

THE BREAKDOWN OF WEAK NULL SINGULARITIES INSIDE BLACK HOLES

MAXIME VAN DE MOORTELE

Abstract

It is widely expected that generic black holes have a nonempty but weakly singular Cauchy horizon, due to mass inflation. Indeed this has been proven by the author in the spherical collapse of a charged scalar field, under decay assumptions of the field in the black exterior which are conjectured to be generic. A natural question then arises: can this weakly singular Cauchy horizon close off the space-time, or does the weak null singularity necessarily “break down,” giving way to a different type of singularity? The main result of this paper is to prove that the Cauchy horizon cannot ever “close off” the space-time. As a consequence, the weak null singularity breaks down and transitions to a stronger singularity for which the area-radius r extends to 0.

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1. Introduction

The characterization of singularities inside black holes is a fundamental problem in General Relativity, which is related to the fate of in-falling observers and the very validity of the principle of determinism. “Strong” singularities, for which the area-radius r extends to 0, are already present in the interior of the Schwarzschild black hole and raised immense interest (see [1] and its vast developments and see, e.g., the review [2]).

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For many years, it was believed that a generic black hole interior is necessarily delimited by a singularity which is everywhere strong and space-like. It is now well understood that the above belief is *false*. Indeed, it has been proven in [18] that all dynamical black holes settling down to Kerr possess a Cauchy horizon in the black hole interior, that is, a null boundary spanned by spheres of nonzero radius. Additionally, a Cauchy horizon on which $r > 0$ necessarily occurs for dynamical charged black holes settling down to Reissner–Nordström in spherical symmetry, as proven in [14], [37]; at the heuristic level, this phenomenon is explained by the repulsive mechanism provided by the Maxwell field in the charged case, or provided by angular momentum in the non-spherically-symmetric case.

While the future boundary components on which $r = 0$ are (strong) singularities, in contrast Cauchy horizons are *not always* singular: for instance the Reissner–Nordström Cauchy horizon is smoothly regular. Nevertheless, the pioneering works [22], [33]–[35] suggested that the Cauchy horizon of *generic* dynamical black holes is, in fact, a *weak null singularity*. This phenomenon is known as “mass inflation.” It has been proven indeed (see [15], [30], [37], [39]) that the Cauchy horizon of charged spherically symmetric black holes is weakly singular, under assumptions which are conjectured to be generic; retrieving these assumptions remains an important open problem. Weak null singularities have been constructed in vacuum (see [29]) and are conjecturally present in the interior of generic perturbations of Kerr black holes.

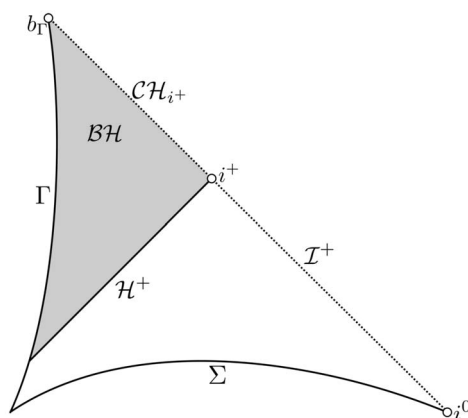
In view of the mathematical evidence in favor of the generic character of weak null singularities, the *necessity* of the occurrence in collapse of $r = 0$ singularities on part of the boundary, ironically, becomes subject to questioning: do weak null singularities necessarily “break down” in finite retarded time, and does a new type of (presumably stronger) singularity take over? Or, on the contrary, is it possible in some cases that they subsist up to the center of symmetry and close off the space-time, as depicted in the Penrose diagram of Figure 1? This is not a moot point, since in the *two-ended* asymptotically flat case, the Cauchy horizon closes off the space-time for a large class of dynamical solutions, as later depicted in Figure 4 (see also Section 1.7).

In the present paper, we carry out the first global study of the black hole interior in what can be seen as the simplest model in which this question makes sense, namely the gravitational collapse of a charged scalar field, governed by the Einstein–Maxwell–Klein–Gordon equations in spherical symmetry:

$$\text{Ric}_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}^{EM} + \mathbb{T}_{\mu\nu}^{KG}, \quad (1.1)$$

$$\mathbb{T}_{\mu\nu}^{EM} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}\right), \quad (1.2)$$

$$\mathbb{T}_{\mu\nu}^{KG} = 2\left(\Re(D_\mu\phi\overline{D_\nu\phi}) - \frac{1}{2}(g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} + m^2|\phi|^2)g_{\mu\nu}\right), \quad (1.3)$$


$$\nabla^\mu F_{\mu\nu} = \frac{q_0}{2} i (\phi \overline{D_\nu \phi} - \overline{\phi} D_\nu \phi), \quad F = dA, \quad (1.4)$$

which feature a scalar field ϕ of charge $q_0 \neq 0$ and of mass $m^2 \geq 0$ and $D_\mu = \nabla_\mu + iq_0 A_\mu$ is the gauge derivative. This model has been extensively studied (see [23], [24], [27], [28], [32]). In the present paper, we consider solutions with one-ended asymptotically flat initial data, diffeomorphic to $\mathbb{R}^3$, as they model black holes arising from gravitational collapse.

Our main result in this context is summarized in layman's terms as the following.

THEOREM

In the spherical collapse of a charged scalar field, weak null singularities necessarily break down.

Therefore, a weakly singular Cauchy horizon can never close off the space-time, so the Penrose diagram of Figure 1 is ruled out in the presence of a weak null singularity. As a consequence of the necessary breakdown of weak null singularities, we obtain a proof of the “ $r = 0$ singularity conjecture” (see [27]): a one-ended black hole which does not have a “locally naked singularity” always features an “ $r = 0$ singularity,” in addition to a weakly singular Cauchy horizon, and its Penrose diagram is given by Figure 2. To prove this result, we make use of an earlier classification of possible Penrose diagrams [27]. As a consequence, we show there exists a so-called “first singularity” where $r = 0$, that is, a Terminal Indecomposable Past (associated

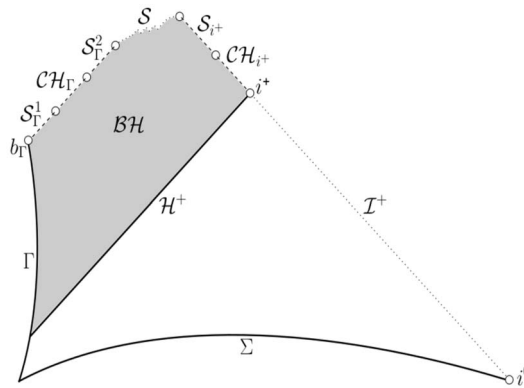


Figure 3. General Penrose diagram of a one-ended charged spherically symmetric black hole, [27].

- or the Cauchy horizon is an isometric copy of the Reissner–Nordström Cauchy horizon.

In the latter case, we prove that the Cauchy horizon cannot close off the space-time either, due to rigidity properties established via a different argument. Our assumptions are comparable to those used in [37] to prove the nonemptiness of the Cauchy horizon and backed up by multiple numerical studies (see Section 1.2). Moreover, the decay assumptions that we use are quite weak since we require much less information on the outgoing data than what is conjectured to be generic.

We now return to the main result (Theorem A), assuming mass inflation. Our proof relies on quantitative estimates in which the center of symmetry Γ plays a role of utmost importance (recall that it is the presence of a regular center Γ that distinguishes gravitational collapse space-times from the two-ended case considered in [17]). The most significant quantity that we control is the Maxwell field, which is dynamical as it interacts with the charged scalar field. As we explained earlier, if this interaction was not present in the equations—like for the Dafermos model—no regular one-ended solution could exist, as the center of symmetry Γ would be singular, which is impossible. Thus, we emphasize that the quantitative estimates of this Maxwell field, in particular *near the center*, are a fundamental aspect of the problem which cannot be overlooked by any serious attempts to establish the generic character of $r = 0$ singularities.

In Section 1.1, we give a detailed description of the matter model and the Einstein–Maxwell–Klein–Gordon equations, and we enumerate all the possible a priori Penrose diagrams, following [27]. Then we state our main result and discuss its assumptions in Section 1.2. We mention previous works on uncharged models

in Section 1.3. In Section 1.4, we explain the $r = 0$ singularity conjecture and its relation to first singularities. We continue with a presentation of other major problems, solved or unsolved, in charged collapse in Section 1.5. The few heuristic and numerical previous works on one-ended black holes are mentioned in Section 1.6. In Section 1.7, we mention prior results on two-ended black holes and emphasize the contrast with the one-ended case. Finally in Section 1.8, we give a short outline of the proof and of the paper.

1.1. *A priori boundary characterization of one-ended space-times*

We consider the Einstein–Maxwell–Klein–Gordon equations, namely the Einstein equations in the presence of a charged scalar field (which we also allow to be massive if $m^2 \neq 0$ or massless if $m^2 = 0$), that is, (1.1), (1.2), (1.3), (1.4), (1.5), where $D := \nabla + iq_0 A$ is the gauge derivative, ∇ is the Levi-Civita connection of g , and A is the potential one-form. We emphasize that the Klein–Gordon mass $m^2 \geq 0$ is allowed to be zero, but the coupling constant $q_0 \neq 0$ is not.

Some a priori information can be derived for this system in spherical symmetry from “soft estimates” only involving the null condition satisfied by the nonlinearity. This work was carried out by Kommemi in [27], who gave an inventory of the vast a priori possibilities for the interior structure of the black hole. To determine which boundary components are empty or singular, one must go beyond such “a priori estimates,” and a precise analysis of the equations is required, which we undertake in the present work. Note that in the sequel, *Penrose diagrams* themselves will be used to convey important information regarding the space-time geometry, (see Figure 2 and see later Figure 4, Figure 5, and the discussion of spherically symmetric space-times in Section 2.1 and in [27]). We now present the preliminary result of Kommemi.

THEOREM 0 (Theorem 1.1 of [27])

We consider the maximal development $(M = \mathcal{Q}^+ \times_r \mathcal{S}^2, g_{\mu\nu}, \phi, F_{\mu\nu})$ of smooth, spherically symmetric, containing no anti-trapped surface, one-ended asymptotically flat initial data satisfying the Einstein–Maxwell–Klein–Gordon system, where $r : \mathcal{Q}^+ \rightarrow [0, +\infty)$ is the area-radius function. Then the Penrose diagram of $\mathcal{Q}^+$ is given by Figure 3, with boundary $\Sigma \cup \Gamma$ in the sense of manifold-with-boundary—where Σ is space-like, and Γ , the center of symmetry, is time-like with $r|_\Gamma = 0$ —and boundary $\mathcal{B}^+$ induced by the manifold ambient $\mathbb{R}^{1+1}$:

$$\mathcal{B}^+ = b_\Gamma \cup \mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \cup \mathcal{S}_{i^+} \cup \mathcal{CH}_{i^+} \cup i^+ \cup \mathcal{I}^+ \cup i^0,$$

where i^0 is space-like infinity, $\mathcal{I}^+$ is null infinity, i^+ is time-like infinity (see [27] for details), and

- (1) $\mathcal{CH}_{i^+}$ is a connected (possibly empty) half-open null ingoing segment emanating from i^+ . The area-radius function r extends as a strictly positive function on $\mathcal{CH}_{i^+}$, except maybe at its future endpoint.
- (2) $\mathcal{S}_{i^+}$ is a connected (possibly empty) half-open null ingoing segment emanating (but not including) from the endpoint of $\mathcal{CH}_{i^+} \cup i^+$. r extends continuously to zero on $\mathcal{S}_{i^+}$.
- (3) b_Γ is the center end-point, that is, the unique future limit point of Γ in $\overline{\mathcal{Q}^+} - \mathcal{Q}^+$.
- (4) $\mathcal{S}_\Gamma^1$ is a connected (possibly empty) half-open null outgoing segment emanating from b_Γ . r extends continuously to zero on $\mathcal{S}_\Gamma^1$.
- (5) $\mathcal{CH}_\Gamma$ is a connected (possibly empty) half-open null outgoing segment emanating from the future end-point of $b_\Gamma \cup \mathcal{S}_\Gamma^1$. r extends as a strictly positive function on $\mathcal{CH}_\Gamma$, except maybe at its future endpoint.
- (6) $\mathcal{S}_\Gamma^2$ is a connected (possibly empty) half-open null outgoing segment emanating from the future end-point of $\mathcal{CH}_\Gamma$. r extends continuously to zero on $\mathcal{S}_\Gamma^2$.
- (7) $\mathcal{S}$ is a connected (possibly empty) achronal curve that does not intersect null rays emanating from b_Γ or i^+ . r extends continuously to zero on $\mathcal{S}$.

We also define the black hole region $\mathcal{BH} := \mathcal{Q}^+ \setminus J^-(\mathcal{I}^+)$ and the event horizon $\mathcal{H}^+ = \overline{J^-(\mathcal{I}^+)} \setminus J^-(\mathcal{I}^+) \subset \mathcal{Q}^+$.

Remark 1

$\mathcal{S}$ is the only boundary component including “first singularities” (see Section 1.1 for a discussion).

In the Penrose diagram, every point in M represents a sphere. At each sphere, one can define the outgoing null derivative of the area-radius function r . We define the regular region, denoted $\mathcal{R}$, as the set of points for which the outgoing null derivative of r is strictly positive; the trapped region, denoted $\mathcal{T}$, as the set of points for which the outgoing null derivative of r is strictly negative; and the apparent horizon, denoted $\mathcal{A}$, as the set of points for which the outgoing null derivative of r is zero.¹

An important feature of charged matter models like the Einstein–Maxwell–Klein–Gordon system is that they permit space-times with a regular center $\Gamma \subset \{r = 0\}$ (corresponding to constant-time slices with one asymptotically flat end). Indeed, in spherical symmetry, the Maxwell field can be written in terms of a scalar function Q defined on $\mathcal{Q}^+$ as

$$F_{\mu\nu} = \frac{Q}{r^2} \cdot \Omega^2 du \wedge dv,$$

¹The definitions of $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{A}$ rely on a sign, and thus are independent of the null vectors normalization (see Section 2).

in any double null coordinate system (u, v) on $\mathcal{Q}^+$, where we defined the null lapse to be $\Omega^2 = -2g(\partial_u, \partial_v)$.

In the case $q_0 = 0$, the right-hand-side of (1.4) vanishes, and this implies that $Q \equiv e$ is a constant function. If $e \neq 0$, then $F_{\mu\nu}$ diverges at the center $\Gamma = \{r = 0\}$; thus, no one-ended smooth solution is possible in the uncharged matter case (e.g., in the Dafermos model). In the setting of charged matter, Q is no longer a constant function, and one-ended regular solutions are available, provided that we impose boundary conditions at the center Γ , as detailed in Section 2.

1.2. First version of the main results and discussion of the hypothesis

In this section, we state our main result, namely that a weakly singular Cauchy horizon cannot close off the space-time. For the consequences on the $r = 0$ singularity conjecture and the generic existence of first singularities, see Section 1.4.

1.2.1. Theorem assuming the existence of a weak null singularity

We present our main theorem, which does not require any quantitative assumption. Namely, *we only assume the existence of a weak null singularity*, that is, the blowup of the mass and the boundedness of the matter fields over one outgoing trapped cone reaching the Cauchy horizon. We give a first version of the theorem, which will be made precise in Section 3.

THEOREM A

For initial data as in Theorem 0, assume there exists a trapped cone $\{u_1\} \times [v_1, v_{\max})$ with $(u_1, v_{\max}) \in \mathcal{CH}_{i+}$, on which the Hawking mass ρ blows up, while ϕ and Q are bounded:

$$\lim_{v \rightarrow v_{\max}} \rho(u_1, v) = +\infty, \quad \sup_{v \leq v_{\max}} |\phi|(u_1, v) + |Q|(u_1, v) < +\infty. \quad (1.6)$$

Then

$$\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset,$$

that is, $\mathcal{CH}_{i+} \cup \mathcal{S}_{i+}$ cannot close off the space-time at b_Γ (i.e., the Penrose diagram of Figure 1 is impossible).

The theorem relies on two assumptions: the first one is the blowup of the Hawking mass toward (at least) *one* sphere on the Cauchy horizon. This blowup, which is conjectured to be generic, results from the blue-shift effect of radiation at the Cauchy horizon. This effect is localized near i^+ inside the black hole, in the sense that no knowledge of the global structure of space-time is necessary to obtain it, as it only depends on the asymptotic structure of the event horizon (see [16], [39]). Of course,

the mass blowup of (1.6) is not satisfied for the Reissner–Nordström solution: this is simply due to the absence of radiation in the Reissner–Nordström space-time, which is static. The other assumption is the boundedness of ϕ and Q , again over the same outgoing light cone, which is expected to hold generically as well, due to local stability estimates near time-like infinity (see [37] and [25]).

1.2.2. Theorem with assumptions on the event horizon

Instead of making assumptions on the behavior of one given light cone in the black hole interior, one can also make assumptions on the event horizon $\mathcal{H}^+$ to prove that the Cauchy horizon $\mathcal{CH}_{i+}$ does not close the space-time. We give a first version of the theorem, which will be made precise later in Section 3.

THEOREM B

We assume that the black hole event horizon $\mathcal{H}^+$ settles quantitatively toward a subextremal Reissner–Nordström event horizon. Then $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$.

This theorem uses the following result of the author: assuming the quantitative exterior stability, we proved in [39] that

- (1) either the Hawking mass blows up on the Cauchy horizon, while ϕ and Q are controlled,²
- (2) or the Cauchy horizon $\mathcal{CH}_{i+}$ is an isometric copy of the Reissner–Nordström one.

If the first option is true, then the assumptions of Theorem A are satisfied, which implies that the Cauchy horizon cannot close the space-time: $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$. If the second option is true, then we prove that it is impossible to connect an isometric copy of the Reissner–Nordström Cauchy horizon to a space-time for which $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$. To do this, we make use of an argument showing that focusing cannot occur in the vicinity of the Cauchy horizon, together with geometric properties (see Section 1.8).

1.2.3. Conjectured decay rates on the event horizon

In this subsection, we discuss the conjectured decay rates at which a black hole is expected to settle toward a subextremal Reissner–Nordström black hole for large times, and the previous works on the subject in the physics literature. In our upcoming Theorem B, we will assume these rates are satisfied on the event horizon of the dynamical black hole.

²We cannot work under the assumption that (ϕ, Q) are bounded, since this is not true in general (see [25]). However, we have proved in [39] that (ϕ, Q) do not blow up too fast, which is what we mean by “controlled.”

In [23], Hod and Piran provided a heuristic argument, based on asymptotic matching, to formulate the correct decay of charged scalar fields on charged spherically symmetric black holes. The main difference with uncharged fields is that the decay rate now depends on the black hole charge e , as opposed to the universal rate prescribed by Price's law (see [19], [21], [31], [36]) in the uncharged case. The results of [23] are confirmed by the numerics of Oren and Piran [32], providing further evidence that the rate depends on $q_0 e$, the adimensional black hole charge.

CONJECTURE 1.1 (Decay of charged scalar fields, Hod and Piran [23], Oren and Piran [32])

Among all the data admissible and sufficiently regular and decaying from Theorem 0, there exists a generic subclass for which if the maximal future development has $\mathcal{Q}^+ \cap J^-(\mathcal{I}^+) \neq \emptyset$, we have, in the charged massless case $q_0 \neq 0$, $m^2 = 0$:

$$|\phi|_{|\mathcal{H}^+}(v) \sim v^{-2+\delta(q_0 e)}, \quad |D_v \phi|_{|\mathcal{H}^+}(v) \sim v^{-2+\delta(q_0 e)},$$

where e is asymptotic charge of the black hole at time-like infinity, $\delta(q_0 e) := 1 - \Re(\sqrt{1 - 4(q_0 e)^2}) \in [0, 1)$, and v is a null coordinate defined by the gauge choice (2.23).

Note that the upper bound corresponding to Conjecture 1.1 was retrieved rigorously in [38], on a fixed Reissner–Nordström background, for small charge $q_0 e$ and for a rate $p = 2 - \delta(q_0 e) + o(\sqrt{|q_0 e|})$ as $q_0 e \rightarrow 0$.

Now we turn to the case of a massive scalar field. In [26], Koyama and Tomimatsu considered the case of a massive, uncharged scalar field and provided a heuristic argument, also based on asymptotic matching, to support that massive fields decay polynomially at a very weak rate and with oscillations. Their tails were later confirmed by the numerics of Burko and Khanna [5]. For the case of a massive, charged scalar field, it was argued by Konoplya and Zhidenko [28] that the late-time tail must be identical, as they claim that the asymptotic behavior of massive scalar field is universal.

CONJECTURE 1.2 (Decay of massive scalar fields, [5], [26], [28])

Among all the data admissible, sufficiently regular, and decaying from Theorem 0, there exists a generic subclass for which if the maximal future development has $\mathcal{Q}^+ \cap J^-(\mathcal{I}^+) \neq \emptyset$, we have, in the massive uncharged case $m^2 \neq 0$, $q_0 = 0$ (see [5], [26]), and in the massive charged case $m^2 \neq 0$, $q_0 \neq 0$ (see [28]):

$$|\phi|_{|\mathcal{H}^+}(v) \sim |\sin|(mv + o(v)) \cdot v^{-\frac{5}{6}}, \quad |D_v \phi|_{|\mathcal{H}^+}(v) \sim |\sin|(mv + o(v)) \cdot v^{-\frac{5}{6}},$$

where v is a null coordinate defined by the gauge choice (2.23).

1.2.4. Comments on the assumptions of Theorem A and Theorem B

The assumptions we make for Theorem B (see Theorem 3.3 for the details) are compatible with the tails of Conjecture 1.1 and Conjecture 1.2 but require much less information. We only assume, for some $s > \frac{3}{4}$, that

$$|\phi|_{\mathcal{H}^+}(v) + |D_v \phi|_{\mathcal{H}^+}(v) \lesssim v^{-s},$$

$$\int_v^{+\infty} |D_v \phi|_{\mathcal{H}^+}^2(v') dv' \gtrsim v^{1-2s}.$$

Remarks

Notice that $1 > \frac{5}{6} > \frac{3}{4}$, so this decay is compatible with both Conjecture 1.1 and Conjecture 1.2.

Note that we require an L^2 -averaged (energy) polynomial lower bound (which is weaker than a pointwise bound) to also account for the potential oscillations prescribed by Conjecture 1.2.

As for Theorem A, assumption (1.6) can be weakened if we allow global, but soft, assumptions. More precisely, the mildest assumption that will suffice for our result, instead of the mass blowup, is some integrability condition (4.7) over (only) *one* outgoing cone, in addition to the assumption that the entire Cauchy horizon is trapped (see Theorem 3.2 for a precise statement). In turn, we prove that this scenario occurs if one only assumes mass blowup on one cone, due to the propagation of blowup (see Section 4.6). Thus, we emphasize that the blowup of the Hawking mass is just a *sufficient* condition, but it is not necessary to prove that the Cauchy horizon cannot close the space-time. In the asymptotically flat setting, mass inflation occurs (see [33], [39]), so the sufficiency of these weaker assumptions does not matter. However, in other settings such as the Einstein equations with a positive cosmological constant, mass inflation is not generically expected (see [13], [20]), but the weaker assumption may be still satisfied; thus we expect our argument to be useful in this setting as well.

1.3. The models of Christodoulou and Dafermos and their generalization

In this section, we present two submodels of the Einstein–Maxwell–Klein–Gordon equations. The first one is the uncharged spherically symmetric model studied by Christodoulou [9]–[12], governed by the Einstein–scalar-field equations, that is, the system (1.1), (1.2), (1.3), (1.4), (1.5), in the special case $F \equiv 0$, $m^2 = 0$. While this model is suitable to study gravitational collapse, as one-ended solutions are allowed, it does not permit the formation of Cauchy horizons due to the absence of any repulsive mechanism such as angular momentum or charge. Therefore, this model is ill-suited to understand weak null singularities, as there is no Cauchy horizon emanating from time-like infinity i^+ .

The second model, featuring a Maxwell field with *uncharged* matter, was studied by Dafermos [14]–[17] and is governed by the Einstein–Maxwell–(uncharged)–scalar-field equations, that is, the system (1.1), (1.2), (1.3), (1.4), (1.5), in the special case $q_0 = 0$, $m^2 = 0$. This model allows Cauchy horizons to form and provides a good setting to understand the formation of weak null singularities and their *local* aspects. Yet, the Dafermos model is in turn restricted by the topology of its initial data, necessarily *two-ended*. This is because the Maxwell field, which is static due to the absence of charged matter, is singular in the one-ended case. Therefore, the Dafermos model is inappropriate to study the *global* aspects of gravitational collapse, including the necessary breakdown of weak null singularities, due to the absence of a center Γ .

The Einstein–Maxwell–Klein–Gordon equations in spherical symmetry that we study in the present paper generalize both the Christodoulou and the Dafermos model, and are free from the above restrictions, as one-ended black holes with Cauchy horizon are allowed in principle. In fact, the Einstein–Maxwell–Klein–Gordon system is one³ of the only spherically symmetric models which is elaborate enough to formulate the breakdown of weak null singularities in a nontrivial way.

An important preliminary step, before proving the result of our present paper, is to establish that the Cauchy horizon is indeed *always* nonempty⁴ in the Einstein–Maxwell–Klein–Gordon model (see [37]). As the dynamics of charged scalar fields in the exterior are more intricate than their uncharged counterparts, new difficulties arise. These difficulties were nonetheless overcome by the author in [37], [39], where it was also shown that the Cauchy horizon $\mathcal{CH}_{i^+}$ is weakly singular.

We now briefly present this result, after mentioning previous works for context.

1.3.1. The Einstein-scalar-field equations in spherical symmetry

The uncharged gravitational collapse has been analyzed in great detail by Christodoulou. Recall that in this case, no Cauchy horizon is allowed to emanate from time-like infinity i^+ due to the absence of charge or angular momentum. We sum up Christodoulou’s main results on the black hole interior.

THEOREM 1.1 (Christodoulou, Einstein-scalar-field in spherical symmetry [9], [10], [12])

For initial data as in Theorem 0 in the more general BV class, assume that the Maxwell field is trivial $F_{\mu\nu} \equiv 0$ and that the field is massless $m^2 = 0$. Then:

- (1) *There is no Cauchy horizon emanating from time-like infinity: $\mathcal{CH}_{i^+} = \emptyset$.*

³It is also possible to study charged dust, but dust on its own propagates by transport. The dust model is simpler than the Einstein–Maxwell–Klein–Gordon system, but it is thus not expected to reflect as accurately the dynamics of the vacuum Einstein equations [8].

⁴Provided, of course, that the asymptotic charge e of the black hole is nonzero, which is conjecturally the generic case.

- (2) *There is no secondary outgoing null segment emanating from b_Γ and where $r = 0$: $\mathcal{S}_\Gamma^2 = \emptyset$.*
- (3) *Among all the data admissible from Theorem 0, in the BV class, with $F_{\mu\nu} \equiv 0$, $m^2 = 0$, there exists a generic subclass for which if the maximal future development has $\mathcal{Q}^+ \cap J^-(\mathcal{I}^+) \neq \emptyset$, then $\mathcal{S}$ is the only nontrivial component of the boundary $\mathcal{S}_\Gamma^1 = \mathcal{CH}_\Gamma = \emptyset$.*

Notice that the statement $\mathcal{S} \neq \emptyset$ is immediate for the Christodoulou model, where very special monotonicity properties dominate, in the absence of any repulsive mechanism such as angular momentum or charge. This is in contrast with the model considered in the present paper, where the (nontrivial) presence of a Cauchy horizon $\mathcal{CH}_{i+}$ (see Section 1.3.3), together with a more complex model, could, in principle, allow for $\mathcal{CH}_{i+}$ to be the only nonempty boundary component.

1.3.2. The Einstein–Maxwell-scalar-field equations in spherical symmetry

The breakthrough of Dafermos [14] was to realize and prove that the Cauchy horizon is nonempty for dynamical black holes, in a model where a Maxwell field plays the role of angular momentum. This early insight, gained from a spherically symmetric model, paved the way to the monumental work of Dafermos and Luk [18], who recently proved the stability of the Cauchy horizon of Kerr black holes for the vacuum Einstein equations, remarkably in the absence of any symmetry.

THEOREM 1.2 (Dafermos, Einstein–Maxwell-scalar-field in spherical symmetry [14], [15])

Assume that the black hole event horizon $\mathcal{H}^+$ settles quantitatively toward a subextremal Reissner–Nordström event horizon. Then $\mathcal{CH}_{i+} \neq \emptyset$, that is, there exists a nonempty Cauchy horizon emanating from i^+ . Moreover, $\mathcal{CH}_{i+}$ is weakly singular, yet C^0 extendible.

The C^0 extendibility of $\mathcal{CH}_{i+}$, together with a result on the exterior by Dafermos and Rodnianski [19], also falsifies the C^0 version of strong cosmic censorship in spherical symmetry. See Section 1.5 for a discussion of this important conjecture related to determinism.

We emphasize, however, that the theorem of Dafermos is for uncharged scalar fields, and thus it does not apply to the Einstein–Maxwell–Klein–Gordon model—featuring a charged, massive scalar field—that we consider in the present paper.

1.3.3. Nonemptiness of $\mathcal{CH}_{i+}$ for the Einstein–Maxwell–Klein–Gordon model

In the charged and massive case, the model becomes more intricate, and the proof of Dafermos does not carry over. Additionally, the scalar field obeys different dynamics, in particular a weaker decay than in the uncharged case. See Conjecture 1.1 and Conjecture 1.2 and compare with the Price’s law governing uncharged fields [19]. This weak decay of charged/massive fields renders nonlinear stability harder and requires new estimates, established by the author in [37].

THEOREM 1.3 (Einstein–Maxwell–Klein–Gordon in spherical symmetry [37])

Assume that the event horizon $\mathcal{H}^+$ settles quantitatively toward a subextremal Reissner–Nordström event horizon. Then $\mathcal{CH}_{i+} \neq \emptyset$. Moreover, $\mathcal{CH}_{i+}$ is weakly singular.

Remark 2

In this context, the weak null singularity of $\mathcal{CH}_{i+}$ is to be understood as a blowup of some curvature component.

Under the same assumptions, it is also proven in [37] that $\mathcal{CH}_{i+}$ is C^0 extendible in the charged massless case. Therefore, the C^0 version of strong cosmic censorship (see Section 1.5) is also false in this more general setting. The same conclusion was reached for the massive and charged case by the author and Kehle [25], exploiting a novel mechanism based on the oscillations of the perturbations.

1.4. The $r = 0$ singularity conjecture in charged gravitational collapse

Theorem A has consequences on space-time singularities. Under some reasonable additional assumptions, one can prove that a $r = 0$ singularity exists generically in the black hole interior. We discuss this issue in the present section.

1.4.1. First singularities

We define the notion of spherically symmetric “first singularities,” a concept introduced by Dafermos in [16] and further formalized by Kommemi in [27]. First singularities are the boundary points from which nontrivial components emanate. Thus, most of the investigation of the black hole interior relies on the precise understanding of those singularities.

Definition

Let $p \in \overline{\mathcal{Q}^+}$. We say that $J^-(p) \subset \overline{\mathcal{Q}^+}$ is compactly generated if there exists a compact set $X \subset \mathcal{Q}^+$ such that $J^-(p) \subset \Pi(\mathcal{D}_M^+(\Pi^{-1}(X))) \cup J^-(X)$, where $\mathcal{D}_M^+(A)$ is

the domain of dependence of A and $\Pi : M \rightarrow \mathcal{Q}^+$ is the projection onto the Penrose diagram $\mathcal{Q}^+$ (see the definition in Section 2).

Definition 1.1 (See [27])

With the conventions of Theorem 0, we say that $p \in \mathcal{B}^+$ is a first singularity if $J^-(p)$ is compactly generated and if any compactly generated proper causal subset of $J^-(p)$ is of the form $J^-(q)$, $q \in \mathcal{Q}^+$. Then,

- (1) if $p \in \mathcal{B}^+ - \{b_\Gamma\}$, p is a first singularity if and only if there exists $q \in I^-(p) \cap \mathcal{Q}^+ \setminus \{p\}$, such that $J^+(q) \cap J^-(p) \setminus \{p\} \subset \mathcal{Q}^+$, and we say that p is a noncentral first singularity,
- (2) b_Γ is a first singularity if and only if $\mathcal{S} \cup \mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \neq \emptyset$. In that case, we say b_Γ is a central first singularity.

Theorem 0 then has an immediate application on the location of the first singularities.

COROLLARY (Corollary of Theorem 0, [27])

If $\mathcal{S} \neq \emptyset$, there exists at least one noncentral first singularity $p \in \mathcal{S}$, and b_Γ is a central first singularity. Moreover, there are no noncentral first singularities in $\mathcal{B}^+ - \mathcal{S}$. Therefore, if p is a noncentral first singularity, then $r(p) = 0$.

In view of Theorem 0, one can also define a reasonable notion of space-like portion of the boundary.

Definition 1.2

Let $\mathcal{S}' \subset \mathcal{S}$. We say that $\mathcal{S}'$ is space-like if every $p \in \mathcal{S}'$ is a first singularity.

Remark 3

Notice that, even if $\mathcal{S}$ is space-like in the sense of Definition 1.2, it is not clear a priori whether one can, from $\mathcal{S}$, construct and attach a space-like boundary to the $3 + 1$ space-time $\mathcal{Q}^+ \times_r \mathbb{S}^2$, as subtle considerations may be important.

For the uncharged model of Christodoulou, that is, (1.1), (1.2), (1.3), (1.4), (1.5), in the special case $F_{\mu\nu} \equiv 0$, $m^2 = 0$ the analysis of Christodoulou [9], [10], [12] leading to Theorem 1.1 also impacts the structure of first singularities.

PROPOSITION (First singularities for Einstein-scalar-field (see Christodoulou [9], [10], [12] and Dafermos [16]))

Among all the data admissible from Theorem 1.1, there exists a generic subclass for

which if the maximal future development has $\mathcal{Q}^+ \cap J^-(\mathcal{I}^+) \neq \emptyset$, then every point $p \in \mathcal{B}^+$ is a first singularity. In particular, $\mathcal{S}$ is space-like.

1.4.2. The $r = 0$ singularity conjecture

Now, we return to the charged gravitational collapse case, that is, the full system (1.1), (1.2), (1.3), (1.4), (1.5), where $F_{\mu\nu} \neq 0$, $q_0 \neq 0$, and we mention important conjectures formulated in [27]. In view of Definition 1.1, our main result directly implies the following.

COROLLARY 1.4

Under the assumptions of Theorem A or Theorem B, b_Γ is a central first singularity; therefore the set of first singularities is nonempty.

Now, we want to investigate the behavior of the area-radius r at the interior boundary $\mathcal{B}^+$. We introduce an important conjecture stating the existence of a crushing singularity $\mathcal{S} = \{r = 0\}$.

CONJECTURE 1.3 ($r = 0$ singularity conjecture, as formulated in [27])

Among all the data admissible from Theorem 0, there exists a generic subclass for which if the maximal future development has $\mathcal{Q}^+ \cap J^-(\mathcal{I}^+) \neq \emptyset$, then the Penrose diagram is given by Figure 2, that is, $\mathcal{S} \neq \emptyset$, $\mathcal{CH}_{i+} \neq \emptyset$ and $\mathcal{S}_\Gamma^1 = \mathcal{CH}_\Gamma = \mathcal{S}_\Gamma^2 = \emptyset$.

The main content of the conjecture is the statement $\mathcal{S} \neq \emptyset$, which implies that there exists a nontrivial boundary component where $r = 0$. Additionally, $\mathcal{S} \neq \emptyset$ implies the existence of a (noncentral) first singularity, by Definition 1.1. The additional assumption we need to prove Conjecture 1.3 in the present paper is formulated as another conjecture, which has important connections with the Weak Cosmic Censorship conjecture (see Section 1.5).

CONJECTURE 1.4 (Spherical trapped surface conjecture, formulated in [27])

Among all the data admissible from Theorem 0, there exists a generic subclass for which, if the maximal future development has $\mathcal{Q}^+ - J^-(\mathcal{I}^+) \neq \emptyset$, then the apparent horizon $\mathcal{A}$ has a limit point on b_Γ . Moreover, if that is the case, then $\mathcal{S}_\Gamma^1 = \mathcal{CH}_\Gamma = \mathcal{S}_\Gamma^2 = \emptyset$.

Remark 4

Note that the statement $\mathcal{S}_\Gamma^1 = \mathcal{CH}_\Gamma = \mathcal{S}_\Gamma^2 = \emptyset$ corresponds to the absence of “locally naked” singularity emanating from the center b_Γ , a slightly stronger statement than weak cosmic censorship (Conjecture 1.5).

Notice that Conjecture 1.4 is related to the behavior of space-time in the vicinity of b_{Γ} ; therefore, by causality, this behavior cannot be influenced by the late time tail on the event horizon.⁵ In contrast, our results start from data on a fixed outgoing trapped cone, which is itself related to asymptotic behavior on the event horizon ultimately responsible for the existence of a weak null singularity. In fact, by the same principle, the result of [37] and Theorem A are the only emptiness/nonemptiness statements which can be nontrivially obtained from the late time behavior on an outgoing cone, as all the other possible statements would result from purely local considerations.

In the uncharged case $F_{\mu\nu} = m^2 = 0$, Christodoulou proved the validity of Conjecture 1.4, which directly implies Statement 3 of Theorem 1.1 and is also the key ingredient of his proof of the weak cosmic censorship conjecture.

One immediate consequence of Theorem A is the following fact, which was not previously recorded.

THEOREM C

Assume Conjecture 1.4. Then Conjecture 1.3 is true.

The breakdown of weak null singularities (a global property) combined with Conjecture 1.4 (a statement on b_{Γ}) implies there exists a noncentral first singularity $p \in \mathcal{S}$ and that b_{Γ} is a central first singularity. Therefore, the last step to obtain a full geometric understanding of spherical collapse is to prove Conjecture 1.4, which would also imply the instability of naked singularities, a statement known as the weak cosmic censorship conjecture (see Section 1.5).

1.5. Additional related questions in charged gravitational collapse

In this section, we give a brief review of the past works, conjectures, and open problems related to the black hole interior. Ironically, the most prominent subsisting problem in gravitational collapse is related to the existence of singularities which form in the absence of a black hole. Such “naked” singularities are conjectured to be non-generic. This statement—the weak cosmic censorship conjecture—can be formulated in modern terms as such:

CONJECTURE 1.5 (Weak cosmic censorship conjecture for the Einstein–Maxwell–Klein–Gordon–Model)

Among all the data admissible from Theorem 0, there exists a generic subclass for which $\mathcal{I}^+$ is complete.

⁵Except, of course, in the case where $\mathcal{CH}_{i^+}$ closes off the space-time as in Figure 1, but this is precisely the case we disprove in Theorem A.

Notice that in spherical symmetry, it can be proven that if the black hole region is nonempty, then $\mathcal{I}^+$ is future-complete (see [16], [27]). One can also immediately show (see [27]) that Conjecture 1.4 implies Conjecture 1.5.

We now turn to another important problem, the strong cosmic censorship conjecture, which broadly states that general relativity is a deterministic theory.

CONJECTURE 1.6 (Strong cosmic censorship conjecture for the Einstein–Maxwell–Klein–Gordon–Model)

Among all the data admissible from Theorem 0, there exists a generic subclass for which the maximal globally hyperbolic development (M, g) is future inextendible as a suitably regular Lorentzian manifold.

The main obstruction to strong cosmic censorship is the existence of Cauchy horizons, which can be smoothly extendible (e.g., for the Kerr stationary metric or for the Reissner–Nordström static metric). Nevertheless, it was expected that *generic* dynamical Cauchy horizons feature a weak null singularity [33] and, therefore, are C^2 inextendible. In the case of gravitational collapse, additional obstructions related to the center appear, such as the existence of $\mathcal{CH}_\Gamma$, which is however empty generically if Conjecture 1.4 is true. Modulo these issues which are unrelated to weak null singularities, and assuming the quantitative stability of the black hole exterior, the author has obtained a version of Conjecture 1.6.

THEOREM 1.5 (C^2 inextendibility of the Cauchy horizon for the Einstein–Maxwell–Klein–Gordon model (see [37], [39]))

Assume that Conjecture 1.4 is true. Then, under the assumptions of Theorem B, the C^2 version of Conjecture 1.6 holds (i.e., (M, g) is future inextendible as a C^2 Lorentzian manifold).

We also mention the remarkable work of Luk and Oh [30], [31], who provide a comprehensive proof of the C^2 version of strong cosmic censorship for the Einstein–(uncharged)–scalar–field model. Note that in their case, the data are two-ended, and thus there is no obstruction coming from the center of symmetry Γ , in contrast with our model. Note also that the decay of uncharged fields in the exterior is well understood and governed by Price’s law (see [19], [21], [31], [36]). However, Price’s law does not apply to charged scalar fields, which obey more complicated dynamics, and decay is only known in the small charge case. See [38] for the proof of upper bounds that are sharp according to Conjecture 1.1.

We now return to the characterization of the interior boundary. In addition to the weak and strong cosmic censorship conjectures, one problem is left unexplored:

the causal character of $\mathcal{S}$. In particular, one can wonder whether $\mathcal{S}$ is space-like, in the sense of Definition 1.2. The following “space-like singularity” conjecture appears reasonable.

CONJECTURE 1.7

Among all the data admissible from Theorem 0, there exists a generic subclass for which if the maximal future development has $\mathcal{Q}^+ \cap J^-(J^+) \neq \emptyset$, then $\mathcal{S} \neq \emptyset$ is space-like.

Note that, with the approach adopted in the present paper, we do not have any control over the causal character of $\mathcal{S}$, as we use a contradiction argument. Thus, it seems that a different approach is required to investigate this issue.

We conclude this discussion with an interesting open problem: What happens in the interior of the black hole for the Einstein–Maxwell–Klein–Gordon equations in the presence of a positive cosmological constant? Based on the works [13], [20], the Cauchy horizon always exists but is not weakly singular for a certain range of black hole parameters (i.e., the Hawking mass is finite). While it seems reasonable that our approach could be adapted to the cosmological setting, for parameters such that a weak null singularity is present, it would be interesting to see whether the Cauchy horizon may in some cases close off the space-time for parameters such that the Hawking mass is finite.

1.6. Numerical and heuristic previous studies on $r = 0$ singularities

The presence of $r = 0$ singularities during the process of gravitational collapse received a lot of attention from the numerical relativity community. In view of the work of Christodoulou, it is not the existence of those singularities which requires evidence, but the statement that they are generic, for models which allow for the formation of Cauchy horizons. However, it is difficult to validate generic statements numerically, as it requires exploring the whole moduli space of initial data.

The pioneering works of Brady and Smith [3], Burko and Ori [6], and Burko [4] first provided evidence for the occurrence of $r = 0$ singularities inside spherically symmetric Einstein–Maxwell–uncharged–scalar-field black holes. While the uncharged character of the scalar field does not significantly modify the asymptotic behavior, it forces the black hole to be *two-ended* (Section 1.3). In the two-ended case, the occurrence of $r = 0$ singularities is *not* generic (see [17] and Section 1.7).

In contrast, to study the genericity of $r = 0$ singularities in spherical collapse, we must consider charged matter in *one-ended* space-times, as in the present paper. One of the only numerical studies of charged gravitational collapse was carried out by Hod and Piran [24], who considered the Einstein–charged–scalar-field system (i.e.,

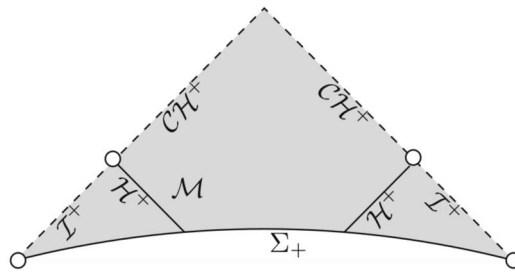


Figure 4. Bifurcating Cauchy horizons in the two-ended case for small data. Figure from [17].

(1.1), (1.2), (1.3), (1.4), (1.5) with $m^2 = 0$). For a particular choice of (global) initial data, including the center, they exhibited a Cauchy horizon, which is singular due to mass inflation, and at later times a singularity toward which the area-radius r extends to zero.

We also mention some heuristics of [7], which attempt to argue in favor of the emergence of a $r = 0$ singularity. It is not clear, however, what the role of the center Γ and of the Maxwell field is in their work.

1.7. Contrast with two-ended black holes, for charged/uncharged matter

In this section, we describe what happens in the case of so-called *two-ended* initial data that have $\mathbb{R} \times \mathbb{S}^2$ topology (as opposed to the one-ended case $\mathbb{R}^3$ considered in the present paper). Two-ended space-times are historically important, in that they have the same topology as Reissner–Nordström or Kerr black holes. In such space-times, however, there is no center (i.e., $\Gamma = \emptyset$), and thus the two-ended data assumption *does not model the global structure of gravitational collapse*. The two-ended setting is radically different from the one-ended case already for the Einstein–Maxwell–scalar-field model: in [17], Dafermos proves that the two-ended analogue of Conjecture 1.3 is *false*: there exists an open set of small, regular initial data for which there is no $r = 0$ singularity, and an outgoing Cauchy horizon branches with an ingoing Cauchy horizon to close off the space-time at a bifurcation sphere as in Figure 4, just like for the Reissner–Nordström solution. For those solutions constructed by Dafermos, the Hawking mass blows up everywhere, at least for a generic subclass inside the open set of data. Thus, weak null singularities *do not necessarily* break down, in contrast with the one-ended case.

We present a result which generalizes [17] to the more elaborate Einstein–Maxwell–Klein–Gordon model. The argument of [17] is easily transposable, and almost no modification is necessary. We provide a sketch of the proof in the [Appendix](#).

THEOREM 1.6

Small scalar field data give rise a bifurcate Cauchy horizon in the two-ended case. Consider (M, g, ϕ, F) a solution of Einstein–Maxwell–Klein–Gordon system (1.1), (1.2), (1.3), (1.4), (1.5) arising from two-ended, spherically symmetric regular initial data.

Assume moreover that the event horizon $\mathcal{H}^+$ settles quantitatively toward a subextremal Reissner–Nordström event horizon and that the scalar field is small; then there are no $r = 0$ singularities: $\mathcal{S} = \emptyset$ and the Penrose diagram is given by Figure 4 (i.e., a bifurcate Cauchy horizon $\mathcal{CH}_{i_1^+} \cup \mathcal{CH}_{i_2^+}$ closes off the space-time).

This result proves that it is imperative to consider the global structure of the space-time for Theorem A to be valid, and, a fortiori, that there is no possible local approach to proving Conjecture 1.3.

1.8. Method of the proof and outline of the paper

The proof of our main result, Theorem A, is by contradiction. We assume that the Penrose diagram is given by Figure 1 (additionally, we may also have $\mathcal{S}_{i^+} \neq \emptyset$ as in Figure 5; see Theorem 0 for a definition of $\mathcal{S}_{i^+}$), where $\mathcal{CH}_{i^+}$ features a weak null singularity at an early time, and we derive a contradiction from these two facts. In our approach, the main mechanism is the breakdown of weak null singularities; the necessary occurrence of $r = 0$ singularities (i.e., $\mathcal{S} \neq \emptyset$) is only obtained as an indirect consequence (assuming the absence of “locally naked singularities,” i.e., $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 = \emptyset$), using the a priori result of Theorem 0.

Our methods are based on quantitative estimates involving the center of symmetry Γ and the control of the Maxwell field by the Hawking mass. These estimates, based on the nonlinear focusing properties of the Einstein equations in the presence of a weak null singularity, are proven on a causal rectangle with a top vertex $p = (u, v) \in \mathcal{A}$ ($\mathcal{A}$ being the apparent horizon), a bottom vertex $q \in \mathcal{T}$ ($\mathcal{T}$ being the trapped region), and a left vertex on the center Γ (see Figure 5). As a consequence, we prove that there exists a trapped ingoing null segment emanating from the top vertex p .

Due to the geometry of the Penrose diagram given by Figure 1, there exist such rectangles where $p = (u, v) \in \mathcal{A}$ is followed by an ingoing *regular* segment. This is in contradiction with the consequence of the focusing estimates, which proves that the Penrose diagram given by Figure 1, where $\mathcal{CH}_{i^+}$ is a weak null singularity, was impossible in the first place.

We outline the rest of the paper. In Section 2, we lay out the geometric framework and the equations in double null coordinates. Then we state precisely our results in Section 3. In Section 4, we provide the proof of Theorem A (i.e., Theorem 3.1 and

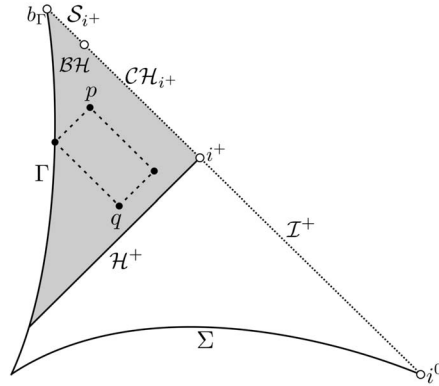


Figure 5. The causal rectangle $J^+(q) \cap J^-(p)$ on which we prove focusing estimates, where $p \in \mathcal{A}$, $q \in \mathcal{I}$.

Theorem 3.2). In Section 5, we give the proof of Theorem B (i.e. Theorem 3.3), using the main result of [39] that we recall. Finally, in the Appendix, we prove Theorem 1.6, implying that the two-ended case analogue of Conjecture 1.3 is false.

2. Geometric framework

The purpose of this section is to provide the precise setup, together with the definition of various geometric quantities, the coordinates, and the equations that we will use throughout the paper.

2.1. Spherically symmetric solution, as given by Theorem 0

From Theorem 0, we obtain (M, g, ϕ, F) , a regular solution of the system (1.1), (1.2), (1.3), (1.4), (1.5), where (M, g) is a Lorentzian manifold of dimension $3 + 1$, ϕ is a complex-valued function on M , and F is a real-valued 2-form on M .

(M, g, ϕ, F) is related to a quadruplet of scalar functions $\{\Omega^2(u, v), r(u, v), \phi(u, v), Q(u, v)\}$, with $(u, v) \in \mathcal{Q} \subset \mathbb{R}^{1+1}$ by

$$\begin{aligned} g &= g_{\mathcal{Q}} + r^2 \cdot (d\theta^2 + \sin(\theta)^2 d\varphi^2) \\ &= -\Omega^2(u, v) du dv + r^2(u, v) \cdot (d\theta^2 + \sin(\theta)^2 d\varphi^2), \\ F(u, v) &= \frac{Q(u, v)}{2r^2(u, v)} \Omega^2(u, v) du \wedge dv. \end{aligned} \tag{2.1}$$

The domain $\mathcal{Q} \subset \mathbb{R}^2$, called the Penrose diagram, is depicted in Figure 3 (for a choice of (u, v) , see Remark 7, such that $\mathcal{Q}$ is a bounded subset of $\mathbb{R}^2$). Note that the boundary components of Theorem 0 can be identified with intervals in $\mathbb{R}^2$ as

$$\begin{aligned}
i^+ &= \{(u_{-\infty}, v_{\infty})\}, \\
\mathcal{CH}_{i^+} &= (u_{-\infty}, u_{\mathcal{CH}_{i^+}}) \times \{v_{\infty}\}, \\
\mathcal{S}_{i^+} &= [u_{\mathcal{CH}_{i^+}}, u(\mathcal{CH}_{i^+} \cup \mathcal{S}_{i^+})) \times \{v_{\infty}\}, \\
b_{\Gamma} &= \{(u(b_{\Gamma}), v(b_{\Gamma}))\} \\
\mathcal{S}_{\Gamma}^1 &= \{u(b_{\Gamma})\} \times (v(b_{\Gamma}), v(\mathcal{S}_{\Gamma}^1)), \\
\mathcal{CH}_{\Gamma} &= \{u(b_{\Gamma})\} \times [v(\mathcal{S}_{\Gamma}^1), v(\mathcal{S}_{\Gamma}^1 \cup \mathcal{CH}_{\Gamma}))], \\
\mathcal{S}_{\Gamma}^2 &= \{u(b_{\Gamma})\} \times [v(\mathcal{S}_{\Gamma}^1 \cup \mathcal{CH}_{\Gamma}), v(\mathcal{S}_{\Gamma}^1 \cup \mathcal{CH}_{\Gamma} \cup \mathcal{S}_{\Gamma}^2)),
\end{aligned}$$

where all these intervals are bounded and possibly degenerate or empty (recall Theorem 0 for the precise definition of these boundary components).

Remark 5

Note that the statement $\mathcal{S} \neq \emptyset$ (obtained under the assumptions of Theorem C) is equivalent to $u(\mathcal{CH}_{i^+} \cup \mathcal{S}_{i^+}) < u(b_{\Gamma})$. The statement $\mathcal{S}_{\Gamma}^1 \cup \mathcal{CH}_{\Gamma} \cup \mathcal{S}_{\Gamma}^2 \cup \mathcal{S} \neq \emptyset$ appearing in Theorem A is equivalent to $v(b_{\Gamma}) < v_{\infty}$.

One can now formulate the Einstein equations (1.1), (1.2), (1.3), (1.4), (1.5) as a system of nonlinear PDEs on Ω^2 , r , ϕ , and Q expressed in the double null coordinate system $(u, v) \in \mathcal{Q}$:

$$\partial_u \partial_v \log(\Omega^2) = -2\Re(D_u \phi \overline{D_v \phi}) + \frac{\Omega^2}{2r^2} + \frac{2\partial_u r \partial_v r}{r^2} - \frac{\Omega^2}{r^4} Q^2, \quad (2.2)$$

$$\partial_u \partial_v r = \frac{-\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + \frac{\Omega^2}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 |\phi|^2, \quad (2.3)$$

$$D_u D_v \phi = -\frac{\partial_v r \cdot D_u \phi}{r} - \frac{\partial_u r \cdot D_v \phi}{r} + \frac{i q_0 Q \Omega^2}{4r^2} \phi - \frac{m^2 \Omega^2}{4} \phi, \quad (2.4)$$

$$\partial_u Q = -q_0 r^2 \Im(\phi \overline{D_u \phi}), \quad (2.5)$$

$$\partial_v Q = q_0 r^2 \Im(\phi \overline{D_v \phi}), \quad (2.6)$$

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = \frac{-r}{\Omega^2} |D_u \phi|^2, \quad (2.7)$$

$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = \frac{-r}{\Omega^2} |D_v \phi|^2, \quad (2.8)$$

where the gauge derivative is defined by $D_{\mu} := \partial_{\mu} + i q_0 A_{\mu}$, and the electromagnetic potential $A_{\mu} = A_u du + A_v dv$ satisfies

$$\partial_u A_v - \partial_v A_u = \frac{Q \Omega^2}{2r^2}. \quad (2.9)$$

Remark 6

(2.7) and (2.8) are the Raychaudhuri equations, and we will use this terminology in the paper.

Remark 7

The double null coordinates (u, v) are not unique and can be re-parametrized as $du' = f_1(u) du$, $dv' = f_2(v) dv$ for functions $f_1 > 0$, $f_2 > 0$. This procedure takes $\mathcal{Q}$ to a different domain $\mathcal{Q}'$ but does not change the system of equations.

Subsequently, we define the Lorentzian gradient of r and introduce the mass ratio μ by the formula

$$1 - \mu := g_{\mathcal{Q}}(\nabla r, \nabla r),$$

where we recall that $g_{\mathcal{Q}}$ is the radial part of g defined in (2.1). We can also define the Hawking mass:

$$\rho := \frac{\mu \cdot r}{2} = \frac{r}{2} \cdot (1 - g_{\mathcal{Q}}(\nabla r, \nabla r)).$$

Notice that the (u, v) coordinate system, we have $g_{\mathcal{Q}}(\nabla r, \nabla r) = \frac{-4\partial_u r \cdot \partial_v r}{\Omega^2}$. We can then define κ :

$$\kappa = \frac{\partial_v r}{1 - \frac{2\rho}{r}} = \frac{-\Omega^2}{4\partial_u r} \in \mathbb{R} \cup \{\pm\infty\}. \quad (2.10)$$

Now we introduce the modified mass ϖ which involves the charge Q :

$$\varpi := \rho + \frac{Q^2}{2r} = \frac{\mu r}{2} + \frac{Q^2}{2r}. \quad (2.11)$$

An elementary computation relates the previously quantities:

$$1 - \frac{2\rho}{r} = 1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2} = \frac{-4\partial_u r \cdot \partial_v r}{\Omega^2} = \kappa^{-1} \cdot \partial_v r. \quad (2.12)$$

On the subextremal Reissner–Nordström space-time g_{RN} of mass $\varpi \equiv M > 0$ and charge $Q \equiv e$ with $0 < |e| < M$, we denote $r_+ = M + \sqrt{M^2 - e^2}$, the area-radius of the event horizon $\mathcal{H}^+$, its surface gravity $2K_+ := \frac{2}{r_+^2}(M - \frac{e^2}{r_+}) > 0$, and $r_- = M - \sqrt{M^2 - e^2}$, the radius of the Cauchy horizon, its surface gravity $2K_- := \frac{2}{r_-^2}(M - \frac{e^2}{r_-}) < 0$.

In view of the definition of new quantities ρ , κ , one can derive new PDEs governing these quantities as reformulations of the system previously written. The first two are variants of the ingoing Raychaudhuri equation (2.7):

$$\partial_u(\kappa^{-1}) = \frac{4r}{\Omega^2} |D_u \phi|^2, \quad (2.13)$$

$$\partial_u(\log(\kappa^{-1})) = \frac{r}{|\partial_u r|} |D_u \phi|^2. \quad (2.14)$$

Next, we can derive two transport equations for the Hawking mass using (2.3), (2.13), (2.8):

$$\partial_u \rho = \frac{-2r^2 \Omega^2}{\partial_v r} |D_u \phi|^2 + \left(\frac{m^2}{2} r^2 |\phi|^2 + \frac{Q^2}{r^2} \right) \cdot \partial_u r, \quad (2.15)$$

$$\partial_v \rho = \frac{r^2}{2\kappa} |D_v \phi|^2 + \left(\frac{m^2}{2} r^2 |\phi|^2 + \frac{Q^2}{r^2} \right) \cdot \partial_v r. \quad (2.16)$$

Now we can reformulate our former equations to put them in a form that is more convenient to use. For instance, the Klein–Gordon wave equation (2.4) can be expressed in different ways, using the commutation relation $[D_u, D_v] = \frac{iq_0 Q \Omega^2}{2r^2}$:

$$D_u(r D_v \phi) = -\partial_v r \cdot D_u \phi + \frac{\Omega^2 \cdot \phi}{4r} \cdot (iq_0 Q - m^2 r^2), \quad (2.17)$$

$$D_v(r D_u \phi) = -\partial_u r \cdot D_v \phi - \frac{\Omega^2 \cdot \phi}{4r} \cdot (iq_0 Q + m^2 r^2). \quad (2.18)$$

We can also rewrite (2.2) and (2.3) as

$$\partial_u \partial_v \log(r \Omega^2) = \frac{\Omega^2}{4r^2} \cdot \left(1 - \frac{3Q^2}{r^2} + m^2 r^2 |\phi|^2 - 8r^2 \Re \left(\frac{D_u \phi}{\Omega^2} \cdot D_v \bar{\phi} \right) \right), \quad (2.19)$$

$$-\partial_u \partial_v \left(\frac{r^2}{2} \right) = \partial_u (-r \partial_v r) = \partial_v (r |\partial_u r|) = \frac{\Omega^2}{4} \cdot \left(1 - \frac{Q^2}{r^2} - m^2 r^2 |\phi|^2 \right). \quad (2.20)$$

2.2. One-ended smooth solutions and regularity conditions on Γ

In view of (2.1), we define $\Gamma := \{(u, v) \in \mathcal{Q}, r(u, v) = 0\}$, the center of symmetry. From Theorem 0, $\Gamma \neq \emptyset$ is time-like.

The smoothness of the solution (M, g, ϕ, F) imposes the following boundary conditions for the geometric quantities:

$$|g_{\mathcal{Q}}(\nabla r, \nabla r)|_{|\Gamma} < +\infty, \quad |\phi|_{|\Gamma} < +\infty, \quad |F_{\mu\nu}|_{|\Gamma} < +\infty. \quad (2.21)$$

The reader can check that the regularity condition (2.21) imposes in particular the following boundary conditions, which are crucial in the present paper and follow from Theorem 0:

$$\rho|_{\Gamma} = 0, \quad r\phi|_{\Gamma} = 0, \quad Q|_{\Gamma} = 0. \quad (2.22)$$

Notice (see Figure 3) also that every future directed ingoing ray must intersect Γ . For a fixed v , we denote $u_{\Gamma}(v)$, the u coordinate of the intersection point,

$(u_\Gamma(v), v) \in \Gamma$. Notice also that every past directed outgoing ray inside the black hole must intersect Γ . For a fixed u , we denote $v_\Gamma(u)$, the v coordinate of the intersection point, $(u, v_\Gamma(u)) \in \Gamma$.

2.3. Trapped region and apparent horizon

We recall that it is assumed in Theorem 0 that there is no antitrapped surface in the initial data, that is, $\partial_u r|_{\Sigma_0} < 0$, and hence $\partial_u r < 0$ in the whole space-time, as a consequence of (2.13). In particular, $\kappa > 0$ and $\partial_v r$ has the same sign as $1 - \frac{2\rho}{r}$.

We define the trapped region $\mathcal{T}$, the regular region $\mathcal{R}$, and the apparent horizon $\mathcal{A}$ as (see [27]):

$$(1) \quad (u, v) \in \mathcal{T} \text{ if and only if } \partial_v r(u, v) < 0 \text{ if and only if } 1 - \frac{2\rho(u, v)}{r(u, v)} < 0,$$

$$(2) \quad (u, v) \in \mathcal{R} \text{ if and only if } \partial_v r(u, v) > 0 \text{ if and only if } 1 - \frac{2\rho(u, v)}{r(u, v)} > 0,$$

$$(3) \quad (u, v) \in \mathcal{A} \text{ if and only if } \partial_v r(u, v) = 0 \text{ if and only if } 1 - \frac{2\rho(u, v)}{r(u, v)} = 0.$$

Note that the no antitrapped surface assumption of Theorem 0 implies, as $r|_\Gamma = 0$, that $\Gamma \subset \mathcal{R}$.

2.4. Double null coordinate choice

We renormalize the coordinate v by the condition $\partial_v r|_{\mathcal{H}^+} = 1 - \frac{2\rho|_{\mathcal{H}^+}}{r|_{\mathcal{H}^+}}$, which is equivalent by (2.12) to

$$\kappa|_{\mathcal{H}^+} \equiv 1, \tag{2.23}$$

$$v(p) = 0, \quad \text{where } \{p\} = \mathcal{H}^+ \cap \Gamma. \tag{2.24}$$

Note that the second condition is simply a normalization of v up to an additive constant, a choice which leaves (2.23) invariant.

As for the choice of u coordinate, it is much less important because we will always write estimates which are independent of the u coordinate choice. For concreteness, we introduce the following u -gauge:

$$u_\Gamma(v) = v. \tag{2.25}$$

Note that by (2.24) we have $\mathcal{H}^+ = \{u = 0\}$.

For convenience, we will keep the notations of Section 2.1 for the boundary components. Note, however, that under the coordinate choice 2.23, 2.25 we now have $u_{-\infty} = 0$, $v_\infty = +\infty$, $u(b_\Gamma) = +\infty$, and $(u, v) \in \mathcal{Q}'$, where $\mathcal{Q}' \subset \mathbb{R}^2$ is an unbounded set in $\mathbb{R}^2$.

2.5. Electromagnetic gauge choice, and gauge invariant estimates

The system of equations (1.1), (1.2), (1.3), (1.4), (1.5) is invariant under the gauge transformation:

$$\begin{aligned}\phi &\rightarrow e^{-iq_0 f} \phi, \\ A &\rightarrow A + df,\end{aligned}$$

where f is a smooth real-valued function. By an easy computation, one can show that the quantities $|\phi|$ and $|D_\mu \phi|$ are gauge invariant. We can then derive a gauge-invariant estimate (see Lemma 2.1 in [38]): for all $u_1 < u_2$, $v_1 < v_2$,

$$\begin{aligned}|\phi|(u_2, v) &\leq |\phi|(u_1, v) + \int_{u_1}^{u_2} |D_u \phi|(u', v) du', \\ |\phi|(u, v_2) &\leq |\phi|(u, v_1) + \int_{v_1}^{v_2} |D_v \phi|(u, v') dv' .\end{aligned}$$

In the present paper, we only use such gauge-invariant estimates, and we will not involve A_μ in any computation.

3. Statement of the main results

We now give a precise statement of our theorems from Section 1.2 for space-times as in Theorem 0.

3.1. Precise version of Theorem A

We start with a precise version of Theorem A, the main result of the paper.

THEOREM 3.1

For initial data as in Theorem 0, assume that there exists an outgoing future cone emanating from $(u_1, v_1) \in \mathcal{T}$ and reaching $\mathcal{CH}_{i+}$ on which ϕ and Q obey the following upper bounds: for all $v \geq v_1$,

$$|\phi|(u_1, v) + |Q|(u_1, v) \leq C \cdot |\log(\rho)| \quad (3.1)$$

for some $C > 0$ and the Hawking mass ρ blows up

$$\lim_{v \rightarrow +\infty} \rho(u_1, v) = +\infty. \quad (3.2)$$

Then

$$\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset.$$

In reality, Theorem 3.1 will be realized as a consequence of a more general theorem, for which we replace the mass blowup by the more relaxed condition (3.3), and we make the soft, but global, assumption that there exists a trapped neighborhood of the Cauchy horizon, at least for sufficiently late times. While these assumptions can seem obscure at first, we prove that they are both satisfied if (3.1) and (3.2) hold, so the following theorem is more general than Theorem 3.1.

THEOREM 3.2

For initial data as in Theorem 0, assume that $\mathcal{CH}_{i+} \neq \emptyset$ and there exists $(u_1, v_1) \in \mathcal{T}$, with $(u_1, +\infty) \in \mathcal{CH}_{i+}$ such that

$$\int_{v_1}^{+\infty} \kappa(u_1, v)(1 + |\phi|^2(u_1, v)) dv < +\infty. \quad (3.3)$$

Denoting $u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, the u coordinate of the future endpoint of $\mathcal{CH}_{i+} \cup \mathcal{S}_{i+}$, assume that there exists $u_0 < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$ such that, for all $u_0 \leq u < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$. Then

$$\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset.$$

Remark 8

Theorem 3.2 is a result of independent interest, as its assumptions are a priori uncorrelated with the blowup (or the boundedness) of the Hawking mass; therefore we may hope that they hold in various settings, in particular in the cosmological case where the blowup of the mass is not expected generically.

As the integral of (3.3) is, in fact, the crucial quantity governing the problem, we will first prove Theorem 3.2 in Section 4 and then deduce Theorem 3.1 in Section 4.6, using the propagation of the Hawking mass blowup proven in [39].

3.2. Precise version of Theorem B

In the next theorem, we replace the integrability assumption (3.3) and the trapped neighborhood assumption of Theorem 3.2 by a decay assumption on the event horizon $\mathcal{H}^+$ (at the expected rates; see Section 1.2).

THEOREM 3.3

We normalize v by the gauge condition (2.23). For some $s > \frac{3}{4}$, we assume on $\mathcal{H}^+$, for all $v \geq v_0$

$$|\phi|_{|\mathcal{H}^+}(v) + |D_v \phi|_{|\mathcal{H}^+}(v) \lesssim v^{-s}, \quad (3.4)$$

$$\int_v^{+\infty} |D_v \phi|_{|\mathcal{H}^+}^2(v') dv' \gtrsim v^{-p} \quad (3.5)$$

for some $2s - 1 \leq p \leq \min\{2s, 6s - 3\}$. On the ingoing cone, we assume a red-shift estimate

$$|D_u \phi|(u, v_0) \lesssim |\partial_u r|(u, v_0) \quad (3.6)$$

for all $u \leq u_0$. Additionally, assume that a subextremal Reissner–Nordström event horizon is approached, that is, on $\mathcal{H}^+$

$$0 < \limsup_{v \rightarrow +\infty} \frac{|Q|_{\mathcal{H}^+}(v)}{r|_{\mathcal{H}^+}(v)} < 1. \quad (3.7)$$

Then

$$\mathcal{S}_\Gamma^1 \cup \mathcal{C}\mathcal{H}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset.$$

Remark 9

The decay rates that we assume for (3.4) and (3.5) are conjectured to hold in the black hole exterior for generic Cauchy data (see Section 1.2.3). Red-shift bounds such as (3.6) are also conjectured to hold and reflect the fact that the event horizon $\mathcal{H}^+$ is a regular hypersurface for the black hole metric and that ϕ is also regular across $\mathcal{H}^+$.

4. Proof of Theorem 3.1 and Theorem 3.2

4.1. The strategy to prove Theorem 3.1 and Theorem 3.2

Theorem 3.1, with its stronger version Theorem 3.2, corresponds to Theorem A. Before starting their proof, we give in this subsection an account of the strategy that we use in the rest of Section 4.

4.1.1. The logic of the proof of Theorem 3.1 and Theorem 3.2

In this section, we outline the proof that, under the assumptions of Theorem 3.1 and Theorem 3.2, $\mathcal{S}_\Gamma^1 \cup \mathcal{C}\mathcal{H}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$. The proof is divided in three steps and follows a contradiction argument. Nonetheless, it relies on constructive focusing estimates (see Section 4.1.2) which are valid independently and subsist even after the contradiction has been established.

- (1) *We consider focusing properties on a large class of causal rectangles with a vertex on the center (Section 4.4).*

We consider any causal rectangle of the form $[u_1, u] \times [v_\Gamma(u), v] \subset \mathcal{Q}^+$, with $p = (u, v) \in \mathcal{A}$ and $q = (u_1, v_\Gamma(u)) \in \mathcal{T}$ as in Figure 5.

Assuming that the following focusing condition holds on the $\{u_1\} \times [v_\Gamma(u), v]$ side of the rectangle, for a small $\delta > 0$,

$$\int_{v_\Gamma(u)}^v \kappa(u_1, v') \cdot (1 + |\phi|^2(u_1, v')) dv' \leq \delta, \quad (4.1)$$

we prove that the ingoing segment emanating from (u, v) is trapped, that is, there exists $\eta > 0$ with $(u, u + \eta) \times \{v\} \subset \mathcal{T}$.

- (2) *We construct one rectangle with a vertex on the center, using the trapped neighborhood assumption (Section 4.5).*

For this step, we assume for now that there exists a neighborhood of the Cauchy horizon $\mathcal{C}\mathcal{H}_{i+}$ in the trapped region.

Then, we work by contradiction and assume that $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$. Then, we essentially establish that a connected component of $\mathcal{A}$ must terminate on b_Γ , via a soft argument using the trapped neighborhood of $\mathcal{CH}_{i+}$ and the geometry resulting from $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$ (see the Penrose diagram of Figure 1).

As a result, we can construct one causal rectangle of the form $[u_1, u] \times [v_\Gamma(u), v]$, with $(u, v) \in \mathcal{A}$ and $(u_1, v_\Gamma(u)) \in \mathcal{T}$, such that (4.1) holds—a consequence of the blowup of the Hawking mass (1.6)—and yet $(u, u + \epsilon) \times \{v\} \subset \mathcal{R}$ for some small $\epsilon > 0$. This constitutes a *contradiction*, using the result of Step 1 that $(u, u + \eta) \times \{v\} \subset \mathcal{T}$ for $\eta > 0$.

- (3) *We show the existence of a trapped neighborhood of $\mathcal{CH}_{i+}$ from the blowup of the Hawking mass (Section 4.6).*

In this final step, we prove, independently of the contradiction argument, and using (1.6), that there exists a trapped neighborhood of $\mathcal{CH}_{i+}$. This follows from the propagation of the blowup of the Hawking mass over $\mathcal{CH}_{i+}$, a result proven in [39]. Therefore, every outgoing cone over $\mathcal{CH}_{i+}$ eventually satisfies $\frac{2\rho}{r} > 1$, and hence is trapped.

4.1.2. Novel focusing estimates, the key ingredient in the proof of Theorem 3.1

The proof of Theorem 3.1 and Theorem 3.2, in particular step 1, relies on focusing estimates near the center, which are valid in any circumstance, independently of the proof of Theorem A. As a consequence of these estimates, the Hawking mass controls a very large flux of radiation. Thus, the charge, which is controlled by a smaller flux, is dominated by the Hawking mass.

- (1) *The Hawking mass ρ controls an exponential flux of radiation.*

In the regular region $\mathcal{R}$, the combination of (2.14) with (2.16) gives a focusing estimate as follows (ignoring for now the r weights).

- Integrating (2.14) in u gives an estimate of the schematic form

$$|\phi|^2 \lesssim \log(\kappa^{-1}). \quad (4.2)$$

- Using (4.2), (2.16) gives an estimate of the schematic form

$$\partial_v \rho \gtrsim e^{|\phi|^2} \cdot |D_v \phi|^2. \quad (4.3)$$

- Integrating (4.3) in v from⁶ the center where $\rho = 0$ shows that an exponential flux of radiation is controlled by ρ .

Note that we omitted various terms, some depending on the data on the outgoing cone $\{u_1\} \times [v_1, v_{\max}]$, and r factors. The main novel ingredient through which we obtain this control is an a priori radiation flux estimate (see (4.6)).

⁶Note that the entire domain of integration is contained in the regular region $\mathcal{R}$; see Section 2.1 for more details.

- (2) *The charge Q is dominated by the Hawking mass ρ .*

By the Maxwell equation (2.6), we also have an estimate on the charge Q of the following form, using also (4.3),

$$|\partial_v Q| \lesssim |\phi| \cdot |D_v \phi| \ll \partial_v \rho,$$

which we integrate from the center where $Q = 0$ and $\rho = 0$. Thus, the charge is dominated by the Hawking mass

$$|Q|(u, v) \ll \rho(u, v).$$

- (3) *We exploit the focusing of the Einstein equations in the trapped region.*

In fact, the actual estimate which is obtained in step 1 is of the more specific form:

$$|Q|(u, v) \lesssim \rho(u, v) \cdot \int_{v_\Gamma(u)}^{+\infty} \kappa \cdot (1 + |\phi|^2(u_1, v')) dv'.$$

From the fact that (u, v) is in the regular region/apparent horizon, $\rho(u, v) \leq 2r(u, v)$ (see Section 2). Moreover, for any small $\delta > 0$, there exists v large enough such that (4.1) is valid, using the blowup assumption (1.6).

Thus, combining the three inequalities, one can deduce that $|Q|(u, v) \leq C(M, e, q_0, m^2) \cdot \delta \cdot r(u, v) < r(u, v)$, which then implies in the massless $m^2 = 0$ case, from (2.20) and for v large enough, with (u, v) in the regular region,

$$\partial_u(r \partial_v r)(u, v) < 0. \quad (4.4)$$

In turn, if $(u, v) \in \mathcal{A}$, this estimate directly implies that $(u, u + \eta) \times \{v\} \subset \mathcal{T}$ for some small $\eta > 0$.

This is essentially the content of Theorem 4.4 with an estimate of the charge proven Lemma 4.5. The massive case $m^2 \neq 0$ can be also be treated: we then need an additional estimate for the massive term provided by Lemma 4.6.

4.2. Two elementary calculus lemmata

We start with two elementary computations, stated here for convenience. The first one is simply a one-dimensional functional inequality, which is important to handle the potential blowup in (4.6). The second one is a simple second order polynomial equation, which is useful to sort out the right smallness of δ required to apply Theorem 4.4 and Lemma 4.6.

LEMMA 4.1

Assume that for some $u > u_1$, $v > v_1$, $\{u\} \times [v_1, v] \subset \mathcal{R} \cup \mathcal{A}$, and $\{u_1\} \times [v_1, v] \subset \mathcal{T} \cup \mathcal{A}$.

Then, defining $r_1 := r(u_1, v_1) > 0$, for all nonnegative functions f , we have the following estimate:

$$\int_{v_1}^v r(u, v') \log\left(\frac{r^{-1}(u, v')}{r^{-1}(u_1, v')}\right) f(u, v') dv' \leq r_1 \int_{v_1}^v f(u, v') dv'. \quad (4.5)$$

Proof

Since $\{u_1\} \times [v_1, v] \subset \mathcal{T} \cup \mathcal{A}$, $r(u_1, v') \leq r_1$ for all $v_1 \leq v' \leq v$, and since $\partial_u r \leq 0$, $r(u, v) \leq r(u_1, v) \leq r_1$. Also, since $\{u\} \times [v_1, v] \subset \mathcal{R} \cup \mathcal{A}$, for all $v_1 \leq v' \leq v$, $r(u, v') \leq r(u, v) \leq r_1$.

As a consequence of the inequalities, we have the estimate defining $x(v') = \frac{r(u, v')}{r_1} \in (0, 1]$, for all $v_1 \leq v' \leq v$:

$$r(u, v') \log\left(\frac{r^{-1}(u, v')}{r^{-1}(u_1, v')}\right) \leq r_1 \cdot \frac{r(u, v')}{r_1} \log\left(\frac{r_1}{r(u, v')}\right) = r_1 \cdot x \log(x^{-1}).$$

The function $x \rightarrow x \log(x^{-1})$ is increasing on $(0, e^{-1})$ and decreasing on $(e^{-1}, 1]$, with a maximum at $x = e^{-1}$, whose value is $e^{-1} \leq 1$. This gives (4.5) immediately. $\square$

LEMMA 4.2

If $0 < \delta < \frac{1}{32m^2r_1(1+m^2r_1^2)}$, then

$$m^2 \cdot \sqrt{8r_1^3 \cdot \delta} \cdot \left(1 + \sqrt{\frac{2\delta}{r_1}}\right) < \frac{1}{2}.$$

Proof

Set $y = \sqrt{\frac{2\delta}{r_1}}$. If $y_{\pm}$ are the roots of the polynomial second degree equation $y(1+y) = \frac{1}{4m^2r_1^2}$, we must obtain $y < y_+$. We see by standard methods that

$$y_+ = \frac{-1 + \sqrt{1 + m^{-2}r_1^{-2}}}{2} = \frac{m^{-2}r_1^{-2}}{2(1 + \sqrt{1 + m^{-2}r_1^{-2}})} \geq \frac{m^{-2}r_1^{-2}}{4\sqrt{1 + m^{-2}r_1^{-2}}}.$$

Therefore, it is sufficient that $y = \sqrt{\frac{2\delta}{r_1}} < \frac{1}{4m^2r_1^2\sqrt{1 + m^{-2}r_1^{-2}}}$ or, equivalently, $\delta < \frac{1}{32m^2r_1(1 + m^2r_1^2)}$. $\square$

4.3. An ingoing a priori estimate on ϕ in the entire space-time

Now, we establish a elementary focusing estimate, which relates the scalar field to the flux of ingoing radiation, quantified by κ , only using the Raychaudhuri equation. This estimate is important for Section 4.4.

LEMMA 4.3

For all $v \in \mathbb{R}$, $u_1 \leq u \leq u_\Gamma(v)$, the following estimate is true:

$$|\phi|^2(u, v) \leq 2|\phi|^2(u_1, v) + 2 \log\left(\frac{\kappa^{-1}(u, v)}{\kappa^{-1}(u_1, v)}\right) \cdot \log\left(\frac{r^{-1}(u, v)}{r^{-1}(u_1, v)}\right). \quad (4.6)$$

Proof

We estimate $\phi(u, v)$ with respect to $\phi(u_1, v)$. Using Cauchy–Schwarz and the ingoing Raychaudhuri equation (2.14),

$$\begin{aligned} |\phi|(u, v) &\leq |\phi|(u_1, v) + \int_{u_1}^u |D_u \phi|(u', v) du' \\ &\leq |\phi|(u_1, v) + \left(\int_{u_1}^u \frac{r |D_u \phi|^2}{|\partial_u r|}(u', v) du' \right)^{\frac{1}{2}} \left(\int_{u_1}^u \frac{|\partial_u r|}{r}(u', v) du' \right)^{\frac{1}{2}} \\ &\leq |\phi|(u_1, v) + \left(\log\left(\frac{\kappa^{-1}(u, v)}{\kappa^{-1}(u_1, v)}\right) \right)^{\frac{1}{2}} \cdot \left(\log\left(\frac{r^{-1}(u, v)}{r^{-1}(u_1, v)}\right) \right)^{\frac{1}{2}}, \end{aligned}$$

an estimate we can square, using $(a + b)^2 \leq 2a^2 + 2b^2$, which immediately gives (4.6). $\square$

4.4. The key estimate on a late rectangle with a vertex on the center

Now, we get to the heart of the proof of Theorem 3.2. We establish focusing estimates, which will later be revealed, in Section 4.5, to be incompatible with the Cauchy horizon closing off the space-time. We will work on causal rectangles of the form $J^-(p) \cap J^+(q)$, as in Figure 5, for $p = (u, v) \in \mathcal{A}$ and $q = (u_1, v_\Gamma(u)) \in \mathcal{T} \cup \mathcal{A}$.

THEOREM 4.4

Let $(u, v) \in \mathcal{A}$. We define the area-radius of the past vertex $r_1 := r(u_1, v_\Gamma(u))$. We make the following assumptions on the causal rectangle $J^-(p) \cap J^+(q) = [u_1, u] \times [v_\Gamma(u), v]$ for $p = (u, v)$ and $q = (u_1, v_\Gamma(u))$, as in Figure 5:

- (1) $(u_1, v_\Gamma(u)) \in \mathcal{T} \cup \mathcal{A}$; therefore for all $v_\Gamma(u) \leq v' \leq v$, $(u_1, v') \in \mathcal{T} \cup \mathcal{A}$.
- (2) For all $v_\Gamma(u) \leq v' < v$, $(u, v') \in \mathcal{R} \cup \mathcal{A}$.
- (3) We have the following (gauge invariant) estimate on the past outgoing boundary of the rectangle:

$$\int_{v_\Gamma(u)}^v \kappa(u_1, v') \cdot (1 + |\phi|^2(u_1, v')) dv' \leq \delta. \quad (4.7)$$

Then there exists $\delta_1(q_0, m^2, r_1) > 0$, which we can choose to be $\delta_1 = \min\{\frac{1}{4q_0^2 r_1}, \frac{1}{32m^2 r_1(1+m^2 r_1^2)}\}$ such that, if $\delta \leq \delta_1$, we have $\partial_u(r \partial_v r)(u, v) < 0$. Therefore, a small

future ingoing segment emanating from (u, v) is included in the trapped region; there exists $\eta > 0$ such that $(u, u + \eta) \times \{v\} \subset \mathcal{T}$.

We start with the main ingredient of the proof of Theorem 4.4: the control of the charge, in particular near the center. The following lemma is probably the most important result in the present paper.

LEMMA 4.5

Under the assumption of Theorem 4.4, we have the following estimate at the top vertex (u, v) :

$$\frac{Q^2}{r^2}(u, v) \leq 2r_1 \cdot q_0^2 \cdot \delta < \frac{1}{2}, \quad (4.8)$$

where for the last inequality, we chose $\delta < \frac{1}{4r_1 \cdot q_0^2}$.

Proof

Using (2.6), we have $|\partial_v Q| \leq |q_0| \cdot r^2 \cdot |\phi| \cdot |D_v \phi|$, which we integrate on $\{u\} \times [v_\Gamma(u), v]$, in the notations of Theorem 4.4. Using the fact that $Q(u, v_\Gamma(u)) = 0$ and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |Q|(u, v) &\leq |q_0| \int_{v_\Gamma(u)}^v r^2 |\phi| |D_v \phi|(u, v') dv' \\ &\leq |q_0| \left(\int_{v_\Gamma(u)}^v \frac{r^2}{2\kappa} |D_v \phi|^2 \right)^{\frac{1}{2}} \left(\int_{v_\Gamma(u)}^v 2r^2 \kappa |\phi|^2(u, v') dv' \right)^{\frac{1}{2}}. \end{aligned} \quad (4.9)$$

Since $\{u\} \times [v_\Gamma(u), v] \subset \mathcal{R} \cup \mathcal{A}$ by assumption, $\partial_v r(u, v') \geq 0$ for all $v' \in [v_\Gamma(u), v]$. Thus, by (2.16), we have $\frac{r^2}{2\kappa} |D_v \phi|^2(u, v') \leq \partial_v \rho(u, v')$ for all $v' \in [v_\Gamma(u), v]$; hence, combining with (4.9) and using $\rho(u, v_\Gamma(u)) \geq 0$, we get

$$\begin{aligned} |Q|(u, v) &\leq |q_0| \int_{v_\Gamma(u)}^v r^2 |\phi| |D_v \phi|(u, v') dv' \\ &\leq |q_0| \cdot \rho^{\frac{1}{2}}(u, v) \cdot \left(\int_{v_\Gamma(u)}^v 2r^2 \kappa |\phi|^2(u, v') dv' \right)^{\frac{1}{2}}. \end{aligned} \quad (4.10)$$

Now we estimate the term under the square-root, using (4.6) to control $|\phi|^2(u, v')$:

$$\begin{aligned}
 & \int_{v_\Gamma(u)}^v 2r^2 \kappa |\phi|^2(u, v') dv' \\
 & \leq 2r(u, v) \int_{v_\Gamma(u)}^v r \kappa |\phi|^2(u, v') dv' \\
 & \leq 4r(u, v) \int_{v_\Gamma(u)}^v r(u, v') \kappa(u, v') |\phi|^2(u_1, v') dv' \\
 & \quad + 4r(u, v) \int_{v_\Gamma(u)}^v r(u, v') \log\left(\frac{r^{-1}(u, v')}{r^{-1}(u_1, v')}\right) \kappa(u, v') \log\left(\frac{\kappa^{-1}(u, v')}{\kappa^{-1}(u_1, v')}\right) dv' \\
 & \leq 4r_1 \cdot r(u, v) \cdot \left(\int_{v_\Gamma(u)}^v \kappa(u_1, v') |\phi|^2(u_1, v') dv' \right. \\
 & \quad \left. + \int_{v_\Gamma(u)}^v \kappa(u, v') \cdot \log\left(\frac{\kappa^{-1}(u, v')}{\kappa^{-1}(u_1, v')}\right) dv' \right),
 \end{aligned}$$

where we used $r(u, v') \leq r(u, v)$ for the first inequality (since $\{u\} \times [v_\Gamma(u), v] \subset \mathcal{R} \cup \mathcal{A}$) and for the last inequality, $r(u, v') \leq r_1$ for the first term (since $r(u, v) \leq r_1$, see the proof of Lemma 4.1), and (4.5) with $f(v') = \kappa(u, v') \log(\frac{\kappa^{-1}(u, v')}{\kappa^{-1}(u_1, v')})$ for the second term.

Now, notice that $f(v') = \kappa(u_1, v') \cdot \frac{\log(x)}{x}$ for $x = \frac{\kappa^{-1}(u, v')}{\kappa^{-1}(u_1, v')}$. Then, we use the fact that $\frac{\log(x)}{x} \leq e^{-1} \leq 1$ for any $x \in [1, +\infty)$, applied to $x = \frac{\kappa^{-1}(u, v')}{\kappa^{-1}(u_1, v')} \geq 1$. Therefore,

$$\begin{aligned}
 & \int_{v_\Gamma(u)}^v 2r^2 \kappa |\phi|^2(u, v') dv' \\
 & \leq 2r(u, v) \int_{v_\Gamma(u)}^v r \kappa |\phi|^2(u, v') dv' \\
 & \leq 4r_1 \cdot r(u, v) \cdot \int_{v_\Gamma(u)}^v \kappa(u_1, v') (1 + |\phi|^2(u_1, v')) dv' \\
 & \leq 4r_1 \cdot \delta \cdot r(u, v),
 \end{aligned} \tag{4.11}$$

where we used (4.7) in the last inequality. Thus, squaring (4.10) and dividing by r^2 , we get

$$\begin{aligned}
 \frac{Q^2(u, v)}{r^2(u, v)} & \leq q_0^2 \frac{(\int_{v_\Gamma(u)}^v r^2 |\phi| |D_v \phi|(u, v') dv')^2}{r^2(u, v)} \leq q_0^2 \cdot \rho(u, v) \cdot \frac{4r_1 \delta}{r(u, v)} \\
 & = 2r_1 \cdot \delta \cdot q_0^2 \cdot \frac{2\rho(u, v)}{r(u, v)} = 2r_1 \cdot \delta \cdot q_0^2,
 \end{aligned} \tag{4.12}$$

where for the last inequality, we used $\frac{2\rho(u,v)}{r(u,v)} = 1$ since $(u, v) \in \mathcal{A}$. This concludes the proof. $\square$

In particular, Lemma 4.5 provides immediately a proof of Theorem 4.4 in the massless case $m^2 = 0$, as in this case, we see by (2.20) that $\partial_u(r\partial_v r)(u, v) = -\frac{\Omega^2}{4}(1 - \frac{Q^2}{r^2})(u, v) < -\frac{\Omega^2(u,v)}{8} < 0$ since $\Omega^2(u, v) > 0$.

Now, we turn to the crucial estimate to handle the massive term when $m^2 \neq 0$.

LEMMA 4.6

Under the assumption of Theorem 4.4, we have the following estimate in the top vertex (u, v) :

$$m^2 r^2 |\phi|^2(u, v) \leq m^2 \cdot \sqrt{8r_1^3 \cdot \delta} \cdot \left(1 + \sqrt{\frac{2\delta}{r_1}}\right) < \frac{1}{2}, \quad (4.13)$$

where for the last inequality, we took $\delta < \frac{1}{32m^2 r_1(1+m^2 r_1^2)}$.

Proof

We integrate $\partial_v(r^2|\phi|^2)$ on $\{u\} \times [v_\Gamma(u), v]$. Using the fact that $r^2|\phi|^2(u, v_\Gamma(u)) = 0$, we get

$$r^2|\phi|^2(u, v) \leq 2 \int_{v_\Gamma(u)}^v r^2|\phi| |D_v \phi|(u, v') dv' + 2 \int_{v_\Gamma(u)}^v r \partial_v r |\phi|^2(u, v') dv',$$

where we used the identity $\partial_v(|\phi|^2) = 2\Re(\bar{\phi} D_v \phi)$. For the first term, we use (4.12), which we proved in Lemma 4.5, and for the second term, we use the inequality $\partial_v r(u, v') = (1 - \frac{2\rho}{r})\kappa(u, v') \leq \kappa(u, v')$, which holds for all $v_\Gamma(u) \leq v' \leq v$ by (2.12) and because $\{u\} \times [v_\Gamma(u), v] \subset \mathcal{R}$, and hence $\partial_v r(u, v') \geq 0$. Thus, we get

$$\begin{aligned} r^2|\phi|^2(u, v) &\leq 2\sqrt{2} \cdot \sqrt{r_1} \cdot r(u, v) \cdot \sqrt{\delta} + 2 \int_{v_\Gamma(u)}^v r \kappa |\phi|^2(u, v') dv' \\ &\leq r_1^{\frac{3}{2}} \cdot \sqrt{8\delta} + 2 \int_{v_\Gamma(u)}^v r \kappa |\phi|^2(u, v') dv', \end{aligned}$$

and we already proved (see (4.11)) that $2 \int_{v_\Gamma(u)}^v r \kappa |\phi|^2(u, v') dv' \leq 4r_1 \cdot \delta$. Thus, combining everything,

$$m^2 r^2 |\phi|^2(u, v) \leq m^2 \cdot \sqrt{8r_1^3 \cdot \delta} \cdot \left(1 + \sqrt{\frac{2\delta}{r_1}}\right) < \frac{1}{2},$$

where we chose $\delta < \frac{1}{32m^2 r_1(1+m^2 r_1^2)}$ for the last estimate, by Lemma 4.2. This concludes the proof. $\square$

In the case $m^2 \neq 0$, choosing $\delta < \delta_1$ with δ_1 defined in the statement of Theorem 4.4, the combination of Lemma 4.5 and Lemma 4.6 concludes the proof of Theorem 4.4 since we have $\partial_u(r\partial_v r)(u, v) = -\frac{\Omega^2}{4}(1 - \frac{Q^2}{r^2} - m^2 r^2 |\phi|^2)(u, v) < 0$ by (2.20). To finish the proof, notice that $u \rightarrow r\partial_v r(u, v)$ is a C^1 function on $[u, u_\Gamma(v))$, and since $r\partial_v r(u, v) = 0$, there exists $\eta > 0$ such that for all $u' \in (u, u + \eta)$, $r\partial_v r(u', v) < 0$; thus $\partial_v r(u', v) < 0$, which proves that $(u, u + \eta) \times \{v\} \subset \mathcal{T}$.

4.5. Existence of arbitrarily late half-diamonds in the regular region

We start with a geometric result. We construct half-diamonds with specific causal properties, and on which we will later apply the key estimate of Section 4.4. To carry out this construction, we work by contradiction, assuming that $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$ (i.e., we assume by contradiction that the Cauchy horizon closes off the space-time in b_Γ as depicted in Figure 1).

PROPOSITION 4.7

Assume, as in Theorem 3.2, that there exists $u_0 < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$ such that, for all $u \in [u_0, u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+}))$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$. Assume also $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$.

Then, for all $v \geq v(u_0)$, there exists $u_{\mathcal{A}+}(v) \in [u_0, u_\Gamma(v))$ such that $(u_{\mathcal{A}+}(v), v) \in \mathcal{A}$ and:

- (1) The future ingoing cone emanating from $(u_{\mathcal{A}+}(v), v)$ lies in the regular region (i.e., $(u_{\mathcal{A}+}(v), u_\Gamma(v)) \times \{v\} \subset \mathcal{R}$).
- (2) The past outgoing cone emanating from $(u_{\mathcal{A}+}(v), v)$ lies in the marginally regular region (i.e., $\{u_{\mathcal{A}+}(v)\} \times [v_\Gamma(u_{\mathcal{A}+}(v)), v] \subset \mathcal{R} \cup \mathcal{A}$).
- (3) $(u_{\mathcal{A}+}(v), v)$ approaches b_Γ as $v \rightarrow +\infty$. Denoting $u(b_\Gamma)$, the u coordinate of b_Γ , we have

$$\lim_{v \rightarrow +\infty} u_{\mathcal{A}+}(v) = u(b_\Gamma). \quad (4.14)$$

Therefore, we also have, given that Γ is time-like (cf. Figure 1),

$$\lim_{v \rightarrow +\infty} v_\Gamma(u_{\mathcal{A}+}(v)) = +\infty. \quad (4.15)$$

Proof

By assumption, $\mathcal{CH}_{i+} \cup \mathcal{S}_{i+} \neq \emptyset$ and, since $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$, then $u(b_\Gamma) = u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$. By the trapped neighborhood assumption, for all $u_0 \leq u < u(b_\Gamma)$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$, and therefore, by the monotonicity of the Raychaudhuri equation (2.8), $\{u\} \times [v(u), +\infty) \subset \mathcal{T}$.

For any $v \geq v(u_0)$, we know that $(u_0, v) \in \mathcal{T}$. Also $(u_\Gamma(v), v) \in \Gamma \subset \mathcal{R}$, so because $u \rightarrow \partial_v r(u, v)$ is continuous on $[u_0, u_\Gamma(v)]$, there exists $u' \in (u_0, u_\Gamma(v))$

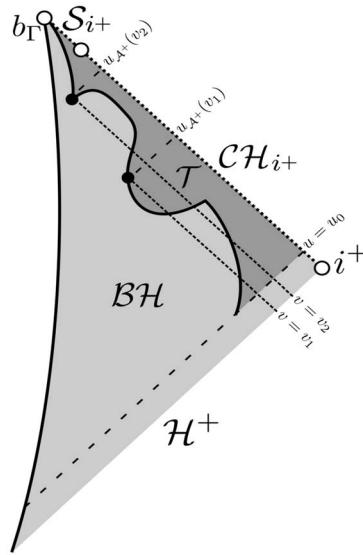


Figure 6. Illustration of the proof of Proposition 4.7.

such that $\partial_v r(u', v) = 0$ by the intermediate value theorem. Hence $(u', v) \in \mathcal{A}$. Define $u_{\mathcal{A}+}(v) := \max\{u' \in (u_0, u_{\Gamma}(v)), \partial_v r(u', v) = 0\}$. By continuity, $u_{\mathcal{A}+}(v)$ is well defined and $(u_{\mathcal{A}+}(v), v) \in \mathcal{A}$. By definition (and continuity again), for all $u' \in (u_{\mathcal{A}+}(v), u_{\Gamma}(v))$, $\partial_v r(u', v) > 0$; thus $(u_{\mathcal{A}+}(v), u_{\Gamma}(v)) \times \{v\} \subset \mathcal{R}$.

$v \rightarrow u_{\mathcal{A}+}(v)$ is a nondecreasing function with values on $[u_0, u(b_{\Gamma})]$. Indeed, if $v < v'$, then by the monotonicity of the Raychaudhuri equation (2.8), $\partial_v r(u_{\mathcal{A}+}(v), v') \leq 0$; thus $u_{\mathcal{A}+}(v) \leq u_{\mathcal{A}+}(v')$. This means that $u_{\mathcal{A}+}(v)$ converges to a limit value $u_l \in [u_0, u(b_{\Gamma})]$ as $v \rightarrow +\infty$. We claim that $u_l = u(b_{\Gamma})$. If not, then the region $\{u > u_l, v \geq v(u_0)\} \subset \mathcal{R}$ by definition, obviously contradicting the trapped neighborhood assumption.

Therefore, we chose $u = u_{\mathcal{A}+}(v)$ and the claims 1 and 3 are satisfied, and only claim 2 remains.

For claim 2, notice that $\{u_{\mathcal{A}+}(v)\} \times [v_{\Gamma}(u), v] \subset \mathcal{R} \cup \mathcal{A}$, using again the monotonicity of the Raychaudhuri equation (2.8). $\square$

Now, combining Theorem 4.4 and Proposition 4.7, we prove Theorem 3.2.

COROLLARY 4.8

Assume, as in Theorem 3.2, that there exists $(u_1, v_1) \in \mathcal{T}$ with $(u_1, +\infty) \in \mathcal{CH}_{i+}$ such that

$$\int_{v_1}^{+\infty} \kappa(u_1, v)(1 + |\phi|^2(u_1, v)) dv < +\infty,$$

and that there exists $u_0 < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$ such that, for all $u_0 \leq u < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$. Then $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$.

Proof

First, define $r_1 = r(u_1, v_1)$ and then $\delta_1(q_0, m^2, r_1)$ by the expression given in the statement of Theorem 4.4. Using the integrability assumption, we see that there exists $v_0 > v_1$ large enough such that, if $v \geq v_0$,

$$\int_v^{+\infty} \kappa(u_1, v)(1 + |\phi|^2(u_1, v)) dv < \delta_1(q_0, m^2, r_1).$$

Then we proceed by contradiction. Assume that $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$. Then, for all $v > v_0$ large enough (recall that v_0 is defined right above), by Proposition 4.7, there exists $u_{\mathcal{A}}(v) > u_1$ such that $(u_{\mathcal{A}}(v), v) \in \mathcal{A}$, $[u, u_\Gamma(v)] \times \{v\} \subset \mathcal{R}$, $\{u\} \times [v_\Gamma(u), v] \subset \mathcal{R} \cup \mathcal{A}$ and $v_\Gamma(u) > v_1$ (i.e., a causal rectangle $J^-(p) \cap J^+(q)$ as in Figure 5) with $p = (u, v)$, $q = (u_1, v_\Gamma(u))$.

We choose such a $V > v_0 > v_1$ large enough so that $v_\Gamma(u_{\mathcal{A}}(V)) > v_0 > v_1$ (this is possible by the Claim 3 of Proposition 4.7). Therefore, we have in particular that

$$\int_{v_\Gamma(u_{\mathcal{A}}(V))}^{+\infty} \kappa(u_1, v)(1 + |\phi|^2(u_1, v)) dv < \delta_1(q_0, m^2, r_1).$$

Notice also that $r(u_1, v_\Gamma(u_{\mathcal{A}}(V))) \leq r_1$ as $v_\Gamma(u_{\mathcal{A}}(V)) > v_1$ and $\{u_1\} \times [v_1, \infty) \subset \mathcal{T}$; thus, $\delta_1(q_0, m^2, r_1) \leq \delta_1(q_0, m^2, r(u_1, v_\Gamma(u_{\mathcal{A}}(V))))$ as $x \rightarrow \delta_1(q_0, m^2, x)$ is a decreasing function, which also implies that

$$\int_{v_\Gamma(u_{\mathcal{A}}(V))}^{+\infty} \kappa(u_1, v)(1 + |\phi|^2(u_1, v)) dv < \delta_1(q_0, m^2, r(u_1, v_\Gamma(u_{\mathcal{A}}(V)))).$$

Thus, the assumptions of Theorem 4.4 are satisfied on the rectangle $[u_1, u] \times [v_\Gamma(u), V]$.

Hence $(u, u + \eta) \times \{V\} \subset \mathcal{T}$, which contradicts $(u, u_\Gamma(V)) \times \{V\} \subset \mathcal{R}$, and the corollary is proven. $\square$

The proof of Corollary 4.8 thus also concludes the proof of Theorem 3.2.

4.6. Proof of Theorem 3.1 and propagation of the mass blowup

We now start with the stronger assumptions of Theorem 3.1. We will simply prove that those assumptions imply the assumptions of Theorem 3.2, which we can then

apply. For this, we need to use a result from [39], which propagates the blowup of the Hawking mass, provided that ϕ and Q are controlled.

LEMMA 4.9 (Propagation of the Hawking mass blowup, [39])

If there exists $u_1 < u_{\mathcal{CH}_{i+}}$ (recall the definition of $u_{\mathcal{CH}_{i+}}$ from Section 2.1) such that

$$\lim_{v \rightarrow +\infty} \rho(u_1, v) = +\infty,$$

$v_1 \in \mathbb{R}$ large enough, and a constant $D > 0$ such that for all $v \geq v_1$,

$$|\phi|^2(u_1, v) + |Q|(u_1, v) \leq D \cdot |\log(\rho)|(u_1, v), \quad (4.16)$$

then, for all $u_1 \leq u_2 < u_{\mathcal{CH}_{i+}}$,

$$\lim_{v \rightarrow +\infty} \rho(u_2, v) = +\infty.$$

Thus, for all $u_1 \leq u < u_{\mathcal{CH}_{i+}}$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$.

Proof

This statement is proven in [39], but we give a streamlined version of the argument for the benefit of the reader.

First, fix $u_2 < u_{\mathcal{CH}_{i+}}$. Note that, by definition (see Theorem 0), for v_1 large enough, there exist constants $0 < r_0^- < r_0^+$ such that $r_0^- < r(u, v) < r_0^+$ for all $(u, v) \in [u_1, u_2] \times [v_1, +\infty]$. Fix $\alpha = \frac{1}{4}$ and $\eta_0 > \eta > 0$; we bootstrap in the region $[u_1, u_B] \times [v_1, +\infty]$ and $u_B \in [u_1, u_2]$:

$$|\phi|^2 + Q^2 \leq \rho^{2\alpha}, \quad (4.17)$$

$$\frac{-\partial_v r}{\Omega^2} \geq \eta. \quad (4.18)$$

From the assumptions of the lemma and (2.8), it is clear that the set of space-time points at which (4.17) and (4.18) are true is nonempty for a small $\eta > 0$ and taking v_1 larger if necessary.

Then, using (2.15) together with bootstrap (4.17), we have for some $C'(C, M, e, m^2, r_0^+, r_0^-) > 0$ that

$$\partial_u \rho \geq \frac{2r^2 \Omega^2}{-\partial_v r} |D_u \phi|^2 - C' \cdot \rho^{2\alpha} \cdot |\partial_u r| \geq -C' \cdot \rho^{2\alpha} \cdot |\partial_u r|,$$

where for the last lower bound we just used $\frac{-\partial_v r}{\Omega^2} \geq 0$, as a soft consequence of (4.18). Since $0 < \alpha < \frac{1}{2}$, it is clear that

$$\partial_u (\rho^{1-2\alpha})(u, v) \geq -(1-2\alpha) \cdot C' \cdot |\partial_u r|(u, v).$$

Thus integrating, we get that for all $(u, v) \in [u_1, u_B] \times [v_1, +\infty)$,

$$\begin{aligned}\rho^{1-2\alpha}(u, v) &\geq \rho^{1-2\alpha}(u_1, v) - (1 - 2\alpha)C' \cdot r(u_1, v) \\ &\geq \rho^{1-2\alpha}(u_1, v) - (1 - 2\alpha)C' \cdot r_+(M, e) \\ &\geq \rho^{1-2\alpha}(u_1, v) - C'',\end{aligned}$$

where $C''(C, M, e, m^2, r_0^+, r_0^-) > 0$. Note that to get the second inequality we have used $r(u_1, v) \leq r(0, v) = r|_{\mathcal{H}^+}(v) \leq r_+(M, e)$. From there, we get

$$\rho(u, v) \geq \rho(u_1, v) \cdot [1 - C'' \cdot \rho^{2\alpha-1}(u_1, v)]^{\frac{1}{1-2\alpha}}.$$

Now, remembering that $\lim_{v \rightarrow +\infty} \rho(u_1, v) = +\infty$ and Taylor-expanding gives us the following inequality, since $2\alpha - 1 < 0$, for some $C''' > 0$,

$$\rho(u, v) \geq \rho(u_1, v) \cdot (1 - C''' \cdot \rho^{2\alpha-1}(u_1, v)). \quad (4.19)$$

From this we obtain the blowup of the mass. Thus, there exists $v'_1 > v_1$ such that for all $u_1 \leq u \leq u_B$, $v \geq v'_1$, $2\rho(u, v) > r_+$; thus $(u, v) \in \mathcal{T}$. Therefore, by (2.8), $\frac{-\partial_v r}{\Omega^2}(u, v) \geq \eta_1 > 0$ for all $v \geq v'_1$ and $u_1 \leq u \leq u_B$, defining $\eta_1 := \sup_{u \in [u_1, u_2]} \frac{-\partial_v r}{\Omega^2}(u, v'_1)$. Thus we retrieve bootstrap (4.18) if $0 < \eta < \eta_1$.

Now we need to retrieve bootstrap (4.17). For this, consider (2.15) and write, under bootstrap (4.17) and (4.18),

$$\frac{r^2 \cdot |\partial_v r|}{2\Omega^2} \cdot |D_u \phi|^2(u, v) \leq C' \cdot |\partial_u r| \rho^{2\alpha}(u, v) + \partial_u \rho(u, v),$$

which is also equivalent, using (2.12), to

$$\begin{aligned}\frac{r|D_u \phi|^2}{2|\partial_u r|}(u, v) &\leq \frac{C' \cdot |\partial_u r| \rho^{2\alpha}(u, v)}{2\rho(u, v) - r(u, v)} + \frac{\partial_u \rho(u, v)}{2\rho(u, v) - r(u, v)} \\ &\leq C' \cdot |\partial_u r| \rho^{-1+2\alpha}(u, v) + \partial_u \log(\rho)(u, v),\end{aligned} \quad (4.20)$$

where we have used $2\rho(u, v) - r(u, v) \geq \rho(u, v)$ on $[u_1, u_B] \times [v_1, +\infty]$ for v_1 large enough, since ρ tends to $+\infty$.

Thus, we get, integrating (4.20) and using (4.19), that for all $u_1 \leq u \leq u_B$,

$$\int_{u_1}^u \frac{r|D_u \phi|^2}{2|\partial_u r|}(u', v) du' \lesssim \rho^{-1+2\alpha}(u_1, v) + \log\left(\frac{\rho(u, v)}{\rho(u_1, v)}\right). \quad (4.21)$$

We can now use Cauchy–Schwarz to estimate ϕ as follows:

$$\begin{aligned}
& \left| \phi(u, v) - e^{-iq_0 \int_{u_1}^u A_u(u', v) du'} \phi(u_1, v) \right| \\
& \leq \int_{u_1}^u |D_u \phi|(u', v) du' \\
& \leq \left(\int_{u_1}^u \frac{r |D_u \phi|^2}{2 |\partial_u r|} (u', v) du' \right)^{\frac{1}{2}} \left(\int_{u_1}^u \frac{2 |\partial_u r|}{r} (u', v) du' \right)^{\frac{1}{2}}, \tag{4.22}
\end{aligned}$$

which gives, combining (4.21) and (4.22),

$$|\phi|(u, v) \lesssim |\phi|(u_1, v) + \left(\log \left(\frac{\rho(u, v)}{\rho(u_1, v)} \right) \right)^{\frac{1}{2}} \lesssim \log(\rho(u, v))^{\frac{1}{2}},$$

where we used (4.16) and (4.19). A similar estimate is true for Q , using (2.5) so we get for some constant $C > 0$ (independent of v_1) and for all $u_1 \leq u \leq u_2$, $v \geq v_1$,

$$|\phi|^2(u, v) + Q^2(u, v) \leq C \log(\rho(u, v)).$$

Since $\lim_{v \rightarrow +\infty} \log(\rho) \rho^{-2\alpha}(u, v) = 0$, and taking v_1 larger if necessary, we have improved bootstrap (4.17), which concludes the proof. $\square$

Later, we will use this lemma to obtain a trapped neighborhood of $\mathcal{CH}_{i+}$. Now we prove Theorem 3.1.

COROLLARY 4.10

For initial data as in Theorem 0, assume that there exists an outgoing future cone emanating from $(u_1, v_1) \in \mathcal{T}$ with $(u_1, +\infty) \in \mathcal{CH}_{i+}$ and $C > 0$ such that for all $v \geq v_1$,

$$|\phi|(u_1, v) + |Q|(u_1, v) \leq C \cdot |\log(\rho)|, \tag{4.23}$$

and the Hawking mass blows up:

$$\lim_{v \rightarrow +\infty} \rho(u_1, v) = +\infty. \tag{4.24}$$

Then (3.3) is satisfied, and there exists $u_0 < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$ such that, for all $u_0 \leq u < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$.

Proof

To prove (3.3), we use (2.12) as $\kappa = \frac{|\partial_v r|}{\frac{2\rho}{r} - 1}$. Then we get, using (4.23), that

$$\kappa \cdot (1 + |\phi|^2) \leq |\partial_v r| \cdot \frac{1 + C^2 \cdot [\log(\rho)]^2}{\frac{2\rho}{r} - 1},$$

and by (4.24), there exists $C' > 0$ such that

$$\frac{1 + C^2 \cdot [\log(\rho(u_1, v))]^2}{\frac{2\rho(u_1, v)}{r} - 1} \leq C'$$

for all v large enough. Then, we have

$$\begin{aligned} \int_{v_1}^{+\infty} \kappa \cdot (1 + |\phi|^2)(u_1, v') dv' &\leq C' \int_{v_1}^{+\infty} (-\partial_v r)(u_1, v') dv' \\ &\leq C' \cdot r(u_1, v_1) < +\infty. \end{aligned}$$

For the trapped neighborhood, we divide the proof in two cases. If $\mathcal{S}_{i+} = \emptyset$ (recall the definition of $\mathcal{S}_{i+}$ from Theorem 0), then for all $u < u_{\mathcal{CH}_{i+}}$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$ by Lemma 4.9 and $u_{\mathcal{CH}_{i+}} = u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, so there is nothing more to do. If $\mathcal{S}_{i+} \neq \emptyset$, for every $u_{\mathcal{CH}_{i+}} < u < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, $r(u, v) \rightarrow 0$ as $v \rightarrow +\infty$. Since for all $(u, v) \in \mathcal{Q}^+ - \Gamma$, $r(u, v) > 0$, $\{u\} \times [v_0, +\infty) \not\subseteq \mathcal{R} \cup \mathcal{A}$ for any $v_0 > v_\Gamma(u)$, so there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$, which concludes the proof, choosing $u_0 > u_{\mathcal{CH}_{i+}}$. $\square$

Thus, the assumptions of Theorem 3.2 are satisfied, and this concludes the proof of Theorem 3.1.

5. Proof of Theorem 3.3

5.1. The logic of the proof of Theorem 3.3

In this section, we state three more steps to prove Theorem 3.3, using Theorem 3.1. Thus, we only assume polynomial decay of the scalar field on the event horizon. This allows us to invoke a result from [39] which proves that there exists a neighborhood of $\mathcal{CH}_{i+} \cup \mathcal{S}_{i+}$ inside the trapped region and that, moreover, either (1.6) is satisfied, or $\mathcal{CH}_{i+}$ is “of static type,” meaning it is an isometric copy of the Reissner–Nordström Cauchy horizon (step 1). Then, we prove that a static type $\mathcal{CH}_{i+}$ cannot be connected to $\mathcal{S}_{i+}$ (step 2) or to b_Γ (step 3), which is sufficient to obtain the conclusion $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$. (Recall that all these boundary components and their notations were defined in Theorem 0.)

- (1) *Starting from assumptions on the event horizon: the dichotomy of [39] (Section 5.2).*

We use the main result of [39]: under the assumptions on the event horizon stated in Theorem 3.3, there exists a trapped neighborhood of $\mathcal{CH}_{i+} \cup \mathcal{S}_{i+}$, and one can classify the Cauchy horizon $\mathcal{CH}_{i+}$:

- (a) $\mathcal{CH}_{i+}$ is of static type: then ϕ , $r - r_-(M, e)$, $Q - e$, $\varpi - M$ extend to zero on $\mathcal{CH}_{i+}$, meaning that $\mathcal{CH}_{i+}$ is an isometric copy of the Reissner–Nordström Cauchy horizon with parameters M and e .
- (b) $\mathcal{CH}_{i+}$ is of dynamical type: for all outgoing light cone $\{u\} \times [v_0, +\infty)$ to the future of the event horizon we have the following asymptotic estimates, in the gauge (2.23) and for v_0 large enough:

$$\kappa(u, v) \lesssim C(u) \cdot e^{-\alpha(M, e) \cdot v}, \quad (5.1)$$

$$|\phi|(u, v) \lesssim C(u) \cdot v. \quad (5.2)$$

- (c) $\mathcal{CH}_{i+}$ is of mixed type: there exists an outgoing light cone $\{u_1\} \times [v_0, +\infty)$ such that the asymptotic estimates (5.1) and (5.2) are true on any outgoing light cone to the future of $\{u_1\} \times [v_0, +\infty)$.

Now, either possibility 1b or possibility 1c hold, in which case there exists a trapped outgoing cone on which (3.1), (3.2) are true and thus $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$ by Theorem 3.1; or possibility 1a holds, and the Cauchy horizon $\mathcal{CH}_{i+}$ is of static type. The rest of the discussion will thus focus on the case where $\mathcal{CH}_{i+}$ is of static type.

In the rest of the proof, we establish, using radically different techniques, that in the latter case [possibility 1a], $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$ is still true. The core of the argument relies on the impossibility to connect a static Cauchy horizon to the other boundary components in presence when $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$.

- (2) *Impossibility to connect a static $\mathcal{CH}_{i+}$ to a nontrivial $\mathcal{S}_{i+}$ (Section 5.3 as well).*

We establish that, if $\mathcal{CH}_{i+}$ is of static type, $\mathcal{S}_{i+} = \emptyset$. This follows from a more general result (cf. Corollary 5.4 and Remark 10) that we establish: if there exists a trapped neighborhood of the endpoint of $\mathcal{CH}_{i+}$ and $\mathcal{S}_{i+} \neq \emptyset$, then the area-radius r must extend to a continuous function on that neighborhood, including at the endpoint where $r = 0$. The continuity of r results from the wave equation (2.20) with a regular right-hand side (because we are in the trapped region) and the propagation of singularities. This is incompatible with a static Cauchy horizon, on which r is a strictly positive constant: $r \equiv r_-(M, e) > 0$.

To complete the proof, the remaining task is to show, using a standard bootstrap method, that there exists a trapped neighborhood of the endpoint of $\mathcal{CH}_{i+}$ if $\mathcal{CH}_{i+}$ is of static type (Proposition 5.3).

- (3) *Impossibility to connect a static $\mathcal{CH}_{i+}$ to b_Γ (Section 5.3).*

To conclude the argument, we must show that it is impossible to have $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \cup \mathcal{S}_{i+} = \emptyset$ and $\mathcal{CH}_{i+}$ of static type. This follows directly from

the estimates of Proposition 5.3, which in particular conclude that there exists a space-time rectangle below the Cauchy horizon where r is lower bounded, which is incompatible with the presence of a center Γ where $r = 0$.

5.2. The classification of the Cauchy horizon proven in [39]

Now, we set the preliminaries to the proof of Theorem 3.3. For this, we recall a theorem proven in [39], under the same assumptions as Theorem 3.3: the main result is precisely the existence of a trapped neighborhood of $\mathcal{CH}_{i+}$ together with some rigidity results, which are related to the blowup or the finiteness of the Hawking mass.

THEOREM 5.1 (Classification of the Cauchy horizon (Theorem 3.5 of [39]))

Under the assumptions of Theorem 3.3, $\mathcal{CH}_{i+} \neq \emptyset$ and there exists a neighborhood of $\mathcal{CH}_{i+}$ inside the trapped region: for all $u < u_{\mathcal{CH}_{i+}}$, there exists $v(u) \in \mathbb{R}$ such that $\{u\} \times [v(u), +\infty) \subset \mathcal{T}$. Moreover, there is an alternative between two possibilities:

- (1) *$\mathcal{CH}_{i+}$ is of dynamical or mixed type in the language of [39] and, as a consequence, there exists $u_1 < u_{\mathcal{CH}_{i+}}$, $\alpha(M, e) > 0$, $C(u_1) > 0$, $v_1 > 0$ such that $(u_1, v_1) \in \mathcal{T}$ and for all $v \geq v_1$:*

$$\kappa(u_1, v) \leq C \cdot e^{-\alpha v}, \quad (5.3)$$

$$|\phi|(u_1, v) \leq C \cdot v^{1-s}. \quad (5.4)$$

In particular, (3.3) and the other assumptions of Theorem 3.2 are satisfied.

- (2) *$\mathcal{CH}_{i+}$ is of static type in the language of [39]; then $r - r_-(e, M)$, ϕ , $D_u \phi$, $\varpi - M$, $Q - e$, $\partial_v \log(\Omega^2) - 2K_-$ all extend continuously to 0 on $\mathcal{CH}_{i+}$, where $2K_-(M, e) := \frac{2}{r_-^2} [M - \frac{e^2}{r_-}] < 0$ is the surface gravity of the Reissner–Nordström Cauchy horizon. Moreover, there exist $u_1 < u_{\mathcal{CH}_{i+}}$, $C(u_1) > 1$, $v_1(u_1) > 0$ such that $(u_1, v_1) \in \mathcal{T}$, and we have the following estimates for all $v \geq v_1$:*

$$C^{-1} \cdot e^{2.01K_-v} \leq \Omega^2(u_1, v) \leq C \cdot e^{1.99K_-v}, \quad (5.5)$$

$$|\phi|(u_1, v) + |D_v \phi|(u_1, v) \leq C \cdot v^{-s}, \quad (5.6)$$

$$\begin{aligned} & \left| \varpi(u_1, v) - M \right| + \left| Q(u_1, v) - e \right| + \left| r(u_1, v) - r_-(M, e) \right| \\ & \leq C \cdot v^{1-2s}, \end{aligned} \quad (5.7)$$

$$\left| \partial_v \log(\Omega^2)(u_1, v) - 2K_- \right| \leq C \cdot v^{1-2s}, \quad (5.8)$$

$$\left| \kappa^{-1}(u_1, v) - 1 \right| \leq C \cdot v^{1-2s}, \quad (5.9)$$

$$|\partial_v r|(u_1, v) \leq C \cdot v^{-2s}, \quad (5.10)$$

$$\int_v^{+\infty} |\partial_v r|(u_1, v') dv' \geq C^{-1} \cdot v^{-p}. \quad (5.11)$$

Now, we prove a small corollary, which is an easy consequence of Theorem 5.1 but is not, strictly speaking, proven in [39]. This corollary proves the trapped neighborhood assumption in Theorem 3.2, under the assumptions of Theorem 3.3.

COROLLARY 5.2

Under the assumptions of Theorem 3.3, there exists $u_0 < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$ such that, for all $u_0 \leq u < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$.

Proof

If $\mathcal{S}_{i+} = \emptyset$, then we apply Theorem 5.1 and the result follows. If $\mathcal{S}_{i+} \neq \emptyset$, then we argue exactly as in the proof of Corollary 4.10: for every $u \in \mathcal{CH}_{i+} < u < u(\mathcal{CH}_{i+} \cup \mathcal{S}_{i+})$, $r(u, v) \rightarrow 0$ as $v \rightarrow +\infty$; thus $\{u\} \times [v_0, +\infty) \not\subseteq \mathcal{R} \cup \mathcal{A}$ for any $v_0 > v_\Gamma(u)$, so there exists $v(u)$ such that $(u, v(u)) \in \mathcal{T}$. $\square$

5.3. The proof of Theorem 3.3

We are now ready to prove Theorem 3.3, that is, that $\mathcal{S}_\Gamma^1 \cup \mathcal{CH}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$ under assumptions on the event horizon. First, we establish stability estimates with data on a static Cauchy horizon and an outgoing cone. Those estimates are extremely strong (a consequence of the rigidity of Cauchy horizons of static type).

PROPOSITION 5.3

Assume that for some $u_1 < u_2 < u_{\mathcal{CH}_{i+}}$, $r - r_-(e, M)$, ϕ , $D_u \phi$, $\varpi - M$, $Q - e$ extend continuously to 0 on $\mathcal{CH}_{i+} \cap [u_1, u_2]$ and moreover estimates (5.5), (5.6), (5.7), (5.8), (5.9), (5.10) are true on $\{u_1\} \times [v_1, +\infty) \subset \mathcal{T}$ for some $v_1 > 1$ and a constant $C > 0$. Then there exists $v'_1 = v'_1(C, M, e, q_0, m^2) > v_1$ such that the following are true:

- (1) *The rectangle belongs to the space-time: $[u_1, u_2] \times [v'_1, +\infty) \subset \mathcal{Q}^+$.*
- (2) *The rectangle is trapped: $[u_1, u_2] \times [v'_1, +\infty) \subset \mathcal{T}$, and the following estimate is true for all $u \in [u_1, u_2]$:*

$$-r \partial_v r(u, v'_1) \geq A > 0, \quad (5.12)$$

where $A(C, M, e, q_0, m^2, p) > 0$ is a constant independent of u_1 and u_2 .

- (3) *Moreover, for all $\epsilon > 0$, there exists $\tilde{v}(\epsilon, C, M, e, q_0, m^2)$ such that, if $v'_1 > \tilde{v}$, the following estimates are true for all $(u, v) \in [u_1, u_2] \times [v'_1, +\infty)$:*

$$|r(u, v) - r_-(M, e)| \leq \epsilon. \quad (5.13)$$

Then, choosing $\epsilon \leq \frac{r_-}{2}$, there exists $v'_1(C, M, e, q_0, m^2)$ such that for all $u_1 < u_2 < u \in \mathcal{H}_{i+}$, $[u_1, u_2] \times [v'_1, +\infty) \cap \Gamma = \emptyset$.

Proof

To fix the ideas and without loss of generality, we fix $v_1 = 2$. In the proof, we consider characteristic initial data on

$$\mathcal{CH}_{i+} \cap \{u_1 \leq u \leq u_2\} \cup \{u_1\} \times [2, +\infty),$$

satisfying (5.5)–(5.11). Whenever necessary, we will restrict the domain of evolution to $\{v \geq v'_1\}$, where $v'_1(C, M, e, q_0, m^2, u_1) \geq v_1 = 2$ will be chosen adequately. The v -gauge will be chosen as (2.23). The u -gauge will be chosen later (note that (2.25) does not make sense, as we consider a “local problem” on a space-time rectangle that does not a priori intersect Γ).

For some $D'(C, M, e, q_0, m^2) > 0$ to be determined later, we formulate the following bootstrap assumptions:

$$|\phi| + |D_v \phi| \leq 4C \cdot v^{-s}, \quad (5.14)$$

$$|D_u \phi| \leq D' \cdot \Omega^2 \cdot v^{-s}, \quad (5.15)$$

$$|Q| \leq 10|e|, \quad (5.16)$$

$$r \geq \frac{r_-}{2}, \quad (5.17)$$

$$|\partial_v r| \leq 4C \cdot v^{-2s}, \quad (5.18)$$

$$3K_-(M, e) \leq \partial_v \log(\Omega^2) \leq K_-(M, e), \quad (5.19)$$

$$\frac{1}{2} \leq \kappa^{-1} \leq 2, \quad (5.20)$$

$$\int_{u_1}^{u_B} \Omega^2 du \leq e^{K-v}. \quad (5.21)$$

Note that (5.14)–(5.21) are all invariant with respect to the choice of u -gauge. We define the bootstrap set

$$\mathcal{B} := \{u_B \geq u_1, [u_1, u_B] \times [v'_1, +\infty) \subset \mathcal{Q}^+,$$

$$\text{and (5.14)–(5.21) are satisfied for all } (u, v) \in [u_1, u_B] \times [v'_1, +\infty)\}.$$

Note that, because of (5.5)–(5.10), $\mathcal{B}$ is nonempty for v'_1 large enough. We will show that $u_B = u_2$ by proving that (5.14)–(5.21) are satisfied with a smaller constant.

Now integrate (2.13) using the boundedness of r and bootstrap (5.15), (5.20): $|\partial_u(\log(\kappa^{-1}))| \lesssim |\partial_u r| \cdot v^{-2s}$, which we can integrate to get, for v'_1 large enough,

$$|\kappa^{-1} - 1| \leq C \cdot v^{1-2s} + \tilde{D}(M, e, q_0, m^2, C) \cdot v^{-2s} \leq \frac{1}{100}. \quad (5.22)$$

In particular, bootstrap (5.20) is retrieved. Note that (5.22) also shows that κ^{-1} extends to 1 on $\mathcal{CH}_i^+$.

Integrate (2.2) in u , using the bound $|\partial_u \partial_v \log(\Omega^2)| \lesssim \Omega^2$ which comes from (5.22), and bootstraps (5.15), (5.16), (5.17), (5.18); we get by (5.21),

$$|\partial_v \log(\Omega^2)(u, v) - \partial_v \log(\Omega^2)(u_1, v)| \lesssim e^{K-v} \leq \frac{|K_-|}{100}, \quad (5.23)$$

where we took v'_1 large enough. In particular, bootstrap (5.19) is retrieved, in view of (5.8) (again for v'_1 large enough).

Now define $\Omega_{CH}^2(u, v) := \frac{\Omega^2(u, v)}{\Omega^2(u_1, v)}$. Since (5.23) shows that $\partial_v \log(\Omega_{CH}^2(u, v))$ is integrable, it means that $\Omega_{CH}^2(u, v)$ admits an extension $\Omega_{CH}^2(u) > 0$ to $\mathcal{CH}_i^+$, and moreover

$$\left| \log\left(\frac{\Omega_{CH}^2(u, v)}{\Omega_{CH}^2(u)}\right) \right| \lesssim e^{K-v}. \quad (5.24)$$

In parallel, note that by (2.3), we have

$$\frac{-\partial_u(\partial_v r)}{-\partial_u r} = \frac{\kappa}{r} \left[\frac{2\varpi}{r} - \frac{Q^2}{r^2} + m^2 r^2 |\phi|^2 \right]. \quad (5.25)$$

Recall that, by assumption, (r, ϖ, Q, ϕ) extends to $(r_-(M, e), M, e, 0)$ on $\mathcal{CH}_i^+$; thus $\frac{-\partial_u(\partial_v r)}{-\partial_u r}$ extends to $2K_- = [\frac{2M}{r_-^2} - \frac{Q^2}{r_-^3}] < 0$.

Note that the above also shows that $\frac{-\partial_u r}{\Omega^2(u_1, v)}$ extends to a finite nonzero limit (namely $\Omega_{CH}^2(u)$) on $\mathcal{CH}_i^+$. Hence the above extension of $\frac{-\partial_u(\partial_v r)}{-\partial_u r}$ is also equivalent to the fact that the quantity

$$-\partial_u \left(\frac{\partial_v r(u, v) - \partial_v r(u_1, v)}{\Omega^2(u_1, v)} \right)$$

extends to $2K_- \cdot \Omega_{CH}^2(u) < 0$; hence $\Omega^2(u_1, 2) \cdot \frac{\partial_v r(u, v) - \partial_v r(u_1, v)}{\Omega^2(u_1, v)}$ also extends to a finite function $F(u) \geq 0$ on $\mathcal{CH}_i^+$ (note that we multiplied $\frac{\partial_v r(u, v) - \partial_v r(u_1, v)}{\Omega^2(u_1, v)}$ by the constant $\Omega^2(u_1, 2)$ so that $F(u)$ is gauge independent). Now choose the u -gauge to be

$$\Omega_{CH}^2(u) = F(u) + 1. \quad (5.26)$$

In view of the above, we have that $\partial_u \log(\Omega_{CH}^2)$ is constant on $\mathcal{CH}_i^+$, more precisely:

$$\partial_u \log(\Omega_{CH}^2) \equiv 2K_- \cdot \Omega^2(u_1, 2) < 0, \quad (5.27)$$

from which we obtain the following estimate for some $D(u_1) > 0$:

$$\int_{u_1}^{u_B} \Omega_{CH}^2(u) du \leq \frac{\Omega_{CH}^2(u_1)}{2|K_-|\Omega^2(u_1, 2)} = D(u_1). \quad (5.28)$$

Combining (5.28) with (5.24), we get, using (5.5),

$$\int_{u_1}^{u_B} \Omega^2(u, v) du \leq D(M, e, u_1) \cdot \Omega^2(u_1, v) \leq e^{1.99K_-v} \leq \frac{1}{2}e^{K_-v} \quad (5.29)$$

for $v'_1(C, M, e, q_0, m^2, u_1)$ large enough, which retrieves bootstrap (5.21).

Then, using (5.22), we write $|\partial_u r| = \frac{\kappa_-}{4} \cdot \Omega^2 \leq \Omega^2$, which we integrate using (5.7) and (5.29):

$$|r(u, v) - r_-| \leq 2C \cdot v^{1-2s}, \quad (5.30)$$

where we took v'_1 large enough. Bootstrap (5.17) is then retrieved.

Now we integrate (2.17) using (5.22), bootstraps (5.14), (5.15), (5.16), and the r boundedness, and we get, for some constant $D(M, e, q_0, m^2, C) > 0$ and using the upper bound $\partial_v \log(\Omega^2) \leq 1.99K_-$,

$$|D_v(rD_u\phi)| \leq D \cdot \Omega^2 \cdot v^{-s} = \frac{D \cdot v^{-s}}{-\partial_v \log(\Omega^2)} \cdot (-\partial_v \Omega^2) \leq \frac{D \cdot v^{-s}}{1.99|K_-|} \cdot (-\partial_v \Omega^2),$$

which we integrate from the future $\{v = +\infty\}$ where $D_u\phi|_{\mathcal{CH}_{i+}} = 0$ by assumption, also using bootstrap (5.17) and integration by parts:

$$|D_u\phi| \leq \frac{2D \cdot v^{-s}}{1.99|K_-| \cdot r_-} \Omega^2. \quad (5.31)$$

This retrieves bootstrap (5.15) for $D' > \frac{2D}{1.99|K_-| \cdot r_-}$. Additionally, one can integrate this estimate in u , using (5.22), (5.29) to obtain

$$|\phi| \leq 2C \cdot v^{-s}, \quad (5.32)$$

where we took v'_1 large enough. Using (2.5), with bootstraps (5.14) and (5.31), we have $|\partial_u Q| \lesssim \Omega^2$; thus by (5.29),

$$|Q(u, v) - e| \leq 2C \cdot v^{1-2s}, \quad (5.33)$$

where we took v'_1 large enough. This retrieves bootstrap (5.16).

Then using the upper and lower bounds on r , the upper bound on Q , and (5.35), we see using (2.17) that $|D_u(rD_v\phi)| \lesssim \Omega^2 \cdot v^{-s}$, which we can integrate, using (5.6), (5.29), and the largeness of v'_1 to get

$$|D_v\phi|(u, v) \leq 2C \cdot v^{-s}, \quad (5.34)$$

which we combine with (5.32) to close bootstrap (5.14).

Now, we integrate (2.20) in the u direction, using (5.10):

$$\begin{aligned} |r\partial_v r(u, v) - r\partial_v r(u_1, v)| &\leq e^{1.98K-v}, \\ |r\partial_v r(u, v)| &\leq 2r_- \cdot C \cdot v^{-2s} + e^{1.98K-v} \leq 4r_- \cdot C \cdot v^{-2s}, \end{aligned} \quad (5.35)$$

where we took v'_1 large enough. (5.35) together with bootstrap (5.17) allows us to retrieve bootstrap (5.18).

Thus we have closed all the bootstraps, and the first claim of the proposition follows.

Finally, we estimate ρ using (2.15) as $|\partial_u \rho| \lesssim \Omega^2$, using (5.35), (5.31), (5.22), (5.32), (5.29), and the upper bounds on r and Q . Integrating, we obtain, for v'_1 large enough,

$$|\rho(u, v) - \rho(u_1, v)| \leq e^{1.98K-v}.$$

Recall also that $\varpi = \rho + \frac{Q^2}{2r}$. Thus combining with (5.30) and (5.33), the upper-lower bounds on r and Q , and (5.7), we also get, taking v'_1 large enough,

$$|\varpi(u, v) - M| \leq 2C \cdot v^{1-2s}. \quad (5.36)$$

From (5.30), (5.33), and (5.36), the third claim of the proposition follows immediately.

Recall that $\{u_1\} \times [v_1, +\infty) \subset \mathcal{T}$ by assumption. By (5.35) we have for all $(u, v) \in [u_1, u_2] \times [v'_1, +\infty)$,

$$-r\partial_v r(u, v) \geq r|\partial_v r|(u_1, v) - e^{1.98K-v}.$$

Now, note that by (5.11) there exists $C' > 0$ and a sequence $v_n \rightarrow +\infty$ such that

$$r|\partial_v r|(u_1, v_n) \geq C' \cdot (v_n)^{-p-1}.$$

Therefore, there exists N (independent of u_1 and u_2) large enough (so that the term $C' \cdot (v_n)^{-p-1}$ dominates $e^{1.98K-v_n}$) such that for all $n \geq N$ and for all $u \in [u_1, u_2]$,

$$-r\partial_v r(u, v_n) \geq \frac{C'}{2} \cdot (v_n)^{-p-1} > 0. \quad (5.37)$$

Therefore, $[u_1, u_2] \times \{v_N\} \subset \mathcal{T}$. By the monotonicity of (2.8), $[u_1, u_2] \times [v_N, +\infty) \subset \mathcal{T}$. Therefore, choosing $v'_1 > v_N$, the entire space-time rectangle is trapped. Choosing $v'_1 = v_n$ for some $n \geq N$ gives (5.12), and the second claim of the proposition follows. $\square$

Using Proposition 5.3, we now show that a static Cauchy horizon cannot be connected to $\mathcal{S}_i^+$.

COROLLARY 5.4

Assume that alternative 2 holds, that is, $\mathcal{CH}_{i+}$ is of static type. Then $\mathcal{S}_{i+} = \emptyset$.

Proof

Suppose for the sake of contradiction that $\mathcal{S}_{i+} \neq \emptyset$.

Set $\epsilon > 0$ and take u_1 as in the statement of Theorem 5.1, alternative 2. By Proposition 5.3, there exists $v'_1(\epsilon)$ such that for all $(u, v) \in [u_1, u\mathcal{CH}_{i+}) \times [v'_1, +\infty)$, (5.12) and (5.13) hold for some $A > 0$. Additionally, by continuity of the functions $u \rightarrow r(u, v)$ and $u \rightarrow -r\partial_v r(u, v)$ for every fixed $v \geq v'_1$, we also have that (5.13) holds for all $(u, v) \in [u_1, u\mathcal{CH}_{i+}] \times [v'_1, +\infty)$ and (5.12) holds for all $u \in [u_1, u\mathcal{CH}_{i+}]$. In particular, taking ϵ smaller if necessary, we have $r(u, v) \geq \frac{r_-}{2}$ for all $(u, v) \in [u_1, u\mathcal{CH}_{i+}] \times [v'_1, +\infty)$. Also, as a consequence of (5.12), $(u\mathcal{CH}_{i+}, v'_1) \in \mathcal{T}$.

Since $\mathcal{T}$ is an open set in the topology of $\mathcal{Q}^+$, there exists $\eta > 0$ such that $[u\mathcal{CH}_{i+}, u\mathcal{CH}_{i+} + \eta] \times \{v'_1\} \subset \mathcal{T}$; thus by the monotonicity of the Raychaudhuri equation (2.8) we have that $[u\mathcal{CH}_{i+}, u\mathcal{CH}_{i+} + \eta] \times [v'_1, +\infty) \subset \mathcal{T}$.

For every fixed $u \in [u\mathcal{CH}_{i+}, u\mathcal{CH}_{i+} + \eta]$, $v \rightarrow r(u, v)$ has a limit $r_{CH}(u) \geq 0$ as $v \rightarrow +\infty$, by monotonicity. Now, we integrate (2.20) in both u and v over the space-time rectangle $[u\mathcal{CH}_{i+}, u\mathcal{CH}_{i+} + \eta] \times [v'_1, +\infty)$ to get a “four points” estimate:

$$\begin{aligned} & \frac{r^2(u\mathcal{CH}_{i+} + \eta, v'_1)}{2} + \frac{r_{CH}^2(u\mathcal{CH}_{i+})}{2} - \frac{r^2(u\mathcal{CH}_{i+}, v'_1)}{2} - \frac{r_{CH}^2(u\mathcal{CH}_{i+})}{2} \\ &= \int_{u\mathcal{CH}_{i+}}^{u\mathcal{CH}_{i+} + \eta} \int_{v'_1}^{+\infty} \frac{\Omega^2}{4} \left(1 - \frac{Q^2}{r^2} - m^2 r^2 |\phi|^2 \right) (u', v') du' dv' \\ &\leq \int_{u\mathcal{CH}_{i+}}^{u\mathcal{CH}_{i+} + \eta} \int_{v'_1}^{+\infty} \frac{\Omega^2(u', v')}{4} du' dv' \\ &\leq \int_{u\mathcal{CH}_{i+}}^{u\mathcal{CH}_{i+} + \eta} \frac{\Omega^2(u', v'_1)}{4|\partial_v r|(u', v'_1)} \int_{v'_1}^{+\infty} |\partial_v r|(u', v') dv' du' \\ &\leq \int_{u\mathcal{CH}_{i+}}^{u\mathcal{CH}_{i+} + \eta} \frac{\Omega^2(u', v'_1) \cdot r(u', v'_1)}{4|\partial_v r|(u', v'_1)} du' \leq D \cdot \eta, \end{aligned}$$

where, from the second line to the third, we used the monotonicity of the Raychaudhuri equation (2.8), and for the last inequality we set

$$D := \left\| \frac{\Omega^2 \cdot r}{4|\partial_v r|} \right\|_{L^\infty([u\mathcal{CH}_{i+}, u\mathcal{CH}_{i+} + \eta] \times \{v'_1\})} < +\infty$$

because the solution is smooth. Since, by monotonicity, we know $r_{CH}^2(u\mathcal{CH}_{i+} + \eta) \leq r_{CH}^2(u\mathcal{CH}_{i+})$, the previous estimate also implies

$$|r_{CH}^2(u\mathcal{C}\mathcal{H}_{i+} + \eta) - r_{CH}^2(u\mathcal{C}\mathcal{H}_{i+})| \leq 2C \cdot \eta + |r^2(u\mathcal{C}\mathcal{H}_{i+} + \eta, v'_1) - r^2(u\mathcal{C}\mathcal{H}_{i+}, v'_1)|.$$

Thus, by continuity of the function $u \rightarrow r^2(u, v'_1)$ on $[u\mathcal{C}\mathcal{H}_{i+}, u\mathcal{C}\mathcal{H}_{i+} + \eta]$, we see that

$$\lim_{\eta \rightarrow 0} r_{CH}^2(u\mathcal{C}\mathcal{H}_{i+} + \eta) = r_{CH}^2(u\mathcal{C}\mathcal{H}_{i+}) \geq \frac{r_-}{2}.$$

This contradicts that for η small enough, $r_{CH}^2(u\mathcal{C}\mathcal{H}_{i+} + \eta) = 0$, by definition of $\mathcal{S}_{i+}$. Thus, $\mathcal{S}_{i+} = \emptyset$. $\square$

Remark 10

Note that we proved a result of independent interest: if $\mathcal{S}_{i+} \neq \emptyset$ and there exists a trapped neighborhood $\mathcal{V}$ of the endpoint of $\mathcal{C}\mathcal{H}_{i+}$ in $\mathcal{Q}^+$, then r extends to a continuous function on $\mathcal{V} \cap (\mathcal{C}\mathcal{H}_{i+} \cup \mathcal{S}_{i+})$; in particular r must be continuous at the endpoint of $\mathcal{C}\mathcal{H}_{i+}$. Since we just used $\mathbb{T}_{vv} \geq 0$ and $\mathbb{T}_{uv} \geq 0$, the result is true in general for any matter model which satisfies the null energy condition.

We now conclude the proof of Theorem 3.3. Under the assumptions of Theorem 3.3, either alternative 1 holds or alternative 2 holds, by Theorem 5.1. If alternative 1 holds, then the assumptions of Theorem 3.2 are satisfied, also using Corollary 5.2; thus $\mathcal{S}_\Gamma^1 \cup \mathcal{C}\mathcal{H}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$. If alternative 2 holds, then $\mathcal{S}_{i+} = \emptyset$ by Corollary 5.4. Assume by contradiction that $\mathcal{S}_\Gamma^1 \cup \mathcal{C}\mathcal{H}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} = \emptyset$. Then, $\mathcal{C}\mathcal{H}_{i+}$ closes off the space-time, and its endpoint is b_Γ ; thus $u\mathcal{C}\mathcal{H}_{i+} = u(b_\Gamma)$. By Theorem 5.1, alternative 2, the assumptions of Proposition 5.3 are satisfied; thus there exists $v'_1 \in \mathbb{R}$ such that $[u_1, u\mathcal{C}\mathcal{H}_{i+}) \times [v'_1, +\infty) \cap \Gamma = \emptyset$. This obviously leads to a contradiction, and thus $\mathcal{S}_\Gamma^1 \cup \mathcal{C}\mathcal{H}_\Gamma \cup \mathcal{S}_\Gamma^2 \cup \mathcal{S} \neq \emptyset$. Therefore, the proof Theorem 3.3 is complete.

Appendix. Cauchy horizons can close off a two-ended space-time

In this section, we give a brief sketch of the proof of Theorem 1.6, mostly based on the strategy of [17].

THEOREM A.1

For some $s > \frac{1}{2}$ and $\epsilon > 0$, we assume that the right event horizon $\mathcal{H}_1^+$ is a future-affine-complete outgoing cone, that for all $v \geq v_0$,

$$|\phi|_{|\mathcal{H}_1^+}(v) + |D_v \phi|_{|\mathcal{H}_1^+}(v) \leq \epsilon \cdot v^{-s}, \quad (\text{A.1})$$

and that the following red-shift estimates hold:

$$|D_u \phi|(u, v_0) \leq \epsilon \cdot |\partial_u r|(u, v_0) \quad (\text{A.2})$$

for all $u \leq u_0$. Additionally, assume that a subextremal Reissner–Nordström black hole is approached on $\mathcal{H}_1^+$, that is,

$$0 < \limsup_{v \rightarrow +\infty} \frac{|Q|_{|\mathcal{H}_1^+(v)}}{r_{|\mathcal{H}_1^+(v)}} < 1. \quad (\text{A.3})$$

We make the same assumptions on the left event horizon $\mathcal{H}_2^+$. Then there exists an $\epsilon_0 > 0$ such that, if $\epsilon < \epsilon_0$, then the Penrose diagram is the one of Figure 4, that is, $\mathcal{C}\mathcal{H}_1^+ \cup \mathcal{C}\mathcal{H}_2^+$ close off the interior and $\mathcal{S} = \emptyset$.

Proof

The hard work, already established in [37], is to prove first the local result $\mathcal{C}\mathcal{H}_1^+ \neq \emptyset$, $\mathcal{C}\mathcal{H}_2^+ \neq \emptyset$ together with stability estimates. The Cauchy stability argument of [17] can be immediately adapted ([17, Proposition 9.1]). Now, one must generalize [17, Theorem 7], which states that r is lower bounded, implying immediately the result. The only place where the Einstein equations are used in this result is in equation (36), ([17, p. 747]), to establish that, in some region,

$$\partial_u \log(|\partial_v r|) \leq \frac{|\partial_u r|}{r}. \quad (\text{A.4})$$

This implication is not true in our model, as by (2.3), $\partial_u \partial_v r = \frac{-\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + \frac{\Omega^2}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 |\phi|^2$, and thus

$$\partial_u \log(|\partial_v r|) \leq \frac{|\partial_u r|}{r} + \frac{\Omega^2}{|\partial_v r|} \cdot \left(\frac{1}{4r} - \frac{Q^2}{4r^3} - \frac{m^2 r |\phi|^2}{2} \right);$$

and this estimate is valid in the region of interest, where $\partial_u r < 0$ and $\partial_v r < 0$. However, the new term does not create any problem as, by Cauchy stability, one still has $\frac{1}{4r} - \frac{Q^2}{4r^3} - \frac{m^2 r |\phi|^2}{2} < 0$, which is the analog of equation (36). Therefore (A.4) is true, and from there one can follow the argument of [17] to the letter. $\square$

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Department of Mathematics, Princeton University, Princeton, New Jersey 08544, USA;

mmooritel@princeton.edu