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GLOBAL L_p ESTIMATES FOR KINETIC KOLMOGOROV–FOKKER–PLANCK EQUATIONS IN DIVERGENCE FORM*

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Abstract. We present a priori estimates and unique solvability results in the mixed-norm Lebesgue spaces for the kinetic Kolmogorov–Fokker–Planck (KFP) equation in divergence form. The leading coefficients are bounded uniformly nondegenerate with respect to the velocity variable v and satisfy a vanishing mean oscillation (VMO) type condition. We consider the L_2 case separately and treat more general equations, which include the relativistic KFP equation. This paper is a continuation of our previous work on L_p estimates for KFP equations in nondivergence form.

Key words. kinetic Kolmogorov–Fokker–Planck equations, mixed-norm Sobolev estimates, Muckenhoupt weights, vanishing mean oscillation coefficients

MSC codes. 35K70, 35H10, 35B45, 34A12

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1. Introduction and main results. For any integer $d \geq 1$, let $\mathbb{R}^d$ be a Euclidean space of points $(x_1, \ldots, x_d)$, and for $T \in (-\infty, \infty]$, we set $\mathbb{R}^d_T = (-\infty, T) \times \mathbb{R}^{d-1}$. Throughout the paper, z is the triple (t, x, v), where $t \in \mathbb{R}$, and $x, v \in \mathbb{R}^d$.

The goal of this paper is to prove the a priori estimates and unique solvability results for the kinetic Kolmogorov–Fokker–Planck (KFP) equation in divergence form given by

$$\partial_t u - v \cdot D_x u - D_{v_i}(a^{ij}(z) D_{v_j} u) + \operatorname{div}_v(\bar{b} u) + b \cdot D_v u + cu + \lambda u = \operatorname{div}_v \vec{f} + g. \tag{1.1}$$

1.1. Notation and assumptions. For $x_0 \in \mathbb{R}^d$, $z_0 \in \mathbb{R}^{1+2d}$, and $r, R > 0$, we introduce

$$\begin{aligned}
B_r(x_0) &= \{\xi \in \mathbb{R}^d : |\xi - x_0| < r\},\\
Q_{r,R}(z_0) &= \{z : -r^2 < t - t_0 < 0, |v - v_0| < r, |x - x_0 + (t - t_0) v_0|^{1/3} < R\},\\
\widetilde{Q}_{r,R}(z_0) &= \{z : |t - t_0|^{1/2} < r, |v - v_0| < r, |x - x_0 + (t - t_0) v_0|^{1/3} < R\},\\
Q_r(z_0) &= Q_{r,r}(z_0), \quad \widetilde{Q}_r(z_0) = \widetilde{Q}_{r,r}(z_0), \quad Q_r = Q_r(0), \quad \widetilde{Q}_r = \widetilde{Q}_r(0).
\end{aligned}$$

For $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and a Lebesgue measurable set, we denote by $(f)_A$ or $⨍_A f\,dx$ the average of f over A. Furthermore, for $c > 0$, $T \in (-\infty, \infty]$, and $f \in L_{1,\mathrm{loc}}(\mathbb{R}^{1+2d}_T)$, we introduce variants of maximal and sharp functions

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$$
(1.2)\qquad
\begin{aligned}
\mathbb{M}_{c,T} f(z_0) &= \sup_{r>0, z_1 \in \overline{\mathbb{R}^{1+2d}_T}: z_0 \in Q_{r,cr}(z_1)} \fint_{Q_{r,cr}(z_1)} |f(z)|\, dz, \quad \mathcal{M}_T f := \mathbb{M}_{1,T} f,\\
f^{\#}_T(z_0) &= \sup_{r>0, z_1 \in \overline{\mathbb{R}^{1+2d}_T}: z_0 \in Q_r(z_1)} \fint_{Q_r(z_1)} |f(z) - (f)_{Q_r(z_1)}|\, dz.
\end{aligned}
$$

We impose the following assumption on the coefficients.

Assumption 1.1. The coefficients $a(z) = (a^{ij}(z), i, j = 1, \ldots, d)$ are bounded measurable functions such that for some $\delta \in (0,1)$,

$$
\delta |\xi|^2 \le a^{ij}(z) \xi_i \xi_j, \quad |a^{ij}(z)| \le \delta^{-1} \quad \forall \xi \in \mathbb{R}^d,\, z \in \mathbb{R}^{1+2d}.
$$

The next assumption can be viewed as a kinetic $VMO_{x,v}$ (vanishing mean oscillation) assumption with respect to

$$
(1.3)\qquad \rho(z, z_0) = \max\{|t - t_0|^{1/2}, |x - x_0 + (t - t_0) v_0|^{1/3}, |v - v_0|\},
$$

which satisfies all the properties of the quasi-metric except the symmetry. It is analogous to the VMO_x condition from the theory of nondegenerate parabolic equations with rough coefficients (see Chapter 6 of [24]).

Assumption 1.2. (γ_0) There exists $R_0 \in (0,1)$ such that for any z_0 and $r \in (0, R_0]$,

$$
\operatorname{osc}_{x,v}(a, Q_r(z_0)) \le \gamma_0,
$$

where

$$
\begin{aligned}
&\operatorname{osc}_{x,v}(a, Q_r(z_0))\\
&\quad = \fint_{(t_0 - r^2, t_0)} \fint_{D_r(z_0,t) \times D_r(z_0,t)} |a(t,x,v) - a(t,x',v')|\, dx dv dx' dv'\, dt,
\end{aligned}
$$

and

$$
D_r(z_0, t) = \{(x,v) : |x - x_0 + (t - t_0) v_0|^{1/3} < r, |v - v_0| < r\}.
$$

Remark 1.3. In this remark, we give examples of when Assumption 1.2 is satisfied. Throughout the remark, $\omega : [0,\infty) \to [0,\infty)$ is an increasing function such that $\omega(0+) = 0$.

Anisotropic $VMO_{x,v}$ condition:

$$
(1.4)\qquad
\begin{aligned}
&osc'_{x,v}(a, r) := \sup_{t,x,v} r^{-8d}\\
&\quad \times \int_{x,x' \in B_{r^3}(x)} \int_{v,v' \in B_r(v)} |a(t,x,v) - a(t,x',v')|\, dx dx' dv dv' \le \omega(r).
\end{aligned}
$$

Since $\operatorname{osc}_{x,v}(a, Q_r(z_0)) \le \operatorname{osc}'_{x,v}(a, r)$, if the anisotropic $VMO_{x,v}$ condition holds, then for any $\gamma_0 \in (0,1)$, Assumption 1.2 (γ_0) holds.

Continuity with respect to the anisotropic distance $\operatorname{dist}((x,v),(x',v')) := |x - x'|^{1/3} + |v - v'|$: For any t, x, x', v, v',

$$
|a(t,x,v) - a(t,x',v')| \le \omega\big(\operatorname{dist}((x,v),(x',v'))\big).
$$

Note that if this condition holds, then (1.4) is true, and therefore, for any $\gamma_0 \in (0,1)$, Assumption 1.2 (γ_0) is satisfied.

Assumption 1.4. The functions $b=(b^i, i=1,\dots,d)$, $\bar{b}=(\bar{b}^i, i=1,\dots,d)$, and c are bounded measurable on $\mathbb{R}^{1+2d}$, and they satisfy the condition

$$|b|+|\bar{b}|+|c|\le L$$

for some constant $L>0$.

1.2. Function spaces. Below we define the mixed-norm Lebesgue and Sobolev spaces. In all these definitions, $G\subset\mathbb{R}^{1+2d}$ is an open set, and $p, r_1, \dots, r_d, q>1$ are numbers.

DEFINITION 1.5. *We say that w is a weight on $\mathbb{R}^d$ if w is a locally integrable function that is positive almost everywhere. Let $w_i, i=0,1,\dots,d$, be weights on $\mathbb{R}$. By $L_{p,r_1,\dots,r_d,q}(G,w)$ with*

$$w=w(t,v)=w_0(t)w_1(v_1)\cdots w_d(v_d), \tag{1.5}$$

we denote the space of all Lebesgue measurable functions on $\mathbb{R}^{1+2d}$ such that

$$\|f\|_{L_{p,r_1,\dots,r_d,q}(G,w)} = \left|\int_{\mathbb{R}}\left|\dots\left|\int_{\mathbb{R}}\left|\int_{\mathbb{R}^d}|f|^p(z)1_G(z)\,dx\right|^{\frac{r_1}{p}} w_1(v_1)dv_1\right|^{\frac{r_2}{r_1}}\dots w_d(v_d)dv_d\right|^{\frac{q}{r_d}} w_0(t)dt\right|^{\frac{1}{q}},$$

and for $\alpha\in(-1,p-1)$, we set $L_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T, |x|^\alpha\prod_{i=1}^d w_i(v_i))$ to be the weighted mixed-norm Lebesgue space with the norm

$$\|f\|_{L_{p;r_1,\dots,r_d}(G,|x|^\alpha\prod_{i=1}^d w_i(v_i))} = \left|\int_{\mathbb{R}}\dots\left|\int_{\mathbb{R}}\left|\int_{\mathbb{R}^{d+1}}|f|^p(z)1_G(z)|x|^\alpha\,dxdt\right|^{\frac{r_1}{p}} w_1(v_1)dv_1\right|^{\frac{r_2}{r_1}}\dots w_d(v_d)dv_d\right|^{\frac{1}{r_d}}. \tag{1.6}$$

Furthermore, for a vector-valued function $\vec{f}=(f_1,\dots,f_d)$, we write

$$\vec{f}\in L_{p,r_1,\dots,r_d,q}(G,w)\left(or\ L_{p;r_1,\dots,r_d}(G,|x|^\alpha\prod_{i=1}^d w_i(v_i))\right)$$

if each component f_i is in $L_{p,r_1,\dots,r_d,q}(G,w)$ $\left(or\ L_{p;r_1,\dots,r_d}(G,|x|^\alpha\prod_{i=1}^d w_i(v_i))\right)$.

Throughout this paper, $w=w(t,v)$ is a weight on $\mathbb{R}^{1+d}$.

DEFINITION 1.6. *By $\mathbb{H}^{-1}_{p,r_1,\dots,r_d,q}(G,w)$ we denote the set of all functions u on G such that there exist $\vec{f}, g\in L_{p,r_1,\dots,r_d,q}(G,w)$ satisfying*

$$u=div_v\vec{f}+g. \tag{1.7}$$

The norm is given by

$$\|u\|_{\mathbb{H}^{-1}_{p,r_1,\dots,r_d,q}(G,w)}=\inf\left(\|\vec{f}\|_{L_{p,r_1,\dots,r_d,q}(G,w)}+\|g\|_{L_{p,r_1,\dots,r_d,q}(G,w)}\right),$$

where the infimum is taken over all $\vec{f}$ and g satisfying (1.7).

Here is the definition of the kinetic (ultraparabolic) Sobolev spaces. The first one is designed to treat the divergence form equations, whereas the second works with the KFP equations in nondivergence form.

Definition 1.7. *By $\mathbb{S}_{p,r_1,\ldots,r_d,q}(G,w)$ we denote the Banach space of all functions u such that $u, D_v u \in L_{p,r_1,\ldots,r_d,q}(G,w)$, and $(\partial_t - v\cdot D_x)u \in \mathbb{H}^{-1}_{p,r_1,\ldots,r_d,q}(G,w)$. The norm is defined as follows:*

$$\|u\|_{\mathbb{S}_{p,r_1,\ldots,r_d,q}(G,w)} = \|u\| + \|D_v u\| + \|\partial_t u - v\cdot D_x u\|_{\mathbb{H}^{-1}_{p,r_1,\ldots,r_d,q}(G,w)},$$

where $\|\cdot\| = \|\cdot\|_{L_{p,r_1,\ldots,r_d,q}(G,w)}$.

Definition 1.8. *Let $S_{p,r_1,\ldots,r_d,q}(G,w)$ be the Banach space of functions u such that u, $D_v u$, $D_v^2 u$, $(\partial_t - v\cdot D_x)u \in L_{p,r_1,\ldots,r_d,q}(G,w)$, and the norm is given by*

$$\|u\|_{S_{p,r_1,\ldots,r_d,q}(G,w)} = \||u| + |D_v u| + |D_v^2 u| + |\partial_t u - v\cdot D_x u|\|_{L_{p,r_1,\ldots,r_d,q}(G,w)}.$$

If $w\equiv 1$ or $p = q = r_1 = r_2 = \cdots = r_d$, we drop w or $q, r_1,\ldots,r_d$ from the above notation.

We define the spaces

$$\mathbb{H}^{-1}_{p;r_1,\ldots,r_d}\left(G, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right), \quad \mathbb{S}_{p;r_1,\ldots,r_d}\left(G, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right),$$
$$\text{and } S_{p;r_1,\ldots,r_d}\left(G, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right)$$

in the same way.

By $\mathcal{S}(\mathbb{R}^d)$ we denote the set of Schwartz functions and by $C_0^\infty(\mathbb{R}^d)$ denote the set of all smooth compactly supported functions on $\mathbb{R}^d$.

Definition 1.9. *We write $u\in C_0(\mathbb{R}^d)$ if u is a continuous function vanishing at infinity. For $k\in\{1,2,\ldots\}$, by $C_0^k(\mathbb{R}^d)$, we mean the subspace of $C_0(\mathbb{R}^d)$ of functions such that $D^j u\in C_0(\mathbb{R}^d), j = 1,\ldots,k$.*

1.3. Main results.

1.3.1. L_p theory for KFP equations with VMO coefficients.

Denote

$$\mathcal{P} = \partial_t - v\cdot D_x - D_{v_i}(a^{ij} D_{v_j}). \tag{1.8}$$

Definition 1.10. *For $T\in(-\infty,\infty]$, we say that $u\in\mathbb{S}_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ is a solution to* (1.1) *if the identity* (1.1) *holds in the space $\mathbb{H}^{-1}_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, that is, both sides of* (1.1) *belong to $\mathbb{H}^{-1}_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ and coincide as distributions. Furthermore, for $-\infty < S < T \le \infty$,*

$$u\in\mathbb{S}_{p,r_1,\ldots,r_d,q}((S,T)\times\mathbb{R}^{2d}, w)$$

is a solution to the Cauchy problem

$$\mathcal{P}u + div_v(\bar{b}u) + b^i D_{v_i} u + cu = div_v \vec{f} + g, \quad u(S,\cdot) = 0 \tag{1.9}$$

if there exists $\widetilde{u}\in\mathbb{S}_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ such that $\widetilde{u} = u$ on $(S,T)\times\mathbb{R}^{2d}$, $\widetilde{u} = 0$ on $(-\infty,S)\times\mathbb{R}^{2d}$, and the equality

$$\mathcal{P}u + div_v(\bar{b}u) + b^i D_{v_i} u + cu = div_v \vec{f} + g$$

holds in $\mathbb{H}^{-1}_{p,r_1,\ldots,r_d,q}((S,T)\times\mathbb{R}^{2d}, w)$. Similarly, we define a solution in the space $\mathbb{S}_{p;r_1,\ldots,r_d}((S,T)\times\mathbb{R}^{2d}, |x|^\alpha\prod_{i=1}^d w_i(v_i))$.

Remark 1.11. Testing the identity (1.1) with a function $\phi \in C_0^\infty(\mathbb{R}_T^{1+2d})$, one obtains the following standard weak formulation of the KFP equation (1.1) (cf. [5], [27]):

$$-\int(\partial_t\phi - v\cdot D_x\phi)u\,dz + \int(a^{ij}D_{v_j}\phi + b_i)D_{v_i}u\,dz$$
$$+\int(-\bar{b}\cdot D_v\phi + c + \lambda)u\,dz = \int(-\vec{f}\cdot D_v\phi + g\phi)\,dz.$$

Definition 1.12 (A_p-weight). *For a number $p>1$, we write $w\in A_p(\mathbb{R}^d)$ if w is a weight on $\mathbb{R}^d$ such that*

$$[w]_{A_p(\mathbb{R}^d)} := \sup_{x_0\in\mathbb{R}^d, r>0}\left(⨍_{B_r(x_0)} w(x)\,dx\right) \times \left(⨍_{B_r(x_0)} w^{-1/(p-1)}(x)\,dx\right)^{p-1} < \infty. \tag{1.10}$$

Remark 1.13. An example of an $A_p(\mathbb{R}^d)$-weight is $w(x)=|x|^\alpha, \alpha\in(-d, d(p-1))$ (see, for instance, [17, Example 7.1.7]).

Definition 1.14. *For $s\in\mathbb{R}$, the fractional Laplacian $(-\Delta_x)^s$ is defined as a Fourier multiplier with the symbol $|\xi|^{2s}$. Furthermore, when $s\in(0,1/2)$, for any Lipschitz function $u\in\cup_{p\in[1,\infty]}L_p(\mathbb{R}^d)$, the following pointwise formula is valid:*

$$(-\Delta_x)^s u(x) = c_{d,s}\int_{\mathbb{R}^d}\frac{u(x)-u(x+y)}{|y|^{d+2s}}\,dy, \tag{1.11}$$

where $c_{d,s}$ is a constant depending only on d and s. When $s\in[1/2,1)$ and u is bounded and $C^{1,1}$, the formula still holds provided that the integral is understood as the principal value. For $s\in(0,1)$ and $u\in L_p(\mathbb{R}^d)$, $(-\Delta_x)^s u$ is understood as a distribution given by

$$((-\Delta_x)^s u, \phi) = (u, (-\Delta_x)^s\phi), \quad \phi\in C_0^\infty(\mathbb{R}^d). \tag{1.12}$$

To prove that (1.12) *defines a distribution, one needs to use the fact that*

$$|(-\Delta_x)^s\phi(z)| \le N(d,\phi)(1+|x|)^{-d-2s}, \quad \phi\in C_0^\infty(\mathbb{R}^d). \tag{1.13}$$

Furthermore, by (1.13), *for any $\alpha\in(-d-2sp, d(p-1))$,*

$$(-\Delta_x)^s\phi\in L_{p/(p-1)}(\mathbb{R}^d, |x|^{-\alpha/(p-1)}),$$

so that (1.12) *defines the distribution $(-\Delta_x)^s u$ for any $u\in L_p(\mathbb{R}^d,|x|^\alpha)$. For a detailed discussion of the fractional Laplacians, we refer the reader to* [37].

Convention. By $N=N(\cdots)$, we denote a constant depending only on the parameters inside the parentheses. A constant N might change from line to line. Sometimes, when it is clear what parameters N depends on, we omit them.

Theorem 1.15. *Let*
- *p, $r_1,\dots,r_d$, $q>1$, $K\ge 1$ be numbers, $T\in(-\infty,\infty]$;*
- *$w_i, i=0,\dots,d$, be weights on $\mathbb{R}$ such that*

$$[w_0]_{A_q(\mathbb{R})}, [w_i]_{A_{r_i}(\mathbb{R})} \le K, \quad i=1,\dots,d; \tag{1.14}$$

– *w be defined by* (1.5)*;*
– *Assumptions* 1.1 *and* 1.4 *hold.*

There exists a constant

$$\gamma_0 = \gamma_0(d, \delta, p, r_1, \dots, r_d, q, K) > 0$$

such that if Assumption 1.2 (γ_0) *holds, then, the following assertions are valid:*

(i) *There exists a constant*

$$\lambda_0 = \lambda_0(d, \delta, p, r_1, \dots, r_d, q, K, L, R_0) > 1$$

such that for any $\lambda \geq \lambda_0$, $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, and $g, \vec{f} \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ satisfying (1.1)*, one has*

$$\lambda^{1/2}\|u\| + \|D_v u\| + \|(-\Delta_x)^{1/6} u\| \leq N\lambda^{-1/2}\|g\| + N\|\vec{f}\|, \tag{1.15}$$

where $R_0 \in (0,1)$ is the constant in Assumption 1.2 (γ_0)*,*

$$\|\cdot\| = \|\cdot\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)}, \quad \text{and} \quad N = N(d, \delta, p, r_1, \dots, r_d, q, K).$$

(ii) *For any $\lambda \geq \lambda_0$, $\vec{f}, g \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$,* (1.1) *has a solution $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ (see Definition* 1.10*), and the uniqueness holds in the class of $\mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$-solutions (see also Remark* 1.18*). Here λ_0 is the constant from the assertion* (i)*.*

(iii) *For any numbers $-\infty < S < T < \infty$ and $\vec{f}, g \in L_{p,r_1,\dots,r_d,q}((S,T) \times \mathbb{R}^{2d}, w)$,* (1.9) *has a unique solution $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}((S,T) \times \mathbb{R}^{2d}, w)$. In addition,*

$$\|u\| + \|D_v u\| + \|(-\Delta_x)^{1/6} u\| \leq N\|\vec{f}\| + N\|g\|,$$

where

$$\|\cdot\| = \|\cdot\|_{L_{p,r_1,\dots,r_d,q}((S,T)\times\mathbb{R}^{2d}, w)} \text{ and } N = N(d, \delta, p, r_1, \dots, r_d, q, K, L, R_0, T - S).$$

(iv) *Let $\alpha \in (-1, p-1)$. The assertions* (i)$-$(iii) *also hold in the case when*

$$\vec{f}, g \in L_{p;r_1,\dots,r_d}\left(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right),$$
$$u \in \mathbb{S}_{p;r_1,\dots,r_d}\left(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right).$$

Furthermore, one needs to take into account the dependence of constants γ_0, λ_0, N on α and remove the dependence on q.

Remark 1.16. The assertion (iii) is derived from (ii) by using an exponential multiplier (see, for example, [24, Theorem 2.5.3]).

Remark 1.17. By viewing an elliptic equation as a steady state parabolic equation, we can obtain the corresponding results for elliptic equations when the coefficients and data are independent of the temporal variable. See, for example, the proof of [23, Theorem 2.6].

Remark 1.18. It is an interesting problem to investigate Liouville-type results for KFP equations in divergence form. See, for example, [13], [21], and the survey papers [4], [22], which contain references to other relevant articles.

Remark 1.19. It would be interesting to see if the method of the present paper and [15] can be used to extend the weighted mixed-norm estimates (1.15) and (2.8) in [15] to more general degenerate Kolmogorov equations studied in the articles [4], [5], [6], [7], [9], [11], [26], and others. We point out that in [11], the L_p estimate similar to (2.8) of [15] was established for the degenerate Kolmogorov equation with the leading coefficients depending only on the temporal variable. We also mention the papers [6] and [26], where the properties of the fundamental solution and the Schauder estimates are studied for the degenerate Kolmogorov equation with variable coefficients a^{ij} that are merely measurable in t.

To the best of our knowledge, Theorem 1.15 provides the first global a priori L_p estimate with $p \neq 2$ for kinetic KFP equations in divergence form with nonsmooth coefficients (see section 1.6). We also prove the first unique solvability result in $\mathbb{S}_p$ space for (1.1) in the case of the variable coefficients a^{ij}. To the best of our knowledge, the imposed assumption on the leading coefficients a^{ij} (see Assumption 1.2) is weaker than assumptions in the existing literature (see section 1.6).

To prove Theorem 1.15, we use the results and techniques of [15], which are based on Krylov's kernel-free approach to nondegenerate parabolic equations (see [24, Chapters 4–7]). The main part of the argument is the mean oscillation estimates of $(-\Delta_x)^{1/6}u$, $\lambda^{1/2}u$, and $D_v u$ in the case when the coefficients a^{ij} are independent of the x and v variables. Our proof of these inequalities does not involve the fundamental solution of the KFP operator. Instead, we use the scaling properties of the KFP equation combined with localized L_p estimates and a pointwise formula for fractional Laplacians in order to get mean oscillation estimates of solutions. By using the method of frozen coefficients, we generalize the aforementioned mean oscillation estimates to the case when a^{ij} also depend on x and v. Once such inequalities are established, the a priori estimates are obtained by using the variants Hardy–Littlewood and Fefferman–Stein theorems (see Theorem A.3).

1.4. L_2 theory for the kinetic equations with bounded measurable coefficients. Let $x \in \mathbb{R}^d$ and $\mathsf{v} \in \mathbb{R}^{d_1}$ for some $d_1 = \{1, 2, \dots\}$, let α be a mapping from $\mathbb{R}^{d_1}$ to $\mathbb{R}^d$, and let $a, b, \bar{b}, c, \vec{f}, g$ be functions of t, x, v.

We consider the equation

$$\mathcal{P}_\alpha u + \operatorname{div}_{\mathsf{v}}(\bar{b}u) + b \cdot D_{\mathsf{v}} u + (c + \lambda)u = \operatorname{div}_{\mathsf{v}} \vec{f} + g, \tag{1.16}$$

where

$$\mathcal{P}_\alpha u = \partial_t u + \alpha(\mathsf{v}) \cdot D_x u - D_{\mathsf{v}_i}(a^{ij} D_{\mathsf{v}_j} u).$$

Assumption 1.20. The function α is such that for some $\theta \in (0, 1]$,

$$\sup_{\mathsf{v} \neq \mathsf{v}'} \frac{|\alpha(\mathsf{v}) - \alpha(\mathsf{v}')|}{|\mathsf{v} - \mathsf{v}'|^\theta} < \infty.$$

Remark 1.21. Here we give examples of the equations of type (1.16) that appear in the existing literature.

Kinetic equations: $d_1 = d$ and $\alpha(\mathsf{v}) = \pm \mathsf{v}$ or $\alpha = \frac{\pm \mathsf{v}}{(1+|\mathsf{v}|^2)^{1/2}}$. In the second case, (1.16) with such α can be viewed as a relativistic counterpart of (1.1).

The Mumford equation. Another example comes from computer vision. In [30], Mumford considered the operator

$$\partial_t u + \cos(\mathsf{v}) D_{x_1} u + \sin(\mathsf{v}) D_{x_2} u - D_{\mathsf{v}}^2 u, \; t, \mathsf{v}, x_1, x_2 \in \mathbb{R},$$

which is an operator of the KFP type. For the discussion of certain PDE aspects of this operator, see [21].

DEFINITION 1.22. *For* $\alpha : \mathbb{R}^{d_1} \to \mathbb{R}^d$ *and an open set* $G \subset \mathbb{R}^{1+d+d_1}$*, we say that* $u \in \mathfrak{S}_2(G, \alpha)$ *if* $u, D_{\mathsf{v}} u \in L_2(G)$*, and* $\partial_t u + \alpha(\mathsf{v}) \cdot D_x u \in \mathbb{H}_2^{-1}(G)$*.*

Here is the $\mathfrak{S}_2(\mathbb{R}^{1+d+d_1}_T, \alpha)$ unique solvability result for (1.16).

THEOREM 1.23. *Let* $T \in (-\infty, \infty]$*, let* a*,* b*,* $\bar b$ *be functions satisfying Assumptions* 1.1 *and* 1.4 *(with* $\mathbb{R}^{1+2d}$ *replaced with* $\mathbb{R}^{1+d+d_1}$*), and let* $\alpha : \mathbb{R}^{d_1} \to \mathbb{R}^d$ *satisfy Assumption* 1.20*. Then,*

(i) *there exists* $\lambda_0 = \lambda_0(d, d_1, \delta, L) > 0$ *such that for any* $\lambda \ge \lambda_0$*, and functions* $u \in \mathfrak{S}_2(\mathbb{R}^{1+d+d_1}_T, \alpha)$*,* $\vec f, g \in L_2(\mathbb{R}^{1+d+d_1}_T)$ *satisfying* (1.16)*, we have*

$$\lambda^{1/2} \|u\| + \|D_{\mathsf{v}} u\| \le N \|\vec f\| + N \lambda^{-1/2} \|g\|, \tag{1.17}$$

where

$$N = N(d, d_1, \delta), \quad \| \cdot \| = \| \cdot \|_{L_2(\mathbb{R}^{1+d+d_1}_T)}.$$

In addition, for any $\vec f, g \in L_2(\mathbb{R}^{1+d+d_1}_T)$ *and* $\lambda \ge \lambda_0$*,* (1.16) *has a unique solution* $u \in \mathfrak{S}_2(\mathbb{R}^{1+d+d_1}_T, \alpha)$*.*

(ii) *For any numbers* $S < T$ *and* $\vec f, g \in L_2((S,T) \times \mathbb{R}^{d+d_1})$*, the Cauchy problem*

$$\mathcal{P}_\alpha u + div_{\mathsf{v}}(\bar b u) + b \cdot D_{\mathsf{v}} u + cu = div_{\mathsf{v}} \vec f + g, \quad u(S, \cdot) = 0,$$

has a unique solution $u \in \mathfrak{S}_2((S,T) \times \mathbb{R}^{d+d_1}, \alpha)$*; furthermore,*

$$\|u\| + \|D_v u\| \le N \|\vec f\| + N \|g\|,$$

where

$$N = N(d, d_1, \delta, L, T - S), \quad \| \cdot \| = \| \cdot \|_{L_2((S,T) \times \mathbb{R}^{d+d_1})}.$$

COROLLARY 1.24. *In the case when* $d_1 = d$ *and* $\alpha = -\mathsf{v}$*, which corresponds to the kinetic KFP equation, in addition to* (1.17)*, we have*

$$\|(-\Delta_x)^{1/6} u\|_{L_2(\mathbb{R}^{1+2d}_T)} \le N \|\vec f\|_{L_2(\mathbb{R}^{1+2d}_T)} + N \lambda^{-1/2} \|g\|_{L_2(\mathbb{R}^{1+2d}_T)},$$

where $N = N(d, \delta)$*.*

Proof. Note that the identity

$$\begin{aligned} &\partial_t u - \mathsf{v} \cdot D_x u - \Delta_{\mathsf{v}} u + \mathrm{div}_{\mathsf{v}}(\bar b u) + b \cdot D_{\mathsf{v}} u + (c + \lambda) u \\ &\quad = D_{\mathsf{v}_i}((a^{ij} - \delta_{ij}) D_{\mathsf{v}_j} u) + \mathrm{div}_{\mathsf{v}} \vec f + g \end{aligned}$$

is true. Here Δ_{v} is the Laplacian in the v variable. Then, by Theorem 1.15,

$$\begin{aligned} &\|(-\Delta_x)^{1/6} u\|_{L_2(\mathbb{R}^{1+2d}_T)} \\ &\quad \le N \|(a^{ij} - \delta_{ij}) D_{\mathsf{v}_j} u\|_{L_2(\mathbb{R}^{1+2d}_T)} + N \|\vec f\|_{L_2(\mathbb{R}^{1+2d}_T)} + N \lambda^{-1/2} \|g\|_{L_2(\mathbb{R}^{1+2d}_T)}. \end{aligned}$$

This, combined with Assumption 1.1 and (1.17), gives the desired estimate. ∎

The results of Theorem 1.23 and Corollary 1.24 are not surprising; however, the present authors have not seen such assertions in their full generality in the existing literature.

1.5. Motivations.

Filtering. Stochastic partial differential equation (SPDEs) in divergence form appear naturally in the theory of partially observable diffusion processes. In particular, the unnormalized conditional probability density of the unobservable component of the diffusion process with respect to the observable one satisfies a linear SPDE called the Dunkan–Mortensen–Zakai equation (see, for example, [35]). In [25], Krylov showed that the L_p theory of SPDEs can be used to deduce certain regularity properties of the unnormalized density. For the Langevin-type diffusion processes, such a program was carried out in [38] (see also [32]). In particular, the authors developed the Besov regularity theory for the equation

$$du = [v \cdot D_x u - a^{ij} D_{v_i v_j} u + b \cdot D_v u + f]\, dt + [\sigma^k \cdot D_v u + g^k]\, dw^k,$$

where $w_k, k \geq 1$, is a sequence of independent standard Wiener processes. In the same paper [38], they used that regularity theory to show that the unnormalized conditional probability density is a continuous function. We believe that Theorem 1.15 is useful in developing the theory of stochastically forced KFP equations in *divergence form* in the case when the forcing term g belongs to the L_p space with respect to the probability measure and t, x, v.

Kinetic theory. The nonlinear Landau equation is an important model of collisional plasma which has been studied extensively (see, for example, [3], [29], [16], [20]). A linearized version of this equation has the form of (1.1). Recently, there has been an interest in developing the L_p theory of KFP equations with rough coefficients. Such results are useful for establishing the well-posedness of diffusive kinetic equations in bounded domains with the specular reflection boundary condition (see [14]) and for the conditional regularity problem (see [16]).

1.6. Related works.

Divergence form equations. Many articles on KFP equations in divergence form are concerned with the local boundedness, Harnack inequality (including a nonhomogeneous version), and Hölder continuity of solutions to (1.1) (see [5], [33], [39], [16], [29], [18], [19], [40]). See also the references in [4].

It seems that there are very few works on the Sobolev space theory of (1.1). Previously, an interior L_p estimate of $D_v u$ for (1.1) was established in [27] under the assumption that $u, \vec{f} \in L_{p,\text{loc}}$, $u, D_v u, (\partial_t - v \cdot D_x) u \in L_{2,\text{loc}}$, and $g \equiv 0$ by using the explicit representation of the fundamental solution of the operator $\mathcal{P}$ (see (1.8)) and singular-integral techniques. In addition, in the same work, it was showed that if p is large enough, then u is locally Hölder continuous with respect to ρ (see (1.3)). The authors of [27] imposed the VMO condition with respect to ρ on the coefficients a^{ij}, which is stronger than Assumption 1.2. It can be seen from (1.3) that such an assumption might not be satisfied even when the coefficients $a^{ij} = a^{ij}(x, v)$ are smooth, bounded, and independent of t. A similar result in the ultraparabolic Morrey spaces was proved in [34]. We point out that the papers [27] and [34] are concerned with the operators that are more general than $\mathcal{P}$. We also mention a recent paper [2] which studies the L_2-regularity theory, the trend toward equilibrium, and enhanced dissipation for the KFP equation in divergence form with $a^{ij} = \delta_{ij}$.

Nondivergence form equations. For a thorough review of the classical theory for the generalized KFP equations, we refer the reader to [4]. An overview of the literature on the Sobolev theory for KFP equations in nondivergence form can be found in the recent paper [15]. We also mention briefly the following papers:

- The article [7] on the interior S_p estimate with leading coefficients of class VMO with respect to ρ;

- the articles [8], [11], [9], [31] where the global S_p estimate is proved under the following assumptions: either the leading coefficients are constant or independent of x, v, or they are continuous with respect to ρ;
- the article [15], where the present authors proved the global S_p estimate and unique solvability results with the coefficients satisfying Assumption 1.2.

Ultra-analyticity and Gevrey regularity. Finally, we would like to mention the works [12], [10], [28] on the ultra-analyticity and Gevrey regularity for the KFP and Landau equations.

1.7. Organization of the paper. In section 2 , we prove the main result in the $\mathbb{S}_p$ space in the case when the coefficients a^{ij} are independent of x and v. We then extend the a priori estimate to the weighted mixed-norm kinetic spaces in section 3 so that the reader interested only in the constant coefficient case need only read the sections 1, 2, and 3. We prove the main results for the equations with the variable coefficients $a^{ij} = a^{ij}(z)$ in section 4.

2. $\mathbb{S}_p$ estimate for the model equation.

Denote

$$P_0 = \partial_t - v \cdot D_x - a^{ij}(t) D_{v_i v_j},$$

where the coefficients a^{ij} satisfy Assumption 1.1.

The goal of this section is to prove Theorem 1.15 with L_p in place of the weighted mixed-norm Lebesgue space, $\mathcal{P} = P_0$ (see (1.8)), and without the lower-order terms (see Theorem 2.1). We do this by using the duality argument and the S_p estimate taken from [15], which we state below (see Theorem 2.3).

THEOREM 2.1. *Let $p > 1$ be a number, and let $T \in (-\infty, \infty]$. Then, the following assertions hold.*

(i) *For any number $\lambda > 0$, $u \in \mathbb{S}_p(\mathbb{R}^{1+2d}_T)$, and $\vec{f}, g \in L_p(\mathbb{R}^{1+2d}_T)$, the equation*

$$P_0 u + \lambda u = div_v \vec{f} + g \tag{2.1}$$

has a unique solution $u \in \mathbb{S}_p(\mathbb{R}^{1+2d}_T)$, and, in addition,

$$\begin{aligned} &\lambda^{1/2} \|u\|_{L_p(\mathbb{R}^{1+2d}_T)} + \|D_v u\|_{L_p(\mathbb{R}^{1+2d}_T)} + \|(-\Delta_x)^{1/6} u\|_{L_p(\mathbb{R}^{1+2d}_T)} \\ &\quad \le N(d, \delta, p)(\|\vec{f}\|_{L_p(\mathbb{R}^{1+2d}_T)} + \lambda^{-1/2} \|g\|_{L_p(\mathbb{R}^{1+2d}_T)}). \end{aligned} \tag{2.2}$$

(ii) *For any finite numbers $\lambda \ge 0$, $S < T$, and $\vec{f}, g \in L_p((S,T) \times \mathbb{R}^{2d})$, the Cauchy problem*

$$P_0 u + \lambda u = div_v \vec{f} + g, \quad u(S, \cdot) \equiv 0,$$

has a unique solution $u \in \mathbb{S}_p((S,T) \times \mathbb{R}^{2d})$ (see Definition 1.10); furthermore,

$$\begin{aligned} &(1 + \lambda^{1/2}) \|u\|_{L_p((S,T)\times\mathbb{R}^{2d})} + \|D_v u\|_{L_p((S,T)\times\mathbb{R}^{2d})} + \|(-\Delta_x)^{1/6} u\|_{L_p((S,T)\times\mathbb{R}^{2d})} \\ &\quad \le N(\|\vec{f}\|_{L_p((S,T)\times\mathbb{R}^{2d})} + \lambda^{-1/2} \|g\|_{L_p((S,T)\times\mathbb{R}^{2d})}), \end{aligned}$$

where $N = N(d, \delta, p, T - S)$.

COROLLARY 2.2. *For any $u \in \mathbb{S}_p(\mathbb{R}^{1+2d}_T)$, one has $(-\Delta_x)^{1/6} u \in L_p(\mathbb{R}^{1+2d}_T)$, and, in addition,*

$$\|(-\Delta_x)^{1/6} u\|_{L_p(\mathbb{R}^{1+2d}_T)} \le N \|u\|_{\mathbb{S}_p(\mathbb{R}^{1+2d}_T)},$$

where $N = N(d, p) > 0$.

Proof. To prove the result, we set $a^{ij} = \delta_{ij}$, find $\vec{f}$ and g in $L_p(\mathbb{R}^{1+2d}_T)$ such that

$$\partial_t u - v \cdot D_x u - \Delta_v u + u = \operatorname{div}_v \vec{f} + g$$

in $\mathbb{H}^{-1}_p(\mathbb{R}^{1+2d}_T)$, and apply Theorem 2.1. ∎

Here is the main result of [15] in the case when $\mathcal{P} = P_0$, which will also be used in the next section.

THEOREM 2.3. *Let $p > 1$ be a number. Then, the following assertions hold.*

(i) *For any number $\lambda \geq 0$, $T \in (-\infty, \infty]$, and $u \in S_p(\mathbb{R}^{1+2d}_T)$, one has*

$$\lambda \|u\| + \lambda^{1/2} \|D_v u\| + \|D^2_v u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v (-\Delta_x)^{1/6} u\| \leq N(d, p, \delta) \|P_0 u + \lambda u\|,$$

where $\|\cdot\| = \|\cdot\|_{L_p(\mathbb{R}^{1+2d}_T)}$.

(ii) *For any $\lambda > 0$, $T \in (-\infty, \infty]$, and $f \in L_p(\mathbb{R}^{1+2d}_T)$, the equation*

$$P_0 u + \lambda u = f$$

has a unique solution $u \in S_p(\mathbb{R}^{1+2d}_T)$.

(iii) *For any finite numbers $S < T$ and $f \in L_p((S,T) \times \mathbb{R}^{2d})$, the Cauchy problem*

$$P_0 u = f, \quad u(S, \cdot) \equiv 0,$$

has a unique solution $u \in S_p((S,T) \times \mathbb{R}^{2d})$. In addition,

$$\|u\| + \|D_v u\| + \|D^2_v u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v (-\Delta_x)^{1/6} u\| \leq N \|f\|,$$

where

$$\|\cdot\| = \|\cdot\|_{L_p((S,T) \times \mathbb{R}^{2d})}, \quad N = N(d, \delta, p, T - S).$$

Remark 2.4. The above theorem follows from [15, Theorem 2.6] and the scaling property of the operator P_0 (see Lemma 3.10).

The following lemma implies the uniqueness part of Theorem 2.1 (ii).

LEMMA 2.5. *Let $p > 1$, $\lambda > 0$ be numbers, let $T \in (-\infty, \infty]$, and let $u \in \mathbb{S}_p(\mathbb{R}^{1+2d}_T)$ satisfy $P_0 u + \lambda u = 0$. Then, $u \equiv 0$.*

Proof. Let $\eta = \eta(x, v) \in C^\infty_0(B_1 \times B_1)$ be a function with the unit integral. For $h \in L_{1,\mathrm{loc}}(\mathbb{R}^{2d})$, we denote

$$h_{(\varepsilon)}(x, v) = \varepsilon^{-(3/2)d} \int h(x', v') \eta((x - x')/\varepsilon^{1/2}, (v - v')/\varepsilon) \, dx' dv'.$$

Then, u satisfies the equation

$$P_0 u_{(\varepsilon)} + \lambda u_{(\varepsilon)} = g_\varepsilon, \tag{2.3}$$

where

$$g_\varepsilon(z) = \varepsilon^{1/2} \int u(t, x - \varepsilon^{1/2} x', v - \varepsilon v') v' \cdot D_x \eta(x', v') \, dx' dv'.$$

Note that by the Minkowski inequality,

$$\|g_\varepsilon\|_{L_p(\mathbb{R}^{1+2d}_T)} \le N\varepsilon^{1/2}\|u\|_{L_p(\mathbb{R}^{1+2d}_T)}. \tag{2.4}$$

Then, it follows from (2.3) that $(\partial_t - v\cdot D_x)u_{(\varepsilon)} \in L_p(\mathbb{R}^{1+2d}_T)$, and therefore, $u_{(\varepsilon)} \in S_p(\mathbb{R}^{1+2d}_T)$. Hence, by Theorem 2.3 and (2.4),

$$\lambda\|u_{(\varepsilon)}\|_{L_p(\mathbb{R}^{1+2d}_T)} \le N\|g_\varepsilon\|_{L_p(\mathbb{R}^{1+2d}_T)} \le N\varepsilon^{1/2}\|u\|_{L_p(\mathbb{R}^{1+2d}_T)}.$$

Taking the limit as $\varepsilon\to 0$ in the above inequality, we prove the assertion. ∎

The following result is needed for the duality argument in the proof of Theorem 2.1. For the proof, see Lemma 5.12 of [15].

Lemma 2.6. *For any numbers $\lambda \ge 0$ and $p > 1$, the set $(P_0+\lambda)C_0^\infty(\mathbb{R}^{1+2d})$ is dense in $L_p(\mathbb{R}^{1+2d})$.*

Here is the a priori estimate (2.2) in the case when $g\equiv 0$ and $\vec f$ is smooth and compactly supported.

Lemma 2.7. *Let $\lambda > 0, p > 1$ be numbers, and let $\vec f \in C_0^\infty(\mathbb{R}^{1+2d})$. Let u be the unique solution in $S_p(\mathbb{R}^{1+2d})$ to (2.1) with $g\equiv 0$. Then, one has*

$$\begin{aligned}\lambda^{1/2}\|u\|_{L_p(\mathbb{R}^{1+2d})} &+ \|(-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}^{1+2d})}\\ &+ \|D_v u\|_{L_p(\mathbb{R}^{1+2d})} \le N(d,\delta,p)\|\vec f\|_{L_p(\mathbb{R}^{1+2d})}.\end{aligned}$$

Proof. The proof is by a duality argument. We denote $q = p/(p-1)$ and fix some $U\in C_0^\infty(\mathbb{R}^{1+2d})$.

Estimate of $(-\Delta_x)^{1/6}u$. Note that for any multi-index α, one has $P_0 D_x^\alpha u = \operatorname{div}_v D_x^\alpha \vec f$, and hence by Theorem 2.3, $D_x^\alpha u \in L_p(\mathbb{R}^{1+2d})$. In addition, note that by (1.13) for any $U\in C_0^\infty(\mathbb{R}^{1+2d})$, $(-\Delta_x)^{1/6}U \in C^\infty_{\text{loc}}(\mathbb{R}^{1+2d})\cap L_1(\mathbb{R}^{1+2d})$. Then, by this and integration by parts, we have

$$\begin{aligned} I &= \int \big((-\Delta_x)^{1/6}u\big)(-\partial_t U + v\cdot D_x U - a^{ij}(t)D_{v_iv_j}U + \lambda U)\,dz\\ &= \int \big((\partial_t - v\cdot D_x - a^{ij}(t)D_{v_iv_j} + \lambda)u\big)\big((-\Delta_x)^{1/6}U\big)\,dz\\ &= \int (\operatorname{div}_v \vec f)((-\Delta_x)^{1/6}U)\,dz = -\int \vec f\cdot D_v(-\Delta_x)^{1/6}U\,dz.\end{aligned}$$

By Hölder's inequality, Theorem 2.3, and the change of variables $t\to -t$, $x\to -x$,

$$|I| \le N\|\vec f\|_{L_p(\mathbb{R}^{1+2d})}\|-\partial_t U + v\cdot D_x U - a^{ij}D_{v_iv_j}U + \lambda U\|_{L_q(\mathbb{R}^{1+2d})}. \tag{2.5}$$

Furthermore, by Lemma 2.6 and the aforementioned change of variables, $(-\partial_t + v\cdot D_x - a^{ij}D_{v_iv_j} + \lambda)C_0^\infty(\mathbb{R}^{1+2d})$ is dense in $L_q(\mathbb{R}^{1+2d})$. This, combined with (2.5), implies the desired estimate for $(-\Delta_x)^{1/6}u$.

Estimate of $D_v u$. Integrating by parts gives

$$\begin{aligned}&\int (D_v u)(-\partial_t U + v\cdot D_x U - a^{ij}(t)D_{v_iv_j}U + \lambda U)\,dz\\ &\quad = \int (D_v^2 U)\vec f\,dz - \int u D_x U\,dz =: J_1 + J_2.\end{aligned}$$

As before, it suffices to show that $|J_1|+|J_2|$ is dominated by the right-hand side of (2.5). By Hölder's inequality, Theorem 2.3, and the same change of variables, we get

$$\begin{aligned}|J_1| &\le \|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}\|D_v^2 U\|_{L_q(\mathbb{R}^{1+2d})}\\ &\le N\|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}\|-\partial_t U + v\cdot D_x U - a^{ij}(t) D_{v_i v_j} U + \lambda U\|_{L_q(\mathbb{R}^{1+2d})}.\end{aligned}$$

Next, note that

$$J_2 = -\int ((-\Delta_x)^{1/6} u)\,\mathcal{R}_x(-\Delta_x)^{1/3} U\,dz,$$

where $\mathcal{R}_x$ is the Riesz transform in the x variable. Then, by the L_p estimate of $(-\Delta_x)^{1/6}u$, the L_q boundedness of the Riesz transform, and Theorem 2.3, we obtain

$$\begin{aligned}|J_2| &\le N\|(-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}^{1+2d})}\|(-\Delta_x)^{1/3}U\|_{L_q(\mathbb{R}^{1+2d})}\\ &\le N\|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}\|-\partial_t U + v\cdot D_x U - a^{ij} D_{v_i v_j} U + \lambda U\|_{L_q(\mathbb{R}^{1+2d})}.\end{aligned}$$

The estimate is proved.

Estimate of u. As above, we consider

$$\mathcal{I} := \int u(-\partial_t U + v\cdot D_x U - a^{ij} D_{v_i v_j} U + \lambda U)\,dz = -\int \vec{f}\cdot D_v U\,dz.$$

Then, by Hölder's inequality and Theorem 2.3,

$$\begin{aligned}|\mathcal{I}| &\le \|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}\|D_v U\|_{L_q(\mathbb{R}^{1+2d})}\\ &\le N\lambda^{-1/2}\|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}\|-\partial_t U + v\cdot D_x U - a^{ij} D_{v_i v_j} U + \lambda U\|_{L_q(\mathbb{R}^{1+2d})}.\end{aligned}$$

This implies the desired estimate. ∎

Proof of Theorem 2.1. By Remark 1.16, we only need to prove assertion (i).

(i) The uniqueness follows from Lemma 2.5. To prove the existence, let $u_1 \in S_p(\mathbb{R}^{1+2d}_T)$ be the unique solution to the equation (see Theorem 2.3)

$$P_0 u_1 + \lambda u_1 = g. \tag{2.6}$$

By the same theorem and the interpolation inequality (see Lemma A.5),

$$\begin{aligned}&\lambda\|u_1\|_{L_p(\mathbb{R}^{1+2d}_T)} + \lambda^{1/2}\|D_v u_1\|_{L_p(\mathbb{R}^{1+2d}_T)} + \lambda^{1/2}\|(-\Delta_x)^{1/6}u_1\|_{L_p(\mathbb{R}^{1+2d}_T)}\\ &\quad\le N\|g\|_{L_p(\mathbb{R}^{1+2d}_T)}.\end{aligned}$$

Subtracting (2.6) from (2.1), we may assume that $g \equiv 0$. We will consider the cases $T = \infty$ and $T < \infty$ separately.

Case $T = \infty$. We take a sequence of functions $\vec{f_n} \in C_0^\infty(\mathbb{R}^{1+2d})$ such that $f_n \to f$ in $L_p(\mathbb{R}^{1+2d})$. By Theorem 2.3, there exists a unique solution $u_n \in S_p(\mathbb{R}^{1+2d})$ to the equation

$$(P_0 + \lambda)u_n = \operatorname{div}_v \vec{f_n}. \tag{2.7}$$

By Lemma 2.7, we have

$$\begin{aligned}&\lambda^{1/2}\|u_n\|_{L_p(\mathbb{R}^{1+2d})} + \|D_v u_n\|_{L_p(\mathbb{R}^{1+2d})} + \|(-\Delta_x)^{1/6}u_n\|_{L_p(\mathbb{R}^{1+2d})}\\ &\quad\le N(d,\delta,p)\|\vec{f_n}\|_{L_p(\mathbb{R}^{1+2d})},\\ &\quad\quad\|\partial_t u_n - v\cdot D_x u_n\|_{\mathbb{H}_p^{-1}(\mathbb{R}^{1+2d})} \le N(d,\delta,p,\lambda)\|\vec{f_n}\|_{L_p(\mathbb{R}^{1+2d})}.\end{aligned} \tag{2.8}$$

Furthermore, by the same lemma,

$$u_n, n \geq 1, \quad (-\Delta_x)^{1/6} u_n,\, n \geq 1,$$

are Cauchy sequences in $\mathbb{S}_p(\mathbb{R}^{1+2d})$ and $L_p(\mathbb{R}^{1+2d})$, respectively. Hence, there exists a function $u \in \mathbb{S}_p(\mathbb{R}^{1+2d})$ such that u_n, $(-\Delta_x)^{1/6} u_n$ converge to u and $(-\Delta_x)^{1/6} u$, respectively. Passing to the limit in (2.7) and (2.8), we prove the existence and the inequality (2.2).

Case $T < \infty$. Let $\widetilde{u} \in \mathbb{S}_p(\mathbb{R}^{1+2d})$ be the unique solution to the equation

$$P_0 \widetilde{u} + \lambda \widetilde{u} = \mathrm{div}_v \vec{f} 1_{t<T}.$$

We conclude that $u := \widetilde{u}$ is a solution of class $\mathbb{S}_p(\mathbb{R}^{1+2d}_T)$ to (2.1), and the estimate (2.2) holds. The theorem is proved. ∎

3. Mixed-norm estimate for the model equation.

In this section, we consider the case when the coefficients a^{ij} are independent of x, v, and the lower-order terms are absent. The goal is to prove the a priori estimates in the weighted mixed-norm spaces by establishing a mean oscillation estimate of $(-\Delta_x)^{1/6} u$, $\lambda^{1/2} u$, and $D_v u$ for $u \in \mathbb{S}_p(\mathbb{R}^{1+2d}_T)$ solving (2.1). To this end, we split u into a $P_0 + \lambda$-caloric part and the remainder. To bound the former, we use the method of section 5 of [15]. The remainder is handled by using a localized version of the $\mathbb{S}_p$ estimate in Theorem 2.1 (see Lemma 3.3).

Theorem 3.1. *Invoke the assumptions of Theorem* 1.15 *and assume, additionally,* $b \equiv 0 \equiv \bar{b}$, $c \equiv 0$. *Let* $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, $\vec{f}, g \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ *be functions such that*

$$P_0 u + \lambda u = div_v \vec{f} + g.$$

Then, for any $\lambda > 0$, *the estimate* (1.15) *is valid. Furthermore, in the case when* $g \equiv 0$ *and* $\lambda \equiv 0$, (1.15) *also holds. In addition, the same inequalities hold with* $\mathbb{S}_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i))$, $L_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i))$ *in place of* $\mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, $L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, *respectively, and with* $N = N(d, \delta, p, r_1, \dots, r_d, \alpha, K)$.

The next result is derived from the above theorem in the same way as Corollary 2.2 from Theorem 2.1.

Corollary 3.2 (cf. Corollary 2.2). *For any* $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, *one has* $(-\Delta_x)^{1/6} u \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, *and, in addition,*

$$\|(-\Delta_x)^{1/6} u\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)} \leq N \|u\|_{\mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)},$$

where $N = N(d, p, r_1, \dots, r_d, q, K) > 0$. *A similar assertion holds for* $u \in \mathbb{S}_{p;r_1,\dots,r_d}$ $(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i))$.

In the next lemma, we establish the estimate of the aforementioned "remainder term."

Lemma 3.3. *Let* $\lambda \geq 0$ *be a number, and let* $\vec{f} \in L_p(\mathbb{R}^{1+2d}_0)$ *be a function vanishing outside* $(-1, 0) \times \mathbb{R}^d \times B_1$. *Let* $u \in \mathbb{S}_p((-1, 0) \times \mathbb{R}^{2d})$ *be the unique solution to the equation (see Theorem* 2.1 (ii)*)*

$$P_0 u + \lambda u = div_v \vec{f} + g, \quad u(-1, \cdot) = 0. \tag{3.1}$$

Then, for any $R \geq 1$, one has

(3.2)
$$\|(1+\lambda^{1/2})|u| + |D_v u|\|_{L_p((-1,0)\times B_{R^3}\times B_R)} \leq N(d,\delta,p)\sum_{k=0}^{\infty} 2^{-k(k-1)/4} R^{-k} \||\vec{f}| + \lambda^{-1/2}|g|\|_{L_p(Q_{1,2^{k+1}R})},$$

(3.3)
$$\left(|(-\Delta_x)^{1/6}u|^p\right)^{1/p}_{Q_{1,R}} \leq N(d,\delta,p)\sum_{k=0}^{\infty} 2^{-k}\left((|\vec{f}|^p)^{1/p}_{Q_{1,2^{k+1}R}} + \lambda^{-1/2}(|g|^p)^{1/p}_{Q_{1,2^{k+1}R}}\right).$$

Proof. We follow the proof of Lemma 5.2 of [15] very closely. By considering the equation satisfied by $U := ue^{-t}$, without loss of generality, we may assume that $\lambda \geq 1$.

Estimate of $u, D_v u$. We denote

$$\vec{f}_0 := \vec{f}1_{x\in B_{(2R)^3}},\ \vec{f}_k := \vec{f}1_{x\in B_{(2^{k+1}R)^3}\setminus B_{(2^k R)^3}},\ k\in\{1,2,\ldots\},\ \text{so that } \vec{f} = \sum_{k=0}^{\infty}\vec{f}_k,$$

and we define $g_k, k \geq 0$, in a similar way. By Theorem 2.1 (ii), there exists a unique solution $u_k \in \mathbb{S}_p((-1,0)\times\mathbb{R}^{2d})$ to (3.1) with $\vec{f}_k$ and g_k in place of $\vec{f}$ and g, respectively, and, in addition, one has

$$\|\lambda^{1/2}|u_k| + |D_v u_k|\|_{L_p((-1,0)\times\mathbb{R}^{2d})} \leq N\||\vec{f}_k| + \lambda^{-1/2}|g_k|\|_{L_p((-1,0)\times\mathbb{R}^{2d})}. \tag{3.4}$$

In addition, by Theorem 2.1 (ii),

$$u = \lim_{n\to\infty}\sum_{k=0}^{n} u_k \quad \text{in } L_p((-1,0)\times\mathbb{R}^{2d}),$$

and a similar identity holds for $D_v u$.

Next, let $\zeta_j = \zeta_j(x,v) \in C_0^\infty(B_{(2^{j+1}R)^3}\times B_{2^{j+1}R})$, $j = 0,1,2,\ldots$, be a sequence of functions such that $\zeta_j = 1$ on $B_{(2^{j+1/2}R)^3}\times B_{2^{j+1/2}R}$ and

$$|\zeta_j| \leq 1, \quad |D_v\zeta_j| \leq N2^{-j}R^{-1},$$
$$|D_v^2\zeta_j| \leq N2^{-2j}R^{-2}, \quad |D_x\zeta_j| \leq N2^{-3j}R^{-3}.$$

For $k \geq 1$ and $j = 0,1,\ldots,k-1$, we set $u_{k,j} = u_k\zeta_j$, which satisfies

$$P_0 u_{k,j} + \lambda u_{k,j} = u_k P_0\zeta_j + \operatorname{div}_v(\vec{f}_k\zeta_j) - \vec{f}_k\cdot D_v\zeta_j + g_k\zeta_j - 2(aD_v\zeta_j)\cdot D_v u_k. \tag{3.5}$$

Observe that for such j, $\vec{f}_k\zeta_j \equiv 0$, $\vec{f}_k\cdot D_v\zeta_j \equiv 0$, and $g_k\zeta_j = 0$. Then, by Theorem 2.1 (ii) and the fact that $\lambda \geq 1$,

$$\begin{aligned}
&\|\lambda^{1/2}|u_k| + |D_v u_k|\|_{L_p((-1,0)\times B_{(2^j R)^3}\times B_{2^j R})} \\
&\quad \leq N\lambda^{-1/2}2^{-j}R^{-1}\||u_k| + |D_v u_k|\|_{L_p((-1,0)\times B_{(2^{j+1}R)^3}\times B_{2^{j+1}R})} \\
&\quad \leq N2^{-j}R^{-1}\|\lambda^{1/2}|u_k| + |D_v u_k|\|_{L_p((-1,0)\times B_{(2^{j+1}R)^3}\times B_{2^{j+1}R})}.
\end{aligned}$$

By using induction, the above inequality, and (3.4), we obtain

$$\begin{aligned}
&\|\lambda^{1/2}|u_k| + |D_v u_k|\|_{L_p((-1,0)\times B_{R^3}\times B_R)} \\
&\quad \leq N^k 2^{-k(k-1)/2}R^{-k}\||\vec{f}_k| + \lambda^{-1/2}|g_k|\|_{L_p((-1,0)\times\mathbb{R}^{2d})} \\
&\quad \leq N2^{-k(k-1)/4}R^{-k}\||\vec{f}| + \lambda^{-1/2}|g|\|_{L_p(Q_{1,2^{k+1}R})}.
\end{aligned}$$

This, combined with (3.4) with $k = 0$, gives (3.2).

Estimate of $(-\Delta_x)^{1/6}u$. Recall that $u\zeta_0$ satisfies (3.5) with $k=0$. Then, by Theorem 2.1 (ii) and (3.2) with $2R$ in place of R, one has

$$\begin{aligned}&\|(-\Delta_x)^{1/6}(u\zeta_0)\|_{L_p((-1,0)\times\mathbb{R}^{2d})}\\&\quad\le N\sum_{k=0}^{\infty}2^{-k(k-1)/4}(2R)^{-k}\||\vec f|+\lambda^{-1/2}|g|\|_{L_p(Q_{1,2^{k+1}(2R)})}.\end{aligned}\tag{3.6}$$

Now we only need to bound the commutator term. Let u_ε be a mollification of u in the x variable. Then, $(-\Delta_x)^{1/6}u_\varepsilon$ is given by (1.11) with $s=1/6$. Due to the fact that $\zeta_0=1$ in $B_{(2^{1/2}R)^3}\times B_{2^{1/2}R}$, for any $z\in Q_{1,R}$,

$$\begin{aligned}&|\zeta_0(-\Delta_x)^{1/6}u_\varepsilon-(-\Delta_x)^{1/6}(u_\varepsilon\zeta_0)|(z)\\&\quad\le N(d)\int_{|y|>(2^{3/2}-1)R^3}|u_\varepsilon|(t,x+y,v)|y|^{-d-1/3}\,dy.\end{aligned}$$

Then, by Lemma A.1, we get

$$\begin{aligned}&\|\zeta_0(-\Delta_x)^{1/6}u_\varepsilon-(-\Delta_x)^{1/6}(u_\varepsilon\zeta_0)\|_{L_p(Q_{1,R})}\\&\quad\le N(d)R^{-1}\sum_{k=0}^{\infty}2^{-k-3dk/p}\|u_\varepsilon\|_{L_p(Q_{1,2^kR})}.\end{aligned}$$

Passing to the limit as $\varepsilon\to 0$, we may replace u_ε with u in the above inequality. Furthermore, by (3.2), the right-hand side is less than

$$N(d,\delta,p)R^{-1}\sum_{j=0}^{\infty}2^{-k-3dk/p}\sum_{k=0}^{\infty}2^{-j(j-1)/4}(2^kR)^{-j}\||\vec f|+\lambda^{-1/2}|g|\|_{L_p(Q_{1,2^{k+j+1}R})}.$$

Switching the order of summation and changing the index $k\to k+j$, we may replace the double sum with

$$\sum_{k=0}^{\infty}2^{-k-3dk/p}\||\vec f|+\lambda^{-1/2}|g|\|_{L_p(Q_{1,2^{k+1}R})}.$$

This, combined with (3.6), gives the desired estimate (3.3). ∎

Here is the mean oscillation estimate of a $P_0+\lambda$-caloric part.

PROPOSITION 3.4. *Let $p>1$, $\lambda\ge 0$, $r>0$, $\nu\ge 2$ be numbers, let $z_0\in\overline{\mathbb{R}^{1+2d}_T}$, and let $u\in\mathbb{S}_p((t_0-(2\nu r)^2,t_0)\times\mathbb{R}^{2d})$ be a function such that $P_0u+\lambda u=0$ in $(t_0-(\nu r)^2,t_0)\times\mathbb{R}^d\times B_{\nu r}(v_0)$. Then, one has*

$$\begin{aligned}J_1&:=\left(|(-\Delta_x)^{1/6}u-((-\Delta_x)^{1/6}u)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}\\&\le N\nu^{-1}(|(-\Delta_x)^{1/6}u|^p)^{1/p}_{Q_{\nu r}(z_0)},\\J_2&:=\lambda^{1/2}\left(|u-(u)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}+\left(|D_vu-(D_vu)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}\\&\le N\nu^{-1}\lambda^{1/2}(|u|^p)^{1/p}_{Q_{\nu r}(z_0)}+N\nu^{-1}(|D_vu|^p)^{1/p}_{Q_{\nu r}(z_0)}\\&\quad+N\nu^{-1}\sum_{k=0}^{\infty}2^{-2k}(|(-\Delta_x)^{1/6}u|^p)^{1/p}_{Q_{\nu r,2^k\nu r}(z_0)},\end{aligned}$$

where $N=N(d,\delta,p)$.

3.1. Proof of Proposition 3.4. The next two lemmas are taken from [15]. The first one, Lemma 3.5, is proved by localizing Theorem 2.3 (i). The second lemma follows from the global L_p estimate of $(-\Delta_x)^{1/3}u$ in Theorem 2.3 (i) and the local estimate of $D_v u$ in Lemma 3.5.

LEMMA 3.5 (interior S_p estimate; see Lemma 6.4 of [15]). *Let $p>1$, $\lambda \ge 0$, and $r_1, r_2, R_1, R_2 > 0$ be numbers such that $r_1 < r_2$ and $R_1 < R_2$. Let $u \in S_{p,loc}(\mathbb{R}^{1+2d}_0)$, and denote $f = P_0 u + \lambda u$. Then, there exists a constant $N = N(d,\delta,p) > 0$ such that*

$$\begin{aligned}
&\lambda \|u\|_{L_p(Q_{r_1,R_1})} + (r_2-r_1)^{-1}\|D_v u\|_{L_p(Q_{r_1,R_1})} \\
&\quad + \|D_v^2 u\|_{L_p(Q_{r_1,R_1})} + \|\partial_t u - v\cdot D_x u\|_{L_p(Q_{r_1,R_1})} \\
&\quad \le N\|f\|_{L_p(Q_{r_2,R_2})} + N((r_2-r_1)^{-2} + r_2(R_2-R_1)^{-3})\|u\|_{L_p(Q_{r_2,R_2})}.
\end{aligned}$$

LEMMA 3.6 (Caccioppoli-type inequality; see Lemma 6.5 of [15]). *Let $\lambda \ge 0$, $0 < r < R \le 1$, and $p > 1$ be numbers, and let $u \in S_{p,loc}(\mathbb{R}^{1+2d}_0)$ be a function such that $P_0 u + \lambda u = 0$ in Q_1. Then, there exists a constant $N = N(d,\delta,p,r,R)$ such that*

$$\|D_x u\|_{L_p(Q_r)} \le N\|u\|_{L_p(Q_R)}. \tag{3.7}$$

Remark 3.7. In the interior estimates in the aforementioned [15, Lemma 6.4], there are no terms involving λu and $\partial_t u - v\cdot D_x u$. By following the proof of that lemma and using the global S_p estimate (see Theorem 2.3), one can, indeed, add these terms to the left-hand sides of the a priori estimates.

Furthermore, the Caccioppoli inequality in [15, Lemma 6.5] is stated only in the case when $\lambda = 0$. Nevertheless, the same argument yields (3.7) in the case when $\lambda > 0$.

The next lemma is a key ingredient of the proof of Proposition 3.4.

LEMMA 3.8 (cf. Lemma 6.6 of [15]). *Let $p \in (1,\infty)$, and let $u \in \mathbb{S}_p((-4,0)\times\mathbb{R}^d\times B_2)$ be a function such that $P_0 u + \lambda u = 0$ in $(-1,0)\times\mathbb{R}^d\times B_1$. Then, the following assertions hold.*

(i) *The functions $u, (-\Delta_x)^{1/6}u \in S_{p,loc}((-1,0)\times\mathbb{R}^d\times B_1)$. Furthermore,*

$$(P_0+\lambda)u = 0, \quad (P_0+\lambda)(-\Delta_x)^{1/6}u = 0 \quad \text{a.e. in } (-1,0)\times\mathbb{R}^d\times B_1. \tag{3.8}$$

(ii) *For any $r \in (0,1)$, we have*

$$\|D_x u\|_{L_p(Q_r)} \le N\sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/6}u|^p)^{1/p}_{Q_{1,2^k}}, \tag{3.9}$$

where $N = N(d,\delta,p,r)$.

Proof. Multiplying $u \in \mathbb{S}_p((-4,0)\times\mathbb{R}^d\times B_2)$ by a cutoff function $\phi = \phi(t,v)$ and using Corollary 2.2, we conclude that $(-\Delta_x)^{1/6}u \in L_p((-1,0)\times\mathbb{R}^d\times B_1)$, so that the series on the right-hand side of (3.9) converges.

(i) Let u_ε be the mollification of u in the x variable. First, we will show that u_ε is sufficiently regular. We fix some $r_0 \in (0,1)$. We claim that for any $k = \{0,1,2,\dots\}$,

$$D_x^k \xi \in L_p((-r_0^2,0)\times\mathbb{R}^d\times B_{r_0}) \quad \text{for } \xi = u_\varepsilon, \partial_t u_\varepsilon, D_v^2 u_\varepsilon. \tag{3.10}$$

To show this, we note that

$$P_0 u_\varepsilon + \lambda u_\varepsilon = 0 \quad \text{in} \quad (-1,0)\times\mathbb{R}^d\times B_1.$$

We fix $x \in \mathbb{R}^d$ and write

$$\partial_t u_\varepsilon - a^{ij} D_{v_i v_j} u_\varepsilon + \lambda u_\varepsilon = v \cdot D_x u_\varepsilon =: \mathsf{f} \quad \text{in } (-1,0) \times B_1.$$

For $f \in L_{1,\text{loc}}(\mathbb{R}^d)$, let $f^\varkappa$ be a mollification of f in the v variable. Then, $u_\varepsilon^\varkappa$ satisfies

$$\partial_t u_\varepsilon^\varkappa - a^{ij} D_{v_i v_j} u_\varepsilon^\varkappa + \lambda u_\varepsilon^\varkappa = \mathsf{f}^\varkappa \quad \text{in } (-r_0^2, 0) \times B_{r_0}, \quad \varkappa \in (0, 1 - r_0).$$

Note that $(t,v) \to \mathsf{f} \in L_p((-1,0) \times B_1)$, and then $\partial_t u_\varepsilon^\varkappa, D_v^2 u_\varepsilon^\varkappa \in L_p((-r_0^2, 0) \times B_{r_0})$. By the interior estimate for nondegenerate parabolic equations (cf. Lemma 2.4.4 in [24]), for any $r_1 \in (0, r_0)$,

$$\begin{aligned} &\|\lambda |u_\varepsilon^\varkappa(\cdot, x, \cdot)| + |\partial_t u_\varepsilon^\varkappa(\cdot, x, \cdot)| + |D_v^2 u_\varepsilon^\varkappa(\cdot, x, \cdot)|\|_{L_p((-r_1^2, 0) \times B_{r_1})} \\ &\quad \le N(d, \delta, p, r_0, r_1) \| |\mathsf{f}^\varkappa(\cdot, x, \cdot)| + |u_\varepsilon^\varkappa(\cdot, x, \cdot)| \|_{L_p((-r_0^2, 0) \times B_{r_0})}. \end{aligned}$$

Raising the above inequality to the power p and integrating over $x \in \mathbb{R}^d$, we get

$$\begin{aligned} &\|\lambda |u_\varepsilon^\varkappa| + |\partial_t u_\varepsilon^\varkappa| + |D_v^2 u_\varepsilon^\varkappa|\|_{L_p((-r_1^2, 0) \times \mathbb{R}^d \times B_{r_1})} \\ &\quad \le N \| |\mathsf{f}^\varkappa| + |u_\varepsilon^\varkappa| \|_{L_p((-r_0^2, 0) \times \mathbb{R}^d \times B_{r_0})} \le N \| |u_\varepsilon| + |D_x u_\varepsilon| \|_{L_p((-1, 0) \times \mathbb{R}^d \times B_1)}, \end{aligned}$$

where $N = N(d, \delta, p, r_0, r_1)$. Passing to the limit as $\varkappa \to 0$, we conclude that (3.10) holds with $k = 0$. In the case when $k \ge 1$, we use the method of finite-difference quotients combined with the above argument.

Next, by (3.10) with $k = 0$, $u_\varepsilon \in S_p((-r_0^2, 0) \times \mathbb{R}^d \times B_{r_0})$. Then, by the interior S_p estimate (see Lemma 3.5), for any $r_1 \in (0, r_0)$ and $x_0 \in \mathbb{R}^d$,

$$\| |\partial_t u_\varepsilon - v \cdot D_x u_\varepsilon| + |D_v^2 u_\varepsilon| \|_{L_p(Q_{r_1}(0, x_0, 0))} \le N \|u\|_{L_p(Q_{r_0}(0, x_0, 0))},$$

where $N = N(d, \delta, r_1, r_0)$. Passing to the limit as $\varepsilon \to 0$, we prove that $u \in S_{p,\text{loc}}((-1,0) \times \mathbb{R}^d \times B_1)$ and that $(P_0 + \lambda) u = 0$ a.e. in $(-1, 0) \times \mathbb{R}^d \times B_1$.

To prove the second part of assertion (i), we note that by (3.10) and the Sobolev embedding theorem, for a.e. $t, v \in ((-r_0^2, 0) \times B_{r_0})$ and the same ξ,

$$\xi(t, \cdot, v) \in C_0^k(\mathbb{R}^d), \quad k \ge 1 \tag{3.11}$$

(see Definition 1.9). Therefore, by the pointwise formula (1.11),

$$(-\Delta_x)^{1/6} \xi(t, \cdot, v) \in C_0^k(\mathbb{R}^d)$$

is a well-defined function, and

$$(-\Delta_x)^{1/6} A u_\varepsilon(t, \cdot, v) = A (-\Delta_x)^{1/6} u_\varepsilon(t, \cdot, v), \quad A = \partial_t, D_v^2.$$

Then, $(P_0 + \lambda)(-\Delta_x)^{1/6} u_\varepsilon = 0$ a.e. in $(-r_0^2, 0) \times \mathbb{R}^d \times B_{r_0}$. As above, by using the interior S_p estimate and a limiting argument, we prove the part of assertion (i) about $(-\Delta_x)^{1/6} u$.

(ii) In what follows, we follow the argument of Lemma 6.6 of [15]. Let $r_0 \in (r, 1)$, and let $\zeta \in C_0^\infty(\widetilde{Q}_{r_0})$ be a function taking values in $[0,1]$ such that $\zeta = 1$ on $\widetilde{Q}_r$. We split $D_x u_\varepsilon$ as follows:

$$\zeta^2 D_x u_\varepsilon = \zeta (\mathcal{L} u_\varepsilon + \text{Comm}),$$

where

$$\mathcal{L}u_\varepsilon = \mathcal{R}_x(-\Delta_x)^{1/3}(\zeta(-\Delta_x)^{1/6}u_\varepsilon),$$
$$\text{Comm} = \zeta D_x u_\varepsilon - \mathcal{R}_x(-\Delta_x)^{1/3}(\zeta(-\Delta_x)^{1/6}u_\varepsilon),$$

and $\mathcal{R}_x$ is the Riesz transform.

Estimate of $\mathcal{L}u$. Denote $h = (-\Delta_x)^{1/6}u_\varepsilon$. Then, by assertion (i), $\zeta h \in S_p(\mathbb{R}^{1+2d}_0)$ satisfies the identity

$$(P_0 + \lambda)(\zeta h) = hP_0\zeta - 2(aD_v\zeta)\cdot D_v h \quad \text{in } \mathbb{R}^{1+2d}_0. \tag{3.12}$$

By the L_p-boundedness of the Riesz transform and Theorem 2.3 applied to (3.12),

$$\begin{aligned}\|\mathcal{L}u_\varepsilon\|_{L_p(\mathbb{R}^{1+2d}_0)} &\le N(d,p)\|(-\Delta_x)^{1/3}(\zeta h)\|_{L_p(\mathbb{R}^{1+2d}_0)}\\ &\le N(d,p,\delta)\||hP_0\zeta| + |(aD_v\zeta)\cdot D_v h|\|_{L_p(\mathbb{R}^{1+2d}_0)}.\end{aligned} \tag{3.13}$$

Furthermore, by (3.8) and the interior gradient estimate in Lemma 3.5, we get

$$\|(aD_v\zeta)\cdot D_v h\|_{L_p(\mathbb{R}^{1+2d}_0)} \le N\|h\|_{L_p(Q_{r_0})}, \tag{3.14}$$

where $N = N(d,\delta,p,r,r_0)$.

Commutator estimate. We denote

$$\mathcal{A} = D_x(-\Delta_x)^{-1/6}.$$

By Lemma A.2, this operator can be extended to $C^1_0(\mathbb{R}^d)$ functions as follows:

$$\mathcal{A}\phi(x) = \text{p.v.}\int \phi(x-y)\frac{y}{|y|^{d+5/3}}\,dy.$$

Furthermore, by the same lemma, for any $\phi \in C^2_0(\mathbb{R}^d)$, one has $A(-\Delta_x)^{1/6}\phi \equiv D_x\phi$. Then, since $u_\varepsilon(t,\cdot,v) \in C^2_0(\mathbb{R}^d)$ (see (3.11)), for a.e $(t,v) \in (-1,0)\times B_1$,

$$\begin{aligned}\text{Comm}(z) &= \zeta\mathcal{A}h(z) - \mathcal{A}(\zeta h)(z)\\ &= \text{p.v.}\int h(t,x-y,v)\big(\zeta(t,x,v) - \zeta(t,x-y,v)\big)\,\frac{y}{|y|^{d+5/3}}\,dy\\ &= \left(\int_{|y|\le 1}\cdots + \int_{|y|>1}\cdots\right) =: I_1(z) + I_2(z).\end{aligned}$$

By the mean-value theorem and the Minkowski inequality,

$$\|I_1\|_{L_p(Q_r)} \le N(d,p)\|h\|_{L_p(Q_{1,2})}. \tag{3.15}$$

Next, for any $z \in Q_r$, we have

$$|I_2|(z) \le 2\int_{|y|>1}|h(t,x-y,v)|\frac{dy}{|y|^{d+2/3}}.$$

Then, by Lemma A.1,

$$\|I_2\|_{L_p(Q_r)} \le N(d,p)\sum_{k=0}^{\infty}2^{-2k}(|h|^p)^{1/p}_{Q_{1,2^k}}. \tag{3.16}$$

Combining (3.13)–(3.16) and passing to the limit as $\varepsilon \to 0$, we prove assertion (ii). ◻

The next lemma is about estimates for $P_0 + \lambda$-caloric functions.

LEMMA 3.9. *Let* $p \in (1,\infty)$, *and let* $u \in \mathbb{S}_{p,loc}((-4,0)\times\mathbb{R}^d\times B_2)$ *(or* $S_{p,loc}((-1,0)\times\mathbb{R}^d \times B_1)$*) be a function such that* $P_0 u + \lambda u = 0$ *in* $(-1,0)\times\mathbb{R}^d\times B_1$. *Then, for any* $j\in\{0,1\}$, $l,m\in\{0,1,\dots\}$, *the following assertions hold.*

(i) *For any* $1/2 \le r < R \le 1$,

$$(1+\lambda)\|\partial_t^j D_x^l D_v^m u\|_{L_p(Q_r)} \le N(d,\delta,p,j,l,m,r,R)\|u\|_{L_p(Q_R)}. \tag{3.17}$$

Furthermore,

$$(1+\lambda)\|\partial_t^j D_x^l D_v^m u\|_{L_\infty(Q_r)} \le N(d,\delta,p,j,l,m,r,R)\|u\|_{L_p(Q_R)}. \tag{3.18}$$

(ii) *If, additionally,* $u \in \mathbb{S}_p((-4,0)\times\mathbb{R}^d\times B_1)$ *and* $j+l+m\ge 1$, *then*

$$\begin{aligned}&\|\partial_t^j D_x^l D_v^m u\|_{L_p(Q_{1/2})}\\ &\quad\le N(d,\delta,p,j,l,m)\left(\||D_v u| + \lambda^{1/2}|u|\|_{L_p(Q_1)} + \sum_{k=0}^{\infty} 2^{-2k}\left(|(-\Delta_x)^{1/6}u|^p\right)^{1/p}_{Q_{1,2^k}}\right).\end{aligned} \tag{3.19}$$

As in assertion (i), *we may replace the left-hand side of* (3.19) *with*

$$\|\partial_t^j D_x^l D_v^m u\|_{L_\infty(Q_{1/2})}.$$

Proof. (i) By Lemma 3.8 (i), $u \in S_{p,\mathrm{loc}}((-1,0)\times\mathbb{R}^d\times B_1)$. In what follows, we follow the argument of Lemma 5.6 (i) in [15].

Case $l,j=0$. First, we prove that for any $r\in(1/2,R)$ and $m=\{0,1,2,\dots\}$,

$$\lambda\|D_v^m u\|_{L_p(Q_r)} + \|D_v^{m+1}u\|_{L_p(Q_r)} \le N(d,\delta,p,r,m)\|u\|_{L_p(Q_R)}. \tag{3.20}$$

To prove this, we use an induction argument. Note that (3.20) with $m=0$ follows directly from Lemma 3.5. In the rest of the argument, we do some formal calculations. To make the argument rigorous, one needs to use the method of finite-difference quotient. For $m>0$, we fix some multi-index α of order m. Then, by the product rule,

$$(P_0+\lambda)(D_v^\alpha u) = \sum_{\widetilde{\alpha}:\,\widetilde{\alpha}<\alpha,|\widetilde{\alpha}|=m-1} c_{\widetilde{\alpha}} D_v^{\widetilde{\alpha}} D_x^{\alpha-\widetilde{\alpha}} u, \tag{3.21}$$

where $c_{\widetilde{\alpha}}$ is a constant. Next, for any $r_1\in(r,R)$, by Lemma 3.5, we have

$$\begin{aligned}&\lambda\|D_v^m u\|_{L_p(Q_r)} + \|D_v^{m+1}u\|_{L_p(Q_r)}\\ &\quad\le N\|D_v^{m-1}D_x u\|_{L_p(Q_{r_1})} + N\|D_v^m u\|_{L_p(Q_{r_1})}.\end{aligned} \tag{3.22}$$

Observe that for any multi-index β,

$$(P_0+\lambda)(D_x^\beta u) = 0 \quad \text{in } (-1,0)\times\mathbb{R}^d\times B_1. \tag{3.23}$$

Then, by the induction hypothesis and Lemma 3.6, for any $r_2\in(r_1,R)$, we have

$$\|D_v^{m-1}D_x u\|_{L_p(Q_{r_1})} \le N\|D_x u\|_{L_p(Q_{r_2})} \le N\|u\|_{L_p(Q_R)}.$$

This, combined with (3.22) and the induction hypothesis, implies (3.20).

Case $j=0$. Combining (3.23), (3.20), and Lemma 3.6, we obtain (3.17) with $j=0$.

Case $j=1$. By (3.21) and (3.23), for any multi-indexes $\alpha\neq 0$ and β, the function

$$U = D_v^\alpha D_x^\beta u \tag{3.24}$$

satisfies the identity

$$P_0 U + \lambda U = \sum_{\tilde\alpha:\,\tilde\alpha<\alpha, |\tilde\alpha| = |\alpha|-1} c_{\tilde\alpha} D_v^{\tilde\alpha} D_x^{\alpha+\beta-\tilde\alpha} u \quad \text{in } (-1,0)\times\mathbb{R}^d\times B_1. \tag{3.25}$$

Then, by Lemma 3.5 and (3.17) with $j=0$, we conclude that

$$\begin{aligned}(1+\lambda)\|\partial_t U\|_{L_p(Q_r)} &\le (1+\lambda)\big(\|\partial_t U - v\cdot D_x U\|_{L_p(Q_r)} + r\|D_x U\|_{L_p(Q_r)}\big)\\ &\le N(1+\lambda)(\|U\|_{L_p(Q_{r_1})} + \|D_v^{|\alpha|-1} D_x^{1+|\beta|} u\|_{L_p(Q_{r_1})}\\ &\quad + \|D_x U\|_{L_p(Q_r)}) \le N\|u\|_{L_p(Q_R)},\end{aligned} \tag{3.26}$$

where $N = N(d,\delta,|\alpha|,|\beta|,p,r,R)$. In the case $\alpha=0$, the above argument yields the same bound $(1+\lambda)\|\partial_t U\|_{L_p(Q_r)} \le N\|u\|_{L_p(Q_R)}$. Thus, (3.17) with $j=1$ is also valid.

Next, note that the second assertion with $j=0$ follows from (3.17) and the Sobolev embedding theorem. To prove the estimate with $j=1$, we use (3.25), (3.18) with $j=0$, and Lemma 3.5:

$$\begin{aligned}(1+\lambda)\|\partial_t U\|_{L_\infty(Q_r)} &\le (1+\lambda)\||D_x U| + |D_v^2 U| + \lambda|U|\|_{L_\infty(Q_r)}\\ &\le N(1+\lambda)\|u\|_{L_p(Q_{R/2+1/4})} \le N\|u\|_{L_p(Q_R)}.\end{aligned} \tag{3.27}$$

(ii) It suffices to show the validity of the estimate

$$\begin{aligned}&\|\partial_t^j D_x^l D_v^m u\|_{L_p(Q_{1/2})}\\ &\quad \le N(d,\delta,R,j,l,m)(\||D_x u| + |D_v u| + \lambda^{1/2}|u|\|_{L_p(Q_R)}), R\in(1/2,1],\end{aligned} \tag{3.28}$$

because the desired assertion follows from (3.28) and Lemma 3.8 (ii). To prove (3.28), we will consider four cases.

Case 1: $l\ge 1$. Note that by (3.17) and (3.23), one has for $1/2<r<r_1<R$,

$$\|\partial_t^j D_x^l D_v^m u\|_{L_p(Q_r)} \le N\|D_x^l u\|_{L_p(Q_{r_1})}.$$

Hence, (3.28) holds in the case $l=1$. If $l\ge 2$, we use Lemma 3.6.

Case 2: $l=0, m\ge 1, j=0$. By using an induction argument (see (3.22)) and Lemma 3.6 as in the proof of assertion (i), one can show that

$$\|D_v^m u\|_{L_p(Q_r)} \le N\||D_x u| + |D_v u|\|_{L_p(Q_R)}. \tag{3.29}$$

Case 3: $l=0, m\ge 1, j=1$. By (3.26) with $|\alpha|=m$ and $\beta=0$,

$$\|\partial_t D_v^m u\| \le N\||D_v^m u| + |D_v^{m-1} D_x u| + |D_x u|\|_{L_p(Q_{r_1})}.$$

Now (3.28) follows from (3.29) and (3.28) with $l=1$ (see Case 1).

Case 4: $l=0, m=0, j=1$. Since $\partial_t u = v\cdot D_x u + a^{ij}(t) D_{v_i v_j} u - \lambda u$ in $(-1,0)\times\mathbb{R}^d\times B_1$, by using (3.29) with $m=2$, we get

$$\|\partial_t u\|_{L_p(Q_r)} \le N\||D_x u| + |D_v u| + \lambda|u|\|_{L_p(Q_{R_1})},\ R_1\in(R,1). \tag{3.30}$$

Let $R_2\in(R_1,1)$. By (3.17), we may replace the term $\lambda\|u\|_{L_p(Q_{R_1})}$ with $\lambda^{1/2}\|u\|_{L_p(Q_{R_2})}$ on the right-hand side of (3.30), and, thus, (3.28) holds.

Finally, the second part of the assertion in the case when $j=0$ follows from (3.19) and the Sobolev embedding theorem.

In the case when $j=1$, we invoke (3.24)–(3.25). By (3.27) and the $L_\infty(Q_{1/2})$ estimate of D_xU and D_v^2U proved in the previous paragraph, we get

$$\begin{aligned}\|\partial_t U\|_{L_\infty(Q_{1/2})} &\le \||D_xU|+|D_v^2U|+\lambda|U|\|_{L_\infty(Q_{1/2})}\\ &\le N\||D_xu|+|D_vu|+\lambda^{1/2}|u|\|_{L_p(Q_1)}+\lambda\|U\|_{L_\infty(Q_{1/2})}.\end{aligned}$$

By (3.18), we replace the last term with $\lambda^{1/2}\|u\|_{L_p(Q_1)}$. The assertion is proved. ∎

The next result follows from direct computations.

Lemma 3.10 (scaling property of P_0). *Let $p\in[1,\infty]$, $T\in(-\infty,\infty]$, and $u\in\mathbb{S}_{p,loc}(\mathbb{R}^{1+2d}_T)$. For any $z_0\in\mathbb{R}^{1+2d}_T$, denote*

$$\begin{aligned}&\widetilde{z}=(r^2t+t_0,r^3x+x_0-r^2tv_0,rv+v_0),\quad \widetilde{u}(z)=u(\widetilde{z}),\\ &Y=\partial_t-v\cdot D_x,\quad \widetilde{P}_0=\partial_t-v\cdot D_x-a^{ij}(r^2t+t_0)D_{v_iv_j}.\end{aligned}\tag{3.31}$$

Then,

$$Y\widetilde{u}(z)=r^2(Yu)(\widetilde{z}),\quad \widetilde{P}_0\widetilde{u}(z)=r^2(Pu)(\widetilde{z}).$$

Proof of Proposition 3.4. Let $\widetilde{u}$ and $\widetilde{P}_0$ be the function and the operator from Lemma 3.10 defined with νr in place of r. Then, by the same lemma,

$$\widetilde{P}_0\widetilde{u}+\lambda(\nu r)^2\widetilde{u}=0\quad\text{in } (-1,0)\times\mathbb{R}^d\times B_1,$$

and for any $c>0$, and $A=(-\Delta_x)^{1/6}$ or D_v,

$$\begin{aligned}&(|Au|^p)^{1/p}_{Q_{\nu r,c\nu r}(z_0)}=(\nu r)^{-1}(|A\widetilde{u}|^p)^{1/p}_{Q_{1,c}},\\ &\left(|Au-(Au)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}=(\nu r)^{-1}\left(|A\widetilde{u}-(A\widetilde{u})_{Q_{1/\nu}}|^p\right)^{1/p}_{Q_{1/\nu}}.\end{aligned}\tag{3.32}$$

Next, by Lemma 3.8 (i), $(-\Delta_x)^{1/6}\widetilde{u}\in S_{p,\text{loc}}((-1,0)\times\mathbb{R}^d\times B_1)$, and

$$(\widetilde{P}_0+\lambda(\nu r)^2)(-\Delta_x)^{1/6}\widetilde{u}=0\quad\text{a.e. in } (-1,0)\times\mathbb{R}^d\times B_1,$$

and then, by Lemma 3.9 (i) with u replaced with $(-\Delta_x)^{1/6}\widetilde{u}$, for any $\nu\ge 2$, we get

$$\begin{aligned}&\left(|(-\Delta_x)^{1/6}\widetilde{u}-((-\Delta_x)^{1/6}\widetilde{u})_{Q_{1/\nu}}|^p\right)^{1/p}_{Q_{1/\nu}}\\ &\quad\le\sup_{z_1,z_2\in Q_{1/\nu}}|(-\Delta_x)^{1/6}\widetilde{u}(z_1)-(-\Delta_x)^{1/6}\widetilde{u}(z_2)|\\ &\quad\le N(d,\delta,p)\nu^{-1}\left(\fint_{Q_1}|(-\Delta_x)^{1/6}\widetilde{u}|^p\,dz\right)^{1/p}.\end{aligned}$$

Combining this with (3.32), we prove the estimate for $(-\Delta_x)^{1/6}u$.

Arguing as above and using Lemma 3.9 (ii), for any $\nu\ge 2$, we obtain

$$\begin{aligned}&\lambda^{1/2}\nu r\left(|\widetilde{u}-(\widetilde{u})_{Q_{1/\nu}}|^p\right)^{1/p}_{Q_{1/\nu}}+\left(|D_v\widetilde{u}-(D_v\widetilde{u})_{Q_{1/\nu}}|^p\right)^{1/p}_{Q_{1/\nu}}\\ &\quad\le N\nu^{-1}\left(\lambda^{1/2}\nu r\,(|\widetilde{u}|^p)^{1/p}_{Q_1}+(|D_v\widetilde{u}|^p)^{1/p}_{Q_1}+\sum_{k=0}^{\infty}2^{-2k}(|(-\Delta_x)^{1/6}\widetilde{u}|^p)^{1/p}_{Q_{1,2^k}}\right).\end{aligned}$$

Dividing both sides of the above inequality by νr and using (3.32) yield the desired estimate. ∎

3.2. Proof of Theorem 3.1. The following mean oscillation estimate plays a crucial role in the proofs of Theorems 3.1 and 1.15.

PROPOSITION 3.11. *Let* $p > 1$, $r > 0$, $\nu \ge 2$ *be numbers,* $T \in (-\infty, \infty]$, $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$, *and let* $u \in \mathbb{S}_p(\mathbb{R}^{1+2d}_T)$, $\vec{f}, g \in L_p(\mathbb{R}^{1+2d}_T)$ *be functions such that*

$$P_0 u + \lambda u = div_v \vec{f} + g.$$

Then, there exists a constant $N = N(d, \delta, p) > 0$ *such that*

$$\begin{aligned} I_1 &:= \left(|(-\Delta_x)^{1/6} u - ((-\Delta_x)^{1/6} u)_{Q_r(z_0)}|^p \right)^{1/p}_{Q_r(z_0)} \\ &\le N\nu^{-1} (|(-\Delta_x)^{1/6} u|^p)^{1/p}_{Q_{\nu r}(z_0)} \\ &\quad + N\nu^{(4d+2)/p} \sum_{k=0}^{\infty} 2^{-k} \big((|\vec{f}|^p)^{1/p}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)} + \lambda^{-1/2} (|g|^p)^{1/p}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)} \big), \end{aligned}$$

$$\begin{aligned} I_2 &:= \lambda^{1/2} \left(|u - (u)_{Q_r(z_0)}|^p \right)^{1/p}_{Q_r(z_0)} + \left(|D_v u - (D_v u)_{Q_r(z_0)}|^p \right)^{1/p}_{Q_r(z_0)} \\ &\le N\nu^{-1} \lambda^{1/2} (|u|^p)^{1/p}_{Q_{\nu r}(z_0)} + N\nu^{-1} (|D_v u|^p)^{1/p}_{Q_{\nu r}(z_0)} \\ &\quad + N\nu^{-1} \sum_{k=0}^{\infty} 2^{-2k} (|(-\Delta_x)^{1/6} u|^p)^{1/p}_{Q_{\nu r, 2^k \nu r}(z_0)} \\ &\quad + N\nu^{(4d+2)/p} \sum_{k=0}^{\infty} 2^{-k} \big((|\vec{f}|^p)^{1/p}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)} + \lambda^{-1/2} (|g|^p)^{1/p}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)} \big), \end{aligned}$$

Proof. Estimate of I_1. We fix some function $\phi = \phi(t, v) \in C_0^\infty((t_0 - (2\nu r)^2, t_0 + (2\nu r)^2) \times B_{2\nu r}(v_0))$ such that $\phi = 1$ on $(t_0 - (\nu r)^2, t_0) \times B_{\nu r}(v_0)$. By Theorem 2.1 (ii), the Cauchy problem (see Definition 1.10)

$$P_0 u_0 = \operatorname{div}_v(\vec{f}\phi) + g\phi, \quad u_0(t_0 - (2\nu r)^2, \cdot) \equiv 0,$$

has a unique solution $u_0 \in \mathbb{S}_p((t_0 - (4\nu r)^2, t_0) \times \mathbb{R}^{2d})$. To obtain a mean oscillation estimate of $(-\Delta_x)^{1/6} u_0$, we use the argument of Proposition 3.4. Let $\widetilde{u}_0$ and $\widetilde{P}_0$ be the function and the operator from Lemma 3.10 defined with $2\nu r$ in place of r and with u_0 replaced by u. We define the functions $\widetilde{f}_i, i = 1, \dots, d$ and $\widetilde{g}$ by (3.31). Then, we have

$$\widetilde{P}_0 \widetilde{u}_0 + \lambda (2\nu r)^2 \widetilde{u}_0 = (2\nu r) D_{v_i} \widetilde{f}_i + (2\nu r)^2 \widetilde{g} \quad \text{in } (-1, 0) \times \mathbb{R}^d \times B_1.$$

By Lemma 3.3 with
- $(2\nu r)^2 \lambda$ in place of λ,
- $2\nu r \widetilde{f}_i$ in place of $f_i, i = 1, \dots, d$,
- $(2\nu r)^2 \widetilde{g}$ in place of g,

for any $R \ge 1$,

$$\begin{aligned} &(|(-\Delta_x)^{1/6} \widetilde{u}_0|^p)^{1/p}_{Q_{1,R}} \\ &\quad \le N(d, p, \delta)(2\nu r) \sum_{k=0}^{\infty} 2^{-k} \left(\sum_{i=1}^{d} (|\widetilde{f}_i|^p)^{1/p}_{Q_{1, 2^{k+1} R}} + \lambda^{-1/2} (|\widetilde{g}|^p)^{1/p}_{Q_{1, 2^{k+1} R}} \right). \end{aligned}$$

By dividing both sides of the above inequality by $2\nu r$ and using (3.32) with νr replaced by $2\nu r$, for any $R \geq 1$, we obtain

$$(|(-\Delta_x)^{1/6} u_0|^p)^{1/p}_{Q_{\nu r, R\nu r}(z_0)} \leq N \sum_{k=0}^{\infty} 2^{-k} F_k(R), \tag{3.33}$$

$$(|(-\Delta_x)^{1/6} u_0|^p)^{1/p}_{Q_{r, Rr}(z_0)} \leq N \nu^{(4d+2)/p} \sum_{k=0}^{\infty} 2^{-k} F_k(R), \tag{3.34}$$

where

$$F_k(R) = (|\vec{f}|^p)^{1/p}_{Q_{2\nu r, 2^{k+1} R(2\nu r)}(z_0)} + \lambda^{-1/2} (|g|^p)^{1/p}_{Q_{2\nu r, 2^{k+1} R(2\nu r)}(z_0)}.$$

Next, note that the function $u_h := u - u_0 \in \mathbb{S}_p((t_0 - (2\nu r)^2) \times \mathbb{R}^{2d})$ satisfies

$$P_0 u_h = \operatorname{div}_v(\vec{f}(1-\phi)) + g(1-\phi) \quad \text{in } (t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^{2d}.$$

Since $\vec{f}(1-\phi)$ and $g(1-\phi)$ vanish inside $(t_0 - (\nu r)^2, t_0) \times \mathbb{R}^d \times B_{\nu r}$, by Proposition 3.4 and (3.33) with $R = 1$,

$$\begin{aligned}
&\left(|(-\Delta_x)^{1/6} u_h - ((-\Delta_x)^{1/6} u_h)_{Q_r(z_0)}|^p \right)^{1/p}_{Q_r(z_0)} \leq N \nu^{-1} (|(-\Delta_x)^{1/6} u_h|^p)^{1/p}_{Q_{\nu r}(z_0)} \\
&\quad \leq N \nu^{-1} (|(-\Delta_x)^{1/6} u|^p)^{1/p}_{Q_{\nu r}(z_0)} + N \nu^{-1} \sum_{k=0}^{\infty} 2^{-k} F_k(1).
\end{aligned}$$

Finally, the mean oscillation estimate of $(-\Delta_x)^{1/6} u$ follows from the above inequality and (3.34) with $R = 1$.

Estimate of I_2. First, by Lemma 3.3 and the scaling argument presented above,

$$\lambda^{1/2} (|u_0|^p)^{1/p}_{Q_{\nu r}(z_0)} + (|D_v u_0|^p)^{1/p}_{Q_{\nu r}(z_0)} \leq N \sum_{k=0}^{\infty} 2^{-k^2/8} F_k(1), \tag{3.35}$$

$$\lambda^{1/2} (|u_0|^p)^{1/p}_{Q_r(z_0)} + (|D_v u_0|^p)^{1/p}_{Q_r(z_0)} \leq N \nu^{(4d+2)/p} \sum_{k=0}^{\infty} 2^{-k^2/8} F_k(1). \tag{3.36}$$

Hence, as above, by (3.36), it remains to estimate I_2 with u replaced by u_h.

Next, by Proposition 3.4, we get

$$\begin{aligned}
&\lambda^{1/2} \left(|u_h - (u_h)_{Q_r(z_0)}|^p \right)^{1/p}_{Q_r(z_0)} + \left(|D_v u_h - (D_v u_h)_{Q_r(z_0)}|^p \right)^{1/p}_{Q_r(z_0)} \\
&\leq N \nu^{-1} \lambda^{1/2} (|u|^p)^{1/p}_{Q_{\nu r}(z_0)} + N \nu^{-1} (|D_v u|^p)^{1/p}_{Q_{\nu r}(z_0)} \\
&\quad + N \nu^{-1} \sum_{j=0}^{\infty} 2^{-2j} (|(-\Delta_x)^{1/6} u|^p)^{1/p}_{Q_{\nu r, 2^j \nu r}(z_0)} \\
&\quad + N \nu^{-1} (A_1 + A_2),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sum_{j=0}^{\infty} 2^{-2j} (|(-\Delta_x)^{1/6} u_0|^p)^{1/p}_{Q_{\nu r, 2^j \nu r}(z_0)}, \\
A_2 &= \lambda^{1/2} (|u_0|^p)^{1/p}_{Q_{\nu r}(z_0)} + (|D_v u_0|^p)^{1/p}_{Q_{\nu r}(z_0)}.
\end{aligned}$$

By (3.33) with $R = 2^j$, we get

$$A_1 \le N \sum_{j=0}^{\infty} 2^{-2j} \sum_{k=0}^{\infty} 2^{-k} F_k(2^j).$$

Using the fact that $F_k(2^j) = F_{k+j}(1)$ and changing the index of summation $k \to k + j$ yield

$$A_1 \le N \sum_{j=0}^{\infty} 2^{-j} F_j(1).$$

Finally, note that the term A_2 is estimated in (3.35). The lemma is proved. □

Proof of Theorem 3.1. In the first two steps below, we assume, additionally, that

$$(-\Delta_x)^{1/6} u \in L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w). \tag{3.37}$$

We will remove this assumption in Step 3.

Step 1: *Estimate of a localized function.* By Lemma A.4 and the self-improving property of the A_p-weights (see, for instance, Corollary 7.2.6 of [17]), there exists a number

$$p_0 = p_0(d, p, r_1, \ldots, r_d, q, K), \quad 1 < p_0 < \min\{p, r_1, \ldots, r_d, q\},$$

such that

$$L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w) \subset L_{p_0,\mathrm{loc}}(\mathbb{R}^{1+2d}_T), \tag{3.38}$$

$$w_0 \in A_{q/p_0}(\mathbb{R}), \quad w_i \in A_{r_i/p_0}(\mathbb{R}), \quad i = 1, \ldots, d. \tag{3.39}$$

Let $\phi \in C_0^\infty(\mathbb{R}^{1+2d})$ be a function such that $\phi = 1$ on $\widetilde{Q}_1$, and denote

$$\phi_n(z) = \phi(t/n^2, x/n^3, v/n), \quad u_n = u\phi_n, \quad \vec{f}_n = \vec{f}\phi_n. \tag{3.40}$$

Observe that u_n satisfies

$$P_0 u_n + \lambda u_n = \mathrm{div}_v(\vec{f}_n) + g_n,$$

where

$$g_n = g\phi_n - \vec{f} \cdot D_v \phi_n + u P_0 \phi_n - 2(a D_v \phi_n) \cdot D_v u. \tag{3.41}$$

Note that

$$\vec{f}_n, g_n \in L_{p_0}(\mathbb{R}^{1+2d}_T), \quad u_n \in \mathbb{S}_{p_0}(\mathbb{R}^{1+2d}_T).$$

We now use Proposition 3.11 and conclude that for any $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$,

$$\begin{aligned} ((-\Delta_x)^{1/6} u_n)^{\#}_T(z_0) &\le N \nu^{-1} \mathcal{M}_T^{1/p_0} |(-\Delta_x)^{1/6} u_n|^{p_0}(z_0) \\ &+ N \nu^{(4d+2)/p_0} \sum_{k=0}^{\infty} 2^{-k} \big(\mathbb{M}^{1/p_0}_{2^{k+1},T} |\vec{f}_n|^{p_0}(z_0) + \lambda^{-1/2} \mathbb{M}^{1/p_0}_{2^{k+1},T} |g_n|^{p_0}(z_0)\big), \end{aligned} \tag{3.42}$$

$$
\begin{aligned}
(3.43)\qquad & \lambda^{1/2}(u_n)^{\#}_T(z_0) + (D_v u_n)^{\#}_T(z_0) \\
&\le N\nu^{-1}\lambda^{1/2}\mathcal{M}_T^{1/p_0}|u_n|^{p_0}(z_0) + N\nu^{-1}\mathcal{M}_T^{1/p_0}|D_v u_n|^{p_0}(z_0) \\
&\quad + N\nu^{-1}\sum_{k=0}^{\infty} 2^{-2k}\mathbb{M}_{2^k,T}^{1/p_0}|(-\Delta_x)^{1/6}u_n|^{p_0}(z_0) \\
&\quad + N\nu^{(4d+2)/p_0}\sum_{k=0}^{\infty} 2^{-k}\big(\mathbb{M}_{2^{k+1},T}^{1/p_0}|\vec{f}_n|^{p_0}(z_0) + \lambda^{-1/2}\mathbb{M}_{2^{k+1},T}^{1/p_0}|g_n|^{p_0}(z_0)\big),
\end{aligned}
$$

where $\mathbb{M}_{c,T} f$ and $\mathcal{M}_T f$ are defined as in (1.2). We take the $\|\cdot\|$-norm on both sides of (3.42)–(3.43). Then we use Theorem A.3 with p/p_0, q/p_0, $r_i/p_0 > 1, i = 1,\ldots,d$, combined with (3.39). By this and the Minkowski inequality, we obtain

$$
\begin{aligned}
(3.44)\qquad & \|(-\Delta_x)^{1/6}u_n\| \le N\nu^{-1}\|(-\Delta_x)^{1/6}u_n\| \\
&\quad + N\nu^{(4d+2)/p_0}(\|\vec{f}_n\| + \lambda^{-1/2}\|g_n\|), \\
(3.45)\qquad & \lambda^{1/2}\|u_n\| + \|D_v u_n\| \le N\nu^{-1}(\lambda^{1/2}\|u_n\| + \|D_v u_n\|) \\
&\quad + N\nu^{-1}\|(-\Delta_x)^{1/6}u_n\| + N\nu^{(4d+2)/p_0}(\|\vec{f}_n\| + \lambda^{-1/2}\|g_n\|).
\end{aligned}
$$

Taking $\nu \ge 2+4N$, we cancel the term $\|(-\Delta_x)^{1/6}u_n\|$ on the right-hand side of (3.44) and obtain

$$
(3.46)\qquad \|(-\Delta_x)^{1/6}u_n\| \le N(\|\vec{f}_n\| + \lambda^{-1/2}\|g_n\|).
$$

By using the last inequality, (3.45), and our choice of ν, we prove

$$
(3.47)\qquad \lambda^{1/2}\|u_n\| + \|D_v u_n\| \le N(\|\vec{f}_n\| + \lambda^{-1/2}\|g_n\|).
$$

Step 2: *Limiting argument.* By (3.47), (3.41), and the construction of ϕ_n (see (3.40)), we have

$$
\begin{aligned}
&\|\lambda^{1/2}|u| + |D_v u|\|_{L_{p,r_1,\ldots,r_d,q}(\widetilde{Q}_n\cap\mathbb{R}^{1+2d}_T)} \\
&\quad \le N\|\vec{f}\| + N\lambda^{-1/2}\|g\| + Nn^{-1}\lambda^{-1/2}(\|\vec{f}\| + \|D_v u\| + \|u\|).
\end{aligned}
$$

Passing to the limit as $n\to\infty$, we prove the estimate (1.15) for u and $D_v u$.

Next, note that due to (1.13), and Hölder's inequality for any $\eta \in C_0^\infty(\mathbb{R}^{1+2d}_T)$,

$$
(-\Delta_x)^{1/6}\eta \in L_* := L_{p^*,r_1^*,\ldots,r_d^*,q^*}(\mathbb{R}^{1+2d}_T, w_*),
$$

where $p^*, r_1^*,\ldots,r_d^*, q^*$ are Hölder's conjugates relative to $p, r_1,\ldots,r_d, q$ and

$$
w_*(t,v) = w_0^{-1/(q-1)}(t)\prod_{i=1}^d w_i^{-1/(r_i-1)}(v_i).
$$

Then, by this and the convergence $u_n \to u$ in $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, we have

$$
\left|\int_{\mathbb{R}^{1+2d}_T}((-\Delta_x)^{1/6}u)\eta\,dz\right| \le \|\eta\|_{L_*}\varlimsup_{n\to\infty}\|(-\Delta_x)^{1/6}u_n\|.
$$

The above inequality combined with (3.46) gives (1.15) for $(-\Delta_x)^{1/6}u$.

Step 3: *removing the assumption* (3.37). Let u_ε be the convolution of u in the x variable with $\varepsilon^{-d}\zeta(\cdot/\varepsilon)$, where ζ is a smooth cutoff function with the unit integral. We note that

$$P_0 u_\varepsilon + \lambda u_\varepsilon = \operatorname{div}_v \vec{f}_\varepsilon + g_\varepsilon.$$

Furthermore, by (1.13), $(-\Delta_x)^{1/6}\zeta(\cdot/\varepsilon)$ satisfies the condition of Lemma A.7. Hence, due to the identity

$$(-\Delta_x)^{1/6} u_\varepsilon = \varepsilon^{-d} u * (-\Delta_x)^{1/6}\zeta(\cdot/\varepsilon)$$

and Lemma A.7, the condition (3.37) holds with u replaced by u_ε. Then, by what was proved above and Lemma A.7,

$$\begin{aligned}(3.48)\qquad &\|\lambda^{1/2}|u_\varepsilon| + |D_v u_\varepsilon| + |(-\Delta_x)^{1/6} u_\varepsilon|\| \\ &\le N\||\vec{f}_\varepsilon| + \lambda^{-1/2}|g_\varepsilon|\| \le N\||\vec{f}| + \lambda^{-1/2}|g|\|.\end{aligned}$$

By using a duality argument as in Step 2 and (3.48), we conclude that (3.37) and (1.15) hold for $\lambda > 0$.

Step 4: case $g \equiv 0$, $\lambda = 0$. For any $\lambda > 0$, we have

$$P_0 u + \lambda u = \operatorname{div}_v \vec{f} + \lambda u.$$

Then, by (1.15) with $\lambda > 0$, and λu in place of g,

$$\|\lambda^{1/2}|u| + |D_v u| + |(-\Delta_x)^{1/6} u|\| \le N\|f\| + \lambda^{1/2}\|u\|.$$

Taking the limit as $\lambda \downarrow 0$, we prove the desired bound.

To prove the assertion for the space $L_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i))$, we follow the above argument, only modifying the proof of the estimate for $(-\Delta_x)^{1/6}u$. In particular, in Steps 2–3, due to (1.13), for any $\alpha \in (-1, p-1)$ and any $\eta \in C_0^\infty(\mathbb{R}^{1+2d}_T)$, one has

$$(-\Delta_x)^{1/6}\eta \in L_{p^*;r_1^*,\dots,r_d^*}\left(\mathbb{R}^{1+2d}_T, |x|^{-\alpha/(p-1)} \prod_{i=1}^d w_i^{-1/(r_i-1)}(v_i)\right),$$

where the latter is defined by (1.6) with

$$p^*,\ r_1^*, \dots, r_d^*,\ -\alpha/(p-1),\ w_1^{-1/(r_1-1)}, \dots, w_d^{-1/(r_d-1)}$$

in place of $p, r_1, \dots, r_d, \alpha, w_1, \dots, w_d$, respectively. The theorem is proved. ◻

4. Proof of Theorem 1.15.

4.1. Proof of assertion (i). In this section, we prove the main result for the KFP equation in the space $\mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$. The assertion (iv) of Theorem 1.15 is proved along the lines of this section (see Remark 4.3). We start by proving a mean oscillation estimate, which generalizes the one in Proposition 3.11.

LEMMA 4.1. *Let* $\lambda \ge 0$, $\gamma_0 > 0$, $\nu \ge 2$, $p_1 \in (1,\infty)$, $\alpha \in (1, 3/2)$ *be numbers,* $T \in (-\infty, \infty]$, *and let* R_0 *be the constant in Assumption* 1.2 (γ_0). *Let* $u \in \mathbb{S}_{p_1}(\mathbb{R}^{1+2d}_T)$, $\vec{f}, g \in L_{p_1}(\mathbb{R}^{1+2d}_T)$ *be functions such that*

$$\mathcal{P}u + \lambda u = div_v \vec{f} + g. \qquad (4.1)$$

Then, under Assumptions 1.1–1.2 (γ_0)*, there exists a sequence of positive numbers* $\{c_k, k \geq 0\}$ *such that*

$$\sum_{k=0}^{\infty} c_k \leq N_0(d, p_1, \alpha),$$

and for any $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$ *and* $r \in (0, R_0/(4\nu))$,

$$\begin{aligned}
&\lambda^{1/2}\Big(|u-(u)_{Q_r(z_0)}|^{p_1}\Big)^{1/p_1}_{Q_r(z_0)} + \Big(|D_v u - (D_v u)_{Q_r(z_0)}|^{p_1}\Big)^{1/p_1}_{Q_r(z_0)} \\
&\quad \leq N\nu^{-1}\lambda^{1/2}(|u|^{p_1})^{1/p_1}_{Q_{\nu r}(z_0)} + N\nu^{-1}(|D_v u|^{p_1})^{1/p_1}_{Q_{\nu r}(z_0)} \\
&\qquad + N\nu^{-1}\sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/6}u|^{p_1})^{1/p_1}_{Q_{\nu r, 2^k \nu r}(z_0)} \\
&\qquad + N\nu^{(4d+2)/p_1}\sum_{k=0}^{\infty} 2^{-k}\big((|\vec{f}|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)} + \lambda^{-1/2}(|g|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)}\big) \\
&\qquad + N\nu^{(4d+2)/p_1}\gamma_0^{(\alpha-1)/(\alpha p_1)}\sum_{k=0}^{\infty} c_k(|D_v u|^{p_1\alpha})^{1/(p_1\alpha)}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)},
\end{aligned}$$

$$\begin{aligned}
&\Big(|(-\Delta_x)^{1/6}u - ((-\Delta_x)^{1/6}u)_{Q_r(z_0)}|^{p_1}\Big)^{1/p_1}_{Q_r(z_0)} \\
&\quad \leq N\nu^{-1}(|(-\Delta_x)^{1/6}u|^{p_1})^{1/p_1}_{Q_{\nu r}(z_0)} \\
&\qquad + N\nu^{(4d+2)/p_1}\sum_{k=0}^{\infty} 2^{-k}\big((|\vec{f}|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)} + \lambda^{-1/2}(|g|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)}\big) \\
&\qquad + N\nu^{(4d+2)/p_1}\gamma_0^{(\alpha-1)/(\alpha p_1)}\sum_{k=0}^{\infty} c_k(|D_v u|^{p_1\alpha})^{1/(p_1\alpha)}_{Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0)},
\end{aligned}$$

where $N = N(d, \delta, p_1, \alpha)$.

Proof. Clearly, we may assume that $D_v u \in L_{p_1\alpha}(Q_{2\nu r, 2^{k+1}(2\nu r)}(z_0))$ for any $k \geq 0$. Thanks to Lemma 3.10, we may also assume that $z_0 = 0$.

We introduce

$$\bar{a}(t) = (a(t,\cdot,\cdot))_{B_{(2\nu r)^3}\times B_{2\nu r}} \quad \text{and} \quad \bar{P}_0 = \partial_t - v\cdot D_x - \bar{a}^{ij}D_{v_i v_j}.$$

Observe that u satisfies $\bar{P}_0 u + \lambda u = \operatorname{div}_v(\vec{f} + (a-\bar{a})D_v u) + g$. By this and Proposition 3.11,

$$\begin{aligned}
&\lambda^{1/2}\Big(|u-(u)_{Q_r(z_0)}|^{p_1}\Big)^{1/p_1}_{Q_r} + \Big(|D_v u - (D_v u)_{Q_r(z_0)}|^{p_1}\Big)^{1/p_1}_{Q_r} \\
&\quad \leq N\nu^{-1}(|D_v u|^{p_1})^{1/p_1}_{Q_{\nu r}} + N\nu^{-1}\sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/6}u|^{p_1})^{1/p_1}_{Q_{\nu r, 2^k \nu r}} \\
&\qquad + N\nu^{(4d+2)/p_1}\sum_{k=0}^{\infty} 2^{-k}\big((|\vec{f}|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}} + \lambda^{-1/2}(|g|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}}\big) \\
&\qquad + N\nu^{(4d+2)/p_1}\sum_{k=0}^{\infty} 2^{-k}(|a-\bar{a}|^{p_1}|D_v u|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}}.
\end{aligned}$$

Using Hölder's inequality with α and $\alpha_1 := \alpha/(\alpha-1)$ gives

$$I := (|a-\bar{a}|^{p_1}\,|D_v u|^{p_1})^{1/p_1}_{Q_{2\nu r, 2^{k+1}(2\nu r)}}$$
$$\le (|a-\bar{a}|^{p_1\alpha_1})^{1/(p_1\alpha_1)}_{Q_{2\nu r, 2^{k+1}(2\nu r)}} (|D_v u|^{p_1\alpha})^{1/(p_1\alpha)}_{Q_{2\nu r, 2^{k+1}(2\nu r)}} =: I_1^{1/(p_1\alpha_1)} I_2^{1/(p_1\alpha)}.$$

Due to the boundedness of the function a, we have

$$I_1 \le N(|a-\bar{a}|)_{Q_{2\nu r, 2^{k+1}(2\nu r)}}.$$

Furthermore, since $2\nu r \le R_0/2$, by Lemma A.6 with $c = 2^{k+1}$,

$$I_1 \le N 2^{3k}\gamma_0,$$

and then,

$$2^{-k} I_1^{1/(p_1\alpha_1)} \le N 2^{-k+3k/(p_1\alpha_1)} \gamma_0^{1/(p_1\alpha_1)}.$$

We set $c_k = 2^{-k+3k/(p_1\alpha_1)}, k \ge 0$, and note that $\sum_k c_k < \infty$, since $\alpha_1 > 3$. The estimate for $(-\Delta_x)^{1/6}u$ is established in the same way. The lemma is proved. ∎

In the next lemma, we prove the a priori estimate (1.15) with $b \equiv \bar{b} \equiv 0$, $c \equiv 0$, and compactly supported $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, $\vec{f}, g \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$.

Lemma 4.2. *Let*
- $\lambda > 0$, $p, r_1, \dots, r_d, q > 1, K \ge 1$ *be numbers*, $T \in (-\infty, \infty]$*;*
- $w_i, i = 0, 1, \dots, d$, *be weights on* $\mathbb{R}$ *satisfying* (1.14)*;*
- *Assumption* 1.1 *be satisfied;*
- *the functions* $u \in \mathbb{S}_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, $\vec{f}, g \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ *have compact supports and satisfy* (4.1).

Then, there exists a number $\gamma_0 = \gamma_0(d, \delta, p, r_1, \dots, r_d, q, K) > 0$ *such that, under Assumption* 1.2 (γ_0)*, we have*

$$\lambda^{1/2}\|u\| + \|D_v u\| + \|(-\Delta_x)^{1/6}u\| \le N\|\vec{f}\| + N\lambda^{-1/2}\|g\| + N\lambda^{-1/2}R_0^{-2}\|u\|, \tag{4.2}$$

where $\|\cdot\| = \|\cdot\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)}$, $N = N(d, \delta, p, r_1, \dots, r_d, q, K)$, *and* $R_0 \in (0,1)$ *is the constant in Assumption* 1.2 (γ_0).

Proof. Step 1: *estimate of a function with a small support in t.* Let $R_1, \gamma_0 > 0$ be numbers which we will choose later. We assume, additionally, that u vanishes outside $(s-(R_0R_1)^2, s) \times \mathbb{R}^{2d}$ for some $s \in \mathbb{R}$. The small support in time restriction will be removed in Step 2.

Let p_0 be the number satisfying (3.38)–(3.39). Then, since $u, \vec{f}, g$ have compact supports, we have $u, D_v u, \vec{f}, g \in L_{p_0}(\mathbb{R}^{1+2d}_T)$, and then by (4.1), $\partial_t u - v\cdot D_x u \in \mathbb{H}^{-1}_{p_0}(\mathbb{R}^{1+2d}_T)$, so that $u \in \mathbb{S}_{p_0}(\mathbb{R}^{1+2d}_T)$. By Corollary 2.2, $(-\Delta_x)^{1/6}u \in L_{p_0}(\mathbb{R}^{1+2d}_T)$.

We fix some $\nu \ge 2$, $\alpha \in (1, \min\{3/2, p_0\})$, and denote $p_1 = p_0/\alpha$, so that $p_0 = \alpha p_1$. If $4\nu r \ge R_0$, then by Hölder's inequality with α and $\alpha_1 = \alpha/(\alpha-1)$, for any function $h \in L_{\alpha p_1, \mathrm{loc}}(\mathbb{R}^{1+2d}_T)$ vanishing outside $(s-(R_0R_1)^2, s) \times \mathbb{R}^{2d}$ and $z \in \mathbb{R}^{1+2d}_T$,

$$(|h-(h)_{Q_r(z)}|^{p_1})^{1/p_1}_{Q_r(z)} \le 2(|h|^{p_1})^{1/p_1}_{Q_r(z)}$$
$$\le 2(I_{(s-(R_0R_1)^2,s)})^{1/(p_1\alpha_1)}_{Q_r(z)} (|h|^{p_1\alpha})^{1/(p_1\alpha)}_{Q_r(z)}$$
$$\le 2(R_0R_1r^{-1})^{2/(p_1\alpha_1)} \mathcal{M}_T^{1/(p_1\alpha)}|h|^{p_1\alpha}(z)$$
$$\le N\nu^{2/(p_1\alpha_1)} R_1^{2/(p_1\alpha_1)} \mathcal{M}_T^{1/(p_1\alpha)}|h|^{p_1\alpha}(z).$$

In the case when $4\nu r < R_0$, we use Lemma 4.1, which is applicable, since $\vec{f}, g \in L_{p_1}(\mathbb{R}^{1+2d}_T)$, and $u \in \mathbb{S}_{p_1}(\mathbb{R}^{1+2d}_T)$. Combining these cases, we get, in $\overline{\mathbb{R}^{1+2d}_T}$,

$$\begin{aligned}
&\lambda^{1/2}(u)^{\#}_T + (D_v u)^{\#}_T \\
&\quad \le N\nu^{2/(p_1\alpha_1)} R_1^{2/(p_1\alpha_1)} (\lambda^{1/2}\mathcal{M}_T^{1/(p_1\alpha)}|u|^{p_1\alpha} + \mathcal{M}_T^{1/(p_1\alpha)}|D_v u|^{p_1\alpha}) \\
&\qquad + N\nu^{-1}(\lambda^{1/2}\mathcal{M}_T^{1/p_1}|u|^{p_1} + \mathcal{M}_T^{1/p_1}|D_v u|^{p_1}) \\
&\qquad + N\nu^{-1}\sum_{k=0}^{\infty} 2^{-2k}\mathbb{M}^{1/p_1}_{2^k,T}|(-\Delta_x)^{1/6}u|^{p_1} \\
&\qquad + N\nu^{(4d+2)/p_1}\gamma_0^{1/(p_1\alpha_1)}\sum_{k=0}^{\infty} c_k \mathbb{M}^{1/(p_1\alpha)}_{2^{k+1},T}|D_v u|^{p_1\alpha} \\
&\qquad + N\nu^{(4d+2)/p_1}\sum_{k=0}^{\infty} 2^{-k}(\mathbb{M}^{1/p_1}_{2^{k+1},T}|\vec{f}|^{p_1} + \lambda^{-1/2}\mathbb{M}^{1/p_1}_{2^{k+1},T}|g|^{p_1}),
\end{aligned}$$

and

$$\begin{aligned}
&((-\Delta_x)^{1/6}u)^{\#}_T \\
&\quad \le N\nu^{2/(p_1\alpha_1)} R_1^{2/(p_1\alpha_1)}\mathcal{M}_T^{1/(p_1\alpha)}|(-\Delta_x)^{1/6}u|^{p_1\alpha} \\
&\qquad + N\nu^{-1}\mathcal{M}_T^{1/p_1}|(-\Delta_x)^{1/6}u|^{p_1} \\
&\qquad + N\nu^{(4d+2)/p_1}\gamma_0^{1/(p_1\alpha_1)}\sum_{k=0}^{\infty} c_k \mathbb{M}^{1/(p_1\alpha)}_{2^{k+1},T}|D_v u|^{p_1\alpha} \\
&\qquad + N\nu^{(4d+2)/p_1}\sum_{k=0}^{\infty} 2^{-k}(\mathbb{M}^{1/p_1}_{2^{k+1},T}|\vec{f}|^{p_1} + \lambda^{-1/2}\mathbb{M}^{1/p_1}_{2^{k+1},T}|g|^{p_1}),
\end{aligned}$$

where $(f)^{\#}_T$, $\mathbb{M}_{c,T} f$, and $\mathcal{M}_T f$ are defined as in (1.2). We take the $\|\cdot\|$-norm of both sides of the above inequalities and use the Minkowski inequality. Then, by (3.39) with $p_0 = p_1\alpha$ and Theorem A.3 with

$$p/(p_1\alpha), r_1/(p_1\alpha), \dots r_d/(p_1\alpha), q/(p_1\alpha) > 1,$$

we obtain

$$\begin{aligned}
\lambda^{1/2}\|u\| + \|D_v u\| &\le N(\nu^{-1} + \nu^{2/(p_1\alpha_1)} R_1^{2/(p_1\alpha_1)})(\lambda^{1/2}\|u\| + \|D_v u\|) \\
&\quad + N\nu^{-1}\|(-\Delta_x)^{1/6}u\| + N\nu^{(4d+2)/p_1}\gamma_0^{1/(p_1\alpha_1)}\|D_v u\| \\
&\quad + N\nu^{(4d+2)/p_1}(\|\vec{f}\| + \lambda^{-1/2}\|g\|),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\|(-\Delta_x)^{1/6}u\| &\le N(\nu^{-1} + \nu^{2/(p_1\alpha_1)} R_1^{2/(p_1\alpha_1)})\|(-\Delta_x)^{1/6}u\| \\
&\quad + N\nu^{(4d+2)/p_1}\gamma_0^{1/(p_1\alpha_1)}\|D_v u\| + N\nu^{(4d+2)/p_1}(\|\vec{f}\| + \lambda^{-1/2}\|g\|).
\end{aligned} \tag{4.4}$$

Taking $\nu \ge 2 + 4N$ first, then choosing $R_1, \gamma_0 > 0$ sufficiently small such that

$$N\nu^{(4d+2)/p_1}\gamma_0^{1/(p_1\alpha_1)} + N\nu^{2/(p_1\alpha_1)} R_1^{2/(p_1\alpha_1)} < 1/4,$$

and using the fact that $(-\Delta_x)^{1/6}u \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ (see Corollary 3.2), we obtain from (4.4) that

$$\|(-\Delta_x)^{1/6}u\| \le (1/2)\|D_v u\| + N\nu^{(4d+2)/p_1}(\|\vec{f}\| + \lambda^{-1/2}\|g\|). \tag{4.5}$$

By this, (4.3), and our choice of ν, R_1, and γ_0, we get

$$\lambda^{1/2}\|u\| + \|D_v u\| \le (5/8)(\lambda^{1/2}\|u\| + \|D_v u\|) + N\nu^{(4d+2)/p_1}(\|\vec{f}\| + \lambda^{-1/2}\|g\|),$$

which implies

$$\lambda^{1/2}\|u\| + \|D_v u\| \le N(\|\vec{f}\| + \lambda^{-1/2}\|g\|). \tag{4.6}$$

This, combined with (4.5), gives

$$\|(-\Delta_x)^{1/6}u\| \le N(\|\vec{f}\| + \lambda^{-1/2}\|g\|). \tag{4.7}$$

Step 2: *partition of unity.* Let $\zeta \in C_0^\infty((-(R_0R_1)^2, 0))$ be a nonnegative function such that

$$\int \zeta^q(t)\,dt = 1, \quad |\zeta'| \le N_0(R_0R_1)^{-2-2/q}. \tag{4.8}$$

Observe that for any $t \in \mathbb{R}_T$ and $U \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, by (4.8),

$$\begin{aligned}&\|U(t,\cdot)\|^q_{L_{p,r_1,\dots,r_d}(\mathbb{R}^{2d},\Pi_{i=1}^d w_i)}\\ &\quad = \int_{\mathbb{R}} \|U(t,\cdot)\|^q_{L_{p,r_1,\dots,r_d}(\mathbb{R}^{2d},\Pi_{i=1}^d w_i)}\zeta^q(t-s)\,ds.\end{aligned}$$

Multiplying the above identity by w_0 and integrating over $(-\infty, T]$, we get

$$\|U\|^q = \int_{\mathbb{R}} \|U\zeta(\cdot - s)\|^q\,ds. \tag{4.9}$$

Next, note that for any $s \in \mathbb{R}$, the function $u_s(z) := u(z)\zeta(t-s)$ vanishes outside $(s-(R_0R_1)^2, s)$ and satisfies the equation

$$\mathcal{P}u_s(z) + \lambda u_s(z) = \operatorname{div}_v(\vec{f}(z)\zeta(t-s)) + g(z)\zeta(t-s) + u\zeta'(t-s).$$

By (4.6) and (4.7) proved in Step 1,

$$\begin{aligned}&\lambda^{1/2}\|u_s\| + \|D_v u_s\| + \|(-\Delta_x)^{1/6}u_s\|\\ &\quad \le N\|\vec{f}\zeta(\cdot - s)\| + N\lambda^{-1/2}\|g\zeta(\cdot - s)\| + N(R_0R_1)^{-2-2/q}\lambda^{-1/2}\|u\xi(\cdot - s)\|,\end{aligned}$$

where $\xi \in C_0^\infty(\mathbb{R})$ is a nonnegative function such that $\xi = 1$ on the support of ζ, and $\int \xi^q(t)\,dt = N_1(R_0R_1)^2$. Raising the above inequality to the power q, integrating over $s \in \mathbb{R}$, and using (4.9) and our choice of R_1, we prove (4.2). ◻

Remark 4.3. A version of Lemma 4.2 holds in the case when

$$\begin{aligned}&u \in \mathbb{S}_{p;r_1,\dots,r_d}\left(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right),\\ &\vec{f}, g \in L_{p;r_1,\dots,r_d}\left(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right),\end{aligned}$$

and these functions have compact supports. To prove the result, one needs to follow the proof of Lemma 4.2 but use the partition of unity in v_d instead of t. We give a few details. First, repeating the argument of Step 1, we prove the estimate (4.2) for

u satisfying (4.1) and vanishing outside $\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^{d-1}_{v_1,\dots,v_d} \times (s - R_0R_1, s)$ for some $s \in \mathbb{R}$. Furthermore, we fix a nonnegative function $\zeta \in C_0^\infty((-R_0R_1, 0))$ such that

$$\int \zeta^{r_d}\, dv_d = 1, \quad |\zeta'| < N(R_0R_1)^{-1-1/r_d}$$

and denote $u_s(z) = u(z)\zeta(v_d - s)$. The function u_s satisfies the identity

$$\begin{aligned}\mathcal{P}u_s(z) + \lambda u_s(z) = D_{v_i}[f^i(z)\zeta(v_d - s) - a^{id}(z)\zeta'(v_d - s)u(z)]\\ - f_d(z)\zeta'(v_d - s) + g\zeta(v_d - s) - a^{dj}(z)\zeta'(v_d - s)D_{v_j}u(z).\end{aligned}$$

As in the proof of Lemma 4.2, we obtain

$$\begin{aligned}&\lambda^{1/2}\|u\| + \|D_v u\| + \|(-\Delta_x)^{1/6}u\|\\ &\quad \le N\||\vec{f}| + R_0^{-1}|u| + \lambda^{-1/2}|g|\| + N\lambda^{-1/2}R_0^{-1}\||D_v u| + |f_d|\|.\end{aligned}$$

Taking λ sufficiently large, we may erase the terms involving u from the right-hand side of the above inequality.

Proof of Theorem 1.15 (i). First, we consider the case when $b \equiv \bar{b} \equiv 0$ and $c \equiv 0$. We will focus on the case when the weight is independent of the x variable, since in the remaining case, the proof is the same. Let ϕ_n, u_n be the functions defined by (3.40). Note that

$$\mathcal{P}u_n + \lambda u_n = \operatorname{div}_v \mathsf{f}_n + \mathsf{g}_n,$$

where

$$\begin{aligned}&\mathsf{f}_n = \vec{f}\phi_n - (aD_v\phi_n)u,\\ &\mathsf{g}_n = g\phi_n - \vec{f}\cdot D_v\phi_n + u(\partial_t\phi_n - v\cdot D_x\phi_n) - (aD_v\phi_n)\cdot D_v u,\end{aligned}$$

and, furthermore, the functions $\mathsf{f}_n, \mathsf{g}_n, u_n$ are compactly supported, and

$$\mathsf{f}_n, \mathsf{g}_n, u_n \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w).$$

Then, by Lemma 4.2, there exist $\gamma_0 = \gamma_0(d, \delta, p, r_1, \dots, r_d, q, K) > 0$ such that if Assumption 1.2 (γ_0) holds, then for any $\lambda > 0$,

$$\begin{aligned}&\|\lambda^{1/2}|u_n| + |D_v u_n| + |(-\Delta_x)^{1/6}u_n|\|\\ &\quad \le N\|\mathsf{f}_n\| + N\lambda^{-1/2}\||\mathsf{g}_n| + R_0^{-2}|u_n|\|,\\ &\quad \le \|\vec{f}\| + N\lambda^{-1/2}(\|g\| + R_0^{-2}\|u\|) + Nn^{-1}(1 + \lambda^{-1/2})\|u\|\\ &\qquad + Nn^{-1}\lambda^{-1/2}\||D_v u| + |\vec{f}|\|,\end{aligned}$$

where $N = N(d, \delta, p, r_1, \dots, r_d, q, K)$, and $R_0 \in (0,1)$ is the number in Assumption 1.2 (γ_0). Using a limiting argument as in Step 2 of the proof of Theorem 3.1, we conclude that

$$\|\lambda^{1/2}|u| + |D_v u| + |(-\Delta_x)^{1/6}u|\| \le N\|f\| + N\lambda^{-1/2}\|g\| + N\lambda^{-1/2}R_0^{-2}\|u\|.$$

Taking $\lambda > 2NR_0^{-2}$ so that $\lambda^{1/2} - N\lambda^{-1/2}R_0^{-2} > \lambda^{1/2}/2$, we prove (1.15).

In the general case, we rewrite (1.1) as

$$\mathcal{P}u + \lambda u = G + \operatorname{div}_v F, \quad G = g - b\cdot D_v u - cu, \quad F = \vec{f} - \bar{b}u.$$

Then, by (1.15) with F and G in place of $\vec{f}$ and g, we get

$$\begin{aligned}&\lambda^{1/2}\|u\|+\|D_v u\|+\|(-\Delta_x)^{1/6}u\|\\&\quad\le N\lambda^{-1/2}\|g\|+NL\lambda^{-1/2}(\|D_v u\|+\|u\|)+N\|\vec{f}\|+NL\|u\|.\end{aligned}$$

Taking $\lambda\ge 1+4(NL)^2$, we may drop the terms involving u on the right-hand side of the above inequality. The assertion (i) is proved. ◻

4.2. Proof of Theorem 1.15 (ii) and (iii). As we pointed out in Remark 1.16, the assertion (iii) follows directly from (ii). To prove the latter, we first establish the unique solvability result in $L_p(\mathbb{R}^{1+2d})$ spaces.

PROPOSITION 4.4. *Theorem* 1.15 (ii) *is satisfied in the case when* $p=r_1=\cdots=r_d=q$, $w\equiv 1$, *and* $T=\infty$.

Proof. The assertion follows from the method of continuity, Theorem 1.15 (i), and Theorem 2.1 (i). ◻

The following is a decay estimate for the solution to (1.1) with the compactly supported right-hand side, which is analogous to Lemma 3.3. This result is needed for establishing the existence part in Theorem 1.15 (ii).

LEMMA 4.5. *Invoke the assumptions of Proposition* 4.4, *and let* $\lambda_0=\lambda_0(d,\delta,p,L)>1$ *be the constant from the statement of this result. Assume, additionally, that* $\vec{f}$ *and* g *vanish outside* $\widetilde{Q}_R$ *for some* $R\ge 1$, *and let* $u\in\mathbb{S}_p(\mathbb{R}^{1+2d})$ *be the solution to* (1.1), *which exists and is unique due to the aforementioned proposition. Then, for any* $\lambda\ge\lambda_0$ *and* $j\in\{0,1,2,\dots\}$,

$$\begin{aligned}&\lambda^{1/2}\|u\|_{L_p(\widetilde{Q}_{2^{j+1}R}\setminus\widetilde{Q}_{2^jR})}+\|D_v u\|_{L_p(\widetilde{Q}_{2^{j+1}R}\setminus\widetilde{Q}_{2^jR})}\\&\quad\le N2^{-j(j-1)/4}R^{-j}\big(\|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}+\lambda^{-1/2}\|g\|_{L_p(\mathbb{R}^{1+2d})}\big),\end{aligned}$$

where $N=N(d,\delta,p,L)$.

Proof. The proof is similar to that of Lemma 7.4 in [15]. First, by Theorem 1.15 (i), we have

$$\lambda^{1/2}\|u\|_{L_p(\mathbb{R}^{1+2d})}+\|D_v u\|_{L_p(\mathbb{R}^{1+2d})}\le N\|\vec{f}\|_{L_p(\mathbb{R}^{1+2d})}+N\lambda^{-1/2}\|g\|_{L_p(\mathbb{R}^{1+2d})}. \tag{4.10}$$

Let $\eta_j, j\in\{0,1,2,\dots\}$, be a sequence of smooth functions such that $\eta_j=0$ in $\widetilde{Q}_{2^jR}$, $\eta_j=1$ outside $\widetilde{Q}_{2^{j+1}R}$,

$$\begin{aligned}&|\eta_j|\le 1,\quad |D_v\eta_j|\le N2^{-j}R^{-1},\quad |D_v^2\eta_j|\le N2^{-2j}R^{-2},\\&|D_x\eta_j|\le N2^{-3j}R^{-3},\quad |\partial_t\eta_j|\le N2^{-2j}R^{-2}.\end{aligned}\tag{4.11}$$

Note that $u_j=u\eta_j$ satisfies the equation

$$\begin{aligned}&\mathcal{P}u_j+\operatorname{div}_v(\bar{b}u_j)+b^iD_{v_i}u_j+cu_j+\lambda u_j=\operatorname{div}_v[-u\,(aD_v\eta_j)]\\&\quad-(aD_v\eta_j)\cdot D_v u+u(\partial_t\eta_j-v\cdot D_x\eta_j+b\cdot D_v\eta_j+\bar{b}\cdot D_v\eta_j)\end{aligned}$$

because $\vec{f}$ and g vanish outside $\widetilde{Q}_R$. Then, by the a priori estimate in Theorem 1.15 (i), (4.11), and the fact that $\lambda>1$, we get

$$\begin{aligned}&\|\lambda^{1/2}|u|+|D_v u|\|_{L_p(\widetilde{Q}_{2^{j+2}R}\setminus\widetilde{Q}_{2^{j+1}R})}\\&\quad\le N\|u\,(aD_v\eta_j)\|_{L_p(\mathbb{R}^{1+2d})}+N\lambda^{-1/2}\|(aD_v\eta_j)\cdot D_v u\|_{L_p(\mathbb{R}^{1+2d})}\\&\qquad+N\lambda^{-1/2}\|u(\partial_t\eta_j-v\cdot D_x\eta_j+(b+\bar{b})\cdot D_v\eta_j)\|_{L_p(\mathbb{R}^{1+2d})}\\&\quad\le N2^{-j}R^{-1}\|\lambda^{1/2}|u|+|D_v u|\|_{L_p(\widetilde{Q}_{2^{j+1}R}\setminus\widetilde{Q}_{2^jR})}.\end{aligned}$$

Iterating the above estimate and using (4.10), we prove the assertion. ◻

Proof of Theorem 1.15 (ii). The uniqueness follows from Theorem 1.15 (i). To prove the existence part, we follow the proof of Theorem 2.5 of [15]. Next, we delineate the argument.

First, we consider the case $T = \infty$. By using the reverse Hölder inequality for the A_p-weights and the scaling property of the A_p-weights (see, for instance, Chapter 7 in [17]), one can show that there exists a sufficiently large number $p_1 = p_1(d, p, r_1, \ldots, r_d, q, K) \in (1, \infty)$ such that, for any $h \in L_{p_1,\mathrm{loc}}(\mathbb{R}^{1+2d})$, one has

$$\|h\|_{L_{p,r_1,\ldots,r_d,q}(\widetilde{Q}_R, w)} \le N R^{\kappa} \|h\|_{L_{p_1}(\widetilde{Q}_R)}, \tag{4.12}$$

where $\kappa, N > 0$ are independent of R and h. In addition, the above inequality also holds with $\widetilde{Q}_{2R} \setminus \widetilde{Q}_R$ in place of $\widetilde{Q}_R$.

Next, let $\vec{f}_n$, $g_n, n \ge 1$, be sequences of $C_0^\infty(\mathbb{R}^{1+2d})$ functions converging to $\vec{f}$ and g in $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}, w)$, respectively. Then, by Proposition 4.4, for any n, the equation

$$\mathcal{P} u_n + \mathrm{div}_v(\bar{b} u_n) + b \cdot D_v u_n + (c + \lambda) u_n = \mathrm{div}_v \vec{f}_n + g_n \tag{4.13}$$

has a unique solution $u_n \in \mathbb{S}_{p_1}(\mathbb{R}^{1+2d})$. Fix any n, and let $R = R_n \ge 1$ be a constant such that $\vec{f}_n$ and g_n vanish outside $\widetilde{Q}_R$. Then, by (4.12) combined with Lemma 4.5, for any $j \in \{0, 1, 2, \ldots\}$,

$$\begin{aligned}
&\|\lambda^{1/2}|u_n| + |D_v u_n|\|_{L_{p,r_1,\ldots,r_d,q}(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R}, w)} \\
&\quad \le N (2^j R)^{\kappa} \|\lambda^{1/2}|u_n| + |D_v u_n|\|_{L_{p_1}(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})} \\
&\quad \le N (2^j R)^{\kappa} 2^{-j(j-1)/4} R^{-j} \big(\|\vec{f}_n\|_{L_{p_1}(\mathbb{R}^{1+2d})} + \lambda^{-1/2} \|g_n\|_{L_{p_1}(\mathbb{R}^{1+2d})}\big).
\end{aligned}$$

The above inequality implies that $u_n \in \mathbb{S}_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}, w)$. Hence, by Theorem 1.15 (i), $u_n, n \ge 1$, is a Cauchy sequence in $\mathbb{S}_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}, w)$ and has a limit u. Passing to the limit in (4.13), we conclude the existence of the unique solution to (1.1).

The case $T < \infty$ is treated as in the proof of Theorem 2.1 (i). ∎

5. Proof of Theorem 1.23.

In the next two lemmas, we prove energy identities for the operator

$$Y u := \partial_t u + \alpha(\mathsf{v}) \cdot D_x u.$$

For $T \in (-\infty, \infty]$, let $H_2^1(\mathbb{R}_T^{1+d+d_1})$ be the space of functions $u \in L_2(\mathbb{R}_T^{1+d+d_1})$ such that $D_{\mathsf{v}} u \in L_2(\mathbb{R}_T^{1+d+d_1})$, and let $\langle \cdot, \cdot \rangle_T$ be the duality pairing between $\mathbb{H}_2^{-1}(\mathbb{R}_T^{1+d+d_1})$ and $H_2^1(\mathbb{R}_T^{1+d+d_1})$ given by

$$\langle f, g \rangle_T = \int_{-\infty}^{T} \int_{\mathbb{R}^d} [f(t, x, \cdot), g(t, x, \cdot)] \, dx dt, \tag{5.1}$$

where

$$[f, g] = \int_{\mathbb{R}^{d_1}} \big((1 - \Delta_{\mathsf{v}})^{-1/2} f\big) \big((1 - \Delta_{\mathsf{v}})^{1/2} g\big) \, d\mathsf{v}.$$

LEMMA 5.1. *Let $u \in H_2^1(\mathbb{R}^{1+d+d_1})$ be a function such that $Y u \in \mathbb{H}_2^{-1}(\mathbb{R}^{1+d+d_1})$. Then,*

$$\langle Y u, u \rangle_\infty = 0.$$

Proof. For a distribution h on $\mathbb{R}^{1+d+d_1}$, a cutoff function $\eta \in C_0^\infty(\mathbb{R}^{1+d+d_1})$ with the unit integral, and $\varepsilon > 0$, we denote

$$h_\varepsilon(t,x,\mathsf{v}) = \varepsilon^{-(1+\theta d/2+d_1)}(h, \eta((t-\cdot)/\varepsilon, (x-\cdot)/\varepsilon^{\theta/2}, (\mathsf{v}-\cdot)/\varepsilon)), \tag{5.2}$$

where (h,η) is the action of h on η. For the sake of convenience, we omit $\mathbb{R}^{1+d+d_1}$ in the notation of functional spaces and write $\langle \cdot,\cdot\rangle = \langle \cdot,\cdot\rangle_\infty$. First, we split $\langle Yu, u\rangle$ as follows:

$$\begin{aligned}\langle Yu, u\rangle &= \langle Yu_\varepsilon, u_\varepsilon\rangle + \langle Yu - (Yu)_\varepsilon, u\rangle \\ &\quad + \langle (Yu)_\varepsilon - Yu_\varepsilon, u\rangle + \langle Yu_\varepsilon, u - u_\varepsilon\rangle \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Since u_ε is a smooth function vanishing at infinity, one has $I_1 = 0$. Furthermore, by the properties of Bessel potential spaces (see, for example, Theorem 13.9.2 in [24]),

$$|I_2| \le \|Yu - (Yu)_\varepsilon\|_{\mathbb{H}_2^{-1}} \|u\|_{H_2^1} \to 0 \quad \text{as } \varepsilon \to 0.$$

Next, note that

$$\begin{aligned}&\big((Yu)_\varepsilon - Yu_\varepsilon\big)(t,x,\mathsf{v}) \\ &\quad = \varepsilon^{-\theta/2} \int (\alpha(\mathsf{v} - \varepsilon \mathsf{v}') - \alpha(\mathsf{v})) \cdot D_x \eta(t', x', \mathsf{v}') u(t - \varepsilon t', x - \varepsilon^{\theta/2} x', \mathsf{v} - \varepsilon \mathsf{v}')\, dx'\, d\mathsf{v}'\, dt',\end{aligned}$$

and, then, by the Minkowski inequality and Assumption 1.20, we have

$$\|(Yu)_\varepsilon - Yu_\varepsilon\|_{L_2} \le \varepsilon^{\theta/2} \|u\|_{L_2} \to 0 \tag{5.3}$$

as $\varepsilon \to 0$, which gives

$$I_3 \to 0 \quad \text{as } \varepsilon \to 0.$$

Next, by duality,

$$I_4 \le \|Yu_\varepsilon\|_{\mathbb{H}_2^{-1}} \|u - u_\varepsilon\|_{H_2^1}.$$

By (5.3), for sufficiently small $\varepsilon > 0$, the first factor on the right-hand side is bounded by

$$\|(Yu)_\varepsilon\|_{\mathbb{H}_2^{-1}} + \|(Yu)_\varepsilon - Yu_\varepsilon\|_{\mathbb{H}_2^{-1}} \le \|Yu\|_{\mathbb{H}_2^{-1}} + \|u\|_{L_2}.$$

Thus, by this and the fact that

$$\|u - u_\varepsilon\|_{H_2^1} \to 0 \quad \text{as } \varepsilon \to 0,$$

we conclude that $I_4 \to 0$ as $\varepsilon \to 0$. The lemma is proved. ∎

LEMMA 5.2. *Let $T \in \mathbb{R}$, $u \in H_2^1(\mathbb{R}_T^{1+d+d_1})$, and $Yu \in \mathbb{H}_2^{-1}(\mathbb{R}_T^{1+d+d_1})$. Then, for a.e. $s \in (-\infty, T]$,*

$$\langle Yu, u\rangle_s = (1/2)\|u(s,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})},$$

where $\langle \cdot,\cdot\rangle_s$ is defined as in (5.1).

Proof. We extend u by 0 for $t>T$. Let ξ be a smooth function on $\mathbb{R}$ defined by

$$\begin{cases}\xi(t)=0, & t\le 1,\\ \xi(t)\in(0,1), & t\in(1,2),\\ \xi(t)=1, & t\ge 2.\end{cases}$$

We fix some $s\in(-\infty,T]$ and denote $\xi_\varepsilon(\cdot)=\xi((s-\cdot)/\varepsilon)$, $u_\varepsilon=u\xi_\varepsilon$. It follows that $u_\varepsilon\in H^1_2(\mathbb{R}^{1+d+d_1})$ and $Yu_\varepsilon=\xi_\varepsilon(Yu)+u\xi'_\varepsilon\in\mathbb{H}^{-1}_2(\mathbb{R}^{1+d+d_1})$. Then, by Lemma 5.1,

$$\langle Yu_\varepsilon,u_\varepsilon\rangle_s=0,$$

which gives

$$\langle (Yu)\xi_\varepsilon,u_\varepsilon\rangle_s=-(1/2)\int_{\mathbb{R}^{1+d+d_1}_s}u^2(\xi^2_\varepsilon)'\,dxd\mathsf{v}dt. \tag{5.4}$$

The integral on the left-hand side of (5.4) equals

$$\int_{\mathbb{R}^{1+d}_s}[(Yu)(t,x,\cdot),u(t,x,\cdot)]\xi^2_\varepsilon(t)\,dxdt\to\int_{\mathbb{R}^{1+d}_s}[(Yu)(t,x,\cdot),u(t,x,\cdot)]\,dxdt$$

as $\varepsilon\to 0$ by the dominated convergence theorem.

Note that $\int_{-\infty}^s(\xi^2_\varepsilon)'\,dt=-1$. Then, the right-hand side of (5.4) is equal to

$$\frac12\|u(s,\cdot,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})}-\frac12\int_{-\infty}^s(\|u(t,\cdot,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})}-\|u(s,\cdot,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})})(\xi^2_\varepsilon(t))'\,dt.$$

The last term is bounded by

$$N\varepsilon^{-1}\int_{s-2\varepsilon}^{s-\varepsilon}\big|\|u(t,\cdot,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})}-\|u(s,\cdot,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})}\big|\,dt.$$

By the Lebesgue differentiation theorem, the above expression converges to 0 as $\varepsilon\to 0$ for a.e. $s\in(-\infty,T]$. ∎

Proof of Theorem 1.23. First, note that by Remark 1.16, we only need to prove the assertion (i).

(i) By pairing both sides of (1.16) with $2u$ and using Lemma 5.2, the Cauchy–Schwarz inequality, and Assumptions 1.1 and 1.4, for a.e. $s\in(-\infty,T]$, we obtain

$$\begin{aligned}&\|u(s,\cdot,\cdot)\|^2_{L_2(\mathbb{R}^{d+d_1})}+\delta\|D_\mathsf{v}u\|^2_{L_2(\mathbb{R}^{1+d+d_1}_s)}+(\lambda-N_1)\|u\|^2_{L_2(\mathbb{R}^{1+d+d_1}_s)}\\&\quad\le N\|\vec f\|^2_{L_2(\mathbb{R}^{1+d+d_1}_s)}+N\lambda^{-1}\|g\|^2_{L_2(\mathbb{R}^{1+d+d_1}_s)},\end{aligned} \tag{5.5}$$

where $N_1=N_1(d,d_1,\delta,L)$ and $N=N(d,d_1,\delta)$. Taking $\lambda\ge 2N_1$, we may replace $\lambda-N_1$ with $\lambda/2$. Finally, by this and the fact that (5.5) holds for a.e. $s\in(-\infty,T]$, the desired estimate (1.17) is valid, which also implies the uniqueness part of assertion (i).

To prove the existence, due to the method of continuity and the a priori estimate (1.17), we only need to prove that $(Y-\Delta_\mathsf{v}+\lambda)C^\infty(\mathbb{R}^{1+d+d_1})$ is dense in $\mathbb{H}^{-1}_2(\mathbb{R}^{1+d+d_1})$ for $\lambda>0$. Assume the opposite is true. Then, by duality, there exists a nonzero $u\in H^1_2(\mathbb{R}^{1+d+d_1})$ such that the equality

$$-Yu-\Delta_\mathsf{v}u+\lambda u=0$$

holds in the sense of distributions. Mollifying the above equation with the mollifier defined in (5.2) gives

$$-Yu_\varepsilon - \Delta_{\mathsf{v}} u_\varepsilon + \lambda u_\varepsilon = (Yu)_\varepsilon - Yu_\varepsilon.$$

Then, replacing t with $-t$ in the a priori estimate (1.17) and using (5.3), we get

$$\lambda \|u_\varepsilon\|_{L_2(\mathbb{R}^{1+d+d_1})} \le N\varepsilon^{\theta/2}\|u\|_{L_2(\mathbb{R}^{1+d+d_1})}.$$

Passing to the limit as $\varepsilon \to 0$ in the above inequality, we conclude $u \equiv 0$, which gives a contradiction. The theorem is proved. ☐

Appendix A.

LEMMA A.1 (Lemma A.1 in [15]). *Let $\sigma > 0$, $R > 0$, $p \ge 1$ be numbers, and let $f \in L_{p,loc}(\mathbb{R}^d)$. Denote*

$$g(x) = \int_{|y|>R^3} f(x+y)|y|^{-(d+\sigma)}\,dy.$$

Then,

$$(|g|^p)^{1/p}_{B_{R^3}} \le N(d,\sigma)R^{-3\sigma}\sum_{k=0}^{\infty} 2^{-3k\sigma}(|f|^p)^{1/p}_{B_{(2^kR)^3}}.$$

LEMMA A.2. *Let $s \in (0, 1/2)$. Then, the following assertions hold.*

(i) *One has*

$$D_x(-\Delta_x)^{-s}u(x) = N(d,s)\, p.v. \int u(x-y)\frac{y}{|y|^{d-2s+2}}\,dy,\ u \in \mathcal{S}(\mathbb{R}^d). \tag{A.1}$$

This formula also holds for $u \in C^1_0(\mathbb{R}^d)$ (see Definition 1.9).

(ii) *For any $u \in C^2_0(\mathbb{R}^d)$,*

$$\big(D_x(-\Delta_x)^{-s}\big)\big((-\Delta_x)^s u\big) \equiv D_x u. \tag{A.2}$$

Proof. It is well known that for any $u \in \mathcal{S}(\mathbb{R}^d)$ (see, for example, Chapter 5 of [36]),

$$(-\Delta_x)^{-s}u(x) = N_0(d,s)\int u(x-y)\frac{1}{|y|^{d-2s}}\,dy.$$

Differentiating under the integral's sign and integrating by parts, we obtain

$$\begin{aligned} N_0^{-1}D_x(-\Delta_x)^{-s}u(x) &= \int D_x u(x-y)\frac{1}{|y|^{d-2s}}\,dy \\ &= -\lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon} D_y u(x-y)\frac{1}{|y|^{d-2s}}\,dy = -(d-2s)\lim_{\varepsilon\downarrow 0}\int_{|y|>\varepsilon} u(x-y)\frac{y}{|y|^{d-2s+2}}\,dy, \end{aligned}$$

which proves the first part of assertion (i).

Next, since $y|y|^{-d+2s-2}$ is an odd function, we have

$$\begin{aligned} \left|\int_{|y|>\varepsilon} u(x-y)\frac{y}{|y|^{d-2s+2}}\,dy\right| &\le \int_{|y|>1}|u(x-y)|\frac{dy}{|y|^{d-2s+1}}\,dy \\ &+ \int_{\varepsilon<|y|<1}|u(x-y)-u(x)|\frac{|y|}{|y|^{d-2s+2}}\,dy \le N(d,s)\|u\|_{C^1(\mathbb{R}^d)}. \end{aligned} \tag{A.3}$$

This bound combined with a limiting argument enables us to extend the formula (A.1) for $u \in C_0^1(\mathbb{R}^d)$.

(ii) First, for any $u \in C_0^2(\mathbb{R}^d)$, by (A.3) and (1.11),

$$\begin{aligned} &\|D_x(-\Delta_x)^{-s}\big((-\Delta_x)^s u\big)\|_{L_\infty(\mathbb{R}^d)} \\ &\quad \le N(d,s)\|(-\Delta_x)^s u\|_{C^1(\mathbb{R}^d)} \le N(d,s)\|u\|_{C^2(\mathbb{R}^d)}, \end{aligned} \tag{A.4}$$

so that the left-hand side of (A.2) is well defined. Furthermore, note that (A.2) holds if $u \in \mathcal{S}(\mathbb{R}^d)$. Then, the desired assertion follows from (A.4) and a limiting argument. ∎

THEOREM A.3 (Corollaries 3.2 and 3.5 of [15]). *Let $c \ge 1$, $K \ge 1$, $p, q, r_1, \dots, r_d > 1$ be numbers, $T \in (-\infty, \infty]$, and $f \in L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, where w is given by* (1.5), *and $w_i, i = 0, 1, \dots, d$ satisfy* (1.14). *Then, the following assertions hold.*

(i) *(Hardy–Littlewood type theorem)*

$$\|\mathbb{M}_{c,T} f\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)} \le N(d,p,q,r_1,\dots,r_d,K)\|f\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)}.$$

(ii) *(Fefferman–Stein type theorem)*

$$\|f\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)} \le N(d,p,q,r_1,\dots,r_d,K)\|f^{\#}_{c,T}\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}_T, w)}.$$

(iii) *For $\alpha \in (-1, p-1)$, the above inequalities also hold in the space*

$$L_{p;r_1,\dots,r_d}\left(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i)\right)$$

with $N = N(d,p,r_1,\dots,r_d,K,\alpha)$.

LEMMA A.4 (Lemma A.2 in [15]). *Let $p > 1, K \ge 1$ be numbers, $w \in A_p(\mathbb{R}^d)$ be such that $[w]_{A_p(\mathbb{R}^d)} \le K$, and $f \in L_p(\mathbb{R}^d, w)$. Then, there exists a number $p_0 = p_0(d,p,K) > 1$ such that $f \in L_{p_0,loc}(\mathbb{R}^d)$.*

LEMMA A.5. *Let $p \in (1,\infty)$ and $u \in L_p(\mathbb{R}^d)$ be a function such that $(-\Delta)^{1/3}u \in L_p(\mathbb{R}^d)$. Then, for any $\varepsilon > 0$,*

$$\|(-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}^d)} \le N\varepsilon\|(-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}^d)} + N\varepsilon^{-1}\|u\|_{L_p(\mathbb{R}^d)},$$

where $N = N(d,p)$.

Proof. It follows from the Hormander–Mikhlin multiplier theorem that $u \in H^{1/3}_p(\mathbb{R}^d)$, where the latter is the Bessel potential space (see the definition, for example, in Chapter 13 of [24]). Then, by the Hormander–Mikhlin multiplier theorem and the properties of the Bessel potential space (see, for example, [24]),

$$\begin{aligned} \|(-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}^d)} &\le N\|(1-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}^d)} \le N\|(1-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}^d)} \\ &\le N\|(-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}^d)} + N\|u\|_{L_p(\mathbb{R}^d)}. \end{aligned}$$

Now the desired assertion follows from the scaling argument. ∎

LEMMA A.6 (Lemma 7.2 in [15]). *Let $\gamma_0 > 0$ be a number and R_0 be the constant in Assumption* 1.2 (γ_0). *Let $r \in (0, R_0/2)$, $c > 0$ be numbers. Then, one has*

$$I := \fint_{Q_{r,cr}} |a(t,x,v) - (a(t,\cdot,\cdot))_{B_{r^3}\times B_r}|\,dz \le N(d)c^3\gamma_0.$$

LEMMA A.7. *Let $p \in (1,\infty), \alpha \in (-d, d(p-1))$ be numbers, $u \in L_p(\mathbb{R}^d, |x|^\alpha)$, and ξ be a measurable function satisfying the bound*

$$|\xi(y)| \le N_0(1+|y|)^{-d-\beta}, y \in \mathbb{R}^d,$$

for some $\beta > 0$. Let $\xi_\varepsilon = \varepsilon^{-d}\xi(\cdot/\varepsilon)$. Then, $u * \xi_\varepsilon \in L_p(\mathbb{R}^d, |x|^\alpha)$, and

$$\|u * \xi_\varepsilon\|_{L_p(\mathbb{R}^d,|x|^\alpha)} \le N(d,p,\alpha,\beta,N_0)\|u\|_{L_p(\mathbb{R}^d,|x|^\alpha)}. \tag{A.5}$$

Furthermore, if we assume, additionally, that $\xi \in C_0^\infty(\mathbb{R}^d)$ is a function with the unit integral, then $u * \xi_\varepsilon \to u$ in $L_p(\mathbb{R}^d, |x|^\alpha)$.

Proof. Note that for any $x \in \mathbb{R}^d$,

$$\begin{aligned} |u * \xi_\varepsilon(x)| &\le N \fint_{|y|<1} |u(x-\varepsilon y)|\,dy \\ &+ N\sum_{k=0}^{\infty} 2^{-\beta k} \fint_{2^k<|y|<2^{k+1}} |u(x-\varepsilon y)|\,dy \le NMu(x), \end{aligned} \tag{A.6}$$

where $N = N(d, N_0, \beta)$, and M is the usual Hardy–Littlewood maximal function. Since $|x|^\alpha, \alpha \in (-d, d(p-1))$ is an $A_p(\mathbb{R}^d)$-weight (see Remark 1.13), (A.5) follows from a version of the Hardy–Littlewood maximal inequality in weighted Lebesgue spaces (see [1]).

To prove the second assertion, we note that $u*\xi_\varepsilon$ converges to u as $\varepsilon \to 0$ a.e. due to Lemma A.4 and the Lebesgue differentiation theorem. Now the claim follows from (A.6) and the dominated convergence theorem. ∎

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