



Cohesive powers of structures

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Abstract

A cohesive power of a structure is an effective analog of the classical ultrapower of a structure. We start with a computable structure, and consider its effective power over a cohesive set of natural numbers. A cohesive set is an infinite set of natural numbers that is indecomposable with respect to computably enumerable sets. It plays the role of an ultrafilter, and the elements of a cohesive power are the equivalence classes of certain partial computable functions determined by the cohesive set. Thus, unlike many classical ultrapowers, a cohesive power is a countable structure. In this paper we focus on cohesive powers of graphs, equivalence structures, and computable structures with a single unary function satisfying various properties, which can also be viewed as directed graphs. For these computable structures, we investigate the isomorphism types of their cohesive powers, as well as the properties of cohesive powers when they are not isomorphic to the original structure.

Keywords Cohesive power · Computable structure · Graph · Equivalence structure · Partial injection structure · Two-to-one structure

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1 Introduction and preliminaries

The classical ultraproduct construction is one of the major tools for constructing new models of theories, such as nonstandard models of arithmetic and analysis. The construction has found important and elegant applications in logic and other areas of mathematics. For example, the compactness theorem for the first-order logic can be established directly using an ultraproduct construction. Given a family of structures

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$(\mathcal{A}_i)_{i \in I}$ for the same first-order language L , indexed by a set I , with the domains $(A_i)_{i \in I}$, the ultraproduct is defined using an ultrafilter on I .

More precisely, a *filter* F over I is a nonempty set of subsets of I such that F is closed under the intersection of two (hence finitely many) sets and under the superset relation. A filter over I is proper if it is not equal to the power set of I . A filter U is an *ultrafilter* if for every subset X of I , either X or its complement $I - X$ belongs to U . It follows that an ultrafilter is a *maximal* proper filter.

Given an ultrafilter U over I , two functions $f, g \in \prod_{i \in I} A_i$ are U -equivalent if the set of all inputs on which their values are equal belongs to U . The elements of the ultraproduct $\prod_U A_i$ are the corresponding equivalence classes $[f]$. The ultrafilter is also used to give interpretations of the symbols in L . For each atomic formula $\alpha(x_1, \dots, x_n)$ that has at most one symbol from L , by definition, $\alpha([f_1], \dots, [f_n])$ holds in $\prod_U A_i$ if and only if the set $\{i : \mathcal{A}_i \models \alpha(f_1(i), \dots, f_n(i))\}$ belongs to U . Using the properties of an ultrafilter we can show that the ultraproduct structure is well-defined.

Łoś's theorem, the fundamental theorem of ultraproducts, establishes the satisfiability of a first-order formula of an arbitrary complexity in an ultraproduct $\prod_U \mathcal{A}_i$ given similarly as in the above definition (see [7]). If all structures $\mathcal{A}_i$ are the same structure $\mathcal{A}$, then the ultraproduct is the *ultrapower* $\prod_U \mathcal{A}$. We say that two structures for the same language are *elementarily equivalent* if they satisfy the same first-order sentences (i.e., have the same first-order theory). An important consequence of Łoś's theorem is that a structure and its ultrapower are elementarily equivalent.

An ultrafilter is *principal* if it contains a singleton. It follows that a non-principal ultrafilter over an infinite I contains all co-finite subsets of I . If U is a principal ultrafilter such that $\{i_0\} \in U$, then $\prod_U \mathcal{A}_i$ is isomorphic to $\mathcal{A}_{i_0}$. Hence the construction of nontrivial ultraproducts requires non-principal ultrafilters. The existence of non-principal ultrafilters cannot be proved in ZF (see [2]). However, the axiom of choice proves not just the existence of a non-principal ultrafilter over a given infinite I , but a stronger statement known as the ultrafilter lemma. The *ultrafilter lemma* says that every proper filter on I is contained in an ultrafilter on I . The ultrafilter lemma is strictly weaker than the axiom of choice.

In this paper, we will consider an effective product (called *cohesive product*) of structures, a computability-theoretic analog of the ultraproduct construction, which does not depend on the axiom of choice. That is, a cohesive product is based on a more constructive approach by using, instead of an ultrafilter on the index set ω , the set of all supersets of a *cohesive* set, where cohesive sets come naturally in computability theory. A cohesive set is an infinite set of natural numbers, which is indecomposable with respect to computably enumerable sets. There are continuum many cohesive sets. Some cohesive sets (countably many of them) are the complements of maximal sets. *Maximal* sets are co-atoms in the lattice $\mathcal{E}^*$ of computably enumerable sets modulo finite sets. Cohesive and maximal sets have been studied extensively (see [8, 32]).

Starting with infinite countable structures, the classical ultraproduct construction typically produces an uncountable structure. An ultrafilter is *countably complete* if it is closed under countable intersections. An ultrafilter is countably incomplete if it has a countable subset of sets the intersection of which is empty, or equivalently it is not countably complete. While the principal ultrafilters are countably complete,

the existence of non-principal countably complete ultrafilters is a strong axiom of infinity which is not provable in ZFC. In particular, there is no countably complete non-principal ultrafilter on ω . Frayne, Morel and Scott [22] proved that if U is a countably incomplete ultrafilter, then $\Pi_U \mathcal{A}_i$ is either finite or of cardinality at least $2^{\aleph_0}$. Hence such an ultraproduct $\Pi_U \mathcal{A}_i$ is never countably infinite. However, cohesive product construction always produces a countable structure. That is, we start with a uniformly computable sequence of structures for the same computable language, and in their direct product consider only *partial computable* sequences modulo a fixed cohesive set. In some cases partial computable functions can be replaced by (total) computable functions.

The motivation for effective powers dates back to Skolem's construction of a countable non-standard model of arithmetic, where instead of building a structure from all functions on natural numbers he used only arithmetical functions (see also [14]). Skolem's idea was further developed in the study of models of fragments of arithmetic by Feferman, Scott, and Tennenbaum [20], Lerman [26], Hirschfeld [24], Hirschfeld and Wheeler [25], and McLaughlin [28–30]. Nelson [31] investigated recursive saturation of effective ultraproducts (see also [17]). While for a countable language, ultrapowers over countably incomplete ultrafilters are $\aleph_1$ -saturated, cohesive powers of decidable (that is, strongly constructive) structures are recursively saturated.

Dimitrov [10, 12] discovered an effective power construction of fields while working on a long-standing open problem posed by Downey and Remmel in [19]. The problem is to find natural orbits under automorphisms of the lattice $\mathcal{L}^*$ of computably enumerable vector spaces over a computable field, modulo finite subspaces. Maximal vector spaces play an important role in the study of this lattice. In particular, Dimitrov [12] used the effective powers of the field $\mathbb{Q}$ of rational numbers to characterize principal filters determined by maximal vector spaces over $\mathbb{Q}$. In [11], Dimitrov introduced the notion of a cohesive power of a computable structure in general, and established a restricted version of Łoś's theorem for cohesive powers. Although a cohesive power of a computable structure $\mathcal{A}$ may not even be elementarily equivalent to $\mathcal{A}$, additional decidability in the structure $\mathcal{A}$ plays a significant role in increasing preservation of satisfiability of more complex sentences in the cohesive power (see [17]). In [15], Dimitrov and Harizanov found further applications of cohesive powers of $\mathbb{Q}$ to the automorphisms of the lattice $\mathcal{L}^*$.

The cohesive power construction allows us to obtain countable models with interesting properties. While the ultrapowers of isomorphic structures over a fixed ultrafilter are isomorphic, it is possible for isomorphic computable structures to have cohesive powers, over a fixed cohesive set, which are not even elementarily equivalent. However, if C is a cohesive set, and computable structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are computably isomorphic, then $\Pi_C \mathcal{A}_1 \cong \Pi_C \mathcal{A}_2$ (see [11, 17]). A computable structure $\mathcal{A}$ is called *computably categorical* if every computable isomorphic structure is computably isomorphic to $\mathcal{A}$. For example, the usual ordered set of natural numbers, $(\omega, <)$, is a computable structure that is not computably categorical. Let ω also stand for the order type of natural numbers, let ζ stand for the order type of integers, and η for the order type of rational numbers. It was shown in [17, 18] that if a computable $\mathcal{A}$ is computably isomorphic to $(\omega, <)$, then for any cohesive set C , the cohesive power $\Pi_C \mathcal{A}$ is a linear ordering of order type $\omega + \zeta \eta$. On the other hand, if C is a cohesive set with a

computably enumerable complement, then there is a computable isomorphic copy $\mathcal{M}$ of $(\omega, <)$ such that $\Pi_C \mathcal{M}$ is of order type $\omega + \eta$. It was shown in [17] that many more order types can be obtained as cohesive powers of computable isomorphic copies of $(\omega, <)$. In this paper, we will focus on cohesive powers of computable structures from several other classes that were studied in computable structure theory, in particular, in the context of computable categoricity and categoricity at higher levels of arithmetical hierarchy.

We will next state precise definitions and some properties of the relevant notions from computability theory and computable structure theory, many of which were mentioned above.

Cohesive sets. The complement of a set $X \subseteq \omega$ is denoted by $\overline{X}$. We write $\subseteq^*$ for inclusion up to finitely many elements. By c.e. we abbreviate computably enumerable.

A set $C \subseteq \omega$ is *cohesive* if C is infinite and for every c.e. set W , either $W \cap C$ or $\overline{W} \cap C$ is finite. Hence if $W \cap C$ is infinite, then $C \subseteq^* W$, and if $\overline{W} \cap C$ is infinite, then $C \subseteq^* \overline{W}$.

Clearly, an infinite subset of a cohesive set is cohesive. It follows that if a cohesive set C is contained in the union of finitely many c.e. sets, up to finitely many elements, then it is contained in one of them, up to finitely many elements. That is, because C must have an infinite intersection with at least one of the finitely many c.e. sets in the union. It can be shown that every infinite set of natural numbers has a cohesive subset. Hence there are $2^{\aleph_0}$ cohesive sets. Some cohesive sets are the complements of c.e. sets.

A set $E \subseteq \omega$ is *maximal* iff E is c.e. and $\overline{E}$ is cohesive.

Computable structures. If L is the language of a structure $\mathcal{A}$ with domain A , then L_A is the language L expanded by adding a constant symbol for every $a \in A$, and $\mathcal{A}_A = (\mathcal{A}, a)_{a \in A}$ is the corresponding expansion of $\mathcal{A}$ to L_A . The *atomic diagram* of $\mathcal{A}$ is the set of all atomic and negations of atomic sentences of L_A true in $\mathcal{A}_A$.

A countable structure $\mathcal{A}$ for a computable language L is *computable* if its domain A is a computable set and its atomic diagram is computable or, equivalently, its functions and relations are uniformly computable.

The *elementary diagram* (or complete diagram) of $\mathcal{A}$, denoted by $D^c(\mathcal{A})$, is the set of all first-order sentences of L_A that are true in $\mathcal{A}_A$. A Σ_n^0 *diagram* of $\mathcal{A}$ is the set of all Σ_n^0 sentences in $D^c(\mathcal{A})$. A structure is *decidable* (or strongly constructive) if its domain is computable and its elementary diagram is computable.

A structure is *n-decidable* if its domain is computable and its Σ_n^0 diagram is computable. In particular, computable structures are the same as 0-decidable structures.

Cohesive powers. We will first give a definition of the cohesive product of computable structures, which was introduced in [13]. By $\simeq$ we denote the equality of partial functions.

Definition 1 Let L be a computable language. Let $(\mathcal{A}_i)_{i \in \omega}$ be a uniformly computable sequence of computable structures in L , with uniformly computable sequence of domains $(A_i)_{i \in \omega}$. Let $C \subseteq \omega$ be a cohesive set. The *cohesive product* $\mathcal{B}$ of $\mathcal{A}_i$ over C , in symbols $\mathcal{B} = \prod_C \mathcal{A}_i$, is a structure defined as follows.

1. Let $D = \{\psi \mid \psi : \omega \rightarrow \bigcup_{i \in \omega} A_i, \psi \text{ is a partial computable function, } C \subseteq^* \text{dom}(\psi), \text{ and if } \psi(i) \downarrow \text{ then } \psi(i) \in A_i\}$. For $\psi_1, \psi_2 \in D$, let $\psi_1 =_C \psi_2$ iff $C \subseteq^* \text{dom}(\psi_1 \oplus \psi_2)$.

- $\{i : \psi_1(i) \downarrow = \psi_2(i) \downarrow\}$. The domain of $\prod_C \mathcal{A}_i$ is the quotient set $D/=_C$ and we will denote it here by B .
2. If $f \in L$ is an n -ary function symbol, then f^B is an n -ary function on B such that for every $[\psi_1], \dots, [\psi_n] \in B$, we have

$$f^B([\psi_1], \dots, [\psi_n]) = [\psi] \text{ iff } (\forall i \in \omega) \left[\psi(i) \simeq f^{\mathcal{A}_i}(\psi_1(i), \dots, \psi_n(i)) \right].$$

3. If $R \in L$ is an m -ary relation symbol, then R^B is an m -ary relation on B such that for every $[\psi_1], \dots, [\psi_m] \in B$,

$$R^B([\psi_1], \dots, [\psi_m]) \text{ iff } C \subseteq^* \{i \in \omega \mid R^{\mathcal{A}_i}(\psi_1(i), \dots, \psi_m(i))\}.$$

4. If $c \in L$ is a constant symbol, then c^B is the equivalence class (with respect to $=_C$) of the computable function $g : \omega \rightarrow \bigcup_{i \in \omega} A_i$ such that $g(i) = c^{\mathcal{A}_i}$, for each $i \in \omega$.

If $\mathcal{A}_i = \mathcal{A}$ for every $i \in \omega$, then $\prod_C \mathcal{A}_i$ is called the *cohesive power* of $\mathcal{A}$ over C and is denoted by $\prod_C \mathcal{A}$.

If C is co-c.e., then for every $[\psi] \in \prod_C \mathcal{A}_i$ there is a computable function f such that $[f] = [\psi]$. That is, for $(a_i)_i \in \prod_i A_i$ which is a fixed computable sequence, define

$$f(i) = \begin{cases} \psi(i) & \text{if } \psi(i) \downarrow \text{ first,} \\ a_i & \text{if } i \text{ is enumerated into } \overline{C} \text{ first.} \end{cases}$$

An *embedding* of a structure $\mathcal{A}$ into a structure $\mathcal{B}$ is an isomorphism between $\mathcal{A}$ and a substructure of $\mathcal{B}$. A structure $\mathcal{A}$ naturally embeds into its cohesive power $\mathcal{B} = \prod_C \mathcal{A}$. For $a \in A$ let $[c_a] \in B$ be the equivalence class of the constant function c_a such that $c_a(i) = a$ for every $i \in \omega$. The function $d : A \rightarrow B$ such that $d(a) = [c_a]$ is called the *canonical embedding* of $\mathcal{A}$ into $\mathcal{B}$. If $\mathcal{A}$ is a finite structure, then $\prod_C \mathcal{A} \cong \mathcal{A}$ since for every element $[\psi]$ from $\prod_C \mathcal{A}$, the partial function ψ is eventually constant on C .

In [13, 17] the authors provide variants of Łoś's theorem for cohesive products of uniformly computable and more generally uniformly n -decidable structures. For example, every Σ_{n+3}^0 sentence true in an n -decidable structure is also true in its cohesive powers. In particular, we have the following theorem for cohesive powers of computable structures.

Theorem 2 (Dimitrov [11]) *Let $\mathcal{B} = \prod_C \mathcal{A}$ be a cohesive power of a computable structure $\mathcal{A}$. Let C be a cohesive set.*

1. *If σ is a Π_2^0 (or Σ_2^0) sentence in L , then $\mathcal{B} \models \sigma$ iff $\mathcal{A} \models \sigma$.*
2. *If σ is a Π_3^0 sentence in L , then $\mathcal{B} \models \sigma$ implies $\mathcal{A} \models \sigma$. By contrapositive, if σ is a Σ_3^0 sentence in L , then $\mathcal{A} \models \sigma$ implies $\mathcal{B} \models \sigma$.*

The converse of part (2) in the previous theorem does not hold. The first such counterexample was produced by Feferman, Scott and Tennenbaum in their result in [20] that no cohesive power of the standard model of arithmetic is a model of Peano arithmetic. There is a Π_3^0 sentence (involving Kleene's T predicate) that is true in the standard model of arithmetic $\mathcal{N}$ but is false in every cohesive power of $\mathcal{N}$ (see

[26]). More recently, in [17], the authors produced natural examples of such sentences concerning linear orders. On the other hand, if $\mathcal{A}$ is a decidable structure, then $\mathcal{A}$ and $\prod_C \mathcal{A}$ i.e., they are elementarily equivalent (see [11, 31]).

An equivalence structure $\mathcal{A} = (A, E^{\mathcal{A}})$ consists of a set A with a binary relation $E^{\mathcal{A}}$ that is reflexive, symmetric, and transitive. An equivalence structure $\mathcal{A}$ is *computable* if A is a computable set and $E^{\mathcal{A}}$ is a computable relation. An application of Theorem 2 is that if $\mathcal{A}$ is a computable equivalence structure, then $\prod_C \mathcal{A}$ is also an equivalence structure. That is because the theory of equivalence structures is Π_1^0 -axiomatizable. Similarly, a cohesive power of a computable field is a field. In [16], Dimitrov, Harizanov, Miller, and Mourad investigated cohesive powers of the field $\mathbb{Q}$ of rational numbers over co-maximal sets. For example, they proved that two cohesive powers of $\mathbb{Q}$ over co-maximal sets are isomorphic iff the maximal sets have the same many-one degree.

Computable formulas. A computable (infinitary) language is more expressive than the usual finitary first-order language. For a computable ordinal α , Ash defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$ recursively and simultaneously and together with their Gödel numbers. For the natural numbers we roughly have the following classification of formulas. Computable Σ_0 and Π_0 formulas are just the finitary quantifier-free formulas. For $n > 1$, a *computable Π_n formula* is a c.e. conjunction of formulas $\forall \bar{u} \phi(\bar{x}, \bar{u})$, where ϕ is a computable Σ_m formula for some $m < n$. Dually, a *computable Σ_n formula* is a c.e. disjunction of formulas $\exists \bar{v} \theta(\bar{y}, \bar{v})$, where θ is a computable Π_m formula for some $m < n$.

For more on computability theory see [8, 32] and on computable structure theory see [1, 21].

By $\langle k, n \rangle$ we will denote a computable bijection from ω^2 onto ω , which is strictly increasing with respect to each coordinate and such that $k, n \leq \langle k, n \rangle$. For a set $P \subseteq \omega^2$, let $\langle P \rangle = \{\langle k, n \rangle : (k, n) \in P\}$.

This paper is a greatly expanded version of the preliminary work in [23] to appear in the proceedings following the Fall Western Sectional Meeting of the AMS during October 23–24, 2021. There, we studied cohesive powers of certain graphs and equivalence structures. For example, we showed that every computable graph can be embedded into a cohesive power of a strongly locally finite graph. Here, we also investigate cohesive powers of computable structures with a single unary function satisfying various properties, called injection structures, two-to-one structures, and (2,0)-to-one structures. We further study cohesive powers of partial injection structures viewed as relational structures. We characterize the isomorphism types of cohesive powers of these computable structures, and use computable (infinitary) language to describe the properties of cohesive powers when they are not isomorphic to the original structure.

2 Cohesive powers of graphs

A *graph* (or undirected graph) (V, E) is a nonempty set V of vertices with a symmetric binary relation E (also called the edge relation), so it can be axiomatized by the following universal sentence:

$$\forall x \forall y [E(x, y) \Rightarrow E(y, x)].$$

Hence, by Theorem 2 a cohesive power of a graph is a graph.

If $(x, y) \in E$, then vertices x and y are adjacent to each other. The *degree of a vertex* is the number of vertices it is adjacent to. A graph G is called *locally finite* if the degree of each vertex in G is finite. A graph G is *strongly locally finite* if all connected components of G are finite. In [9], a criterion was obtained for computable categoricity of certain strongly locally finite computable graphs.

The *disjoint union* of graphs (V_1, E_1) and (V_2, E_2) where $V_1 \cap V_2 = \emptyset$ is a graph $(V_1 \cup V_2, E_1 \cup E_2)$. Hence there are no edges between V_1 and V_2 . We write $(V_1, E_1) \sqcup (V_2, E_2)$ and also view it as a decomposition into a disjoint union. If $V_1 \cap V_2 = \emptyset$, then a union of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is any graph $G = (V_1 \cup V_2, E)$ such that $E_1 \cup E_2 \subseteq E$ and $E \upharpoonright (V_i \times V_i) = E_i$ for $i = 1, 2$. We simply write $G = G_1 \sqcup G_2$.

The following result demonstrates the universal feature with respect to embeddability into cohesive powers of certain computable graphs.

Theorem 3 *Let G be a computable graph. Let C be a cohesive set. Then there is a computable, strongly locally finite graph A such that $\Pi_C A$ is isomorphic to the union $G \cup H$ for some graph H or $H = \emptyset$.*

Proof If G is finite, then G is strongly locally finite and $G \cong \Pi_C G$.

Let $\mathbb{N}^+ = \omega - \{0\}$. Now, assume that a graph $G = (V, R)$ is infinite and fix a computable enumeration f of its vertices $V = \{f(n) : n \in \mathbb{N}^+\}$. Using this enumeration we will build a computable graph A with domain $\mathbb{N}^+$ and edge set E . Consider vertices $a, b \in \mathbb{N}^+$. If a, b can be written as $a = \frac{k(k+1)}{2} + m$ and $b = \frac{k(k+1)}{2} + n$ for some k, m, n such that $1 \leq m, n \leq k+1$, then let

$$(a, b) \in E \Leftrightarrow (f(m), f(n)) \in R.$$

If there are no k, m, n as above, then let $(a, b) \notin E$. Note that $\frac{k(k+1)}{2} = 1 + 2 + \dots + k$, so the idea is to divide the natural numbers into segments of lengths 1, 2, Clearly, A is a computable, strongly locally finite graph.

For $n \in \mathbb{N}^+$ we define functions ψ_n by $\psi_n(x) = \frac{x(x+1)}{2} + n$. Hence each $[\psi_n]$ is an element of the cohesive power $\Pi_C A$. Consider the subgraph $\mathcal{S}$ of A with the vertex set $\{[\psi_n] : n \in \mathbb{N}^+\}$. Consider a function $\rho : \{[\psi_n] : n \in \mathbb{N}^+\} \rightarrow V$ defined by $\rho([\psi_n]) = f(n)$. Then we can show that ρ is a graph isomorphism (see [23]), so $\mathcal{S}$ is isomorphic to G . Hence $\Pi_C A$ is isomorphic to the union $G \cup H$ for some graph H . $\square$

If a computable graph G is locally finite, we have a stronger result.

Theorem 4 *Let G be an infinite computable graph that is locally finite. Let C be a cohesive set. Then there is a computable, strongly locally finite graph A such that $\Pi_C A$ is isomorphic to the disjoint union $G \sqcup H$ for some graph H .*

Proof Let $G = (V, R)$. Let f, A, ψ_n, ρ be defined as in the proof of the previous theorem. Let φ be a partial computable function such that $[\varphi] \in \Pi_C A$ and $E([\varphi], [\psi_m])$ for some $m \geq 1$. Then $C \subseteq^* \{i \in \omega : \varphi(i) \downarrow \wedge (\varphi(i), \psi_m(i)) \in E\}$. Since G is locally finite, we have that $\{i \in \omega : \varphi(i) \downarrow \wedge (\varphi(i), \psi_m(i)) \in E\}$ is the following finite disjoint union of c.e. sets:

$$\bigsqcup_{n: f(m)Rf(n)} \{i \in \omega : \varphi(i) = \frac{i(i+1)}{2} + n \text{ where } 1 \leq m, n \leq i+1\}.$$

Since C is cohesive, there is some n_0 such that

$$C \subseteq^* \{i : \varphi(i) \downarrow = \frac{i(i+1)}{2} + n_0\} \subseteq \{i : \varphi(i) \downarrow = \psi_{n_0}(i)\}.$$

Hence $[\varphi] = [\psi_{n_0}]$. Thus, $\Pi_C \mathcal{A}$ is isomorphic to the disjoint union $G \sqcup (\Pi_C \mathcal{A} - \{[\psi_m] : m \geq 1\})$. $\square$

3 Cohesive powers of equivalence structures

Let $\mathcal{A} = (A, E^{\mathcal{A}})$ be an equivalence structure $\mathcal{A}$. The equivalence class of $a \in A$ is

$$\text{equiv}^{\mathcal{A}}(a) = \{x \in A : x E^{\mathcal{A}} a\}.$$

We generally omit the superscript when it can be inferred from the context.

Definition 5 (i) Let $\mathcal{A}$ be an equivalence relation. The *character* of $\mathcal{A}$ is the set

$$\chi(\mathcal{A}) = \{\langle k, n \rangle : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

(ii) We say that $\mathcal{A}$ has *bounded character* if there is some finite k such that all finite equivalence classes of $\mathcal{A}$ have size at most k .

Clearly, two countable equivalence structures are isomorphic if they have the same character and the same number of infinite equivalence classes.

For a set X , by $\text{card}(X)$ or $|X|$ we denote the size of X . Let

$$\text{Inf}^{\mathcal{A}} = \{a : \text{equiv}^{\mathcal{A}}(a) \text{ is infinite}\} \text{ and } \text{Fin}^{\mathcal{A}} = \{a : \text{equiv}^{\mathcal{A}}(a) \text{ is finite}\}.$$

The following lemma from [4] gives us some important complexities.

Lemma 6 For any computable equivalence structure $\mathcal{A}$:

- (a) $\{\langle k, a \rangle : \text{card}(\text{equiv}^{\mathcal{A}}(a)) \leq k\}$ is a Π_1^0 set, and $\{\langle k, a \rangle : \text{card}(\text{equiv}^{\mathcal{A}}(a)) \geq k\}$ is a Σ_1^0 set;
- (b) $\text{Inf}^{\mathcal{A}}$ is a Π_2^0 set, and $\text{Fin}^{\mathcal{A}}$ is a Σ_2^0 set;
- (c) $\chi(\mathcal{A})$ is a Σ_2^0 set.

We say that a subset K of ω is a *character* if there is some equivalence structure with character K . This is the same as saying that $K \subseteq \langle (\omega - \{0\}) \times (\omega - \{0\}) \rangle$, and for all $n > 0$ and k ,

$$\langle k, n+1 \rangle \in K \Rightarrow \langle k, n \rangle \in K.$$

It was shown in [4] that for any Σ_2^0 character K , there is a computable equivalence structure $\mathcal{A}$ with character K , which has infinitely many infinite equivalence classes while $\text{Fin}^{\mathcal{A}}$ is a Π_1^0 set.

Theorem 7 [4] Let $\mathcal{A}$ be a computable equivalence structure. The structure $\mathcal{A}$ is computably categorical iff it is one of the following types:

1. $\mathcal{A}$ has only finitely many finite equivalence classes;
2. $\mathcal{A}$ has finitely many infinite classes, bounded character, and at most one finite k such that there are infinitely many classes of size k .

Hence if $\mathcal{A}$ is an equivalence structure as in the previous theorem and $\mathcal{D}$ is a computable structure isomorphic to $\mathcal{A}$, then for any cohesive set C we have $\Pi_C \mathcal{A} \cong \Pi_C \mathcal{D}$.

Proposition 8 *Let $\mathcal{A}$ be a computable equivalence structure. Let C be a cohesive set and let $\mathcal{B} = \Pi_C \mathcal{A}$.*

- (a) *Then $\chi(\mathcal{B}) = \chi(\mathcal{A})$.*
- (b) *If $\mathcal{A}$ has infinitely many infinite equivalence classes, then $\mathcal{B}$ has infinitely many infinite equivalence classes. If $\mathcal{A}$ has exactly n infinite equivalence classes, then $\mathcal{B}$ has at least n infinite equivalence classes.*
- (c) *If $\mathcal{A}$ has infinitely many infinite equivalence classes, then $\Pi_C \mathcal{A} \cong \mathcal{A}$.*
- (d) *If $\mathcal{A}$ has finitely many finite equivalence classes and no infinite equivalence classes, then $\Pi_C \mathcal{A} \cong \mathcal{A}$.*

Proof (a) The character of an equivalence structure is definable by a Σ_2^0 sentence (see [4]).

- (b) This holds since there is an embedding of $\mathcal{A}$ into $\mathcal{B}$.
- (c) This follows from (a) and (b). That is, if $\mathcal{A}$ has infinitely many infinite equivalence classes, then $\Pi_C \mathcal{A}$ also has infinitely many infinite equivalence classes, and since $\mathcal{A}$ and $\Pi_C \mathcal{A}$ have the same character, they are isomorphic.
- (d) This is true since $\mathcal{A}$ is a finite structure.

□

Theorem 9 *Let $\mathcal{A} = (A, E)$ be a computable equivalence structure. Let C be a cohesive set. If $\mathcal{A}$ has a bounded character, then $\Pi_C \mathcal{A} \cong \mathcal{A}$.*

Proof Since the character of $\mathcal{A}$ is bounded, if it is nonempty, let $k \in \omega$ be the largest size of a finite equivalence class of $\mathcal{A}$. If the character of $\mathcal{A}$ is empty, let $k = 0$. Recall that $\mathcal{A}$ and $\Pi_C \mathcal{A}$ have the same character. If $\mathcal{A}$ has infinitely many infinite equivalence classes, then $\mathcal{A}$ and $\Pi_C \mathcal{A}$ are isomorphic. Thus, assume that $\mathcal{A}$ has at most finitely many infinite equivalence classes.

If $\mathcal{A}$ has no infinite equivalence classes, then $\mathcal{A}$ satisfies the following Π_1^0 sentence, saying that there are no $k + 1$ pairwise distinct elements that are equivalent:

$$\neg(\exists x_1) \cdots (\exists x_{k+1}) [\bigwedge_{1 \leq i < j \leq k+1} (x_i \neq x_j \wedge x_i E x_j)]$$

Since $\Pi_C \mathcal{A}$ satisfies the same sentence, it has no infinite equivalence classes, so $\mathcal{A}$ and $\Pi_C \mathcal{A}$ are isomorphic.

Thus, assume that $\mathcal{A}$ has m infinite equivalence classes, where $m \in \omega$ and $m > 0$. Hence $\mathcal{A}$ satisfies the following Σ_2^0 sentence, saying that there are exactly m equivalence classes with at least $k + 1$ elements, hence infinite. We will use notation x_i^l for variables.

$$(\exists x_1^1) \cdots (\exists x_{k+1}^1) \cdots (\exists x_1^m) \cdots (\exists x_{k+1}^m) (\forall y_1) \cdots (\forall y_{k+1}) [\bigwedge_{\substack{1 \leq l \leq m \\ 1 \leq i < j \leq k+1}} (x_i^l \neq x_j^l) \wedge (\bigwedge_{1 \leq i < j \leq k+1} (y_i \neq y_j \wedge y_i E y_j) \Rightarrow \bigvee_{l=1, \dots, m} y_l E x_1^l)]$$

Hence $\Pi_C \mathcal{A}$ satisfies the same sentence, so it has exactly m infinite equivalence classes, so it is isomorphic to $\mathcal{A}$. $\square$

Theorem 10 *Let $\mathcal{A} = (A, E)$ be a computable equivalence structure. Let C be a cohesive set. If $\mathcal{A}$ has an unbounded character, then $\Pi_C \mathcal{A}$ has infinitely many infinite equivalence classes.*

Hence if $\mathcal{A}$ has an unbounded character and finitely many (possibly zero) infinite equivalence classes, then $\Pi_C \mathcal{A} \not\cong \mathcal{A}$.

Proof Without loss of generality, we will assume that $A = \omega$. Hence we can order the elements in A by the usual ordering of the natural numbers. Let $\mathcal{B} = \Pi_C \mathcal{A}$, and $\mathcal{B} = (B, E_B)$. Choose a computable sequence of elements in A :

$$a(0), a(1), a(2), a(3), \dots$$

such that they all belong to distinct equivalence classes, and for every k , the equivalence class of $a(k)$ has $> k$ elements. Since E is computable and the character of $\mathcal{A}$ is unbounded, such a sequence can be obtained by enumerating A and checking the conditions. We can think of $a(\langle m, i \rangle)$ as a representative of the equivalence class in $\mathcal{A}$ where a partial function $\psi_{m,*}$ might take its i^{th} coordinate value (for $i \geq 0$). That is, define partial computable functions $\psi_{m,n}(i)$, for $m, i \geq 0$ and $n \geq 1$, as follows:

$$\psi_{m,n}(i) = \begin{cases} c_n & \text{if } c_n \text{ is the } n^{th} \text{ element in } eqv(a(\langle m, i \rangle)) \text{ if it exists;} \\ \uparrow & \text{otherwise.} \end{cases}$$

Every $\psi_{m,n}$ is defined for all except possibly finitely many initial values, so $C \subseteq^* dom(\psi_{m,n})$. Hence $[\psi_{m,n}] \in B$.

Fix m . Let $n_1 \neq n_2$. Then we have that $[\psi_{m,n_1}] \neq [\psi_{m,n_2}]$ since $|eqv(a(\langle m, i \rangle))| > \langle m, i \rangle \geq i$, so starting with some i , $eqv(a(\langle m, i \rangle))$ will have the n_1^{th} and the n_2^{th} elements and they will be distinct. Also, $[\psi_{m,n_1}] E_B [\psi_{m,n_2}]$ since the values of ψ_{m,n_1} and ψ_{m,n_2} , when defined, are from the same equivalence class in $\mathcal{A}$. Hence the equivalence class of $[\psi_{m,1}]$ is infinite.

Now, let $m_1 \neq m_2$. Then $\neg([\psi_{m_1,1}] E_B [\psi_{m_2,1}])$ since the values of $\psi_{m_1,1}$ and $\psi_{m_2,1}$, when defined, are from different equivalence classes in $\mathcal{A}$. $\square$

Corollary 11 *Let $\mathcal{A} = (A, E)$ be a computable equivalence structure. Let C be a cohesive set. Then $\Pi_C \mathcal{A} \cong \mathcal{A}$ iff $\mathcal{A}$ has a bounded character or infinitely many infinite equivalence classes.*

It was shown in [3] that the following model-theoretic result holds for the equivalence structures. Let the formulas $\gamma_k(x)$ state that the equivalence class of x has at least k elements, where $k \in \omega - \{0\}$. Then the language of equivalence relations $\{E\}$ expanded with unary predicates $\{\gamma_k : k \geq 1\}$ has quantifier elimination; i.e., every first-order formula in the original language is logically equivalent to a quantifier-free formula in the new language. Since the formulas $\gamma_k(x)$ are Σ_1^0 formulas, it follows that a computable equivalence structure $\mathcal{A}$ and its cohesive powers satisfy the same

first-order sentences. However, in some cases, the distinction can be made by using computable (infinitary) sentences.

Corollary 12 *Let $\mathcal{A} = (A, E)$ be a computable equivalence structure with unbounded character and no infinite equivalence classes. Let C be a cohesive set. Then there is a computable infinitary Σ_3 sentence α such that $\Pi_C \mathcal{A} \models \alpha$ and $\mathcal{A} \models \neg \alpha$.*

Proof We have that $\Pi_C \mathcal{A}$ has an infinite equivalence class, while $\mathcal{A}$ does not. Let α be the sentence saying that there is an infinite equivalence class:

$$\exists x \bigwedge_{n \in \omega} [\exists y_1 \cdots \exists y_n \bigwedge_{1 \leq i < j \leq n} (y_i \neq y_j \wedge x E y_i)].$$

Hence α is a computable Σ_3 sentence true in $\Pi_C \mathcal{A}$ but false in $\mathcal{A}$. $\square$

4 Cohesive powers of injection structures

We will now study structures with a single unary function. An *injection structure* $\mathcal{A} = (A, f)$ consists of a non-empty set A with a one-to-one function $f : A \rightarrow A$. Let $f^0(a) =_{\text{def}} a$. Given $a \in A$, the *orbit of a under f* is defined as

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \omega)[f^n(a) = b \vee f^n(b) = a]\}.$$

We have that the size of an orbit $\mathcal{O}_f(a)$ is $k \geq 1$ if and only if $f^k(a) = a$ and $(\forall t < k)[f^t(a) \neq a]$. Hence the property that $\text{card}(\mathcal{O}_f(a)) = k$ is computable.

By analogy with the character of an equivalence structure, we define the *character* $\chi(\mathcal{A})$ of an injection structure $\mathcal{A}$ as follows:

$$\chi(\mathcal{A}) = \{\langle k, n \rangle : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}.$$

Hence $\langle k, n \rangle \in \chi(\mathcal{A})$ if and only if

$$(\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i=1}^n \text{card}(\mathcal{O}_f(x_i)) = k \wedge \bigwedge_{i \neq j} (\forall t < k)[f^t(x_i) \neq x_j] \right).$$

By $\text{ran}(f)$ we denote the range of f , $\text{ran}(f) = f(A)$. An injection structure (A, f) may have two types of infinite orbits: Z -orbits, which are isomorphic to $(\mathbb{Z}, S)$ and in which every element is in $\text{ran}(f)$, and ω -orbits, which are isomorphic to (ω, S) and have the form $\mathcal{O}_f(a) = \{f^n(a) : n \in \omega\}$ for some $a \notin \text{ran}(f)$. Thus, injection structures are characterized by the number of orbits of size k for each finite k , and by the number of orbits of types Z and of type ω .

For every computable injection structure $\mathcal{A} = (A, f)$, we have the following arithmetic complexity of important relations:

- (a) $\{(k, a) : a \in \text{ran}(f^k)\}$ is a Σ_1^0 set,
- (b) $\{(a, k) : \text{card}(\mathcal{O}_f(a)) \geq k\}$ is a Δ_1^0 set,
- (c) $\{a : \mathcal{O}_f(a) \text{ is infinite}\}$ is a Π_1^0 set,
- (d) $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$ is a Π_2^0 set,
- (e) $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ is a Σ_2^0 set, and
- (f) $\chi(\mathcal{A})$ is a Σ_1^0 set.

It was shown in [5] that for any c.e. character K , there is a computable injection structure $\mathcal{A} = (\omega, f)$ with character K and any specified finite or countably infinite number of orbits of types ω and Z and for which $\text{ran}(f)$ is computable and $\{a : \mathcal{O}_f(a) \text{ is finite}\}$ is computable.

Theorem 13 [5] *A computable injection structure $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ has finitely many infinite orbits.*

Hence if $\mathcal{A}$ is an injection structure with finitely many infinite orbits and $\mathcal{D}$ is a computable structure isomorphic to $\mathcal{A}$, then for any cohesive set C we have $\Pi_C \mathcal{A} \cong \Pi_C \mathcal{D}$.

By $f^{-1}(b)$ we will denote the unique a such that $f(a) = b$ if it exists, in symbols $f^{-1}(b) \downarrow = a$; otherwise $f^{-1}(b)$ is not defined which we also denote by $f^{-1}(b) \uparrow$. For $n \geq 1$, we denote by f^{-n} a partial function $(f^{-1})^n$.

Since the injection structures have the following axiom

$$\forall x \forall y [f(x) = f(y) \Rightarrow x = y],$$

we have that a cohesive power of a computable injection structure is an injection structure. We would like to determine the isomorphism types of such cohesive powers.

Proposition 14 *Let $\mathcal{A} = (A, f)$ be a computable injection structure. Let C be a cohesive set, and $\mathcal{B} = \Pi_C \mathcal{A}$.*

- (a) *Then $\chi(\mathcal{B}) = \chi(\mathcal{A})$.*
- (b) *The structures $\mathcal{A}$ and $\mathcal{B}$ have the same number of ω -orbits.*
- (c) *If $\mathcal{A}$ has bounded character and no infinite orbits, then $\mathcal{A} \cong \mathcal{B}$.*

Proof (a) The character of a computable injection structure is definable by a Σ_1^0 sentence.

- (b) Let $n \in \omega - \{0\}$. We can say that there are $\geq n$ many ω -orbits by the following Σ_2^0 sentence $\sigma_n: (\exists x_1) \cdots (\exists x_n)(\forall y)[\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \bigwedge_{1 \leq i \leq n} x_i \neq f(y)]$. Thus, having $< n$ many ω -orbits can be expressed by a Π_2^0 sentence, $\neg \sigma_n$. Hence if $\mathcal{A}$ has no ω -orbits, $\mathcal{B}$ has no ω -orbits; and if $\mathcal{A}$ has exactly n many ω -orbits, $\mathcal{B}$ has exactly n many ω -orbits. If $\mathcal{A}$ has infinitely many ω -orbits, then for every $n \geq 1$, $\mathcal{A} \models \sigma_n$ and hence $\mathcal{B} \models \sigma_n$, so $\mathcal{B}$ has infinitely many ω -orbits. The last conclusion also follows from the fact that $\mathcal{A}$ can be embedded into $\mathcal{B}$.
- (c) Since the character of $\mathcal{A}$ is bounded, let $k \in \omega - \{0\}$ be the largest size of a finite orbit of $\mathcal{A}$.

Since $\mathcal{A}$ has no infinite orbits, $\mathcal{A}$ satisfies the following Π_1^0 sentence, saying that there are no orbits of size $k + 1$:

$$\neg(\exists x)[\bigwedge_{1 \leq i \leq k+1} f^i(x) \neq x].$$

Thus, $\mathcal{B}$ satisfies the same sentence and, since $\mathcal{B}$ has the same character as $\mathcal{A}$, it is isomorphic to $\mathcal{A}$. $\square$

Theorem 15 *Let $\mathcal{A}$ be a computable injection structure with unbounded character. Let C be a cohesive set, and $\mathcal{B} = \Pi_C \mathcal{A}$. Then $\mathcal{B}$ has infinitely many Z -orbits.*

Hence if $\mathcal{A}$ has unbounded character and finitely many Z -orbits, then $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.

Proof Without loss of generality, we will assume that $A = \omega$. Choose a computable sequence of elements in A :

$$a(0), a(1), a(2), a(3), \dots$$

such that they all belong to distinct finite orbits, and for every k , the orbit of $a(k)$ has $> k$ elements. Since f is computable and the character of $\mathcal{A}$ is unbounded, such a sequence can be obtained by simultaneously enumerating the elements of A and checking the conditions.

For every natural number m and an integer $z \in \mathbb{Z}$, we will define a computable function $\psi_{m,z}$ as follows:

$$\psi_{m,z}(i) = f^z(a(\langle m, i \rangle)).$$

Each $[\psi_{m,z}]$ is in B . Fix m . Let integers z_1, z_2 be such that $z_1 \neq z_2$. Starting with some i_0 , the orbit of $a(\langle m, i \rangle)$ will be so large that for $i \geq i_0$, we will have

$$f^{z_1}(a(\langle m, i \rangle)) \neq f^{z_2}(a(\langle m, i \rangle)),$$

hence $[\psi_{m,z_1}] \neq [\psi_{m,z_2}]$. In addition, $[\psi_{m,z_1}]$ and $[\psi_{m,z_2}]$ belong to the same orbit in B since $f^{z_2-z_1}([\psi_{m,z_1}]) = [\psi_{m,z_2}]$. Hence the set $\{[\psi_{m,z}] : z \in \mathbb{Z}\}$ forms a Z -orbit. On the other hand, if $m_1 \neq m_2$, then for every i , $\langle m_1, i \rangle \neq \langle m_2, i \rangle$, so $a(\langle m_1, i \rangle)$ and $a(\langle m_2, i \rangle)$ belong to different orbits. Hence $[\psi_{m_1,0}]$ and $[\psi_{m_2,0}]$ belong to different orbits in B , so there are infinitely many Z -orbits. $\square$

Theorem 16 Let $\mathcal{A}$ be a computable injection structure with an infinite orbit. Let C be a cohesive set, and $B = \Pi_C \mathcal{A}$. Then B has infinitely many Z -orbits.

Hence if $\mathcal{A}$ has an infinite orbit, but has at most finitely many Z -orbits, then $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.

Proof Assume that $\mathcal{A}$ has an infinite orbit. Let a be an element of such an infinite orbit. For every natural number m and every integer z , we define a partial computable function $\psi_{m,z}$ by $\psi_{m,z}(n) = f^{m(2n+1)+n-1+z}(a)$. Since the domain of each ψ is co-finite, we have that $[\psi_{m,z}] \in B$. For any m and z_1, z_2 such that $z_1 \neq z_2$, we have that $\{n : \psi_{m,z_1}(n) \downarrow = \psi_{m,z_2}(n) \downarrow\} = \emptyset$ since the orbit of a is infinite; hence $[\psi_{m,z_1}] \neq [\psi_{m,z_2}]$. We also have that $f([\psi_{m,z}]) = [f \circ \psi_{m,z}] = [\psi_{m,z+1}]$. Hence for every m we have a Z -orbit $\{[\psi_{m,z}] : z \in \mathbb{Z}\} = \mathcal{O}([\psi_{m,0}])$.

Now, assume that $m_1 \neq m_2$. Since for any $k \in \omega$, we have $f^k([\psi_{m_1,0}]) \neq [\psi_{m_2,0}]$, it follows that $\mathcal{O}([\psi_{m_1,0}]) \neq \mathcal{O}([\psi_{m_2,0}])$. Hence B has infinitely many Z -orbits. $\square$

Corollary 17 Let $\mathcal{A} = (A, f)$ be a computable injection structure. Let C be a cohesive set. Then $\Pi_C \mathcal{A} \cong \mathcal{A}$ iff $\mathcal{A}$ has a bounded character and no infinite orbits, or $\mathcal{A}$ has infinitely many Z -orbits.

It was shown in [5] that the following model-theoretic result holds for the injection structures. Let the formulas $\gamma_k(x)$ state that $(\exists y)[f^k(y) = x]$. Then in the language of injection structures $\{f\}$ expanded with unary predicates $\{\gamma_k : k \geq 1\}$ we have quantifier elimination; i.e., every first-order formula in the original language is logically equivalent to a quantifier-free formula in the new language. Since the formulas $\gamma_k(x)$ are Σ_1^0 formulas, it follows that a computable injection structure $\mathcal{A}$ and its cohesive powers satisfy the same first-order sentences. However, in some cases, the distinction can be made by using computable (infinitary) sentences.

Corollary 18 *If $\mathcal{A}$ is a computable injection structure with an unbounded character and no infinite orbits, then there is a computable (infinitary) Σ_2 sentence α such that $\mathcal{B} \models \alpha$ and $\mathcal{A} \not\models \alpha$.*

Proof Let α say that there is an infinite orbit:

$$\exists x \bigwedge_{k \in \omega} [f^{(k)}(x) \neq x].$$

□

5 Cohesive powers of two-to-one structure

We will now investigate cohesive powers of two-to-one structures that were introduced and studied in [6] from the computability-theoretic point of view with focus on the complexity of isomorphisms between these structures.

Definition 19 A *two-to-one structure* $\mathcal{A} = (A, f)$ consists of a non-empty domain A with a single unary function $f : A \rightarrow A$ such that for every $a \in A$ we have $\text{card}(f^{-1}(a)) = 2$.

We will also denote a two-to-one structure as 2:1 structure, and often identify it with its directed graph $G_{\mathcal{A}}$ with vertex set A and edges $(a, f(a))$ for $a \in A$.

Definition 20 For $a \in A$, the orbit of a is:

$$\mathcal{O}_f(a) = \{x \in A : (\exists n, m \in \omega) [f^n(x) = f^m(a)]\}.$$

That is, the orbit of a is the set of elements of A , which belong to the same connected component of $G_{\mathcal{A}}$ to which a belongs.

Let B be a full binary tree with its nodes pointing toward the root. We can show that there are two types of orbits in a 2:1 structure: Z -chains, and k -cycles for $k \geq 1$. A Z -chain consists of a directed one-to-one basic sequence of nodes ordered as integers, with a tree B attached to every node of the basic sequence as follows: there is a connecting edge, which is not part of the basic sequence but points toward the sequence, to which the root of B is attached. A k -cycle is a directed one-to-one cycle of size k such that a tree B is attached to each node of the cycle via a connecting edge, which is not part of the cycle but points toward the cycle, to which the root of B is attached. Hence all tree edges point toward the cycle. For pictures illustrating orbits see Sect. 1 in [6].

Lemma 21 [6] (i) *The predicate “ $\mathcal{O}_f(a)$ is a k -cycle” is Σ_1^0 .*

(ii) *The predicate “ $\mathcal{O}_f(a)$ is a Z -chain” is Π_1^0 .*

Two countable 2:1 structures are isomorphic if they have the same number of k -cycles for every $k \geq 1$, and the same number of Z -chains.

2:1 structures have the following axioms:

$$\forall y \exists x_1 \exists x_2 [x_1 \neq x_2 \wedge f(x_1) = y \wedge f(x_2) \Rightarrow y]$$

$$\forall x_1 \forall x_2 \forall x_3 [(f(x_1) = f(x_2) \wedge f(x_2) = f(x_3)) \Rightarrow (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3)].$$

Hence a cohesive power of a computable 2:1 structure is a 2:1 structure. We would like to determine the isomorphism types of these cohesive powers.

Theorem 22 Let $\mathcal{A}$ be a computable $2 : 1$ structure. Let C be a cohesive set, and $\mathcal{B} = \Pi_C \mathcal{A}$.

- (i) The cohesive power $\mathcal{B}$ has the same number of k -cycles, for any $k \geq 1$, as $\mathcal{A}$ does.
- (ii) The cohesive power $\mathcal{B}$ has infinitely many Z -chains.

Hence if $\mathcal{A}$ has at most finitely many Z -chains, then $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.

Proof (i) The property that a $2:1$ structure has at least n many k -cycles, where $n, k \geq 1$ can be expressed by an existential sentence $\theta_{n,k}$:

$$(\exists x_1) \cdots (\exists x_n) \left[\bigwedge_{1 \leq m \leq n} (f^k(x_m) = x_m \wedge (\bigwedge_{1 \leq l < k} f^l(x_m) \neq x_m)) \right. \\ \left. \wedge \bigwedge_{(1 \leq i < j \leq n) \& (1 \leq l < k)} f^l(x_i) \neq x_j \right].$$

Hence both $\mathcal{A}$ and its cohesive power $\mathcal{B}$ satisfy the same such sentences, so they have the same number of k -cycles.

- (ii) Fix a natural ordering on the domain A . We will “abuse” the notation and by f^{-1} denote the unary function on A , which for every a chooses the smaller of the two elements that f maps into a . Hence f^{-z} will be defined for every integer z where, as usual, $f^0(a) = a$.

Since $\mathcal{A}$ always contains a full binary tree component $\mathcal{T}$, we can define a computable function $g : \omega \rightarrow A$, which chooses elements $g(n)$ on $\mathcal{T}$ that are spaced apart so that $f^z(g(n))$ where $|z| \leq n$ do not “interfere” for different n ’s. More precisely, if $n_1 \neq n_2$ or $z_1 \neq z_2$, then $f^{z_1}(g(n_1)) \neq f^{z_2}(g(n_2))$ where $|z_1| \leq n_1$ and $|z_2| \leq n_2$. Equivalently,

$$|f^z(g(n)) : -n \leq z \leq n \wedge 0 \leq n \leq m| = (m+1)^2.$$

Our goal is to use this property to define partial computable functions $\psi_{m,z}$ for natural numbers m and integers z , such that $\psi_{m,*}$ ’s witness that there are infinitely many Z -chains.

A partial function $\psi_{m,z} : \omega \rightarrow A$ is defined as follows:

$$\psi_{m,z}(x) = \begin{cases} f^z(g(\langle m, x \rangle)) & \text{if } |z| \leq \langle m, x \rangle; \\ \uparrow & \text{otherwise.} \end{cases}$$

It follows that $[\psi_{m,z}] \in B$ for every m, z . Furthermore, $f([\psi_{m,z}]) = [\psi_{m,z+1}]$, so $\{[\psi_{m,z}] : z \in \mathbb{Z}\}$ is a subset of a Z -chain. For any pair of natural numbers m_1 and m_2 such that $m_1 \neq m_2$ and arbitrary k_1, k_2 , we have $f^{k_1}([\psi_{m_1,0}]) \neq f^{k_2}([\psi_{m_2,0}])$, so $[\psi_{m_1,0}]$ and $[\psi_{m_2,0}]$ belong to different Z -chains. Hence $\mathcal{B}$ has infinitely many Z -chains. $\square$

Corollary 23 Let $\mathcal{A}$ be a computable $2 : 1$ structure. Let C be a cohesive set. Then $\mathcal{A} \cong \Pi_C \mathcal{A}$ iff $\mathcal{A}$ has infinitely many Z -chains.

Corollary 24 Let $\mathcal{A}$ be a computable $2 : 1$ structure with no Z -chains. Let C be a cohesive set, and $\mathcal{B} = \Pi_C \mathcal{A}$. Then there is a computable (infinitary) Σ_2 sentence α such that $\mathcal{B} \models \alpha$ and $\mathcal{A} \not\models \alpha$.

Proof Let α say that there is a Z -chain:

$$\exists x \bigwedge_{l \geq 0 \& k > 0} (f^{(l+k)}(x) \neq f^l(x)).$$

□

6 Cohesive powers of $(2,0):1$ structures

We will now investigate the class of $(2,0):1$ structures, which includes $2:1$ structures. They were introduced and studied in [6] from the computability-theoretic point of view with focus on the complexity of isomorphisms between these structures.

Definition 25 A $(2, 0) : 1$ structure is a structure with a single unary function, $\mathcal{A} = (A, f)$ where $f : A \rightarrow A$, such that for every $a \in A$, we have $\text{card}(f^{-1}(a)) \in \{0, 2\}$.

As usual, a $(2,0):1$ structure $\mathcal{A}$ is often identified with its directed graph $G(A, f)$, and the orbit of a is defined to be the set of all points in A which belong to the connected component of $G(A, f)$ containing a . The orbits of $(2,0):1$ structures can be k -cycles for $k \geq 1$, Z -chains, or ω -chains. A k -cycle consists of a directed one-to-one cycle of size k such that for each node of the cycle there is a connecting edge, which is not part of the cycle and pointing toward the cycle, to which the root of a binary tree is attached with all tree edges pointing toward the cycle. Here, a tree can be finite or infinite and it satisfies the condition that $\text{card}(f^{-1}(a)) \in \{0, 2\}$.

A Z -chain consists of a directed one-to-one basic sequence of nodes ordered as integers, with a binary tree attached to every node of the basic sequence as follows: there is a connecting edge, which is not part of the basic sequence but points toward the sequence, to which the root of a tree is attached. Hence each element of a k -cycle or a Z -chain also has a binary branching tree attached to it as its root and with all edges directed toward the root. An ω -chain consists of a directed basic sequence of nodes ordered as natural numbers such that for every node except the first one there is a binary branching tree attached as above, with connecting edge pointing toward the basic sequence and the tree edges pointing toward the root. Hence the first node of the basic sequence does not belong to the range of f . For pictures illustrating orbits see Sect. 1 in [6].

Let (A, f) be a $(2,0):1$ structure and $a \in A$. The *length* of a is defined as:

$$l(a) = \sup\{n + 1 : |\{a, f(a), f^2(a), \dots, f^n(a)\}| = n + 1\}.$$

It is the longest “non-cycling” directed path starting with a . It can be finite or infinite.

Definition 26 Let $\mathcal{A} = (A, f)$ be a $(2, 0) : 1$ structure. We will assume below that $k, n \in \omega - \{0\}$.

(a) The *cycle character* of $\mathcal{A}$ is

$$\chi_{\text{cycle}}(\mathcal{A}) = \{\langle k, n \rangle : \mathcal{A} \text{ has } \geq n \text{ many } k\text{-cycles}\}.$$

(b) The *path character* of $\mathcal{A}$ is

$$\chi_{\text{path}}(\mathcal{A}) = \{\langle k, n \rangle : \mathcal{A} \text{ has } \geq n \text{ many } a \text{ such that } l(a) = k\}.$$

(c) The *endpath character* of $\mathcal{A}$ is

$$\chi_{\text{endpath}}(\mathcal{A}) = \{\langle k, n \rangle : \mathcal{A} \text{ has } \geq n \text{ many } a \notin f(A) \text{ such that } l(a) = k\}.$$

We say that a character is bounded if there is an upper bound on k .

(2,0):1 structures have the following axioms:

$$\forall y \forall x \exists z [f(x) \neq y \vee (z \neq x \wedge f(z) = y)] \text{ and}$$

$$\forall x_1 \forall x_2 \forall x_3 [(f(x_1) = f(x_2) \wedge f(x_2) = f(x_3)) \Rightarrow (x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3)].$$

Hence a cohesive power of a computable (2,0):1 structure is a (2,0):1 structure.

Proposition 27 *Let $\mathcal{A}$ be a computable (2, 0) : 1 structure. Let C be a cohesive set, and let $\mathcal{B} = \prod_C \mathcal{A}$. Then $\mathcal{A}$ and $\mathcal{B}$ have the same cycle character, path character, and endpath character.*

Proof Let $k, n \geq 1$. Then $\langle k, n \rangle \in \chi_{\text{cycle}}(\mathcal{A})$ can be expressed by a Σ_1^0 sentence as in the proof of Theorem 22 (i).

Furthermore, $\langle k, n \rangle \in \chi_{\text{path}}(\mathcal{A})$ can be expressed by the following Σ_1^0 sentence:

$$(\exists x_1) \cdots (\exists x_n) [\bigwedge_{1 \leq m \leq n} ((\bigwedge_{0 \leq l < s < k} f^l(x_m) \neq f^s(x_m)) \wedge$$

$$\bigvee_{1 \leq l < k} (f^k(x_m) = f^l(x_m))) \wedge \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j].$$

Finally, $\langle k, n \rangle \in \chi_{\text{endpath}}(\mathcal{A})$ can be expressed by the following Σ_2^0 sentence:

$$(\exists x_1) \cdots (\exists x_n) [\bigwedge_{1 \leq m \leq n} ((\bigwedge_{0 \leq l < s < k} f^l(x_m) \neq f^s(x_m)) \wedge$$

$$\bigvee_{1 \leq l < k} (f^k(x_m) = f^l(x_m)) \wedge (\forall y)(f(y) \neq x_m)) \wedge \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j].$$

By Theorem 2, it follows that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences above for every pair $\langle k, n \rangle$, so $\mathcal{A}$ and $\mathcal{B}$ have the same characters. $\square$

Theorem 28 *Let C be a cohesive set.*

- (i) *Let $\mathcal{A}$ be a computable (2, 0) : 1 structure with bounded path character and no infinite orbits. Then $\Pi_C \mathcal{A} \cong \mathcal{A}$.*
- (ii) *Let $\mathcal{A}$ be a computable (2, 0) : 1 structure with unbounded path character or with an infinite orbit. Then $\Pi_C \mathcal{A}$ has infinitely many Z -chains.*

Hence if a computable (2, 0) : 1 structure $\mathcal{A}$ has unbounded path character and only finitely many (including zero) Z -chains, we have $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.

Proof (i) Let $\mathcal{B} = \Pi_C \mathcal{A}$. Let $M \in \omega$ be the least upper bound for the length of (finite) paths in $\mathcal{A}$. Since $\mathcal{A}$ does not have infinite orbits, it satisfies the following Π_1^0 sentence: $(\forall x) \bigvee_{0 \leq m < n \leq M} (f^m(x) = f^n(x))$. By Theorem 2, $\mathcal{B}$ also satisfies this sentence, which implies that $\mathcal{B}$ has no infinite orbits. Since we can completely describe finite orbits by Σ_2^0 sentences, it follows that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

- (ii) Fix a natural ordering on the domain A . We will denote by f^{-1} a partially computable unary function on A , which for every a chooses the smaller of the two elements that f maps into a , if $\text{card}(f^{-1}(a)) = 2$, and is undefined otherwise. Hence we have f^{-z} for every integer z , where $f^0(a) = a$. By the assumption about $\mathcal{A}$, we can define a computable function $g : \omega \rightarrow A$, which chooses elements $g(n)$ in A such that for every $m \in \omega$, we have $|\{f^z(g(n)) : f^z(g(n)) \downarrow \wedge 0 \leq n \leq m \wedge -n \leq z \leq n\}| = (m+1)^2$. We now proceed similarly as in (ii) in the proof of Theorem 22 to show that there are infinitely many Z -chains. $\square$

The following theorem focuses on ω -chains.

Theorem 29 *Let C be a cohesive set. Let $\mathcal{A}$ be a computable $(2, 0) : 1$ structure with bounded endpath character and with finitely many (including 0) elements in $A - f(A)$ of infinite length. Then $\prod_C \mathcal{A}$ and $\mathcal{A}$ have the same number of ω -chains.*

Proof Let $\mathcal{B} = \prod_C \mathcal{A}$. Let $M \in \omega$ be the least upper bound for the endpath character of $\mathcal{A}$; that is, $M = \max\{k : \langle k, n \rangle \in \chi_{\text{endpath}}(\mathcal{A})\}$.

Suppose that in $\mathcal{B}$ we choose $[\psi] \in B - f(B)$ such that $[\psi]$ is an element of an ω -chain. We will show that $[\psi]$ belongs to the range of the canonical embedding function of $\mathcal{A}$ into $\mathcal{B}$.

Consider a co-c.e. set $W = \{i \in \omega : \psi(i) \uparrow \vee l(\psi(i)) > M\}$, which is infinite. We claim that $C \subseteq^* W$. To establish the claim, assume otherwise, hence $C \subseteq^* \overline{W}$. We have that $\overline{W} = \{i \in \omega : \psi(i) \downarrow \wedge l(\psi(i)) \leq M\}$ so it is a finite union of c.e. sets Y_j for $j \in \{1, \dots, M\}$ where

$$Y_j = \bigcup_{0 \leq k < j} \{i \in \omega : \psi(i) \downarrow \wedge f^j(\psi(i)) \downarrow = f^k(\psi(i)) \downarrow \wedge \bigwedge_{0 \leq m < n < j} f^m(\psi(i)) \downarrow \neq f^n(\psi(i)) \downarrow\}.$$

Hence, since C is cohesive, for some j_0, k_0 such that $j_0 > k_0$ we have that

$$C \subseteq^* \{i \in \omega : \psi(i) \downarrow \wedge f^{j_0}(\psi(i)) \downarrow = f^{k_0}(\psi(i)) \downarrow \wedge \bigwedge_{0 \leq m < n < j_0} f^m(\psi(i)) \downarrow \neq f^n(\psi(i)) \downarrow\},$$

which will imply that $f^{j_0}[\psi] = f^{k_0}[\psi]$, contradicting the fact that $[\psi]$ is an element of an ω -chain. Hence

$$C \subseteq^* W = \{i \in \omega : \psi(i) \uparrow \vee l(\psi(i)) > M\}.$$

Since $C \subseteq^* \{i \in \omega : \psi(i) \downarrow\}$, we have

$$\begin{aligned} C &\subseteq^* \{i \in \omega : \psi(i) \downarrow \wedge l(\psi(i)) > M\} = \\ &\{i \in \omega : \psi(i) \downarrow \wedge l(\psi(i)) > M \wedge \psi(i) \in f(A)\} \cup \\ &\{i \in \omega : \psi(i) \downarrow \wedge l(\psi(i)) > M \wedge \psi(i) \in A - f(A)\}. \end{aligned}$$

We can further show that $C \subseteq^* \{i \in \omega : \psi(i) \downarrow \wedge l(\psi(i)) > M \wedge \psi(i) \in A - f(A)\}$ since, otherwise, there is $[\tau] \in B$ such that $f([\tau]) = [\psi]$, contradicting the fact that $[\psi] \in B - f(B)$. For $a \in A$, let $X_a = \{i \in \omega : \psi(i) \downarrow = a\}$.

Hence, $C \subseteq^* \bigcup \{X_a : a \in (A - f(A)) \wedge l(a) = \infty\}$. By assumption, the union in the previous formula is a finite union, so for some a_0 , we have

$$C \subseteq^* \{i \in \omega : \psi(i) \downarrow = a_0\}.$$

Thus, $[\psi]$ belongs to the range of the canonical embedding. It follows that the number of ω -chains is the same in $\mathcal{A}$ and $\mathcal{B}$. $\square$

For the following result we require more decidability in a computable structure.

Definition 30 [6] A computable $(2, 0) : 1$ structure $\mathcal{A} = (A, f)$ is said to be *highly computable* if $\text{ran}(f)$ is a computable set.

Theorem 31 *Let C be a cohesive set. Let $\mathcal{A}$ be a highly computable $(2, 0) : 1$ structure with unbounded endpath character or with infinitely many elements in $A - f(A)$ of infinite length. Then $\prod_C \mathcal{A}$ has infinitely many ω -chains.*

Hence if $\mathcal{A}$ has only finitely many ω -chains, we have $\mathcal{A} \not\cong \prod_C \mathcal{A}$.

Proof Let $B = \prod_C \mathcal{A}$. Let $g : \omega \rightarrow A$ be a computable function such that $g(\omega) \subseteq A - f(A)$ and for every $m \in \omega$ we have

$$|\{f^k(g(n)) : f^k(g(n)) \downarrow \wedge 0 \leq n \leq m \wedge 0 \leq k \leq n\}| = \frac{(m+1)(m+2)}{2}.$$

A partial function $\psi_{m,n} : \omega \rightarrow A$ for $m, n \in \omega$ is defined as follows:

$$\psi_{m,n}(x) = \begin{cases} f^n(g(\langle m, x \rangle)) & \text{if } 0 \leq n \leq \langle m, x \rangle; \\ \uparrow & \text{otherwise.} \end{cases}$$

It follows that every $[\psi_{m,n}] \in B$. If $n_1 \neq n_2$, then $[(\psi_{m,n_1})] \neq [(\psi_{m,n_2})]$. Furthermore, $[(\psi_{m,0})] \in B - f(B)$ and $f([\psi_{m,n}]) = [\psi_{m,n+1}]$, so $\{[\psi_{m,n}] : n \in \omega\}$ is a subset of an ω -chain. For any pair of natural numbers m_1, m_2 such that $m_1 \neq m_2$ and arbitrary k_1, k_2 , we have $f^{k_1}([\psi_{m_1,0}]) \neq f^{k_2}([\psi_{m_2,0}])$, so $[\psi_{m_1,0}]$ and $[\psi_{m_2,0}]$ belong to different ω -chains. Hence B has infinitely many ω -chains. $\square$

Corollary 32 *Let $\mathcal{A}$ be a highly computable $(2, 0) : 1$ structure with unbounded end-path character and no ω -chains. Let C be a cohesive set, and $B = \prod_C \mathcal{A}$. Then there is a computable Σ_2 sentence α such that $B \models \alpha$ and $\mathcal{A} \not\models \alpha$.*

Proof The property that there is an ω -chain can be expressed by a computable (infinite) Σ_2 sentence α :

$$\exists x (\bigwedge_{y \in A} (x \neq f(y)) \wedge \bigwedge_{l \geq 0 \wedge k > 0} (f^{(l+k)}(x) \neq f^n(x)))$$

$\square$

7 Cohesive powers of partial injection structures

A *partial injection structure* (A, f) consists of a set A and a partial function $f : A \rightarrow A$ such that if $x, y \in \text{dom}(f)$ and $x \neq y$, then $f(x) \neq f(y)$. We will call f a *partial injection*. As usual, we write $f(x) \downarrow$ to denote that $x \in \text{dom}(f)$, and $f(x) \uparrow$ to denote that $x \notin \text{dom}(f)$. Also, $f(x) \downarrow = y$ stands for $f(x) \downarrow$ and $f(x) = y$. Partial inverse function f^{-1} is defined naturally. For $z \in \mathbb{Z}$, f^z is defined as the usual composition of partial functions. Partial injection structures and their computability-theoretic properties, including complexity of their isomorphisms, were studied by Marshall in [27]. She calls a partial injection structure (A, f) a *partial computable injection structure* if A is a computable set and f is a partial computable function.

In order to make a partial injection structure (A, f) into a first-order structure, we will consider it as a relational structure $\mathcal{A} = (A, G_f)$, where G_f is the graph of f :

$$G_f = \{(x, y) : x \in \text{dom}(f) \wedge f(x) = y\}.$$

Having this framework in mind, we can still write (A, f) .

Definition 33 We say that a partial computable injection structure (A, f) is a *computable partial injection structure* if G_f is a computable binary relation. Hence (A, G_f) is a computable structure.

Proposition 34 *Let (A, f) be a partial computable injection structure.*

- (i) *If $\text{ran}(f)$ is computable, then G_f is computable.*
- (ii) *If $\text{dom}(f)$ is computable, then G_f is computable.*

Proof (i) Without loss of generality, we may assume that $A = \omega$. Given a pair (x, y) , first determine whether $y \in \text{ran}(f)$. If $y \notin \text{ran}(f)$, then $(x, y) \notin G_f$. If $y \in$

- $\text{ran}(f)$, then run a Turing machine program P_f for computing f simultaneously on $0, 1, 2, \dots$ by adding more and more inputs and computation steps (although finitely many at every stage) until we find z such that the program P_f halts on z and outputs y . If $x = z$ then $(x, y) \in G_f$, and if $x \neq z$ then $(x, y) \notin G_f$.
- (ii) Given a pair (x, y) , first determine whether $x \in \text{dom}(f)$, and if that is the case compute $f(x)$.

□

The domain of a partial computable injection function with computable range does not have to be computable. For example, for the halting set K , consider a computable $1-1$ enumeration $g : \omega \rightarrow K$. Let $f : \omega \rightarrow \omega$ be defined as:

$$f(x) = \begin{cases} g^{-1}(x) & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

Then f is a partial computable injection with $\text{ran}(f) = \omega$ and $\text{dom}(f) = K$.

Let (A, f) be a partial injection structure. The orbit of a is defined to be:

$$O_f(a) = \{b : \exists n \in \omega (f^n(a) \downarrow = b \vee f^n(b) \downarrow = a)\}.$$

There are five kinds of orbits. Finite orbits may be k -cycles or k -chains for $k \geq 1$. A k -chain is of the form $\{x_i : 1 \leq i \leq k\}$ where $x_i \neq x_j$ for $1 \leq i < j \leq k$ and $x_1 \in A - \text{ran}(f)$, $x_{i+1} = f(x_i)$, and $x_k \notin \text{dom}(f)$. Infinite orbits may be $\mathbb{Z}$ -chains, ω -chains, or ω^* -chains. An ω^* -chain is of the form $\{x_i : i \in \omega\}$ where $x_0 \in A - \text{dom}(f)$, $x_i = f(x_{i+1})$, and $x_i \neq x_j$ for $i \neq j$.

Definition 35 Let $\mathcal{A} = (A, f)$ be a partial injection structure. In the following definitions we will assume that $k, n \in \omega - \{0\}$.

(a) The *cycle character* of $\mathcal{A}$ is

$$\chi_{\text{cycle}}(\mathcal{A}) = \{\langle k, n \rangle : \mathcal{A} \text{ has } \geq n \text{ many } k\text{-cycles}\}.$$

(b) The *finite chain character* of $\mathcal{A}$ is

$$\chi_{\text{path}}(\mathcal{A}) = \{\langle k, n \rangle : \mathcal{A} \text{ has } \geq n \text{ many } k\text{-chains}\}.$$

We say that a character in the previous definition is bounded if there is an upper bound on the size k . Two countable partial injection structures are isomorphic if and only if they have the same cycle character, the same finite chain character, and the same number of $\mathbb{Z}$ -chains, ω -chains and ω^* -chains.

Let $\mathcal{A} = (A, f)$ be a computable partial injection structure. Let C be a cohesive set, and $\mathcal{B} = \Pi_C \mathcal{A}$. Then $f^{\mathcal{B}}([\psi]) \downarrow$ if $C \subseteq^* \{i \in \omega : f(\psi(i)) \downarrow\}$ and $f^{\mathcal{B}}([\psi]) \uparrow$ otherwise. Similarly, $f^{\mathcal{B}}([\psi]) = [\phi]$ if and only if $C \subseteq^* \{i \in \omega : f(\psi(i)) \downarrow = \phi(i) \downarrow\}$. We often omit the superscript in $f^{\mathcal{B}}$.

The following proposition is based on Theorem 2.

Theorem 36 Let $\mathcal{A} = (A, f)$ be a computable partial injection structure. Let C be a cohesive set. Then the cohesive power $\Pi_C \mathcal{A}$ is a partial injection structure that has the same cycle character and finite chain character as $\mathcal{A}$.

Proof Being a partial injection structure can be described by the following Π_1^0 sentences:

$$\begin{aligned} & \forall x \forall y \forall z [(f(x) = z \wedge f(y) = z) \Rightarrow x = y] \\ & \forall x \forall y \forall z [(f(x) = y \wedge f(x) = z) \Rightarrow y = z]. \end{aligned}$$

Furthermore,

$\langle k, n \rangle \in \chi_{\text{cycle}}(\mathcal{A})$ iff
 $\exists x_1 \cdots \exists x_n [\bigwedge_{1 \leq i \leq n} (f^k(x_i) = x_i \wedge \bigwedge_{1 \leq l < k} f^l(x_i) \neq x_i) \wedge \bigwedge_{1 \leq i < j \leq n \wedge 1 \leq l < k} f^l(x_i) \neq x_j],$

which is a Σ_1^0 sentence.

Also,

$\langle k, n \rangle \in \chi_{\text{path}}(\mathcal{A})$ iff
 $\exists x_1 \cdots \exists x_n \exists y_1 \cdots \exists y_n \forall y [\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \bigwedge_{1 \leq i \leq n} (f(y) \neq x_i \wedge f^{k-1}(x_i) = y_i \wedge f(y_i) \neq y)],$

which is a Σ_2^0 sentence. $\square$

In some cases, the cohesive power is isomorphic to the original structure.

Theorem 37 *Let $\mathcal{A}$ be a computable partial injection structure with bounded cycle character, bounded finite chain character, and no infinite orbits. Let C be a cohesive set. Then $\mathcal{A} \cong \Pi_C \mathcal{A}$.*

Proof Let $\mathcal{A} = (A, f)$ and $\mathcal{B} = \Pi_C \mathcal{A}$. Let M be the maximum size of finite orbits. Since $\mathcal{A}$ has no infinite orbits, it satisfies the following Π_2^0 sentence:

$$\neg \exists x \exists y (f^{M+1}(x) = y \wedge \bigwedge_{1 \leq i \leq M} (f^i(x) \neq x)).$$

Hence $\mathcal{B}$ satisfies the same sentence, so it has no infinite orbits. Thus, together with Theorem 36, we have $\mathcal{A} \cong \mathcal{B}$. $\square$

If the conditions of the previous theorem are not satisfied, the cohesive power will have infinitely many Z -chains.

Theorem 38 *Let $\mathcal{A}$ be a computable partial injection structure with unbounded cycle character, or with unbounded finite chain character, or with an infinite orbit. Let C be a cohesive set. Then $\Pi_C \mathcal{A}$ has infinitely many Z -chains.*

Hence if $\mathcal{A}$ has only finitely many Z -chains, then $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.

Proof The proof is similar to the proof of part (ii) of Theorem 22. $\square$

The cohesive power always has at least as many orbits of a certain fixed type as $\mathcal{A}$. In some cases the number of ω -chains and ω^* -chains is the same.

Theorem 39 *Let $\mathcal{A}$ be a computable partial injection structure with bounded finite chain character. Let C be a cohesive set. Then $\Pi_C \mathcal{A}$ has the same number of ω -chains and the same number of ω^* -chains as $\mathcal{A}$.*

Proof Let $\mathcal{A} = (A, f)$ and $\mathcal{B} = \Pi_C \mathcal{A}$. Let M be the maximum size of finite chains. The property that $\mathcal{A}$ has $\geq n$ many ω -chains can be expressed by the following Σ_2^0 sentence:

$$\exists x_1 \cdots \exists x_n \exists y_1 \cdots \exists y_n \forall z [\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \bigwedge_{1 \leq i \leq n} (f^M(x_i) = y_i \wedge f(z) \neq x_i)].$$

Hence $\mathcal{A}$ and $\mathcal{B}$ have the same number of ω -chains.

Similarly, the property that $\mathcal{A}$ has $\geq n$ many ω^* -chains can be expressed by the following Σ_2^0 sentence:

$$\exists x_1 \cdots \exists x_n \exists y_1 \cdots \exists y_n \forall z [\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \bigwedge_{1 \leq i \leq n} (f^M(y_i) = x_i \wedge f(x_i) \neq z)].$$

$\square$

Theorem 40 Let $\mathcal{A} = (A, f)$ be a computable partial injection structure with unbounded finite chain character. Let C be a cohesive set.

- (i) Assume that $\text{ran}(f)$ is computable. Then $\Pi_C \mathcal{A}$ has infinitely many ω -chains. Hence if $\mathcal{A}$ has only finitely many ω -chains, then $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.
- (ii) Assume that $\text{dom}(f)$ is computable. Then $\Pi_C \mathcal{A}$ has infinitely many ω^* -chains. Hence if $\mathcal{A}$ has only finitely many ω^* -chains, then $\mathcal{A} \not\cong \Pi_C \mathcal{A}$.

Proof (i) The proof is similar to the proof of Theorem 31.

- (ii) Let $g : \omega \rightarrow (A - \text{dom}(f))$ be a computable function such that if natural numbers m_1, m_2, n_1, n_2 satisfy the property that $m_1 \neq m_2$ or $n_1 \neq n_2$, then $f^{-n_1}(g(m_1)) \neq f^{-n_2}(g(m_2))$ where $n_1 \leq m_1$ and $n_2 \leq m_2$. Such a function exists since the chain character is unbounded and $\text{dom}(f)$ is computable.

We define a partial function $\psi_{m,n} : \omega \rightarrow A$ for $m, n \in \omega$ as follows:

$$\psi_{m,n}(x) = \begin{cases} f^{-n}(g(\langle m, x \rangle)) & \text{if } n \leq \langle m, x \rangle; \\ \uparrow & \text{otherwise.} \end{cases}$$

It follows that $[\psi_{m,n}] \in \Pi_C \mathcal{A}$ since $\text{dom}(\psi_{m,n})$ is cofinite. For $m \in \omega$, we can show that $\{[\psi_{m,n}] : n \in \omega\}$ forms an ω^* -chain, and for $m_1 \neq m_2$ we have that $[\psi_{m_1,0}]$ and $[\psi_{m_2,0}]$ belong to different ω^* -chains. Hence $\Pi_C \mathcal{A}$ has infinitely many ω^* -chains. $\square$

8 Concluding remarks

In this paper, we focus on the isomorphism types of cohesive powers of certain computable structures with a single binary relation, such as graphs, equivalence structures, and partial injection structures. This adds to the previous study of the cohesive powers of the ordered set of natural numbers, $(\omega, <)$, and other natural linear orderings in [17, 18]. Here, we also investigate cohesive powers of computable structures with a unary function that is one-to-one, two-to-one, and $(2, 0) : 1$, which can be identified with the directed graphs they induce. It will be worthwhile to investigate the isomorphism types of cohesive powers of other directed graphs induced by functions. Some structures in the classes we consider are isomorphic to all of their cohesive powers. It was previously known that this is also true for finite structures, ordered set of rationals, random graph, and the countable atomless Boolean algebra. Some structures in the classes we consider are not isomorphic to their cohesive powers, having properties that distinguish them and that can be described by computable (infinitary) sentences.

Our goal is to further develop the theory of cohesive powers and, more generally, cohesive products of effective structures by investigating their algebraic, computability-theoretic, and syntactic properties. We would like to include more complicated algebraic structures such as semigroups, groups, rings, and fields. Cohesive powers of computable fields will extend the earlier study of cohesive powers of the field of rationals, $\mathbb{Q} = (Q, +, \cdot)$, in [16] and will have further applications in the study of the lattice of c.e. vector spaces and their automorphisms, thus generalizing results in [15].

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