

On Parabolic and Elliptic Equations with Singular or Degenerate Coefficients

HONGJIE DONG & TUOC PHAN

ABSTRACT. We study parabolic and elliptic equations of both divergence and non-divergence form in the half space $\{x_d > 0\}$ whose coefficients are the product of x_d^α , and uniformly nondegenerate bounded measurable matrix-valued functions, where $\alpha \in (-1, \infty)$. As such, the coefficients are singular or degenerate near the boundary of the half space. For equations with the conormal or Neumann boundary condition, we prove the existence, uniqueness, and regularity of solutions in weighted Sobolev spaces and mixed-norm weighted Sobolev spaces when the coefficients are only measurable in the x_d direction and have small mean oscillation in the other directions in small cylinders. Our results are new even in the special case when the coefficients are constants.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the existence, uniqueness, and regularity estimates of solutions in Sobolev spaces to a class of parabolic (and elliptic) equations in the upper half space, whose coefficients can be singular or degenerate on the boundary of the upper half space in a way which may not satisfy the classical Muckenhoupt A_2 condition.

Throughout the paper, let $\Omega_T = (-\infty, T) \times \mathbb{R}_+^d$ be a space-time domain, where $T \in (-\infty, +\infty]$, $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times \mathbb{R}_+$ is the upper-half space. Let $(a_{ij}) : \Omega_T \rightarrow \mathbb{R}^{d \times d}$ be a matrix of measurable coefficients, which satisfies the following ellipticity and boundedness conditions: there is a constant $\kappa \in (0, 1)$ such that

$$(1.1) \quad \kappa |\xi|^2 \leq a_{ij}(t, x) \xi_i \xi_j \quad \text{and} \quad |a_{ij}(t, x)| \leq \kappa^{-1}$$

for every $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$ and $(t, x) = (t, x', x_d) \in \Omega_T$. Here, we do not impose the symmetry condition on (a_{ij}) . Let $\alpha \in (-1, \infty)$ be a fixed number. We investigate the conormal boundary value problem

$$(1.2) \quad \begin{cases} x_d^\alpha (u_t + \lambda u) - \operatorname{Di}[x_d^\alpha (a_{ij}(t, x) D_j u - F_i)] = \sqrt{\lambda} x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (a_{dj}(t, x) D_j u - F_d) = 0 \end{cases}$$

in Ω_T , where $F = (F_1, F_2, \dots, F_d) : \Omega_T \rightarrow \mathbb{R}^d$ and $f : \Omega_T \rightarrow \mathbb{R}$ are given measurable functions in suitable weighted Lebesgue spaces, and $\lambda \geq 0$ is a parameter. It is worth noting that the weight x_d^α satisfies the Muckenhoupt A_2 condition only if $\alpha \in (-1, 1)$. As a special case of our main results, for the model equation

$$(1.3) \quad \begin{cases} x_d^\alpha u_t - \operatorname{div}[x_d^\alpha (\nabla u - F)] = x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (D_d u - F_d) = 0 \end{cases}$$

in the upper-half parabolic cylinder Q_2^+ and for $\alpha \in (-1, \infty)$, we obtain the local boundary weighted estimate

$$\begin{aligned} & \left(\int_{Q_1^+} [|u|^p + |Du|^p] x_d^\alpha dz \right)^{1/p} \\ & \leq N \int_{Q_2^+} [|u| + |Du|] x_d^\alpha dz + N \left(\int_{Q_2^+} |F|^p x_d^\alpha dz \right)^{1/p} \\ & \quad + N \left(\int_{Q_2^+} |f|^{p^*} x_d^\alpha dz \right)^{1/p^*} \end{aligned}$$

for every $p \in (1, \infty)$, where $p^* \in [1, p)$ depending on α , p , and d as in (2.3)–(2.4) below and $N > 0$ is a constant depending on d , α , p , and p^* . Equation (1.3) is related to the extension problem of the fractional heat operator (see, e.g., [1, 32, 36]), and our result in this special case is already new.

We also consider the parabolic equation in non-divergence form

$$(1.4) \quad a_0(t, x) u_t - a_{ij}(t, x) D_{ij} u(t, x) - \frac{\alpha}{x_d} a_{dj}(t, x) D_j u(t, x) + \lambda c_0(t, x) u = f$$

in Ω_T with the boundary condition

$$(1.5) \quad \lim_{x_d \rightarrow 0^+} x_d^\alpha a_{dj}(t, x) D_j u(t, x', x_d) = 0,$$

where $a_0, c_0 : \Omega_T \rightarrow \mathbb{R}$ are measurable functions satisfying

$$(1.6) \quad \kappa \leq a_0(t, x), \quad c_0(t, x) \leq \kappa^{-1}, \quad (t, x) \in \Omega_T.$$

In this case, we impose an additional structural condition on the leading coefficients a_{ij} :

$$(1.7) \quad a_{dj}(t, x) = 0, \quad j = 1, 2, \dots, d-1,$$

or $a_{dj} = \lambda_j a_{dd}$ for $j = 1, 2, \dots, d-1$ with constants λ_j , which can be reduced to (1.7) after the change of variables $y_j = x_j - \lambda_j x_d$ for $j = 1, 2, \dots, d-1$ and $y_d = x_d$. We note that this condition is satisfied for a large class of equations (see, e.g., [1, 3, 14, 16]). Unlike (1.2), the equation (1.4) has extra coefficients a_0 and c_0 . The main reason we introduce them in (1.4) is for convenience because in the proofs of main results for (1.4)–(1.5), we divide both sides of (1.4) by a_{dd} to use the hidden divergence structure of the equation. Nevertheless, with a_0 and c_0 the equation (1.4) is slightly more general. Of course, in view of (1.6), by dividing both sides of (1.4) by a_0 or c_0 , one can always assume one of them to be the identity.

The interest of studying equations with singular or degenerate coefficients comes from both pure mathematics and applied problems. As examples, we refer the reader to [1, 3] for problems about fractional heat and fractional Laplace equations, [14] for problems arising in mathematical finance, [16, 19, 31] for the problems in geometric PDEs, and [5, 37] for problems from porous media. See also [13, 26, 30, 33] for some other classical studies of equations with singular degenerate coefficients. To put this work in perspectives, let us recall some known related results. In [11], we considered a class of parabolic equations in divergence form with a general weight

$$(1.8) \quad a_0(x_d)u_t - \frac{1}{\mu(x_d)}D_i[\mu(x_d)(a_{ij}D_j u - F_i)] + \lambda u = f$$

in the half space $\{x_d > 0\}$ with conormal boundary condition:

$$(1.9) \quad \lim_{x_d \rightarrow 0^+} \mu(x_d)(a_{dj}D_j u - F_d) = 0.$$

Here, (a_{ij}) satisfies (1.1), $a_0 \in [\kappa, \kappa^{-1}]$, $\lambda \geq 0$, and the weight μ satisfies the A_2 condition and a relaxed A_1 -type condition away from the boundary. This, in particular, includes the A_2 weights $\mu(x_d) = x_d^\alpha$ for any $\alpha \in (-1, 1)$. We obtained the local and global weighted Calderón-Zygmund type estimates for (1.8)–(1.9) with respect to the weight μ , under the condition that the coefficients are only measurable in the x_d direction and have small mean oscillation in the other directions in small cylinders (partially VMO) with respect to the considered weight. The proofs in [11] carry over to systems under the usual strong ellipticity condition. In [12], we studied the corresponding non-divergence form scalar equations (1.4), where $\alpha \in (-1, 1)$ and a_0, c_0 satisfy (1.6). Under the condition that a_0 , a_{ij} , and c_0 are partially VMO with respect to the weight x_d^α , we obtained weighted mixed-norm $W_p^{1,2}$ estimates and solvability. Among others, the results of this paper extend the

results in [11, 12] to the full range of exponent $\alpha \in (-1, \infty)$ in which the weight $\mu(x_d) = x_d^\alpha$ may no longer be in the class of the commonly used A_2 weights. It is worth noting that even for divergence-form equations, in contrast to [11], the proofs below only work for scalar equations because the Moser iteration is used (cf. Lemma 4.3). For other related work in this direction, we refer the reader to the references in [11, 12]. More discussions can be found in Remark 2.9 below.

The class of partially VMO coefficients was first introduced by Kim and Krylov [21, 22] for non-degenerate elliptic and parabolic equations that are in non-divergence form. Divergence-form elliptic and parabolic equations with non-degenerate partially VMO coefficients were later studied in [6, 7]. This type of equations arises from the problems of linearly elastic laminates and composite materials, for example, in homogenization of layered materials (see, e.g., [4]). We also refer the reader to [8–10] for extensions to second-order and higher-order systems with or without weights.

We apply a mean oscillation argument, which was used in [27] for non-degenerate parabolic equations with coefficients which are VMO in the space variables. In the case of partially VMO coefficients, the main difficulty is that, since they are merely measurable in x_d , it is only possible to estimate the mean oscillation of $D_{x'}u$, not the full gradient Du . Therefore, one needs to bound $D_d u$ by $D_{x'}u$. An idea in [6, 7] is to break the “symmetry” of the coordinates so that t and x_d are distinguished from x' by using a delicate re-scaling argument. Another idea is to estimate the mean oscillation of $a_{dd}D_d u$ instead of $D_d u$, and apply a generalized Fefferman-Stein theorem established in [28]. In [8], a new method was developed, in which the key step is to estimate $\mathcal{U} := a_{dj}D_j u$ and $D_i u$, $i = 1, \dots, d-1$, instead of the full gradient of u . By using this argument, one was able to bypass the scaling argument mentioned above and greatly simplified the proof. In this paper, we adapt this method to singular/degenerate equations.

In our main results, Theorems 2.2, 2.4, and 2.7 below, we obtain the unique solvability (1.2) and (1.4)–(1.5) in weighted Sobolev spaces and in mixed-norm weighted Sobolev spaces. Local boundary estimates for solutions of these equations are also obtained in Corollaries 2.3 and 2.10. To the best of our knowledge, these results are new even in the elliptic case and in the unmixed-norm case with constant coefficients a_{ij} , a_0 , and c_0 .

The proofs of the main theorems are based on an idea in [8] mentioned above and the perturbation technique. To implement the method, we first consider equations whose coefficients depend only on x_d , and prove various results on the existence, uniqueness, and regularity of solutions to this class of equations. For this, we establish the L_∞ estimate of weak solutions by applying the Moser iteration, and then derive Lipschitz- and Schauder-type estimates. In particular, to estimate the L_∞ norms of $D_d u$ and $\mathcal{U}$, we use a bootstrap argument. Schauder-type estimates for elliptic equations similar to (2.8) were proved recently in [34] when the matrix (a_{ij}) is symmetric, Hölder in all variables, and satisfies a structural condition that the hyperplane $\{x_d = 0\}$ is invariant with respect to (a_{ij}) , that is, $a_{jd} = d_{aj} = 0$ for $j = 1, \dots, d-1$. The proof in [34] uses a Liouville-type

theorem and a compactness argument. Our proof in Section 4 is more direct and works for more general operators. For the local estimates Corollaries 2.3 and 2.10, we prove a parabolic embedding (see Lemma 3.1) by using a generalized Hardy-Littlewood-Sobolev inequality in [18], which seems to be new in the weighted setting and is of independent interest. We also note that in contrast to the previous work such as [2, 11, 13] in which the A_2 weights are commonly assumed as the weighted Poincaré inequality is needed, we do not use the weighted Poincaré inequality in the proof. In fact, as pointed out in [34], when $\alpha \geq 1$, such an inequality is not valid.

For simplicity, in this paper we choose not to consider lower-order terms. The results still hold for equations

$$x_d^\alpha(u_t - b_i D_i u - cu + \lambda u) - D_i[x_d^\alpha(a_{ij} D_j u + \hat{b}_i u - F_i)] = \sqrt{\lambda} x_d^\alpha f$$

and

$$a_0 u_t - a_{ij} D_{ij} u - \left(\frac{\alpha}{x_d} a_{dj} + b_i \right) D_j u(t, x) - cu + \lambda c_0 u = f,$$

where b_i , $\hat{b}_i$, and c are bounded measurable functions. To see this, it suffices to move the terms $b_i D_i u$ and cu to the righthand side of the equations, absorb $\hat{b}_i u$ to F_i , and take a sufficiently large λ . (See, e.g., [27] for details.) By using the weighted embedding results such as Lemma 3.1 below, it is also possible to consider unbounded lower-order coefficients. We refer the reader to the recent interesting work [20, 24, 25, 29] and the references therein.

The remaining part of the paper is organized as follows. In the next section, we introduce some notation and state the main results of the paper. In Section 3, we prove two weighted embedding results that are needed in the paper as well as a result on the existence and uniqueness of L_2 -solutions. In Section 4, we study equations whose coefficients depend only on x_d . We prove the existence, uniqueness, and regularity estimates of solutions in $\mathcal{H}_p^1(\Omega_T, \mu)$ after we obtain the L_∞ , Lipschitz-, and Schauder-type estimates for solutions to homogeneous equations. Finally, in Section 5, we provide the proofs of Theorems 2.2 and 2.4, and Corollaries 2.3 and 2.10.

2. NOTATION AND MAIN THEOREMS

2.1. Notation. For $r > 0$, $z_0 = (t_0, x_0)$ with $x_0 = (x'_0, x_{0d}) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $t_0 \in \mathbb{R}$, we define $B_r(x_0)$ to be the ball in $\mathbb{R}^d$ of radius r centered at x_0 , $Q_r(z_0)$ to be the parabolic cylinder of radius r centered at z_0 , that is,

$$Q_r(z_0) = (t_0 - r^2, t_0) \times B_r(x_0),$$

and $B_r^+(x_0)$ and $Q_r^+(z_0)$ to be the upper-half ball and cylinder of radius r centered at x_0 and z_0 , respectively:

$$\begin{aligned} B_r^+(x_0) &= \{x = (x_d, x') \in \mathbb{R}^d : x_d > 0, |x - x_0| < r\}, \\ Q_r^+(z_0) &= (t_0 - r^2, t_0) \times B_r^+(x_0). \end{aligned}$$

When $x_0 = 0$ and $t_0 = 0$, for simplicity of notation we drop x_0, z_0 and write B_r, B_r^+, Q_r , and Q_r^+ , etc. We also define $B'(x'_0)$ and $Q'(z'_0)$ to be the ball and the parabolic cylinder in $\mathbb{R}^{d-1}$ and $\mathbb{R}^d$, where $z'_0 = (t_0, x'_0)$.

For $p \in (1, \infty)$, $-\infty \leq S < T \leq +\infty$, and $\mathcal{D} \subset \mathbb{R}_+^d$, let $L_p((S, T) \times \mathcal{D}, \mu)$ be the weighted Lebesgue space consisting of measurable functions g on $(S, T) \times \mathcal{D}$ such that its norm is as follows:

$$\|g\|_{L_p((S,T) \times \mathcal{D}, \mu)} = \left(\int_{(S,T) \times \mathcal{D}} |g(t, x)|^p \mu(dz) \right)^{1/p} < \infty,$$

where $\mu(dz) = x_d^\alpha dx dt$. For $p, q \in (1, \infty)$, the weights $\omega_0 = \omega_0(t)$, and $\omega_1 = \omega_1(x)$, we define $L_{q,p}(\Omega_T, \omega d\mu)$ to be the weighted mixed-norm Lebesgue space on Ω_T equipped with the norm

$$\|f\|_{L_{q,p}(\Omega_T, \omega d\mu)} = \left(\int_0^T \left(\int_{\mathbb{R}_+^d} |f(t, x)|^p \omega_1(x) \mu(dx) \right)^{q/p} \omega_0(t) dt \right)^{1/q},$$

where $\omega(t, x) = \omega_0(t)\omega_1(x)$. We also define

$$\begin{aligned} \mathbb{H}_{q,p}^{-1}((S, T) \times \mathcal{D}, \omega d\mu) \\ = \left\{ g : g = D_i F_i + F_0/x_d + f \text{ for some } f \in L_{q,p}((S, T) \times \mathcal{D}, \omega d\mu), \right. \\ \left. F = (F_0, \dots, F_d) \in L_{q,p}((S, T) \times \mathcal{D}, \omega d\mu)^{d+1} \right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_{q,p}^1((S, T) \times \mathcal{D}, \omega d\mu) = \\ = \{g : g, Dg \in L_{q,p}((S, T) \times \mathcal{D}, \omega d\mu), g_t \in \mathbb{H}_{q,p}^{-1}((S, T) \times \mathcal{D}, \omega d\mu)\}, \end{aligned}$$

which are equipped with the norms

$$\begin{aligned} \|g\|_{\mathbb{H}_{q,p}^{-1}((S,T) \times \mathcal{D}, \omega d\mu)} = \inf \left\{ \|F\|_{L_{q,p}((S,T) \times \mathcal{D}, \omega d\mu)} + \|f\|_{L_{q,p}((S,T) \times \mathcal{D}, \omega d\mu)} : \right. \\ \left. g = D_i F_i + F_0/x_d + f \right\} \end{aligned}$$

and

$$\begin{aligned} \|g\|_{\mathcal{H}_{q,p}^1((S,T) \times \mathcal{D}, \omega d\mu)} = \|g\|_{L_{q,p}((S,T) \times \mathcal{D}, \omega d\mu)} + \|Dg\|_{L_{q,p}((S,T) \times \mathcal{D}, \omega d\mu)} \\ + \|g_t\|_{\mathbb{H}_{q,p}^{-1}((S,T) \times \mathcal{D}, \omega d\mu)}. \end{aligned}$$

When $p = q$, we simply write $\mathcal{H}_p^1(\Omega_T, \omega d\mu) = \mathcal{H}_{p,p}^1(\Omega_T, \omega d\mu)$. Similar notation is also used for other spaces. When $\omega \equiv 1$, we have $L_{q,p}(\Omega_T, \omega d\mu) = L_{q,p}(\Omega_T, \mu)$, and similarly for other function spaces.

We say $u \in \mathcal{H}_{q,p}^1((S, T) \times \mathcal{D}, \omega \, d\mu)$ is a weak solution of (1.2) in $(S, T) \times \mathcal{D}$ if

$$\begin{aligned} & \int_{(S,T) \times \mathcal{D}} (-u \partial_t \varphi + \lambda u \varphi) \mu(dz) + \int_{(S,T) \times \mathcal{D}} (a_{ij} D_j u - F_i) D_i \varphi \mu(dz) \\ &= \lambda^{1/2} \int_{(S,T) \times \mathcal{D}} f(z) \varphi(z) \mu(dz) \end{aligned}$$

for any $\varphi \in C_0^\infty((S, T) \times (\mathcal{D} \cup (\bar{\mathcal{D}} \cap \partial \mathbb{R}^d)))$.

We use the notation $a_+ = \max\{a, 0\}$ and $a_- = \max\{-a, 0\}$ for $a \in \mathbb{R}$ so that $a = a_+ - a_-$. Finally, for a set $\Omega \subset \mathbb{R}^{d+1}$ and any integrable function f on Ω with respect to some Borel measure ω , we write

$$\oint_{\Omega} f \omega(dz) = \frac{1}{\omega(\Omega)} \int_{\Omega} f \omega(dz), \quad \text{where } \omega(\Omega) = \int_{\Omega} \omega(dz).$$

2.2. Main theorems. As in [11, 12], we impose the following partially-VMO condition on the leading coefficients.

Assumption 2.1 (γ_0, R_0). For any $r \in (0, R_0]$ and $z_0 = (z'_0, x_d) \in \mathbb{R}^d \times \bar{\mathbb{R}}_+$, we have

$$\sup_{i,j} \oint_{Q_r^+(z_0)} |a_{ij}(t, x) - [a_{ij}]_{r,z_0}(x_d)| \mu(dz) \leq \gamma_0,$$

where $\mu(dz) = x_d^\alpha dt \, dx$, $[a_{ij}]_{r,z_0}(x_d)$ is the average of a_{ij} with respect to (t, x') in $Q'_r(z'_0)$:

$$[a_{ij}]_{r,z_0}(x_d) = \oint_{Q'_r(z'_0)} a_{ij}(t, x', x_d) \, dx' \, dt.$$

In the special case that the coefficients (a_{ij}) only depend on the x_d variable, no regularity assumption is required on them, as Assumption 2.1 (γ_0, R_0) is always satisfied.

Our first main result is about the existence, uniqueness, and global regularity estimates of solutions to the divergence-form equation (1.2).

Theorem 2.2. Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, and $p \in (1, \infty)$. Then, there exist $\gamma_0 = \gamma_0(d, \kappa, \alpha, p) \in (0, 1)$ and $\lambda_0 = \lambda_0(d, \kappa, \alpha, p) \geq 0$ such that the following assertions hold. Suppose that (1.1) and Assumption 2.1 (γ_0, R_0) are satisfied. If $u \in \mathcal{H}_p^1(\Omega_T, \mu)$ is a weak solution of (1.2) for some $\lambda \geq \lambda_0 R_0^{-2}$, $f \in L_p(\Omega_T, \mu)$, and $F \in L_p(\Omega_T, \mu)^d$, then we have

$$(2.1) \quad \|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_p(\Omega_T, \mu)} \leq N \|F\|_{L_p(\Omega_T, \mu)} + N \|f\|_{L_p(\Omega_T, \mu)},$$

where $N = N(d, \kappa, \alpha, p) > 0$. Moreover, for any $\lambda > \lambda_0 R_0^{-2}$, $f \in L_p(\Omega_T, \mu)$, and $F \in L_p(\Omega_T, \mu)^d$, there exists a unique weak solution $u \in \mathcal{H}_p^1(\Omega_T, \mu)$ to (1.2).

In the next result, we give a local boundary estimate in a half cylinder. Consider

$$(2.2) \quad \begin{cases} x_d^\alpha u_t - \operatorname{Di}(x_d^\alpha(a_{ij}D_j u - F_i)) = x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha(a_{dj}D_j u - F_d) = 0 \end{cases}$$

in Q_2^+ . Let $p \in [1, \infty)$ and $p^* \in [1, p)$ satisfy

$$(2.3) \quad \begin{cases} \frac{d+2+\alpha_+}{p^*} \leq 1 + \frac{d+2+\alpha_+}{p} & \text{when } p^* > 1 \\ \frac{d+2+\alpha_+}{p^*} < 1 + \frac{d+2+\alpha_+}{p} & \text{when } p^* = 1 \end{cases}$$

if $d \geq 2$ or $\alpha = 0$, and

$$(2.4) \quad \begin{cases} \frac{4+\alpha_+}{p^*} \leq 1 + \frac{4+\alpha_+}{p} & \text{when } p^* > 1 \\ \frac{4+\alpha_+}{p^*} < 1 + \frac{4+\alpha_+}{p} & \text{when } p^* = 1 \end{cases}$$

if $d = 1$ and $\alpha \neq 0$. Note that the condition on p^* is used in a weighted parabolic Sobolev embedding result. (See Lemma 3.1 below.)

Corollary 2.3. *Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, $1 < p_0 < p < \infty$, and $p^* \in [1, p)$ satisfy (2.3)–(2.4). Then, there is $\gamma_0 = \gamma_0(d, \kappa, \alpha, p_0, p) \in (0, 1)$ such that the following assertion holds. Suppose (1.1) and Assumption 2.1 (γ_0, R_0) are satisfied. If $u \in \mathcal{H}_{p_0}^1(Q_2^+, \mu)$ is a weak solution of (2.2), $F \in L_p(Q_2^+, \mu)^d$, and $f \in L_{p^*}(Q_2^+, \mu)$, then $u \in \mathcal{H}_p^1(Q_1^+, \mu)$ and*

$$(2.5) \quad \begin{aligned} \|u\|_{L_p(Q_1^+, \mu)} + \|Du\|_{L_p(Q_1^+, \mu)} \\ \leq N\|u\|_{L_1(Q_2^+, \mu)} + N\|Du\|_{L_1(Q_2^+, \mu)} \\ + N\|F\|_{L_p(Q_2^+, \mu)} + N\|f\|_{L_{p^*}(Q_2^+, \mu)}, \end{aligned}$$

where $N = N(d, \kappa, \alpha, p_0, p, p^*, R_0) > 0$.

We conjecture that for any $d \geq 1$ and $\alpha \in (-1, \infty)$, the above corollary still holds when p^* satisfies (2.3).

In this paper, we also show that Theorem 2.2 can be extended to the setting of weighted mixed-norm spaces. The result is of interest because the inhomogeneous terms F and f could behave anisotropically. For $p \in (1, \infty)$, a locally integrable function $\omega : \mathbb{R}_+^d \rightarrow \mathbb{R}_+$ is said to be in the $A_p(\mathbb{R}_+^d, \mu)$ Muckenhoupt class of weights if

$$[\omega]_{A_p(\mathbb{R}_+^d, \mu)} := \sup_{r>0, x \in \mathbb{R}_+^d} \left(\int_{B_r^+(x)} \omega(y) \mu(dy) \right)$$

$$\times \left(\int_{B_r^+(x)} \omega(y)^{1/(1-p)} \mu(dy) \right)^{p-1} < \infty.$$

Similarly, a locally integrable function $\omega : \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be in the $A_p(\mathbb{R})$ Muckenhoupt class of weights if

$$[\omega]_{A_p(\mathbb{R})} := \sup_{r>0, t \in \mathbb{R}} \left(\int_{t-r^2}^{t+r^2} \omega(s) ds \right) \left(\int_{t-r^2}^{t+r^2} \omega(s)^{1/(1-p)} ds \right)^{p-1} < \infty.$$

Theorem 2.4. Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, $p, q, K \in (1, \infty)$, $\omega_0 \in A_q(\mathbb{R})$, $\omega_1 \in A_p(\mathbb{R}_+^d, \mu)$, and $\omega = \omega_0(t)\omega_1(x)$, such that

$$[\omega_0]_{A_q(\mathbb{R})} \leq K, \quad [\omega_1]_{A_p(\mathbb{R}_+^d, \mu)} \leq K.$$

Then, there exist

$$\gamma_0 = \gamma_0(d, \kappa, \alpha, p, q, K) \in (0, 1) \quad \text{and} \quad \lambda_0 = \lambda_0(d, \kappa, \alpha, p, q, K) \geq 0,$$

such that the following assertions hold. Suppose that (1.1) and Assumption 2.1 (γ_0, R_0) are satisfied. If $u \in \mathcal{H}_{q,p}^1(\Omega_T, \omega d\mu)$ is a weak solution of (1.2) for some $\lambda \geq \lambda_0 R_0^{-2}$, $f \in L_{q,p}(\Omega_T, \omega d\mu)$, and $F \in L_{q,p}(\Omega_T, \omega d\mu)^d$, then we have

$$(2.6) \quad \begin{aligned} & \|Du\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \sqrt{\lambda} \|u\|_{L_{q,p}(\Omega_T, \omega d\mu)} \\ & \leq N \|F\|_{L_{q,p}(\Omega_T, \omega d\mu)} + N \|f\|_{L_{q,p}(\Omega_T, \omega d\mu)}, \end{aligned}$$

where $N = N(d, \kappa, \alpha, p, q, K) > 0$.

Moreover, for any $\lambda > \lambda_0 R_0^{-2}$, $f \in L_{q,p}(\Omega_T, \omega d\mu)$, and $F \in L_{q,p}(\Omega_T, \omega d\mu)^d$, there exists a unique weak solution $u \in \mathcal{H}_{q,p}^1(\Omega_T, \omega d\mu)$ to (1.2).

Next, we state the main results for non-divergence form equations. Besides the regularity assumption on (a_{ij}) as in Assumption 2.1, we impose similar conditions on the coefficients a_0 and c_0 .

Assumption 2.5 (γ_0, R_0). For any $r \in (0, R_0]$ and $z_0 = (z'_0, x_d) \in \mathbb{R} \times \overline{\mathbb{R}_+^d}$, we have

$$\begin{aligned} & \sup_{i,j} \int_{Q_r^+(z_0)} |a_{ij}(t, x) - [a_{ij}]_{r,z_0}(x_d)| \mu(dz) \\ & + \int_{Q_r^+(z_0)} (|a_0(t, x) - [a_0]_{r,z_0}(x_d)| + |c_0(t, x) - [c_0]_{r,z_0}(x_d)|) \mu(dz) \leq \gamma_0, \end{aligned}$$

where $[a_{ij}]_{r,z_0}(x_d)$, $[a_0]_{r,z_0}(x_d)$, and $[c_0]_{r,z_0}(x_d)$ are, respectively, the average of a_{ij} , a_0 , and c_0 with respect to (t, x') in $Q'_r(z'_0)$ as defined in Assumption 2.1.

We also need the following definition which is used in a weighted Hardy inequality (cf. [12, Lemma 2.2]).

Definition 2.6. Let $\alpha \in (-1, \infty)$ and $p \in (1, \infty)$; we then say that the weight $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is in $M_p(\mu)$ if

$$[\omega]_{M_p(\mu)} = \sup_{r>0} \left(\int_r^\infty \gamma^{-p(\alpha+1)} \omega(\gamma) \mu(d\gamma) \right)^{1/p} \\ \times \left(\int_0^r \omega(\gamma)^{-1/(p-1)} \mu(d\gamma) \right)^{1-1/p} < \infty,$$

where $\mu(d\gamma) = \gamma^\alpha d\gamma$ for $\gamma \in \mathbb{R}_+$.

Define $W_{q,p}^{1,2}(\Omega_T, \omega d\mu)$ to be the weighted mixed-norm Sobolev space equipped with the norm

$$\|u\|_{W_{q,p}^{1,2}(\Omega_T, \omega d\mu)} = \|u\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \|u_t\|_{L_{q,p}(\Omega_T, \omega d\mu)} \\ + \|Du\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \|D^2u\|_{L_{q,p}(\Omega_T, \omega d\mu)}.$$

When $p = q$ and $\omega \equiv 1$, we can write $W_p^{1,2}(\Omega_T, \mu) = W_{p,p}^{1,2}(\Omega_T, d\mu)$. A function $u \in W_{q,p}^{1,2}(\Omega_T, \omega d\mu)$ is said to be a strong solution to (1.4) if it satisfies the equation almost everywhere. Our main result for the non-divergence form equation (1.4)–(1.5) is the following theorem.

Theorem 2.7. Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, $p, q, K \in (1, \infty)$. Let $\omega_0 \in A_q(\mathbb{R})$, $\omega_1 \in A_p(\mathbb{R}^{d-1})$, $\omega_2 \in A_p(\mathbb{R}_+, \mu) \cap M_p(\mu)$, and $\omega(t, x) = \omega_0(t)\omega_1(x')\omega_2(x_d)$, such that

$$[\omega_0]_{A_q(\mathbb{R})} \leq K, \quad [\omega_1]_{A_p(\mathbb{R}^{d-1})} \leq K, \quad [\omega_2]_{A_p(\mathbb{R}_+, \mu)} \leq K, \quad [\omega_2]_{M_p(\mu)} \leq K.$$

Then, there exist

$$\gamma_0 = \gamma_0(d, \kappa, \alpha, p, q, K) \in (0, 1) \quad \text{and} \quad \lambda_0 = \lambda_0(d, \kappa, \alpha, p, q, K) \geq 0$$

such that the following assertions hold. Suppose that (1.1), (1.6), (1.7), and Assumption 2.5 (γ_0, R_0) are satisfied. If $u \in W_{q,p}^{1,2}(\Omega_T, \omega d\mu)$ is a strong solution of (1.4)–(1.5) with $f \in L_{q,p}(\Omega_T, \omega d\mu)$ and $\lambda \geq \lambda_0 R_0^{-2}$, then

$$\|u_t\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \|D^2u\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \|D_d u / x_d\|_{L_{q,p}(\Omega_T, \omega d\mu)} \\ + \sqrt{\lambda} \|Du\|_{L_{q,p}(\Omega_T, \omega d\mu)} + \lambda \|u\|_{L_{q,p}(\Omega_T, \omega d\mu)} \leq N \|f\|_{L_{q,p}(\Omega_T, \omega d\mu)},$$

where $N = N(d, \kappa, \alpha, p, q, K) > 0$. Moreover, for any $f \in L_{q,p}(\Omega_T, \omega d\mu)$ and $\lambda > \lambda_0 R_0^{-2}$, there is a unique strong solution $u \in W_{q,p}^{1,2}(\Omega_T, \omega d\mu)$ of (1.4)–(1.5).

Remark 2.8. As a typical example, in Theorem 2.7 we can take the power weight $\omega_2(x_d) = x_d^\beta$. It is easily seen that for any $\beta \in (-\alpha - 1, (\alpha + 1)(p - 1))$, we have $\omega_2 \in A_p(\mathbb{R}_+, \mu) \cap M_p(\mu)$. In the special case when we have $\alpha = 0$ and $\beta \in (-1, p - 1)$, a similar result was proved in [23] when the coefficients are measurable in the time variable and have small mean oscillations in the spatial variables, by using a different argument.

Remark 2.9. It is worth highlighting that in [11] the authors studied (1.8) with general $\mu \in A_2(\mathbb{R}_+)$, and the corresponding equations in non-divergence form were also studied in [12] when $\mu = x_d^\alpha$ with $\alpha \in (-1, 1)$. Similar requirements of the A_2 -condition can be found in [2, 13], for example. Theorems 2.2, 2.4, and 2.7 cover the case when $\mu(x_d) = x_d^\alpha$, which is not in $A_2(\mathbb{R}_+)$ when $\alpha \in [1, \infty)$, and this seems to be the first time that the L_p -theory is developed for equations with non- A_2 coefficients. The key ingredient is our newly developed results on Lipschitz and Schauder estimates in Section 4.1 for solutions of homogeneous equations. Specifically, motivated by [34], to obtain such results we prove a weighted parabolic embedding theorem in Lemma 3.1 and apply it with Moser's iteration argument to establish the boundedness of the solutions (see Lemma 4.3 below). Then, to derive the estimates of higher-order derivatives of the solutions, we use an anisotropic Sobolev embedding theorem, a special structure of the weight, together with an iteration method (see Proposition 4.4 below). Our method works for equations with coefficients that are measurable in the x_d -variable. Similar Schauder estimates were also obtained recently in [34] for elliptic equations, the proof of which relies on Liouville-type theorems and special invariant structure of the coefficients, and also requires that the coefficients be sufficiently smooth.

Once Lipschitz and Schauder estimates in Section 4 and Theorem 2.2 are proved, Theorem 2.7 can be proved by using the same argument as in [12]. To keep the paper within a reasonable length, we skip the proof of Theorem 2.7 and refer the reader to [12] for details.

Much as in Corollary 2.3, we also obtain the following local boundary estimate for solutions of (1.4) in Q_2^+ .

Corollary 2.10. *Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, and $1 < p_0 < p < \infty$. Then, there exists $\gamma_0 = \gamma_0(d, \kappa, \alpha, p_0, p) \in (0, 1)$ such that the following assertion holds. Suppose that (1.1), (1.6), (1.7), and Assumption 2.5 (γ_0, R_0) are satisfied. If $u \in W_{p_0}^{1,2}(Q_2^+, \mu)$ is a strong solution of*

$$\begin{cases} a_0 u_t - a_{ij} D_{ij} u - \frac{\alpha}{x_d} a_{dd} D_d u + c_0 u = f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha a_{dd} D_d u = 0 \end{cases}$$

in Q_2^+ , and $f \in L_p(Q_2^+, \mu)$, then we have $u \in W_p^{1,2}(Q_1^+, \mu)$ and

$$(2.7) \quad \|u\|_{W_p^{1,2}(Q_1^+, \mu)} \leq N \|u\|_{W_1^{1,2}(Q_2^+, \mu)} + N \|f\|_{L_p(Q_2^+, \mu)},$$

where $N = N(d, \kappa, \alpha, p_0, p, R_0) > 0$.

Using the above results for parabolic equations, we can directly derive similar results for elliptic equations by viewing solutions to elliptic equations as steady-state solutions of the corresponding parabolic equations. (See, for example, the proofs of [27, Theorem 2.6] and [12, Theorem 1.2].) We only present here a result of the local boundary estimate for weak solutions. Consider

$$(2.8) \quad \begin{cases} -\operatorname{Di}(x_d^\alpha [a_{ij}(x) D_j u - F_i]) = x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (a_{dj}(x) D_j u - F_d) = 0 \end{cases}$$

in B_2^+ , where $a_{ij} : B_2^+ \rightarrow \mathbb{R}$, $F = (F_1 F_2, \dots, F_d) : B_2^+ \rightarrow \mathbb{R}^d$ and $f : B_2^+ \rightarrow \mathbb{R}$ are given measurable functions. In this time-independent case, (1.1) and Assumption 2.1 can be stated similarly. For each $p \in (1, \infty)$, suppose that $\hat{p} \in [1, p)$ satisfies

$$(2.9) \quad \begin{cases} \frac{d + \alpha_+}{\hat{p}} \leq 1 + \frac{d + \alpha_+}{p} & \text{when } \hat{p} > 1, \\ \frac{d + \alpha_+}{\hat{p}} < 1 + \frac{d + \alpha_+}{p} & \text{when } \hat{p} = 1. \end{cases}$$

For $\Omega \subset \mathbb{R}^d$, $W_p^1(\Omega, \mu)$ denotes the weighted Sobolev space consisting of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ such that $u, Du \in L_p(\Omega, \mu)$.

Corollary 2.11. *Let $\alpha \in (-1, \infty)$, $\kappa \in (0, 1)$, $R_0 \in (0, \infty)$, $1 < p_0 < p < \infty$, and $\hat{p} \in [1, p)$ satisfy (2.9). Then, there exists $\gamma_0 = \gamma_0(d, \kappa, \alpha, p_0, p) \in (0, 1)$ such that the following assertion holds. Suppose that (1.1) and Assumption 2.1 (γ_0, R_0) are satisfied. If $u \in W_{p_0}^1(B_2^+, \mu)$ is a weak solution of (2.8), $F \in L_p(B_2^+, \mu)^d$, and $f \in L_{\hat{p}}(B_2^+, \mu)$, then $u \in W_p^1(B_1^+, \mu)$ and*

$$\|u\|_{W_p^1(B_1^+, \mu)} \leq N \|u\|_{W_1^1(B_2^+, \mu)} + N \|F\|_{L_p(B_2^+, \mu)} + N \|f\|_{L_{\hat{p}}(B_2^+, \mu)},$$

where $N = N(d, \kappa, \alpha, p_0, p, \hat{p}, R_0) > 0$.

The proof of Corollary 2.11 is similar to that of Corollary 2.3 by using the corresponding weighted embedding inequality (see Remark 3.2 (ii)). Therefore, we also omit it.

3. WEIGHTED SOBOLEV INEQUALITIES AND L_2 -SOLUTIONS

Our first result in this section is a weighted parabolic embedding lemma which will be used in the proof of Corollary 2.3. The range of q^* below is optimal when $d \geq 2$. However, when $d = 1$, we impose a slightly stronger condition. In view of the classical parabolic Sobolev embedding when $\alpha = 0$, we conjecture that this condition can be relaxed.

Lemma 3.1 (Weighted parabolic embedding). *Let $\alpha \in (-1, \infty)$ and let also $q, q^* \in (1, \infty)$ satisfy*

$$(3.1) \quad \begin{cases} \frac{d+2+\alpha_+}{q} \leq 1 + \frac{d+2+\alpha_+}{q^*} & \text{if } d \geq 2, \\ \frac{4+\alpha_+}{q} \leq 1 + \frac{4+\alpha_+}{q^*} & \text{if } d = 1. \end{cases}$$

Then, for any $v \in \mathcal{H}_q^1(Q_2^+, \mu)$, we have

$$(3.2) \quad \|v\|_{L_{q^*}(Q_2^+, \mu)} \leq N \|v\|_{\mathcal{H}_q^1(Q_2^+, \mu)},$$

where $N = N(d, \alpha, q, q^) > 0$ is a constant. The result still holds when $q^* = \infty$ and the inequality in (3.1) is strict.*

Proof. Note that the case when $d = 1$ follows by considering $v(t, x_1) = v(t, x_1, x_2)$ with a dummy variable x_2 and using the result when $d = 2$. Hence, we only need to prove (3.2) when $d \geq 2$. It suffices to consider the case when $q^* > q$. Without loss of generality, we may assume that

$$(3.3) \quad v_t = D_i G_i + G_0/x_d + g$$

in Q_2^+ in the weak sense and

$$\|v\|_{L_q(Q_2^+, \mu)} + \|Dv\|_{L_q(Q_2^+, \mu)} + \|G\|_{L_q(Q_2^+, \mu)} + \|g\|_{L_q(Q_2^+, \mu)} \leq 1,$$

where $G = (G_0, G_1, \dots, G_d)$. Let $\tilde{Q} = Q_{1/2}(0, 0, \dots, 0, \frac{3}{2})$ and $\psi \in C_0^\infty(\tilde{Q})$ with unit integral. For any $(t, x) \in Q_2^+$, by the fundamental theorem of calculus,

$$(3.4) \quad \begin{aligned} & v(t, x) - c \\ &= \int_{\tilde{Q}} \int_0^1 \left(v_t(t(1-\theta^2) + s\theta^2, x(1-\theta) + y\theta) 2\theta(s-t) \right. \\ & \quad \left. + (Dv)(t(1-\theta^2) + s\theta^2, x(1-\theta) + y\theta) \cdot (y-x) \right) \psi(s, y) d\theta ds dy \\ &:= I_1 + I_2, \end{aligned}$$

where

$$c = \int_{\tilde{Q}} v(s, y) \psi(s, y) ds dy.$$

Let $\hat{x} = x(1-\theta) + y\theta$ and $\tau = t(1-\theta^2) + s\theta^2$. Clearly,

$$(3.5) \quad (|\hat{x} - x|^2 + |\tau - t|)^{1/2} = (|x - y|^2 + |t - s|)^{1/2} \theta \leq N\theta.$$

It then follows from (3.3) that

$$(3.6) \quad I_1 = 2 \int_{\tilde{Q}} \int_0^1 \left(g(\tau, \hat{x}) + (D_i G_i)(\tau, \hat{x}) + \frac{G_0(\tau, \hat{x})}{\hat{x}_d} \right) \theta(s-t) \psi(s, y) \, d\theta \, ds \, dy.$$

Since $y \in \tilde{Q}$, we have $y_d \geq 1$ and thus

$$(3.7) \quad |x - \hat{x}| = |x - y| \theta \leq N \theta y_d \leq N \hat{x}_d \quad \text{and} \quad x_d \leq N \hat{x}_d.$$

Moreover,

$$(D_i G_i)(\tau, \hat{x}) = D_{y_i} G_i(\tau, \hat{x}) \theta^{-1}.$$

Therefore, from (3.6) and integration by parts, we deduce

$$(3.8) \quad |I_1| \leq N \int_{\tilde{Q}} \int_0^1 \left(|g(\tau, \hat{x})| \theta + |G(\tau, \hat{x})| + |G_0(\tau, \hat{x})| |x - \hat{x}|^{-1} \theta \right) |s - t| \, d\theta \, ds \, dy.$$

Combining (3.4) and (3.8), we obtain

$$\begin{aligned} & |v(t, x) - c| \\ & \leq N \int_{\tilde{Q}} \int_0^1 (|g(\tau, \hat{x})| \theta + |G(\tau, \hat{x})| + |G_0(\tau, \hat{x})| |x - \hat{x}|^{-1} \theta + |Dv(\tau, \hat{x})|) \, d\theta \, ds \, dy \\ & \leq N \int_{Q_2^+} \int_0^1 \theta^{-d-2} (|g(\tau, \hat{x})| \theta + |G(\tau, \hat{x})| + |G_0(\tau, \hat{x})| |x - \hat{x}|^{-1} \theta \\ & \quad + |Dv(\tau, \hat{x})|) \chi_{\{|x - \hat{x}|^2 + |t - \tau|^{1/2} \leq N \theta, x_d \leq N \hat{x}_d\}} \, d\theta \, d\tau \, d\hat{x} \\ & \leq N \int_{Q_2^+} \left(|g(\tau, \hat{x})| (|x - \hat{x}|^2 + |t - \tau|)^{-d/2} \right. \\ & \quad + |G_0(\tau, \hat{x})| |x - \hat{x}|^{-1} (|x - \hat{x}|^2 + |t - \tau|)^{-d/2} \\ & \quad \left. + (|G(\tau, \hat{x})| + |Dv(\tau, \hat{x})|) (|x - \hat{x}|^2 + |t - \tau|)^{-(d+1)/2} \right) \chi_{\{x_d \leq N \hat{x}_d\}} \, d\tau \, d\hat{x}, \end{aligned}$$

where we used $dy = \theta^{-d} d\hat{x}$, $d\tau = \theta^{-2} ds$, (3.5), and (3.7) in the third inequality.

We apply Young's inequality for convolutions with respect to the time variable to get that, for any $x \in B_2^+$,

$$(3.9) \quad \|v(\cdot, x) - c\|_{L_{q^*}((-4,0))} \leq N \int_{B_2^+} \left(\|g(\cdot, \hat{x})\|_{L_q((-4,0))} |x - \hat{x}| + \|G(\cdot, \hat{x})\|_{L_q((-4,0))} \right. \\ \left. + \|Dv(\cdot, \hat{x})\|_{L_q((-4,0))} \right) |x - \hat{x}|^{-d-1+2/\ell} \chi_{\{x_d \leq N \hat{x}_d\}} \, d\hat{x},$$

where $\ell \in (1, \infty)$ satisfies

$$(3.10) \quad \frac{1}{\ell} + \frac{1}{q} = 1 + \frac{1}{q^*}$$

and $\ell d > 2$ which always holds because $\ell > 1$ and $d \geq 2$. Similarly, we use Young's inequality in x' to get that for any $x_d \in (0, 2)$,

$$(3.11) \quad \|v(\cdot, \cdot, x_d) - c\|_{L_{q^*}(Q'_2)} \\ \leq N \int_0^2 \left(\|g(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} |x_d - \hat{x}_d| + \|G(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right. \\ \left. + \|Dv(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right) |x_d - \hat{x}_d|^{-(d+1)(1-1/\ell)} \chi_{\{x_d \leq N\hat{x}_d\}} d\hat{x}_d,$$

where we used $(d+1-2/\ell)\ell > d-1$ which holds true as $\ell > 1$. In the sequel, we discuss two cases: $\alpha \geq 0$ and $\alpha \in (-1, 0)$.

Case I. $\alpha \geq 0$. We first consider the case when $q^* < \infty$. Multiplying both sides of (3.11) by x_d^{α/q^*} , we get

$$x_d^{\alpha/q^*} \|v(\cdot, \cdot, x_d) - c\|_{L_{q^*}(Q'_2)} \\ \leq N x_d^{\alpha/q^*} \int_0^2 \left(\|g(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} |x_d - \hat{x}_d| + \|G(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right. \\ \left. + \|Dv(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right) |x_d - \hat{x}_d|^{-(d+1)(1-1/\ell)} \chi_{\{x_d \leq N\hat{x}_d\}} d\hat{x}_d \\ \leq N \int_0^2 \left(\|g(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} |x_d - \hat{x}_d| + \|G(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right. \\ \left. + \|Dv(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right) \hat{x}_d^{\alpha/q} |x_d - \hat{x}_d|^{-(d+1)(1-1/\ell)+\alpha/q^*-\alpha/q} d\hat{x}_d.$$

Since both x_d and $\hat{x}_d$ are bounded, we can apply the Hardy-Littlewood-Sobolev inequality for fractional integration in x_d to obtain

$$(3.12) \quad \|v - c\|_{L_{q^*}(Q_2^+, \mu)} \leq N \|g\|_{L_q(Q_2^+, \mu)} + N \|G\|_{L_q(Q_2^+, \mu)} \\ + N \|Dv\|_{L_q(Q_2^+, \mu)},$$

provided that

$$\frac{d+1}{1-1/\ell} - \frac{\alpha}{q^*} + \frac{\alpha}{q} \leq 1 + \frac{1}{q^*} - \frac{1}{q}.$$

From (3.10), we see that this condition is equivalent to (3.1).

When $q^* = \infty$, we have $\ell = p = q/(q-1)$. Thus, if the inequality (3.1) is strict, we also get (3.12) by using Hölder's inequality. From (3.12) and the definition of c , we easily get (3.2).

Case II: $\alpha \in (-1, 0)$. For the case $q^* < \infty$, we will apply the generalized Hardy-Littlewood-Sobolev inequality (see [18, Theorem 6] or Theorem B of [35]) to conclude (3.12), which gives (3.2). Indeed, in terms of the notation in Theorem 6 of [18], we choose

$$r = q, \quad s = \frac{q^*}{q^* - 1}, \quad h = \frac{\alpha}{q^*}, \quad k = -\frac{\alpha}{q^*}, \quad \lambda = 2 - \frac{1}{s} - \frac{1}{r}.$$

Then, it is easily seen that the conditions in there are satisfied. Let

$$\begin{aligned} f(\hat{x}_d) = & \left(\|g(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} + \|G(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right. \\ & \left. + \|Dv(\cdot, \cdot, \hat{x}_d)\|_{L_q(Q'_2)} \right) \chi_{(0,2)}(\hat{x}_d). \end{aligned}$$

As both x_d and $\hat{x}_d$ are bounded, if

$$(3.13) \quad (d+1)(1-1/\ell) \leq \lambda - h - k,$$

then by (3.11) we see that for any $g \in L_s((0, 2))$,

$$\begin{aligned} & \left| \int_0^2 x_d^{\alpha/q^*} \|v(\cdot, \cdot, x_d) - c\|_{L_{q^*}(Q'_2)} g(x_d) dx_d \right| \\ & \leq N \int_0^2 \int_0^2 \frac{f(\hat{x}_d) \hat{x}_d^h |g(x_d)|}{\hat{x}_d^h |x_d - \hat{x}_d|^{\lambda-h-k} x_d^k} d\hat{x}_d dx_d. \end{aligned}$$

From this, we apply [18, Theorem 6] to get

$$\begin{aligned} & \left| \int_0^2 x_d^{\alpha/q^*} \|v(\cdot, \cdot, x_d) - c\|_{L_{q^*}(Q'_2)} g(x_d) dx_d \right| \\ & \leq N \left(\int_0^2 f^q(\hat{x}_d) \hat{x}_d^{hq} d\hat{x}_d \right)^{1/q} \|g\|_{L_s((0,2))} \\ & \leq N \left(\int_0^2 f^q(\hat{x}_d) \hat{x}_d^\alpha d\hat{x}_d \right)^{1/q} \|g\|_{L_s((0,2))}, \end{aligned}$$

where we used the fact that $\hat{x}_d^{hq} \leq N \hat{x}_d^\alpha$ for any $\hat{x}_d \in (0, 2)$ because $\alpha < 0$. Then, by the duality, we obtain (3.12) when $q^* < \infty$. Because of (3.10), the condition (3.13) is equivalent to

$$(3.14) \quad \frac{d+2}{q} \leq 1 + \frac{d+2}{q^*},$$

which is (3.1) when $\alpha < 0$. When $q^* = \infty$ and the inequality (3.14) is strict, we also have (3.12) by using Hölder's inequality. The lemma is proved. $\square$

Remark 3.2.

- (i) In view of the additional factors in the g terms in (3.9) and (3.11), it is possible to relax the integrability condition on g in Lemma 3.1: we only need $g \in L_{\bar{q}}(Q_2^+, \mu)$, where $\bar{q} \in (1, q)$ satisfies

$$\frac{d+2+\alpha_+}{\bar{q}} \leq 2 + \frac{d+2+\alpha_+}{q^*} \quad \text{when } d \geq 2.$$

However, this will not be used in the proofs of our main results.

- (ii) In the time-independent case, (3.9) is not needed. Therefore, with a minor modification of the proof, we also have the embedding

$$\|u\|_{L_{q^*}(B_2^+, \mu)} \leq N \|u\|_{W_q^1(B_2^+, \mu)} \quad \text{for all } u \in W_q^1(B_2^+, \mu)$$

with $q, q^* \in (1, \infty)$ satisfying

$$\frac{d+\alpha_+}{q} \leq 1 + \frac{d+\alpha_+}{q^*}.$$

The result still holds when $q > d + \alpha_+$ and $q^* = \infty$. (See Theorem 6 in [17] for a different proof in a more general setting.)

We also need a weighted parabolic embedding result for functions in the energy space, which will be used in the proof of Lemma 4.3 when we apply the Moser iteration.

Lemma 3.3. *Let $\alpha \in (-1, \infty)$, $\ell_0 = (d + \alpha_+ + 2)/(d + \alpha_+)$ if $d + \alpha_+ > 2$ and $\ell_0 \in (1, 2)$ be any number if $d + \alpha_+ \leq 2$. Then, there exists a constant $N = N(d, \ell_0, \alpha)$ such that*

$$\begin{aligned} \left(\int_{Q_r^+(z_0)} |u(t, x)|^{2\ell_0} \mu(dz) \right)^{1/\ell_0} &\leq N \sup_{t \in (t_0 - r^2, t_0)} \int_{B_r^+(x_0)} |u(t, x)|^2 \mu(dx) \\ &\quad + Nr^2 \int_{Q_r^+(z_0)} |Du(t, x)|^2 \mu(dz), \end{aligned}$$

for every $z_0 = (t_0, x_0) \in \overline{\mathbb{R}_+^{d+1}}$, $r > 0$, and

$$u \in L_\infty((t_0 - r^2, t_0); L_2(B_r^+(x_0), \mu)) \cap L_2((t_0 - r^2, t_0); W_2^1(B_r^+(x_0), \mu)).$$

Proof. Let $\Gamma = (t_0 - r^2, t_0)$, and let $\kappa_0 = 2/(2 - \ell_0) \in (1, \infty)$. By Remark 3.2 (ii) (see also [34, Theorem 2.4]) and after rescaling, we have the following weighted Sobolev inequality:

$$\begin{aligned} \left(\int_{B_r^+(x_0)} |u(t, x)|^{\kappa_0} \mu(dx) \right)^{1/\kappa_0} &\leq Nr \left(\int_{B_r^+(x_0)} |Du(t, x)|^2 \mu(dx) \right)^{1/2} \\ &\quad + N \left(\int_{B_r^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{1/2}, \end{aligned}$$

where $N = N(d, \ell_0, \alpha) > 0$. This together with Hölder's inequality gives

$$\begin{aligned} & \int_{B_r^+(x_0)} |u(t, x)|^{2\ell_0} \mu(dx) \\ & \leq \left(\int_{B_r^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{1-2/\kappa_0} \left(\int_{B_r^+(x_0)} |u(t, x)|^{\kappa_0} \mu(dx) \right)^{2/\kappa_0} \\ & \leq N \left(\sup_{t \in \Gamma} \int_{B_r^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{1-2/\kappa_0} \\ & \quad \times \left(r^2 \int_{B_r^+(x_0)} |Du(t, x)|^2 \mu(dx) + \int_{B_r^+(x_0)} |u(t, x)|^2 \mu(dx) \right). \end{aligned}$$

Now, by integrating with respect to t on Γ and using Young's inequality, we obtain

$$\begin{aligned} \int_{Q_r^+(z_0)} |u(t, x)|^{2\ell_0} \mu(dz) & \leq N \left(\sup_{t \in \Gamma} \int_{B_r^+(x_0)} |u(t, x)|^2 \mu(dx) \right)^{\ell_0} \\ & \quad + Nr \left(\int_{Q_r^+(z_0)} |Du(t, x)|^2 \mu(dz) \right)^{\ell_0}. \end{aligned}$$

The lemma is then proved. $\square$

Finally, we conclude this section with the following useful result on the existence and uniqueness of L_2 -solutions of a class of equations that are slightly more general than (1.2). The result is considered as a special case of Theorem 2.2 when $p = 2$, but no regularity requirements are imposed on the coefficients.

Lemma 3.4. *Let $\alpha \in (-1, \infty)$, $\lambda > 0$, and let (a_{ij}) , a_0 , and c_0 be measurable functions defined on Ω_T such that (1.1) and (1.6) are satisfied. Then, for each $F \in L_2(\Omega_T, \mu)^d$ and $f \in L_2(\Omega_T, \mu)$, there exists a unique weak solution $u \in \mathcal{H}_2^1(\Omega_T, \mu)$ to*

$$(3.15) \quad \begin{cases} x_d^\alpha (a_0(t, x) u_t + \lambda c_0(t, x) u) - \\ \quad - D_i (x_d^\alpha [a_{ij}(t, x) D_j u - F_i]) = \sqrt{\lambda} x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (a_{dj}(t, x) D_j u - F_d) = 0 \end{cases}$$

in Ω_T . Moreover,

$$(3.16) \quad \|Du\|_{L_2(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_2(\Omega_T, \mu)} \leq N \|F\|_{L_2(\Omega_T, \mu)} + N \|f\|_{L_2(\Omega_T, \mu)},$$

where $N = N(\kappa)$.

Proof. We first prove the *a priori* estimate (3.16). Let $u \in \mathcal{H}_2^1(\Omega_T, \mu)$ be a weak solution of (3.15). By multiplying the equation (3.15) with u and using

integration by parts and (1.1), we obtain

$$\begin{aligned} & \sup_{t \in (-\infty, T)} \int_{\mathbb{R}_+^d} |u(t, x)|^2 \mu(dx) + \int_{\Omega_T} |Du|^2 \mu(dz) + \lambda \int_{\Omega_T} |u(z)|^2 \mu(dz) \\ & \leq N \int_{\Omega_T} |F(z)| |Du(z)| \mu(dz) + N\lambda^{1/2} \int_{\Omega_T} |f(z)| |u(z)| \mu(dz). \end{aligned}$$

Then, by Young's inequality, we obtain (3.16).

From (3.16), we see that the uniqueness follows. Now, to prove the existence of solution, for each $k \in \mathbb{N}$, let

$$(3.17) \quad \hat{Q}_k = (-k^2, \min\{k^2, T\}) \times B_k^+.$$

We consider the equation

$$(3.18) \quad x_d^\alpha (a_0 u_t + \lambda c_0 u) - D_i(x_d^\alpha (a_{ij} D_j u - F_i)) = \lambda^{1/2} x_d^\alpha f \quad \text{in } \hat{Q}_k$$

with the boundary conditions

$$(3.19) \quad u = 0 \text{ on } \partial_p \hat{Q}_k \setminus \{x_d = 0\} \quad \text{and} \quad \lim_{x_d \rightarrow 0^+} x_d^\alpha (a_{dj} D_j u - F_d) = 0,$$

where $\partial_p \hat{Q}_k$ is the parabolic boundary of $\hat{Q}_k$. By Galerkin's method, for each k , there exists a unique weak solution $u_k \in \mathcal{H}_2^1(\hat{Q}_k, \mu)$ to (3.18)–(3.19). By taking $u_k = 0$ on $\Omega_T \setminus \hat{Q}_k$, we also have

$$\begin{aligned} & \sup_{t \in (-\infty, T)} \|u_k(t, \cdot)\|_{L_2(\mathbb{R}_+^d, \mu)} + \|Du_k\|_{L_2(\Omega_T, \mu)} + \lambda^{1/2} \|u_k\|_{L_2(\Omega_T, \mu)} \\ & \leq N \|F\|_{L_2(\Omega_T, \mu)} + N \|f\|_{L_2(\Omega_T, \mu)}. \end{aligned}$$

By the weak compactness, there is a subsequence which is still denoted by $\{u_k\}$ and $u \in \mathcal{H}_2^1(\Omega_T, \mu)$ such that

$$u_k \rightharpoonup u, \quad Du_k \rightharpoonup Du$$

weakly in $L_2(\Omega_T, \mu)$. By taking the limit in the weak formulation of solutions, it is easily seen that u is a weak solution of (1.2). The lemma is thus proved. $\square$

4. EQUATIONS WITH SIMPLE COEFFICIENTS

Throughout this section, let $\bar{a}_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times d}$ be measurable functions which satisfy the ellipticity and boundedness conditions: there is a constant $\kappa \in (0, 1)$ such that

$$(4.1) \quad \kappa |\xi|^2 \leq \bar{a}_{ij}(x_d) \xi_i \xi_j, \text{ and } |\bar{a}_{ij}(x_d)| \leq \kappa^{-1}, \quad \forall \xi \in \mathbb{R}^d, x_d \in \mathbb{R}_+.$$

Let $\bar{a}_0, \bar{c}_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be measurable functions satisfying

$$(4.2) \quad \kappa \leq \bar{a}_0(x_d), \quad \bar{c}_0(x_d) \leq \kappa^{-1} \quad \text{for } x_d \in \mathbb{R}_+.$$

We study (1.2) in which the coefficients a_{ij} are replaced with $\bar{a}_{ij}$. More precisely, we consider

$$(4.3) \quad \begin{cases} x_d^\alpha (\bar{a}_0(x_d)u_t + \lambda \bar{c}_0(x_d)u) - \\ \quad - D_i(x_d^\alpha (\bar{a}_{ij}(x_d)D_j u - F_i)) = \sqrt{\lambda} x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (\bar{a}_{dj}(x_d)D_j u - F_d) = 0 \end{cases}$$

in Ω_T . The above equation is slightly different from (1.2) as there are coefficients $\bar{a}_0$ and $\bar{c}_0$ instead of the identity. We do not need this generality for the proofs of our main results for the divergence-form equation (1.2). However, the results below for (4.3) are needed in the proofs of the main results for the non-divergence form equation (1.4) as in [12].

The main result of this section is the following theorem, which is a weak version of Theorem 2.2.

Theorem 4.1. *Let $\alpha \in (-1, \infty)$, $p \in (1, \infty)$, and $\lambda > 0$. Suppose that (4.1) and (4.2) are satisfied. Then, for each $F \in L_p(\Omega_T, \mu)^d$ and $f \in L_p(\Omega_T, \mu)$, there exists a unique solution $u \in \mathcal{H}_p^1(\Omega_T, \mu)$ of (4.3). Moreover,*

$$(4.4) \quad \|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda} \|u\|_{L_p(\Omega_T, \mu)} \leq N \|F\|_{L_p(\Omega_T, \mu)} + N \|f\|_{L_p(\Omega_T, \mu)},$$

where $N = N(d, \alpha, \kappa, p)$.

The rest of the section is devoted to the proof of this theorem. We need some preliminaries to prove it.

4.1. Lipschitz and Schauder estimates for homogeneous equations. Let $\lambda \geq 0$, $z_0 = (t_0, x_0) \in \mathbb{R}_+^{d+1}$ and $r > 0$. We study (4.3) in $Q_r^+(z_0)$ when $F = 0$, $f = 0$, that is, the homogeneous parabolic equation

$$(4.5) \quad -x_d^\alpha (\bar{a}_0(x_d)u_t + \lambda \bar{c}_0(x_d)u) + D_i(x_d^\alpha \bar{a}_{ij}(x_d)D_j u) = 0$$

in $Q_r^+(z_0)$ with the homogeneous conormal boundary condition

$$(4.6) \quad x_d^\alpha \bar{a}_{dj}(x_d)D_j u = 0 \quad \text{if } B_r(x_0) \cap \partial \mathbb{R}_+^d \neq \emptyset.$$

Our goal is to derive Lipschitz and Schauder estimates for (4.5)–(4.6). We begin with the following lemma.

Lemma 4.2 (Caccioppoli-type inequality). Let $r > 0$, $z_0 = (t_0, x_0) \in \overline{\mathbb{R}_+^{d+1}}$, and $u \in \mathcal{H}_2^1(Q_r^+(z_0), \mu)$ be a weak solution to (4.5)–(4.6). Then, we have

$$\int_{Q_{r/2}^+(z_0)} (|Du|^2 + \lambda|u|^2) \mu(dz) \leq Nr^{-2} \int_{Q_r^+(z_0)} |u|^2 \mu(dz)$$

and

$$\int_{Q_{r/2}^+(z_0)} |u_t|^2 \mu(dz) \leq Nr^{-2} \int_{Q_r^+(z_0)} (|Du|^2 + \lambda|u|^2) d\mu(dz),$$

where $N(d, \alpha, \kappa) > 0$.

Proof. The proof is more or less standard. For the first inequality, we test the equation with $u\zeta^2$, where $\zeta \in C_0^\infty$ is a smooth function, $\zeta = 1$ in $Q_{r/2}(z_0)$, and $\zeta = 0$ near the parabolic boundary $\partial_p Q_r(z_0)$. For the second inequality, we test the equation with $u_t\zeta^2$, and then use the fact that u_t satisfies the same equation as u and the first inequality applied to u_t . (See, e.g., the proof of [8, Lemma 3.3].) We omit the details. $\square$

Next, we prove the local boundedness of solutions of (4.5)–(4.6).

Lemma 4.3 (Local boundedness estimate). Let $r > 0$, $z_0 \in \overline{\mathbb{R}_+^{d+1}}$, and $u \in \mathcal{H}_2^1(Q_r^+(z_0), \mu)$ be a weak solution to (4.5)–(4.6). Then, we have

$$\|u\|_{L_\infty(Q_{r/2}^+(z_0))} \leq N \left(\int_{Q_r^+(z_0)} |u(t, x)|^2 \mu(dz) \right)^{1/2},$$

where $N = N(d, \alpha, \kappa) > 0$.

Proof. We use the Moser iteration. For elliptic equations, a similar argument was also used in [34]. By a scaling, we only need to prove the lemma when $r = 1$. For each $R, \rho \in (0, 1]$ with $\rho < R$, let $\phi \in C_0^\infty((t_0 - R^2, t_0 + R^2) \times B_R(x_0))$ be a cut-off function satisfying

$$\begin{aligned} \phi &= 1 \quad \text{in } Q_\rho(z_0), \\ 0 &\leq \phi \leq 1, \\ |D\phi|^2 + |\partial_t \phi| &\leq \frac{N(d)}{(R - \rho)^2} \quad \text{in } Q_R(z_0). \end{aligned}$$

Let $w = u_+$. For $\beta \geq 2$, using $\phi^2 w^{\beta-1}$ as a test function for the equation (4.5) and using (4.1), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{B_R^+(x_0)} \bar{a}_0(x_d) w^\beta \phi^2 \mu(dx) + \frac{4\kappa(\beta-1)}{\beta} \int_{B_R^+(x_0)} |D(w^{\beta/2})|^2 \phi^2 \mu(dx) \\ & \leq 2\beta \int_{B_R^+(x_0)} \bar{a}_0(x_d) w^\beta \phi |\phi_t| \mu(dx) \\ & \quad + 4d\kappa^{-1} \int_{B_R^+(x_0)} |D(w^{\beta/2})| |D\phi| \phi w^{\beta/2} \mu(dx), \end{aligned}$$

where we used the fact that $\lambda \bar{c}_0(x_d) u \phi^2 w^{\beta-1} \geq 0$. As $\beta \geq 2$, we have $(\beta-1)/\beta \geq \frac{1}{2}$. It then follows that

$$\begin{aligned} & \frac{d}{dt} \int_{B_R^+(x_0)} \bar{a}_0(x_d) w^\beta \phi^2 \mu(dx) + 2\kappa \int_{B_R^+(x_0)} |D(w^{\beta/2})|^2 \phi^2 \mu(dx) \\ & \leq 2\beta \int_{B_R^+(x_0)} \bar{a}_0(x_d) w^\beta \phi |\phi_t| \mu(dx) \\ & \quad + 4d\kappa^{-1} \int_{B_R^+(x_0)} |Dw^{\beta/2}| |D\phi| \phi w^{\beta/2} \mu(dx). \end{aligned}$$

By applying Young's inequality to the last term and then cancelling similar terms, we have

$$\begin{aligned} & \frac{d}{dt} \int_{B_R^+(x_0)} \bar{a}_0(x_d) w^\beta \phi^2 \mu(dx) + \int_{B_R^+(x_0)} |D(w^{\beta/2} \phi)|^2 \mu(dx) \\ & \leq N\beta \int_{B_R^+(x_0)} w^\beta (|\phi_t| + |D\phi|^2) \mu(dx), \end{aligned}$$

where $N = N(d, \kappa)$ and we used (4.2). Integrating this estimate with respect to t on $(t_0 - R^2, t_0)$ and using (4.2) again, we find that

$$\begin{aligned} & \sup_{t \in (t_0 - R^2, t_0)} \int_{B_R^+(x_0)} w^\beta \phi^2 \mu(dx) + R^2 \int_{Q_R^+(z_0)} |D(w^{\beta/2} \phi)|^2 \mu(dz) \\ & \leq \frac{N(d, \kappa)\beta}{(R - \rho)^2} \int_{Q_R^+(z_0)} w^\beta \mu(dz). \end{aligned}$$

From this estimate and Lemma 3.3, it follows that

$$\begin{aligned} (4.7) \quad & \left(\int_{Q_\rho^+(z_0)} w^{\beta \ell_0} \mu(dz) \right)^{1/(\beta \ell_0)} \\ & \leq \left(\frac{N}{R - \rho} \right)^{2/\beta} \beta^{1/\beta} \left(\int_{Q_R^+(z_0)} w^\beta \mu(dz) \right)^{1/\beta}. \end{aligned}$$

We now choose a sequence of radii

$$r_0 = 1, \quad r_{k+1} = \frac{r_k + 1/2}{2},$$

and a sequence of exponents

$$\beta_0 = 2, \quad \beta_{k+1} = \beta_k \ell_0, \quad k = 0, 1, 2, \dots,$$

such that

$$\lim_{k \rightarrow \infty} r_k = \frac{1}{2}, \quad \lim_{k \rightarrow \infty} \beta_k = \infty,$$

and

$$r_k - r_{k+1} = \frac{1}{2^{k+2}}, \quad k = 0, 1, 2, \dots$$

By applying (4.7) with $R = r_k$, $\rho = r_{k+1} < R$, and $\beta = \beta_k$, we have

$$\begin{aligned} & \left(\int_{Q_{r_{k+1}}^+(z_0)} w^{\beta_{k+1}} \mu(dz) \right)^{1/\beta_{k+1}} \\ & \leq (4N)^{2/\beta_k} 2^{2k/\beta_k} \beta_k^{1/\beta_k} \left(\int_{Q_{r_k}(z_0)} w^{\beta_k} \mu(dz) \right)^{1/\beta_k}. \end{aligned}$$

By iterating this estimate, we obtain

$$(4.8) \quad \left(\int_{Q_{r_{k+1}}^+(z_0)} w^{\beta_{k+1}} \mu(dz) \right)^{1/\beta_{k+1}} \leq M_k \left(\int_{Q_1^+(z_0)} w^2 \mu(dz) \right)^{1/2},$$

where

$$M_k = (4N)^{\sum_{j=0}^k 2/\beta_j} 2^{\sum_{j=0}^k 2j/\beta_j} \prod_{j=0}^k \beta_j^{1/\beta_j}.$$

As

$$\sum_{j=0}^{\infty} \frac{2}{\beta_j} < \infty, \quad \sum_{j=0}^{\infty} \frac{2j}{\beta_j} < \infty, \quad \text{and} \quad \prod_{j=0}^{\infty} \beta_j^{1/\beta_j} < \infty,$$

we conclude that $\{M_k\}_k$ is convergent. Therefore, by sending $k \rightarrow \infty$, we deduce from (4.8) that

$$\|u_+\|_{L_\infty(Q_{1/2}^+(z_0))} \leq N \left(\int_{Q_1^+(z_0)} u_+^2(t, x) \mu(dz) \right)^{1/2}.$$

We can get a similar estimate for $u_- = \max\{-u, 0\}$, with the same argument. Hence,

$$\|u\|_{L_\infty(Q_{1/2}^+(z_0))} \leq N \left(\int_{Q_1^+(z_0)} |u(t, x)|^2 \mu(dz) \right)^{1/2}.$$

The lemma is proved. $\square$

We recall that for $\beta \in (0, 1]$ and each parabolic cylinder $Q \subset \mathbb{R}^{d+1}$, the β -Hölder semi-norm of a function f in Q is defined as

$$[f]_{C^{\beta/2, \beta}(Q)} = \sup_{\substack{(t, x), (s, y) \in Q \\ (t, x) \neq (s, y)}} \frac{|f(t, x) - f(s, y)|}{|t - s|^{\beta/2} + |x - y|^\beta}.$$

The following proposition is the key step of the proof.

Proposition 4.4. *Let $q \in (1, 2]$, $r > 0$, $z_0 \in \overline{\mathbb{R}_+^{d+1}}$, and $u \in \mathcal{H}_2^1(Q_r^+(z_0), \mu)$ be a weak solution to (4.5)–(4.6). Then, we have*

$$(4.9) \quad \begin{aligned} & \|Du\|_{L_\infty(Q_{r/2}^+(z_0))} + \sqrt{\lambda}\|u\|_{L_\infty(Q_{r/2}^+(z_0))} \\ & \leq N \left(\int_{Q_r^+(z_0)} (|Du|^q + \lambda^{q/2}|u|^q) \mu(dz) \right)^{1/q} \end{aligned}$$

and

$$(4.10) \quad \begin{aligned} & [D_{x'}u]_{C^{1/2,1}(Q_{r/2}^+(z_0))} + [\mathcal{U}]_{C^{1/2,1}(Q_{r/2}^+(z_0))} + \sqrt{\lambda}[u]_{C^{1/2,1}(Q_{r/2}^+(z_0))} \\ & \leq Nr^{-1} \left(\int_{Q_r^+(z_0)} (|Du|^q + \lambda^{q/2}|u|^q) \mu(dz) \right)^{1/q}, \end{aligned}$$

where $\mathcal{U} = \bar{a}_{dj}(x_d)D_ju$ and $N = N(d, \alpha, \kappa, q)$.

Proof. First of all, whenever the lemma is proved for $q = 2$, the case $q \in (1, 2)$ follows by a standard iteration (see, e.g., [15, pp. 80–82]). Therefore, we only consider the case when $q = 2$. As before, we may assume $r = 1$. The bound of $\|u\|_{L_\infty(Q_{r/2}^+(z_0))}$ follows from Lemma 4.3. Since $D_{x'}u$ and u_t satisfy the same equation as u , from Lemmas 4.3 again we have

$$\|D_{x'}u\|_{L_\infty(Q_{1/2}^+(z_0))} \leq N \left(\int_{Q_{2/3}^+(z_0)} |D_{x'}u|^2 \mu(dz) \right)^{1/2}$$

and

$$\|u_t\|_{L_\infty(Q_{1/2}^+(z_0))} \leq N \left(\int_{Q_{2/3}^+(z_0)} |u_t|^2 \mu(dz) \right)^{1/2}.$$

To make this rigorous, we need to use the finite-difference quotient and pass to the limit. These together with Lemma 4.2 give

$$(4.11) \quad \begin{aligned} & \|D_{x'}u\|_{L_\infty(Q_{1/2}^+(z_0))} + \|u_t\|_{L_\infty(Q_{1/2}^+(z_0))} \\ & \leq N \left(\int_{Q_1^+(z_0)} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2}. \end{aligned}$$

Moreover, again from Lemma 4.2, we also have for any $i, j = 0, 1, 2, \dots$ satisfying $i + j \geq 1$

$$(4.12) \quad \int_{Q_{1/2}^+(z_0)} (|\partial_t^i D_{x'}^j u|^2 + |\partial_t^i D_{x'}^j Du|^2) \mu(dz) \\ \leq N \int_{Q_1^+(z_0)} (|Du|^2 + \lambda|u|^2) \mu(dz),$$

where $N = N(d, \kappa, i, j)$.

Next, we estimate $D_d u$. We first consider the boundary estimate and, without loss of generality, we take $z_0 = 0$. We use a bootstrap argument. Since $\mathcal{U} = \bar{a}_{dj}(x_d) D_j u$, from the equation we have

$$D_d(x_d^\alpha \mathcal{U}) = x_d^\alpha \left(\bar{a}_0(x_d) u_t + \lambda \bar{c}_0(x_d) u - \sum_{i=1}^{d-1} D_i(\bar{a}_{ij} D_j u) \right).$$

By using the boundary condition and Hölder's inequality, we get for any $z \in Q_1^+$

$$(4.13) \quad x_d^\alpha |\mathcal{U}| \leq N \int_0^{x_d} s^\alpha (|u_t(z', s)| + \lambda |u(z', s)| + |DD_{x'} u(z', s)|) ds \\ \leq N \left(\int_0^{x_d} s^\alpha (|u_t(z', s)|^2 + \lambda^2 |u(z', s)|^2 + |DD_{x'} u(z', s)|^2) ds \right)^{1/2} \left(\int_0^{x_d} s^\alpha ds \right)^{1/2}.$$

Thus, when $x_d \in (0, \frac{1}{2}]$, by the Sobolev embedding in the z' variables, (4.12), and Lemma 4.2, for an integer $k \geq (d+1)/4$,

$$x_d^\alpha |\mathcal{U}| \leq N \left(\int_0^{1/2} s^\alpha (|u_t(z', s)|^2 + \lambda^2 |u(z', s)|^2 + |DD_{x'} u(z', s)|^2) ds \right)^{1/2} \left(\int_0^{x_d} s^\alpha ds \right)^{1/2} \\ \leq N \left(\int_0^{1/2} s^\alpha (\|u_t(\cdot, s)\|_{W_2^{k,2k}(Q'_{1/2})}^2 + \lambda^2 \|u(\cdot, s)\|_{W_2^{k,2k}(Q'_{1/2})}^2 + \|DD_{x'} u(\cdot, s)\|_{W_2^{k,2k}(Q'_{1/2})}^2) ds \right)^{1/2} \left(\int_0^{x_d} s^\alpha ds \right)^{1/2} \\ \leq N \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} x_d^{(\alpha+1)/2},$$

which implies that

$$(4.14) \quad |\mathcal{U}| \leq N \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} x_d^{(1-\alpha)/2} \quad \text{in } Q_{1/2}^+.$$

This together with (4.11) gives

$$|Du| \leq N \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} x_d^{-(1-\alpha)/2} \quad \text{in } Q_{1/2}^+.$$

Since $D_{x'}u$ satisfies the same equation, by a covering argument and Lemma 4.2 we have

$$\begin{aligned} (4.15) \quad |DD_{x'}u| &\leq N \left(\int_{Q_{2/3}^+} (|DD_{x'}u|^2 + \lambda|D_{x'}u|^2) \mu(dz) \right)^{1/2} x_d^{-(1-\alpha)/2} \\ &\leq N \left(\int_{Q_1^+} |D_{x'}u|^2 \mu(dz) \right)^{1/2} x_d^{-(1-\alpha)/2} \quad \text{in } Q_{1/2}^+. \end{aligned}$$

Now, we plug (4.11) and (4.15) into (4.13) and use Lemmas 4.3 and 4.2 to get

$$\begin{aligned} |u| &\leq N x_d^{-\alpha} \int_0^{x_d} s^\alpha s^{-(1-\alpha)/2} ds \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} \\ &\leq N x_d^{1-(1-\alpha)/2} \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} \quad \text{in } Q_{1/2}^+, \end{aligned}$$

which improves (4.14). Repeating this procedure, in finitely many steps, we get

$$(4.16) \quad |u| \leq N x_d \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2},$$

and therefore

$$|Du| \leq N \left(\int_{Q_1^+} (|Du|^2 + \lambda|u|^2) \mu(dz) \right)^{1/2} \quad \text{in } Q_{1/2}^+,$$

which gives (4.9) in this case.

In the interior case when $x_{0d} \geq 2r = 2$, the coefficients $\bar{a}_{ij}(x_d) = x_d^\alpha \bar{a}_{ij}(x_d)$ are nondegenerate in $Q_{2/3}(z_0)$ and independent of z' . By using the standard energy estimate (cf. [8, Lemma 3.5]), we also have

$$(4.17) \quad |Du| \leq N \left(\int_{Q_1(z_0)} (|Du|^2 + \lambda|u|^2) dz \right)^{1/2} \quad \text{in } Q_{1/2}(z_0).$$

Since in $Q_1(z_0)$ we have $x_d \sim x_{0d}$ so that $\mu(dz) \sim x_{0d}^\alpha dz$, we also obtain (4.9) in the interior case. Moreover, (4.11) still holds in this case. When $x_{0d} \in (0, 2)$, (4.9) follows from a covering argument and the doubling property of μ .

It remains to prove (4.10). By using (4.9) and (4.11), we obtain the bound of the third term on the lefthand side of (4.10). Since $D_{x'}u$ and u_t satisfy the same equation as u , from (4.9), (4.12), and Lemma 4.2, we have

$$(4.18) \quad \|DD_{x'}u\|_{L_\infty(Q_{1/2}^+(z_0))} + \|Du_t\|_{L_\infty(Q_{1/2}^+(z_0))}$$

$$\begin{aligned} &\leq N \left(\int_{Q_{2/3}^+(z_0)} (|DD_{x'} u|^2 + \lambda |D_{x'} u|^2 + |Du_t|^2 + \lambda |u_t|^2) \mu(dz) \right)^{1/2} \\ &\leq N \left(\int_{Q_1^+(z_0)} (|Du|^2 + \lambda |u|^2) \mu(dz) \right)^{1/2}, \end{aligned}$$

which yields

$$\begin{aligned} &[D_{x'} u]_{C^{1/2,1}(Q_{1/2}^+(z_0))} + \|u_t\|_{L_\infty(Q_{1/2}^+(z_0))} + \|D_{x'} u\|_{L_\infty(Q_{1/2}^+(z_0))} \\ &\leq N \left(\int_{Q_1^+(z_0)} (|Du|^2 + \lambda |u|^2) \mu(dz) \right)^{1/2}. \end{aligned}$$

To estimate $D_d u$, we again discuss two cases. In the boundary case when $z_0 = 0$, from the equation we have

$$(4.19) \quad D_d u = \bar{a}_0 u_t + \lambda \bar{c}_0 u - \sum_{i=1}^{d-1} \bar{a}_{ij} D_{ij} u - \alpha x_d^{-1} u,$$

which together with (4.11), (4.16), (4.18), and Lemma 4.3 gives

$$(4.20) \quad \|D_d u\|_{L_\infty(Q_{1/2}^+(z_0))} \leq N \left(\int_{Q_1^+(z_0)} (|Du|^2 + \lambda |u|^2) \mu(dz) \right)^{1/2}.$$

In the interior case (i.e., when $x_{0d} \geq 2$), by (4.19), (4.11), (4.17), (4.18), and Lemma 4.3, we still get (4.20). This completes the proof of (4.10) and thus the proposition. $\square$

From Lemma 3.4 and Proposition 4.4, we obtain the following solution decomposition.

Proposition 4.5. *Let $z_0 \in \overline{\Omega_T}$ and $r > 0$. Suppose that $F \in L_2(Q_{2r}^+(z_0), \mu)^d$, $f \in L_2(Q_{2r}^+(z_0), \mu)$, and $u \in \mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)$ is a weak solution of (4.3) in $Q_{2r}^+(z_0)$. Then, we can write $u(t, x) = v(t, x) + w(t, x)$ in $Q_{2r}^+(z_0)$, where v and w are functions in $\mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)$ and satisfy*

$$(4.21) \quad \int_{Q_{2r}^+(z_0)} |V|^2 \mu(dz) \leq N \int_{Q_{2r}^+(z_0)} (|F|^2 + |f|^2) \mu(dz)$$

and

$$(4.22) \quad \begin{aligned} \|W\|_{L_\infty(Q_r^+(z_0))}^2 &\leq N \int_{Q_{2r}^+(z_0)} |U|^2 \mu(dz) \\ &\quad + N \int_{Q_{2r}^+(z_0)} (|F|^2 + |f|^2) \mu(dz), \end{aligned}$$

where $N = N(d, \kappa, \alpha)$ and

$$V = |Dv| + \lambda^{1/2} |v|, \quad W = |Dw| + \lambda^{1/2} |w|, \quad U = |Du| + \lambda^{1/2} |u|.$$

Proof. Let $v \in \mathcal{H}_2^1(\Omega_T, \mu)$ be a weak solution of the equation

$$\begin{aligned} & x_d^\alpha (\bar{a}_0(x_d)v_t + \lambda \bar{c}_0(x_d)v) \\ & - \operatorname{D}_i(x_d^\alpha (\bar{a}_{ij}(x_d)\operatorname{D}_j v - F_i(z)\chi_{Q_{2r}^+(z_0)}(z))) \\ & = \lambda^{1/2} x_d^\alpha f(z)\chi_{Q_{2r}^+(z_0)}(z) \quad \text{in } \Omega_T \end{aligned}$$

with the boundary condition

$$\lim_{x_d \rightarrow 0^+} x_d^\alpha (\bar{a}_{dj}(x_d)\operatorname{D}_j v - F_d(z)\chi_{Q_{2r}^+(z_0)}(z)) = 0.$$

Then, (4.21) follows by Lemma 3.4. Now, let $w = u - v$ so $w \in \mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)$ is a weak solution of

$$x_d^\alpha (\bar{a}_0(x_d)w_t + \lambda \bar{c}_0(x_d)w) - \operatorname{D}_i(x_d^\alpha \bar{a}_{ij}(x_d)\operatorname{D}_j w) = 0 \quad \text{in } Q_{2r}^+(z_0)$$

with the boundary condition

$$\lim_{x_d \rightarrow 0^+} x_d^\alpha \bar{a}_{dj}(x_d)\operatorname{D}_j w = 0 \quad \text{if } B_{2r}(x_0) \cap \overline{\partial \mathbb{R}_+^d} \neq \emptyset.$$

By Proposition 4.4 and the triangle inequality, we get (4.22). The proof of the proposition is completed. $\square$

4.2. Proof of Theorem 4.1. We are now ready to give the proof of Theorem 4.1.

Proof. When $p = 2$, Theorem 4.1 follows from Lemma 3.4. Therefore, we only need to consider the cases when $p \in (2, \infty)$ and $p \in (1, 2)$.

Case I. $p \in (2, \infty)$. Let $u \in \mathcal{H}_{2,\text{loc}}^1(\Omega_T, \mu)$ be a weak solution of (4.3). It follows from Proposition 4.5 that for every $z_0 \in \overline{\Omega_T}$ and $r > 0$, we have the decomposition

$$u = v + w \quad \text{in } Q_{2r}^+(z_0),$$

where v and w satisfy (4.21) and (4.22). Then (4.4) follows from the standard real variable argument. (See, for example, [9].) We omit the details.

By (4.4), the uniqueness of solutions follows. Hence, it remains to prove the existence of the solution. Recall the definition (3.17). For $k = 1, 2, \dots$, let $F^{(k)} = F(z)\chi_{\hat{Q}_k}(z)$. Then, $F^{(k)} \in L_2(\Omega_T, \mu)^d \cap L_p(\Omega_T, \mu)^d$, and by the dominated convergence theorem, $F^{(k)} \rightarrow F$ in $L_p(\Omega_T, \mu)$ as $k \rightarrow \infty$. Similarly, we define $\{f^{(k)}\} \subset L_2(\Omega_T, \mu) \cap L_p(\Omega_T, \mu)$. Let $u^{(k)} \in \mathcal{H}_2^1(\Omega_T, \mu)$ be the weak solution of the equation (4.3) with $F^{(k)}$ and $f^{(k)}$ in place of F and f , respectively. The existence of $u^{(k)}$ follows from Lemma 3.4. By the estimate (4.4), we have $u^{(k)} \in \mathcal{H}_p^1(\Omega_T, \mu)$. Moreover, by the strong convergence of $\{F^{(k)}\}$ and $\{f^{(k)}\}$

in $L_p(\Omega_T, \mu)$, we infer that $\{u^{(k)}\}$ is a Cauchy sequence in $\mathcal{H}_p^1(\Omega_T, \mu)$. Let $u \in \mathcal{H}_p^1(\Omega_T, \mu)$ be its limit. Then, by passing to the limit in the weak formulation of solutions, it is easily seen that u is a solution to the equation (4.3).

Case II. $p \in (1, 2)$. We use a duality argument. We first prove the estimate (4.4). Let $q = p/(p-1) \in (2, \infty)$ and let $G \in L_q(\Omega_T, \mu)^d$ and $g \in L_q(\Omega_T, \mu)$. We consider the adjoint problem in $\mathbb{R} \times \mathbb{R}_+^d$

$$(4.23) \quad \begin{cases} x_d^\alpha (-\bar{a}_0 v_t + \lambda \bar{c}_0 v) \\ \quad - D_i(x_d^\alpha (\bar{a}_{ji}(x_d) D_j v - G_i \chi_{(-\infty, T)})) = \lambda^{1/2} x_d^\alpha g \chi_{(-\infty, T)}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (\bar{a}_{jd} D_j v - G_d \chi_{(-\infty, T)}) = 0. \end{cases}$$

By Case 1, there exists a unique solution $v \in \mathcal{H}_q^1(\mathbb{R} \times \mathbb{R}_+^d, \mu)$ of the above equation, which satisfies

$$(4.24) \quad \int_{\mathbb{R} \times \mathbb{R}_+^d} (|Dv|^q + \lambda^{q/2} |v|^q) \mu(dz) \leq N \int_{\Omega_T} (|G|^q + |g|^q) \mu(dz).$$

Moreover, by the uniqueness of solutions, we have $v = 0$ for $t \geq T$. It follows from the equations (4.3) and (4.23) that

$$\int_{\Omega_T} (G \cdot \nabla u + \lambda^{1/2} g u) \mu(dz) = \int_{\Omega_T} (F \cdot \nabla v + \lambda^{1/2} f v) \mu(dz).$$

Therefore, by Hölder's inequality and (4.24),

$$\begin{aligned} & \left| \int_{\Omega_T} (G \cdot \nabla u + \lambda^{1/2} g u) \mu(dz) \right| \\ & \leq \|F\|_{L_p(\Omega, \mu)} \|\nabla v\|_{L_q(\Omega_T, \mu)} + \lambda^{1/2} \|f\|_{L_p(\Omega_T, \mu)} \|v\|_{L_q(\Omega_T, \mu)} \\ & \leq N (\|F\|_{L_p(\Omega, \mu)} + \|f\|_{L_p(\Omega_T, \mu)}) (\|G\|_{L_q(\Omega_T, \mu)} + \|g\|_{L_q(\Omega_T, \mu)}). \end{aligned}$$

From this last estimate, and as G and g are arbitrary, we obtain (4.4).

It now remains to prove the existence of solution $u \in \mathcal{H}_p^1(\Omega_T, \mu)$. We proceed slightly differently from Case 1 and follow the argument in Section 8 of [10]. For $i = 1, 2, \dots, d$ and $k = 1, 2, \dots$, let

$$F_i^{(k)} = \max(-k, \min(k, F_i)) \chi_{\hat{Q}_k}.$$

Then, $F^{(k)} \in L_2(\Omega_T, \mu)^d \cap L_p(\Omega_T, \mu)^d$, and by the dominated convergence theorem, $F^{(k)} \rightarrow F$ in $L_p(\Omega_T, \mu)$ as $k \rightarrow \infty$. Similarly, we define

$$\{f^{(k)}\} \subset L_2(\Omega_T, \mu) \cap L_p(\Omega_T, \mu).$$

By Lemma 3.4, there is a unique weak solution $u^{(k)} \in \mathcal{H}_2^1(\Omega_T, \mu)$ to the equation (4.3) with $F^{(k)}$ and $f^{(k)}$ in place of F and f , respectively. As in Case 1, it suffices to prove that $u^{(k)} \in \mathcal{H}_p^1(\Omega_T, \mu)$. Let us fix a $k \in \mathbb{N}$. Because μ is a doubling measure, there exists $N_0 = N_0(\alpha, d) > 0$ such that

$$(4.25) \quad \mu(\hat{Q}_{2r}) \leq N_0 \mu(\hat{Q}_r), \quad \forall r > 0.$$

Since $u^{(k)} \in \mathcal{H}_2^1(\Omega_T, \mu)$, by Hölder's inequality,

$$(4.26) \quad \|u^{(k)}\|_{L_p(\hat{Q}_{2k}, \mu)} + \|Du^{(k)}\|_{L_p(\hat{Q}_{2k}, \mu)} < \infty.$$

Therefore, it remains to prove that $\|u^{(k)}\|_{L_p(\Omega_T \setminus \hat{Q}_{2k}, \mu)} < \infty$. To this end, for $j \geq 0$, let η_j be such that

$$\eta_j \equiv \begin{cases} 0 & \text{in } \hat{Q}_{2^j k}, \\ 1 & \text{outside } \hat{Q}_{2^{j+1} k}, \end{cases}$$

and $|D\eta_j| \leq C_0 2^{-j}$, $|(\eta_j)_t| \leq C_0 2^{-2j}$, where C_0 is independent of j . Observe that the supports of $F^{(k)}$ and $f^{(k)}$ are in $\hat{Q}_k$, while the supports of η_j are all outside $\hat{Q}_k$. Thus, $\eta_j F_i^{(k)} \equiv \eta_j f^{(k)} \equiv F_i^{(k)} D_i \eta_j \equiv 0$ for every $i = 1, 2, \dots, d$ and $j = 0, 1, \dots$. Because of this, a simple calculation reveals that $w^{(k, \ell)} := u^{(k)} \eta_\ell \in \mathcal{H}_2^1(\Omega_T, \mu)$ is a weak solution of

$$\begin{cases} x_d^\alpha (\bar{a}_0 w_t^{(k, \ell)} + \lambda \bar{c}_0 w^{(k, \ell)}) - \\ \quad - D_i (x_d^\alpha (\bar{a}_{ij} D_j w^{(k, \ell)} - F_i^{(k, \ell)})) = \lambda^{1/2} x_d^\alpha f^{(k, \ell)}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (\bar{a}_{dj} D_j w^{(k, \ell)} - F_d^{(k, \ell)}) = 0 \end{cases}$$

in Ω_T , where

$$\begin{aligned} F_i^{(k, \ell)} &= u^{(k)} \bar{a}_{ij} D_j \eta_\ell, \quad i = 1, 2, \dots, d, \\ f^{(k, \ell)} &= \lambda^{-1/2} (u^{(k)} (\eta_\ell)_t - \bar{a}_{ij} D_j u^{(k)} D_i \eta_\ell). \end{aligned}$$

Now, by applying the estimate (3.16) to the above equation of $w^{(k, \ell)}$, we have

$$\begin{aligned} \|Dw^{(k, \ell)}\|_{L_2(\Omega_T, \mu)} + \sqrt{\lambda} \|w^{(k, \ell)}\|_{L_2(\Omega_T, \mu)} \\ \leq N \|F^{(k, j)}\|_{L_2(\Omega_T, \mu)} + N \|f^{(k, \ell)}\|_{L_2(\Omega_T, \mu)}, \end{aligned}$$

which implies that

$$\begin{aligned} \|Du^{(k)}\|_{L_2(\hat{Q}_{2^{j+2}k} \setminus \hat{Q}_{2^{j+1}k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2^{j+2}k} \setminus \hat{Q}_{2^{j+1}k}, \mu)} \\ \leq N (2^{-j} \|u^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} + \lambda^{-1/2} 2^{-2j} \|u^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)}) \end{aligned}$$

$$\begin{aligned}
& + \lambda^{-1/2} 2^{-j} \|Du^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} \\
& \leq C 2^{-j} (\|Du^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)})
\end{aligned}$$

for every $j \geq 1$, where C also depends on λ , but is independent of j . By iterating the last estimate, we obtain

$$\begin{aligned}
(4.27) \quad & \|Du^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} \\
& \leq C^j 2^{-j(j-1)/2} (\|Du^{(k)}\|_{L_2(\hat{Q}_{2k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2k}, \mu)}).
\end{aligned}$$

Finally, by Hölder's inequality, (4.25), and (4.27), we have

$$\begin{aligned}
& \|Du^{(k)}\|_{L_p(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_p(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} \\
& \leq (\mu(\hat{Q}_{2^{j+1}k}))^{1/p-1/2} (\|Du^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} \\
& \quad + \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)}) \\
& \leq N_0^{j(1/p-1/2)} (\mu(\hat{Q}_{2k}))^{1/p-1/2} C^j 2^{-j(j-1)/2} \times \\
& \quad \times (\|Du^{(k)}\|_{L_2(\hat{Q}_{2k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2k}, \mu)}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \|Du^{(k)}\|_{L_p(\Omega_T \setminus \hat{Q}_{2k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_p(\Omega_T \setminus \hat{Q}_{2k}, \mu)} \\
& = \sum_{j=1}^{\infty} (\|Du^{(k)}\|_{L_p(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)} + \sqrt{\lambda} \|u^{(k)}\|_{L_p(\hat{Q}_{2^{j+1}k} \setminus \hat{Q}_{2^j k}, \mu)}) \\
& \leq N \|Du^{(k)}\|_{L_2(\hat{Q}_{2k}, \mu)} + N \sqrt{\lambda} \|u^{(k)}\|_{L_2(\hat{Q}_{2k}, \mu)} < \infty.
\end{aligned}$$

Using this estimate and (4.26), we infer that $u^{(k)} \in \mathcal{H}_p^1(\Omega_T, \mu)$. The theorem is thus proved. $\square$

5. EQUATIONS WITH PARTIALLY VMO COEFFICIENTS

In this section, we give the proofs of Theorem 2.2, Corollary 2.3, Theorem 2.4, and Corollary 2.10. We begin with the proof of Theorem 2.2.

5.1. Proof of Theorem 2.2. We need the following decomposition result for our proof.

Proposition 5.1. *Let $\gamma_0 \in (0, 1)$, $\alpha \in (-1, \infty)$, $r \in (0, \infty)$, $z_0 \in \overline{\Omega_T}$, and $q \in (2, \infty)$. Suppose $G = |F| + |f| \in L_2(Q_{2r}^+(z_0), \mu)$, and $u \in \mathcal{H}_q^1(Q_{2r}^+(z_0), \mu)$ is a weak solution of (1.2). If Assumption 2.1 (γ_0, R_0) is satisfied and we have that $\text{spt}(u) \subset (s - (R_0 r_0)^2, s + (R_0 r_0)^2) \times \mathbb{R}_+^d$ for some $r_0 > 0$ and $s \in \mathbb{R}$, then*

$$u(t, x) = v(t, x) + w(t, x) \quad \text{in } Q_{2r}^+(z_0),$$

where v and w are functions in $\mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)$ that satisfy

$$(5.1) \quad \begin{aligned} \int_{Q_{2r}^+(z_0)} |V|^2 \mu(dz) &\leq N \int_{Q_{2r}^+(z_0)} |G|^2 \mu(dz) \\ &\quad + N(\gamma_0^{1-2/q} + r_0^{2-4/q}) \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} \end{aligned}$$

and

$$(5.2) \quad \|W\|_{L_\infty(Q_r^+(z_0))}^2 \leq N \int_{Q_{2r}^+(z_0)} |U|^2 \mu(dz) + N \int_{Q_{2r}^+(z_0)} |G|^2 \mu(dz),$$

where

$$V = |Dv| + \sqrt{\lambda}|v|, \quad W = |Dw| + \sqrt{\lambda}|w|, \quad U = |Du| + \sqrt{\lambda}|u|,$$

and $N = N(d, \alpha, \kappa, q)$.

Proof. For $i = 1, 2, \dots, d$, let

$$b_i(t, x) = \chi_{Q_{2r}^+(z_0)}(z)(a_{ij}(t, x) - [a_{ij}]_{2r, z_0}(x_d))D_j u(t, x) - F_i(z)\chi_{Q_{2r}^+(z_0)}(z),$$

where $[a_{ij}]_{2r, z_0}(x_d)$ is defined in Assumption 2.1. Observe that $b_i \in L_2(\Omega_T, \mu)$. In particular, if $r \in (0, R_0/2)$, it follows from Hölder's inequality and Assumption 2.1 (γ_0, R_0) that

$$\begin{aligned} &\int_{Q_{2r}^+(z_0)} |b(z)|^2 \mu(dz) \\ &\leq \left(\int_{Q_{2r}^+(z_0)} |a_{ij} - [a_{ij}]_{2r, z_0}|^{2q/(q-2)} \mu(dz) \right)^{(q-2)/q} \\ &\quad \times \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} + \int_{Q_{2r}^+(z_0)} |F|^2 \mu(dz) \\ &\leq N\gamma_0^{(q-2)/q} \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} + \int_{Q_{2r}^+(z_0)} |F|^2 \mu(dz). \end{aligned}$$

On the other hand, when $r \geq R_0/2$, as

$$\text{spt}(u) \subset (s - (R_0 r_0)^2, s + (R_0 r_0)^2) \times \mathbb{R}_+^d,$$

and by the boundedness of (a_{ij}) in (1.1), we have

$$\int_{Q_{2r}^+(z_0)} |b(z)|^2 \mu(dz)$$

$$\begin{aligned}
&\leq N(\kappa) \left(\int_{Q_{2r}^+(z_0)} \chi_{(s-(R_0 r_0)^2, s+(R_0 r_0)^2)}(t) \mu(dz) \right)^{(q-2)/q} \\
&\quad \times \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} + N \int_{Q_{2r}^+(z_0)} |F|^2 \mu(dz) \\
&\leq N \left(\frac{R_0 r_0}{r} \right)^{2(q-2)/q} \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} + N \int_{Q_{2r}^+(z_0)} |F|^2 \mu(dz) \\
&\leq N r_0^{2(q-2)/q} \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} + N \int_{Q_{2r}^+(z_0)} |F|^2 \mu(dz).
\end{aligned}$$

Hence, for every $r \in (0, \infty)$ we have

$$\begin{aligned}
(5.3) \quad &\int_{Q_{2r}^+(z_0)} |b(z)|^2 \mu(dz) \\
&\leq N(r_0^{2(q-2)/q} + \gamma_0^{(q-2)/q}) \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} \\
&\quad + N \int_{Q_{2r}^+(z_0)} |F|^2 \mu(dz).
\end{aligned}$$

Now, let $v \in \mathcal{H}_2^1(\Omega_T, \mu)$ be a weak solution in Ω_T of

$$\begin{cases} x_d^\alpha (\partial_t v + \lambda v) - \text{Di}(x_d^\alpha ([a_{ij}]_{2r, z_0}(x_d) D_j v + b_i)) = \lambda^{1/2} x_d^\alpha f \chi_{Q_{2r}^+(z_0)}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha ([a_{dj}]_{2r, z_0}(x_d) D_j v + b_d) = 0. \end{cases}$$

By Lemma 3.4 and (5.3), we have

$$\begin{aligned}
(5.4) \quad &\int_{Q_{2r}^+(z_0)} |V|^2 \mu(dz) \leq N \int_{Q_{2r}^+(z_0)} (|b|^2 + |f|^2) \mu(dz) \\
&\leq N(\gamma_0^{1-2/q} + r_0^{2-4/q}) \left(\int_{Q_{2r}^+(z_0)} |Du|^q \mu(dz) \right)^{2/q} \\
&\quad + N \int_{Q_{2r}^+(z_0)} |G|^2 \mu(dz),
\end{aligned}$$

which yields (5.1). Let $w = u - v \in \mathcal{H}_2^1(Q_{2r}^+(z_0), \mu)$, which is a weak solution of

$$x_d^\alpha (w_t + \lambda w) - \text{Di}(x_d^\alpha [a_{ij}]_{2r, z_0}(x_d) D_j w) = 0 \quad \text{in } Q_{2r}^+(z_0)$$

with the boundary condition

$$\lim_{x_d \rightarrow 0^+} x_d^\alpha [a_{dj}]_{2r, z_0}(x_d) D_j w = 0 \quad \text{if } B_{2r}(x_0) \cap \overline{\partial \mathbb{R}_+^d} \neq \emptyset.$$

Then, we apply Proposition 4.4 to conclude that

$$\begin{aligned}\|W\|_{L_\infty(Q_r^+(z_0))} &\leq N\left(\int_{Q_{2r}^+(z_0)} |W|^2 \mu(dz)\right)^{1/2} \\ &\leq N\left(\int_{Q_{2r}^+(z_0)} |U|^2 \mu(dz)\right)^{1/2} + N\left(\int_{Q_{2r}^+(z_0)} |V|^2 \mu(dz)\right)^{1/2}.\end{aligned}$$

From this and (5.4), we obtain (5.2). $\square$

Proof of Theorem 2.2. It suffices to consider the case $p \in (2, \infty)$, as the case $p \in (1, 2)$ can be proved by using the duality argument as in the proof of Theorem 4.1. We first prove the *a priori* estimate (2.1) for each weak solution $u \in \mathcal{H}_p^1(\Omega_T, \mu)$ of (1.2). We suppose $\lambda > 0$. Assume for a moment that

$$\text{spt}(1_{t < T} u) \subset (s - (R_0 r_0)^2, s + (R_0 r_0)^2) \times \mathbb{R}_+^d$$

with some $s \in (-\infty, T)$ and $r_0 \in (0, 1)$. We claim that (2.1) holds if y_0 and r_0 are sufficiently small depending on d, α, κ , and p . Let $q \in (2, p)$ be fixed. Applying Proposition 5.1, for each $r > 0$ and $z_0 \in \overline{\Omega_T}$, we can write $u(t, x) = v(t, x) + w(t, x)$ in $Q_{2r}^+(z_0)$, where v and w satisfy (5.1) and (5.2). Then, it follows from the standard real variable argument (see, e.g., [9]) that

$$\begin{aligned}\|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda}\|u\|_{L_p(\Omega_T, \mu)} &\leq N(y_0^{1-2/q} + r_0^{2-4/q})\|Du\|_{L_p(\Omega_T, \mu)} \\ &\quad + N\|F\|_{L_p(\Omega_T, \mu)} + N\|f\|_{L_p(\Omega_T, \mu)}\end{aligned}$$

for $N = N(d, \alpha, \kappa, p)$. From this, and by choosing y_0 and r_0 sufficiently small so that $N(y_0^{1-2/q} + r_0^{2-4/q}) < \frac{1}{2}$, we obtain (2.1).

We now remove the additional assumption that

$$\text{spt}(1_{t < T} u) \subset (s - (R_0 r_0)^2, s + (R_0 r_0)^2) \times \mathbb{R}_+^d$$

by using a partition of unity argument. Let

$$\xi = \xi(t) \in C_0^\infty(-(R_0 r_0)^2, (R_0 r_0)^2)$$

be a standard non-negative cut-off function satisfying

$$(5.5) \quad \int_{\mathbb{R}} \xi^p(s) ds = 1, \quad \int_{\mathbb{R}} |\xi'(s)|^p ds \leq \frac{N}{(R_0 r_0)^{2p}}.$$

For any $s \in (-\infty, \infty)$, let $u^{(s)}(z) = u(z)\xi(t - s)$ for $z = (t, x) \in \Omega_T$. Then $u^{(s)} \in \mathcal{H}_p^1(\Omega_T, \mu)$ is a weak solution of

$$\begin{cases} x_d^\alpha(u_t^{(s)} + \lambda u^{(s)}) - D_i(x_d^\alpha(a_{ij}D_j u^{(s)} - F_i^{(s)})) = \lambda^{1/2} x_d^\alpha f^{(s)}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha(a_{dj}D_j u^{(s)} - F_d^{(s)}) = 0 \end{cases}$$

in Ω_T , where

$$F^{(s)}(z) = \xi(t-s)F(z), \quad f^{(s)}(z) = \xi(t-s)f(z) + \lambda^{-1/2}\xi'(t-s)u(z).$$

As $\text{spt}(1_{t < T}u^{(s)}) \subset (s - (R_0r_0)^2, s + (R_0r_0)^2) \times \mathbb{R}_+^d$, we can apply the estimate we just proved and infer that

$$\|Du^{(s)}\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda}\|u^{(s)}\|_{L_p(\Omega_T, \mu)} \leq N\|F^{(s)}\|_{L_p(\Omega_T, \mu)} + N\|f^{(s)}\|_{L_p(\Omega_T, \mu)}.$$

Integrating with respect to s , we get

$$(5.6) \quad \int_{\mathbb{R}} (\|Du^{(s)}\|_{L_p(\Omega_T, \mu)}^p + \lambda^{p/2}\|u^{(s)}\|_{L_p(\Omega_T, \mu)}^p) ds \\ \leq N \int_{\mathbb{R}} (\|F^{(s)}\|_{L_p(\Omega_T, \mu)}^p + \|f^{(s)}\|_{L_p(\Omega_T, \mu)}^p) ds.$$

It follows from the Fubini theorem and (5.5) that

$$\int_{\mathbb{R}} \|Du^{(s)}\|_{L_p(\Omega_T, \mu)}^p ds = \int_{\Omega_T} \int_{\mathbb{R}} |Du(z)|^p \xi^p(t-s) ds \mu(dz) = \|Du\|_{L_p(\Omega_T, \mu)}^p.$$

Similarly,

$$\int_{\mathbb{R}} \|u^{(s)}\|_{L_p(\Omega_T, \mu)}^p ds = \|u\|_{L_p(\Omega_T, \mu)}^p, \\ \int_{\mathbb{R}} \|F^{(s)}\|_{L_p(\Omega_T, \mu)}^p ds = \|F\|_{L_p(\Omega_T, \mu)}^p.$$

Since r_0 depends only on d , α , κ , and p , from the definition of $f^{(s)}$, (5.5), and the Fubini theorem, we have

$$\left(\int_{\mathbb{R}} \|f^{(s)}\|_{L_p(\Omega, \mu)}^p ds \right)^{1/p} \leq N\|f\|_{L_p(\Omega_T, \mu)} + NR_0^{-2}\lambda^{-1/2}\|u\|_{L_p(\Omega_T, \mu)}$$

for $N = N(d, \alpha, \kappa, p)$. Collecting these estimates, we infer from (5.6) that

$$\|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda}\|u\|_{L_p(\Omega_T, \mu)} \\ \leq N\|F\|_{L_p(\Omega_T, \mu)} + N\|f\|_{L_p(\Omega_T, \mu)} + NR_0^{-2}\lambda^{-1/2}\|u\|_{L_p(\Omega_T, \mu)}$$

with $N = N(d, \alpha, \kappa, p)$. Now we choose $\lambda_0 = 2N$. For $\lambda \geq \lambda_0 R_0^{-2}$, we have $NR_0^{-2}\lambda^{-1/2} \leq \sqrt{\lambda}/2$, and therefore

$$\|Du\|_{L_p(\Omega_T, \mu)} + \sqrt{\lambda}\|u\|_{L_p(\Omega_T, \mu)} \\ \leq N\|F\|_{L_p(\Omega_T, \mu)} + N\|f\|_{L_p(\Omega_T, \mu)} + \frac{\sqrt{\lambda}}{2}\|u\|_{L_p(\Omega_T, \mu)},$$

which yields (1.2).

Finally, the solvability of solution $u \in \mathcal{H}_p^1(\Omega_T, \mu)$ can be obtained by the method of continuity using the solvability of the equation

$$\begin{cases} x_d(u_t + \lambda u) - D_i(x_d^\alpha D_i u - F_i) = \lambda^{1/2} x_d^\alpha f, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha (D_d u - F_d) = 0 \end{cases}$$

in Ω_T , which is proved in Theorem 4.1. The proof is now completed. $\square$

5.2. Proof of Corollary 2.3. We adapt an idea in [24]. Let $p_1 > p_0$ satisfy

$$(5.7) \quad \begin{cases} \frac{d+2+\alpha_+}{p_0} \leq 1 + \frac{d+2+\alpha_+}{p_1} & d \geq 2, \\ \frac{4+\alpha_+}{p_0} \leq 1 + \frac{4+\alpha_+}{p_1} & d = 1. \end{cases}$$

Then, it follows from Lemma 3.1 that

$$(5.8) \quad \|u\|_{L_{p_1}(Q_2^+, \mu)} \leq N \|u\|_{\mathcal{H}_{p_0}^1(Q_2^+, \mu)} < \infty.$$

We split the proof into two cases.

Case I: We have $p \leq p_1$. Observe that we only need to consider the case $p^* \leq p_0$, because otherwise we can proceed with the proof with p_0 in place of p^* and then apply Hölder's inequality (noting that in this case (2.3) and (2.4) still hold with p_0 in place of p^*). Let $\eta \in C_0^\infty((-4, 4) \times B_2)$ be such that $\eta \equiv 1$ on Q_1 . A direct calculation yields that $u\eta \in \mathcal{H}_{p_0}^1(\Omega_0, \mu)$ satisfies

$$(5.9) \quad \begin{cases} x_d^\alpha((u\eta)_t + \lambda u\eta) - D_i(x_d^\alpha(a_{ij}D_j(u\eta) - \tilde{F}_i)) = x_d^\alpha \tilde{f}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha(a_{dj}D_j(u\eta) - \tilde{F}_d) = 0 \end{cases}$$

in $(-4, 0) \times \mathbb{R}_+^d$, with the zero initial condition $(u\eta)(-4, \cdot) = 0$, where

$$\tilde{F}_i = F_i\eta - a_{ij}uD_j\eta, \quad \tilde{f} = f\eta + \lambda u\eta + u\eta_t - D_i\eta(a_{ij}D_ju - F_i),$$

and $\lambda > \lambda_0 R_0^{-2}$.

Let $q = p/(p-1)$, $q_0 = p_0/(p_0-1)$, and $G = (G_1, \dots, G_d)$, $g \in C_0^\infty(Q_1^+)$ satisfying

$$\|G\|_{L_q(Q_1^+, \mu)} = \|g\|_{L_q(Q_1^+, \mu)} = 1.$$

By Theorem 2.2, there is a weak solution $v \in \mathcal{H}_{q_0}^1((-4, 0) \times \mathbb{R}_+^d, \mu)$ to

$$(5.10) \quad \begin{cases} -x_d^\alpha(v_t - \lambda v) - D_i(x_d^\alpha(a_{ji}D_jv - G_i)) = \sqrt{\lambda}x_d^\alpha g, \\ \lim_{x_d \rightarrow 0^+} (x_d^\alpha(a_{jd}D_jv - G_d)) = 0 \end{cases}$$

in $(-4, 0) \times \mathbb{R}_+^d$, with the zero terminal condition $v(0, \cdot) = 0$. Since $q \leq q_0$ and G and g are compactly supported, following the proof of Theorem 4.1 (Case 2), we have $v \in H_q^1((-4, 0) \times \mathbb{R}_+^d, \mu)$. Moreover, by Theorem 2.2, we have

$$(5.11) \quad \sqrt{\lambda} \|v\|_{L_q((-4, 0) \times \mathbb{R}_+^d, \mu)} + \|Dv\|_{L_q((-4, 0) \times \mathbb{R}_+^d, \mu)} \leq N.$$

Testing (5.9) and (5.10) with v and $u\eta$, respectively, we get

$$\int_{Q_1^+} (\nabla u \cdot G + \sqrt{\lambda} u g) d\mu(z) = \int_{Q_2^+} (\nabla v \cdot \tilde{F} + v \tilde{f}) d\mu(z),$$

which together with Hölder's inequality gives

$$(5.12) \quad \left| \int_{Q_1^+} (\nabla u \cdot G + \sqrt{\lambda} u g) d\mu(z) \right| = \\ \leq \|Dv\|_{L_q(Q_2^+, \mu)} \|\tilde{F}\|_{L_p(Q_2^+, \mu)} + \|v\|_{L_{q^*}(Q_2^+, \mu)} \|\tilde{f}\|_{L_{p^*}(Q_2^+, \mu)},$$

where $q^* = p^*/(p^* - 1)$. From (5.10), we see that $v \in \mathcal{H}_q^1(Q_2^+, \mu)$ satisfies

$$\begin{cases} -x_d^\alpha v_t - D_i(x_d^\alpha(a_{ji}D_j v - G_i)) = x_d^\alpha \tilde{g}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha(a_{jd}D_j v - G_d) = 0 \end{cases}$$

in Q_2 , where $\tilde{g} = -\lambda v + \sqrt{\lambda}g$. When $\alpha \neq 0$, by (2.3)–(2.4), q^* satisfies the condition (3.1) in Lemma 3.1. Then, by using Lemma 3.1 and (5.11), we get

$$(5.13) \quad \|v\|_{L_{q^*}(Q_2^+, \mu)} \leq N\|v\|_{L_q(Q_2^+, \mu)} + N\|Dv\|_{L_q(Q_2^+, \mu)} \\ + N\|v_t\|_{\mathcal{H}_q^{-1}(Q_2^+, \mu)} \\ \leq N + N\|G\|_{L_q(Q_2^+, \mu)} + N\|\tilde{g}\|_{L_q(Q_2^+, \mu)} \\ \leq N\sqrt{\lambda}.$$

When $\alpha = 0$, by the usual unweighted parabolic Sobolev embedding, we still get (5.13). It then follows from (5.12), (5.11), (5.13), and the arbitrariness of G and g that

$$(5.14) \quad \|Du\|_{L_p(Q_1^+, \mu)} + \sqrt{\lambda}\|u\|_{L_p(Q_1^+, \mu)} \\ \leq N\|\tilde{F}\|_{L_p(Q_2^+, \mu)} + N\sqrt{\lambda}\|\tilde{f}\|_{L_{p^*}(Q_2^+, \mu)} \\ \leq N(\sqrt{\lambda} + 1)\|F\|_{L_p(Q_2^+, \mu)} + N\|u\|_{L_p(Q_2^+, \mu)} + N\sqrt{\lambda}\|f\|_{L_{p^*}(Q_2^+, \mu)} \\ + N\sqrt{\lambda}(\lambda + 1)\|u\|_{L_{p^*}(Q_2^+, \mu)} + N\sqrt{\lambda}\|Du\|_{L_{p^*}(Q_2^+, \mu)},$$

where N is independent of λ . Now, as $p^* \leq p_0$ and (5.8), the terms in the righthand side of (5.14) are all finite, so as $p^* < p$, we conclude (2.5) from (5.14) by using Hölder's inequality and a standard iteration argument for a sufficiently large λ (see, e.g., [15, pp. 80–82]). The corollary is proved when $p \leq p_1$.

Case II: $p > p_1$. We use an iterative argument. Let p_1 be as in (5.7). For each $k \geq 2$, let $p_k > p_{k-1}$ satisfy

$$\begin{cases} \frac{d+2+\alpha_+}{p_{k-1}} \leq 1 + \frac{d+2+\alpha_+}{p_k} & d \geq 2, \\ \frac{4+\alpha_+}{p_{k-1}} \leq 1 + \frac{4+\alpha_+}{p_k} & d = 1. \end{cases}$$

Using the result we just proved in [Case 1](#), we obtain (2.5) with p replaced by p_1 . From this and with some obvious modifications, we may assume now that $u \in \mathcal{H}_{p_1}^1(Q_2^+, \mu)$. If $p \leq p_2$, we apply [Case 1](#) with p_1 in place of p_0 , and then obtain (2.5). Otherwise, we use [Case 1](#) with p_1 in place of p_0 to obtain (2.5) with p_2 in place of p . Then, we repeat the process. After a finite number of steps, we reach (2.5). The proof is completed.

5.3. Proof of Theorem 2.4. It follows from Corollary 2.3 and Proposition 4.4 that, for any $q_0 \in (1, 2)$, if $v \in \mathcal{H}_{p_0}^1(Q_r^+(z_0), \mu)$ is a weak solution of (4.5)–(4.6), we have

$$(5.15) \quad \begin{aligned} & [D_{x'} v]_{C^{1/2,1}(Q_{r/2}^+(z_0))} + [\mathcal{V}]_{C^{1/2,1}(Q_{r/2}^+(z_0))} + \sqrt{\lambda} [v]_{C^{1/2,1}(Q_{r/2}^+(z_0))} \\ & \leq N r^{-1} \left(\int_{Q_r^+(z_0)} |Dv|^{q_0} + \lambda^{q_0/2} |v|^{q_0} \mu(dz) \right)^{1/q_0}, \end{aligned}$$

where $\mathcal{V} = \bar{a}_{dj}(x_d) D_j v$. By using (5.15), Theorem 2.2, and a decomposition argument as in the proof of Proposition 5.1, we have the following mean oscillation estimate: if $\text{spt}(1_{t < T} u) \subset (s - (R_0 r_0)^2, s + (R_0 r_0)^2) \times \mathbb{R}_+^d$ for some $s \in \mathbb{R}$, then, for any $\tau \leq \frac{1}{30}$ and $z_0 \in \overline{\Omega_T}$,

$$\begin{aligned} & \int_{Q_{\tau r}^+(z_0)} |D_{x'} u - (D_{x'} u)_{Q_{\tau r}^+(z_0)}| \\ & \quad + |u - (u)_{Q_{\tau r}^+(z_0)}| + \sqrt{\lambda} |u - (u)_{Q_{\tau r}^+(z_0)}| \mu(dz) \\ & \leq N \tau^{-(d+2+\alpha_+)} r_0^{2(1-1/q_0)} \left(\int_{Q_r^+(z_0)} |Du|^{q_0} \mu(dz) \right)^{1/q_0} \\ & \quad + N \tau^{-(d+2+\alpha_+)/q_0} \left(\int_{Q_r^+(z_0)} (|F|^{q_0} + |f|^{q_0}) \mu(dz) \right)^{1/q_0} \\ & \quad + N \tau \left(\int_{Q_r^+(z_0)} |Du|^{q_0} + \lambda^{q_0/2} |u|^{q_0} \mu(dz) \right)^{1/q_0} \\ & \quad + N \tau^{-(d+2+\alpha_+)/q_0} \gamma_0^{1/(q_0 v_1)} \left(\int_{Q_r^+(z_0)} |Du|^{q_0 v_2} \mu(dz) \right)^{1/(q_0 v_2)}, \end{aligned}$$

where $\nu_1 \in (1, \infty)$, $\nu_2 = \nu_1/(\nu_1 - 1)$, and $\mathcal{U} = a_{dj}D_j u$. Here, we used the notation

$$(g)_{Q_r^+(z_0)} = \int_{Q_r^+(z_0)} g(z) \mu(dz)$$

for a function g defined in $Q_r^+(z_0)$. The *a priori* estimate (2.6) then follows from the mean oscillation estimate, the reverse Hölder's inequality for A_p weights, the weighted mixed-norm Fefferman-Stein type theorems on sharp functions, and the weighted mixed-norm Hardy-Littlewood maximal function theorem. (See, e.g., Corollary 2.6, 2.7, and Section 7 of [10] for details.) The solvability in weighted mixed-norm Sobolev spaces then follows from the estimate (2.6) and an approximate argument by using the solvability result in Theorem 2.2. We omit the details and refer the reader to [10, Section 8].

5.4. Proof of Corollary 2.10. We first assume

$$(5.16) \quad \frac{d+3+\alpha_+}{p_0} < 1 + \frac{d+3+\alpha_+}{p}.$$

Let $\eta \in C_0^\infty((-4, 4) \times B_2)$ be an even function with respect to x_d such that $\eta \equiv 1$ on Q_1 . A direct calculation yields that $w := u\eta \in W_{p_0}^{1,2}(\Omega_0, \mu)$ satisfies

$$(5.17) \quad \begin{cases} a_0 w_t - a_{ij}D_{ij}w - \frac{\alpha}{x_d}a_{dd}D_d w + \lambda c_0 w = \tilde{f}, \\ \lim_{x_d \rightarrow 0^+} x_d^\alpha a_{dd}D_d w = 0 \end{cases}$$

in $(-4, 0) \times \mathbb{R}_+^d$, with the zero initial condition $w(-4, \cdot) = 0$, where

$$\begin{aligned} \tilde{f} = & f\eta + (a_0\eta_t - a_{ij}D_{ij}\eta - \alpha a_{dd}D_d\eta/x_d + (\lambda - 1)c_0\eta)u \\ & - (a_{ij} + a_{ji})D_i\eta D_j u, \end{aligned}$$

$\lambda > \lambda_0 R_0^{-2}$ is a fixed number, and λ_0 is the constant from Theorem 2.7 with $q = p$ and $\omega \equiv K = 1$. It follows from Lemma 3.1 and (5.16) that

$$(5.18) \quad \|u\|_{L_p(Q_2^+, \mu)} + \|Du\|_{L_p(Q_2^+, \mu)} \leq N\|u\|_{W_{p_0}^{1,2}(Q_2^+, \mu)}.$$

By using Theorem 2.7 with $q = p$ and $\omega \equiv K = 1$, (5.17) has a unique solution $v \in W_p^{1,2}(\Omega_0, \mu)$. Since $\tilde{f}$ is compactly supported, as in Case 2 of the proof of Theorem 4.1, we have $v \in W_{p_0}^{1,2}(\Omega_0, \mu)$. Now by the uniqueness of $W_{p_0}^{1,2}(\Omega_0, \mu)$ -solutions to (5.17), we conclude that $u\eta = v \in W_p^{1,2}(\Omega_0, \mu)$. Furthermore, by Theorem 2.7 and (5.18),

$$\|u\|_{W_p^{1,2}(Q_1^+, \mu)} \leq N\|\tilde{f}\|_{W_p^{1,2}(Q_2^+, \mu)} \leq N\|f\|_{W_p^{1,2}(Q_2^+, \mu)} + N\|u\|_{W_{p_0}^{1,2}(Q_2^+, \mu)},$$

which, together with Hölder's inequality and a standard iteration argument, yields (2.7) under the additional condition (5.16).

Finally, for general $p \in (p_0, \infty)$, the result follows from an induction argument by taking a sequence of increasing exponents p_j , $j = 1, \dots, n$, such that $p_n = p$ and

$$\frac{(d+3+\alpha_+)}{p_{j-1}} < 1 + \frac{(d+3+\alpha_+)}{p_j} \quad \text{for } j = 1, \dots, n.$$

Acknowledgements. The authors would like to thank the referee for the careful reading and helpful comments.

The first author is supported in part by a Simons Fellows Award (no. 007638), the National Science Foundation (grant no. DMS-2055244), and the Charles Simonyi Endowment at the Institute for Advanced Study.

REFERENCES

- [1] A. BANERJEE and N. GAROFALO, *Monotonicity of generalized frequencies and the strong unique continuation property for fractional parabolic equations*, Adv. Math. **336** (2018), 149–241. <https://dx.doi.org/10.1016/j.aim.2018.07.021>. MR3846151
- [2] D. CAO, T. MENGESHA, and T. PHAN, *Weighted- $W^{1,p}$ estimates for weak solutions of degenerate and singular elliptic equations*, Indiana Univ. Math. J. **67** (2018), no. 6, 2225–2277. <https://dx.doi.org/10.1512/iumj.2018.67.7533>. MR3900368
- [3] L. A. CAFFARELLI and L. SILVESTRE, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7–9, 1245–1260. <https://dx.doi.org/10.1080/03605300600987306>. MR2354493
- [4] M. CHIPOT, D. KINDERLEHRER, and G. VERGARA-CAFFARELLI, *Smoothness of linear laminates*, Arch. Rational Mech. Anal. **96** (1986), no. 1, 81–96. <https://dx.doi.org/10.1007/BF00251414>. MR853976
- [5] P. DASKALOPOULOS, R. HAMILTON, and K. LEE, *All time C^∞ -regularity of the interface in degenerate diffusion: A geometric approach*, Duke Math. J. **108** (2001), no. 2, 295–327. <https://dx.doi.org/10.1215/S0012-7094-01-10824-7>. MR1833393
- [6] H. DONG, *Solvability of parabolic equations in divergence form with partially BMO coefficients*, J. Funct. Anal. **258** (2010), no. 7, 2145–2172. <https://dx.doi.org/10.1016/j.jfa.2010.01.003>. MR2584743
- [7] H. DONG and D. KIM, *Elliptic equations in divergence form with partially BMO coefficients*, Arch. Ration. Mech. Anal. **196** (2010), no. 1, 25–70. <https://dx.doi.org/10.1007/s00205-009-0228-7>. MR2601069
- [8] ———, *Parabolic and elliptic systems in divergence form with variably partially BMO coefficients*, SIAM J. Math. Anal. **43** (2011), no. 3, 1075–1098. <https://dx.doi.org/10.1137/100794614>. MR2800569
- [9] ———, *Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains*, J. Funct. Anal. **261** (2011), no. 11, 3279–3327. <https://dx.doi.org/10.1016/j.jfa.2011.08.001>. MR2835999
- [10] ———, *On L_p -estimates for elliptic and parabolic equations with A_p weights*, Trans. Amer. Math. Soc. **370** (2018), no. 7, 5081–5130. <https://dx.doi.org/10.1090/tran/7161>. MR3812104
- [11] H. DONG and T. PHAN, *Regularity for parabolic equations with singular or degenerate coefficients*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, Paper No. 44, 39 pp. <https://dx.doi.org/10.1007/s00526-020-01876-5>. MR4204570

- [12] ———, *Weighted mixed-norm L_p -estimates for elliptic and parabolic equations in non-divergence form with singular coefficients*, Rev. Mat. Iberoam. **37** (2021), no. 4, 1413–1440. <https://dx.doi.org/10.4171/rmi/1233>. MR4269403
- [13] E. B. FABES, C. E. KENIG, and R. P. SERAPIONI, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116. <https://dx.doi.org/10.1080/03605308208820218>. MR643158
- [14] P. M. N. FEEHAN and C. A. POP, *Schauder a priori estimates and regularity of solutions to boundary-degenerate elliptic linear second-order partial differential equations*, J. Differential Equations **256** (2014), no. 3, 895–956. <https://dx.doi.org/10.1016/j.jde.2013.08.012>. MR3128929
- [15] M. GIAQUINTA, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993. MR1239172
- [16] V. V. GRUŠIN, *A certain class of elliptic pseudodifferential operators that are degenerate on a sub-manifold*, Mat. Sb. (N.S.) **84 (126)** (1971), 163–195 (Russian). MR0283630
- [17] P. HAJŁASZ, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), no. 4, 403–415. <https://dx.doi.org/10.1007/BF00275475>. MR1401074
- [18] G. H. HARDY and J. E. LITTLEWOOD, *Some properties of fractional integrals. I*, Math. Z. **27** (1928), no. 1, 565–606. <https://dx.doi.org/10.1007/BF01171116>. MR1544927
- [19] H. JIAN and X.-J. WANG, *Bernstein theorem and regularity for a class of Monge-Ampère equations*, J. Differential Geom. **93** (2013), no. 3, 431–469. <http://dx.doi.org/10.4310/jdg/1361844941>. MR3024302
- [20] B. KANG and H. KIM, *On L^p -resolvent estimates for second-order elliptic equations in divergence form*, Potential Anal. **50** (2019), no. 1, 107–133. <https://dx.doi.org/10.1007/s11118-017-9675-1>. MR3900848
- [21] D. KIM and N. V. KRYLOV, *Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others*, SIAM J. Math. Anal. **39** (2007), no. 2, 489–506. <https://dx.doi.org/10.1137/050646913>. MR2338417
- [22] ———, *Parabolic equations with measurable coefficients*, Potential Anal. **26** (2007), no. 4, 345–361. <https://dx.doi.org/10.1007/s11118-007-9042-8>. MR2300337
- [23] D. KIM, H. DONG, and H. ZHANG, *Neumann problem for non-divergence elliptic and parabolic equations with BMO_x coefficients in weighted Sobolev spaces*, Discrete Contin. Dyn. Syst. **36** (2016), no. 9, 4895–4914. <https://dx.doi.org/10.3934/dcds.2016011>. MR3541508
- [24] D. KIM, S. RYU, and K. WOO, *Parabolic equations with unbounded lower-order coefficients in Sobolev spaces with mixed norms*, J. Evol. Equ. **22** (2022), no. 1, Paper No. 9, 40 pp. <https://dx.doi.org/10.1007/s00028-022-00761-2>. MR4387945
- [25] H. KIM and T.-P. TSAI, *Existence, uniqueness, and regularity results for elliptic equations with drift terms in critical weak spaces*, SIAM J. Math. Anal. **52** (2020), no. 2, 1146–1191. <https://dx.doi.org/10.1137/19M1282969>. MR4075335
- [26] M. V. KELDYŠ, *On certain cases of degeneration of equations of elliptic type on the boundary of a domain*, Doklady Akad. Nauk SSSR (N.S.) **77** (1951), 181–183 (Russian). MR0042031
- [27] N. V. KRYLOV, *Parabolic and elliptic equations with VMO coefficients*, Comm. Partial Differential Equations **32** (2007), no. 1–3, 453–475. <https://dx.doi.org/10.1080/03605300600781626>. MR2304157
- [28] ———, *Second-order elliptic equations with variably partially VMO coefficients*, J. Funct. Anal. **257** (2009), no. 6, 1695–1712. <https://dx.doi.org/10.1016/j.jfa.2009.06.014>. MR2540989
- [29] ———, *Elliptic equations with VMO $a, b \in L_d$, and $c \in L_{d/2}$* , Trans. Amer. Math. Soc. **374** (2021), no. 4, 2805–2822. <https://dx.doi.org/10.1090/tran/8282>. MR4223034
- [30] J.-L. LIONS and E. MAGENES, *Problèmes aux limites non homogènes et applications. Vol. 1*, Travaux et Recherches Mathématiques, vol. 17, Dunod, Paris, 1968 (French). MR0247243

- [31] N. Q. LE and O. SAVIN, *Schauder estimates for degenerate Monge-Ampère equations and smoothness of the eigenfunctions*, Invent. Math. **207** (2017), no. 1, 389–423. <https://dx.doi.org/10.1007/s00222-016-0677-1>. MR3592760
- [32] K. NYSTRÖM and O. SANDE, *Extension properties and boundary estimates for a fractional heat operator*, Nonlinear Anal. **140** (2016), 29–37. <https://dx.doi.org/10.1016/j.na.2016.02.027>. MR3492726
- [33] O. A. OLEĬNIK and E. V. RADKEVIČ, *Second Order Equations with Nonnegative Characteristic Form*, Plenum Press, New York-London, 1973. Translated from the Russian by Paul C. Fife. MR0457908
- [34] Y. SIRE, S. TERRACINI, and S. VITA, *Liouville type theorems and regularity of solutions to degenerate or singular problems part I: Even solutions*, Comm. Partial Differential Equations **46** (2021), no. 2, 310–361. <https://dx.doi.org/10.1080/03605302.2020.1840586>. MR4207950
- [35] E. M. STEIN and G. WEISS, *Fractional integrals on n -dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514. <https://dx.doi.org/10.1512/iumj.1958.7.57030>. MR0098285
- [36] P. R. STINGA and J. L. TORREA, *Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation*, SIAM J. Math. Anal. **49** (2017), no. 5, 3893–3924. <https://dx.doi.org/10.1137/16M1104317>. MR3709888
- [37] C. WANG and Z. XIN, *Optimal Hölder continuity for a class of degenerate elliptic problems with an application to subsonic-sonic flows*, Comm. Partial Differential Equations **36** (2011), no. 5, 873–924. <https://dx.doi.org/10.1080/03605302.2010.535074>. MR2769112

HONGJIE DONG:

Division of Applied Mathematics Brown University

182 George Street

Providence, RI 02912

USA

E-MAIL: Hongjie.Dong@brown.edu

TUOC PHAN:

Department of Mathematics

University of Tennessee

227 Ayres Hall

1403 Circle Drive

Knoxville, TN 37996-1320

USA

E-MAIL: phan@utk.edu

KEY WORDS AND PHRASES: Singular and degenerate parabolic equations, Muckenhoupt weights, Calderón-Zygmund estimates, weighted Sobolev spaces.

2020 MATHEMATICS SUBJECT CLASSIFICATION: 35K65, 35K67, 35D10, 35R11.

Received: November 14, 2021.