

PROOF OF THE MICHAEL-SIMON-SOBOLEV INEQUALITY USING OPTIMAL TRANSPORT

SIMON BRENDLE AND MICHAEL EICHMAIR

ABSTRACT. We give an alternative proof of the Michael-Simon-Sobolev inequality using techniques from optimal transport. The inequality is sharp for submanifolds of codimension 2.

1. INTRODUCTION

In this paper, we use techniques from optimal transport to prove the following result.

Theorem 1. *Let $n \geq 2$ and $m \geq 1$ be integers. Let $\rho : [0, \infty) \rightarrow (0, \infty)$ be a continuous function with $\int_{\bar{B}^{n+m}} \rho(|\xi|^2) d\xi = 1$, where $\bar{B}^{n+m} = \{\xi \in \mathbb{R}^{n+m} : |\xi| \leq 1\}$ denotes the closed unit ball in $\mathbb{R}^{n+m}$. Let*

$$(1) \quad \alpha = \sup_{z \in \mathbb{R}^n} \int_{\{y \in \mathbb{R}^m : |z|^2 + |y|^2 \leq 1\}} \rho(|z|^2 + |y|^2) dy.$$

Let Σ be a compact n -dimensional submanifold of $\mathbb{R}^{n+m}$, possibly with boundary $\partial\Sigma$. Then

$$(2) \quad |\partial\Sigma| + \int_{\Sigma} |H| \geq n \alpha^{-\frac{1}{n}} |\Sigma|^{\frac{n-1}{n}},$$

where H denotes the mean curvature vector of Σ .

The proof of Theorem 1 is based on an optimal mass transport problem between the submanifold Σ and the unit ball in $\mathbb{R}^{n+m}$, the latter equipped with a rotationally invariant measure. A notable feature is that this transport problem is between spaces of different dimensions.

In Theorem 1, we are free to choose the density ρ . For $m \geq 2$, it is convenient to choose the density ρ so that nearly all of the mass of the measure $\rho(|\xi|^2) d\xi$ on $\bar{B}^{n+m}$ is concentrated near the boundary. This recovers the main result of [2].

Corollary 2. *Let $n \geq 2$ and $m \geq 2$ be integers. Let Σ be a compact n -dimensional submanifold of $\mathbb{R}^{n+m}$, possibly with boundary $\partial\Sigma$. Then*

$$(3) \quad |\partial\Sigma| + \int_{\Sigma} |H| \geq n \left(\frac{(n+m) |B^{n+m}|}{m |B^m|} \right)^{\frac{1}{n}} |\Sigma|^{\frac{n-1}{n}},$$

The first-named author was supported by the National Science Foundation under grant DMS-2103573 and by the Simons Foundation. The second-named author was supported by the START-Project Y963 of the Austrian Science Fund.

where H denotes the mean curvature vector of Σ .

Note that the constant in (3) is sharp for $m = 2$.

Earlier proofs of the non-sharp version of the inequality were obtained by Allard [1], Michael and Simon [8], and Castillon [4]. In particular, the Michael-Simon-Sobolev inequality implies an isoperimetric inequality for minimal surfaces. We refer to [3] for a recent survey on geometric inequalities for minimal surfaces.

Finally, we refer to [5], [6], [7] for some of the earlier work on optimal transport and its applications to geometric inequalities.

2. PROOF OF THEOREM 1

Let Σ be a compact n -dimensional submanifold of $\mathbb{R}^{n+m}$, possibly with boundary $\partial\Sigma$. We denote by g the Riemannian metric on Σ and by $d(\cdot, \cdot)$ the Riemannian distance. For each point $x \in \Sigma$, we denote by $II(x) : T_x\Sigma \times T_x\Sigma \rightarrow T_x^\perp\Sigma$ the second fundamental form of Σ . As usual, the mean curvature vector $H(x) \in T_x^\perp\Sigma$ is defined as the trace of the second fundamental form.

We first consider the special case when $|\Sigma| = 1$. Let μ denote the Riemannian measure on Σ . We define a Borel measure ν on the unit ball $\bar{B}^{n+m}$ by

$$\nu(G) = \int_G \rho(|\xi|^2) d\xi$$

for every Borel set $G \subset \bar{B}^{n+m}$. With this understood, μ is a probability measure on Σ and ν is a probability measure on $\bar{B}^{n+m}$. Let $\mathcal{J}$ denote the set of all pairs (u, h) such that u is an integrable function on Σ , h is an integrable function on $\bar{B}^{n+m}$, and

$$(4) \quad u(x) - h(\xi) - \langle x, \xi \rangle \geq 0$$

for all $x \in \Sigma$ and all $\xi \in \bar{B}^{n+m}$. By Theorem 5.10 (iii) in [11], we can find a pair $(u, h) \in \mathcal{J}$ which maximizes the functional

$$(5) \quad \int_{\bar{B}^{n+m}} h d\nu - \int_{\Sigma} u d\mu.$$

In fact, the result in [11] shows that the maximizer (u, h) may be chosen in such a way that h is Lipschitz continuous and

$$(6) \quad u(x) = \sup_{\xi \in \bar{B}^{n+m}} (h(\xi) + \langle x, \xi \rangle)$$

for all $x \in \Sigma$.

Note that our notation differs from the one in [11]. In our setting, the space X is the unit ball $\bar{B}^{n+m}$ equipped with the measure ν ; the space Y is the submanifold Σ equipped with the Riemannian measure μ ; the cost function is given by $c(x, \xi) = -\langle x, \xi \rangle$ for $x \in \Sigma$ and $\xi \in \bar{B}^{n+m}$; the function ψ in [11] corresponds to the function $-h$; and the function ϕ in [11] corresponds to the function $-u$ in this paper. The fact that ψ can be chosen to be a

c -convex function implies that h is Lipschitz continuous (see [11], Definition 5.2). The fact that ϕ can be taken as the c -transform of ψ corresponds to the statement (6) above (see [11], Definition 5.2).

It follows from (6) that u is the restriction to Σ of a convex function on $\mathbb{R}^{n+m}$ which is Lipschitz continuous with Lipschitz constant at most 1. In particular, u is Lipschitz continuous with Lipschitz constant at most 1. Moreover, u is semiconvex with a quadratic modulus of semiconvexity (see [11], Definition 10.10 and Example 10.11).

Lemma 3. *Let E be a compact subset of Σ . Moreover, suppose that G is a compact subset of $\bar{B}^{n+m}$ such that $u(x) - h(\xi) - \langle x, \xi \rangle > 0$ for all $x \in E$ and all $\xi \in \bar{B}^{n+m} \setminus G$. Then $\mu(E) \leq \nu(G)$.*

Proof. For every positive integer j , we define a compact set $G_j \subset \bar{B}^{n+m}$ by

$$G_j = \{\xi \in \bar{B}^{n+m} : \exists x \in E \text{ with } u(x) - h(\xi) - \langle x, \xi \rangle \leq j^{-1}\}.$$

We define an integrable function u_j on Σ by $u_j = u - j^{-1} \cdot 1_E$. Moreover, we define an integrable function h_j on $\bar{B}^{n+m}$ by $h_j = h - j^{-1} \cdot 1_{G_j}$. Using (4), it is straightforward to verify that

$$u_j(x) - h_j(\xi) - \langle x, \xi \rangle \geq 0$$

for all $x \in \Sigma$ and all $\xi \in \bar{B}^{n+m}$. Therefore, $(u_j, h_j) \in \mathcal{J}$ for each j . Since the pair (u, h) maximizes the functional (5), we obtain

$$\int_{\bar{B}^{n+m}} h_j d\nu - \int_{\Sigma} u_j d\mu \leq \int_{\bar{B}^{n+m}} h d\nu - \int_{\Sigma} u d\mu$$

for each j . This implies $\mu(E) \leq \nu(G_j)$ for each j .

Finally, we pass to the limit as $j \rightarrow \infty$. Note that $G_{j+1} \subset G_j$ for each j . Since E is compact and u is continuous, we obtain

$$\bigcap_{j=1}^{\infty} G_j \subset \{\xi \in \bar{B}^{n+m} : \exists x \in E \text{ with } u(x) - h(\xi) - \langle x, \xi \rangle \leq 0\} \subset G.$$

Putting these facts together, we conclude that

$$\mu(E) \leq \lim_{j \rightarrow \infty} \nu(G_j) \leq \nu(G).$$

This completes the proof of Lemma 3.

Let us fix a large positive constant K such that $|\langle x - \bar{x}, y \rangle| \leq K d(x, \bar{x})^2$ for all points $x, \bar{x} \in \Sigma$ and all $y \in T_{\bar{x}}^{\perp} \Sigma$ with $|y| \leq 1$. For each point $\bar{x} \in \Sigma$, we define

$$\partial u(\bar{x}) = \{z \in T_{\bar{x}} \Sigma : u(x) - u(\bar{x}) - \langle x - \bar{x}, z \rangle \geq -K d(x, \bar{x})^2 \text{ for all } x \in \Sigma\}.$$

We refer to $\partial u(\bar{x})$ as the subdifferential of u at the point $\bar{x}$.

Lemma 4. *Fix a point $\bar{x} \in \Sigma$ and let $\xi \in \bar{B}^{n+m}$. Let $\xi^{\tan}$ denote the orthogonal projection of ξ to the tangent space $T_{\bar{x}} \Sigma$. If $u(\bar{x}) - h(\xi) - \langle \bar{x}, \xi \rangle = 0$, then $\xi^{\tan} \in \partial u(\bar{x})$.*

Proof. By assumption,

$$u(\bar{x}) - h(\xi) - \langle \bar{x}, \xi \rangle = 0.$$

Since

$$u(x) - h(\xi) - \langle x, \xi \rangle \geq 0$$

for all $x \in \Sigma$, it follows that

$$(7) \quad u(x) - u(\bar{x}) - \langle x - \bar{x}, \xi \rangle \geq 0$$

for all $x \in \Sigma$. Using the fact that $\xi - \xi^{\tan} \in T_{\bar{x}}^\perp \Sigma$ and $|\xi - \xi^{\tan}| \leq |\xi| \leq 1$, we obtain

$$(8) \quad \langle x - \bar{x}, \xi - \xi^{\tan} \rangle \geq -K d(x, \bar{x})^2$$

by our choice of K . Combining (7) and (8), we conclude that

$$(9) \quad u(x) - u(\bar{x}) - \langle x - \bar{x}, \xi^{\tan} \rangle \geq -K d(x, \bar{x})^2.$$

Therefore, $\xi^{\tan} \in \partial u(\bar{x})$. This completes the proof of Lemma 4.

By Rademacher's theorem, u is differentiable almost everywhere. At each point where u is differentiable, the norm of its gradient is at most 1. By Alexandrov's theorem (see Theorem 14.1 and Theorem 14.25 in [11]), u admits a Hessian in the sense of Alexandrov at almost every point.

In the following, we fix a point $\bar{x} \in \Sigma \setminus \partial \Sigma$ with the property that u admits a Hessian in the sense of Alexandrov at $\bar{x}$. Let $\hat{u}$ be a smooth function on Σ such that $|u(x) - \hat{u}(x)| \leq o(d(x, \bar{x})^2)$ as $x \rightarrow \bar{x}$.

Let us fix a small positive real number $\bar{r}$ so that $\frac{\sqrt{n}}{2} \bar{r} < d(\bar{x}, \partial \Sigma)$ and $\frac{\sqrt{n}}{2} \bar{r}$ is smaller than the injectivity radius at $\bar{x}$.

For each $r \in (0, \bar{r})$, we denote by $\hat{\omega}(r)$ the smallest nonnegative real number ω with the property that $|z - \nabla^\Sigma \hat{u}(x)| \leq \omega$ whenever $x \in \Sigma$, $z \in \partial u(x)$, and $d(x, \bar{x}) \leq \frac{\sqrt{n}}{2} r$.

Lemma 5. *The function $\hat{\omega} : (0, \bar{r}) \rightarrow [0, \infty)$ is monotone increasing and $\lim_{r \rightarrow 0} \frac{\hat{\omega}(r)}{r} = 0$.*

Proof. The first statement follows immediately from the definition. The second property follows from the basic properties of the Alexandrov Hessian; see [11], Theorem 14.25 (i'). This completes the proof of Lemma 5.

For each $r \in (0, \bar{r})$, we denote by $\hat{\delta}(r)$ the smallest nonnegative real number δ with the property that $D_\Sigma^2 \hat{u}(x) - \langle H(x), \xi \rangle \geq -\delta g$ whenever $x \in \Sigma$, $\xi \in \bar{B}^{n+m}$, $u(x) - h(\xi) - \langle x, \xi \rangle = 0$, and $d(x, \bar{x}) \leq \frac{\sqrt{n}}{2} r$.

Lemma 6. *The function $\hat{\delta} : (0, \bar{r}) \rightarrow [0, \infty)$ is monotone increasing and $\lim_{r \rightarrow 0} \hat{\delta}(r) = 0$.*

Proof. The first statement follows immediately from the definition. To prove the second statement, we argue by contradiction. Suppose that $\limsup_{r \rightarrow 0} \hat{\delta}(r) > 0$. Then we can find a positive real number δ_0 , a sequence of points $x_j \in \Sigma$, and a sequence $\xi_j \in \bar{B}^{n+m}$ with the following properties:

- $x_j \rightarrow \bar{x}$.
- $u(x_j) - h(\xi_j) - \langle x_j, \xi_j \rangle = 0$ for each j .
- For each j , the first eigenvalue of $D_\Sigma^2 \hat{u}(x_j) - \langle II(x_j), \xi_j \rangle$ is less than $-\delta_0$.

After passing to a subsequence, we may assume that the sequence ξ_j converges to $\bar{\xi} \in \bar{B}^{n+m}$. Since $\hat{u}$ is a smooth function, it follows that the first eigenvalue of $D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), \bar{\xi} \rangle$ is strictly negative. Moreover,

$$u(\bar{x}) - h(\bar{\xi}) - \langle \bar{x}, \bar{\xi} \rangle = 0.$$

Since

$$u(x) - h(\bar{\xi}) - \langle x, \bar{\xi} \rangle \geq 0$$

for all $x \in \Sigma$, it follows that

$$u(x) - u(\bar{x}) - \langle x - \bar{x}, \bar{\xi} \rangle \geq 0$$

for all $x \in \Sigma$. Since $|u(x) - \hat{u}(x)| \leq o(d(x, \bar{x})^2)$ as $x \rightarrow \bar{x}$, we conclude that

$$\hat{u}(x) - \hat{u}(\bar{x}) - \langle x - \bar{x}, \bar{\xi} \rangle \geq -o(d(x, \bar{x})^2)$$

as $x \rightarrow \bar{x}$. This implies $D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), \bar{\xi} \rangle \geq 0$. This is a contradiction. This completes the proof of Lemma 6.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_{\bar{x}}\Sigma$. For each $r \in (0, \bar{r})$, we consider the cube

$$W_r = \left\{ z \in T_{\bar{x}}\Sigma : \max_{1 \leq i \leq n} |\langle z, e_i \rangle| \leq \frac{1}{2} r \right\}.$$

We denote by

$$E_r = \exp_{\bar{x}}(W_r) \subset \left\{ x \in \Sigma : d(x, \bar{x}) \leq \frac{\sqrt{n}}{2} r \right\}$$

the image of the cube W_r under the exponential map. We further define

$$A_r = \{(x, y) : x \in E_r, y \in T_x^\perp \Sigma, |\nabla^\Sigma \hat{u}(x)|^2 + |y|^2 \leq (1 + \hat{\omega}(r))^2, \\ D_\Sigma^2 \hat{u}(x) - \langle II(x), y \rangle \geq -\hat{\delta}(r) g\}.$$

Clearly, E_r is a compact subset of Σ and A_r is a compact subset of the normal bundle of Σ . We define a smooth map $\Phi : T^\perp \Sigma \rightarrow \mathbb{R}^{n+m}$ by

$$\Phi(x, y) = \nabla^\Sigma \hat{u}(x) + y$$

for $x \in \Sigma$ and $y \in T_x^\perp \Sigma$. Moreover, we denote by

$$G_r = \{\xi \in \bar{B}^{n+m} : \exists (x, y) \in A_r \text{ with } |\xi - \Phi(x, y)| \leq \hat{\omega}(r)\}$$

the intersection of $\bar{B}^{n+m}$ with the tubular neighborhood of $\Phi(A_r)$ of radius $\hat{\omega}(r)$. Clearly, G_r is a compact subset of $\bar{B}^{n+m}$.

Lemma 7. *Let $r \in (0, \bar{r})$. Then $u(x) - h(\xi) - \langle x, \xi \rangle > 0$ for all $x \in E_r$ and all $\xi \in \bar{B}^{n+m} \setminus G_r$.*

Proof. We argue by contradiction. Suppose that there is a point $x \in E_r$ and a point $\xi \in \bar{B}^{n+m} \setminus G_r$ such that $u(x) - h(\xi) - \langle x, \xi \rangle = 0$. Let $\xi^{\tan}$ denote the orthogonal projection of ξ to the tangent space $T_x \Sigma$. By Lemma 4, $\xi^{\tan} \in \partial u(x)$. Since $d(x, \bar{x}) \leq \frac{\sqrt{n}}{2} r$, it follows that

$$|\xi^{\tan} - \nabla^\Sigma \hat{u}(x)| \leq \hat{\omega}(r)$$

by definition of $\hat{\omega}(r)$. Let $y = \xi - \xi^{\tan} \in T_x^\perp \Sigma$. Then

$$|\xi - \Phi(x, y)| = |\xi - \nabla^\Sigma \hat{u}(x) - y| = |\xi^{\tan} - \nabla^\Sigma \hat{u}(x)| \leq \hat{\omega}(r).$$

Using the triangle inequality, we obtain

$$\sqrt{|\nabla^\Sigma \hat{u}(x)|^2 + |y|^2} = |\Phi(x, y)| \leq |\xi| + \hat{\omega}(r) \leq 1 + \hat{\omega}(r).$$

Finally, since $d(x, \bar{x}) \leq \frac{\sqrt{n}}{2} r$, it follows that

$$D_\Sigma^2 \hat{u}(x) - \langle II(x), y \rangle = D_\Sigma^2 \hat{u}(x) - \langle II(x), \xi \rangle \geq -\hat{\delta}(r) g$$

by definition of $\hat{\delta}(r)$. To summarize, we have shown that $(x, y) \in A_r$ and $|\xi - \Phi(x, y)| \leq \hat{\omega}(r)$. Consequently, $\xi \in G_r$, contrary to our assumption. This completes the proof of Lemma 7.

Lemma 8. *Let $r \in (0, \bar{r})$. Then $\mu(E_r) \leq \nu(G_r)$.*

Proof. This follows by combining Lemma 3 and Lemma 7.

Proposition 9. *Fix a point $\bar{x} \in \Sigma \setminus \partial \Sigma$ with the property that u admits a Hessian in the sense of Alexandrov at $\bar{x}$. Let $\hat{u}$ be a smooth function on Σ such that $|u(x) - \hat{u}(x)| \leq o(d(x, \bar{x})^2)$ as $x \rightarrow \bar{x}$. Let*

$$S = \{y \in T_{\bar{x}}^\perp \Sigma : |\nabla^\Sigma \hat{u}(\bar{x})|^2 + |y|^2 \leq 1, D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle \geq 0\}.$$

Then

$$1 \leq \int_S \det(D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle) \rho(|\nabla^\Sigma \hat{u}(\bar{x})|^2 + |y|^2) dy.$$

Proof. In the following, we fix an arbitrary positive integer j . We define

$$S_j = \{y \in T_{\bar{x}}^\perp \Sigma : |\nabla^\Sigma \hat{u}(\bar{x})|^2 + |y|^2 \leq 1 + j^{-1}, D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle \geq -j^{-1} g\}.$$

For each $r \in (0, \bar{r})$, we decompose the normal space $T_{\bar{x}}^\perp \Sigma$ into compact cubes of size r . Let $\mathcal{Q}_r$ denote the collection of all the cubes in this decomposition. Moreover, we denote by $\mathcal{Q}_{r,j} \subset \mathcal{Q}_r$ the set of all cubes in $\mathcal{Q}_r$ that are contained in the set S_j . We define a smooth map

$$\Psi : W_r \times T_{\bar{x}}^\perp \Sigma \rightarrow \mathbb{R}^{n+m}, (z, y) \mapsto \Phi(\exp_{\bar{x}}(z), P_z y),$$

where $P_z : T_{\bar{x}}^\perp \Sigma \rightarrow T_{\exp_{\bar{x}}(z)}^\perp \Sigma$ denotes the parallel transport along the geodesic $t \mapsto \exp_{\bar{x}}(tz)$ (see [9], pp. 114–115). Since $\lim_{r \rightarrow 0} \hat{\omega}(r) = 0$ and $\lim_{r \rightarrow 0} \hat{\delta}(r) = 0$, we obtain

$$\Phi(A_r) \subset \bigcup_{Q \in \mathcal{Q}_{r,j}} \Psi(W_r \times Q),$$

provided that r is sufficiently small (depending on j). This implies

$$\begin{aligned} G_r &= \{\xi \in \bar{B}^{n+m} : \exists (x, y) \in A_r \text{ with } |\xi - \Phi(x, y)| \leq \hat{\omega}(r)\} \\ &\subset \bigcup_{Q \in \mathcal{Q}_{r,j}} \{\xi \in \bar{B}^{n+m} : \exists (z, y) \in W_r \times Q \text{ with } |\xi - \Psi(z, y)| \leq \hat{\omega}(r)\}, \end{aligned}$$

provided that r is sufficiently small (depending on j).

We next observe that

$$|\det D\Psi(0, y)| = |\det D\Phi(\bar{x}, y)| = |\det(D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle)|$$

for all $y \in T_{\bar{x}}^\perp \Sigma$. Hence, if r is sufficiently small (depending on j), then we obtain

$$\begin{aligned} &\nu(\{\xi \in \bar{B}^{n+m} : \exists (z, y) \in W_r \times Q \text{ with } |\xi - \Psi(z, y)| \leq \hat{\omega}(r)\}) \\ (10) \quad &\leq r^n \int_Q [|\det(D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle)| \rho(|\nabla^\Sigma \hat{u}(\bar{x})|^2 + |y|^2) + j^{-1}] dy \end{aligned}$$

for each cube $Q \in \mathcal{Q}_{r,j}$. To justify (10), we argue as in the proof of the classical change-of-variables formula (see [10], pp. 150–156). We also use the fact that $\lim_{r \rightarrow 0} \frac{\hat{\omega}(r)}{r} = 0$.

Summation over all cubes $Q \in \mathcal{Q}_{r,j}$ gives

$$\begin{aligned} &\nu(G_r) \\ &\leq \sum_{Q \in \mathcal{Q}_{r,j}} \nu(\{\xi \in \bar{B}^{n+m} : \exists (z, y) \in W_r \times Q \text{ with } |\xi - \Psi(z, y)| \leq \hat{\omega}(r)\}) \\ &\leq r^n \int_{S_j} [|\det(D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle)| \rho(|\nabla^\Sigma \hat{u}(\bar{x})|^2 + |y|^2) + j^{-1}] dy, \end{aligned}$$

provided that r is sufficiently small (depending on j).

On the other hand, Lemma 8 implies that $\mu(E_r) \leq \nu(G_r)$ for each $r \in (0, \bar{r})$. Thus, we conclude that

$$\begin{aligned} 1 &= \limsup_{r \rightarrow 0} r^{-n} \mu(E_r) \\ &\leq \limsup_{r \rightarrow 0} r^{-n} \nu(G_r) \\ &\leq \int_{S_j} [|\det(D_\Sigma^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle)| \rho(|\nabla^\Sigma \hat{u}(\bar{x})|^2 + |y|^2) + j^{-1}] dy. \end{aligned}$$

Finally, we pass to the limit as $j \rightarrow \infty$. Note that $S_{j+1} \subset S_j$ for each j . Moreover, $\bigcap_{j=1}^{\infty} S_j = S$. This gives

$$1 \leq \int_S |\det(D_{\Sigma}^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle)| \rho(|\nabla^{\Sigma} \hat{u}(\bar{x})|^2 + |y|^2) dy.$$

Since $D_{\Sigma}^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle \geq 0$ for all $y \in S$, the assertion follows. This completes the proof of Proposition 9.

Corollary 10. *Fix a point $\bar{x} \in \Sigma \setminus \partial\Sigma$ with the property that u admits a Hessian in the sense of Alexandrov at $\bar{x}$. Let $\hat{u}$ be a smooth function on Σ such that $|u(x) - \hat{u}(x)| \leq o(d(x, \bar{x})^2)$ as $x \rightarrow \bar{x}$. Then*

$$n \alpha^{-\frac{1}{n}} \leq \Delta_{\Sigma} \hat{u}(\bar{x}) + |H(\bar{x})|,$$

where α is defined by (1).

Proof. We argue by contradiction. If the assertion is false, then there exists a real number $\hat{\alpha} > \alpha$ such that

$$\Delta_{\Sigma} \hat{u}(\bar{x}) + |H(\bar{x})| \leq n \hat{\alpha}^{-\frac{1}{n}}.$$

Let

$$S = \{y \in T_{\bar{x}}^{\perp} \Sigma : |\nabla^{\Sigma} \hat{u}(\bar{x})|^2 + |y|^2 \leq 1, D_{\Sigma}^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle \geq 0\}.$$

The arithmetic-geometric mean inequality gives

$$0 \leq \det(D_{\Sigma}^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle) \leq \left(\frac{\Delta_{\Sigma} \hat{u}(\bar{x}) - \langle H(\bar{x}), y \rangle}{n} \right)^n \leq \hat{\alpha}^{-1}$$

for all $y \in S$. Using Proposition 9, we obtain

$$\begin{aligned} 1 &\leq \int_S \det(D_{\Sigma}^2 \hat{u}(\bar{x}) - \langle II(\bar{x}), y \rangle) \rho(|\nabla^{\Sigma} \hat{u}(\bar{x})|^2 + |y|^2) dy \\ &\leq \int_S \hat{\alpha}^{-1} \rho(|\nabla^{\Sigma} \hat{u}(\bar{x})|^2 + |y|^2) dy \\ &\leq \hat{\alpha}^{-1} \alpha. \end{aligned}$$

In the last step, we have used the definition of α ; see (1). Thus $\hat{\alpha} \leq \alpha$, contrary to our assumption. This completes the proof of Corollary 10.

After these preparations, we may now complete the proof of Theorem 1. Corollary 10 implies that

$$(11) \quad n \alpha^{-\frac{1}{n}} \leq \Delta_{\Sigma} u + |H|$$

almost everywhere, where $\Delta_{\Sigma} u$ denotes the trace of the Alexandrov Hessian of u . The distributional Laplacian of u may be decomposed into its singular and absolutely continuous part. By Alexandrov's theorem (see Theorem 14.1 in [11]), the density of the absolutely continuous part is given by the

trace of the Alexandrov Hessian of u . The singular part of the distributional Laplacian of u is nonnegative since u is semiconvex. This implies

$$(12) \quad \int_{\Sigma} \eta \Delta_{\Sigma} u \leq - \int_{\Sigma} \langle \nabla^{\Sigma} \eta, \nabla^{\Sigma} u \rangle$$

for every nonnegative smooth function $\eta : \Sigma \rightarrow \mathbb{R}$ that vanishes in a neighborhood of $\partial\Sigma$. Combining (11) and (12), we obtain

$$\begin{aligned} n \alpha^{-\frac{1}{n}} \int_{\Sigma} \eta &\leq \int_{\Sigma} \eta \Delta_{\Sigma} u + \int_{\Sigma} \eta |H| \\ &\leq - \int_{\Sigma} \langle \nabla^{\Sigma} \eta, \nabla^{\Sigma} u \rangle + \int_{\Sigma} \eta |H| \\ &\leq \int_{\Sigma} |\nabla^{\Sigma} \eta| + \int_{\Sigma} \eta |H| \end{aligned}$$

for every nonnegative smooth function $\eta : \Sigma \rightarrow \mathbb{R}$ that vanishes in a neighborhood of $\partial\Sigma$. By a straightforward limiting procedure, this implies

$$n \alpha^{-\frac{1}{n}} |\Sigma| \leq |\partial\Sigma| + \int_{\Sigma} |H|.$$

This completes the proof of Theorem 1 in the special case when $|\Sigma| = 1$. The general case follows by scaling.

3. PROOF OF COROLLARY 2

In this final section, we explain how Corollary 2 follows from Theorem 1. Assume that $n \geq 2$ and $m \geq 2$. We can find a sequence of continuous functions $\rho_j : [0, \infty) \rightarrow (0, \infty)$ such that $\int_{\bar{B}^{n+m}} \rho_j(|\xi|^2) d\xi = 1$,

$$\sup_{[0, 1-j^{-1}]} \rho_j \leq o(1),$$

and

$$\sup_{[1-j^{-1}, 1]} \rho_j \leq \frac{2j}{(n+m)|B^{n+m}|} + o(j)$$

as $j \rightarrow \infty$. For each point $z \in \mathbb{R}^n$, we obtain

$$\begin{aligned} &\int_{\{y \in \mathbb{R}^m : |z|^2 + |y|^2 \leq 1\}} \rho_j(|z|^2 + |y|^2) dy \\ &\leq |B^m| (1 - |z|^2 - j^{-1})_+^{\frac{m}{2}} \sup_{[0, 1-j^{-1}]} \rho_j \\ &\quad + |B^m| \left[(1 - |z|^2)_+^{\frac{m}{2}} - (1 - |z|^2 - j^{-1})_+^{\frac{m}{2}} \right] \sup_{[1-j^{-1}, 1]} \rho_j \\ &\leq |B^m| \sup_{[0, 1-j^{-1}]} \rho_j + \frac{m}{2} |B^m| j^{-1} \sup_{[1-j^{-1}, 1]} \rho_j. \end{aligned}$$

In the last step, we have used the fact that $m \geq 2$. This implies

$$\sup_{z \in \mathbb{R}^n} \int_{\{y \in \mathbb{R}^m : |z|^2 + |y|^2 \leq 1\}} \rho_j(|z|^2 + |y|^2) dy \leq \frac{m |B^m|}{(n+m)|B^{n+m}|} + o(1)$$

as $j \rightarrow \infty$. Therefore, the assertion follows from Theorem 1.

REFERENCES

- [1] W. Allard, *On the first variation of a varifold*, Ann. of Math. 95, 417–491 (1972)
- [2] S. Brendle, *The isoperimetric inequality for a minimal submanifold in Euclidean space*, J. Amer. Math. Soc. 34, 595–603 (2021)
- [3] S. Brendle, *Minimal hypersurfaces and geometric inequalities*, Ann. Fac. Sci. Toulouse Math. 32, 179–201 (2023)
- [4] P. Castillon, *Submanifolds, isoperimetric inequalities and optimal transportation*, J. Funct. Anal. 259, 79–103 (2010)
- [5] D. Cordero-Erausquin, R.J. McCann, and M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math. 146, 219–257 (2001)
- [6] W. Gangbo and R.J. McCann, *Shape recognition via Wasserstein distance*, Quart. Appl. Math. 58, 705–737 (2000)
- [7] R.J. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. 11, 589–608 (2001)
- [8] J.H. Michael and L.M. Simon, *Sobolev and mean value inequalities on generalized submanifolds of $\mathbb{R}^n$* , Comm. Pure Appl. Math. 26, 361–379 (1973)
- [9] B. O’Neill, *Semi-Riemannian geometry*, Academic Press, 1983
- [10] W. Rudin, *Real and Complex Analysis*, Third edition, McGraw Hill, 1987
- [11] C. Villani, *Optimal Transport, Old and New*, Springer-Verlag, 2009

COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK NY 10027, USA

UNIVERSITY OF VIENNA, FACULTY OF MATHEMATICS, OSKAR-MORGENSTERN-PLATZ
1, 1090 VIENNA, AUSTRIA, ORCID: 0000-0001-7993-9536