ which, thanks to Grönwall's lemma, proves that the two solutions must be equal. For the sake of brevity, we omit the details. □

## 4 Global Existence for Small Data with Critical Regularity

In this section, we will obtain the global existence of solutions to system (1.1) in Theorem 1.3. From now on, we define the density and the temperature by the form

$$a := \frac{1}{\rho} - 1, \quad \tau := \theta - 1.$$

Then system (1.1) can be rewritten as

$$\begin{cases} \partial_t a - \kappa_1 \Delta a + \kappa_1 \Delta \tau = F, \\ \kappa_2 \partial_t \tau - (\kappa_1^2 + \bar{\kappa}_3)\Delta\tau + \kappa_1^2 \Delta a = G, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} F &= -2\kappa_1 \frac{\tau+1}{1+a}|\nabla a|^2 + 2\kappa_1 \nabla a \cdot \nabla \tau - \kappa_1 a \Delta\tau + \kappa_1 \tau \Delta a, \\ G &= 2\kappa_1^2 \frac{(\tau+1)^2}{(1+a)^2}|\nabla a|^2 - (3\kappa_1^2 + \kappa_1\kappa_2)\frac{(\tau+1)}{1+a}\nabla a \cdot \nabla\tau \\ &\quad - \kappa_1^2 \frac{\tau^2 + 2\tau}{1+a}\Delta a + \kappa_1^2 \frac{a}{1+a}\Delta a + \bar{\kappa}_3 a \Delta \tau + \kappa_1^2 \tau \Delta\tau \\ &\quad + (1+a)\nabla \cdot (\widetilde{\kappa}_3(\tau)\nabla\tau) + \kappa_1(\kappa_1+\kappa_2)|\nabla\tau|^2. \end{aligned} \tag{4.2}$$

Proving the global existence result is based on the following variant of Banach's fixed point theorem, for the proof, we refer, e.g., to [19]:

**Lemma 4.1** Let $X$ be a reflexive Banach space or let $X$ have a separable pre-dual. Let $K$ be a convex, closed and bounded subset of $X$ and assume that $X$ is embedded into a Banach space Y. Let $\Phi: X \longrightarrow X$ map $K$ into $K$ and assume that there exists $c < 1$ such that

$$\|\Phi(x) - \Phi(y)\|_Y \le c\|x-y\|_Y, \quad x, y \in K.$$

Then there exists a unique fixed point of $\Phi$ in $K$.

Based on the natural scaling of system (1.1), we choose our working space to be

$$E(T) := \left\{ u \in \mathcal{C}\left([0,T], \dot{B}^{3/2}_{2,1}\right), \quad \nabla^2 u \in L^1\left(0,T; \dot{B}^{3/2}_{2,1}\right)\right\}, \quad T > 0,$$

with the norm

$$\|u\|_{E(T)} := \|u\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})} + \|\nabla^2 u\|_{L^1_T(\dot{B}^{3/2}_{2,1})}.$$

The following proposition quantifies the smoothing effect of the linear system of (4.1):

**Proposition 4.2** Let us consider the initial data $(a_0, \tau_0)$ in $\dot{B}^s_{2,1}(\mathbb{R}^3)$ with regularity $s \le \frac{3}{2}$. Introducing a pair of forces $(F, G)$ in $L^1_t(\dot{B}^s_{2,1}(\mathbb{R}^3))$, we denote by $(a, \tau)$ the unique solution of the following linear parabolic system:

$$\begin{cases} \partial_t a - \kappa_1 \Delta a + \kappa_1 \Delta \tau = F, \\ \kappa_2 \partial_t \tau - (\kappa_1^2 + \bar{\kappa}_3)\Delta\tau + \kappa_1^2 \Delta a = G. \end{cases} \tag{4.3}$$

Then $(a, \tau)$ belongs to $L^\infty_t(\dot{B}^s_{2,1}(\mathbb{R}^3))$, and the pairs $(\partial_t a, \partial_t \tau)$ and $(\Delta a, \Delta\tau)$ belong to $L^1_t(\dot{B}^s_{2,1}(\mathbb{R}^3))$. Furthermore, there exists a positive constant $C$ depending only on $\kappa_1, \kappa_2$ and $\bar{\kappa}_3$ such that

$$\begin{aligned} &\|(a,\tau)\|_{L^\infty_t(\dot{B}^s_{2,1})} + \|(\partial_t a, \partial_t \tau)\|_{L^1_t(\dot{B}^s_{2,1})} + \|(\Delta a, \Delta \tau)\|_{L^1_t(\dot{B}^s_{2,1})} \\ &\le C(\kappa_1, \kappa_2, \bar{\kappa}_3)\Big(\|(a_0, \tau_0)\|_{L^\infty_t(\dot{B}^s_{2,1})} + \|(F,G)\|_{L^1_t(\dot{B}^s_{2,1})}\Big). \end{aligned} \tag{4.4}$$

**Proof** We first apply the homogeneous dyadic block $\dot{\Delta}_q$ to system (4.3), and multiply both equations by $\dot{\Delta}_q a$ and $\dot{\Delta}_q \tau$, respectively, and integrate over $\mathbb{R}^3$. We then get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_q a\|_{L^2}^2+\kappa_1\|\nabla\dot{\Delta}_q a\|_{L^2}^2=\kappa_1\int_{\mathbb{R}^3}\nabla\dot{\Delta}_q\tau\cdot\nabla\dot{\Delta}_q a\,\mathrm{d}x+\int_{\mathbb{R}^3}\dot{\Delta}_q F\ \dot{\Delta}_q a\,\mathrm{d}x, \tag{4.5}$$

and

$$\frac{1}{2}\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_q \tau\|_{L^2}^2+(\kappa_1^2+\bar{\kappa}_3)\|\nabla\dot{\Delta}_q \tau\|_{L^2}^2=\kappa_1^2\int_{\mathbb{R}^3}\nabla\dot{\Delta}_q\tau\cdot\nabla\dot{\Delta}_q a\,\mathrm{d}x+\int_{\mathbb{R}^3}\dot{\Delta}_q G\ \dot{\Delta}_q \tau\,\mathrm{d}x. \tag{4.6}$$

By the Hölder and Cauchy inequalities, the first term on the right side of (4.5) can be bounded by

$$\kappa_1\int_{\mathbb{R}^3}\nabla\dot{\Delta}_q\tau\cdot\nabla\dot{\Delta}_q a\,\mathrm{d}x\le\frac{1}{2}\kappa_1\|\nabla\dot{\Delta}_q\tau\|_{L^2}^2+\frac{1}{2}\kappa_1\|\nabla\dot{\Delta}_q a\|_{L^2}^2.$$

Plugging the above inequality into (4.5), and adding the linear combination of the resulting inequality with (4.6), one has that

$$\begin{aligned}&\frac{1}{2}\kappa_1(1+\delta)\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_q a\|_{L^2}^2+\frac{1}{2}\kappa_2\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\Delta}_q \tau\|_{L^2}^2+\delta\kappa_1^2\|\nabla\dot{\Delta}_q a\|_{L^2}^2+(\bar{\kappa}_3-\frac{1}{2}\delta\kappa_1^2)\|\nabla\dot{\Delta}_q \tau\|_{L^2}^2\\&\le\kappa_1(1+\delta)\|\dot{\Delta}_q F\|_{L^2}\|\dot{\Delta}_q a\|_{L^2}+\|\dot{\Delta}_q G\|_{L^2}\|\dot{\Delta}_q \tau\|_{L^2},\end{aligned} \tag{4.7}$$

where $\delta$ is a small positive number. Setting that $f_q^2=\kappa_1(1+\delta)\|\dot{\Delta}_q a\|_{L^2}^2+\kappa_2\|\dot{\Delta}_q \tau\|_{L^2}^2$ and that $\kappa=\min\{\frac{\delta\kappa_1}{1+\delta},\frac{\bar{\kappa}_3-\frac{1}{2}\delta\kappa_1^2}{\kappa_2}\}$, we then have, by Bernstein's inequality, that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}f_q^2+\kappa 2^{2q}f_q^2\le C_{\kappa_1,\kappa_2}(\|\dot{\Delta}_q F\|_{L^2}+\|\dot{\Delta}_q G\|_{L^2})f_q. \tag{4.8}$$

To finish this, we multiply the above inequality by $2^{2qs}$, and denote that

$$g_q=2^{qs}\sqrt{\kappa_1(1+\delta)\|\dot{\Delta}_q a\|_{L^2}^2+\kappa_2\|\dot{\Delta}_q \tau\|_{L^2}^2},$$

so we then get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}g_q^2+\kappa 2^{2q}g_q^2\le C(\kappa_1,\kappa_2)2^{qs}(\|\dot{\Delta}_q F\|_{L^2}+\|\dot{\Delta}_q G\|_{L^2})g_q. \tag{4.9}$$

Using $h_q^2=g_q^2+\epsilon^2$, integrating over $[0,t]$ and then letting $\epsilon$ tend to 0, we infer that

$$g_q(t)+\kappa 2^{2q}\int_0^t g_q(\tau)\,\mathrm{d}\tau\le g_q(0)+C(\kappa_1,\kappa_2)2^{qs}\int_0^t(\|\dot{\Delta}_q F\|_{L^2}+\|\dot{\Delta}_q G\|_{L^2})\mathrm{d}\,\tau. \tag{4.10}$$

We finally conclude that

$$\|(a,\tau)\|_{L_t^\infty(\dot{B}_{2,1}^s)}+\|(a,\tau)\|_{L_t^1(\dot{B}_{2,1}^{s+2})}\le C(\kappa_1,\kappa_2)\Big(\|(a_0,\tau_0)\|_{L_t^\infty(\dot{B}_{2,1}^s)}+\|(F,G)\|_{L_t^1(\dot{B}_{2,1}^s)}\Big). \tag{4.11}$$

Combining (4.11) with the equations of $(a,\tau)$ finishes the proof of this proposition. □

Our construction of the global solution relies on a combination of Proposition 4.2 with Lemma 4.1. To this end, for any given $T>0$, we define the set $K(T)$ by

$$K(T):=\{(b,\tau)\in E(T)\times E(T),\ b(0)=a_0,\ \tau(0)=\widetilde{\theta}_0 \text{ and } \|(b,\tau)\|_{E(T)}\le c\}$$

for some suitable small positive constants $c$, which will be determined shortly. Next, given $(b,\tau)\in K(T)$, we define the mapping $\Phi(b,\tau):=(a,\tau)$, where $(a,\tau)$ is defined as the unique solution of the corresponding linearized problem of (4.1):

$$\begin{cases}\partial_t a-\kappa_1\Delta a+\kappa_1\Delta\tau=F(b,\tau),\\ \kappa_2\partial_t\tau-(\kappa_1^2+\bar{\kappa}_3)\Delta\tau+\kappa_1^2\Delta a=G(b,\tau),\\ (a,\widetilde{\theta})\mid_{t=0}=(a_0,\tau_0).\end{cases} \tag{4.12}$$

Here

$$\begin{aligned}F(b,\tau)&=-2\kappa_1\frac{\tau+1}{1+b}|\nabla b|^2+2\kappa_1\nabla b\cdot\nabla\tau-\kappa_1 b\Delta\tau+\kappa_1\tau\Delta b,\\ G(b,\tau)&=2\kappa_1^2\frac{(\tau+1)^2}{(1+b)^2}|\nabla b|^2-(3\kappa_1^2+\kappa_1\kappa_2)\frac{(\tau+1)}{1+b}\nabla b\cdot\nabla\tau\\ &\quad-\kappa_1^2\frac{\tau^2+2\tau}{1+b}\Delta b+\kappa_1^2\frac{b}{1+b}\Delta b+\bar{\kappa}_3 b\Delta\tau+\kappa_1^2\tau\Delta\tau\\ &\quad+(1+b)\nabla\cdot(\widetilde{\kappa}_3(\tau)\nabla\tau)+\kappa_1(\kappa_1+\kappa_2)|\nabla\tau|^2.\end{aligned}\tag{4.13}$$

Following Proposition 4.2, we easily obtain that

$$\|\Phi(b,\tau)\|_{E(T)}\le C\Big(\|(a_0,\tau_0)\|_{\dot{B}^{3/2}_{2,1}}+\|F(b,\tau)\|_{L^1_T(\dot{B}^{3/2}_{2,1})}+\|G(b,\tau)\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\Big).\tag{4.14}$$

In order to prove that $\Phi(K(T))\subset K(T)$ under the smallness condition on $a_0$ and $\tau_0$, one needs to bound the right side of (4.14). We ignore $\kappa_1$, $\kappa_2$, and $\bar{\kappa}_3$, as they are fixed constants.

For the first term in (4.13), we rewrite it as

$$\frac{\tau+1}{1+b}|\nabla b|^2=m_1(b)|\nabla b|^2(\tau+1)+|\nabla b|^2(\tau+1),$$

where $m_1(b):=\frac{1}{1+b}-1$, satisfying that $m_1(0)=0$. By Lemma 1.6 in [6] and the continuity of the product in Besov spaces ([1, Chapter 2]), we get that

$$\begin{aligned}\left\|\frac{\tau+1}{1+b}|\nabla b|^2\right\|_{L^1_T(\dot{B}^{3/2}_{2,1})}&\le C\left(1+\|b\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\right)\|\nabla b\|^2_{L^2_T(\dot{B}^{3/2}_{2,1})}\left(\|\tau\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}+1\right)\\ &\le C\,(1+c)^2\,c^2.\end{aligned}$$

Similarly, we have that

$$\|\nabla b\cdot\nabla\tau\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\le Cc^2,\qquad \|b\cdot\Delta\tau\|_{L^1_T(\dot{B}^{3/2}_{2,1})}+\|\tau\Delta b\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\le Cc^2.$$

Combining the above estimates, we find that

$$\|F(b,\tau)\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\le C\,(1+c)^2\,c^2.$$

The terms in $G$ can be bounded in essentially the same way. For the first term in $G$, we rewrite it as

$$\frac{(\tau+1)^2}{(1+b)^2}|\nabla b|^2=m_2(b)|\nabla b|^2(\tau+1)^2+|\nabla b|^2(\tau+1)^2,$$

where $m_2(b):=\frac{1}{(1+b)^2}-1$, satisfying that $m_2(0)=0$. We then infer that

$$\begin{aligned}\left\|\frac{(\tau+1)^2}{(1+b)^2}|\nabla b|^2\right\|_{L^1_T(\dot{B}^{3/2}_{2,1})}&\le C\left(1+\|b\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\right)\|\nabla b\|^2_{L^2_T(\dot{B}^{3/2}_{2,1})}\left(\|\tau\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}+1\right)^2\\ &\le C\,(1+c)^3\,c^2.\end{aligned}$$

The term $\frac{\tau+1}{1+b}\nabla b\cdot\nabla\tau$ is handled the same as $\frac{\tau+1}{1+b}|\nabla b|^2$. The third term in $G$ can be rewritten as

$$\frac{\tau^2+2\tau}{1+b}\Delta b=m_1(b)\tau\Delta b(\tau+2)+\tau\Delta b(\tau+2),$$

which is estimated in the same way. The fourth term of $G$ is, in fact, $-m_1(b)\Delta b$, so that

$$\left\|\frac{b}{1+b}\Delta b\right\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\le C\|b\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}\|\Delta b\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\le Cc^2.$$

Likewise,

$$\|b\Delta\tau\|_{L^1_T(\dot B^{3/2}_{2,1})}+\|\tau\Delta\tau\|_{L^1_T(\dot B^{3/2}_{2,1})}\le Cc^2.$$

The seventh term of $G$ becomes $(1+b)\widetilde{\kappa}_3'(\tau)|\nabla\tau|^2+(1+b)\widetilde{\kappa}_3(\tau)\Delta\tau$, where $\widetilde{\kappa}_3(0)=0$ and $\widetilde{\kappa}_3'$ is bounded by assumption. The $L^1_T(\dot B^{3/2}_{2,1})$-norm for this term is controlled by $C(1+c)c^2$. The last term in $G$ is similarly bounded by $Cc^2$. Thus, we have that

$$\|G(b,\tau)\|_{L^1_T(\dot B^{3/2}_{2,1})}\le C\,(1+c)^3\,c^2.$$

Finally, combining the above estimate with the one for $F$ with (4.14), we obtain that

$$\|\Phi(b,\tau)\|_{E(T)}\le C\|(a_0,\tau_0)\|_{\dot B^{3/2}_{2,1}}+C(1+c)^3c^2. \tag{4.15}$$

This implies $\Phi(K(T))\subset K(T)$, provided that

$$c\le\min\{1,\frac{1}{16C}\}\quad\text{and}\quad\|(a_0,\tau_0)\|_{\dot B^{3/2}_{2,1}}\le\frac{1}{2C}c. \tag{4.16}$$

Next, we will prove that, for any $T>0$, the map $\Phi(b,\tau)$ is contractive on $K(T)$. Indeed, for $(v_i,\tau_i)\in K(T)$, let $(a_i,\tau_i)=\Phi(v_i,\tau_i)$ for $i=1,2$. Moreover, we set that $\bar a=a_1-a_2$ and that $\bar\theta=\widetilde\theta_1-\widetilde\theta_2$. Then $(\bar a,\bar\theta)$ satisfies the equation

$$\begin{cases}\partial_t\bar a-\kappa_1\Delta\bar a+\kappa_1\Delta\bar\theta=\delta F,\\ \kappa_2\partial_t\bar\theta-(\kappa_1^2+\bar\kappa_3)\Delta\bar\theta+\kappa_1^2\Delta\bar a=\delta G,\\ (\bar a,\bar\theta)\mid_{t=0}=(0,0),\end{cases} \tag{4.17}$$

where $\delta F=F(b_1,\tau_1)-F(b_2,\tau_2)$ and $\delta G=G(b_1,\tau_1)-F(b_2,\tau_2)$. Applying Proposition 4.2 yields that

$$\|(\bar a,\bar\theta)\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+\|(\Delta\bar a,\Delta\bar\theta)\|_{L^1_T(\dot B^{3/2}_{2,1})}\le C\left(\|\delta F\|_{L^1_T(\dot B^{3/2}_{2,1})}+\|\delta G\|_{L^1_T(B^{3/2}_{2,1})}\right). \tag{4.18}$$

Now, $\delta F$ can be rewritten as

$$\begin{aligned}\delta F=&-2\kappa_1\left(\frac{1}{1+b_1}-\frac{1}{1+b_2}\right)|\nabla b_2|^2(\tau_2+1)-2\kappa_1\frac{1}{1+b_1}\nabla\delta b\cdot\nabla b_2(\tau_2+1)\\&-2\kappa_1\frac{1}{1+b_1}\nabla b_1\cdot\nabla\delta b\ (\tau_2+1)-2\kappa_1\frac{1}{1+b_1}|\nabla b_1|^2\delta\tau+2\kappa_1\nabla\delta b\cdot\nabla\tau_2\\&+2\kappa_1\nabla b_1\cdot\nabla\delta\tau-\kappa_1\delta b\Delta\tau_2-\kappa_1b_1\Delta\delta\tau+\kappa_1\delta\tau\Delta b_2+\kappa_1\tau_1\Delta\delta b,\end{aligned} \tag{4.19}$$

where $\delta b=b_1-b_2,\delta\tau=\tau_1-\tau_2$. Moreover, for the first term, we also have that

$$\frac{1}{1+b_2}-\frac{1}{1+b_1}=\frac{1}{(1+b_2)(1+b_1)}\delta b=(m_1(b_1)+1)(m_1(b_2)+1)\delta b. \tag{4.20}$$

Then we can estimate the terms in $\delta F$ analogously to the terms in (4.13) to obtain that

$$\begin{aligned}\|\delta F\|_{L^1_T(\dot B^{3/2}_{2,1})}\le\,& C(\|b_1\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)(\|b_2\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\|\delta b\|_{L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla b_2\|^2_{L^2_T(\dot B^{3/2}_{2,1})}\\&\times(\|\tau_2\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\\&+C(\|b_1\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\|\nabla\delta b\|_{L^2_T(\dot B^{3/2}_{2,1})}\|\nabla b_2\|_{L^2_T(\dot B^{3/2}_{2,1})}(\|\tau_2\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\\&+C(\|b_1\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\|\nabla\delta b\|_{L^2_T(\dot B^{3/2}_{2,1})}\|\nabla b_1\|_{L^2_T(\dot B^{3/2}_{2,1})}(\|\tau_2\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\\&+C(\|b_1\|_{L^\infty_T(\dot B^{3/2}_{2,1})}+1)\|\nabla b_1\|^2_{L^2_T(\dot B^{3/2}_{2,1})}\|\delta\tau\|_{L^\infty_T(\dot B^{3/2}_{2,1})}\\\le\,& C(1+c)^3c\big(\|\delta b\|_{E(T)}+\|\delta\tau\|_{E(T)}\big).\end{aligned}$$

A similar methodology is applied for the terms in $\delta G$. We write

$$\delta G = \kappa_1^2 J_1 - (3\kappa_1^2 + \kappa_1\kappa_2)J_2 - \kappa_1^2 J_3 + \kappa_1^2 J_4 + \bar{\kappa}_3 J_5 + \kappa_1^2 J_6 + J_7 + \kappa_1(\kappa_1+\kappa_2)J_8.$$

Each of the $J_i$ terms correspond to the difference operator $\delta$ applied to each respective term in the expression for $G$ in (4.13). More specifically, we have that

$$\begin{aligned}
J_1 &= \left(\frac{1}{1+b_1} - \frac{1}{1+b_2}\right)\left(\frac{(\tau_2+1)^2}{1+b_2} + \frac{(\tau_2+1)^2}{1+b_1}\right)|\nabla b_2|^2 \\
&\quad + \frac{\delta\tau(\tau_1+\tau_2+2)}{(1+b_1)^2}|\nabla b_2|^2 + \frac{(\tau_1+1)^2}{(1+b_1)^2}\nabla\delta b\cdot(\nabla b_1 + \nabla b_2), \\
J_2 &= \left(\frac{1}{1+b_1} - \frac{1}{1+b_2}\right)(\tau_2+1)\nabla b_2\cdot\nabla\tau_2 + \frac{\delta\tau}{1+b_1}\nabla b_2\cdot\nabla\tau_2 \\
&\quad + \frac{\tau_1+1}{1+b_1}\nabla\delta b\cdot\nabla\tau_2 + \frac{\tau_1+1}{1+b_1}\nabla b_1\cdot\nabla\delta\tau_2, \\
J_3 &= \left(\frac{1}{1+b_1} - \frac{1}{1+b_2}\right)(\tau_2^2+2\tau_2)\Delta b_2 + \frac{\delta\tau(\tau_1+\tau_2+2)}{1+b_1}\Delta b_2 + \frac{\tau_1^2+2\tau_1}{1+b_1}\Delta\delta b, \\
J_4 &= \left(\frac{b_1}{1+b_1} - \frac{b_2}{1+b_2}\right)\Delta b_2 + \frac{b_1}{1+b_1}\Delta\delta b, \\
J_5 &= \delta b\Delta\tau_2 + b_1\Delta\delta\tau, \\
J_6 &= \delta\tau\Delta\tau_2 + \tau_1\Delta\delta\tau, \\
J_7 &= \delta b(\widetilde{\kappa}_3(\tau_2)\Delta\tau_2 + \widetilde{\kappa}_3'(\tau)|\nabla\tau_2|^2) \\
&\quad + (1+b_1)(\widetilde{\kappa}_3(\tau_1) - \widetilde{\kappa}_3(\tau_2))\Delta\tau_2 + (1+b_1)\widetilde{\kappa}_3(\tau_1)\Delta\delta\tau \\
&\quad + (1+b_1)(\widetilde{\kappa}_3'(\tau_1) - \widetilde{\kappa}_3'(\tau_2))|\nabla\tau_2|^2 + (1+b_1)\widetilde{\kappa}_3'(\tau_1)\nabla\delta\tau\cdot(\nabla\tau_1+\nabla\tau_2), \\
J_8 &= \nabla\delta\tau\cdot(\nabla\tau_1+\nabla\tau_2).
\end{aligned}$$

The estimate for each $J_i$ is similar to those already established, keeping in mind the continuity of the products in Besov spaces, Lemma 1.6 in [6], and the identity (4.20) to control several terms in $J_1$, $J_2$, $J_3$ and $J_4$. The only subtle point is that, to estimate $J_7$, we need to invoke the mean-value theorem to write

$$|\widetilde{\kappa}_3(\tau_1) - \widetilde{\kappa}_3(\tau_2)| \le C|\delta\tau|, \qquad |\widetilde{\kappa}_3'(\tau_1) - \widetilde{\kappa}_3'(\tau_2)| \le C|\delta\tau|,$$

where $C$ above depends on the upper bound for $|\widetilde{\kappa}_3'|$ and $|\widetilde{\kappa}_3''|$. Since this upper bound exists by assumption, we can infer that

$$\|\delta G\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \le C(1+c)^5 c\left(\|\delta b\|_{E(T)} + \|\delta\tau\|_{E(T)}\right).$$

Therefore,

$$\|(\bar{a},\bar{\theta})\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})} + \|(\Delta\bar{a},\Delta\bar{\theta})\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \le C(1+c)^5 c\left(\|\delta b\|_{E(T)} + \|\delta\tau\|_{E(T)}\right).$$

If we additionally assume that $c \le \frac{1}{64C}$, we then have, for all $T>0$, that

$$\|(\bar{a},\bar{\theta})\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})} + \|(\Delta\bar{a},\Delta\bar{\theta})\|_{L^1_T(\dot{B}^{3/2}_{2,1})} \le \frac{1}{2}\left(\|\delta b\|_{E(T)} + \|\delta\tau\|_{E(T)}\right).$$

Thus, $\Phi$ is contractive, as a mapping from $E(T)$ to $E(T)$. The proof of Theorem 1.3 follows from Lemma 4.1.

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