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The Insulated Conductivity Problem with p-Laplacian

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Abstract

We study the insulated conductivity problem with closely spaced insulators embedded in a homogeneous matrix where the current-electric field relation is the power law $J = |E|^{p-2}E$. The gradient of solutions may blow up as ε, the distance between insulators, approaches to 0. We prove an upper bound of the gradient to be of order $\varepsilon^{-\alpha}$, where $\alpha = 1/2$ when $p \in (1, n+1]$ and any $\alpha > n/(2(p-1))$ when $p > n+1$. We provide examples to show that this exponent is almost optimal in 2D. Additionally, in dimensions $n \geqq 3$, for any $p > 1$, we prove another upper bound of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$, and show that $\beta \nearrow 1/2$ as $n \to \infty$.

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1. Introduction and Main results

We investigate the phenomenon of electric field concentration in high-contrast composites. Such a phenomenon can occur when two approaching inclusions possess material properties that differ significantly from the background matrix (see e.g. [8,21,28]). The study of this area originated from [4], where the problem with inclusions closely located in a linear background medium was studied numerically. In this paper, we study the scenario in which the inclusions are insulators, and the background matrix follows the current-electric field relation described by the power law

$$J = \sigma|E|^{p-2}E, \quad p > 1, \tag{1.1}$$

where J, E, and σ denote current, electric field, and conductivity, respectively. Physically, such power law can occur in various materials, including dielectrics,

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plastic moulding, plasticity phenomena, viscous flows in glaciology, electro-rheological and thermo-rheological fluids; see, e.g., [3,17,20,23,29,30], the second paragraph of [7] and the references therein.

Let us describe the mathematical setup: let $n \geqq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary containing two $C^{1,1}$ open sets $\mathcal{D}_1$ and $\mathcal{D}_2$ with $\mathrm{dist}(\mathcal{D}_1 \cup \mathcal{D}_2, \partial\Omega) > c > 0$. Let

$$\varepsilon := \mathrm{dist}(\mathcal{D}_1, \mathcal{D}_2),$$

$\widetilde{\Omega} := \Omega \backslash \overline{(\mathcal{D}_1 \cup \mathcal{D}_2)}$, and $\sigma = \mathbb{1}_{\widetilde{\Omega}}$. The voltage potential u satisfies the p-Laplace equation with $p > 1$

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{D}_i,\ i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\varphi \in C^{1,1}(\partial\Omega)$ is given, and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the inner normal vector on $\partial\mathcal{D}_1 \cup \partial\mathcal{D}_2$.

Our goal is to quantitatively analyze the concentration of the electric field $E = -Du$ between the inclusions, and this is a challenging problem even in the linear case when $p = 2$. While the optimal blow-up rate for the linear case in two dimensions was captured about two decades ago in [1,2], the optimal rate in dimensions $n \geqq 3$ was only recently identified in [12,13]. This optimal rate is linked to the first non-zero eigenvalue of an elliptic operator on $\mathbb{S}^{n-2}$, which is determined by the principal curvatures of the inclusions. This phenomenon is completely different from the perfect conductivity problem, where the optimal blow-up rates do not depend on the curvatures of the inclusions (see [1,2,5]). For other earlier work on the linear insulated conductivity problem, we refer the reader to [6,24,25,34,35]. In the case when the current-electric field relation is given by (1.1), Gorb and Novikov in [18] and Ciraolo and Sciammetta in [10] studied the field concentration when $\mathcal{D}_1$ and $\mathcal{D}_2$ are perfect conductors. They proved that, for $n \geqq 2$,

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leqq \begin{cases} C\varepsilon^{-\frac{n-1}{2(p-1)}} & p > \dfrac{n+1}{2}, \\ C\varepsilon^{-1}|\log \varepsilon|^{\frac{1}{1-p}} & p = \dfrac{n+1}{2}, \\ C\varepsilon^{-1} & p < \dfrac{n+1}{2}. \end{cases}$$

These bounds were shown to be optimal in their respective papers. However, the phenomenon of electric field concentration between insulators has not been studied before.

Before stating our main results, let us introduce some notation. We denote $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$. By choosing a coordinate system properly, we can assume that near the origin, the part of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$, denoted by Γ_+ and Γ_-, are respectively the graphs of two $C^{1,1}$ functions in terms of x'. That is,

$$\Gamma_+ = \left\{x_n = \frac{\varepsilon}{2} + h_1(x'),\ |x'| < 1\right\}, \quad \Gamma_- = \left\{x_n = -\frac{\varepsilon}{2} + h_2(x'),\ |x'| < 1\right\},$$

where h_1 and h_2 are $C^{1,1}$ functions satisfying

$$h_1(0') = h_2(0') = 0, \quad D_{x'}h_1(0') = D_{x'}h_2(0') = 0, \tag{1.3}$$

$$c_1|x'|^2 \leqq h_1(x') - h_2(x') \quad \text{for } \ 0 < |x'| < 1, \tag{1.4}$$

$$\|h_1\|_{C^{1,1}} \leqq c_2, \quad \|h_2\|_{C^{1,1}} \leqq c_2, \tag{1.5}$$

with some positive constants c_1, c_2. For $x_0 \in \widetilde{\Omega}$, $0 < r \leqq 1$, we denote

$$\Omega_{x_0,r} := \left\{(x', x_n) \in \widetilde{\Omega} : \ -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), \ |x' - x_0'| < r\right\},$$

and $\Omega_r := \Omega_{0,r}$. We use $B_r(x_0)$ to denote the open ball of radius r centered at x_0 and we set

$$B_r = B_r(0), \quad \Omega_r(x_0) = \widetilde{\Omega} \cap B_r(x_0).$$

By classical $C^{1,\alpha}$ estimates for the p-Laplace equation and the maximum principle (see e.g. [26,33]), the solution $u \in W^{1,p}(\widetilde{\Omega})$ of (1.2) satisfies

$$\|u\|_{L^\infty(\widetilde{\Omega})} + \|u\|_{C^{1,\alpha}(\widetilde{\Omega}\setminus\Omega_{1/2})} \leqq C\|\varphi\|_{C^{1,1}(\partial\Omega)} \tag{1.6}$$

for some constants $\alpha = \alpha(n, p) \in (0, 1)$ and $C > 0$ independent of ε, φ, and u. As such, we focus on the following problem near the origin:

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = 0 & \text{in } \Omega_1, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_+ \cup \Gamma_-. \end{cases} \tag{1.7}$$

For any domain $\mathcal{D}$, we denote the oscillation of u in $\mathcal{D}$ by

$$\operatorname*{osc}_{\mathcal{D}} u := \sup_{\mathcal{D}} u - \inf_{\mathcal{D}} u.$$

Our first main result is the following pointwise gradient estimate of order $\varepsilon^{-1/2}$ for any $p > 1$ and $n \geqq 2$:

Theorem 1.1. *Let h_1, h_2 be $C^{1,1}$ functions satisfying (1.3)–(1.5), $p > 1$, $n \geqq 2$, $\varepsilon \in (0, 1)$, and $u \in W^{1,p}(\Omega_1)$ be a solution of (1.7). Then there exists a constant $C > 0$ depending only on n, p, c_1, and c_2, such that for any $x \in \Omega_{1/2}$ and $\eta = \frac{1}{4}(\varepsilon + |x'|^2)^{\frac{1}{2}}$,*

$$|Du(x)| \leqq C(\varepsilon + |x'|^2)^{-\frac{1}{2}} \operatorname*{osc}_{\Omega_{x,\eta}} u. \tag{1.8}$$

When $n \geqq 3$, we improve the upper bound in Theorem 1.1 to the order of $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$.

Theorem 1.2. *Let h_1, h_2 be $C^{1,1}$ functions satisfying (1.3)–(1.5), $p > 1$, $n \geqq 3$, $\varepsilon \in (0, 1)$, and $u \in W^{1,p}(\Omega_1)$ be a solution of (1.7). Then there exist positive constants C and β depending only on n, p, c_1, and c_2, such that for any $x \in \Omega_{1/2}$,*

$$|Du(x)| \leqq C(\varepsilon + |x'|^2)^{-1/2+\beta} \operatorname*{osc}_{\Omega_1} u. \tag{1.9}$$

When $p > n+1$, we derive a more explicit upper bound, under an additional assumption that h_1 and h_2 are C^2 strictly convex and strictly concave, respectively. That is, for some positive constants κ_1 and κ_2,

$$\kappa_1 I_{n-1} \leqq D^2 h_1(x') \leqq \kappa_2 I_{n-1}, \quad \kappa_1 I_{n-1} \leqq -D^2 h_2(x') \leqq \kappa_2 I_{n-1} \quad \text{for } 0 \leqq |x'| < 1. \tag{1.10}$$

Our pointwise gradient estimate of order $-\frac{n+2\delta}{2(p-1)}$, for any $\delta > 0$ when $p > n+1$, is as follows:

Theorem 1.3. *Let $n \geqq 2$, $p > n+1$, h_1, h_2 be C^2 functions satisfying (1.3) and (1.10), and let $u \in W^{1,p}(\Omega_1)$ be a solution of (1.7). Then for any $\delta > 0$ and $\varepsilon \in (0,1)$, we have*

$$|Du(x)| \leqq C(\varepsilon + |x'|^2)^{-\frac{n+2\delta}{2(p-1)}} \operatorname*{osc}_{\Omega_1} u \quad \textit{for } x \in \Omega_{1/2}, \tag{1.11}$$

where C is a positive constant depending on n, p, δ, κ_1, κ_2, and the modulus of continuity for $D^2 h_1(x')$ and $D^2 h_2(x')$ at $x' = 0$.

Furthermore, we show that when $n = 2$, the blow-up exponents $-1/2$ for $p \leqq 3$ and $-1/(p-1)$ for $p > 3$ obtained in Theorems 1.1 and 1.3 are critical in the following sense:

Theorem 1.4. *For $n = 2$, $p > 1$, $\varepsilon \in (0,1)$, let $\Omega = B_5$, and $\mathcal{D}_1, \mathcal{D}_2$ be the unit balls center at $(0, 1+\varepsilon/2)$ and $(0, -1-\varepsilon/2)$, respectively. Let $\varphi = x_1$ and $u \in W^{1,p}(\widetilde{\Omega})$ be the solution of (1.2). Then for any $\delta > 0$, there exists a positive constant C depending only on p and δ, such that, when $p \in (1,3]$,*

$$\|Du\|_{L^\infty(\widetilde{\Omega} \cap B_{8\sqrt{\varepsilon/\delta}})} \geqq \frac{1}{C} \varepsilon^{\frac{-1+\delta}{2}},$$

and when $p > 3$,

$$\|Du\|_{L^\infty(\widetilde{\Omega} \cap B_{8\sqrt{\varepsilon/\delta}})} \geqq \frac{1}{C} \varepsilon^{\frac{-1+\delta}{p-1}}.$$

Finally, we also establish a blow-up rate of $\varepsilon^{-1/2+\beta}$ for the gradient, for any $p > 1$ and sufficiently large n, with more explicit constant $\beta \in [0, 1/2)$. For this, we impose a further assumption that h_1 and h_2 are $C^{2,\mathrm{Dini}}$ strictly convex and strictly concave respectively, satisfying (1.10) for some positive constants κ_1 and κ_2.

Theorem 1.5. *Let h_1, h_2 be $C^{2,Dini}$ functions satisfying (1.3) and (1.10), $p > 1$, $\beta \in [0, 1/2)$, $\varepsilon \in (0,1)$, and $u \in W^{1,p}(\Omega_1)$ be a solution of (1.7). If n, p, and β satisfy either*

$$p \geqq 2, \quad n \geqq \frac{5(p-1)}{2}\left(\frac{p+1-2\beta(p-1)}{2} + \frac{\kappa_2}{(1-2\beta)\kappa_1}\right) + 1, \tag{1.12}$$

or

$$1 < p < 2, \quad n \geqq \frac{5}{2}\left(\frac{3-2\beta}{2} + \frac{\kappa_2}{(1-2\beta)\kappa_1}\right) + 3 - p, \tag{1.13}$$

then there exist a positive constant C depending only on n, p, β, κ_1, and κ_2, such that

$$|Du(x)| \leqq C\|u\|_{L^\infty(\Omega_1)}(\varepsilon + |x'|^2)^{-\frac{1}{2}+\beta} \quad \textit{for } x \in \Omega_{1/2}. \tag{1.14}$$

Remark 1.6. By (1.12) and (1.13), when $n \to \infty$, β can be chosen arbitrarily close to $1/2$. In view of (1.14), the singularity of Du diminishes as the dimension n increases. We also note that by refining the inequalities in the proof of Theorem 1.5 further, it is possible to improve the lower bounds of n in both (1.12) and (1.13). However, we have decided not to pursue this in the current paper.

The rest of the paper is organized as follows: in the next section, we give the proof of Theorem 1.1 using mean oscillation estimates. In Section 3, we demonstrate the proof of Theorem 1.2 by utilizing a delicate change of variables, an extension argument, and the Krylov–Safonov Harnack inequality. Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4, respectively, for which we construct suitable sub- and super-solutions. In Section 6, we employ a Bernstein type argument to prove Theorem 1.5. Here we use the fact that for any $q \geqq p$, $|Du|^q$ is a subsolution to the normalized p-Laplace equation, as originally observed by Uhlenbeck [32]. Finally, we provide an alternative proof of the gradient estimates of order $\varepsilon^{-1/2}$ in the Appendix by also using the Bernstein type argument.

2. Mean oscillation estimates

In this section, we give the proof of Theorem 1.1 using mean oscillation estimates. We fix a point $x_0 \in \Omega_{1/2}$ and prove (1.8) at $x = x_0$. Note that we can always assume $\varepsilon + |x_0'|^2 \leqq c$, where $c = c(n, p, c_1, c_2) > 0$ could be any sufficiently small constant depending only on n, p, c_1, and c_2. Otherwise, by classical estimates (see [26,33]), (1.8) directly follows. Next, we derive some mean oscillation estimates of Du on a ball $B_r(x_0)$ for different radii r.

2.1. Mean oscillation estimates for small r

We recall a classical interior mean oscillation estimate when $B_r(x_0) \subset \Omega_1$. Estimates of this type, with different exponents involved, were developed in [11, 15,27].

Lemma 2.1. *Let $u \in W^{1,p}(\Omega_1)$ be a solution to (1.7). There exist constants $C > 1$ and $\alpha \in (0, 1)$ depending only on n and p, such that $u \in C^{1,\alpha}(\Omega_1)$ and for every $B_r(x_0) \subset \Omega_1$ and $\rho \in (0, r]$, we have*

$$\left(\fint_{B_\rho(x_0)} |Du - (Du)_{B_\rho(x_0)}|^p\right)^{\frac{1}{p}} \leqq C\left(\frac{\rho}{r}\right)^\alpha \left(\fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|^p\right)^{\frac{1}{p}}.$$

We denote

$$\phi(x_0, r) = \left(\fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|^p \right)^{\frac{1}{p}}.$$

Then Lemma 2.1 also implies

Corollary 2.2. *Under the assumptions of Lemma 2.1, there exists a constant $\mu_1 \in (0, 1)$ depending only on n and p, such that for any $\mu \in (0, \mu_1]$, $B_r(x_0) \subset \Omega_1$, and $K \in \mathbb{N}$, it holds that*

$$\sum_{k=0}^{K+1} \phi(x_0, \mu^k r) \leqq 2\,\phi(x_0, r). \tag{2.1}$$

Proof. We take $\mu_1 \in (0, 1)$ such that $C\mu_1^\alpha = \frac{1}{2}$, where C and α are the same constants as in Lemma 2.1. Replacing r with $\mu^k r$ and setting $\rho = \mu^{k+1} r$, we get

$$\phi(x_0, \mu^{k+1} r) \leqq \frac{1}{2} \phi(x_0, \mu^k r).$$

Summing the above inequality over $k = 0, 1, \ldots, K$, we obtain (2.1). □

2.2. *Mean oscillation estimates for intermediate r*

Next, we consider the case when $B_r(x_0)$ intersects with only one of Γ_+ and Γ_-. In this case, we choose $\hat{x}_0 \in \Gamma_+ \cup \Gamma_-$ such that $\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-) = |\hat{x}_0 - x_0|$ and we derive mean oscillation estimates around $\hat{x}_0$. Note that we can assume $\varepsilon + c_2|x_0'|^2 \leqq 1/4$ and thus by (1.5) and the triangle inequality, $|\hat{x}_0'| \leqq 3/4$. Without loss of generality, we assume $\hat{x}_0 \in \Gamma_-$. Then by (1.4) and (1.5), there exists a constant $c = c(n, c_1, c_2) \in (0, 1/4)$, such that $B(\hat{x}_0, r) \cap \Gamma_+ = \emptyset$ for any $r \in (0, c(\varepsilon + |\hat{x}_0'|^2))$.

We first choose a coordinate $y = (y', y_n)$ such that $y(\hat{x}_0) = 0$, the direction of axis y_n is the upper normal vector at $\hat{x}_0 \in \Gamma_-$, and $\Omega_{R_0}(\hat{x}_0) = \{y \in B_{R_0} : y_n > \chi(y')\}$, where $R_0 = c_3(\varepsilon + |\hat{x}_0'|^2) \in (0, 1/4)$ for some constant $c_3 = c_3(n, c_1, c_2) \in (0, 1/8)$ and $\chi : \{y' \in \mathbb{R}^{n-1} : |y'| < R_0\} \to \mathbb{R}$ is a $C^{1,1}$ function in the coordinate system depending on $\hat{x}_0$ such that

$$\chi(0') = 0, \quad D_{y'}\chi(0') = 0, \quad \|\chi\|_{C^{1,1}} \leqq C\|h_2\|_{C^{1,1}}. \tag{2.2}$$

Then we let

$$z = \Lambda(y) = (y',\ y_n - \chi(y')).$$

Since Γ_- is $C^{1,1}$, by (2.2) there exist constants $C = C(n, c_1, c_2)$, $c_4 = c_4(n, c_1, c_2) \in (0, c_3) \subset (0, 1/8)$, and $R_1 = c_4(\varepsilon + |\hat{x}_0'|^2)$ such that

$$|D_{y'}\chi(y')| \leqq C\,|y'| \leqq 1/2 \quad \text{if} \quad |y'| \leqq 2R_1, \tag{2.3}$$

$$\Omega_{r/2}(\hat{x}_0) \subset \Lambda^{-1}(B_r^+) \subset \Omega_{2r}(\hat{x}_0) \quad \forall r \in (0, 2R_1], \tag{2.4}$$

and thus

$$|D\Lambda(y) - I_n| \leqq C\,|y'| \leqq 1/2 \quad \text{if} \quad |y'| \leqq 2R_1, \tag{2.5}$$

Therefore, there exist positive constants $c(n)$ and $c'(n)$ depending only on n, such that for any $\hat{x}_0 \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leqq 3/4\}$ and $0 < r \leqq c_4(\varepsilon + |\hat{x}_0'|^2)$,

$$c(n)r^n \leqq |\Omega_r(\hat{x}_0)| \leqq c'(n)r^n. \tag{2.6}$$

Note that

$$\det(D\Lambda) \equiv 1. \tag{2.7}$$

Then $u_1(z) := u(\Lambda^{-1}(z))$ satisfies the following equation with conormal boundary condition

$$\begin{cases} -\operatorname{div}_z\left(|A^T D_z u_1|^{p-2} A A^T D_z u_1\right) = 0 & \text{in } B_{R_1}^+, \\ \left(|A^T D_z u_1|^{p-2} A A^T D_z u_1\right)_n = 0 & \text{on } B_{R_1} \cap \partial\mathbb{R}^n_+, \end{cases} \tag{2.8}$$

where we denote

$$A := A(z) := (a_{ij}(z)) := D\Lambda(\Lambda^{-1}(z)).$$

Next we extend u_1 and the coefficient matrix A to the whole ball B_{R_1}. We take the even extension of u_1, a_{nn}, and a_{ij}, $i, j = 1, 2, \ldots, n-1$, with respect to $z_n = 0$, and take the odd extension of a_{in} and a_{ni}, $i = 1, 2, \ldots, n-1$, with respect to $z_n = 0$. We still denote these functions by u_1 and A after the extension. Because of the conormal boundary condition, u_1 satisfies

$$-\operatorname{div}_z\left(\mathbf{A}(z, D_z u_1)\right) = 0 \quad \text{in } B_{R_1}, \tag{2.9}$$

where the nonlinear operator $\mathbf{A}$ is defined as

$$\mathbf{A}(z, \xi) = |A^T\xi|^{p-2} A A^T \xi \qquad \text{for} \quad z \in B_{R_1}, \ \xi \in \mathbb{R}^n.$$

Lemma 2.3. *There exists a constant $C = C(n, p, c_1, c_2) > 0$, such that for any $z \in B_{R_1}$ and $\xi \in \mathbb{R}^n$,*

$$|\mathbf{A}(z, \xi) - |\xi|^{p-2}\xi| \leqq C\,|z'|\,|\xi|^{p-1}. \tag{2.10}$$

Proof. We recall a well-known inequality (see [19, Lemma 2.1]): for any $p > 1$ and $\xi_1, \ \xi_2 \in \mathbb{R}^n$, it holds that

$$c^{-1}\left(|\xi_1|^2 + |\xi_2|^2\right)^{\frac{p-2}{2}} \leqq \frac{\left||\xi_2|^{p-2}\xi_2 - |\xi_1|^{p-2}\xi_1\right|}{|\xi_2 - \xi_1|} \leqq c\left(|\xi_1|^2 + |\xi_2|^2\right)^{\frac{p-2}{2}}, \tag{2.11}$$

where $c = c(n, p) > 1$ is a positive constant. Using (2.11), (2.5), and the triangle inequality, we obtain

$$\begin{aligned} |\mathbf{A}(z, \xi) - |\xi|^{p-2}\xi| &\leqq \left||A^T\xi|^{p-2}(A - I_n)A^T\xi\right| + \left||A^T\xi|^{p-2}A^T\xi - |\xi|^{p-2}\xi\right| \\ &\leqq |A - I_n|\,|A^T\xi|^{p-1} + c\left(|A^T\xi|^2 + |\xi|^2\right)^{\frac{p-2}{2}}|A^T\xi - \xi| \leqq C\,|z'|\,|\xi|^{p-1}. \end{aligned}$$

Thus the proof of (2.10) is completed. □

Assume that $r \in (0, R_1]$. We let $v_1 \in u_1 + W_0^{1,p}(B_r)$ be the unique solution to

$$\begin{cases} -\operatorname{div}_z(|D_z v_1|^{p-2} D_z v_1) = 0 & \text{in } B_r, \\ v_1 = u_1 & \text{on } \partial B_r. \end{cases} \tag{2.12}$$

By testing (2.12) and (2.9) with $v_1 - u_1$ and using (2.10), we have the comparison estimate

$$⨍_{B_r} |D_z u_1 - D_z v_1|^p \leqq C r^{\min\{2,p\}} ⨍_{B_r} |D_z u_1|^p, \tag{2.13}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. For detailed proof of (2.13), see [15, Eq. (4.35)] when $p \in (1, 2)$ and [16, Lemma 3.4] when $p \geqq 2$.

Applying Lemma 2.1 and the comparison estimate (2.13), we have

Lemma 2.4. *Suppose that $u_1 \in W^{1,p}(B^+_{R_1})$ is a solution to (2.8). Then for any $\mu \in (0, 1)$ and $r \in (0, R_1]$, we have*

$$\begin{aligned} &\left(⨍_{B^+_{\mu r}} |D_{z'} u_1 - (D_{z'} u_1)_{B^+_{\mu r}}|^p + |D_{z_n} u_1|^p \right)^{1/p} \\ &\leqq C\mu^\alpha \left(⨍_{B^+_r} |D_{z'} u_1 - (D_{z'} u_1)_{B^+_r}|^p + |D_{z_n} u_1|^p \right)^{1/p} \\ &\quad + C_\mu r^{\theta_p} \left(⨍_{B^+_r} |D_z u_1|^p \right)^{1/p}, \end{aligned} \tag{2.14}$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 2.1, C_μ is a constant depending on μ, n, p, c_1, c_2 and C is a constant depending on n, p, c_1, c_2.

Proof. By Lemma 2.1, (2.13), and the triangle inequality, we have

$$\begin{aligned} &\left(⨍_{B_{\mu r}} |D_z u_1 - (D_z u_1)_{B_{\mu r}}|^p \right)^{1/p} \\ &\leqq C \left(⨍_{B_{\mu r}} |D_z v_1 - (D_z v_1)_{B_{\mu r}}|^p \right)^{1/p} + C \left(⨍_{B_{\mu r}} |D_z u_1 - D_z v_1|^p \right)^{1/p} \\ &\leqq C\mu^\alpha \left(⨍_{B_r} |D_z v_1 - (D_z v_1)_{B_r}|^p \right)^{1/p} + C\mu^{-\frac{n}{p}} \left(⨍_{B_r} |D_z u_1 - D_z v_1|^p \right)^{1/p} \\ &\leqq C\mu^\alpha \left(⨍_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p \right)^{1/p} + C\mu^{-\frac{n}{p}} \left(⨍_{B_r} |D_z u_1 - D_z v_1|^p \right)^{1/p} \\ &\leqq C\mu^\alpha \left(⨍_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p \right)^{1/p} + C_\mu r^{\theta_p} \left(⨍_{B_r} |D_z u_1|^p \right)^{1/p}. \end{aligned} \tag{2.15}$$

Since u_1 is even in z_n, (2.15) directly implies (2.14). The proof is completed. □

We now define

$$\psi(\hat{x}_0, r) = \left(\fint_{\Omega_r(\hat{x}_0)} |D_{y'}u - (D_{y'}u)_{\Omega_r(\hat{x}_0)}|^p + |D_{y_n}u|^p \right)^{1/p}. \tag{2.16}$$

Let $\mu \in (0, 1)$ and $r \in (0, R_1/2]$ be constants. By using change of variables, (2.3), (2.4), (2.7), and the triangle inequality, we have

$$\begin{aligned}
&\left(\fint_{B^+_{\mu r}} |D_{z'}u_1 - (D_{z'}u_1)_{B^+_{\mu r}}|^p + |D_{z_n}u_1|^p \right)^{1/p} \\
&= \left(\fint_{\Lambda^{-1}(B^+_{\mu r})} |D_{y'}u + D_{y_n}u\, D_{y'}\chi - (D_{y'}u + D_{y_n}u\, D_{y'}\chi)_{\Lambda^{-1}(B^+_{\mu r})}|^p + |D_{y_n}u|^p \right)^{1/p} \\
&\geqq C \left(\fint_{\Omega_{\mu r/2}(\hat{x}_0)} |D_{y'}u - (D_{y'}u)_{\Omega_{\mu r/2}(x_0)}|^p + |D_{y_n}u|^p \right)^{1/p} \\
&\quad - C' \left(\fint_{\Omega_{\mu r/2}(\hat{x}_0)} |D_{y_n}u\; D_{y'}\chi|^p \right)^{1/p} \\
&\geqq C\psi(\hat{x}_0, \mu r/2) - C'\mu r \left(\fint_{\Omega_{\mu r/2}(\hat{x}_0)} |Du|^p \right)^{1/p},
\end{aligned} \tag{2.17}$$

where C and C' are positive constants depending on n, p, c_1, and c_2. Similarly,

$$\begin{aligned}
&\left(\fint_{B^+_r} |D_{z'}u_1 - (D_{z'}u_1)_{B^+_r}|^p + |D_{z_n}u_1|^p \right)^{1/p} \\
&\quad \leqq C''\psi(\hat{x}_0, 2r) + C''r \left(\fint_{\Omega_{2r}(\hat{x}_0)} |Du|^p \right)^{1/p},
\end{aligned} \tag{2.18}$$

where C'' is a positive constant depending only on n, p, c_1, and c_2. Therefore, by using (2.17), (2.18), and (2.6), (2.14) implies that

$$\psi(\hat{x}_0, \mu r/2) \leqq C\mu^\alpha \psi(\hat{x}_0, 2r) + C_\mu r^{\theta_p} \left(\fint_{\Omega_{2r}(\hat{x}_0)} |Du|^p \right)^{1/p}.$$

By replacing $\mu/4$ and $2r$ with μ and r respectively, we obtain

$$\psi(\hat{x}_0, \mu r) \leqq C\mu^\alpha \psi(\hat{x}_0, r) + C_\mu r^{\theta_p} \left(\fint_{\Omega_r(\hat{x}_0)} |Du|^p \right)^{1/p}$$

for $\mu \in (0, 1/4)$ and $r \in (0, R_1]$, where we recall that $R_1 = c_4(\varepsilon + |\hat{x}'_0|^2)$. Note that the same argument above also holds when $\hat{x}_0 \in \Gamma_+$. Therefore, using the same argument as in Corollary 2.2, we have

Lemma 2.5. *Suppose that u is a solution to (1.2) and $\hat{x}_0 \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leqq 3/4\}$. Then there exist constants $c_4 \in (0, 1)$ and $C > 0$, both depending* *only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0, 1/4)$ and $r \in (0,\ c_4(\varepsilon + |\hat{x}_0'|^2)]$, it holds that*

$$\psi(\hat{x}_0, \mu r) \leqq C\mu^\alpha \psi(\hat{x}_0, r) + C_\mu r^{\theta_p} \left(\fint_{\Omega_r(\hat{x}_0)} |Du|^p \right)^{1/p},$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 2.1, and ψ is defined in (2.16). Moreover, there exist constants $\mu_2 = \mu_2(n, p, c_1, c_2) \in (0, 1/4)$ and $C_\mu' = C_\mu'(n, p, c_1, c_2, \mu) > 0$, such that for any $\mu \in (0, \mu_2]$ and $K \in \mathbb{N}$, it holds that

$$\sum_{k=0}^{K+1} \psi(\hat{x}_0, \mu^k r) \leqq 2\,\psi(\hat{x}_0, r) + C_\mu' \sum_{k=0}^{K} (\mu^k r)^{\theta_p} \left(\fint_{\Omega_{\mu^k r}(\hat{x}_0)} |Du|^p \right)^{1/p}.$$

2.4. Mean oscillation estimates for large r

Finally, we consider the case when $B_r(x_0)$ could potentially intersects with both Γ_+ and Γ_-. In this case, we assume $x_0 \in \Omega_{1/2}$ and $\frac{c_4}{12}(\varepsilon + |x_0'|^2) \leqq r \leqq c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$, where c_4 is the same constant as in Lemma 2.5 and c_5 is a constant which will be determined later. We define the map $\mathcal{Z} = \tilde{\Lambda}(x)$ by

$$\begin{cases} \mathcal{Z}' = x' - x_0', \\ \mathcal{Z}_n = (h_1(x_0') - h_2(x_0') + \varepsilon)\Big(\dfrac{x_n - h_2(x') + \varepsilon/2}{h_1(x') - h_2(x') + \varepsilon} - \dfrac{1}{2} \Big). \end{cases}$$

Thus $\tilde{\Lambda}$ is invertible in $\Omega_{x_0,1/2}$,

$$Q_{1/2} := \tilde{\Lambda}(\Omega_{x_0,1/2}) = \left\{ (\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < \frac{1}{2},\ |\mathcal{Z}_n| < \frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon) \right\},$$

and

$$\begin{aligned} \tilde{\Gamma}_\pm &:= \tilde{\Lambda}\left(\Gamma_\pm \cap \left\{ x \in \mathbb{R}^n : |x' - x_0'| < \frac{1}{2} \right\} \right) \\ &= \left\{ (\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < \frac{1}{2},\ \mathcal{Z}_n = \pm\frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon) \right\}. \end{aligned}$$

Then $u_2(\mathcal{Z}) := u(\tilde{\Lambda}^{-1}(\mathcal{Z}))$ satisfies the following equation with homogeneous conormal boundary condition

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}} \left(|B^T D_{\mathcal{Z}} u_2|^{p-2} (\det(B))^{-1} B B^T D_{\mathcal{Z}} u_2 \right) = 0 & \text{in } Q_{1/2} \\ \left(|B^T D_{\mathcal{Z}} u_2|^{p-2} (\det(B))^{-1} B B^T\, D_{\mathcal{Z}} u_2 \right)_n = 0 & \text{on } \tilde{\Gamma}_\pm, \end{cases}$$

where we denote

$$B := B(\mathcal{Z}) := (b_{ij}(\mathcal{Z})) := D\tilde{\Lambda}(\tilde{\Lambda}^{-1}(\mathcal{Z})).$$

For $\mathcal{Z} \in Q_{1/2}$, let $x = \tilde{\Lambda}^{-1}(\mathcal{Z})$. Then

$$\begin{aligned}
&b_{ii}(\mathcal{Z}) = 1 \quad \text{for } i \in \{1, 2, \ldots, n-1\},\\
&b_{ij}(\mathcal{Z}) = 0 \quad \text{for } i \neq j,\ i \in \{1, 2, \ldots, n-1\},\ j \in \{1, 2, \ldots, n\},\\
&b_{nj}(\mathcal{Z}) = \frac{h_1(x_0') - h_2(x_0') + \varepsilon}{(h_1(x') - h_2(x') + \varepsilon)^2}\\
&\qquad \cdot \left[D_{x_j} h_2(x') \left(x_n - h_1(x') - \frac{\varepsilon}{2}\right) - D_{x_j} h_1(x') \left(x_n - h_2(x') + \frac{\varepsilon}{2}\right)\right]
\end{aligned}$$

for $j \in \{1, 2, \ldots, n-1\}$, and

$$b_{nn}(\mathcal{Z}) = \frac{h_1(x_0') - h_2(x_0') + \varepsilon}{h_1(x') - h_2(x') + \varepsilon}.$$

Therefore,

$$\det(B(\mathcal{Z})) = b_{nn}(\mathcal{Z}) = \frac{h_1(x_0') - h_2(x_0') + \varepsilon}{h_1(x_0' + \mathcal{Z}') - h_2(x_0' + \mathcal{Z}') + \varepsilon}$$

is a function independent of $\mathcal{Z}_n$. Assume

$$\frac{c_4}{12}(\varepsilon + |x_0'|^2) \leqq r \leqq \frac{1}{4}(\varepsilon + |x_0'|^2)^{\frac{1}{2}} \leqq \frac{1}{2} \tag{2.19}$$

and let $\mathcal{Z}_0 = \tilde{\Lambda}(x_0)$. Then for any $\mathcal{Z} \in Q_{1/2}$ with $|\mathcal{Z}'| \leqq r$ and $x = \tilde{\Lambda}^{-1}(\mathcal{Z})$, by the triangle inequality, we have

$$|x'| \leqq r + |x_0'| \leqq \left(1 + \sqrt{12/c_4}\right) r^{\frac{1}{2}} \quad \text{and} \quad |x'|^2 \geqq \frac{1}{2}|x_0'|^2 - r^2 \geqq \frac{1}{4}(|x_0'|^2 - \varepsilon).$$

Thus, using (1.3), (1.4), and (1.5), we infer that for $j = 1, 2, \ldots, n-1$ and some constant $C > 0$ depending only on n, p, c_1, and c_2,

$$\begin{aligned}
|b_{nj}(\mathcal{Z})| &\leqq 2c_2 \frac{|x'|(h_1(x_0') - h_2(x_0') + \varepsilon)}{h_1(x') - h_2(x') + \varepsilon} \leqq 2c_2 \frac{|x'|(2c_2|x_0'|^2 + \varepsilon)}{c_1|x'|^2 + \varepsilon}\\
&\leqq C|x'| \leqq C r^{\frac{1}{2}} \leqq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}},\\
|b_{nn}(\mathcal{Z}) - 1| &= \left| \frac{\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(h_1(tx' + (1-t)x_0') - h_2(tx' + (1-t)x_0')\big)\mathrm{d}t}{h_1(x') - h_2(x') + \varepsilon} \right|\\
&\leqq 2c_2 \frac{(|x'| + |x_0'|)|x' - x_0'|}{c_1|x'|^2 + \varepsilon} \leqq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}},
\end{aligned}$$

and similarly,

$$\left|\big(\det(B(\mathcal{Z}))\big)^{-1} - 1\right| = \left|\big(b_{nn}(\mathcal{Z})\big)^{-1} - 1\right| \leqq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}. \tag{2.20}$$

Therefore, when (2.19) holds and $\mathcal{Z} \in Q_{1/2}$ with $|\mathcal{Z}'| \leqq r$, we have for some constant $C = C(n, p, c_1, c_2) > 0$,

$$|B(\mathcal{Z}) - I_n| \leqq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}. \tag{2.21}$$

In particular, there exists $c_5 = c_5(n, p, c_1, c_2) \in (0, 1/4)$, such that for any $\frac{c_4}{12}(\varepsilon + |x_0'|^2) \leqq r \leqq c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$ and $\mathcal{Z} \in Q_{1/2}$ with $|\mathcal{Z}'| \leqq r$, it also holds that

$$|B(\mathcal{Z}) - I_n| \leqq 1/2 \quad \text{and} \quad \left|\big(b_{nn}(\mathcal{Z})\big)^{-1} - 1\right| \leqq 1/2. \tag{2.22}$$

Note that we can always assume $\varepsilon + |x_0'|^2$ to be sufficiently small so that $c_4(\varepsilon + |x_0'|^2) \leqq c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$. Next we extend u_2 and B to the whole cylinder $\mathcal{C}_{1/2} := \{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1/2\}$. We take the even extension of u_2, b_{nn}, and $b_{ij}, i, j = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = \frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon)$, and take the odd extension of b_{in} and $b_{ni}, i = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = \frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon)$. Then we take the periodic extension in $\mathcal{Z}_n$ axis, so that the period is equal to $2(h_1(x_0') - h_2(x_0') + \varepsilon)$. We still denote these functions by u_2 and B after the extension. Then because of the conormal boundary condition, u_2 satisfies

$$-\operatorname{div}_{\mathcal{Z}} \big(\mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}} u_2)\big) = 0 \quad \text{in } \mathcal{C}_{1/2}, \tag{2.23}$$

where the nonlinear operator $\mathbf{B}$ is defined as

$$\mathbf{B}(\mathcal{Z}, \xi) = d(\mathcal{Z}')|B^T \xi|^{p-2} B B^T \xi \qquad \text{for} \quad \mathcal{Z} \in \mathcal{C}_{1/2},\ \xi \in \mathbb{R}^n,$$

and

$$d(\mathcal{Z}') := \big(b_{nn}(\mathcal{Z})\big)^{-1} = \frac{h_1(\mathcal{Z}' + x_0') - h_2(\mathcal{Z}' + x_0') + \varepsilon}{h_1(x_0') - h_2(x_0') + \varepsilon}.$$

Similar to (2.10), using (2.20), (2.21), (2.22), and (2.11), we obtain that for any $r \in \big[\frac{c_4}{12}(\varepsilon + |x_0'|^2), c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}\big]$, $\mathcal{Z} \in B_r(\mathcal{Z}_0)$, and $\xi \in \mathbb{R}^n$,

$$|\mathbf{B}(\mathcal{Z}, \xi) - |\xi|^{p-2}\xi| \leqq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}} |\xi|^{p-1}, \tag{2.24}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. Now we let $v_2 \in u_2 + W_0^{1,p}(B_r(\mathcal{Z}_0))$ be the unique solution to

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}} \big(|D_{\mathcal{Z}} v_2|^{p-2} D_{\mathcal{Z}} v_2\big) = 0 & \text{in } B_r(\mathcal{Z}_0), \\ v_2 = u_2 & \text{on } \partial B_r(\mathcal{Z}_0). \end{cases}$$

Using (2.24), similar to (2.13), we have the following comparison estimate

$$\fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2 - D_{\mathcal{Z}} v_2|^p \leqq C\Big(\frac{r}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}\Big)^{\min\{2,p\}} \fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2|^p, \tag{2.25}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2.

We define

$$\tilde{\phi}(x_0, r) = \left(\fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2 - (D_{\mathcal{Z}} u_2)_{B_r(\mathcal{Z}_0)}|^p \right)^{1/p}. \tag{2.26}$$

Then following the same proof as that of Lemma 2.5 with (2.25) in place of (2.13), we have

Lemma 2.6. *Suppose that $x_0 \in \Omega_{1/2}$ and u_2 is a solution to (2.23). Then there exist constants $c_5 \in (0, 1/4)$ and $C > 0$, both depending only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0, 1)$ and $r \in \left[\frac{c_4}{12}(\varepsilon + |x_0'|^2),\ c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}\right]$, it holds that*

$$\tilde{\phi}(x_0, \mu r) \leqq C\mu^\alpha \tilde{\phi}(x_0, r) + C_\mu \left(\frac{r}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}} \right)^{\theta_p} \left(\fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2|^p \right)^{1/p},$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 2.1, c_4 is the same constant as in Lemma 2.5, and $\tilde{\phi}$ is defined in (2.26). Moreover, there exist constants $\mu_3 = \mu_3(n, p, c_1, c_2) \in (0, 1)$ and $C_\mu' = C_\mu'(n, p, c_1, c_2, \mu) > 0$, such that for any $\mu \in (0, \mu_3]$ and $k_1, k_2 \in \mathbb{N}$ satisfying $\frac{c_4}{12}(\varepsilon + |x_0'|^2) \leqq \mu^{k_2} r \leqq \mu^{k_1} r \leqq c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$, it holds that

$$\sum_{k=k_1}^{k_2+1} \tilde{\phi}(x_0, \mu^k r) \leqq 2\tilde{\phi}(x_0, \mu^{k_1} r) + C_\mu' \sum_{k=k_1}^{k_2} \left(\frac{\mu^k r}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}} \right)^{\theta_p} \left(\fint_{B_{\mu^k r}(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2|^p \right)^{\frac{1}{p}}.$$

2.5. *Proof of Theorem 1.1*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We prove the theorem at any point $x_0 \in \Omega_{1/2}$.

Step 1: Notation and choices of constants. We choose $\mu = \frac{1}{6}\min\{\mu_1, \mu_2, \mu_3\}$, where μ_1, μ_2, and μ_3 are the same constants as in Corollary 2.2, Lemma 2.5, and Lemma 2.6. We define $r_j = \frac{c_5}{2}\mu^j(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$ and let j_1, j_2 be the integers such that

$$r_{j_1} \geqq \frac{c_4}{6}(\varepsilon + |x_0'|^2), \quad r_{j_1+1} < \frac{c_4}{6}(\varepsilon + |x_0'|^2),$$

and

$$r_{j_2} \geqq \operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-), \quad r_{j_2+1} < \operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-),$$

where c_4 and c_5 are the same constants as in Lemma 2.5 and Lemma 2.6. Note that we can assume $\varepsilon + |x_0'|^2$ to be sufficiently small so that $\frac{c_5}{2}(\varepsilon + |x_0'|^2)^{\frac{1}{2}} > \frac{c_4}{6}(\varepsilon + |x_0'|^2)$ and thus $j_1 \geqq 0$.

We denote

$$\phi_j = \begin{cases} \left(\fint_{B_{r_j}(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2 - (D_{\mathcal{Z}} u_2)_{B_{r_j}(\mathcal{Z}_0)}|^p \right)^{1/p} & \text{if } \ 0 \leqq j \leqq j_1, \\ \left(\fint_{\Omega_{r_j}(x_0)} |Du - (Du)_{\Omega_{r_j}(x_0)}|^p \right)^{1/p} & \text{if } \ j \geqq j_1 + 1, \end{cases}$$

$$T_j = \begin{cases} \left(\fint_{B_{r_j}(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2|^p \right)^{1/p} & \text{if } \ 0 \leqq j \leqq j_1, \\ \left(\fint_{\Omega_{3r_j}(x_0)} |Du|^p \right)^{1/p} & \text{if } \ j \geqq j_1 + 1, \end{cases}$$

and

$$\mathbf{m}_j = \begin{cases} (D_{\mathcal{Z}} u_2)_{B_{r_j}(\mathcal{Z}_0)} & \text{if } \ 0 \leqq j \leqq j_1, \\ (Du)_{\Omega_{r_j}(x_0)} & \text{if } \ j \geqq j_1 + 1, \end{cases}$$

where u_2, $\mathcal{Z}_0$, and the coordinate $\mathcal{Z}$ are defined in Section 2.4. In the following proof, we use C, C' to denote positive constants depending only on n, p, c_1, and c_2, which may differ from line to line.

Step 2: Preliminary estimates and iterations. Next, we derive some preliminary estimates. First, we show that there exists a constant $c = c(n) > 0$, such that for any $j \geqq j_1 + 1$,

$$|\Omega_{r_j}(x_0)| \geqq c\, r_j^n. \tag{2.27}$$

If $r_j \leqq 2\,\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-)$, then $B_{\frac{1}{2}r_j}(x_0) \subset \Omega_{r_j}(x_0)$ and (2.27) clearly holds. Otherwise, assume $r_j > 2\,\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-)$. Then we choose $\hat{x}_0 \in \Gamma_+ \cup \Gamma_-$ such that $\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-) = |\hat{x}_0 - x_0|$, and thus $\Omega_{\frac{1}{2}r_j}(\hat{x}_0) \subset \Omega_{r_j}(x_0)$. Note that we can assume $\varepsilon + c_2|x_0'|^2 \leqq 1/4$ and thus by (1.5) and the triangle inequality, $|\hat{x}_0'| \leqq 3/4$. Since $j \geqq j_1 + 1$, by the triangle inequality again, we know that $|x_0'| \leqq |\hat{x}_0'| + r_j/2$ and

$$r_j < \frac{c_4}{6}(\varepsilon + |x_0'|^2) \leqq \frac{c_4}{6}\big(\varepsilon + 2|\hat{x}_0'|^2 + \frac{1}{2}r_j^2\big).$$

Since $c_4 \in (0, 1)$ and $r_j \in (0, 1)$, we get

$$r_j < \frac{c_4}{2}(\varepsilon + |\hat{x}_0'|^2).$$

By using (2.6), we have

$$|\Omega_{r_j}(x_0)| \geqq |\Omega_{\frac{1}{2}r_j}(\hat{x}_0)| \geqq c\, r_j^n.$$

Thus, (2.27) holds for every $j \geqq j_1 + 1$.

By (2.27) and Hölder's inequality, for any $j \in \mathbb{N}$, we have

$$|\mathbf{m}_j| \leqq C T_j. \tag{2.28}$$

Note that since $\mu \leqq 1/6$, $\Omega_{3r_{j+1}}(x_0) \subset \Omega_{\frac{1}{2}r_j}(x_0)$ and thus by (2.27), (2.22) and the definition of j_1,

$$T_{j_1+1} \leqq C \left(\fint_{\Omega_{\frac{1}{2}r_{j_1}}(x_0)} |Du|^p \right)^{1/p} \leqq C\, T_{j_1}.$$

Therefore, there exists $c_6 = c_6(n, p, c_1, c_2) > 0$, such that for any $j \in \mathbb{N}$,

$$T_{j+1} \leqq c_6\, T_j. \tag{2.29}$$

By (2.29) and the triangle inequality, for any $j \leqq j_1$, we have

$$T_{j+1} \leqq c_6\, T_j \leqq C\, |\mathbf{m}_j| + C\, \phi_j.$$

For $j \geqq j_1 + 1$, since $\mu \leqq 1/6$, $\Omega_{3r_{j+1}}(x_0) \subset \Omega_{r_j}(x_0)$ and thus by (2.27) and the triangle inequality, we have

$$T_{j+1} \leqq C \left(\fint_{\Omega_{r_j}(x_0)} |Du|^p \right)^{1/p} \leqq C\, |\mathbf{m}_j| + C\, \phi_j.$$

Therefore, there exists $c_7 = c_7(n, p, c_1, c_2) > 0$, such that for any $j \in \mathbb{N}$,

$$T_{j+1} \leqq c_7\, |\mathbf{m}_j| + c_7\, \phi_j. \tag{2.30}$$

For any $0 \leqq k \leqq j_1$, since

$$|\mathbf{m}_k - \mathbf{m}_{k-1}|^p \leqq 2^{p-1} |\mathbf{m}_k - D_{\mathcal{Z}} u_2(\mathcal{Z})|^p + 2^{p-1} |D_{\mathcal{Z}} u_2(\mathcal{Z}) - \mathbf{m}_{k-1}|^p,$$

by taking the average over $\mathcal{Z} \in B_{r_k}(\mathcal{Z}_0)$ and then taking the pth root, we obtain

$$|\mathbf{m}_k - \mathbf{m}_{k-1}| \leqq C\phi_k + C\phi_{k-1}.$$

Then by iterating, we get

$$|\mathbf{m}_j - \mathbf{m}_{j_0}| \leqq C \sum_{k=j_0}^{j} \phi_k, \tag{2.31}$$

for any integers j_0, j satisfying $0 \leqq j_0 \leqq j \leqq j_1$. By (2.31), (2.28), and the triangle inequality, we have

$$|\mathbf{m}_j| \leqq C\, T_{j_0} + C \sum_{k=j_0}^{j} \phi_k, \tag{2.32}$$

for $j_0 \leqq j \leqq j_1$. Similarly, for any integers j, l satisfying $j_1 + 1 \leqq l \leqq j$, we also have

$$|\mathbf{m}_j - \mathbf{m}_l| \leqq C \sum_{k=l}^{j} \phi_k,$$

and

$$|\mathbf{m}_j| \leqq C\, T_l + C \sum_{k=l}^{j} \phi_k. \tag{2.33}$$

For $j \in \{j_0, \dots, j_1\}$, from Lemma 2.6 and (2.32), we know that

$$|\mathbf{m}_j| + \sum_{k=j_0}^{j} \phi_k \leqq C\, T_{j_0} + C \sum_{k=j_0}^{j} \left(\frac{r_k}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}} \right)^{\theta_p} T_k \leqq C\, T_{j_0} + C \sum_{k=j_0}^{j} \mu^{k\theta_p} T_k. \tag{2.34}$$

For $j \in \{j_1 + 1, \dots, j_2\}$, we have $r_j \geqq \operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-)$. Choose $\hat{x}_0 \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leqq 3/4\}$ such that $\operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-) = |\hat{x}_0 - x_0|$, and thus $\Omega_{r_j}(x_0) \subset \Omega_{2r_j}(\hat{x}_0) \subset \Omega_{3r_j}(x_0)$. Then

$$r_j < \frac{c_4}{6}(\varepsilon + |x_0'|^2) \leqq \frac{c_4}{6}(\varepsilon + 2|\hat{x}_0'|^2 + 2r_j^2),$$

which also implies

$$2r_j < c_4(\varepsilon + |\hat{x}_0'|^2) \tag{2.35}$$

since $c_4 \in (0, 1)$ and $r_j \in (0, 1)$.

By (2.35), we can apply Lemma 2.5 at $\hat{x}_0 \in \Gamma_+ \cup \Gamma_-$ and use (2.27) and (2.6) to obtain

$$\begin{aligned} \sum_{k=j_1+1}^{j} \phi_k &\leqq C \sum_{k=j_1+1}^{j} \psi(\hat{x}_0, 2r_k) \leqq C\, Y_{j_1+1} + C \sum_{k=j_1+1}^{j} r_k^{\theta_p} Y_k \\ &\leqq C\, T_{j_1+1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k, \end{aligned} \tag{2.36}$$

where

$$Y_j := \left(\fint_{\Omega_{2r_j}(\hat{x}_0)} |Du|^p \right)^{1/p}.$$

Moreover, from (2.33), (2.30), and (2.36) we also know that

$$\begin{aligned} |\mathbf{m}_j| + \sum_{k=j_1+1}^{j} \phi_j &\leqq C\, T_{j_1+1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k \\ &\leqq C\, |\mathbf{m}_{j_1}| + C\, \phi_{j_1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k \end{aligned} \tag{2.37}$$

holds for any $j \in \{j_1+1, \dots, j_2\}$.

For $j \geqq j_2+1$, from Corollary 2.2, (2.33), and (2.30) we have

$$|\mathbf{m}_j| + \sum_{k=j_2+1}^{j} \phi_k \leqq C\, T_{j_2+1} \leqq C\, |\mathbf{m}_{j_2}| + C\, \phi_{j_2}. \tag{2.38}$$

Combining (2.37) and (2.38), we know that

$$|\mathbf{m}_j| + \sum_{k=j_1+1}^{j} \phi_j \leqq C\, |\mathbf{m}_{j_1}| + C\, \phi_{j_1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k \tag{2.39}$$

holds for any $j \geqq j_1+1$. Note that (2.39) also holds if $j_2 \leqq j_1$ since in that case $r_j \leqq \mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-)$ for any $j \geqq j_1+1$ and thus we can directly use Corollary 2.2 and (2.33) to get (2.39).

Moreover, combining (2.39) and (2.34), we know that

$$|\mathbf{m}_j| + \sum_{k=j_0}^{j} \phi_k \leqq C\, T_{j_0} + C \sum_{k=j_0}^{j} \mu^{k\theta_p} T_k \tag{2.40}$$

holds for any $0 \leqq j_0 \leqq j_1$ and $j \geqq j_0$.

Step 3: A stopping time argument. We choose $j_0 = j_0(n, p, c_1, c_2) \in \mathbb{N}$ sufficiently large such that

$$(c_7+1)\, C \sum_{k=j_0}^{\infty} \mu^{k\theta_p} \leqq \frac{1}{10}, \tag{2.41}$$

where c_7 is the constant in (2.30) and C is the constant in (2.40). Note that we can assume $\varepsilon + |x_0'|^2$ to be sufficiently small so that $\frac{c_5}{2}\mu^{j_0}(\varepsilon + |x_0'|^2)^{\frac{1}{2}} > \frac{c_4}{6}(\varepsilon + |x_0'|^2)$ and thus $j_1 \geqq j_0$. Now we show that

$$|Du(x_0)| \leqq C\, T_{j_0}. \tag{2.42}$$

We consider the following three possibilities.

Case 1: If $|Du(x_0)| \leqq T_{j_0}$, then (2.42) directly follows.

Case 2: If $T_j < |Du(x_0)|$, $\forall j_0 \leqq j \leqq j_3$, and $|Du(x_0)| \leqq T_{j_3+1}$, then by (2.30), we have

$$|Du(x_0)| \leqq T_{j_3+1} \leqq c_7\, |\mathbf{m}_{j_3}| + c_7\, \phi_{j_3}. \tag{2.43}$$

Now applying (2.40) with $j = j_3$, from (2.43) and (2.41), we get

$$|Du(x_0)| \leqq C'\, T_{j_0} + C' \sum_{k=j_0}^{j_3} \mu^{k\theta_p} |Du(x_0)| \leqq C'\, T_{j_0} + \frac{1}{10}|Du(x_0)|,$$

where $C' = c_7\, C$, C is the constant in (2.40). The last inequality directly implies (2.42) as desired.

Case 3: If $T_j < |Du(x_0)|$ for every $j \geqq j_0$, then from (2.40), we infer that for any $j \geqq j_0$,

$$|\mathbf{m}_j| \leqq C\, T_{j_0} + C \sum_{k=j_0}^{j} \mu^{k\theta_p} |Du(x_0)| \leqq C\, T_{j_0} + \frac{1}{10}|Du(x_0)|.$$

Here we used (2.41) in the last inequality. Letting $j \to \infty$ and using the fact that $u \in C^1(\Omega_1)$, we get

$$|Du(x_0)| \leqq C\, T_{j_0} + \frac{1}{10}|Du(x_0)|,$$

which directly implies (2.42). The proof of the inequality (2.42) is completed.

Step 4: Caccioppoli inequality and conclusion. Let $\lambda \in \mathbb{R}$ and ζ be a nonnegative smooth function satisfying $\zeta = 1$ in $B_{r_{j_0}}(\mathcal{Z}_0)$, $|D_{\mathcal{Z}}\zeta| \leqq 2r_{j_0}^{-1}$, and $\zeta = 0$ outside $B_{2r_{j_0}}(\mathcal{Z}_0)$. Since $2r_{j_0} \leqq c_5(\varepsilon+|x_0'|^2)^{\frac{1}{2}}$, using $\zeta^p(u_2-\lambda)$ as a test function in (2.23), by (2.22), Young's inequality, we obtain

$$\begin{aligned}
\frac{1}{2^{p+1}} \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \zeta^p |D_{\mathcal{Z}}u_2|^p &\leqq \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \langle \mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}}u_2), \zeta^p D_{\mathcal{Z}}u_2 \rangle \\
&= -p \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \langle \mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}}u_2), \zeta^{p-1}(u_2-\lambda) D_{\mathcal{Z}}\zeta \rangle \\
&\leqq p2^{p+2} r_{j_0}^{-1} \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \zeta^{p-1} |D_{\mathcal{Z}}u_2|^{p-1} |u_2-\lambda| \\
&\leqq \frac{1}{2^{p+2}} \int_{B_{2r_{j_0}}} \zeta^p |D_{\mathcal{Z}}u_2|^p + c(p)\, r_{j_0}^{-p} \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} |u_2-\lambda|.
\end{aligned}$$

Therefore, we have the following Caccioppoli inequality

$$\int_{B_{r_{j_0}}(\mathcal{Z}_0)} |D_{\mathcal{Z}}u_2|^p \leqq c(p)\, r_{j_0}^{-p} \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} |u_2-\lambda|^p, \tag{2.44}$$

where λ is an arbitrary constant and $c(p)$ is a positive constant depending only on p. Since $2r_{j_0} \leqq c_5(\varepsilon+|x_0'|^2)^{\frac{1}{2}} \leqq \frac{1}{4}(\varepsilon+|x_0'|^2)^{\frac{1}{2}} \leqq 1/2$, by choosing $\lambda = (u_2)_{B_{2r_{j_0}}(\mathcal{Z}_0)}$ in (2.44), and using (2.42), we obtain the pointwise blow-up estimate

$$|Du(x_0)| \leqq C(\varepsilon + |x_0'|^2)^{-\frac{1}{2}} \operatorname*{osc}_{\Omega_{x_0,\eta}} u,$$

where $\eta = \frac{1}{4}(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$. □

3. Improved gradient estimates

In this section, we utilize a similar approach of flattening the boundaries and extending the equation, as described in [25], to derive an improved gradient estimate for (1.7) in dimensions $n \geqq 3$. However, in contrast to [25], since our equation is degenerate, we need to exploit the nondivergence form of the normalized equation. Consequently, the argument of flattening the boundaries becomes much more intricate, and unlike in [25], where the De Giorgi-Nash-Moser Harnack inequality is applied, we use the Krylov–Safonov Harnack inequality for nondivergence form equations. Furthermore, there are additional first-order terms that require control over the size of the coefficients.

To prove Theorem 1.2, for $\eta > 0$, we consider the approximating equation

$$\begin{cases} -\operatorname{div}\left((\eta + |Du_\eta|^2)^{\frac{p-2}{2}} Du_\eta\right) = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u_\eta}{\partial \nu} = 0 & \text{on } \partial D_i,\ i = 1, 2, \\ u_\eta = \varphi & \text{on } \partial \Omega. \end{cases} \tag{3.1}$$

Since $\|u_\eta\|_{C^{1,\alpha}(\Omega_1)}$ is bounded independent of η, it suffices to prove (1.9) for u_η. Therefore, we will focus on (3.1) throughout the rest of this section, and denote $u = u_\eta$ for simplicity. Note that u satisfies the normalized p-Laplace equation

$$a^{ij} D_{ij} u = 0 \quad \text{in } \Omega_1,$$

where

$$a^{ij} = \delta_{ij} + (p-2)(\eta + |Du|^2)^{-1} D_i u D_j u \tag{3.2}$$

satisfies

$$\begin{aligned} &(p-1)|\xi|^2 \leqq a^{ij}\xi_i\xi_j \leqq |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \quad \text{when } 1 < p < 2, \\ &|\xi|^2 \leqq a^{ij}\xi_i\xi_j \leqq (p-1)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \quad \text{when } p \geqq 2. \end{aligned} \tag{3.3}$$

For a small r_0 independent of ε, we only need to show (1.9) in Ω_{r_0}, as $|Du|$ is bounded in $\Omega_{1/2} \setminus \Omega_{r_0}$ independent of ε. For any $x \in \Omega_{r_0}$, we estimate $|Du(x)|$ as follows: first we consider the equation in $\Omega_{2r} \backslash \Omega_{r/4}$ for $r \in (\sqrt{\varepsilon}, r_0]$, we perform a suitable change of variables that maps the domain to a flat "annular cylinder". After the change of variables, u will satisfy a second-order uniformly elliptic equation in non-divergence form, and the Neumann boundary condition on the upper and lower boundaries of the domain. Then we obtain a Harnack inequality through the Krylov–Safonov theorem. Together with the maximum principle, this gives the oscillation of u in Ω_r for $r \in (\sqrt{\varepsilon}, r_0]$ with a decay rate $r^{2\beta}$ for some positive ε-independent β. Then the desired estimate on $|Du(x)|$ follows from the decay rate of $\operatorname{osc}_{2(\varepsilon+|x'|^2)^{1/2}} u$ and Theorem 1.1.

Let $r \in (\sqrt{\varepsilon}, r_0]$, where r_0 is an ε -independent constant to be determined later. We define

$$\tilde{h}_i(x') := \begin{cases} h_i(x') & \text{when } |x'| \leqq 2r_0, \\ 0 & \text{when } |x'| > 2r_0 \end{cases}$$

for $i = 1, 2$. We denote

$$Q_{s,t} := \{y = (y', y_n) \in \mathbb{R}^n \;\big|\; |y'| < s, |y_n| < t\},$$

and for $y \in Q_{2r,r^2} \setminus Q_{r/4,r^2}$, we define the map $x = \Phi(y)$ by

$$\begin{cases} x' = y' - g(y), \\ x_n = \dfrac{1}{2}\Big[\dfrac{y_n}{r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')) + \tilde{h}_1(y') + \tilde{h}_2(y')\Big], \end{cases} \tag{3.4}$$

where

$$g(y) = (y_n - r^2)(y_n + r^2)(\Theta y_n + \Xi), \tag{3.5}$$
$$\begin{cases} \Theta = \dfrac{1}{8r^6}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')]D_{y'}[\tilde{h}_1^{\mu}(y') + \tilde{h}_2^{\mu}(y')], \\ \Xi = \dfrac{1}{8r^4}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')]D_{y'}[\tilde{h}_1^{\mu}(y') - \tilde{h}_2^{\mu}(y')], \end{cases}$$

$\tilde{h}_i^{\mu}$ is a mollification of $\tilde{h}_i$ given by

$$\tilde{h}_i^{\mu}(y') := \int_{\mathbb{R}^{n-1}} \tilde{h}_i(y' - \mu z')\varphi(z')\,\mathrm{d}z', \tag{3.6}$$

φ is a positive smooth function with unit integral supported in $B_1 \subset \mathbb{R}^{n-1}$, and

$$\mu = \frac{r^4 - y_n^2}{r} \geqq 0.$$

Here we briefly explain the motivation for defining the map Φ as above: to ensure that $y' = x'$ on $\{y_n = \pm r^2\}$, which is $g\big|_{y_n = \pm r^2} = 0$, we setup the ansatz (3.5) for g. Next, we want the function $v(y) := u(\Phi(y))$ to satisfy the Neumann boundary condition on $\{y_n = \pm r^2\}$, which leads to (see details in Lemma 3.1)

$$\begin{cases} \Big(-D_{y_n} g, \dfrac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y'))\Big) \parallel (-D_{x'}\tilde{h}_1, 1) \quad \text{on } \{y_n = r^2\}, \\ \Big(-D_{y_n} g, \dfrac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y'))\Big) \parallel (-D_{x'}\tilde{h}_2, 1) \quad \text{on } \{y_n = -r^2\}. \end{cases}$$

Using the ansatz (3.5) and solving for Θ and Ξ, we have

$$\begin{cases} \Theta = \dfrac{1}{8r^6}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')]D_{y'}[\tilde{h}_1(y') + \tilde{h}_2(y')], \\ \Xi = \dfrac{1}{8r^4}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')]D_{y'}[\tilde{h}_1(y') - \tilde{h}_2(y')]. \end{cases}$$

Note that the equation of v involves second-order derivatives of Φ, and hence involves third-order derivatives of $\tilde{h}_1$ and $\tilde{h}_2$. However, $\tilde{h}_1$ and $\tilde{h}_2$ are only $C^{1,1}$, so we introduce the mollification (3.6) to overcome the lack of regularities. Here μ is chosen so that $\tilde{h}_i^{\mu} = \tilde{h}_i$ on $\{y_n = \pm r^2\}$, and the coefficients of the equation of v have the desired estimates.

Throughout this section, unless specify otherwise, we use C to denote positive constants that could be different from line to line, and depend only on n, p, c_1, and c_2, where c_1 and c_2 are defined in (1.4) and (1.5), respectively.

Lemma 3.1. *There exists an $r_0 > 0$ independent of ε, such that when $r \in (\sqrt{\varepsilon}, r_0]$ and Φ is given as (3.4), then:*

(a) *There exists a positive constant C independent of ε and r, such that*

$$\frac{I}{C} \leqq D\Phi(y) \leqq CI, \quad y \in Q_{2r,r^2} \setminus Q_{r/4,r^2},$$

and hence Φ is invertible.

(b)

$$Q_{1.9r,r^2} \setminus Q_{0.35r,r^2} \subset \Phi^{-1}(\Omega_{2r} \setminus \Omega_{r/4}),$$

and

$$\Omega_r \setminus \Omega_{r/2} \subset \Phi(Q_{1.1r,r^2} \setminus Q_{0.4r,r^2}).$$

(c) *Let $u \in W^{1,p}(\Omega_{2r} \setminus \Omega_{r/4})$ be a solution of*

$$\begin{cases} -\operatorname{div}\left((\eta + |Du|^2)^{\frac{p-2}{2}} Du\right) = 0 & \text{in } \Omega_{2r} \setminus \Omega_{r/4}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega_{2r} \setminus \Omega_{r/4}} \end{cases} \tag{3.7}$$

for some $\eta > 0$, and $v(y) = u(\Phi(y))$. Then v satisfies an elliptic equation

$$\begin{cases} \tilde{a}^{ij} D_{ij} v(y) + \tilde{b}_i D_i v(y) = 0 & \text{in } Q_{1.9r,r^2} \setminus Q_{0.35r,r^2}, \\ \dfrac{\partial v}{\partial \nu}(y) = 0 & \text{on } \{y_n = \pm r^2\}, \end{cases} \tag{3.8}$$

with

$$\frac{I}{C} \leqq \tilde{a} \leqq CI, \quad |\tilde{b}| \leqq \frac{C}{r}.$$

Proof. By (1.3), we have

$$|D^k_{y'} \tilde{h}_i(y')| \leqq Cr^{2-k}, \quad |D^k_{y'} \tilde{h}^{\mu}_i(y')| \leqq Cr^{2-k} \quad \text{for } i = 1, 2 \text{ and } k = 0, 1, 2. \tag{3.9}$$

Therefore, when $y \in Q_{2r,r^2} \setminus Q_{r/4,r^2}$,

$$\begin{aligned} |D_{y'} x' - I_{(n-1)\times(n-1)}| &= |D_{y'} g(y)| \leqq Cr^2. \\ D_{y_n} x' &= -D_{y_n} g(y) \\ &= -2y_n(\Theta y_n + \Xi) - (y_n^2 - r^4)[(D_{y_n}\Theta) y_n + \Theta + D_{y_n}\Xi], \end{aligned}$$

and

$$|\Theta| \leqq Cr^{-3}, \quad |\Xi| \leqq Cr^{-1}.$$

By (3.6), we have for $y \in Q_{2r,r^2} \backslash Q_{r/4,r^2}$,

$$\begin{aligned}|D_{y_n} D_{y'} \tilde{h}_i^{\mu}(y')| &= \left| -\frac{\partial \mu}{\partial y_n} \int_{\mathbb{R}^{n-1}} D_{y'}^2 \tilde{h}_i(y' - \mu z') z' \varphi(z')\, \mathrm{d}z' \right| \\ &= \left| \frac{2y_n}{r} \int_{\mathbb{R}^{n-1}} D_{y'}^2 \tilde{h}_i(y' - \mu z') z' \varphi(z')\, \mathrm{d}z' \right| \leqq Cr \quad \text{for } i = 1, 2.\end{aligned}$$

Therefore,

$$|D_{y_n} \Theta| \leqq Cr^{-3}, \quad |D_{y_n} \Xi| \leqq Cr^{-1},$$

and hence

$$\begin{aligned}|D_{y_n} x'| \leqq& |2y_n(\Theta y_n + \Xi)| + |(y_n^2 - r^4)[(D_{y_n}\Theta) y_n + \Theta + D_{y_n} \Xi]| \\ \leqq& Cr + Cr^4[r^{-1} + r^{-3} + r^{-1}] \leqq Cr.\end{aligned}$$

By (3.9),

$$|D_{y'} x_n| = \frac{1}{2} \left| \frac{y_n}{r^2} (D_{y'} \tilde{h}_1(y') - D_{y'} \tilde{h}_2(y')) + D_{y'} \tilde{h}_1(y') + D_{y'} \tilde{h}_2(y') \right| \leqq Cr.$$

And lastly,

$$D_{y_n} x_n = \frac{1}{2r^2} (\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')).$$

By (1.4),

$$\frac{1}{C} \leqq D_{y_n} x_n \leqq C.$$

Then (a) follows by shrinking r_0 to be sufficiently small.

Since $g(y) = 0$ when $y = \pm r^2$, Φ maps the upper and lower boundaries of $Q_{2r,r^2} \setminus Q_{r/4,r^2}$ onto the upper and lower boundaries of $\Omega_{2r} \setminus \Omega_{r/4}$, respectively. Then (b) simply follows from the fact that $|g(y)| \leqq Cr^3$, and we can shrink r_0 so that $|g(y)| \leqq r/10$.

To verify (c), note that u is smooth from the classical elliptic theory. We compute by the chain rule,

$$\begin{aligned}D_{x_k} u(x) =& D_{y_i} v(y) D_{x_k} y_i, \\ D_{x_k x_l} u(x) =& D_{y_i} D_{y_j} v(y) D_{x_k} y_i D_{x_l} y_j + D_{y_i} v(y) D_{x_k x_l} y_i.\end{aligned}$$

Recall that $u(x)$ satisfies the equation

$$a^{kl} D_{x_k x_l} u(x) = 0,$$

where the matrix a is given by (3.2). If we define

$$\tilde{a}^{ij} := a^{kl} D_{x_k} y_i D_{x_l} y_j, \quad \tilde{b}^i := a^{kl} D_{x_k x_l} y_i,$$

then $v(y)$ satisfies

$$\tilde{a}^{ij} D_{ij} v(y) + \tilde{b}_i D_i v(y) = 0.$$

Next, we show that v satisfies the Neumann boundary condition on $\{y_n = \pm r^2\}$. We will show the boundary condition on $\{y_n = r^2\}$, as the one on $\{y_n = -r^2\}$ follows similarly. By the chain rule,

$$D_y v(y) \cdot e_n = D_x u(x) \cdot D_y \Phi e_n,$$

where $e_n := (0, \ldots, 0, 1)$. Therefore, it suffices to show that

$$D_y \Phi e_n = \Big(- D_{y_n} g, \frac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y'))\Big) \parallel (-D_{x'}\tilde{h}_1, 1) \quad \text{on } \{y_n = r^2\}. \tag{3.10}$$

Note that when $y_n = r^2$, we have $g(y) = 0$, $y' = x'$, $\mu = 0$, and $\tilde{h}_1^{\mu} = \tilde{h}_1$. Therefore,

$$\begin{aligned} D_{y_n} g =& (y_n + r^2)(\Theta y_n + \Xi)\Big|_{y_n = r^2} \\ =& \frac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')) D_{y'} \tilde{h}_1^{\mu}(y') \\ =& \frac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')) D_{x'} \tilde{h}_1(x'). \end{aligned}$$

This implies (3.10).

Finally, we show that the coefficients $\tilde{a}$ and $\tilde{b}$ satisfy the desired estimates. From part (a), we know that

$$\frac{I}{C} \leqq D_x y = D_x \Phi^{-1}(x) \leqq CI,$$

which together with (3.3) implies that

$$\frac{I}{C} \leqq \tilde{a} \leqq CI.$$

To estimate $\tilde{b}$, we differentiate $\partial y_i/\partial x_k \cdot \partial x_k/\partial y_j = \delta_{ij}$ in x_l. Note that by chain rule, we have

$$\frac{\partial^2 y_i}{\partial x_k \partial x_l}\frac{\partial x_k}{\partial y_j} + \frac{\partial y_i}{\partial x_k}\frac{\partial y_m}{\partial x_l}\frac{\partial^2 x_k}{\partial y_j \partial y_m} = 0.$$

Since $I/C \leqq D_x y \leqq CI$ and $I/C \leqq D_y x \leqq CI$, it suffices to estimate $D_y^2 x$, which is $D_y^2 \Phi(y)$. It is easy to see that

$$\left| \frac{\partial^2 x_n}{\partial y^2} \right| \leqq \frac{C}{r}.$$

To estimate $\partial^2 x'/\partial y^2$, the key terms are

$$D_{y'}^3 \tilde{h}_i^{\mu}(y'), \quad D_{y_n} D_{y'}^2 \tilde{h}_i^{\mu}(y'), \quad D_{y_n}^2 D_{y'} \tilde{h}_i^{\mu}(y'), \quad i = 1, 2.$$

By (3.6) and integration by parts, we have

$$
\begin{aligned}
D_{y'}\tilde{h}_i^{\mu}(y') &= \int_{\mathbb{R}^{n-1}} D_{y'}\tilde{h}_i(y'-\mu z')\varphi(z')\,\mathrm{d}z' \\
&= -\frac{1}{\mu}\int_{\mathbb{R}^{n-1}} D_{z'}\tilde{h}_i(y'-\mu z')\varphi(z')\,\mathrm{d}z' \\
&= \frac{1}{\mu}\int_{\mathbb{R}^{n-1}} \tilde{h}_i(y'-\mu z')D_{z'}\varphi(z')\,\mathrm{d}z'.
\end{aligned}
$$

Then

$$
|D^3_{y'}\tilde{h}_i^{\mu}(y')| \leqq \frac{C\|h_i\|_{C^{1,1}}}{\mu} \leqq \frac{Cr}{r^4-y_n^2}.
$$

Similarly,

$$
\begin{aligned}
D_{y_n}\tilde{h}_i^{\mu}(y') &= \frac{2y_n}{r}\int_{\mathbb{R}^{n-1}} D_{y'}\tilde{h}_i(y'-\mu z')\cdot z'\varphi(z')\,\mathrm{d}z' \\
&= \frac{2y_n}{\mu r}\int_{\mathbb{R}^{n-1}} \tilde{h}_i(y'-\mu z')D_{z'}\cdot(z'\varphi(z'))\,\mathrm{d}z',
\end{aligned}
\tag{3.11}
$$

so

$$
|D_{y_n}D^2_{y'}\tilde{h}_i^{\mu}(y')| = \left|\frac{2y_n}{\mu r}\int_{\mathbb{R}^{n-1}} D^2_{y'}\tilde{h}_i(y'-\mu z')D_{z'}\cdot(z'\varphi(z'))\,\mathrm{d}z'\right| \leqq \frac{Cr^2}{r^4-y_n^2}.
$$

Differentiating the first line of (3.11) in y_n, we have

$$
\begin{aligned}
D^2_{y_n}\tilde{h}_i^{\mu}(y') =& \frac{2}{r}\int_{\mathbb{R}^{n-1}} \sum_{k=1}^{n-1} D_{y_k}\tilde{h}_i(y'-\mu z')z_k\varphi(z')\,\mathrm{d}z' \\
&+ \frac{4y_n^2}{r^2}\int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^{n-1} D_{y_k y_l}\tilde{h}_i(y'-\mu z')z_k z_l\varphi(z')\,\mathrm{d}z' \\
=& \frac{2}{r}\int_{\mathbb{R}^{n-1}} \sum_{k=1}^{n-1} D_{y_k}\tilde{h}_i(y'-\mu z')z_k\varphi(z')\,\mathrm{d}z' \\
&+ \frac{4y_n^2}{\mu r^2}\int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^{n-1} D_{y_l}\tilde{h}_i(y'-\mu z')D_{z_k}\big(z_k z_l\varphi(z')\big)\,\mathrm{d}z'.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|D^2_{y_n}D_{y'}\tilde{h}_i^{\mu}(y')| \leqq& \frac{2}{r}\int_{\mathbb{R}^{n-1}} |D^2_{y'}\tilde{h}_i(y'-\mu z')||z'\varphi(z')|\,\mathrm{d}z' \\
&+ \frac{4y_n^2}{\mu r^2}\int_{\mathbb{R}^{n-1}} |D^2_{y'}\tilde{h}_i(y'-\mu z')||D_{z'}(z'\otimes z'\varphi(z'))|\,\mathrm{d}z' \\
\leqq& \frac{Cr^3}{r^4-y_n^2}.
\end{aligned}
$$

By these estimates above and straightforward computations, we have

$$|D_y^2\Phi(y)| \leqq Cr^{-1},$$

which implies $|\tilde{b}| \leqq Cr^{-1}$. □

Lemma 3.2. *Let r_0 be as in Lemma 3.1, and let $r \in (\sqrt{\varepsilon}, r_0]$. If $u \in W^{1,p}(\Omega_{2r}\backslash\Omega_{r/4})$ is a nonnegative solution of (3.7) for some $\eta > 0$, then,*

$$\sup_{\Omega_r\backslash\Omega_{r/2}} u \leqq C \inf_{\Omega_r\backslash\Omega_{r/2}} u, \tag{3.12}$$

for some constant $C > 0$ depending only on n, p, c_1, and c_2, but independent of ε, η, r, and u.

Proof. We take the change of variable $y = \Phi^{-1}(x)$, where Φ is given as (3.4). Let $v(y) = u(x)$. By Lemma 3.1 (c), v satisfies the equation (3.8).

For $i, j = 1, 2, \ldots, n-1$, we take the even extension of $\tilde{a}^{ij}$, $\tilde{a}^{nn}$, $\tilde{b}^i$, and v with respect to $y_n = r^2$, and take odd extension of $\tilde{a}^{in}$, $\tilde{a}^{ni}$, and $\tilde{b}^n$ with respect to $y_n = r^2$. Then we take the periodic extension (so that the period is equal to $4r^2$). We still denote them by $\tilde{a}$, $\tilde{b}$, and v after the extension. Then v satisfies

$$\tilde{a}^{ij}D_{ij}v(y) + \tilde{b}_i D_i v(y) = 0 \quad \text{in } Q_{1.9r,2r} \setminus Q_{0.35r,2r}.$$

Setting $\bar{a}^{ij}(y) = \tilde{a}^{ij}(ry)$, $\bar{b}^i(y) = r\tilde{b}^i(ry)$, and $\bar{v}(y) = v(ry)$, we see that $\bar{v}$ satisfies

$$\bar{a}^{ij}D_{ij}\bar{v}(y) + \bar{b}_i D_i \bar{v}(y) = 0 \quad \text{in } Q_{1.9,2} \setminus Q_{0.35,2},$$

with

$$\frac{I}{C} \leqq \bar{a} \leqq CI, \quad |\bar{b}| \leqq C.$$

Since $Q_{1.9,2} \setminus Q_{0.35,2}$ is connected when $n \geqq 3$, by the Krylov–Safonov theorem (see Section 4.2 of [22]), we have

$$\sup_{Q_{1.1,1}\backslash Q_{0.4,1}} \bar{v} \leqq C \inf_{Q_{1.1,1}\backslash Q_{0.4,1}} \bar{v}.$$

This implies

$$\sup_{Q_{1.1r,r^2}\backslash Q_{0.4r,r^2}} v \leqq C \inf_{Q_{1.1r,r^2}\backslash Q_{0.4r,r^2}} v.$$

Finally, (3.12) follows by reverting the changes of variables and Lemma 3.1 (b). □

The following estimate on the oscillation of u is a direct consequence of Lemma 3.2.

Corollary 3.3. *For $n \geqq 3$, let $u \in W^{1,p}(\Omega_1)$ be a solution of (3.1) for some $\eta > 0$. Then there exist positive constants C and β, depending only on n, p, c_1, and c_2, such that*

$$\operatorname*{osc}_{\Omega_r} u \leqq C r^{\beta} \operatorname*{osc}_{\Omega_1} u, \quad \forall\, r \in (\sqrt{\varepsilon}, 1/2). \tag{3.13}$$

Proof. It suffices to prove (3.13) for $r \in (\sqrt{\varepsilon}, r_0]$, where r_0 is the same as in Lemma 3.1. Let $\sqrt{\varepsilon} < r \leqq r_0$ and $v = u - \inf_{\Omega_{2r}} u$. Then $v \geqq 0$ in Ω_{2r}. By Lemma 3.2, we have

$$\sup_{\Omega_r \setminus \Omega_{r/2}} v \leqq C_1 \inf_{\Omega_r \setminus \Omega_{r/2}} v,$$

where $C_1 > 1$ is a constant independent of r. Since v satisfies equation (1.7), by the maximum principle,

$$\sup_{\Omega_r \setminus \Omega_{r/2}} v = \sup_{\Omega_r} v, \qquad \inf_{\Omega_r \setminus \Omega_{r/2}} v = \inf_{\Omega_r} v.$$

Therefore,

$$\sup_{\Omega_r} v \leqq C_1 \inf_{\Omega_r} v,$$

which implies

$$\sup_{\Omega_r} u \leqq C_1 \inf_{\Omega_r} u - (C_1 - 1) \inf_{\Omega_{2r}} u.$$

Adding the above inequality with

$$(C_1 - 1) \sup_{\Omega_r} u \leqq (C_1 - 1) \sup_{\Omega_{2r}} u,$$

and dividing both sides by C_1, we have

$$\operatorname*{osc}_{\Omega_r} u \leqq \frac{C_1 - 1}{C_1} \operatorname*{osc}_{\Omega_{2r}} u.$$

Finally, (3.13) follows from iterating the inequality above. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to show (1.9) for $x \in \Omega_{1/8}$ and $\varepsilon \in (0, 1/32)$. By Corollary 3.3, there exist positive constants C and β, depending only on n, p, c_1, and c_2, such that

$$\operatorname*{osc}_{\Omega_{8\eta}} u \leqq C(\varepsilon + |x'|^2)^{\beta} \operatorname*{osc}_{\Omega_1} u,$$

where $\eta = \frac{1}{4}(\varepsilon + |x'|^2)^{\frac{1}{2}}$. Then by Theorem 1.1, we have

$$\begin{aligned} |Du(x)| &\leqq C(\varepsilon + |x'|^2)^{-\frac{1}{2}} \operatorname*{osc}_{\Omega_{x,\eta}} u \\ &\leqq C(\varepsilon + |x'|^2)^{-\frac{1}{2}} \operatorname*{osc}_{\Omega_{8\eta}} u \\ &\leqq C(\varepsilon + |x'|^2)^{-\frac{1}{2}+\beta} \operatorname*{osc}_{\Omega_1} u. \end{aligned}$$

The theorem is proved. $\square$

4. The $p > n+1$ case

In this section, we establish a more explicit gradient estimate for the equation (1.7) when $p > n+1$, with a blow-up rate of order $\varepsilon^{-\alpha}$ for any $\alpha > \frac{n}{2(p-1)}$. Throughout this section, in addition to (1.3) and (1.4), we need to further assume that h_1 and h_2 are C^2 strictly convex and strictly concave functions respectively, satisfying (1.10). Let ν denote the normal vector on $\Gamma_\pm$, pointing upwards and downwards respectively.

To obtain the improved gradient estimate, in the following lemma, we construct a supersolution to show that the oscillation of u enjoys a better decay rate. Then the desired gradient estimate (1.11) follows by using Theorem 1.1.

Lemma 4.1. *Let $n \geqq 2$, $p > n+1$, $\Gamma_+, \Gamma_-, h_1, h_2$ be as above. For any $\delta \in (0, p-n-1)$, let $v(x) = (|x'|^2 + (2+\delta)x_n^2)^{\gamma/2}$. Then for any $\gamma \in (0, \frac{p-n-1-\delta}{p-1})$, there exists a constant $\mu \in (0, 1/2)$ depending only on n, p, δ, γ, κ_1, κ_2, and the modulus of continuity for $D^2h_1(x')$ and $D^2h_2(x')$ at $x' = 0$, such that for any $\varepsilon \in (0, \mu^2/\kappa_2)$,*

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) > 0 & \text{in } \Omega_{\mu/\kappa_2} \setminus \Omega_{\varepsilon/\mu}, \\ \dfrac{\partial v}{\partial \nu} > 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{\mu/\kappa_2}. \end{cases}$$

Proof. We denote $R(x) = (|x'|^2 + (2+\delta)x_n^2)^{1/2}$, so that $v(x) = R(x)^\gamma$. Using Taylor expansion up to order 2, from (1.3) we know that for any $|x'| < 1$.

$$\begin{aligned} h_1(x') &= \left\langle \int_0^1 (1-t)D^2h_1(tx')\mathrm{d}t \cdot x', x' \right\rangle, \\ h_2(x') &= \left\langle \int_0^1 (1-t)D^2h_2(tx')\mathrm{d}t \cdot x', x' \right\rangle. \end{aligned} \tag{4.1}$$

By (4.1) and (1.10), we have

$$\frac{\kappa_1}{2}|x'|^2 \leqq h_1(x') \leqq \frac{\kappa_2}{2}|x'|^2, \quad \frac{\kappa_1}{2}|x'|^2 \leqq h_2(x') \leqq \frac{\kappa_2}{2}|x'|^2 \quad \text{for } |x'| < 1. \tag{4.2}$$

We also note that for any $|x'| < 1$,

$$\langle Dh_1(x'), x' \rangle = \left\langle \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} Dh_1(tx')\mathrm{d}t, x' \right\rangle = \left\langle \int_0^1 D^2h_1(tx')\mathrm{d}t \cdot x', x' \right\rangle, \tag{4.3}$$

and similarly

$$\langle Dh_2(x'), x' \rangle = \left\langle \int_0^1 D^2h_2(tx')\mathrm{d}t \cdot x', x' \right\rangle.$$

Since h_1 and h_2 are C^2, for any $\delta \in (0, p-n-1)$, there is a sufficiently small $r_0 \in (0, 1/2)$ depending only on n, δ, κ_1, and the modulus of continuity for $D^2h_1(x')$ and $D^2h_2(x')$ at $x' = 0$, such that

$$\begin{aligned} |D^2h_1(x') - D^2h_1(0)| &\leqq \frac{\kappa_1\delta}{8+2\delta}, \\ |D^2h_2(x') - D^2h_2(0)| &\leqq \frac{\kappa_1\delta}{8+2\delta} \quad \text{for } |x'| \leqq r_0. \end{aligned} \tag{4.4}$$

Thus by (4.1), (4.3), (4.4), (1.10), and the triangle inequality, we obtain

$$
\begin{aligned}
&(2+\delta)h_1(x') - \left\langle Dh_1(x'), x'\right\rangle \\
&\quad = \left\langle \int_0^1 (2+\delta)(1-t)D^2h_1(tx')\mathrm{d}t \cdot x', x'\right\rangle - \left\langle \int_0^1 D^2h_1(tx')\mathrm{d}t \cdot x', x'\right\rangle \\
&\quad \geqq \left\langle \int_0^1 (2+\delta)(1-t)D^2h_1(0)\mathrm{d}t \cdot x', x'\right\rangle - \frac{2+\delta}{2}\frac{\kappa_1\delta}{8+2\delta}|x'|^2 \\
&\qquad - \left\langle \int_0^1 D^2h_1(0)\mathrm{d}t \cdot x', x'\right\rangle - \frac{\kappa_1\delta}{8+2\delta}|x'|^2 \\
&\quad = \frac{\delta}{2}\langle D^2h_1(0)\cdot x', x'\rangle - \frac{\kappa_1\delta}{4}|x'|^2 \geqq \frac{\kappa_1\delta}{4}|x'|^2 \geqq 0.
\end{aligned} \tag{4.5}
$$

By direct computations, we have

$$
\begin{aligned}
Dv &= \gamma R^{\gamma-2}(x', (2+\delta)x_n), \\
D_{ii}v &= \gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}x_i^2, \quad \text{for } i \in \{1,2,\ldots,n-1\}, \\
D_{ij}v &= \gamma(\gamma-2)R^{\gamma-4}x_ix_j, \quad \text{for } i \neq j,\ i,j \in \{1,2,\ldots,n-1\}, \\
D_{in}v &= \gamma(\gamma-2)R^{\gamma-4}(2+\delta)x_ix_n, \quad \text{for } i \in \{1,2,\ldots,n-1\}, \\
D_{nn}v &= \gamma R^{\gamma-2}(2+\delta) + \gamma(\gamma-2)R^{\gamma-4}(2+\delta)^2x_n^2.
\end{aligned}
$$

On $\Gamma_+ \cap \overline{\Omega}_{r_0}$,

$$
\nu = \frac{1}{\sqrt{1+|Dh_1(x')|^2}}(-Dh_1(x'), 1).
$$

Then by (4.5),

$$
\begin{aligned}
\frac{\partial v}{\partial \nu} =& \frac{\gamma R^{\gamma-2}}{\sqrt{1+(Dh_1(x'))^2}}\Big[-\langle Dh_1(x'), x'\rangle + (2+\delta)\left(\frac{\varepsilon}{2} + h_1(x')\right)\Big] \\
\geqq& \frac{\gamma R^{\gamma-2}}{\sqrt{1+(Dh_1(x'))^2}}\left[\frac{\kappa_1\delta}{4}|x'|^2 + \left(1+\frac{\delta}{2}\right)\varepsilon\right] > 0.
\end{aligned}
$$

A similar computation shows that $\frac{\partial v}{\partial \nu} > 0$ on $\Gamma_- \cap \overline{\Omega}_{r_0}$.

Next, we compute in $\Omega_1 \setminus \{0\}$,

$$
\begin{aligned}
&\operatorname{div}(|Dv|^{p-2}Dv)|Dv|^{4-p} = |Dv|^2\Delta v + (p-2)D_i v D_j v D_{ij} v \\
&\quad = \gamma^2 R^{2\gamma-4}\Big(|x'|^2 + (2+\delta)^2 x_n^2\Big)\Big((n+1+\delta)\gamma R^{\gamma-2} \\
&\qquad + \gamma(\gamma-2)R^{\gamma-4}(|x'|^2 + (2+\delta)^2 x_n^2)\Big) \\
&\qquad + (p-2)\gamma^2 R^{2\gamma-4}|x'|^2\Big(\gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}|x'|^2\Big) \\
&\qquad + 2(p-2)(2+\delta)^2\gamma^2 R^{2\gamma-4}\gamma(\gamma-2)R^{\gamma-4}|x'|^2 x_n^2 \\
&\qquad + (p-2)\gamma^2 R^{2\gamma-4}(2+\delta)^2 x_n^2\Big(\gamma R^{\gamma-2}(2+\delta) + \gamma(\gamma-2)R^{\gamma-4}(2+\delta)^2 x_n^2\Big) \\
&\quad = \gamma^3 R^{3\gamma-8}\Big\{|x'|^4\big[n+\delta+(p-1)(\gamma-1)\big] \\
&\qquad + |x'|^2 x_n^2\big[(2+\delta)(n+\delta+p-1) \\
&\qquad + (2+\delta)^2(n+1+\delta+(p-2)(2+\delta)+2(\gamma-2)(p-1))\big] \\
&\qquad + x_n^4\big[(2+\delta)^3(n+1+\delta+(p-2)(2+\delta)+(\gamma-2)(p-1)(2+\delta))\big]\Big\}.
\end{aligned}
$$

Thus $\operatorname{div}(|Dv|^{p-2}Dv)|Dv|^{4-p}\gamma^{-3}R^{8-3\gamma}$ is a 4th order homogeneous polynomial of $|x'|$ and x_n. Since $\gamma \in (0, \frac{p-n-1-\delta}{p-1})$, we have

$$
n+\delta+(p-1)(\gamma-1) < 0.
$$

Therefore there exists a sufficiently small $\mu_0 \in (0, 1/2)$ depending only on n, p, γ, and δ, such that if $|x_n| \leqq \mu_0|x'|$ and $x \neq 0$, we have

$$
\operatorname{div}(|Dv|^{p-2}Dv) > 0.
$$

We then take $\mu = \min\{\mu_0, \kappa_2 r_0\}$, so that $\mu/\kappa_2 \leqq r_0$ and thus $\frac{\partial v}{\partial \nu} > 0$ on $(\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{\mu/\kappa_2}$. Note that when $x \in \Omega_{\mu/\kappa_2} \setminus \Omega_{\varepsilon/\mu}$, by (4.2) we have

$$
|x_n| \leqq \frac{\varepsilon}{2} + \frac{\kappa_2}{2}|x'|^2 \leqq \mu|x'| \leqq \mu_0|x'|.
$$

This concludes the proof. $\square$

Proof of Theorem 1.3. It suffices to show (1.11) for $\delta \in (0, (p-n-1)/2)$, $x \in \Omega_{\mu/2\kappa_2}$, and $\varepsilon \in (0, \mu^2/(\kappa_2+\kappa_2^2))$, where $\mu \in (0, 1/2)$ is defined in Lemma 4.1 with $\gamma = \frac{p-1-n-2\delta}{p-1} \in (0, 1)$. Without loss of generality, we may assume that $u(0) = 0$ and $\operatorname{osc}_{\Omega_1} u = 1$. By Theorem 1.1,

$$
|u(x))| \leqq C\sqrt{\varepsilon} \quad \text{for } x \in \overline{\Omega}_{\varepsilon/\mu}. \tag{4.6}
$$

Let v be the function defined in Lemma 4.1 with $\gamma = \frac{p-1-n-2\delta}{p-1}$ and $v_1 = v + \sqrt{\varepsilon}$. Note that $u \leqq Cv$ and $-u \leqq Cv$ on $(\{|x'| = \varepsilon/\mu\} \cup \{|x'| = \mu/\kappa_2\}) \cap \overline{\Omega}_1$ for some ε-independent constant C. By the comparison principle, we have $|u| \leqq Cv$ in $\Omega_{\mu/\kappa_2} \setminus \Omega_{\varepsilon/\mu}$. In particular, we have

$$
|u(x)| \leqq C(\varepsilon^2 + |x'|^2)^{\frac{p-1-n-2\delta}{2(p-1)}} + C\varepsilon^{1/2} \quad \text{for } x \in \Omega_{\mu/\kappa_2} \setminus \Omega_{\varepsilon/\mu}. \tag{4.7}
$$

Since $\frac{p-1-n-2\delta}{2(p-1)} \in (0, 1/2)$, by combining (4.6) and (4.7), we obtain

$$|u(x)| \leqq C(\varepsilon + |x'|^2)^{\frac{p-1-n-2\delta}{2(p-1)}} \quad \text{for } x \in \Omega_{\mu/\kappa_2}.$$

This implies that for any $x \in \Omega_{\mu/2\kappa_2}$ and $\varepsilon \in (0, \mu^2/(\kappa_2 + \kappa_2^2))$,

$$\operatorname*{osc}_{\Omega_{x,\eta}} u \leqq C(\varepsilon + |x'|^2)^{\frac{p-1-n-2\delta}{2(p-1)}}, \tag{4.8}$$

where $\eta = \frac{1}{4}(\varepsilon + |x'|^2)^{1/2}$, and C is a constant depending only on n, p, δ, κ_1, κ_2, and the modulus of continuity for $D^2 h_1(x')$ and $D^2 h_2(x')$ at $x' = 0$. Then (1.11) follows from (4.8) and Theorem 1.1. $\square$

5. A two dimensional example

In this section, we provide an example showing that the estimates (1.8) and (1.11) are close to optimal in 2D. In the following and throughout this section, we set our domain $\Omega = B_5 \subset \mathbb{R}^2$, $\mathcal{D}_1$ and $\mathcal{D}_2$ to be the unit balls centered at $(0, 1+\varepsilon/2)$ and $(0, -1-\varepsilon/2)$, respectively. That is,

$$\begin{aligned} \Gamma_+ &= \left\{ x_2 = \frac{\varepsilon}{2} + 1 - \sqrt{1 - x_1^2} \right\}, \\ \Gamma_- &= \left\{ x_2 = -\frac{\varepsilon}{2} - 1 + \sqrt{1 - x_1^2} \right\}, \ x_1 \in (-1, 1). \end{aligned} \tag{5.1}$$

Lemma 5.1. *Let $n = 2$ and Γ_+, Γ_- be as (5.1). For any $\delta \in (0, 1/2)$, $\varepsilon \in (0, \delta/10)$, and $\gamma > \max\{\frac{p-3+\delta}{p-1}, 0\}$, there exists a constant $r_0 \in (0, 1/2)$ depending only on p and δ, such that the function*

$$w(x) := \left[\left(x_1^2 + (2-\delta) x_2^2 \right)^{\frac{\gamma}{2}} - (4\sqrt{\varepsilon/\delta})^\gamma \right]_+$$

satisfies

$$\begin{cases} -\operatorname{div}(|Dw|^{p-2} Dw) \leqq 0 & \text{in } \Omega_{r_0}, \\ \dfrac{\partial w}{\partial \nu} \leqq 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{r_0}. \end{cases} \tag{5.2}$$

Proof. We denote $R = R(x) = \left(x_1^2 + (2-\delta) x_2^2 \right)^{\frac{1}{2}}$ and $v(x) = R(x)^\gamma$. Then

$$Dv = \gamma R^{\gamma-2}(x_1, (2-\delta)x_2).$$

On Γ_+, the upward normal vector $\nu = (-x_1, 1 + \varepsilon/2 - x_2)$. Therefore,

$$\begin{aligned}\frac{\partial v}{\partial \nu} =& \gamma R^{\gamma-2}\Big[- x_1^2 + (2-\delta)x_2\Big(1 + \frac{\varepsilon}{2} - x_2\Big)\Big] \\ =& \gamma R^{\gamma-2}\Big[\Big(x_2 - 1 - \frac{\varepsilon}{2}\Big)^2 - 1 + (2-\delta)x_2\Big(1 + \frac{\varepsilon}{2} - x_2\Big)\Big] \\ =& \gamma R^{\gamma-2}\Big[- (1-\delta)x_2^2 - \delta x_2\Big(1 + \frac{\varepsilon}{2}\Big) + \varepsilon + \frac{\varepsilon^2}{4}\Big].\end{aligned}$$

One can see that $\partial v/\partial \nu < 0$ if $x_2 > \varepsilon/\delta$. Since $x_2 = \frac{\varepsilon}{2} + 1 - \sqrt{1 - x_1^2}$, $|x_1| > 2\sqrt{\varepsilon/\delta}$ implies $x_2 > \varepsilon/\delta$. Note that $w = 0$ on Γ_+ when $|x_1| \leqq 2\sqrt{\varepsilon/\delta}$, therefore $\partial w/\partial \nu \leqq 0$ on Γ_+. The fact that $\partial w/\partial \nu \leqq 0$ on Γ_- follows from a similar argument.

Next, following a similar computation as Lemma 4.1, we have in the region $\{x_1^2 + (2-\delta)x_2^2 > 16\varepsilon/\delta\} \cap \Omega_1$,

$$\begin{aligned}&\operatorname{div}(|Dw|^{p-2}Dw)|Dw|^{4-p} \\ &= \gamma^2 R^{2\gamma-4}\Big(x_1^2 + (2-\delta)^2 x_2^2\Big)\Big((3-\delta)\gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}(x_1^2 + (2-\delta)^2 x_2^2)\Big) \\ &\quad + (p-2)\gamma^2 R^{2\gamma-4} x_1^2\Big(\gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}x_1^2\Big) \\ &\quad + (p-2)\gamma^2 R^{2\gamma-4}(2-\delta)^2 x_2^2\Big((2-\delta)\gamma R^{\gamma-2} + (2-\delta)^2\gamma(\gamma-2)R^{\gamma-4}x_2^2\Big) \\ &\quad + 2(2+\delta)^2(p-2)\gamma^2 R^{2\gamma-4}\gamma(\gamma-2)R^{\gamma-4}x_1^2 x_2^2.\end{aligned}$$

Note that in $\{x_1^2 + (2-\delta)x_2^2 > 16\varepsilon/\delta\} \cap \Omega_1$, $|x_1| > 2\sqrt{\varepsilon/\delta}$. Therefore,

$$|x_2| \leqq \frac{\varepsilon}{2} + 1 - \sqrt{1 - x_1^2} \leqq \frac{\delta}{8}x_1^2 + 1 - \sqrt{1 - x_1^2} \leqq \Big(\frac{\delta}{8} + 1\Big)x_1^2,$$

and thus

$$R^2 - (2-\delta)\Big(\frac{\delta}{8} + 1\Big)^2 R^4 \leqq x_1^2 \leqq R^2.$$

Hence for small $R > 0$, the leading term of $\operatorname{div}(|Dw|^{p-2}Dw)|Dw|^{4-p}$ is given by

$$\gamma^2 R^{2\gamma-2}\Big((3-\delta)\gamma + \gamma(\gamma-2)\Big)R^{\gamma-2} + (p-2)\gamma^2 R^{2\gamma-2}\gamma(\gamma-1)R^{\gamma-2}.$$

If we set

$$(3-\delta) + (\gamma-2) + (p-2)(\gamma-1) > 0 \quad \text{and} \quad \gamma > 0,$$

which implies

$$\gamma > \max\Big\{\frac{p-3+\delta}{p-1}, 0\Big\},$$

then there exists a sufficient small $r_0 \in (0, 1/2)$, depending only on p and δ, such that $\operatorname{div}(|Dw|^{p-2}Dw) \geqq 0$ in Ω_{r_0}. $\square$

Proof of Theorem 1.4. We only need to prove the theorem for any $\delta \in (0, 1/2)$ and $\varepsilon \in (0, r_0^2\delta/64)$, where r_0 is the constant stated in Lemma 5.1. By symmetry and the maximum principle, we have $u(0, x_2) = 0$ for $|x_2| < \varepsilon/2$ and $u(x) > 0$ when $x_1 > 0$.

Next, we use interior Harnack inequality to show that there exists a positive constant C_0 depending only on p and δ such that $u \geqq C_0$ on $\widetilde{\Omega} \cap \{x_1 = r_0\}$. Since $u(x) = x_1$ on ∂B_5, by (1.6), there exists $r_1 \in (4, 5)$ such that $u(r_1, 0) \geqq 4$ and $5 - r_1$ is bounded from below by a positive constant depending only on p. Let $\mu = \min\{r_0^2/10, (5 - r_1)/10\}$ and $N = \lfloor \frac{r_1 - r_0}{\mu} \rfloor$. We then define a chain of balls $\mathcal{B}_k = B_\mu(p_k)$, where $k = 0, 1, \ldots, N$ and $p_k = (r_0 + k\mu, 0)$. Since $B_{2\mu}(p_k) \subset \widetilde{\Omega} \cap \{x_1 > 0\}$, by the Harnack inequality (see e.g. [31]), there exists a constant $C_0' > 0$ depending only on p, such that

$$\max_{\mathcal{B}_k} u \leqq C_0' \min_{\mathcal{B}_k} u$$

holds for any $k = 0, 1, \ldots, N$. Since $(r_1, 0) \in \mathcal{B}_N$, by iteration, we have

$$u(r_0, 0) \geqq (C_0')^{-N-1} u(r_1, 0) \geqq 4(C_0')^{-N-1}. \tag{5.3}$$

Now we set $x_0 = (r_0, 0)$ and perform the flattening and extension of u to u_2 in $\mathcal{C}_{1/2} = \{(\mathcal{Z}_1, \mathcal{Z}_2) \in \mathbb{R}^2 : |\mathcal{Z}_1| < 1/2\}$ as in Section 2.4. Since u_2 is a nonnegative solution to (2.23) when $x_1 > 0$, by the Harnack inequality and a similar iteration argument as above, we obtain that

$$u_2(0, \mathcal{Z}_2) \geqq C_0'' u_2(0, 0) \tag{5.4}$$

holds for any $\mathcal{Z}_2 \in \big(-1 + \sqrt{1 - r_0^2} - \varepsilon/2, 1 - \sqrt{1 - r_0^2} + \varepsilon/2\big)$, where $C_0'' > 0$ is a constant depending only on p and δ. In the original coordinate, (5.4) directly implies that

$$u(x) \geqq C_0'' u(r_0, 0) \tag{5.5}$$

for any $x \in \widetilde{\Omega} \cap \{x_1 = r_0\}$. Combining (5.3) and (5.5), we get

$$u \geqq C_0 \quad \text{on} \quad \widetilde{\Omega} \cap \{x_1 = r_0\},$$

where $C_0 = 4C_0''(C_0')^{-N-1}$ is a positive constant depending only on p and δ.

Let w be the function defined in Lemma 5.1 with

$$\gamma = \begin{cases} \delta & \text{when } 1 < p \leqq 3, \\ \dfrac{p - 3 + 2\delta}{p - 1} & \text{when } p > 3. \end{cases}$$

By the comparison principle, there exists a positive constant C depending only on p and δ, such that $u \geqq \frac{1}{C} w$ in $\Omega_{r_0} \cap \{x_1 > 0\}$. In particular, since $8\sqrt{\varepsilon/\delta} < r_0$, we have

$$u(8\sqrt{\varepsilon/\delta}, 0) \geqq \frac{1}{C} \varepsilon^{\frac{\gamma}{2}}.$$

The desired lower bounds on Du follow from the mean value theorem since $u(0) = 0$. $\square$

Remark 5.2. When $\varepsilon = 0$ and $p > 5$, it can be shown that $w(x) := (x_1^2 + 2x_2^2)^{\gamma/2}$ is also a subsolution satisfying (5.2) for $\gamma = (p-3)/(p-1)$ and some absolute constant $r_0 \in (0, 1)$. Therefore, a similar argument as in the proof of Theorem 1.4 gives $u(x_1, 0) \geqq c_0\, x_1^{(p-3)/(p-1)}$ for some constant $c_0 = c_0(p) > 0$ and any $x_1 \in (0, r_0)$. Thus, for any $r \in (0, 1)$, there exists $x_1 \in (0, r)$ such that

$$D_1 u(x_1, 0) \geqq c\, x_1^{-2/(p-1)},$$

where $c > 0$ is a constant depending only on p.

6. Bernstein type argument

In this section, we adapt the Bernstein type argument used in [34] (see also [9] and [10]) to prove improved gradient estimates for (1.7) in high dimensions. As mentioned before, our proof also relies on the fact that for any $q \geqq p$, $|Du|^q$ is a subsolution to the normalized p-Laplace equation, which was originally observed by Uhlenbeck [32]. In addition to (1.3) and (1.4), we need to further assume that h_1 and h_2 are $C^{2,\mathrm{Dini}}$ (so that u is C^2 at the points where $Du \neq 0$, see [14, Theorem 2.4]), strictly convex and strictly concave respectively, satisfying (1.10).

Let ν denote the normal vector on $\Gamma_\pm$, pointing upwards and downwards respectively. We have the following lemma:

Lemma 6.1. *Let $\Gamma_+, \Gamma_-, h_1, h_2$ be as above, $s \geqq 2$. If u is twice differentiable and $D_\nu u = 0$ on $\Gamma_+ \cup \Gamma_-$, then at any point $x_0 \in \Gamma_+ \cup \Gamma_-$,*

$$s\kappa_1 |Du(x_0)|^s \leqq D_\nu |Du(x_0)|^s \leqq s\kappa_2 |Du(x_0)|^s. \tag{6.1}$$

Proof. We only prove (6.1) at $x_0 \in \Gamma_+$. By a rotation, we may assume that $x_0' = 0$ and $D_{x'} h_1(x_0') = 0$. The normal vector ν on Γ_+ is given by

$$\nu = \frac{1}{\sqrt{1 + |D_{x'} h_1|^2}} (-D_1 h_1, \ldots, -D_{n-1} h_1, 1). \tag{6.2}$$

Then $D_\nu u = 0$ is equivalent to

$$\sum_{j=1}^{n-1} D_j u D_j h_1 - D_n u = 0.$$

Applying D_i to the equation above for $i = 1, \ldots, n-1$, we have at x_0,

$$\sum_{j=1}^{n-1} D_j u D_{ij} h_1 - D_{in} u = 0. \tag{6.3}$$

By direct computation, at x_0,

$$
\begin{aligned}
D_\nu |Du|^s &= s|Du|^{s-2} \sum_{i=1}^{n} D_i u D_{in} u \\
&= s|Du|^{s-2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} D_i u D_j u D_{ij} h_1,
\end{aligned}
$$

where in the second line, we used (6.3) and $D_n u(x_0) = 0$. Then (6.1) follows from (1.10). □

Proof of Theorem 1.5. For convenience, we let $\gamma = 2\beta \in [0, 1)$.

Case 1: For $p \geqq 2$, we consider the quantity

$$
F = Q^{\frac{p-p\gamma}{2}} |Du|^p,
$$

where

$$
Q = \frac{\varepsilon}{\kappa_1} + |x'|^2 - \frac{5\kappa_2}{2(1-\gamma)\kappa_1} x_n^2. \tag{6.4}
$$

We will show by contradiction that F does not achieve its maximum on $(\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{r_0}$ or in Ω_{r_0} for some suitable r_0 which is independent of ε. Therefore, F can only achieve maximum on

$$
\{|x'| = r_0\} \cap \Omega_1,
$$

and (1.14) follows.

If $F^{q/2}$ achieves it maximum at a point x_0, we may assume that $Du(x_0) \neq 0$. First we show that $x_0 \notin \Gamma_+ \cap \overline{\Omega}_{r_0}$. A similar argument applies to $\Gamma_- \cap \overline{\Omega}_{r_0}$. On Γ_+, the normal vector ν is given by (6.2). At x_0, by (6.1) with $s = p$,

$$
\begin{aligned}
D_\nu F =& \frac{p - p\gamma}{2} Q^{\frac{p-p\gamma}{2}-1} D_\nu Q |Du|^p + Q^{\frac{p-p\gamma}{2}} D_\nu |Du|^p \\
\leqq& - \frac{(p-p\gamma) Q^{\frac{p-p\gamma}{2}-1}}{\sqrt{1 + |D_{x'} h_1|^2}} \left[\sum_{j=1}^{n-1} D_j h_1 x_j + \frac{5\kappa_2}{2(1-\gamma)\kappa_1} (\varepsilon/2 + h_1) \right] |Du|^p \\
&+ p\kappa_2 Q^{\frac{p-p\gamma}{2}} |Du|^p.
\end{aligned}
$$

We choose r_0 small enough such that

$$
\frac{-1}{\sqrt{1 + |D_{x'} h_1|^2}} \leqq \frac{-1}{\sqrt{1 + |\kappa_2 x'|^2}} \leqq -\frac{4}{5} \quad \text{for } |x'| < r_0.
$$

By (1.3) and (1.10), we have

$$
\sum_{j=1}^{n-1} D_j h_1 x_j \geqq \kappa_1 |x'|^2 \quad \text{and} \quad h_1 \geqq \frac{1}{2} \kappa_1 |x'|^2.
$$

Therefore,

$$
\begin{aligned}
D_\nu F \leqq & Q^{\frac{p-p\gamma}{2}-1}|Du|^p\bigg(-\frac{4}{5}(p-p\gamma)\Big[\kappa_1|x'|^2+\frac{5\kappa_2}{4(1-\gamma)\kappa_1}(\varepsilon+\kappa_1|x'|^2)\Big] \\
& + p\kappa_2\Big(\frac{\varepsilon}{\kappa_1}+|x'|^2-\frac{5\kappa_2}{2(1-\gamma)\kappa_1}x_n^2\Big)\bigg) \\
= & Q^{\frac{p-p\gamma}{2}-1}|Du|^p\bigg(-\frac{4}{5}(p-p\gamma)\kappa_1|x'|^2-\frac{5p\kappa_2^2}{2(1-\gamma)\kappa_1}x_n^2\bigg)<0.
\end{aligned}
$$

Hence F does not achieve its maximum on $\Gamma_+\cap\overline{\Omega}_{r_0}$. Next, we will show that $x_0\notin\Omega_{r_0}$ by assuming otherwise and showing that $a^{ij}D_{ij}F(x_0)>0$, where

$$
a^{ij}(x)=\delta_{ij}+(p-2)|Du|^{-2}D_iuD_ju \tag{6.5}
$$

is symmetric. Note that

$$
|\xi|^2\leqq a^{ij}\xi_i\xi_j\leqq(p-1)|\xi|^2,\quad\forall\xi\in\mathbb{R}^n, \tag{6.6}
$$

and if u is a solution of (1.7), then $a^{ij}D_{ij}u=0$. Since $Du(x_0)\neq 0$. By the continuity of Du, $Du\neq 0$ in a neighborhood of x_0, and hence a^{ij} is well defined in the neighborhood. By direction computations,

$$
\begin{aligned}
& a^{ij}D_{ij}F \\
& \quad = Q^{\frac{p-p\gamma}{2}}(a^{ij}D_{ij}|Du|^p)+|Du|^p(a^{ij}D_{ij}Q^{\frac{p-p\gamma}{2}})+2a^{ij}(D_i|Du|^p)(D_jQ^{\frac{p-p\gamma}{2}}).
\end{aligned} \tag{6.7}
$$

Next, we estimate the three terms on the right-hand side above. First,

$$
\begin{aligned}
& a^{ij}D_{ij}Q^{\frac{p-p\gamma}{2}} \\
& \quad = a^{ij}\Big[\frac{p-p\gamma}{2}Q^{\frac{p-p\gamma}{2}-1}D_{ij}Q+\frac{p-p\gamma}{2}\Big(\frac{p-p\gamma}{2}-1\Big)Q^{\frac{p-p\gamma}{2}-2}D_iQD_jQ\Big].
\end{aligned} \tag{6.8}
$$

Since $a^{ij}D_{ij}u=0$, applying D_k gives

$$
a^{ij}D_{ijk}u+D_ka^{ij}D_{ij}u=0.
$$

Since

$$
D_ka^{ij}=(p-2)\left[\frac{D_{ik}uD_ju+D_iuD_{jk}u}{|Du|^2}-2\frac{D_iuD_juD_{kl}uD_lu}{|Du|^4}\right],
$$

we have

$$
\begin{aligned}
a^{ij}D_{ijk}uD_ku &= -D_ka^{ij}D_{ij}uD_ku \\
&= -2(p-2)|D|Du||^2+2(p-2)|Du|^{-4}|\Delta_\infty u|^2,
\end{aligned} \tag{6.9}
$$

where $\Delta_\infty u := D_i u D_j u D_{ij} u$. By (6.5) and (6.9), we have

$$\begin{aligned}
a^{ij} D_{ij} |Du|^p =& p a^{ij} \Big[(p-2)|Du|^{p-4} D_{ik}u D_k u D_{jl} u D_l u \\
&+ |Du|^{p-2} D_{ik}u D_{jk}u + |Du|^{p-2} D_k u D_{ikj} u \Big] \\
=& p|Du|^{p-4} \Big[(p-2)|Du|^2 |D|Du||^2 + (p-2)^2 |Du|^{-2} |\Delta_\infty u|^2 \\
&+ |Du|^2 |D^2u|^2 + (p-2)|Du|^2|D|Du||^2 - 2(p-2)|Du|^2|D|Du||^2 \\
&+ 2(p-2)|Du|^{-2}|\Delta_\infty u|^2 \Big] \\
=& p|Du|^{p-4} \Big[p(p-2)|Du|^{-2}|\Delta_\infty u|^2 + |Du|^2|D^2u|^2 \Big]. \qquad (6.10)
\end{aligned}$$

Note that at the point x_0, for any $i = 1, 2, \ldots, n$,

$$0 = D_i F = (D_i Q^{\frac{p-p\gamma}{2}})|Du|^p + Q^{\frac{p-p\gamma}{2}} (D_i |Du|^p). \qquad (6.11)$$

We split the last term on the right-hand side of (6.7) into

$$\frac{2p-1}{p} a^{ij} (D_i |Du|^p)(D_j Q^{\frac{p-p\gamma}{2}}) + \frac{1}{p} a^{ij} (D_i |Du|^p)(D_j Q^{\frac{p-p\gamma}{2}}).$$

For the first term on the right-hand side above, we use (6.11) to substitute $D_i|Du|^p$, and for the second term, we substitute $D_j Q^{\frac{p-p\gamma}{2}}$. Then,

$$\begin{aligned}
&2a^{ij} (D_i |Du|^p)(D_j Q^{\frac{p-p\gamma}{2}}) \\
&= -\frac{2p-1}{p} Q^{-\frac{p-p\gamma}{2}} |Du|^p a^{ij} (D_i Q^{\frac{p-p\gamma}{2}})(D_j Q^{\frac{p-p\gamma}{2}}) \\
&\quad - \frac{1}{p} Q^{\frac{p-p\gamma}{2}} |Du|^{-p} a^{ij} (D_i |Du|^p)(D_j |Du|^p) \\
&= -\frac{2p-1}{p} \frac{(p-p\gamma)^2}{4} Q^{\frac{p-p\gamma}{2}-2} |Du|^p a^{ij} D_i Q D_j Q \\
&\quad - p Q^{\frac{p-p\gamma}{2}} |Du|^{p-4} a^{ij} D_{ik}u D_k u D_{jl}u D_l u \\
&= -\frac{2p-1}{p} \frac{(p-p\gamma)^2}{4} Q^{\frac{p-p\gamma}{2}-2} |Du|^p a^{ij} D_i Q D_j Q \\
&\quad - p Q^{\frac{p-p\gamma}{2}} |Du|^{p-4} (|Du|^2 |D|Du||^2 + (p-2)|Du|^{-2}|\Delta_\infty u|^2), \qquad (6.12)
\end{aligned}$$

where we used (6.5) in the last equality. Therefore, by (6.7), (6.8), (6.10), and (6.12),

$$\begin{aligned}
a^{ij} D_{ij} F =& p Q^{\frac{p-p\gamma}{2}} |Du|^{p-4} \Big[(p-1)(p-2)|Du|^{-2}|\Delta_\infty u|^2 + |Du|^2|D^2u|^2 \\
&- |Du|^2 |D|Du||^2 \Big] + \frac{p-p\gamma}{2} Q^{\frac{p-p\gamma}{2}-1} |Du|^p a^{ij} D_{ij} Q \\
&- \Big[\frac{2p-1}{p} \frac{(p-p\gamma)^2}{4} - \frac{p-p\gamma}{2} \Big(\frac{p-p\gamma}{2} - 1 \Big) \Big] \\
&Q^{\frac{p-p\gamma}{2}-2} |Du|^p a^{ij} D_i Q D_j Q.
\end{aligned}$$

Note that

$$|Du|^2|D|Du||^2 \leqq |Du|^2|D^2u|^2.$$

It remains to show

$$Qa^{ij}D_{ij}Q > \Big[\frac{2p-1}{p}\frac{p-p\gamma}{2} - \Big(\frac{p-p\gamma}{2} - 1\Big)\Big]a^{ij}D_iQD_jQ. \tag{6.13}$$

Recall that Q is given in (6.4). Then

$$DQ = \Big(2x_1, \ldots, 2x_{n-1}, -\frac{5\kappa_2}{(1-\gamma)\kappa_1}x_n\Big), \tag{6.14}$$

and

$$\begin{aligned} a^{ij}D_{ij}Q =& 2n - 2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1} \\ &+ (p-2)|Du|^{-2}\Big(2|D_{x'}u|^2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1}|D_nu|^2\Big) \\ \geqq& 2n - 2 - (p-1)\frac{5\kappa_2}{(1-\gamma)\kappa_1}. \end{aligned}$$

By (6.6) and shrinking r_0 if necessary, we have

$$a^{ij}D_iQD_jQ \leqq 4(p-1)\Big(|x'|^2 + \frac{25\kappa_2^2}{4(1-\gamma)^2\kappa_1^2}x_n^2\Big) < 5(p-1)Q.$$

In order to show (6.13), we only require

$$2n - 2 - (p-1)\frac{5\kappa_2}{(1-\gamma)\kappa_1} \geqq 5(p-1)\Big[\frac{2p-1}{p}\frac{p-p\gamma}{2} - \Big(\frac{p-p\gamma}{2} - 1\Big)\Big],$$

which is equivalent to (1.12) since $\gamma = 2\beta$. This concludes the proof for the case when $p \geqq 2$.

Case 2: For $p \in (1, 2)$, we consider the quantity

$$G = Q^{1-\gamma}|Du|^2,$$

where Q is given in (6.4). From the computation of $D_\nu F$ with $p = 2$, one can see that G does not attain its maximum on $(\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{r_0}$. Next, we assume that G achieves its maximum at $x_0 \in \Omega_{r_0}$. By the computations as in (6.7) and (6.8) with $p = 2$, we have

$$a^{ij}D_{ij}G = Q^{1-\gamma}(a^{ij}D_{ij}|Du|^2) + |Du|^2(a^{ij}D_{ij}Q^{1-\gamma}) + 2a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}), \tag{6.15}$$

where a^{ij} is given in (6.5) with

$$(p-1)|\xi|^2 \leqq a^{ij}\xi_i\xi_j \leqq |\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{6.16}$$

Next, we estimate the three terms on the right-hand side of (6.15). First,

$$a^{ij}D_{ij}Q^{1-\gamma} = a^{ij}\Big[(1-\gamma)Q^{-\gamma}D_{ij}Q - \gamma(1-\gamma)Q^{-1-\gamma}D_iQD_jQ\Big]. \tag{6.17}$$

By (6.5) and (6.9), we have

$$\begin{aligned} a^{ij}D_{ij}|Du|^2 =& a^{ij}\Big[2D_{ijk}uD_ku + 2D_{jk}uD_{ik}u\Big] \\ =& 4(2-p)|D|Du||^2 - 4(2-p)|Du|^{-4}|\Delta_\infty u|^2 \\ &+ 2|D^2u|^2 - 2(2-p)|D|Du||^2 \\ =& 2(2-p)|D|Du||^2 - 4(2-p)|Du|^{-4}|\Delta_\infty u|^2 + 2|D^2u|^2. \end{aligned} \tag{6.18}$$

Note that at the point x_0, for any $i = 1, 2, \ldots, n$,

$$0 = D_iG = (D_iQ^{1-\gamma})|Du|^2 + Q^{1-\gamma}(D_i|Du|^2). \tag{6.19}$$

As before, we split the last term on the right-hand side of (6.15) as

$$\frac{1}{2}a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}) + \frac{3}{2}a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}).$$

For the first term on the right-hand side, we use (6.19) to substitute $D_jQ^{1-\gamma}$, and for the second term, we substitute $D_i|Du|^2$. Then by (6.5),

$$\begin{aligned} 2a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}) =& -\frac{1}{2}a^{ij}(D_i|Du|^2)(D_j|Du|^2)Q^{1-\gamma}|Du|^{-2} \\ &- \frac{3}{2}a^{ij}(D_iQ^{1-\gamma})(D_jQ^{1-\gamma})|Du|^2Q^{-1+\gamma} \\ =& -2Q^{1-\gamma}|Du|^{-2}a^{ij}D_{ik}uD_kuD_{jl}uD_lu \\ &- \frac{3}{2}(1-\gamma)^2|Du|^2Q^{-1-\gamma}a^{ij}D_iQD_jQ \\ =& -2Q^{1-\gamma}|D|Du||^2 + 2(2-p)Q^{1-\gamma}|Du|^{-4}|\Delta_\infty u|^2 \\ &- \frac{3}{2}(1-\gamma)^2|Du|^2Q^{-1-\gamma}a^{ij}D_iQD_jQ. \end{aligned} \tag{6.20}$$

Therefore, by (6.15), (6.17), (6.18), and (6.20),

$$\begin{aligned} a^{ij}D_{ij}G =& Q^{1-\gamma}\Big[(2(2-p)-2)|D|Du||^2 - 2(2-p)|Du|^{-4}|\Delta_\infty u|^2 + 2|D^2u|^2\Big] \\ &+ (1-\gamma)Q^{-\gamma}|Du|^2a^{ij}D_{ij}Q \\ &- \Big[\gamma(1-\gamma) + \frac{3}{2}(1-\gamma)^2\Big]|Du|^2Q^{-1-\gamma}a^{ij}D_iQD_jQ. \end{aligned}$$

Since

$$|Du|^{-4}|\Delta_\infty u|^2 \leqq |D|Du||^2 \leqq |D^2u|^2,$$

it remains to show

$$Qa^{ij}D_{ij}Q > \Big[\gamma + \frac{3}{2}(1-\gamma)\Big]a^{ij}D_iQD_jQ. \tag{6.21}$$

By (6.14), we have

$$\begin{aligned} a^{ij}D_{ij}Q =& 2n-2-\frac{5\kappa_2}{(1-\gamma)\kappa_1} \\ & - (2-p)|Du|^{-2}\Big(2|D_{x'}u|^2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1}|D_nu|^2\Big) \\ \geqq & 2n-2-\frac{5\kappa_2}{(1-\gamma)\kappa_1} - 2(2-p). \end{aligned}$$

By (6.16) and shrinking r_0 if necessary, we have

$$a^{ij}D_iQD_jQ \leqq 4\Big(|x'|^2 + \frac{25\kappa_2^2}{4(1-\gamma)^2\kappa_1^2}x_n^2\Big) < 5Q.$$

In order to show (6.21), we only require

$$2n-2-\frac{5\kappa_2}{(1-\gamma)\kappa_1} - 2(2-p) \geqq 5\Big[\gamma + \frac{3}{2}(1-\gamma)\Big],$$

which is equivalent to (1.13) since $\gamma = 2\beta$. This concludes the proof for the case when $p \in (1, 2)$. □

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Appendix A.

In the appendix, we provide an alternative proof of the gradient estimates of order $\varepsilon^{-1/2}$ using a Bernstein type argument. This proof also requires the assumptions that h_1 and h_2 are $C^{2,\mathrm{Dini}}$ functions and satisfy (1.10) for some $\kappa_1, \kappa_2 > 0$, in addition to (1.3) imposed in Theorem 1.1.

Theorem A.1. *Let h_1, h_2 be $C^{2,Dini}$ functions satisfying (1.10), $p > 1$, $n \geqq 2$, $\varepsilon \in (0, 1)$, and $u \in W^{1,p}(\Omega_1)$ be a solution of (1.7). Then there exists a positive constant C depending only on n, p, κ_1, and κ_2, such that*

$$|Du(x)| \leqq C\|u\|_{L^\infty(\Omega_1)}(\varepsilon + |x'|^2)^{-1/2} \quad \textit{for } x \in \Omega_{1/2}. \tag{A.1}$$

Proof. Without loss of generality, we may assume $\kappa_1 \in (0,1]$ and $\kappa_2 > 1$. The case $p=2$ has been shown in [6,34]. It remains to show the cases when $p>2$ and $p \in (1,2)$.
Case 1: For $p>2$, we consider the quantity $F^{q/2}$, where

$$F = Q|Du|^2 + Au^2, \quad Q = \varepsilon + |x'|^2 - 4\kappa_1^{-1}\kappa_2^2 x_n^2, \tag{A.2}$$

$q \geqq 2$ and A are some positive ε-independent constants to be determined later. Let

$$S_A := \{Q|Du|^2 > 100Au^2\}.$$

We will show that $F^{q/2}$ does not achieve its maximum on $(\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{r_0} \cap S_A$ or in $\Omega_{r_0} \cap S_A$ for some suitable q, A, and r_0. Therefore, $F^{q/2}$ can only achieve its maximum in

$$\Omega_{r_0} \cap \{Q|Du|^2 \leqq 100Au^2\},$$

or on

$$\{|x'| = r_0\} \cap \Omega_1,$$

so (A.1) follows from either case.
First we show that $F^{q/2}$ does not achieve its maximum on $\Gamma_+ \cap \overline{\Omega}_{r_0} \cap S_A$. A similar argument applies to $\Gamma_- \cap \overline{\Omega}_{r_0} \cap S_A$. On Γ_+, the normal vector ν is given by (6.2). Then

$$\begin{aligned} D_\nu F^{q/2} &= \frac{q}{2}F^{q/2-1}(D_\nu Q|Du|^2 + QD_\nu|Du|^2) \\ &= \frac{q}{2}F^{q/2-1}\left(\frac{-2}{\sqrt{1+|D_{x'}h_1|^2}}\left[\sum_{j=1}^{n-1} D_j h_1 x_j + 4\frac{\kappa_2^2}{\kappa_1}(\varepsilon/2 + h_1)\right]|Du|^2 + QD_\nu|Du|^2)\right). \end{aligned}$$

We choose r_0 small enough such that

$$\frac{-2}{\sqrt{1+|D_{x'}h_1|^2}} \leqq \frac{-2}{\sqrt{1+|\kappa_2 x'|^2}} \leqq -1 \quad \text{for } |x'| < r_0.$$

By (1.3) and (1.10), we have

$$\sum_{j=1}^{n-1} D_j h_1 x_j \geqq \kappa_1|x'|^2 \quad \text{and} \quad h_1 \geqq \frac{1}{2}\kappa_1|x'|^2.$$

Therefore, by (6.1) with $s=2$, we have

$$\begin{aligned} &D_\nu F^{q/2} \\ &\leqq \frac{q}{2}F^{q/2-1}|Du|^2\left[-\left(\kappa_1|x'|^2 + 2\frac{\kappa_2^2}{\kappa_1}\varepsilon + 2\kappa_2^2|x'|^2\right) + 2\kappa_2\left(\varepsilon + |x'|^2 - 4\frac{\kappa_2^2}{\kappa_1}x_n^2\right)\right] \\ &= -\frac{q}{2}F^{q/2-1}|Du|^2\left[2\kappa_2\left(\frac{\kappa_2}{\kappa_1} - 1\right)\varepsilon + 2\kappa_2(\kappa_2 - 1)|x'|^2 + \kappa_1|x'|^2 + 8\frac{\kappa_2^2}{\kappa_1}x_n^2\right] \\ &< 0. \end{aligned}$$

Hence $F^{q/2}$ does not achieve its maximum on $\Gamma_+ \cap \overline{\Omega}_{r_0} \cap S_A$. Next, we will show $F^{q/2}$ does not achieve its maximum in $\Omega_{r_0} \cap S_A$ by proving that $a^{ij} D_{ij} F^{q/2} > 0$, where a^{ij} is given in (6.5). By direct computations, we have

$$
\begin{aligned}
a^{ij} D_{ij} F^{q/2} &= \frac{q}{2} F^{q/2-1} a^{ij} D_{ij} F + \frac{q}{2}\Big(\frac{q}{2} - 1\Big) F^{q/2-2} a^{ij} D_i F D_j F,\\
D_i F &= D_i Q |Du|^2 + 2 Q D_{ik} u D_k u + 2 A u D_i u,
\end{aligned}
$$

and

$$
\begin{aligned}
D_{ij} F =& D_{ij} Q |Du|^2 + 2 D_i Q D_k u D_{jk} u + 2 D_j Q D_k u D_{ik} u\\
&+ 2Q(D_{ik} u D_{jk} u + D_k u D_{ijk} u) + 2A(u D_{ij} u + D_i u D_j u).
\end{aligned}
$$

Then by (6.5) and because $a^{ij} D_{ij} u = 0$,

$$
\begin{aligned}
&a^{ij} D_{ij} F\\
&\quad = a^{ij} D_{ij} Q |Du|^2 + 4(p-2)|Du|^{-2} D_i u D_i Q \Delta_\infty u + 4 D_i Q D_k u D_{ik} u + 2Q|D^2 u|^2\\
&\qquad + 2(p-2) Q |D|Du||^2 + 2Q a^{ij} D_{ijk} u D_k u + 2A a^{ij} D_i u D_j u,
\end{aligned}
$$

where $\Delta_\infty u := D_i u D_j u D_{ij} u$. By (6.9),

$$
\begin{aligned}
&a^{ij} D_{ij} F\\
&\quad = a^{ij} D_{ij} Q |Du|^2 + 4(p-2)|Du|^{-2} D_i u D_i Q \Delta_\infty u + 4 D_i Q D_k u D_{ik} u + 2Q|D^2 u|^2\\
&\qquad - 2(p-2) Q |D|Du||^2 + 4(p-2) Q |Du|^{-4} |\Delta_\infty u|^2 + 2A a^{ij} D_i u D_j u.
\end{aligned}
$$

By another direction computation, we have

$$
\begin{aligned}
&a^{ij} D_i F D_j F\\
&\quad = a^{ij} D_i Q D_j Q |Du|^4 + 4Q^2 a^{ij} D_{ik} u D_k u D_{jl} u D_l u + 4A^2 u^2 a^{ij} D_i u D_j u\\
&\qquad + 4Q|Du|^2 a^{ij} D_i Q D_{jl} u D_l u + 4Au|Du|^2 a^{ij} D_i Q D_j u + 8AQu a^{ij} D_{ik} u D_k u D_j u\\
&\quad = a^{ij} D_i Q D_j Q |Du|^4 + 4(p-2) Q^2 |Du|^{-2} |\Delta_\infty u|^2 + 4Q^2 |Du|^2 |D|Du||^2\\
&\qquad + 4(p-1) A^2 u^2 |Du|^2 + 4(p-2) Q D_i u D_i Q \Delta_\infty u + 4 Q D_i Q D_{ik} u D_k u |Du|^2\\
&\qquad + 4A(p-1)|Du|^2 u D_i u D_i Q + 8(p-1) A Q u \Delta_\infty u.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&a^{ij} D_{ij} F^{q/2}\\
&\quad = \frac{q}{2} F^{q/2-2}\big[a^{ij} D_{ij} Q |Du|^2 F + 2Q|D|Du||^2\big((q-2)Q|Du|^2 - (p-2)F\big)\\
&\qquad + 4(p-2) D_i u D_i Q \Delta_\infty u |Du|^{-2}(F + (q-2)Q|Du|^2/2) + 2QF|D^2 u|^2\\
&\qquad + 4 D_i Q D_k u D_{ik} u (F + (q-2)Q|Du|^2/2)\\
&\qquad + 4(p-2) Q |Du|^{-4}(F + (q-2)Q|Du|^2/2)|\Delta_\infty u|^2\\
&\qquad + 2AF a^{ij} D_i u D_j u + (q-2) a^{ij} |Du|^4 D_i Q D_j Q/2 + 2(p-1)(q-2) A^2 u^2 |Du|^2\\
&\qquad + 2A(p-1)(q-2)|Du|^2 u D_i u D_i Q + 4(p-1)(q-2) A Q u \Delta_\infty u\big]. \qquad \text{(A.3)}
\end{aligned}
$$

Note that in S_A,

$$Q|Du|^2 \leqq F \leqq \frac{101}{100} Q|Du|^2. \tag{A.4}$$

By (6.6),

$$(q-2)a^{ij} D_i Q D_j Q/2 \geqq (q-2)|DQ|^2/2 \geqq 0 \quad \text{for } q \geqq 2,$$

and

$$\begin{aligned}
&a^{ij} D_{ij} Q|Du|^2 F + 2AFa^{ij} D_i u D_j u \\
&\quad = [2(n-1) - 8\kappa_1^{-1}\kappa_2^2 + (p-2)|Du|^{-2}(2|D_{x'}u|^2 - 8\kappa_1^{-1}\kappa_2^2|D_n u|^2)]|Du|^2 F \\
&\qquad + 2AFa^{ij} D_i u D_j u \\
&\quad \geqq \Big(2(n-1) - 8\kappa_1^{-1}\kappa_2^2(p-1) + 2A\Big)|Du|^2 F \\
&\quad \geqq \Big(2(n-1) - 8\kappa_1^{-1}\kappa_2^2(p-1) + 2A\Big)|Du|^4 Q.
\end{aligned} \tag{A.5}$$

We choose $q = \frac{101}{100}(p-2) + 2 > p$, so that

$$\begin{aligned}
&2Q|D|Du||^2\big((q-2)Q|Du|^2 - (p-2)F\big) \\
&\quad \geqq 2Q|D|Du||^2\left((q-2) - \frac{101}{100}(p-2)\right)Q|Du|^2 = 0.
\end{aligned}$$

It remains to control

$$\begin{aligned}
&4(p-2)D_i u D_i Q \Delta_\infty u |Du|^{-2}(F + (q-2)Q|Du|^2/2) \\
&\quad + 4D_i Q D_k u D_{ik} u (F + (q-2)Q|Du|^2/2) \\
&\quad + 2A(p-1)(q-2)|Du|^2 u D_i u D_i Q + 4(p-1)(q-2)AQu\Delta_\infty u \\
&=: I + II + III + IV.
\end{aligned}$$

Since $p > 2$, we have

$$\frac{2(p-2)q}{(p-1)(q-2)} = \frac{2\left[\frac{101}{100}(p-2) + 2\right]}{\frac{101}{100}(p-1)} > 2.$$

Fix a constant $B \in (2, \frac{2(p-2)q}{(p-1)(q-2)})$. We shrink r_0 if necessary so that $|DQ|^2 \leqq 8Q$. By Young's inequality and (A.4),

$$\begin{aligned}
|I| &\leqq \left(\frac{p-2}{2} - \frac{B(p-1)(q-2)}{4q}\right)|Du|^{-4}|DQ|^2|\Delta_\infty u|^2(F + (q-2)Q|Du|^2/2) \\
&\quad + C(p)|Du|^2(F + (q-2)Q|Du|^2/2) \\
&\leqq \left(4(p-2) - \frac{2B(p-1)(q-2)}{q}\right)Q|Du|^{-4}|\Delta_\infty u|^2(F + (q-2)Q|Du|^2/2) \\
&\quad + C(p)|Du|^4 Q,
\end{aligned} \tag{A.6}$$

where $C(p)$ is some positive constant depending on p. By Young's inequality and (A.4),

$$
\begin{aligned}
|II| &\leqq 4\Big(\frac{101}{100}+\frac{q-2}{2}\Big)|DQ||D^2u||Du|^3Q \\
&\leqq 4\Big(\frac{101}{100}+\frac{q-2}{2}\Big)^2|DQ|^2|Du|^4+Q^2|D^2u|^2|Du|^2 \\
&\leqq 32\Big(\frac{101}{100}+\frac{q-2}{2}\Big)^2|Du|^4Q+QF|D^2u|^2, \qquad \text{(A.7)}\\
|III| &\leqq 2(p-1)(q-2)A|Du|^3|u||DQ| \\
&\leqq \left(2-\frac{4}{B}\right)(p-1)(q-2)A^2u^2|Du|^2+\frac{4B}{B-2}(p-1)(q-2)|Du|^4Q, \qquad \text{(A.8)}
\end{aligned}
$$

and

$$
\begin{aligned}
|IV| \leqq& \frac{4}{B}(p-1)(q-2)A^2u^2|Du|^2+B(p-1)(q-2)Q^2|Du|^{-2}|\Delta_\infty u|^2 \\
\leqq& \frac{4}{B}(p-1)(q-2)A^2u^2|Du|^2 \\
&+\frac{2B(p-1)(q-2)}{q}Q|Du|^{-4}|\Delta_\infty u|^2(F+(q-2)Q|Du|^2/2). \qquad \text{(A.9)}
\end{aligned}
$$

Now we choose A large such that

$$
\begin{aligned}
&2(n-1)-8\kappa_1^{-1}\kappa_2^2(p-1)+2A-C(p)-32\Big(\frac{101}{100}+\frac{q-2}{2}\Big)^2 \\
&-\frac{4B}{B-2}(p-1)(q-2)>0.
\end{aligned}
$$

Then by (A.3), (A.5), (A.6), (A.7), (A.8), and (A.9), $a^{ij}D_{ij}F^{q/2}>0$ in $\Omega_{r_0}\cap S_A$, and hence $F^{q/2}$ does not achieve its maximum in $\Omega_{r_0}\cap S_A$. This concludes the proof for the case when $p>2$.

Case 2: For $p\in(1,2)$, we consider the quantify F given in (A.2). A similar argument as above shows that F does not achieve maximum on $(\Gamma_+\cup\Gamma_-)\cap\overline{\Omega}_{r_0}\cap S_A$. In $\Omega_{r_0}\cap S_A$, we compute

$$
\begin{aligned}
&a^{ij}D_{ij}F \\
&\quad = a^{ij}D_{ij}Q|Du|^2-4(2-p)|Du|^{-2}D_iuD_iQ\Delta_\infty u+4D_iQD_kuD_{ik}u+2Q|D^2u|^2 \\
&\qquad +2(2-p)Q|D|Du||^2-4(2-p)Q|Du|^{-4}|\Delta_\infty u|^2+2Aa^{ij}D_iuD_ju, \qquad \text{(A.10)}
\end{aligned}
$$

where a^{ij} is given in (6.5) satisfying (6.16). By a direct computation and (6.16), we have

$$
\begin{aligned}
&a^{ij}D_{ij}Q|Du|^2+2Aa^{ij}D_iuD_ju \\
&\quad =[2(n-1)-8\kappa_1^{-1}\kappa_2^2+(p-2)|Du|^{-2}(2|D_{x'}u|^2-8\kappa_1^{-1}\kappa_2^2|D_nu|^2)]|Du|^2 \\
&\qquad +2Aa^{ij}D_iuD_ju \\
&\quad \geqq \Big(2(n-1)-8\kappa_1^{-1}\kappa_2^2-2(2-p)+2A(p-1)\Big)|Du|^2. \qquad \text{(A.11)}
\end{aligned}
$$

Note that

$$|Du|^{-4}|\Delta_\infty u|^2 \leqq |D|Du||^2 \leqq |D^2u|^2.$$

Therefore,

$$\begin{aligned}&2Q|D^2u|^2 + 2(2-p)Q|D|Du||^2 - 4(2-p)Q|Du|^{-4}|\Delta_\infty u|^2\\ &\quad\geqq 2(p-1)Q|D^2u|^2. \end{aligned}\tag{A.12}$$

It remains to control

$$-4(2-p)|Du|^{-2}D_iuD_iQ\Delta_\infty u + 4D_iQD_kuD_{ik}u =: I + II.$$

By Young's inequality and $|DQ|^2 \leqq 8Q$ for small r_0, we have

$$\begin{aligned}|I| \leqq& 4(2-p)|Du|^{-1}|DQ||\Delta_\infty u|\\ \leqq& \frac{p-1}{16}|DQ|^2|Du|^{-4}|\Delta_\infty u|^2 + \frac{64(2-p)^2}{p-1}|Du|^2\\ \leqq& \frac{p-1}{2}Q|Du|^{-4}|\Delta_\infty u|^2 + \frac{64(2-p)^2}{p-1}|Du|^2, \end{aligned}\tag{A.13}$$

and

$$\begin{aligned}|II| \leqq& 4|DQ||Du||D^2u|\\ \leqq& \frac{p-1}{16}|DQ|^2|D^2u|^2 + \frac{64}{p-1}|Du|^2\\ \leqq& \frac{p-1}{2}Q|D^2u|^2 + \frac{64}{p-1}|Du|^2. \end{aligned}\tag{A.14}$$

Now we choose A large such that

$$2(n-1) - 8\kappa_1^{-1}\kappa_2^2 - 2(2-p) + 2A(p-1) - \frac{64(2-p)^2}{p-1} - \frac{64}{p-1} > 0.$$

Then by (A.10), (A.11), (A.12), (A.13), and (A.14), $a^{ij}D_{ij}F > 0$ in $\Omega_{r_0} \cap S_A$, and hence F does not achieve its maximum in $\Omega_{r_0} \cap S_A$. This concludes the proof for the case when $p \in (1, 2)$. □

References

1. Ammari, H., Kang, H., Lee, H., Lee, J., Lim, M.: Optimal estimates for the electric field in two dimensions. *J. Math. Pures Appl. (9)* **88**(4), 307–324, 2007
2. Ammari, H., Kang, H., Lim, M.: Gradient estimates for solutions to the conductivity problem. *Math. Ann.* **332**(2), 277–286, 2005
3. Antontsev, S.N., Rodrigues, J.F.: On stationary thermo-rheological viscous flows. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **52**(1), 19–36, 2006
4. Babuška, I., Andersson, B., Smith, P.J., Levin, K.: Damage analysis of fiber composites. I. Statistical analysis on fiber scale. *Comput. Methods Appl. Mech. Eng.* **172**(1–4), 27–77, 1999

5. Bao, E., Li, Y.Y., Yin, B.: Gradient estimates for the perfect conductivity problem. *Arch. Ration. Mech. Anal.* **193**(1), 195–226, 2009
6. Bao, E., Li, Y.Y., Yin, B.: Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. *Comm. Part. Differ. Equ.* **35**(11), 1982–2006, 2010
7. Brander, T., Ilmavirta, J., Kar, M.: Superconductive and insulating inclusions for linear and non-linear conductivity equations. *Inverse Probl. Imaging* **12**(1), 91–123, 2018
8. Budiansky, B., Carrier, G.F.: High shear stresses in stiff–fiber composites. *J. Appl. Mech.* **51**(4), 733–735, 1984
9. Ciraolo, G., Sciammetta, A.: Gradient estimates for the perfect conductivity problem in anisotropic media. *J. Math. Pures Appl. (9)* **127**, 268–298, 2019
10. Ciraolo, G., Sciammetta, A.: Stress concentration for closely located inclusions in nonlinear perfect conductivity problems. *J. Differ. Equ.* **266**(9), 6149–6178, 2019
11. DiBenedetto, E., Manfredi, J.: On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. *Am. J. Math.* **115**(5), 1107–1134, 1993
12. Dong, H., Li, Y.Y., Yang, Z.: Optimal gradient estimates of solutions to the insulated conductivity problem in dimension greater than two. *J. Eur. Math. Soc.* **(to appear)** (2021). arXiv:2110.11313
13. Dong, H., Li, Y.Y., Yang, Z.: *Gradient estimates for the insulated conductivity problem: The non-umbilical case* (2022). arXiv:2203.10081.
14. Dong, H., Li, Z.: Classical solutions of oblique derivative problem in nonsmooth domains with mean Dini coefficients. *Trans. Am. Math. Soc.* **373**(7), 4975–4997, 2020
15. Duzaar, F., Mingione, G.: Gradient estimates via linear and nonlinear potentials. *J. Funct. Anal.* **259**(11), 2961–2998, 2010
16. Duzaar, F., Mingione, G.: Gradient estimates via non-linear potentials. *Am. J. Math.* **133**(4), 1093–1149, 2011
17. Garroni, A., Kohn, R.V.: Some three-dimensional problems related to dielectric breakdown and polycrystal plasticity. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459**(2038), 2613–2625, 2003
18. Gorb, Y., Novikov, A.: Blow-up of solutions to a p-Laplace equation. *Multiscale Model. Simul.* **10**(3), 727–743, 2012
19. Hamburger, C.: Regularity of differential forms minimizing degenerate elliptic functionals. *J. Reine Angew. Math.* **431**, 7–64, 1992
20. Idiart, M.I.: The macroscopic behavior of power-law and ideally plastic materials with elliptical distribution of porosity. *Mech. Res. Commun.* **35**(8), 583–588, 2008
21. Keller, J.B.: Stresses in narrow regions. *J. Appl. Mech.* **60**(4), 1054–1056, 1993
22. Krylov, N.V.: Nonlinear elliptic and parabolic equations of the second order. *Mathematics and Its Applications (Soviet Series)*, vol. 7. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzytsky [P. L. Buzytskiĭ].
23. Levy, O., Kohn, R.V.: Duality relations for non-Ohmic composites, with applications to behavior near percolation. *J. Stat. Phys.* **90**(1–2), 159–189, 1998
24. Li, Y.Y., Yang, Z.: Gradient estimates of solutions to the conductivity problem with flatter insulators. *Anal. Theory Appl.* **37**(1), 114–128, 2021
25. Li, Y.Y., Yang, Z.: Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two. *Math. Ann.* **385**(3–4), 1775–1796, 2023
26. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.* **12**(11), 1203–1219, 1988
27. Lieberman, G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'seva for elliptic equations. *Comm. Part. Differ. Equ.* **16**(2–3), 311–361, 1991
28. Markenscoff, X.: Stress amplification in vanishingly small geometries. *Comput. Mech.* **19**(1), 77–83, 1996
29. Růžicka, M.: Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, vol. 1748, Springer, Berlin, 2000

30. Suquet, P.-M.: Overall potentials and extremal surfaces of power law or ideally plastic composites. *J. Mech. Phys. Solids* **41**(6), 981–1002, 1993
31. Trudinger, N.S.: On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.* **20**, 721–747, 1967
32. Uhlenbeck, K.: Regularity for a class of non-linear elliptic systems. *Acta Math.* **138**(3–4), 219–240, 1977
33. Vázquez, J.L.: A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.* **12**(3), 191–202, 1984
34. Weinkove, B.: The insulated conductivity problem, effective gradient estimates and the maximum principle. *Math. Ann.* **385**(1–2), 1–16, 2023
35. Yun, K.: An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions. *J. Differ. Equ.* **261**(1), 148–188, 2016

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