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Effective Ultrapowers and Applications

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We give a systematic account of the current state of knowledge of an effective analogue of the ultraproduct construction. We start with a product of a uniformly computable sequence of computable structures indexed by the set of natural numbers. The equality of elements and satisfaction of formulas are defined modulo a subset of the index set, which is cohesive, i.e., indecomposable with respect to computably enumerable sets. We present an analogue of Łoś's theorem for effective ultraproducts and a number of results on definability and isomorphism types of the effective ultrapowers of the field of rational numbers, when the complements of cohesive sets are computably enumerable. These effective ultraproducts arose naturally in the study of the automorphisms of the lattice of computably enumerable vector spaces. Previously, a number of authors considered related constructions in the context of nonstandard models of fragments of arithmetic.

1. Introduction

The idea of establishing an axiomatic system that captures the standard model of arithmetic, $\mathbb{N} = (N, +, \cdot, 0, 1, <)$, has been developed in the late 19th century by Dedekind and Peano [28]. Hilbert asked for proof that such axiomatizations are complete and categorical. Computable axiomatizations give incomplete theories according to Gödel. They are also not categorical. The existence of nonstandard models of *formal arithmetic* (the deductive closure of a fixed computable axiomatization) and of *true arithmetic*, $Th(\mathbb{N})$, can be established using ultrapower constructions. In particular, if F is a non-principal ultrafilter on ω, then the ultrapower $\mathbb{N}^\omega/F$ is not isomorphic to $\mathbb{N}$ and is hence a *nonstandard model of arithmetic*. The ultrafilter construction is very elegant but it uses the Axiom of Choice and also produces uncountable models: $|\mathbb{N}^\omega/F| = 2^{\aleph_0}$.

By the ultrapower construction, various first-order theories of arithmetic are not $2^{\aleph_0}$-categorical. It is natural to ask whether there are countable nonstandard models of arithmetic and if such models can be effective. The first question was answered positively by Skolem in 1934. In [32], Skolem constructed a countable nonstandard model of Peano arithmetic, PA (see also [33]). The second question was answered by Tennenbaum in 1959. He proved that if $\mathcal{M} = (M, +, \cdot, 0, 1, <)$ is a countable model of PA, which is not isomorphic to the standard model $\mathbb{N}$, then $\mathcal{M}$ is not computable.

In his construction of nonstandard countable models of PA, Skolem used a combinatorial lemma (Satz 1 in [32]) to define an equivalence relation $\sim$ on the countable set F_1 of arithmetically definable unary functions. The domain of his nonstandard model $\mathbb{N}^*$ is $F_1/\sim$. Skolem's construction has later been presented as an ultrapower construction (see [2], pp. 236–240) but it is Skolem's original proof that is of particular interest to us. The proof of his Satz 1 provides an idea for the construction of certain (more effective) nonstandard structures. Skolem constructed a total function the range of which is what we now call an *arithmetically indecomposable set* (see [20], p. 429, and Remark 2.3(iii)).

Following Skolem's idea, Feferman, Scott, and Tennenbaum in 1959 considered a quotient structure $\mathcal{R}/C$ as a possible model of arithmetic. Here, $\mathcal{R}$ is the set of all unary (total) computable functions, and C is an r-cohesive set of natural numbers. The domain of $\mathcal{R}/C$ consists of the equivalence classes of the elements in $\mathcal{R}$ modulo an equivalence relation $=_c$ defined using C (see Remark 2.3(i)) as follows:

$$f =_c g \Leftrightarrow C \subseteq^* \{n \in \omega \mid f(n) = g(n)\}.$$

It can be established that $\mathbb{N}$ and $\mathcal{R}/C$ are 2-elementary equivalent. Feferman, Scott, and Tennenbaum [12] proved that $\mathcal{R}/C$ is a model of only a fragment of arithmetic. They gave a specific Π_3^0 sentence σ such that $\mathbb{N} \models \sigma$ but $\mathcal{R}/C \not\models \sigma$. Hirshfeld, McLaughlin, and Wheeler later proved a general result that implies that the structure $\mathcal{R}/C$ has no nontrivial automorphisms (that is, it is rigid). The idea of the proof has been given by Hirshfeld and Wheeler in [18] and formally presented by McLaughlin in [22]. The idea for the construction of $\mathcal{R}/C$ can be generalized to higher levels of arithmetical hierarchy (see Definition 2.2). The structures $\mathcal{R}/C$ and the Skolem's $\mathbb{N}^*$ can be seen as the bottom and the top levels of the hierarchy of such structures, respectively. These structures have been studied as models of fragments of arithmetic by Hirshfeld, Lerman, McLaughlin and Wheeler in [17, 18, 20], and [21–23]. Recursive ultrafilter constructions have also been studied by Nelson in [26], where he provided a sufficient condition for recursive saturation of recursive ultrapowers.

More recently, the notion of *cohesive powers of computable structures* has been presented in [10]. The cohesive powers of fields appear in the characterization of certain filters in the lattice of computably enumerable (c.e.) vector spaces, which has been extensively studied in computable model theory. The vector space V_∞ has been introduced by Metakides and Nerode in [24] as a canonical infinite-dimensional, fully effective vector space over a computable field F. The lattice $\mathcal{L}(V_\infty)$ of c.e. subspaces of V_∞, and its factor lattice modulo finite dimension, $\mathcal{L}^*(V_\infty)$, are examples of modular counterparts to the classical distributive lattices of c.e. sets, $\mathcal{E}$, and c.e. sets modulo finite sets, $\mathcal{E}^*$, in computability theory.

In the characterization of certain principal filters in $\mathcal{L}^*(V_\infty)$ (see [9]), we encountered fields with elements that are the equivalence classes of partial computable functions from ω to F. We noticed that the structure of these fields has certain similarities with the classical ultrapower construction. In [10], we introduced the general notion of a cohesive power of a computable structure. It turns out that, in some instances, the cohesive powers of $\mathbb{N}$ are isomorphic to the structures $\mathcal{R}/C$ introduced by Feferman, Scott, and Tennenbaum. In this chapter, we present results from [3, 6, 10], together with certain generalizations and some new proofs.

In Sec. 2, we give some standard definitions from computability theory. In Sec. 3, we introduce the notion of cohesive products and prove the fundamental theorem of cohesive products, an effective analogue of Łoś's theorem for ultraproducts. We further explore the properties of cohesive products of computable structures and provide several examples. In Sec. 4, we present the following theorem from [3] for a field $\mathbb{Q}$ of rational numbers.

Theorem 4.4: For any two maximal sets M_1 and M_2 with complements $\overline{M_1}$ and $\overline{M_2}$, we have

$$\prod_{\overline{M_1}} \mathbb{Q} \cong \prod_{\overline{M_2}} \mathbb{Q} \text{ iff } M_1 \equiv_m M_2,$$

where $\equiv_m$ stands for m-equivalence.

In Sec. 5, we present a new proof of the following theorem from [3].

Theorem 5.1: If M is a maximal set, then $\prod_{\overline{M}} \mathbb{Q}$ has only trivial automorphisms.

Finally, in Sec. 6, we give an example of the role cohesive powers play in the study of the structure and automorphisms of the lattice $\mathcal{L}^*(V_\infty)$.

2. Computability-Theoretic Background

The following notions come from computability theory. The complement of a set $X \subseteq \omega$ is denoted by $\overline{X}$.

Definition 2.1: (i) An infinite set $C \subseteq \omega$ is *cohesive* iff for every c.e. set W, either $W \cap C$ or $\overline{W} \cap C$ is finite.

(ii) A set $M \subseteq \omega$ is *maximal* iff M is c.e. and $\overline{M}$ is cohesive.

(iii) An infinite set $C \subseteq \omega$ is *r-cohesive* iff for every computable set W, either $W \cap C$ or $\overline{W} \cap C$ is finite.

(iv) A set M is *r-maximal* iff M is c.e. and $\overline{M}$ is *r-cohesive.*

Equivalently, a set $M \subseteq \omega$ is maximal if M is c.e., $\overline{M}$ is infinite, and for every c.e. set E such that $M \subseteq E \subseteq \omega$, either $\omega - E$ or $E - M$ is finite. Maximal sets have been extensively studied within the lattice $\mathcal{E}$ of c.e. sets. For sets A and B, we use $A =^* B$ to denote that A and B differ on at most finitely many elements, and we use $A \subseteq^* B$ to denote that all but finitely many elements of A are also elements of B. For vector spaces, we use the same notation provided "modulo finitely many elements" is replaced by "modulo finite dimension." We will now introduce a more general notion than cohesiveness.

Definition 2.2: Let $\mathcal{F}$ be a countable family of subsets of ω. A set $C \subseteq \omega$ is called *$\mathcal{F}$-indecomposable* if C is infinite, and for every W in $\mathcal{F}$, either $W \cap C$ or $\overline{W} \cap C$ is finite.

Remark 2.3:

(i) If $\mathcal{E}$ is the collection of computably enumerable sets, then $\mathcal{E}$-indecomposable sets are cohesive.

(ii) If $\mathcal{R}$ is the collection of computable sets, then $\mathcal{R}$-indecomposable sets are r-cohesive.

(iii) If $\mathcal{A}$ is the collection of arithmetically definable sets, then $\mathcal{A}$-indecomposable sets will be called *arithmetically indecomposable.*

Definition 2.4: A set $B \subseteq \omega$ is called *quasimaximal* if it is the intersection of finitely many maximal sets. If $B = \bigcap_{i=1}^{n} M_i$ where M_i's are maximal, and for $i \neq j$, $M_i \neq^* M_j$, the number n is called the *rank* of B.

Next, we will introduce the notion of m-reducibility which is stronger than Turing reducibility.

Definition 2.5: Let $A, B \subseteq \omega$.

(i) The set A is m-reducible to B, denoted by $A \leq_m B$, if there is a computable function $f : \omega \to \omega$ such that

$$x \in A \Leftrightarrow f(x) \in B.$$

(ii) The sets A and B have the same *m-degree*, denoted by $A \equiv_m B$ (or $\deg_m(A) = \deg_m(B)$), iff $A \leq_m B$ and $B \leq_m A$.

We will now recall the definitions of a computable and a decidable structure.

Definition 2.6: Let $\mathcal{A}$ be a countable structure for a computable language L.

(i) $\mathcal{A}$ is *computable* if its domain and its atomic diagram are computable.

(ii) $\mathcal{A}$ is *decidable* if its domain and its elementary (complete) diagram are computable.

Clearly, every decidable structure is computable, and the theory of a structure is computable in its elementary diagram.

3. Cohesive Products

Cohesive powers of fields were used in [9] to characterize principal filters of quasimaximal spaces. They motivated the definition of cohesive powers of computable structures in [10]. We now give a more general definition of a cohesive product of computable structures. We will use $\simeq$ to denote the equality of partial functions.

Definition 3.1: Let $\mathcal{A}_i$, for $i \in \omega$, be a uniformly computable sequence of computable structures in a computable language L, and let $C \subseteq \omega$ be a cohesive set. The *cohesive product* $\mathcal{B}$ *of* $\mathcal{A}_i$ *over* C, in symbols $\mathcal{B} = \prod_{C} \mathcal{A}_i$, is a structure defined as follows.

(1) Let

$$D = \{\varphi \mid \varphi : \omega \to \bigcup_{i\in\omega} A_i \text{ is a partial computable function,}$$
$$C \subseteq {}^* dom(\varphi), \text{ and if } \varphi(i)\downarrow \text{ then } \varphi(i) \in A_i\}.$$

For $\varphi_1, \varphi_2 \in D$, let

$$\varphi_1 =_C \varphi_2 \quad \text{iff} \quad C \subseteq^* \{i : \varphi_1(i)\downarrow = \varphi_2(i)\downarrow\}.$$

The domain of $\prod_C \mathcal{A}_i$ is the quotient set $D/_{=_C}$ and is denoted here by B.

(2) If $f \in L$ is an n-ary function symbol, then $f^{\mathcal{B}}$ is an n-ary function on B such that for every $[\varphi_1], \dots, [\varphi_n] \in B$, we have

$$f^{\mathcal{B}}([\varphi_1], \dots, [\varphi_n]) = [\varphi] \quad \text{iff} \quad (\forall i \in \omega)\left[\varphi(i) \simeq f^{\mathcal{A}_i}(\varphi_1(i), \dots, \varphi_n(i))\right].$$

(3) If $P \in L$ is an m-ary predicate symbol, then $P^{\mathcal{B}}$ is an m-ary relation on B such that for every $[\varphi_1], \dots, [\varphi_m] \in B$,

$$P^{\mathcal{B}}([\varphi_1], \dots, [\varphi_m]) \quad \text{iff} \quad C \subseteq^* \{i \in \omega \mid P^{\mathcal{A}_i}(\varphi_1(i), \dots, \varphi_m(i))\}.$$

(4) If $c \in L$ is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class (with respect to $=_C$) of the computable function $g : \omega \to \bigcup_{i\in\omega} A_i$ such that $g(i) = c^{\mathcal{A}_i}$, for each $i \in \omega$.

Definition 3.2: (Dimitrov [10]) If $\mathcal{A}_i = \mathcal{A}$ for $i \in \omega$, then $\prod_C \mathcal{A}_i$ is called the *cohesive power of* $\mathcal{A}$ *over* C and is denoted by $\prod_C \mathcal{A}$.

Theorem 3.3: *(Fundamental theorem of cohesive products)*
(1) If $\tau(y_1, \dots, y_n)$ is a term in L and $[\varphi_1], \dots, [\varphi_n] \in B$, then

$$[\tau^{\mathcal{B}}([\varphi_1], \dots, [\varphi_n])]$$

is the equivalence class of a partial computable function such that

$$\tau^{\mathcal{B}}([\varphi_1], \dots, [\varphi_n])(i) \simeq \tau^{\mathcal{A}_i}(\varphi_1(i), \dots, \varphi_n(i)).$$

(2) If $\Phi(y_1, \dots, y_n)$ is a formula in L, which is a conjunction of Σ^0_1 and Π^0_1 formulas, and $[\varphi_1], \dots, [\varphi_n] \in B$, then

$$\mathcal{B} \models \Phi([\varphi_1], \dots, [\varphi_n]) \text{ iff } C \subseteq^* \{i : \mathcal{A}_i \models \Phi(\varphi_1(i), \dots, \varphi_n(i))\}.$$

(3) If Φ is a Π^0_2 sentence in L, then $C \subseteq^* \{i : \mathcal{A}_i \models \Phi\}$ implies $\mathcal{B} \models \Phi$.

Proof: (1) The proof is straightforward, but we note that we essentially use the fact that the operations in the structures $\mathcal{A}_i$ are uniformly computable.

(2) We proceed by induction.

(2.1) Let $\Phi(y_1, \ldots, y_n) = P(\tau_1(y_1, \ldots, y_n), \ldots, \tau_m(y_1, \ldots, y_n))$ be an atomic formula and suppose $[\psi_i] = \tau_i^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])$. Then

$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$

iff

$\mathcal{B} \models P([\psi_1], \ldots, [\psi_m])$

iff

$C \subseteq^* \{i : \mathcal{A}_i \models P(\psi_1(i), \ldots, \psi_m(i))\}$

iff

$C \subseteq^* \{i : \mathcal{A}_i \models \Phi(\varphi_1(i), \ldots, \varphi_m(i))\}$.

(2.2) Suppose $\Phi(y_1, \ldots, y_n) = \Phi_1(y_1, \ldots, y_n) \wedge \Phi_2(y_1, \ldots, y_n)$ and the claim is true for $\Phi_i(y_1, \ldots, y_n)$, $i = 1, 2$. Then

$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$

iff

$\mathcal{B} \models \Phi_1([\varphi_1], \ldots, [\varphi_n])$ and $\mathcal{B} \models \Phi_2([\varphi_1], \ldots, [\varphi_n])$

iff

$C \subseteq^* \{i : \mathcal{A}_i \models \Phi_1(\varphi_1(i), \ldots, \varphi_n(i))\}$ and
$C \subseteq^* \{i : \mathcal{A}_i \models \Phi_2(\varphi_1(i), \ldots, \varphi_n(i))\}$

iff

$C \subseteq^* \{i : \mathcal{A}_i \models \Phi(\varphi_1(i), \ldots, \varphi_n(i))\}$.

(2.3) Suppose $\Phi(y_1, \ldots, y_n) = \exists y \Psi(y, y_1, \ldots, y_n)$ and $\Psi(y, y_1, \ldots, y_n)$ is a quantifier-free formula for which the claim is true.

(2.3a) Suppose $\mathcal{B} \models \exists y \Psi(y, [\varphi_1], \ldots, [\varphi_n])$ and suppose that the partial computable function φ is such that $\mathcal{B} \models \Psi([\varphi], [\varphi_1], \ldots, [\varphi_n])$. By the inductive hypothesis, $C \subseteq^* \{i : \mathcal{A}_i \models \Psi(\varphi(i), \varphi_1(i), \ldots, \varphi_n(i))\}$ and so

$$C \subseteq^* \{i : \mathcal{A}_i \models \exists y \Psi(y, \varphi_1(i), \ldots, \varphi_n(i))\}.$$

(2.3b) Suppose $C \subseteq^* \{i : \mathcal{A}_i \models \exists y \Psi(y, \varphi_1(i), \ldots, \varphi_n(i))\}$. Since the structures $\mathcal{A}_i$ are uniformly computable and $\Psi(y, y_1, \ldots, y_n)$ is quantifier-free, we can define a partial computable function

$$\varphi(i) = (\mu y \in A_i)\,[\mathcal{A}_i \models \Psi(y, \varphi_1(i), \ldots, \varphi_n(i))].$$

Then

$$\{i : \mathcal{A}_i \models \exists y \Psi(y, \varphi_1(i), \ldots, \varphi_n(i))\} = \{i : \mathcal{A}_i \models \Psi(\varphi(i), \varphi_1(i), \ldots, \varphi_n(i))\}$$

and so $C \subseteq^* \{i : \mathcal{A}_i \models \Psi(\varphi(i), \varphi_1(i), \ldots, \varphi_n(i))\}$. By the inductive hypothesis,

$$\mathcal{B} \models \Psi([\varphi], [\varphi_1], \ldots, [\varphi_n]), \text{ and so } \mathcal{B} \models \exists y \Psi(y, [\varphi_1], \ldots, [\varphi_n]).$$

(2.4) Suppose $\Phi(y_1, \ldots, y_n) = \neg\Psi(y_1, \ldots, y_n)$ where $\Psi(y_1, \ldots, y_n)$ is a Σ^0_1 formula for which the hypothesis is true.

(2.4a) Suppose $\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$ and let

$$W = \{i : \mathcal{A}_i \models \Psi(\varphi_1(i), \ldots, \varphi_n(i))\}.$$

Since $\mathcal{B} \not\models \Psi([\varphi_1], \ldots, [\varphi_n])$, we have $C \not\subseteq^* W$. Since $\Psi(y_1, \ldots, y_n)$ is a Σ^0_1 formula, the structures $\mathcal{A}_i$ are uniformly computable, and φ_j, for $1 \leq j \leq n$, is partial computable, W is a c.e. set. Since C is cohesive, we have $C \cap W =^* \emptyset$. Also, since $C \subseteq^* \bigcap_{j=1}^{n} dom(\varphi_j)$, for almost all $i \in C$ we have

$$\mathcal{A}_i \not\models \Psi(\varphi_1(i), \ldots, \varphi_n(i)).$$

Therefore,

$$C \subseteq^* \{i : \mathcal{A}_i \models \neg\Psi(\varphi_1(i), \ldots, \varphi_n(i))\}.$$

(2.4b) Suppose $C \subseteq^* \{i : \mathcal{A}_i \models \neg\Psi(\varphi_1(i), \ldots, \varphi_n(i))\}$. Then

$$C \cap \{i : \mathcal{A}_i \models \Psi(\varphi_1(i), \ldots, \varphi_n(i))\} =^* \emptyset,$$

and, by the inductive hypothesis, $\mathcal{B} \not\models \Psi([\varphi_1], \ldots, [\varphi_n])$. Therefore,

$$\mathcal{B} \models \neg\Psi([\varphi_1], \ldots, [\varphi_n]).$$

(3) Assume that $C \subseteq^* \{i : \mathcal{A}_i \models \Phi\}$, where $\Phi = \forall y \exists z \Psi(y, z)$ for some quantifier-free formula $\Psi(y, z)$. Let $[\varphi] \in B$. Define a partial computable ψ as follows:

$$\psi(i) = (\mu u \in A_i)[\mathcal{A}_i \models \Psi(\varphi(i), u)].$$

Since $\mathcal{A}_i$'s are uniformly computable, we can deduce that ψ is partial computable. Note that

$$C \subseteq^* dom(\psi) \text{ and } C \subseteq^* \{i : \mathcal{A}_i \models \Psi(\varphi(i), \psi(i))\},$$

because $C \subseteq^* dom(\varphi)$ and $C \subseteq^* \{i : \mathcal{A}_i \models \forall y \exists z \Psi(y, z)\}$. Therefore, $\mathcal{B} \models \Psi([\varphi], [\psi])$, which means that $\mathcal{B} \models \exists z \Psi([\varphi], z)$, and so $\mathcal{B} \models \Phi$. $\square$

Remark 3.4: The statement in Part (2) of Theorem 3.3 can be extended to Boolean combinations of Σ_1^0 and Π_1^0 formulas. (For further improvements see [5].)

Definition 3.5: (Dimitrov [10]) Let $\mathcal{B} = \prod_C \mathcal{A}$ be a cohesive power of $\mathcal{A}$. For $c \in A$, let $[\varphi_c] \in B$ be the equivalence class of the total function φ_c such that $\varphi_c(i) = c$ for every $i \in \omega$. The map $d : A \to B$ such that $d(c) = [\varphi_c]$ is called the *canonical embedding* of $\mathcal{A}$ into $\mathcal{B}$.

Theorem 3.6: *(Dimitrov [10]) Let $\mathcal{B} = \prod_C \mathcal{A}$ be a cohesive power of $\mathcal{A}$.*

(1) If Φ is a Π_3^0 sentence in L, then $\mathcal{B} \models \Phi$ implies $\mathcal{A} \models \Phi$.
(2) If Φ is a Σ_3^0 sentence in L, then $\mathcal{A} \models \Phi$ implies $\mathcal{B} \models \Phi$.
(3) If Φ is a Π_2^0 (or Σ_2^0) sentence in L, then $\mathcal{B} \models \Phi$ iff $\mathcal{A} \models \Phi$.
(4) Let $c_1, \dots, c_n \in A$ and let $\Phi(y_1, \dots, y_n)$ be a Π_2^0 or a Σ_2^0 formula in L. Then

$$\mathcal{A} \models \Phi(c_1, \dots, c_n) \text{ iff } \mathcal{B} \models \Phi([\varphi_{c_1}], \dots, [\varphi_{c_n}]).$$

(5) Let $c_1, \dots, c_n \in A$ and let $\Phi(y_1, \dots, y_n)$ be a Σ_3^0 formula in L. Then

$$\mathcal{A} \models \Phi(c_1, \dots, c_n) \text{ implies } \mathcal{B} \models \Phi([\varphi_{c_1}], \dots, [\varphi_{c_n}]).$$

Proof: (1) Let $\Phi = \forall y \exists z \forall t \Psi(y, z, t)$ where $\Psi(y, z, t)$ is a quantifier-free formula. Let $c \in A$ be arbitrary and let $\varphi_c(i) = c$ for every $i \in \omega$. Let $[\varphi] \in B$ be such that $\mathcal{B} \models \forall t \Psi([\varphi_c], [\varphi], t)$. By (2) of Theorem 3.3 above, we have $C \subseteq^* \{i : \mathcal{A} \models \forall t \Psi(\varphi_c(i), \varphi(i), t)\}$. Then $C \subseteq^* \{i : \mathcal{A} \models \exists z \forall t \Psi(c, z, t)\}$. The set C is infinite and so $\mathcal{A} \models \exists z \forall t \Psi(c, z, t)$. Therefore, $\mathcal{A} \models \Phi$.

(2) This is the contrapositive of (1).

(3) Let $\Phi = \forall y \exists z \Psi(y, z)$ where $\Psi(y, z)$ is a quantifier-free formula.

(3a) The fact that $\mathcal{A} \models \Phi$ whenever $\mathcal{B} \models \Phi$ follows from (1).

(3b) Suppose that $\mathcal{A} \models \Phi$ and let $[\varphi] \in B$ be arbitrary. We have that $C \subseteq^* dom(\varphi) = \{i : \mathcal{A} \models \exists z \Psi(\varphi(i), z)\}$. By (2) of Theorem 3.3, $\mathcal{B} \models \exists z \Psi([\varphi], z)$ and so $\mathcal{B} \models \Phi$.

(4) Let $\mathcal{A}_{\mathcal{K}}$ be the structure $\mathcal{A}$ in the language $L \cup \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ with the constant symbols $\mathbf{c}_1, \dots, \mathbf{c}_n$ interpreted as $c_1, \dots, c_n$, respectively. The result follows from part (3) of this theorem applied to $\mathcal{A}_{\mathcal{K}}$ and $\mathcal{B}_{\mathcal{K}} = \prod_C \mathcal{A}_{\mathcal{K}}$.

(5) It follows from part (2) of this theorem. □

Remark 3.7: If $\mathcal{A}$ is a (dense) linear order (without endpoints), a ring, (an algebraically closed) field (of characteristic p), lattice, (atomless) Boolean algebra, then so is $\prod_C \mathcal{A}$. Each of their theories is Π_2^0-axiomatizable, and so we can apply part (3) of the previous theorem.

Theorem 3.8: *(Dimitrov [10]) Let $\mathcal{A}$ be decidable and $\mathcal{B} = \prod_C \mathcal{A}$.*

(1) If $\Phi(y_1, \ldots, y_n)$ is a formula in L, and $[\varphi_1], \ldots, [\varphi_n] \in B$, then

$$\prod_C \mathcal{A} \models \Phi([\varphi_1], \ldots, [\varphi_n]) \text{ iff } C \subseteq^* \{i : \mathcal{A} \models \Phi(\varphi_1(i), \ldots, \varphi_n(i))\}.$$

(2) If Φ is a sentence, then $\prod_C \mathcal{A} \models \Phi$ iff $\mathcal{A} \models \Phi$.

(3) The canonical map d is an elementary embedding of $\mathcal{A}$ into $\mathcal{B}$.

Proof: (1) Note that for any formula $\Psi(y_1, \ldots, y_n)$ the set $\{(a_1, \ldots, a_n) \in A^n : \mathcal{A} \models \Psi(a_1, \ldots, a_n)\}$ is computable, and hence $\{i : \mathcal{A} \models \Psi(\varphi_1(i), \ldots, \varphi_n(i))\}$ is c.e. Then the steps 2.3 and 2.4 in the proof of Theorem 3.3 can be carried out for any formula Ψ.

(2) and (3) follow immediately from (1). □

Proposition 3.9: *(Dimitrov [10]) Let the structure $\mathcal{A}$ be finite and let $\mathcal{B} = \prod_C \mathcal{A}$. Then $\mathcal{B} \cong \mathcal{A}$.*

Proof: Let $[\varphi] \in B$ be arbitrary. For any $c \in A$, let $I_c = \{i : \varphi(i) = c\}$. Note that the set I_c is c.e. Since $dom(\varphi) = \bigcup_{c \in A} I_c$ and A is finite, for some $c_1 \in A$, the set $I_{c_1} \cap C$ is infinite. Since C is cohesive, we have that $C \subseteq^* I_{c_1}$, and so $[\varphi] = [\varphi_{c_1}]$. Thus, the canonical embedding of $\mathcal{A}$ into $\mathcal{B}$ is onto, and so $\mathcal{B} \cong \mathcal{A}$. □

Proposition 3.10: *(Dimitrov [10]) If there is a computable permutation σ of ω such that $\sigma(C_1) =^* C_2$, then $\prod_{C_1} \mathcal{A} \cong \prod_{C_2} \mathcal{A}$.*

Proof: Let the map $\Phi : \prod_{C_2} \mathcal{A} \to \prod_{C_1} \mathcal{A}$ be such that $\Phi([\psi]) = [\psi \circ \sigma]$.

The proof that Φ is an isomorphism is straightforward. □

Remark 3.11: (1) Let C be co-c.e. Then for every $[\varphi] \in \prod_C \mathcal{A}$ there is a computable function f such that $[f] = [\varphi]$.

(2) If M is a maximal set, then $\prod_{\overline{M}} \mathbb{N}$ and $\mathcal{R}/\sim_{\overline{M}}$ are isomorphic.

Proof: (1) Let $[\varphi] \in \prod_C \mathcal{A}$ be arbitrary, $a \in A$, and define

$$f(n) = \begin{cases} \varphi(n) & \text{if } \varphi(n)\downarrow \text{ first,} \\ a & \text{if } n \text{ is enumerated into } \overline{C} \text{ first.} \end{cases}$$

(2) It follows from (1). □

Remark 3.12: (1) The fields $\mathbb{Z}_{p_i}$ (p_i is the i-th prime number) are uniformly computable and, by part (3) of Theorem 3.3, $\prod_C \mathbb{Z}_{p_i}$ is a field, but of characteristic 0.

(2) Notice that, by Proposition 3.9, $\prod_C \mathbb{Z}_p \cong \mathbb{Z}_p$.

(3) No "nonstandard" element of the cohesive power $\prod_C \mathbb{Q}$ of the field $\mathbb{Q}$ is algebraic over $\mathbb{Q}$ (see [10]).

Let $\overline{\mathbb{Z}}_{p_i}$ be the algebraic closure of $\mathbb{Z}_{p_i}$. By a result of Rabin (Theorem 7 in [29]), each $\overline{\mathbb{Z}}_{p_i}$ has a computable presentation. If these presentations are uniformly computable, then we can form $\mathcal{B} = \prod_C \overline{\mathbb{Z}}_{p_i}$. By part (3) of Theorem 3.3, $\mathcal{B}$ is an algebraically closed field (ACF). The structure $\mathcal{B}$ is a countable model of the theory of ACF and we note that the characteristic of $\mathcal{B}$ is 0. Although the theory of ACF_0 is not $\aleph_0$-categorical, we know that all of its countable models are isomorphic to one of the following

$$\overline{\mathbb{Q}} \prec \overline{\mathbb{Q}(x_1)} \prec \overline{\mathbb{Q}(x_1, x_2)} \prec \cdots \prec \overline{\mathbb{Q}(x_1, x_2, \ldots)}.$$

Note that in the classical case the situation is quite different. If U is a nonprincipal ultrafilter on the set of prime numbers, then the classical ultraproduct $\prod_U \overline{\mathbb{Z}}_{p_i}$ is a model of ACF_0. Since $\left|\prod_U \overline{\mathbb{Z}}_{p_i}\right| = 2^{\aleph_0}$ and ACF_0 is categorical in every $\kappa \geq \aleph_1$, we have $\prod_U \overline{\mathbb{Z}}_{p_i} \cong \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers.

4. Isomorphism Types of Cohesive Powers of $\mathbb{Q}$

The results in this section were obtained by Dimitrov, Harizanov, Miller and Mourad and were originally presented in [3]. We assume that $\mathbb{N}$, $\mathbb{Z}$, and

$\mathbb{Q}$ are the standard computable presentations of the structures of natural numbers, integers, and rational numbers in the language $L = \{+, \cdot, 0, 1\}$. Before giving the characterizations of the isomorphism types of co-maximal powers of $\mathbb{Q}$, we will first prove that $\prod_C \mathbb{N}$ is definable both in $\prod_C \mathbb{Z}$ and $\prod_C \mathbb{Q}$. The definability of $\mathbb{Z}$ in $\mathbb{Q}$ (by a Π_3^0 formula) has been established by J. Robinson in [30]. More recently, Koenigsmann [19] gave a Π_1^0 definition of $\mathbb{Z}$ in $\mathbb{Q}$. He proved that there is a positive integer n and a polynomial $p \in \mathbb{Z}[y, z_1, \ldots, z_n]$ such that

$$y \in \mathbb{Z} \Leftrightarrow \mathbb{Q} \models \forall z_1 \cdots \forall z_n [p(y, z_1, \ldots, z_n) \neq 0].$$

The proof of Proposition 4.1 below makes essential use of Koenigsmann's definition of $\mathbb{Z}$ in $\mathbb{Q}$ and does not work with definitions of higher complexity.

The definability of $\mathbb{N}$ in $\mathbb{Z}$ (by various Σ_1^0 formulas) has been established by R. Robinson in [31]. We will use the formula that defines the natural numbers as sums of squares of four integers. Using these results, we obtain that $\mathbb{N}$ can be defined in $\mathbb{Q}$ as follows:

$$x \in \mathbb{N} \Leftrightarrow \exists y_1 \cdots \exists y_4 \left[\bigwedge_{1 \le i \le 4} y_i \in \mathbb{Z} \wedge x = y_1^2 + y_2^2 + y_3^2 + y_4^2 \right],$$

$$x \in \mathbb{N} \Leftrightarrow \mathbb{Q} \models \exists y_1 \cdots \exists y_4 \forall z_1 \cdots \forall z_n$$
$$\left[\bigwedge_{1 \le i \le 4} p(y_i, z_1, \ldots, z_n) \neq 0 \wedge x = y_1^2 + y_2^2 + y_3^2 + y_4^2 \right],$$

which we will abbreviate as

$$x \in \mathbb{N} \Leftrightarrow \mathbb{Q} \models \exists \overline{y} \forall \overline{z} \theta(x, \overline{y}, \overline{z}).$$

Hence $\theta(x, \overline{y}, \overline{z})$ is the quantifier-free formula above in the language of rings $L = \{+, \cdot, 0, 1\}$.

Proposition 4.1: *(Dimitrov, Harizanov, Miller, and Mourad [3]) The natural embedding of* $\prod_C \mathbb{N}$ *is definable in* $\prod_C \mathbb{Q}$ *by the same formula* $\exists \overline{y} \forall \overline{z} \theta(x, \overline{y}, \overline{z})$ *that defines* $\mathbb{N}$ *in* $\mathbb{Q}$.

Proof: First, assume that for some $[\varphi] \in \prod_C \mathbb{Q}$, we have

$$\prod_C \mathbb{Q} \models \exists \overline{y} \forall \overline{z} \theta([\varphi], \overline{y}, \overline{z}),$$

and that y_i's are such that $y_i = [\psi_i]$ and

$$\prod_C \mathbb{Q} \models \forall \overline{z} \theta([\varphi], \overline{[\psi_i]}, \overline{z}).$$

By Theorem 3.3, we have that

$$C \subseteq^* \{n : \mathbb{Q} \models \forall \overline{z} \theta(\varphi(n), \overline{\psi_i(n)}, \overline{z})\}.$$

Using the definition of $\theta(x, \overline{y}, \overline{z})$ we obtain that $C \subseteq^* \{n : \varphi(n) \in \mathbb{N}\}$, which means that $[\varphi] \in \prod_C \mathbb{N}$.

Now, assume that $[\varphi] \in \prod_C \mathbb{N}$. We will prove that

$$\prod_C \mathbb{Q} \models \exists \overline{y} \forall \overline{z} \theta([\varphi], \overline{y}, \overline{z}).$$

Define the partial computable functions $\xi_i : \omega \to \mathbb{Q}$, for $1 \leq i \leq 4$, as follows. If at stage s we have $\varphi^s(n) = m$ and $m \in \mathbb{N}$, then find the least $(b_1, \ldots, b_4) \in \mathbb{N}^4$ such that $m = \sum_{i=1}^{4} b_i^2$, and let $\xi_i(n) = b_i$.
By the definition of the functions ξ_i, we have that

$$C \subseteq^* \Bigg\{ n : \mathbb{Q} \models \forall z_1 \cdots \forall z_n$$

$$\left[\bigwedge_{1 \leq i \leq 4} p(\xi_i(n), z_1, \ldots, z_n) \neq 0 \wedge \varphi(n) = \sum_{1 \leq i \leq 4} \xi_i(n)^2 \right] \Bigg\},$$

which implies that

$$\prod_C \mathbb{Q} \models \exists \overline{y} \forall \overline{z} \theta([\varphi], \overline{y}, \overline{z}).$$

□

Notation 4.2: We will use $\phi^\dagger(\overline{x})$ to denote the formula $\phi(\overline{x})$ with the scope of its quantifiers restricted from $\mathbb{Q}$ to $\mathbb{N}$ (and from $\prod_C \mathbb{Q}$ to $\prod_C \mathbb{N}$ because of Proposition 4.1).

Proposition 4.3: *(Dimitrov, Harizanov, Miller, and Mourad [3]) The structures $\prod_C \mathbb{Q}$ and $\mathbb{Q}$ are not elementary equivalent.*

Proof: Let T be Kleene's predicate. Recall that $T(e, x, z)$ holds if the e^{th} Turing machine program halts on input x in $\leq z$ steps. By a result by

Feferman, Scott, and Tennenbaum (see Theorem 2.1 in [20]), we know that for

$$\phi = \forall x \exists t \forall e \forall z[(e < x \wedge T(e,x,z)) \Rightarrow z < t],$$

$$\mathbb{N} \models \phi \quad \text{but} \quad \prod_C \mathbb{N} \not\models \phi.$$

We can assume that ϕ is a sentence in $L = \{+, \cdot, 0, 1\}$ since both Kleene's T predicate and $<$ are definable in $\mathbb{N}$. Then $\mathbb{Q} \models \phi^\dagger$ iff $\mathbb{N} \models \phi$, and $\prod_C \mathbb{Q} \not\models \phi^\dagger$ iff $\prod_C \mathbb{N} \not\models \phi$. This finally gives us

$$\mathbb{Q} \not\equiv \prod_C \mathbb{Q}.$$

By $\equiv$ we denoted elementary equivalence of structures. □

The following theorem is the main result in this section.

Theorem 4.4: *(Dimitrov, Harizanov, Miller, and Mourad [3]) Let $M_1 \subseteq \omega$ and $M_2 \subseteq \omega$ be maximal sets.*

(1) If $M_1 \equiv_m M_2$, then $\prod_{\overline{M_1}} \mathbb{Q} \cong \prod_{\overline{M_2}} \mathbb{Q}$.

(2) If $M_1 \not\equiv_m M_2$, then $\prod_{\overline{M_1}} \mathbb{Q} \not\equiv \prod_{\overline{M_2}} \mathbb{Q}$.

Proof: (1) It follows from a result by Hermann [16] that if $M_1 \equiv_m M_2$, then there is a computable permutation σ of ω such that $\sigma(M_1) =^* M_2$. Then, by Proposition 3.10, the map $\Phi : \prod_{\overline{M_2}} \mathbb{Q} \to \prod_{\overline{M_1}} \mathbb{Q}$ such that $\Phi([\psi]) = [\psi \circ \sigma]$ is an isomorphism.

(2) If $M_1 \not\equiv_m M_2$, then $\mathcal{R}/\overline{M_1} \not\equiv \mathcal{R}/\overline{M_2}$, by Theorem 2.3 from [20]. In fact, Lerman provided a specific sentence θ (originally in the language $L_< = \{+, \cdot, 0, 1, <\}$) for which $\mathcal{R}/\overline{M_1} \models \theta$ while $\mathcal{R}/\overline{M_2} \models \neg\theta$. By Remark 3.11, part (2), $\mathcal{R}/\overline{M_i} \cong \prod_{\overline{M_i}} \mathbb{N}$, for $i = 1, 2$, and so $\prod_{\overline{M_1}} \mathbb{N} \not\equiv \prod_{\overline{M_2}} \mathbb{N}$. Since $<$ is definable in $L = \{+, \cdot, 0, 1\}$, we can assume that the sentence θ is equivalent to a sentence in the language L. Thus, for the relativisation $\theta^\dagger$, we have $\prod_{\overline{M_1}} \mathbb{Q} \models \theta^\dagger$, while $\prod_{\overline{M_2}} \mathbb{Q} \models \neg\theta^\dagger$. □

5. Automorphisms of Cohesive Powers of $\mathbb{Q}$

The results in this section were obtained by Dimitrov, Harizanov, Miller and Mourad and were originally presented in [3]. Here, we give a new proof of Theorem 5.1.

Let M be a maximal set and let $C = \overline{M}$. We will prove that the field $\prod_C \mathbb{Q}$ is rigid. In this case, $\prod_C \mathbb{N}$ is a special case of the Δ^0_1 ultrapowers studied by Hirshfeld, Wheeler, and McLaughlin. Specifically, they studied the structures $\mathcal{F}_n/U$, where U is a non-principal ultrafilter in the Boolean algebra of Δ^0_n sets, and $\mathcal{F}_n$ is the set of all total functions with Σ^0_n graphs. In [22] (Theorem 2.11), McLaughlin proved a general result about rigidity of a class of ultrapowers. His result implies that the structure $\mathcal{F}_1/U$, which is the set $\mathcal{F}_1$ of all computable functions modulo a non-principal ultrafilter U in the Boolean algebra of Δ^0_1 sets, is rigid in the language $L_< = \{+, \cdot, 0, 1, <\}$. To apply McLaughlin's result, we first notice that his theorem also holds for the language $L = \{+, \cdot, 0, 1\}$ because of the definability of the relation "$<$". Next, the equivalence relation induced by the cohesive set C is equivalent to the one induced by a Δ^0_1 ultrafilter because

$$U_C =_{def} \{W \mid W \in \Delta^0_1 \text{ and } C \subseteq^* W\}$$

is an ultrafilter in the Boolean algebra Δ^0_1. Finally, the domain of $\prod_C \mathbb{N}$ consists of partial computable functions, while the functions in $\mathcal{F}_1$ are total. However, since the set C is co-c.e., by part (2) of Remark 3.11, $\prod_C \mathbb{N} \cong \mathcal{F}_1/U_C$ and, using McLaughlin's result, we have that $\prod_C \mathbb{N}$ is also rigid.

Theorem 5.1: *(Dimitrov, Harizanov, Miller, and Mourad [3])*

(1) The structure $\prod_C \mathbb{Z}$ is rigid.

(2) The structure $\prod_C \mathbb{Q}$ is rigid.

Proof: We will prove only (2). The idea for the proof of (1) is similar. Suppose that Γ is an automorphism of $\prod_C \mathbb{Q}$, and let $[\varphi] \in \prod_C \mathbb{Q}$ be arbitrary. We will prove that $\Gamma([\varphi]) = [\varphi]$ in the (nontrivial) case when $[\varphi] \neq 0$.

By Proposition 4.1, $\prod_C \mathbb{N}$ is first-order definable in $\prod_C \mathbb{Q}$, and so $\Gamma \restriction_{\prod_C \mathbb{N}}$ is an automorphism of $\prod_C \mathbb{N}$. Since $\prod_C \mathbb{N}$ is rigid, we have $\Gamma \restriction_{\prod_C \mathbb{N}} = id \restriction_{\prod_C \mathbb{N}}$.

Let $\overline{y}$ denote the sequence of variables

$$y_{11}, y_{12}, y_{13}, y_{14}, y_{21}, y_{22}, y_{23}, y_{24}, y_{31}, y_{32}, y_{33}, y_{34}.$$

Also, let

$$\theta^{\#}(x, x_1, x_2, x_3, \overline{y}) =_{def}$$
$$(\forall z_1 \cdots \forall z_n) \left[(x \cdot x_1 + x_2 = x_3) \wedge \bigwedge_{i=1}^{3} (x_i = \sum_{j=1}^{4} y_{ij}^2) \wedge \right.$$
$$\left. \bigwedge_{i=1}^{3} \bigwedge_{j=1}^{4} (p(y_{ij}, z_1, \ldots, z_n) \neq 0) \right].$$

Here, $p(y_{ij}, z_1, \ldots, z_n)$ is the polynomial from the Koenigsmann's formula. The intended interpretation of the Π_1^0 formula $\theta^{\#}$ is that

(i) y_{ij} (for $1 \leq i \leq 3, 1 \leq j \leq 4$) are elements of $\mathbb{Z}$,

(ii) x_i are elements of $\mathbb{N}$,

(iii) for $x \in \mathbb{Q}$ we have

$$x = \frac{x_3 - x_2}{x_1}.$$

We will now construct $[\varphi_i], [\varphi_{ij}] \in \prod_C \mathbb{N}$ (for $1 \leq i \leq 3, 1 \leq j \leq 4$) such that

$$\prod_C \mathbb{Q} \vDash \theta^{\#}([\varphi], [\varphi_1], [\varphi_2], [\varphi_3], \overline{[\varphi_{ij}]}).$$

For each $n \in \omega$ such that $\varphi(n) \downarrow= 0$, let:

$$m_i = 0 \text{ for } 1 \leq i \leq 3, \text{ and}$$
$$m_{ij} = 0 \text{ for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 4.$$

For each $n \in \omega$ such that $\varphi(n) \downarrow\neq 0$, find the least

$$\langle m_1, m_2, m_3 \rangle \in \mathbb{N}^3$$

and

$$\langle m_{11}, m_{12}, m_{13}, m_{14}, m_{21}, m_{22}, m_{23}, m_{24}, m_{31}, m_{32}, m_{33}, m_{34} \rangle \in \mathbb{N}^{12}$$

such that

$$m_1 \neq 0$$
$$\varphi(n) \cdot m_1 + m_2 = m_3, \text{ and}$$
$$m_i = \sum_{j=1}^{4} m_{ij}^2 \text{ for } 1 \leq i \leq 3.$$

Then let $\varphi_i(n) = m_i$ (for $1 \leq i \leq 3$) and $\varphi_{ij}(n) = m_{ij}$ (for $1 \leq i \leq 3$ and $1 \leq j \leq 4$). Note that

$$dom(\varphi) = dom(\varphi_i) = dom(\varphi_{ij}),$$

for each $1 \leq i \leq 3$ and $1 \leq j \leq 4$, and moreover,

$$C \subseteq^* dom(\varphi) = \{n : \mathbb{Q} \vDash \theta^\#(\varphi(n), \varphi_1(n), \varphi_2(n), \varphi_3(n), \overline{\varphi_{ij}(n)})\}.$$

By Theorem 3.3,

$$\prod_C \mathbb{Q} \vDash \theta^\#([\varphi], [\varphi_1], [\varphi_2], [\varphi_3], \overline{[\varphi_{ij}]}).$$

In particular, this implies that $[\varphi]$ is a solution of the linear equation

$$x \cdot [\varphi_1] + [\varphi_2] = [\varphi_3] \text{ in } \prod_C \mathbb{Q}.$$

Since Γ is an automorphism of $\prod_C \mathbb{Q}$, we also have

$$\prod_C \mathbb{Q} \vDash \theta^\#(\Gamma([\varphi]), \Gamma([\varphi_1]), \Gamma([\varphi_2]), \Gamma([\varphi_3]), \overline{\Gamma([\varphi_{ij}])}).$$

Since $[\varphi_i]$ and $[\varphi_{ij}]$ are elements of $\prod_C \mathbb{N}$, we have

$$\Gamma([\varphi_i]) = [\varphi_i] \text{ and } \Gamma([\varphi_{ij}]) = [\varphi_{ij}],$$

for $1 \leq i \leq 3$ and $1 \leq j \leq 4$, and so

$$\prod_C \mathbb{Q} \vDash \theta^\#(\Gamma([\varphi]), [\varphi_1], [\varphi_2], [\varphi_3], \overline{[\varphi_{ij}]}).$$

This implies that $\Gamma([\varphi])$ is also a solution of the linear equation

$$x \cdot [\varphi_1] + [\varphi_2] = [\varphi_3] \text{ in } \prod_C \mathbb{Q}.$$

Therefore, $\Gamma([\varphi]) = [\varphi]$. $\square$

6. An Application of Cohesive Powers to Computable Algebra

Assume that the canonical computable $\aleph_0$-dimensional vector space V_∞ is over the field $\mathbb{Q}$. As usual, we denote the lattice of c.e. subspaces of V_∞ by $\mathcal{L}(V_\infty)$, and the lattice $\mathcal{L}(V_\infty)$ modulo finite dimension by $\mathcal{L}^*(V_\infty)$. If $V \in \mathcal{L}(V_\infty)$, then by $\mathcal{L}^*(V^*, \uparrow)$ we will denote the principal filter of V^* in $\mathcal{L}^*(V_\infty)$. To simplify the notation, we will use $Q_{\mathbf{a}} =_{def} \prod_{\overline{M}} \mathbb{Q}$ to denote the cohesive power of the field $\mathbb{Q}$ over a co-maximal set $\overline{M}$ such that $deg_m(M) = \mathbf{a}$. This notation is justified by Theorem 5.1. In [6], Dimitrov and Harizanov characterized the orbits, in $\mathcal{L}^*(V_\infty)$, of the equivalence classes of the closures of quasimaximal subsets of computable bases. We introduced the notion of an *m-degree type* of a quasimaximal set $E = \bigcap_{i=1}^{n} M_i$ of rank n with respect to a fixed computable basis A of V_∞, denoted by $type_A(E)$. This notion captures the number and the m-degrees of the maximal sets M_i's. In the Main Theorem in [6], we proved that there is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(cl(E_1)^*) = cl(E_2)^*$ if and only if $type_{A_1}(E_1) = type_{A_2}(E_2)$. We now give an example illustrating the notion of the type and apply the theorem to this case.

Example 6.1: Suppose A_1 and A_2 are computable bases of V_∞.

Suppose that, for $j = 1, 2$, the sets $M_1^j \equiv_m M_2^j \equiv_m M_3^j$ are different maximal subsets of A_j and $deg_m(M_i^j) = \mathbf{a}_j$. Let $E_j = \bigcap_{i=1}^{3} M_i^j$. In this case, $type_{A_j}(E_j) = (3; \mathbf{a}_j)$.

By the Main Theorem in [9], $\mathcal{L}^*(cl(E_j)^*, \uparrow) \cong \mathcal{L}(3, Q_{\mathbf{a}_j})$, where $\mathcal{L}(3, Q_{\mathbf{a}_j})$ denotes the lattice of all subspaces of a 3-dimensional vector space over the field $Q_{\mathbf{a}_j} =_{def} \prod_{\overline{M_j}} \mathbb{Q}$, for $j = 1, 2$.

By the Main Theorem in [6], there is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(cl(E_1)^*) = cl(E_2)^*$ if and only if $\mathbf{a}_1 = \mathbf{a}_2$.

In the case of maximal sets, we have the following result.

Theorem 6.2: *(Dimitrov and Harizanov [6]) Let M_1 and M_2 be maximal subsets of computable bases A_1 and A_2 of V_∞, respectively. Then the following are equivalent:*

(1) There is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that

$$\Phi(cl(M_1)^*) = cl(M_2)^*,$$

(2) $deg_m(M_1) = deg_m(M_2)$.

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Since the original submission, there have been additional new results related to cohesive powers and linear orders (see [4, 5]).

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