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Regularity Properties of Passive Scalars with Rough Divergence-Free Drifts

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Abstract

We present sharp conditions on divergence-free drifts in Lebesgue spaces for the passive scalar advection–diffusion equation

$$\partial_t \theta - \Delta\theta + b \cdot \nabla\theta = 0,$$

to satisfy local boundedness, a single-scale Harnack inequality, and upper bounds on fundamental solutions. We demonstrate these properties for drifts b belonging to $L^q_t L^p_x$, where $\frac{2}{q} + \frac{n}{p} < 2$, or $L^p_x L^q_t$, where $\frac{3}{q} + \frac{n-1}{p} < 2$. For steady drifts, the condition reduces to $b \in L^{\frac{n-1}{2}+}$. The space $L^1_t L^\infty_x$ of drifts with 'bounded total speed' is a borderline case and plays a special role in the theory. To demonstrate sharpness, we construct counterexamples whose goal is to transport anomalous singularities into the domain 'before' they can be dissipated.

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1. Introduction

We consider the linear advection–diffusion equation

$$\partial_t \theta - \Delta\theta + b \cdot \nabla\theta = 0. \tag{A-D}$$

The solution $\theta = \theta(x,t)$ is known as a *passive scalar*, and the prescribed divergence-free velocity field $b = b(x,t)$ is known as the *drift*.

Divergence-free drifts arise naturally in the nonlinear PDEs of fluid dynamics. In that context, often the only *a priori* knowledge of the drift is in low regularity. Therefore, it is natural to seek to understand the sharp regularity properties of (A-D) when the drift b is *rough*. A great deal is already known in this direction. In this paper, we give an essentially complete answer to what we consider to be one of the most interesting remaining questions, see Sect. 1.1.

To understand what is 'rough', we recall the scaling symmetry

$$u \to u(\lambda x, \lambda^2 t), \quad b \to \lambda b(\lambda x, \lambda^2 t), \quad \lambda > 0. \tag{1.1}$$

In dimensional analysis, one writes $[x] = L$, $[t] = L^2$, and $[b] = L^{-1}$. The scaling (1.1) identifies the Lebesgue spaces $L^q_t L^p_x$, where $\frac{2}{q} + \frac{n}{p} \leq 1$, as *(sub)critical* spaces for the drift, meaning spaces whose norms do not grow upon 'zooming in' with the scaling symmetry. For example,

$$X = L^\infty_t L^n_x,\ L^2_t L^\infty_x,\ L^{n+2}_{t,x} \tag{1.2}$$

are *critical spaces*, whose norms are dimensionless, i.e., invariant under the symmetry (1.1). Here and throughout, $n \geq 2$ is the spatial dimension.

When b belongs to one of the critical Lebesgue spaces, it is not difficult to adapt the work of De Giorgi, Nash, and Moser [7,29,34] to demonstrate that weak solutions of (A-D) are Hölder continuous and satisfy Harnack's inequality. The above threshold is known to be sharp for continuity within the scale of Lebesgue spaces, see counterexamples in [43,46]. The divergence-free condition even allows access to drifts in the critical spaces

$$X = L^\infty_t L^{-1,\infty}_x, L^\infty_t \mathrm{BMO}^{-1}_x \tag{1.3}$$

considered by [36,38,41]. In these spaces, it is furthermore possible to prove Gaussian upper and lower bounds on fundamental solutions in the spirit of Aronson [2].

For *supercritical* drifts, continuity may fail [43], and we must change our expectations. Nonetheless, a version of the regularity theory may be salvaged due to the divergence-free structure; its crucial role is already visible from the computation[1]

$$\int (b\cdot\nabla\theta)\theta\phi^2 \,\mathrm{d}x\,\mathrm{d}t = \int b\cdot\nabla\left(\frac{\theta^2}{2}\right)\phi^2\,\mathrm{d}x\,\mathrm{d}t = -\int \theta^2(b\cdot\nabla\phi)\phi. \tag{1.4}$$

[1] The divergence-free structure also plays a role in the critical case, but it is more subtle: without this structure, the drift is required to be small in a critical Lebesgue space or Kato class, and local boundedness may depend on the 'profile' of the drift, not merely its norm.

With this well known observation, one may apply Moser's iteration scheme to demonstrate that, when $b \in L^q_t L^p_x$ and $\frac{2}{q} + \frac{n}{p} < 2$, solutions are *locally bounded*, see [35]. Typical examples are

$$X = L^\infty_t L^{\frac{n}{2}+}_x, L^{\frac{n+2}{2}+}_{t,x}. \tag{1.5}$$

Under these conditions (and a weak background assumption), the Harnack inequality persists as a *single-scale Harnack inequality* [19,20]: In the steady case $\theta = \theta(x)$,

$$\sup_{B_R} \theta \leq C_R \inf_{B_R} \theta, \tag{1.6}$$

where C_R may become unbounded as $R \to 0^+$. Whereas a scale-invariant Harnack inequality implies Hölder continuity, it is less well known that a single-scale Harnack may hold in the absence of Hölder continuity. Finally, pointwise upper bounds on fundamental solutions continue to hold, although they have 'fat tails' compared to their Gaussian counterparts [37,47].

One might wonder whether the easy computation (1.4) already yields the sharp conditions. It does not. In the steady case, there is an additional subtle feature, which is not well known and, in our opinion, surprising: Local boundedness continues to hold when $b \in L^{\frac{n-1}{2}+}$. The best of our knowledge, this 'dimension reduction' was first observed in this context by Kontovourkis [21] in his (unpublished) thesis.[2] Heuristically, Kontovourkis' key observation is as follows. Consider the basic L^2 energy estimate in a ball B_r without smooth cut-off. The drift contributes the boundary term

$$\int_{B_r} (b \cdot \nabla\theta)\theta \, \mathrm{d}x = \int_{\partial B_r} \frac{\theta^2}{2} b \cdot n \, \mathrm{d}\sigma, \tag{1.7}$$

where $d\sigma$ is the surface area measure. Since $\nabla\theta \in L^2(B_R)$, on 'many slices' $r \in (R/2, R)$, we have $\nabla\theta \in L^2(\partial B_r)$, with a quantitative bound. Similarly, b belongs to $L^{\frac{n-1}{2}+}(\partial B_r)$ on 'many slices'. Thus, one may exploit Sobolev embedding on the sphere ∂B_r to estimate the boundary term.

The dimension reduction was recently rediscovered by Bella and Schäffner in [5]. There, the authors proved local boundedness and a single-scale Harnack inequality in the context of certain degenerate elliptic PDEs, which we review in Sect. 1.2.

Following the work [21], it has been an interesting problem to understand what dimension reduction holds in the parabolic setting. In particular, is $b \in L^{\frac{n+1}{2}+}_{t,x}$ enough for local boundedness? Very recently, Zhang [48] generalized the work [5]

[2] This kind of dimension reduction itself goes back at least to work [11] of Frehse and Růžička on the steady Navier-Stokes equations in $n = 6$. The 'slicing' was also exploited by Struwe in [42].

to the parabolic setting, and among other things, demonstrated local boundedness under the condition

$$b \in L^p_x L^q_t, \qquad \frac{3}{q} + \frac{n-1}{p} < 2, \quad p \leq q, \tag{1.8}$$

see Corollary 1.5 therein. Crucially, the order of integration in (1.8) is reversed. The condition $b \in L_x^{\frac{n-1}{2}+} L^\infty_t$ implies the elliptic case in [21]. From this condition, we see that, perhaps, one dimension is not 'reduced', but rather hidden into the time variable.

1.1. Main Results

Everything we have discussed so far has been directed toward answering

(Q) What are the optimal conditions on the drift for which the local regularity theory holds?

That is, when do weak solutions satisfy local boundedness and its cousins, Harnack's inequality and pointwise upper bounds on fundamental solutions? In this paper, we give an essentially complete answer to this question in Lebesgue spaces.

Our main results constitute a detailed picture of the local regularity theory for the passive scalar advection–diffusion equation (A-D) with supercritical drifts. To give the complete picture and maximize its usefulness to the reader, we present the known results (appearing with citations) together with our own contributions (without citations). We will explain the novelty of our contributions in detail in Remark 1.3, but we summarize a few key points here.

First, we revisit the condition $b \in L^q_t L^p_x$, $\frac{2}{q} + \frac{n}{p} < 2$. We prove that, without additional structure, it is sharp. The reason for this has to do with a new endpoint case $L^1_t L^\infty_x$, the space of drifts with 'bounded total speed', in the terminology of [44]. In this space, local boundedness holds in a modified form, depending on the profile of b itself rather than its norm in $L^1_t L^\infty_x$. The counterexamples we construct are connected to this space, which plays a special role in the theory.

Second, we consider the condition (1.8), that is, with the opposite order of integration. Under this condition, we prove the parabolic Harnack inequality and pointwise upper bounds on fundamental solutions. To conclude, we present counterexamples demonstrating its sharpness.

We now state the results. Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $\Omega' \subset\subset \Omega$ be a subdomain. Let $I = (S, T]$ and $I' = (S', T'] \subset I$ be finite intervals such that $S < S'$. Let $Q_I = \Omega \times I$ and $Q'_I = \Omega' \times I'$. Let $p, q \in [1, +\infty]$.

Theorem 1.1. *($b \in L^q_t L^p_x$)* (Local boundedness) [35] *If*

$$\zeta := \frac{2}{q} + \frac{n}{p} < 2, \tag{1.9}$$

then we have the following quantitative local boundedness *property: $\theta \in L^1(Q_I) \cap C^\infty(Q_I)$ satisfies the drift-diffusion equation* (A-D) *in Q_I with divergence-free drift $b \in C^\infty(Q_I)$ and*

$$b \in L^q_t L^p_x(Q_I), \tag{1.10}$$

then

$$\sup_{Q_I'} |\theta| \lesssim \|\theta\|_{L^1(Q_I)}, \tag{1.11}$$

where the implied constant depends on n, Ω, Ω', I, I', p, q, and $\|b\|_{L^q_t L^p_x(Q_I)}$.

(Single-scale Harnack) [20] *If, additionally,* $b \in L^2_t H^{-1}_x(Q_I)$ *and* $\theta > 0$, *then we have the following* quantitative Harnack inequality*: If* $I_1, I_2 \subset\subset I$ *are intervals satisfying* $\sup I_1 < \inf I_2$, *then*

$$\sup_{\Omega' \times I_1} \theta \lesssim \inf_{\Omega' \times I_2} \theta, \tag{1.12}$$

where the implied constant depends on n, Ω, Ω', I, I_1, I_2, p, q, $\|b\|_{L^q_t L^p_x(Q_I)}$, *and* $\|b\|_{L^2_t H^{-1}_x(Q_I)}$.

(Bounded total speed) *If* $(p, q) = (\infty, 1)$, *then the above quantitative local boundedness property holds with constants depending on b itself rather than* $\|b\|_{L^1_t L^\infty_x(Q_I)}$. *(The property is false without this adjustment.)*

(Sharpness) *Let* $Q = B_1 \times (0, 1)$. *There exist a smooth divergence-free drift* $b \in C^\infty(Q)$ *belonging to* $L^q_t L^p_x(Q)$ *for all* $(p, q) \in [1, +\infty]^2$ *with* $2/q + n/p = 2$, $(p, q) \neq (\infty, 1)$, *and satisfying the following property. There exists a smooth solution* $\theta \in L^\infty_t L^1_x \cap C^\infty(Q)$ *to the advection–diffusion equation* (A-D) *in Q with*

$$\sup_{B_{1/2} \times (0,T)} |\theta| \to +\infty \text{ as } T \to 1_-.$$

In particular, the above quantitative local boundedness property fails when $2/q + n/p = 2$ *and* $q > 1$.

(Upper bounds on fundamental solutions) [37] *If the divergence-free drift* $b \in C^\infty_0(\mathbb{R}^n \times [0, +\infty))$ *belongs to* $L^q_t L^p_x(\mathbb{R}^n \times \mathbb{R}_+)$ *and* $1 \leq \zeta < 2$, *then the fundamental solution* $\Gamma = \Gamma(x, t; y, s)$ *to the parabolic operator* $L = \partial_t - \Delta + b \cdot \nabla$ *satisfies, when* $p < +\infty$,

$$\Gamma(x, t; 0, 0) \leq Ct^{-\frac{n}{2}} \max\left[\exp\left(-M^{-\frac{1}{1-\alpha}} \frac{|x|^{1+\frac{\alpha}{1-\alpha}}}{Ct^{\frac{\alpha-1/q}{1-\alpha}}}\right), \exp\left(-\frac{|x|^2}{Ct}\right)\right], \tag{1.13}$$

and, when $p = +\infty$,

$$\Gamma(x, t; 0, 0) \lesssim t^{-\frac{n}{2}} \exp\left[-\frac{1}{4Ct}\left(\frac{1}{4}|x| - CMt^{1-\frac{1}{q}}\right)^2\right] \tag{1.14}$$

for all $x_0 \in \mathbb{R}^n$ *and* $t \in \mathbb{R}_+$. *Here,* $C = C(n, p, q) > 0$ *and*

$$\alpha = \frac{2 - \zeta + 2/q}{2}, \quad M = C\|b(\cdot, t)\|_{L^q_t L^p_x(\mathbb{R}^n \times \mathbb{R}_+)}. \tag{1.15}$$

See Fig. 1 for an illustration of Theorem 1.1.

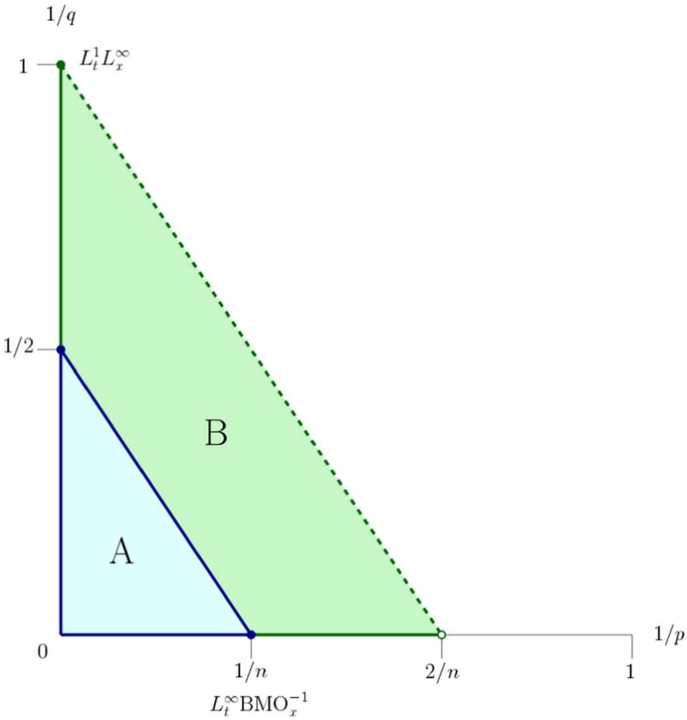

Fig. 1. Divergence-free drift $b \in L^q_t L^p_x$ in dimension $n \geq 2$ (dimension $n = 3$ illustrated above). *Region A* ($2/q + n/p \leq 1$): Local boundedness, Harnack inequality, and Hölder continuity. *Region B* ($1 < 2/q + n/p < 2$ or $(p, q) = (\infty, 1)$): Local boundedness and single-scale Harnack inequality. *Dashed line*: Local boundedness is false

Theorem 1.2. *($b \in L^p_x L^q_t$)* (Local boundedness) [48] *If*

$$\zeta := \frac{3}{q} + \frac{n-1}{p} < 2, \quad p \leq q, \tag{1.16}$$

then we have the following quantitative local boundedness *property: If* $\theta \in L^1(Q_I) \cap C^\infty(Q_I)$ *satisfies the drift-diffusion equation* (A-D) *in* Q_I *with divergence-free drift* $b \in C^\infty(Q_I)$ *and*

$$b \in L^p_x L^q_t(Q_I), \tag{1.17}$$

then

$$\sup_{Q'_I} |\theta| \lesssim \|\theta\|_{L^1(Q_I)}, \tag{1.18}$$

where the implied constant depends on n, Ω, Ω', I, I', p, q, *and* $\|b\|_{L^p_x L^q_t(Q_I)}$.

(Single-scale Harnack) *If, additionally,* $b \in L^2_t H^{-1}_x(Q_I)$ *and* $\theta > 0$, *then we have the following* quantitative Harnack inequality*: If* $I_1, I_2 \subset\subset I$ *are intervals* *satisfying* $\sup I_1 < \inf I_2$, *then*

$$\sup_{\Omega' \times I_1} \theta \lesssim \inf_{\Omega' \times I_2} \theta, \tag{1.19}$$

where the implied constant depends on n, Ω, Ω', I, I_1, I_2, p, q, $\|b\|_{L^p_x L^q_t(Q_I)}$, *and* $\|b\|_{L^2_t H^{-1}_x(Q_I)}$.

(Sharpness, steady case). *Let* $n \geq 3$. *The quantitative local boundedness property fails for steady drifts* $b \in L^{\frac{n-1}{2}}(B)$ *and steady solutions* θ *in the ball* B.

(Sharpness, time-dependent case). *Let* $n \geq 2$ *and* $Q = B_1 \times (0, 1)$. *There exist a smooth divergence-free drift* $b \in C^\infty(Q)$ *belonging to* $L^p_x L^q_t(Q)$ *for all* $p, q \in [1, +\infty]$ *with* $p \leq q$ *and* $3/q + (n-1)/p > 2$ *and satisfying the following property. There exists a smooth solution* $\theta \in L^\infty_t L^1_x \cap C^\infty(Q)$ *to the advection–diffusion equation* (A-D) *in* Q *with*

$$\sup_{B_{1/2} \times (0,T)} |\theta| \to +\infty \quad \text{as } T \to 1_-.$$

In particular, the above quantitative local boundedness property fails when $3/q + (n-1)/p > 2$ *and* $p \leq q$. *Finally, the drift additionally belongs to* $L^q_t L^p_x(Q)$ *for all* $(p, q) \in [1, +\infty]^2$ *with* $2/q + n/p > 2$.

(Upper bounds on fundamental solutions) *If the divergence-free drift* $b \in C^\infty_0(\mathbb{R}^n \times [0, +\infty))$ *belongs to* $L^p_x L^q_t(\mathbb{R}^n \times \mathbb{R}_+)$ *and* $1 \leq \zeta < 2$, *then the fundamental solution* $\Gamma = \Gamma(x, t; y, s)$ *to the parabolic operator* $L = \partial_t - \Delta + b \cdot \nabla$ *satisfies, when* $p < +\infty$,

$$\Gamma(x, t; 0, 0) \leq C t^{-\frac{n}{2}} \max\left[\exp\left(-M^{-\frac{1}{1-\alpha}} \frac{|x|^{1+\frac{\alpha+1/p-1/q}{1-\alpha}}}{C t^{\frac{\alpha-1/q}{1-\alpha}}}\right), \exp\left(-\frac{|x|^2}{Ct}\right)\right], \tag{1.20}$$

and, when $p = +\infty$,

$$\Gamma(x, t; 0, 0) \lesssim t^{-\frac{n}{2}} \exp\left[-\frac{1}{4Ct}\left(\frac{1}{4}|x| - CM|x|^{\frac{1}{q}} t^{1-\frac{1}{q}}\right)^2\right] \tag{1.21}$$

for all $x_0 \in \mathbb{R}^n$ *and* $t \in \mathbb{R}_+$. *Here,* $C = C(n, p, q) > 0$ *and*

$$\alpha = \frac{2 - \zeta + 2/q}{2}, \quad M = C\|b(\cdot, t)\|_{L^p_x L^q_t(\mathbb{R}^n \times \mathbb{R}_+)}. \tag{1.22}$$

See Figs. 2 and 3 for an illustration of Theorem 1.2.

Remark 1.3. (Contributions)

In the positive direction, our main new contributions are the $L^1_t L^\infty_x$ case in Theorem 1.1 and the Harnack inequality and Gaussian-like upper bound for the $L^p_x L^q_t$ cases in Theorem 1.2. However, we actually reprove the known positive results in Theorems 1.1 and 1.2, together with our new results, in a unified and (essentially) self-contained way through a new *form boundedness condition*. This condition encompasses significantly more general drifts than we stated above, see

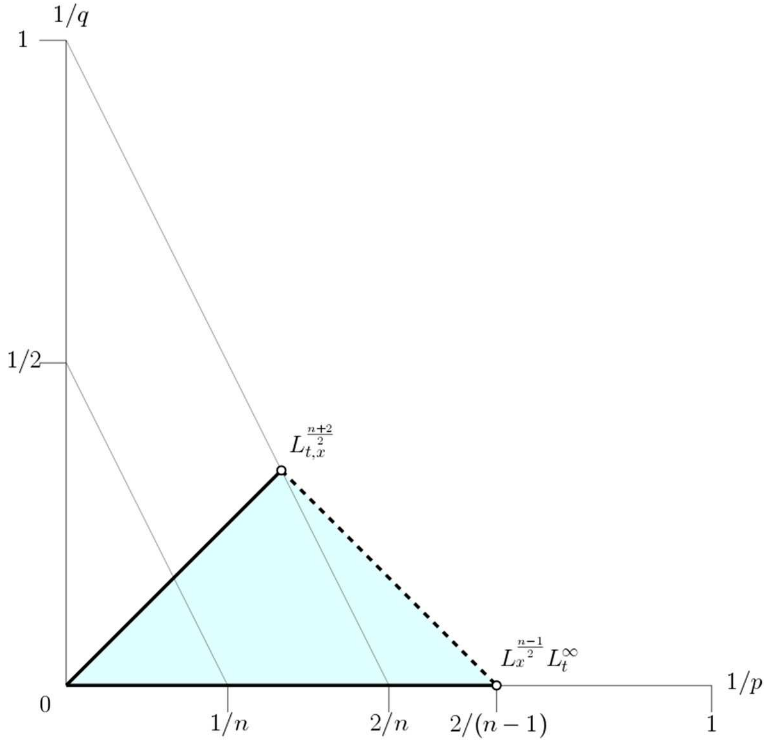

Fig. 2. Divergence-free drift $b \in L_x^p L_t^q$, dimension $n \geq 3$ (dimension $n = 4$ illustrated above). Local boundedness and single-scale Harnack inequality

Sect. 1.2. Moreover, in light of Remark 1.4 below, it is worthwhile to mention that we also give a proof of the Harnack inequality in the $L_t^q L_x^p$ setting. Our condition $b \in L_t^2 H_x^{-1}$ in the Harnack inequality is new; it is used to connect the forward- and backward-in-time regions, see Lemma 2.11. This condition is at the level of making sense of (A-D) in the sense of distributions. For comparison, the background condition in [20] was $b \in L_t^\infty L_x^2$. Finally, the pointwise upper bounds in [37] were only proved with $n \geq 3$; we include the case $n = 2$.

In the direction of sharpness, our main new contribution is to construct counterexamples demonstrating sharpness of the $L_t^q L_x^p$ and $L_x^p L_t^q$ criteria. This is far from obvious, and in the parabolic setting, there are no counterexamples like this in the literature. Our examples are discussed in Sect. 1.3, and as mentioned above, the space $L_t^1 L_x^\infty$ plays a key role in understanding them. Finally, in Proposition 5.4, we give examples demonstrating that our Gaussian-like pointwise upper bounds are optimal in certain regions.

Remark 1.4. At a technical level, there is a gap in the proof of the weak Harnack inequality in [20], see (3.22) therein, where it is claimed that $\log_+(\theta/\mathbf{K})$ is a supersolution. This is related to a step in the proof of Lemma 6.20, p. 124, in Lieberman's book [23], which we had difficulty following, see the first inequality therein. Both of these are related to improving the weak L^1 inequality. We opt to follow Moser's

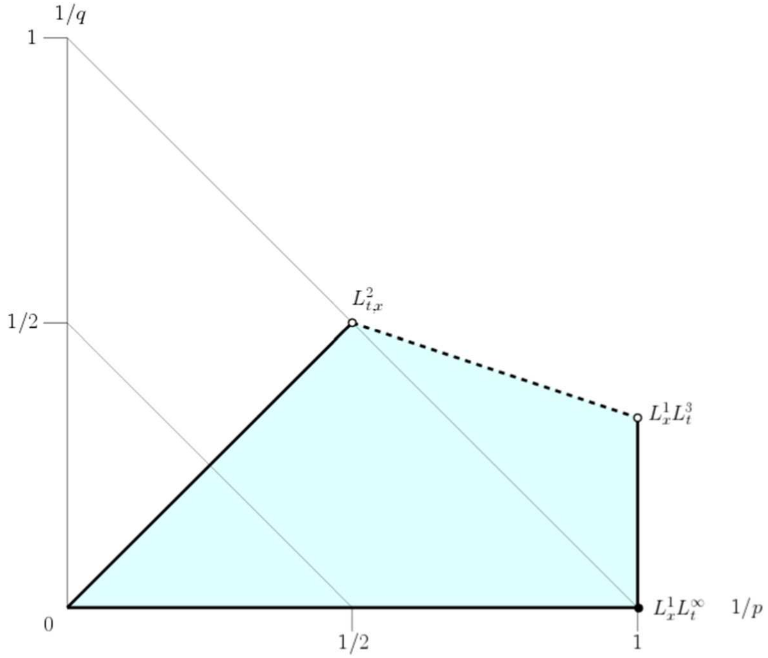

Fig. 3. Divergence-free drift $b \in L^p_x L^q_t$, dimension $n = 2$. Local boundedness and single-scale Harnack inequality

proof in [31] more directly and skip the weak Harnack inequality. In principle, one could directly apply the parabolic John-Nirenberg inequality in [9,29] to obtain the weak Harnack inequality.

Remark 1.5. The local boundedness property and Harnack inequality in Theorems 1.1 and 1.2 can be easily extended to accommodate drifts satisfying $\operatorname{div} b \leq 0$ (with the background assumption $b \in L^2_{t,x}(Q_I)$ in the Harnack inequality). These properties and the fundamental solution estimates can also be extended to divergence-form elliptic operators $\operatorname{div} a\nabla\cdot$ with bounded, uniformly elliptic a.

1.2. Discussion of Dimension Reduction Principle

The 'slicing' described above in the steady setting is more subtle in the time-dependent setting because the anisotropic condition $\theta \in L^\infty_t L^2_x$ does not restrict well to slices in the radial variable r; compare this to the isotropic condition $\nabla\theta \in L^2_{t,x}$. Indeed, to 'slice' in a variable, it seems necessary for that variable to be summed 'last' (that is, on the outside) in the norm. The condition $b \in L^p_x L^q_t$, $p \leq q$, $\frac{3}{q} + \frac{n-1}{p} < 2$, in Theorem 1.2 comes, roughly speaking, from interpolating between the isotropic condition $b \in L^{\frac{n+2}{2}+}_{t,x}$, in which the order of integration may be changed freely, and the dimensionally reduced condition $b \in L^{\frac{n-1}{2}+}_x L^\infty_t$, which implies that

$b \in L^\infty_t L^{\frac{n-1}{2}+}_\sigma(\partial B_r \times I)$ on 'many slices', say, a set of $r \in A \subset (1/2, 1)$ with measure $|A| > 1/4$. Local boundedness under this condition was already observed by X. Zhang in [48, Corollary 1.5], and the counterexamples we construct answer an open question in Remark 1.6 therein.

Our proof of local boundedness and the Harnack inequality is built on a certain *form boundedness condition* (FBC), see Sect. 2, which subsumes a wide variety of possible assumptions on b. For example, in Proposition 2.3, we verify (FBC) not only in the context of Theorems 1.1 and 1.2 but also under the more general conditions

$$b \in L^q_t L^\beta_r L^\gamma_\sigma((B_R \setminus B_{R/2}) \times I), \quad \beta \geq \frac{n}{2}, \quad \frac{2}{q} + \frac{1}{\beta} + \frac{n-1}{\gamma} < 2 \tag{1.23}$$

and

$$b \in L^\kappa_r L^q_t L^p_\sigma((B_R \setminus B_{R/2}) \times I), \quad q \geq \frac{n}{2}, \quad \frac{3}{q} + \frac{n-1}{p} < 2. \tag{1.24}$$

Furthermore, we allow arbitrarily low integrability $\kappa > 0$ in the radial variable; the slicing method does not require high integrability. The norms in the above spaces (1.23) and (1.24) are defined in (2.8) and (2.9) below. Our proof of upper bounds on fundamental solutions is centered on a variant of the form boundedness condition, see Sect. 5 which is partially inspired by the work of Qi S. Zhang [47].

We now describe the work [5], which was generalized to the parabolic setting in [48]. The conditions in [5] are on the ellipticity matrix a, which is allowed to be degenerate. Define

$$\lambda(x) := \inf_{|\xi|=1} \xi \cdot a(x)\xi, \quad \mu(x) := \sup_{|\xi|=1} \frac{|a(x)\xi|^2}{\xi \cdot a(x)\xi}. \tag{1.25}$$

If $n \geq 2$, $p, q \in (1, +\infty]$, and

$$\lambda^{-1} \in L^q(B), \quad \mu \in L^p(B), \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{n-1}, \tag{1.26}$$

then weak solutions of $-\operatorname{div} a\nabla u = 0$ are locally bounded and satisfy a single-scale Harnack inequality. The analogous condition with $\frac{2}{n}$ on the right-hand side is due to Trudinger in [45]. By examples in [13], the right-hand side cannot be improved to $\frac{2}{n-1} + \varepsilon$. Divergence-free drifts b belong to the above framework: Under general conditions, it is possible to realize b as the divergence of an antisymmetric *stream matrix*: $b_i = d_{ij,i}$. Then we have $-\Delta\theta + b \cdot \nabla\theta = -\operatorname{div}[(I+d)\nabla\theta]$, and μ captures the antisymmetric part d. The steady examples we construct in Sect. 4 handle the equality case in (1.26). We mention also the works [3,4].

Earlier, it was hoped that the dimension reduction could be further adapted to treat the case $b \in L^{\frac{n+1}{2}+}_{t,x}$ in the parabolic setting by estimating a half-derivative in time: $|\partial_t|^{1/2}\theta \in L^2_{t,x}$, since this condition is better adapted to slicing than $\theta \in L^\infty_t L^2_x$. On the other hand, our counterexamples rule out this possibility. Half time derivatives in parabolic PDE go back, at least, to [25, Chapter III, Section 4], see [1] for further discussion.

1.3. Discussion of Counterexamples and 'Bounded Total Speed'

Solutions of (A-D) in the whole space evolving from initial data $\theta_0 \in L^1(\mathbb{R}^n)$ become bounded instantaneously. This is captured by the famous Nash estimate [34]

$$\|\theta(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}} \|\theta_0\|_{L^1(\mathbb{R}^n)}, \tag{1.27}$$

where the implied constant is *independent* of the divergence-free drift b. The Nash estimate indicates that a divergence-free drift does not impede smoothing, in the sense of boundedness, of a density, even if the density is initially a Dirac mass.[3] Therefore, for rough drifts, local boundedness must be violated in a different way: The danger is that the drift can 'drag' an anomalous singularity into the domain of observation from outside. There is a competition between the drift, which transports the singularity with some speed, and the diffusion, which smooths the singularity at some rate. Will the singularity, entering from outside, be smoothed before it can be observed inside the domain?

Consider a Dirac mass $\delta_{x=-\vec{e}_1}$, which we seek to transport inside the domain. If one can transport the Dirac mass inside $B_{1/2}$ *instantaneously*, one can violate local boundedness. This can be done easily via the drift $b(x,t) = \delta_{t=0}\vec{e}_1$, which is singular in time. This example already demonstrates the importance of the space $L^1_t L^\infty_x$, whose drifts cannot transport the mass inside arbitrarily quickly.

To improve this example, we seek the most efficient way to transport the Dirac mass. Heuristically, the evolution of the Dirac mass is mostly supported in a ball of radius $R(t) \sim \sqrt{t}$. Therefore, we define our drift b to be $S(t)\vec{e}_1$ restricted to this support. That is, the drift lives on a ball of radius $R(t)$ moving in the x_1-direction at speed $S(t)$. Since we wish to move the Dirac mass instantaneously, we guess that $S(t) \sim 1/t$. A back-of-the-envelope calculation gives

$$\|b\|^q_{L^q_t L^p_x} \sim \int_0^1 S(t)^q R(t)^{\frac{nq}{p}}\,\mathrm{d}t \sim \int_0^1 t^{-q+\frac{nq}{2p}}\,\mathrm{d}t. \tag{1.28}$$

The above quantity is finite when $2/q + n/p > 2$; more care is required to get the borderline cases in Theorem 1.1, see Sect. 4. This heuristic is the basis for our time-dependent counterexamples in Sect. 4, except that we use appropriate subsolutions to keep the compact support property, we glue together many of these Dirac masses, and $S(t)$ must be chosen more carefully.

The elliptic counterexample with $b \in L^{\frac{n-1}{2}}$ is achieved by introducing an ansatz which reduces the problem to counterexamples for the steady Schrödinger equation $-\Delta u + Vu = 0$ in dimension $n-1$. These steady counterexamples are singular on a line through the domain, as they must be to respect the maximum principle.

The time-dependent counterexamples in $L^p_x L^q_t$ seem to be more subtle, and we only exhibit them in the non-borderline cases $\zeta = \frac{3}{q} + \frac{n-1}{p} > 2$ and $p \leq q$. When $\zeta = 2$, we have counterexamples in the cases $p = q = \frac{n+2}{2}$ and $(p,q) =$

[3] By approximation, we may consider the estimate (1.27) also for finite measure initial data.

$(\frac{n-1}{2}, +\infty)$ (the steady example). We believe that local boundedness fails also between these two points, but the counterexamples are yet to be exhibited, see Remark 4.4.

1.4. Further Review of the Existing Literature

Following the seminal works of De Giorgi [7] and Nash [34], Moser introduced his parabolic Harnack inequality [29,30] (see [28] for the elliptic case), whose original proof relied on a parabolic generalization of the John-Nirenberg theorem concerning exponential integrability of BMO functions. Later, Moser published a simplified proof [31], whose basic methods we follow. In [41], Seregin, Silvestre, Šverák, and Zlatoš generalized Moser's methods to accommodate drifts in $L^\infty_t \mathrm{BMO}^{-1}_x$. For recent work on boundary behavior in this setting, see [17,24]. Generalizations to critical Morrey spaces and the supercritical Lebesgue spaces are due to [18–20,35].

The Gaussian estimates on fundamental solutions were discovered by Aronson [2] and were generalized to divergence-free drifts by Osada in [36] ($L^\infty_t L^{-1,\infty}_x$) and Qian and Xi ($L^\infty_t \mathrm{BMO}^{-1}_x$) in [37,38]. Important contributions are due to [47], who developed Gaussian-like upper bounds in the supercritical case $b \in L^{\frac{n}{2}+}(\mathbb{R}^n)$, $n \geq 4$, and [26,33,40], among others. For recent progress on Green's function estimates with sharp conditions on lower order terms, see [8,22,32,39].

The primary examples concerning the regularity of solutions to (A-D) can be found in [41,43,46]. Counterexamples to continuity with time-dependent drifts can be constructed by colliding two discs of $+1$ (subsolution) and -1 (supersolution) with radii $R(t) \sim \sqrt{1-t}$ and speeds $S(t) \sim 1/\sqrt{1-t}$. The parabolic counterexamples with *steady* velocity fields constructed therein are more challenging. See [10,12] for examples in the elliptic setting. We also mention Zhikov's counterexamples [49] to uniqueness when b does not belong to L^2, whereas weak solutions with zero Dirichlet conditions are known to be unique when $b \in L^2$ [47].

For recent counterexamples in the regularity theory of parabolic systems based on self-similarity, see [27].

2. Local Boundedness and Harnack's Inequality

Let b be a smooth, divergence-free vector field defined on $B_{R_0} \times I_0$, where $R_0 > 0$ and I_0 is an open interval. In the sequel, we will use a *form boundedness condition*, which we denote by (FBC):

> There exist constants $M, N, \alpha > 0$, $\varepsilon \in [0, 1/2)$, and $\delta \in (0, 1]$ satisfying the following property. For every $R \in [R_0/2, R_0]$, $\varrho \in [R_0/2, R)$, subinterval $I \subset I_0$, and Lipschitz $u \in W^{1,\infty}(B_R \times I)$, there exists a measurable

set $A = A(\varrho, R, I, u) \subset (\varrho, R)$ with $|A| \geq \delta(R - \varrho)$ and satisfying

$$-\frac{1}{|A|}\iint_{B_A \times I} \frac{|u|^2}{2}(b \cdot n)\,\mathrm{d}x\,\mathrm{d}t \leq \frac{MR_0^\alpha}{\delta^\alpha R_0^2 (R-\varrho)^\alpha}\iint_{(B_R \setminus B_\varrho)\times I} |u|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ N\iint_{(B_R \setminus B_\varrho)\times I} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon \sup_{t \in I}\int_{B_R} |u(x,t)|^2\,\mathrm{d}x, \tag{FBC}$$

where $B_A = \cup_{r \in A}\partial B_r$ and n is the outer unit normal.

The left-hand side of (FBC) appears on the right-hand side of the energy estimates.

In the situations we consider, M may depend on R_0, and we can predict its dependence based on dimensional analysis. For example, since b has dimensions of L^{-1}, the quantity

$$R_0^{1-\frac{2}{q}-\frac{n}{p}} \|b\|_{L_t^q L_x^p (B_{R_0} \times R_0^2 I)}$$

is dimensionless.

In Proposition 2.3, we show that (FBC) is satisfied under the hypotheses of Theorems 1.1 and 1.2.

Notation. In this section, $R_0/2 \leq \varrho < R \leq R_0$ and $-\infty < T < \tau < 0$. Let us introduce the backward parabolic cylinders $Q_{R,T} = B_R \times (T, 0)$. Our working assumptions are that θ is a non-negative Lipschitz function and b is a smooth, divergence-free vector field. To give precise constants, we will frequently use the notation

$$\mathbf{C}(\varrho, \tau, R, T, M, \delta, \alpha) = \frac{1}{\delta^2 (R-\varrho)^2} + \frac{MR_0^\alpha}{\delta^\alpha R_0^2 (R-\varrho)^\alpha} + \frac{1}{\tau - T} \tag{2.1}$$

involving the various parameters from (FBC). Our convention throughout the paper is that all implied constants may depend on n.

Theorem 2.1. (Local boundedness) *Let θ be a non-negative Lipschitz subsolution and b satisfy (FBC) on $Q_{R,T}$. Then, for all $\gamma \in (0, 2]$,*

$$\sup_{Q_{\varrho,\tau}} \theta \leq C(N, \alpha, \varepsilon)^{\frac{1}{\gamma}} \mathbf{C}^{\frac{n+2}{2\gamma}} \|\theta\|_{L^\gamma (Q_{R,T} \setminus Q_{\varrho,\tau})}. \tag{2.2}$$

Theorem 2.2. (Harnack inequality) *Let θ be a non-negative Lipschitz solution on $Q^* = B \times (-T^*, T^*)$. Let $b \in L_t^2 H_x^{-1}(Q^*; \mathbb{R}^n)$ satisfying (FBC) on Q^*. Let $0 < \ell < T^*$ be the time lag. Then*

$$\sup_{B_{1/2} \times (-T^* + \ell, 0)} \theta \lesssim_{N,M,A,T^*,\delta,\alpha,\varepsilon,\ell} \inf_{B_{1/2} \times (\ell, T^*)} \theta, \tag{2.3}$$

where $A = \|b\|^2_{L_t^2 H_x^{-1}(Q^*; \mathbb{R}^n)}$.

2.1. Verifying (FBC)

We verify that (FBC) is satisfied in the setting of the main theorems.

Proposition 2.3. (Verifying FBC) *Let $p, q, \beta, \gamma \in [1, +\infty]$, $\kappa \in (0, +\infty]$, and b be a smooth, divergence-free vector field defined on $B_{R_0} \times I_0$.*

1. If

$$b \in L_t^q L_r^\beta L_\sigma^\gamma((B_{R_0} \setminus B_{R_0/2}) \times I_0), \quad \beta \geq \frac{n}{2}, \quad \zeta := \frac{2}{q} + \frac{1}{\beta} + \frac{n-1}{\gamma} < 2, \tag{2.4}$$

then b satisfies (FBC) with $A = (\varrho, R)$, $\delta = 1$, $N = \varepsilon = 1/4$, and

$$\alpha = \frac{2}{2-\zeta}, \quad M = C\left(R_0^{1-\zeta} \|b\|_{L_t^q L_r^\beta L_\sigma^\gamma((B_{R_0} \setminus B_{R_0/2}) \times I_0)}\right)^{2/(2-\zeta)} + \frac{1}{4}. \tag{2.5}$$

2. If

$$b \in L_r^\kappa L_t^q L_\sigma^p((B_{R_0} \setminus B_{R_0/2}) \times I_0), \quad q \geq \frac{n}{2}, \quad \zeta := \frac{3}{q} + \frac{n-1}{p} < 2, \tag{2.6}$$

then b satisfies (FBC) with $\delta = 1/2$, $N = \varepsilon = 1/4$, and

$$\begin{aligned} \alpha &= \left(\frac{1}{\kappa} - \frac{1}{q} + 1\right)\frac{2}{2-\zeta}, \\ M &= C\left(R_0^{1-\zeta} \|b\|_{L_r^\kappa L_t^q L_\sigma^p((B_{R_0} \setminus B_{R_0/2}) \times I_0)}\right)^{2/(2-\zeta)} + \frac{1}{4}. \end{aligned} \tag{2.7}$$

The above norms are defined by

$$\begin{aligned} &\|b\|_{L_t^q L_r^\beta L_\sigma^\gamma((B_{R_0} \setminus B_{R_0/2}) \times I_0)} \\ &:= \left(\int_{I_0}\left(\int_{R_0/2}^{R_0}\left(\int_{S^{d-1}} |b(r\sigma, t)|^\gamma\, r^{n-1} \mathrm{d}\sigma\right)^{\frac{\beta}{\gamma}} \mathrm{d}r\right)^{\frac{q}{\beta}} \mathrm{d}t\right)^{\frac{1}{q}} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\|b\|_{L_r^\kappa L_t^q L_\sigma^p((B_{R_0} \setminus B_{R_0/2}) \times I_0)} \\ &:= \left(\int_{R_0/2}^{R_0}\left(\int_{I_0}\left(\int_{S^{d-1}} |b(r\sigma, t)|^p\, r^{n-1} \mathrm{d}\sigma\right)^{\frac{q}{p}} \mathrm{d}t\right)^{\frac{\kappa}{q}} \mathrm{d}r\right)^{\frac{1}{\kappa}} \end{aligned} \tag{2.9}$$

with standard modifications when exponents are equal to infinity.

As a corollary, we have

Corollary 2.4. (FBC in $L_x^p L_t^q$) *Let $p, q \in [1, +\infty]$ and b be as above. If*

$$b \in L_x^p L_t^q((B_{R_0} \setminus B_{R_0/2}) \times I_0), \quad p \leq q, \quad \zeta := \frac{3}{q} + \frac{n-1}{p} < 2, \tag{2.10}$$

then b satisfies (FBC) with $\delta = 1/2$, $N = \varepsilon = 1/4$, and

$$\alpha = \left(\frac{1}{p} - \frac{1}{q} + 1\right)\frac{2}{2-\zeta} \quad M = C\left(R_0^{1-\zeta} \|b\|_{L_x^p L_t^q((B_{R_0} \setminus B_{R_0/2}) \times I_0)}\right)^{2/(2-\zeta)} + \frac{1}{4}. \tag{2.11}$$

By Minkowski's inequality, (2.10) is a special case of (2.6) with $\kappa = p$.

Remark 2.5. Condition (2.4) automatically enforces $q \in (1, +\infty]$ and $\gamma \in \left(\frac{n-1}{2}, +\infty\right]$. Condition (2.6) automatically enforces $q \in \left(\frac{n+2}{2}, +\infty\right]$ and $p \in \left(\frac{n-1}{2}, +\infty\right]$.

Proof of Proposition 2.3. First, we rescale $R_0 = 1$. Let $1/2 \leq \varrho < R \leq 1$ and $I \subset I_0$. All norms below are on $B_R \setminus B_\varrho$ unless stated otherwise.

1. Summary of embeddings for u. By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{L^{p_1}} \lesssim \|u\|_{L^2}^{\theta_1}[(R-\varrho)^{-1}\|u\|_{L^2} + \|\nabla u\|_{L^2}]^{1-\theta_1} \tag{2.12}$$

where $1/p_1 = \theta_1/2 + (1-\theta_1)(1/2_n^*)$ with $\theta_1 \in [0,1]$ in dimension $n \geq 3$ and, in dimension $n = 2$, $\theta_1 \in (0,1]$. Here, $2_n^* = 2n/(n-2)$ is the Sobolev exponent and $2_2^* = +\infty$.

Suppose, momentarily, that $n \geq 3$. Then we have the following Gagliardo-Nirenberg inequality on the spheres ∂B_r, $r \in (\varrho, R)$:

$$\|u\|_{L^2_r L^{p_2}_\sigma} \lesssim \|u\|_{L^2}^{\theta_2}[\|u\|_{L^2} + \|\nabla u\|_{L^2}]^{1-\theta_2} \tag{2.13}$$

where $1/p_2 = \theta_2/2 + (1-\theta_2)(1/2_{n-1}^*)$ with $\theta_2 \in [0,1]$ in dimension $n \geq 4$ and, in dimension $n = 3$, $\theta_2 \in (0,1]$. By interpolation between (2.12) and (2.13), we have

$$\|u\|_{L^{\beta_2}_r L^{\gamma_2}_\sigma} \lesssim \|u\|_{L^2}^{\theta_3}[(R-\varrho)^{-1}\|u\|_{L^2} + \|\nabla u\|_{L^2}]^{1-\theta_3} \tag{2.14}$$

whenever $1/\beta_2 + (n-1)/\gamma_2 = \theta_3(n/2) + (1-\theta_3)(n/2-1)$ and $\theta_3 \in (0,1]$.

We now address dimension $n = 2$. Sobolev embedding on the circle bounds

$$\|u\|_{L^2_r C^{1/2}_\sigma} \lesssim \|u\|_{L^2} + \|\nabla u\|_{L^2}\,, \tag{2.15}$$

and the following Gagliardo-Nirenberg inequality on circles (see Remark 2.6) will be useful:

$$\|u\|_{L^{\gamma_2}(B_r)} \lesssim \|u\|_{L^{p_1}(B_r)}^{\theta}\|u\|_{C^{1/2}(B_r)}^{1-\theta}\,, \tag{2.16}$$

whenever $1/\gamma_2 = \theta/p_1 - (1-\theta)/2$, $\gamma_2, p_1 \in [1,+\infty]$, $\theta \in [0,1]$, and $r \in [1/2, 1]$. We can combine (2.12), (2.15), (2.16), and Hölder's inequality in r to recover (2.14) with the same restrictions on the exponents as mentioned below (2.14).

2. Verifying (FBC) for condition (2.4). For any measurable $A \subset (\varrho, R)$, we have

$$\int_{B_A} |b\cdot n||u|^2\,\mathrm{d}x \leq \|b\|_{L^\beta_r L^\gamma_\sigma}\||u|^2\|_{L^{\beta'}_r L^{\gamma'}_r} \leq \|b\|_{L^\beta_r L^\gamma_\sigma}\|u\|^2_{L^{\beta_2}_r L^{\gamma_2}_r}\,, \tag{2.17}$$

where $B_A = \cup_{r\in A}\partial B_r$, $'$ denotes Hölder conjugate, $\beta_2/2 = \beta'$, and $\gamma_2/2 = \gamma'$. By the assumptions, we have $\beta_2 \leq 2_n^*$ (with $\beta_2 < +\infty$ in dimension $n = 2$) and that (β_2, γ_2) is admissible for the interpolation inequality (2.14). Hence, we have

$$\int_{B_A} |b\cdot n||u|^2\,dx \lesssim \|b\|_{L^\beta_r L^\gamma_\sigma}\|u\|_{L^2}^{2\theta_3}\left[(R-\varrho)^{-2}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\right]^{1-\theta_3}. \tag{2.18}$$

We compute

$$\theta_3 = 1 - \frac{n-1}{2\gamma} - \frac{1}{2\beta} = \frac{2-\zeta+2/q}{2}. \tag{2.19}$$

Subsequently, we define

$$\theta_4 := \theta_3 - \frac{1}{q} = \frac{2-\zeta}{2} \geq 0\,. \tag{2.20}$$

By Hölder's inequality in time, we have, with norms on $(B_R \backslash B_\varrho) \times I$,

$$\begin{aligned} &\iint_{B_A \times I} |b \cdot n||u|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \|b\|_{L^q_t L^\beta_r L^\gamma_\sigma} \|u\|^{2\theta_4}_{L^2} \|u\|^{2(\theta_4-\theta_3)}_{L^\infty_t L^2_x} \left[(R-\varrho)^{-2}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\right]^{1-\theta_3} \\ &\lesssim \|b\|_{L^q_t L^\beta_r L^\gamma_\sigma} \|u\|^{2\theta_4}_{L^2} \left[\|u\|^2_{L^\infty_t L^2_x} + (R-\varrho)^{-2}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\right]^{1-\theta_4}. \end{aligned} \tag{2.21}$$

We can set $A = (\varrho, R)$, divide by $|A|$, and split the product with Young's inequality:

$$\begin{aligned} \frac{1}{|A|}\iint_{B_A \times I} |b \cdot n||u|^2 \,\mathrm{d}x\,\mathrm{d}t &\leq C(R-\varrho)^{-\frac{1}{\theta_4}} \|b\|^{\frac{1}{\theta_4}}_{L^q_t L^\beta_r L^\gamma_\sigma} \|u\|^2_{L^2} \\ &+ \frac{1}{100}\left[\|u\|^2_{L^\infty_t L^2_x} + (R-\varrho)^{-2}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}\right]. \end{aligned} \tag{2.22}$$

This completes the proof.

3. Verifying (FBC) *for condition* (2.6). First, we identify *good slices* for b. Specifically, we apply Chebyshev's inequality in r to the integrable function

$$r \mapsto \left\| b|_{I_0 \times \partial B_r} \right\|^\kappa_{L^q_t L^p_\sigma(I_0 \times \partial B_r)} \tag{2.23}$$

to obtain that, on a set $A = A(\varrho, R, I_0)$ of measure $|A| \geq 99(R-\varrho)/100$, we have

$$\begin{aligned} \|b\|_{L^q_r L^q_t L^p_\sigma(B_A \times I_0)} &\lesssim (R-\varrho)^{\frac{1}{q}} \|b\|_{L^\infty_r L^q_t L^p_\sigma(B_A \times I_0)} \\ &\lesssim (R-\varrho)^{\frac{1}{q}-\frac{1}{\kappa}} \|b\|_{L^\kappa_r L^q_t L^p_\sigma((B_R \backslash B_\varrho) \times I_0)}\,. \end{aligned} \tag{2.24}$$

Now we are in the setting of Step 2 with $\beta = q$, $\gamma = p$, and A already chosen, that is, we use (2.21), (2.22), and (2.24) to conclude. □

Remark 2.6. To prove (2.16), we use local coordinates on the sphere and a partition of unity[4] to reduce to functions f on $\mathbb{R}$. Next, we use that $L^p \subset B^{-\frac{1}{p}}_{\infty,\infty}$, $C^{1/2} = B^{1/2}_{\infty,\infty}$, and real interpolation

$$[B^{-\frac{1}{p}}_{\infty,\infty}, B^{1/2}_{\infty,\infty}]_{\theta,1} = B^0_{q,1} \subset L^q \tag{2.25}$$

to demonstrate

$$\|f\|_{L^q} \lesssim \|f\|^\theta_{L^p(B_r)} \|f\|^{1-\theta}_{C^{1/2}(B_r)} \lesssim \varepsilon^{\frac{1}{\theta}} \|f\|_{L^p(B_r)} + \varepsilon^{-\frac{1}{1-\theta}} \|f\|_{C^{1/2}(B_r)}. \tag{2.26}$$

We piece together u from the functions f and optimize in ε to obtain (2.16).

4 Alternatively, one could argue on the flat torus without a partition of unity.

2.2. Proof of Local Boundedness

To begin, we prove Caccioppoli's inequality:

Lemma 2.7. (Caccioppoli inequality) *Under the hypotheses of Theorem 2.1,*

$$\sup_{t\in(\tau,0)}\int_{B_r}|\theta(x,t)|^2\,\mathrm{d}x+\iint_{Q_{\varrho,\tau}}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t\lesssim_{N,\alpha,\varepsilon}\mathbf{C}\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\theta|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.27}$$

Proof of Lemma 2.7. Let $\eta\in C_0^\infty(T,+\infty)$ satisfying $0\le\eta\le 1$ on $\mathbb{R}$, $\eta\equiv 1$ on $(\tau,+\infty)$ and $0\le \mathrm{d}\eta/\mathrm{d}t\lesssim 1/(\tau-T)$. Let $r\in(\varrho,R)$ and $t\in(\tau,0)$. To begin, we multiply by $\theta\eta^2$ and integrate over $B_r\times(T,t)$:

$$\begin{aligned}\frac{1}{2}\int_{B_r}|\theta(x,t)|^2\,\mathrm{d}x+\iint_{B_r\times(\tau,t)}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s&\le\iint_{B_R\times(T,t)}|\theta|^2\frac{\mathrm{d}\eta}{\mathrm{d}s}\eta\,\mathrm{d}x\,\mathrm{d}s\\&+\iint_{\partial B_r\times(T,t)}\Big(\frac{\mathrm{d}\theta}{\mathrm{d}n}\theta\eta^2-\frac{\theta^2}{2}(b\cdot n)\eta^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned} \tag{2.28}$$

Next, we average in the r variable over the set of 'good slices', $A=A(\varrho,R,(T,t),\theta\eta)$, which was defined in (FBC):

$$\begin{aligned}\frac{1}{2}\int_{B_\varrho}|\theta(x,t)|^2\,\mathrm{d}x+\iint_{B_\varrho\times(\tau,t)}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s&\le\frac{C}{\tau-T}\iint_{B_R\times(T,\tau)}|\theta|^2\,\mathrm{d}x\,\mathrm{d}s\\&+\frac{1}{|A|}\iint_{B_A\times(T,t)}\Big(\frac{\mathrm{d}\theta}{\mathrm{d}n}\theta\eta^2-\frac{\theta^2}{2}(b\cdot n)\eta^2\Big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.29}$$

Here $B_A=\cup_{r\in A}\partial B_r$. Let us estimate the term containing $d\theta/dn$:

$$\begin{aligned}\frac{1}{|A|}\iint_{B_A\times(T,t)}\frac{d\theta}{dn}\theta\eta^2\,\mathrm{d}x\,\mathrm{d}s&\le\frac{1}{\delta^2(R-r)^2}\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\theta|^2\,\mathrm{d}x\,\mathrm{d}s\\&+\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.30}$$

To estimate the term containing b, we use (FBC) with $u=\theta\eta$:

$$\begin{aligned}-\frac{1}{|A|}\iint_{B_A\times(T,t)}\frac{\theta^2}{2}(b\cdot n)\eta^2\,\mathrm{d}x\,\mathrm{d}s&\le\frac{M}{\delta^\alpha(R-r)^\alpha}\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\theta|^2\,\mathrm{d}x\,\mathrm{d}s\\&+N\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}s+\varepsilon\sup_{s\in(T,t)}\int_{B_R}|\theta(x,s)|^2\,\mathrm{d}x.\end{aligned} \tag{2.31}$$

Combining everything and applying $\sup_{t\in(T,0)}$, we obtain

$$\begin{aligned}\frac{1}{2}\sup_{t\in(\tau,0)}\int_{B_\varrho}|\theta(x,t)|^2\,\mathrm{d}x+\iint_{Q_{\varrho,\tau}}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t&\le C\times\mathbf{C}\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\theta|^2\,\mathrm{d}x\,\mathrm{d}t\\&+(1+N)\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\nabla\theta|^2\,\mathrm{d}x\,\mathrm{d}t+\varepsilon\sup_{t\in(T,0)}\int_{B_R}|\theta(x,t)|^2\,\mathrm{d}x.\end{aligned} \tag{2.32}$$

By Widman's hole-filling trick, there exists $\gamma := \max\{(N+1)/(N+2), 2\varepsilon\} \in (0,1)$ satisfying

$$\begin{aligned} &\frac{1}{2(N+2)} \sup_{t\in(\tau,0)} \int_{B_\varrho} |\theta(x,t)|^2\,dx + \iint_{Q_{\varrho,\tau}} |\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le C(N) \times \mathbf{C} \iint_{Q_{R,T}\setminus Q_{\varrho,\tau}} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \gamma \iint_{Q_{R,T}} |\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2(N+2)} \sup_{t\in(T,0)} \int_{B_R} |\theta(x,t)|^2 \,\mathrm{d}x. \end{aligned} \tag{2.33}$$

To remove the extra terms on the right-hand side, we use a standard iteration argument on a sequence of scales (progressing 'outward') $\varrho_0 = \varrho$, $\varrho_{k+1} = \varrho + (1-\lambda^{k+1})(R-\varrho)$, $R_k = \varrho_{k+1}$, $\tau_0 = \tau$, $\tau_{k+1} = \tau + (1-\lambda^{2k+2})(T-\tau)$, $T_k = \tau_{k+1}$, $k = 0, 1, 2, \ldots$, where $0 < \lambda < 1$ is defined by the relation $\lambda^{\max(\alpha,2)} = 2\gamma$. The argument is given in [15, p. 191, Lemma 6.1], for example, and we recall it here. Upon iterating, we have

$$\begin{aligned} &\frac{1}{2(N+2)} \sup_{t\in(\tau_k,0)} \int_{B_\varrho} |\theta(x,t)|^2 \,\mathrm{d}\mathrm{d}x + \iint_{Q_{\varrho,\tau}} |\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le C(N) \sum_{j=0}^{k} \gamma^j\, \mathbf{C}(\varrho_j, \tau_j, R_j, T_j, M, \delta, \alpha) \iint_{Q_{R_j,T_j}\setminus Q_{\varrho_j,\tau_j}} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \gamma^{k+1} \iint_{Q_{R_k,T_k}} |\nabla\theta|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma^{k+1}}{2(N+2)} \sup_{t\in(T_k,0)} \int_{B_{R_k}} |\theta(x,t)|^2 \,\mathrm{d}x. \end{aligned} \tag{2.34}$$

We send $k \to +\infty$ and analyze the sum on the right-hand side. By the definition of $\mathbf{C}$ in (2.1) and choice of ρ_k, τ_k, R_k, T_k, it is estimated above by

$$\begin{aligned} &\Bigg(\frac{1}{\delta^2 (R-\varrho)^2} \sum_{j=0}^{\infty} \frac{\gamma^j}{(\lambda^j - \lambda^{j+1})^2} + \frac{M R_0^\alpha}{\delta^\alpha R_0^2 (R-\varrho)^\alpha} \sum_{j=0}^{\infty} \frac{\gamma^j}{(\lambda^j - \lambda^{j+1})^\alpha} \\ &\quad + \frac{1}{T-\tau} \sum_{j=0}^{\infty} \frac{\gamma^j}{\lambda^{2j} - \lambda^{2j+2}} \Bigg) \times \iint_{Q_{R,T}\setminus Q_{\rho,\tau}} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.35}$$

which is summable provided that $\gamma/\lambda^{\max(\alpha,2)} \le 1/2$. This gives the desired Caccioppoli inequality. □

Next, we require a simple corollary.

Corollary 2.8. *(Interpolation inequality) Let* $\chi = 1 + 2/n$. *Then*

$$\left(\iint_{Q_{\varrho,\tau}} |\theta|^{2\chi} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{\chi}} \lesssim_{N,\alpha,\varepsilon} \mathbf{C} \iint_{Q_{R,T}\setminus Q_{\varrho,\tau}} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.36}$$

Proof. Let $0 \le \varphi \in C_0^\infty(B_{(R+\varrho)/2})$ satisfying $\varphi \equiv 1$ on B_ϱ and $|\nabla\varphi| \lesssim 1/(R-\varrho)$. Using (2.27) at an intermediate scale, we find

$$\sup_{t\in(\tau,0)} \int_{B_\varrho} |\theta(x,t)\varphi|^2 \,\mathrm{d}x + \iint_{Q_{\varrho,\tau}} |\nabla(\theta\varphi)|^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim_{N,\alpha,\varepsilon} \mathbf{C} \iint_{Q_{R,T}\setminus Q_{\varrho,\tau}} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.37}$$

Then (2.36) follows from the Gagliardo-Nirenberg inequality on the whole space.

We are now ready to use Moser's iteration.

Proof of Theorem 2.1 (Local boundedness). Let $\beta_k := \chi^k$, where $k = 0, 1, 2, \ldots$. A standard computation implies that θ^{β_k} is also a non-negative Lipschitz subsolution. Hence, it satisfies the Caccioppoli inequality (2.36) with $R_k = \varrho + 2^{-k}(R-\varrho)$, $r_k = R_{k+1}$, $T_k = \tau - 2^{-2k}(\tau - T)$, $\tau_k = T_{k+1}$, $k = 0, 1, 2, \ldots$ (iterating 'inward'). In other words,

$$\|\theta^{2\beta_{k+1}}\|_{L^1(Q_{\varrho_k,\tau_k})}^{\frac{1}{\chi}} \lesssim_{N,\alpha,\varepsilon} \mathbf{C}(\varrho_k, \tau_k, R_k, T_k, M, \delta, \alpha) \|\theta^{2\beta_k}\|_{L^1(Q_{R_k,T_k}\setminus Q_{\varrho_k,\tau_k})}. \tag{2.38}$$

We may expand the domain of integration on the right-hand side as necessary. Define

$$M_0 := \|\theta^2\|_{L^1(Q_{R_0,T_0}\setminus Q_{\varrho_0,\tau_0})} \tag{2.39}$$

and

$$M_{k+1} := \|\theta^2\|_{L^{\beta_{k+1}}(Q_{\varrho_k,\tau_k})} = \|\theta^{2\beta_{k+1}}\|_{L^1(Q_{\varrho_k,\tau_k})}^{\frac{1}{\beta_{k+1}}}, \quad k = 0, 1, 2, \ldots \tag{2.40}$$

Raising (2.38) to $1/\beta_k$ and using Eq. (2.1) defining $\mathbf{C}$, we obtain

$$M_{k+1} \le C(N,\alpha,\varepsilon)^{\frac{1}{\beta_k}} 2^{\frac{\max(2,\alpha)k}{\beta_k}} \mathbf{C}(\varrho, \tau, R, T, M, \delta, \alpha)^{\frac{1}{\beta_k}} M_k. \tag{2.41}$$

Iterating, we have

$$M_{k+1} \le C(N,\alpha,\varepsilon)^{\sum_{j=0}^k \frac{1}{\chi^j}} 2^{\sum_{j=0}^k \frac{\max(2,\alpha)j}{\chi^j}} \mathbf{C}(\varrho, \tau, R, T, M, \delta, \alpha)^{\sum_{j=0}^k \frac{1}{\chi j}} M_0. \tag{2.42}$$

Finally, we send $k \to +\infty$ and substitute $\sum_{j\ge 0} 1/\chi^j = (n+2)/2$ to obtain

$$\|\theta\|_{L^\infty(Q_{\varrho,\tau})} \lesssim_{N,\alpha,\varepsilon} \mathbf{C}^{\frac{n+2}{4}} \|\theta\|_{L^2(Q_{R,T}\setminus Q_{\varrho,\tau})}. \tag{2.43}$$

We now demonstrate how to replace L^2 on the right-hand side of (2.36) with L^γ ($0 < \gamma < 2$). To begin, use the interpolation inequality $\|\theta\|_{L^2} \le \|\theta\|_{L^\gamma}^{\gamma/2} \|\theta\|_{L^\infty}^{1-\gamma/2}$ in (2.43) and split the product using Young's inequality. This gives

$$\|\theta\|_{L^\infty(Q_{\varrho,\tau})} \le C(N,\alpha,\varepsilon)^{\frac{1}{\gamma}} \mathbf{C}^{\frac{n+2}{2\gamma}} \|\theta\|_{L^\gamma(Q_{R,T}\setminus Q_{\varrho,\tau})} + \frac{1}{2}\|\theta\|_{L^\infty(Q_{R,T})}. \tag{2.44}$$

The second term on the right-hand side is removed by iterating outward along a sequence of scales, as in the proof of the Caccioppoli inequality in Lemma 2.7. □

Remark 2.9. (Elliptic case) The analogous elliptic result is

$$\sup_{B_\varrho} \theta \lesssim_{N,\alpha,\varepsilon,\gamma} \mathbf{C}^{\frac{n}{2\gamma}} \|\theta\|_{L^\gamma(B_R \setminus B_\varrho)}, \tag{2.45}$$

where $\mathbf{C}(\varrho, R, M, \delta, \alpha) = 1/[\delta^2(R-\varrho)^2] + MR_0^{\alpha-2}/[\delta^\alpha(R-\varrho)^\alpha]$. The proof is the same except that $\chi = n/(n-2)$ and $\sum 1/\chi^j = n/2$.

2.3. Proof of Harnack Inequality

In this subsection, θ is a strictly positive Lipschitz solution.[5] Then $\log\theta$ is well defined. Let $0 \le \psi \in C_0^\infty(B)$ be a radially decreasing function satisfying $\psi \equiv 1$ on $B_{3/4}$. We use the notation

$$f_{\text{avg}} = \frac{1}{\text{Vol}} \int_{\mathbb{R}^n} f\psi^2 \,\mathrm{d}x, \quad \text{Vol} = \int_{\mathbb{R}^n} \psi^2 \,\mathrm{d}x, \tag{2.46}$$

whenever $f \in L^1_{\text{loc}}(B)$. Let

$$\mathbf{K} = \exp\left(\log\theta(\cdot, 0)\right)_{\text{avg}}. \tag{2.47}$$

whose importance will be made clear in the proof of Lemma 2.11. Define

$$v = \log\left(\frac{\theta}{\mathbf{K}}\right). \tag{2.48}$$

Then $v(\cdot, 0)_{\text{avg}} = 0$. A simple computation yields

$$|\nabla v|^2 = \partial_t v - \Delta v + b \cdot \nabla v. \tag{2.49}$$

That is, v is itself a supersolution, though it may not itself be positive. We crucially exploit that $|\nabla v|^2$ appears on the left-hand side of (2.49). First, we require the following decomposition of the drift:

Lemma 2.10. *(*Decomposition of drift*) We have the following decomposition on $B_{3/4} \times (-T^*, T^*)$.*

$$b = b_1 + b_2, \quad b_1 = -\operatorname{div} a, \quad \operatorname{div} b_2 = 0, \tag{2.50}$$

Here $a : B_{3/4} \times (-T^*, T^*) \to \mathbb{R}^{n\times n}_{\text{anti}}$ is antisymmetric and

$$\|a\|_{L^2(B_{3/4}\times(-T^*,T^*))} + \|b_2\|_{L^2(B_{3/4}\times(-T^*,T^*))} \lesssim \|b\|_{L^2_t H^{-1}_x(Q^*)}, \tag{2.51}$$

where $Q^* = B_1 \times (-T^*, T^*)$.

[5] There is no loss of generality if we replace θ by $\theta + \kappa$ and let $\kappa \to 0^+$.

Proof. Let $\phi \in C_0^\infty(B_1)$ with $\phi \equiv 1$ on $B_{15/16}$. Let $\tilde{b} = \phi b$. Hence, $\|\tilde{b}\|_{L_t^2 H_x^{-1}(Q^*)} \lesssim \|b\|_{L_t^2 H_x^{-1}(Q^*)}$. We may decompose $\tilde{b}(\cdot, t) \in H^{-1}(\mathbb{R}^n)$ into $\tilde{b}_1(\cdot, t) \in \dot{H}^{-1}(\mathbb{R}^n)$, whose Fourier transform is supported outside of B_2, and $\tilde{b}_2(\cdot, t) \in L^2(\mathbb{R}^n)$. Define

$$a_{ij} = \Delta^{-1}(-\partial_j \tilde{b}_{1i} + \partial_i \tilde{b}_{1j}), \quad g = \Delta^{-1}(-\mathrm{div}\, \tilde{b}_1). \tag{2.52}$$

This amounts to performing the Hodge decomposition in $\mathbb{R}^n$ 'by hand'.[6] Clearly, a is antisymmetric, and we have the decomposition

$$-\tilde{b}_1 = \mathrm{div}\, a + \nabla g \tag{2.53}$$

and the estimates

$$\|a(\cdot, t)\|_{L^2(\mathbb{R}^n)} + \|g(\cdot, t)\|_{L^2(\mathbb{R}^n)} \lesssim \|\tilde{b}_1(\cdot, t)\|_{\dot{H}^{-1}(\mathbb{R}^n)}. \tag{2.54}$$

Similarly, we decompose

$$\tilde{b}_2 = \mathbb{P}\tilde{b}_2 + \mathbb{Q}\tilde{b}_2, \tag{2.55}$$

where $\mathbb{P}$ is the Leray (orthogonal) projector onto divergence-free fields, and $\mathbb{Q} = I - \mathbb{P}$ is the orthogonal projector onto gradient fields. We denote $\mathbb{Q}\tilde{b}_2 = \nabla f$.

Since $\tilde{b} = \tilde{b}_1 + \tilde{b}_2 = -\mathrm{div}\, a + \mathbb{P}\tilde{b}_2 + \nabla(f - g)$ is divergence free in $B_{7/8}$ on time slices, we have

$$\Delta(f - g)(\cdot, t) = 0 \text{ in } B_{7/8}, \tag{2.56}$$

and by elliptic regularity, for all $k \geq 0$,

$$\|\nabla(f - g)(\cdot, t)\|_{H^k(B_{3/4})} \lesssim_k \|\tilde{b}_1(\cdot, t)\|_{\dot{H}^{-1}(B_{7/8})} + \|\tilde{b}_2(\cdot, t)\|_{L^2(B_{7/8})}. \tag{2.57}$$

Finally, we define

$$b_2 = \mathbb{P}\tilde{b}_2 + \nabla(f - g) \in L^2_{t,x}(B_{3/4} \times (-T^*, T^*)), \tag{2.58}$$

which satisfies the claimed estimates and is divergence free in $B_{7/8} \times (-T^*, T^*)$.

We now proceed with the proof of Harnack's inequality.

Lemma 2.11. *For all non-zero* $t \in [-T^*, T^*]$, *we write* $I_t = [0, t]$ *if* $t > 0$ *and* $I_t = [t, 0]$ *if* $t < 0$. *Then*

$$-\mathrm{sgn}(t) v(\cdot, t)_{\mathrm{avg}} + \int_{I_t} \left(|\nabla v|^2(\cdot, s) \right)_{\mathrm{avg}} ds \lesssim |t| + \|b\|^2_{L_t^2 H_x^{-1}(B \times I_t)}. \tag{2.59}$$

[6] We are simply exploiting the identity $\Delta = dd^* + d^*d$ on differential k-forms, up to a sign convention, for differential 1-forms $\cong$ vector fields.

Proof. We multiply (2.49) by ψ^2 and integrate over $B \times I_t$:

$$\begin{aligned} &\operatorname{sgn}(t) \int_B (v(x,0) - v(x,t))\, \psi^2 \, \mathrm{d}x + \iint_{B\times I_t} |\nabla v|^2 \psi^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\leq \iint_{B\times I_t} 2\psi \nabla\psi \cdot \nabla v + (b\cdot\nabla v)\psi^2 \, \mathrm{d}x \, \mathrm{d}s. \end{aligned} \tag{2.60}$$

By (2.47), $\int_{\mathbb{R}^n} v(x,0)\psi^2 \, \mathrm{d}x = 0$. The first term on the right-hand side is easily estimated:

$$\iint_{B\times I_t} 2\psi\nabla\psi \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}s \leq \frac{1}{4} \iint_{B\times I_t} |\nabla v|^2 \psi^2 \, \mathrm{d}x \, \mathrm{d}s + C|t|. \tag{2.61}$$

To estimate the term containing b, we require the drift decomposition $b = b_1 + b_2$ from Lemma 2.10. Then

$$\begin{aligned} \iint_{B\times I_t} (b_1 \cdot \nabla v)\psi^2 \, \mathrm{d}x \, \mathrm{d}s &= \iint_{B\times I_t} 2\psi a(\nabla\psi, \nabla v) \, \mathrm{d}x \, \mathrm{d}s \\ &\leq \frac{1}{4} \iint_{B\times I_t} |\nabla v|^2 \psi^2 \, \mathrm{d}x \, \mathrm{d}s + C\|a\|^2_{L^2(B\times I_t)}. \end{aligned} \tag{2.62}$$

and

$$\iint_{B\times I_t} (b_2 \cdot \nabla v)\psi^2 \, \mathrm{d}x \, \mathrm{d}s \leq \frac{1}{4} \iint_{B\times I_t} |\nabla v|^2 \psi^2 \, \mathrm{d}x \, \mathrm{d}s + C\|b_2\|^2_{L^2(B\times I_t)}. \tag{2.63}$$

Recall the estimate (2.51) from the decomposition. Combining (2.60–2.63) and dividing by Vol gives (2.59).

In that follows, we write $v = v_+ - v_-$, where $v_+, v_- \geq 0$. We also use the notation

$$A^+ = \int_0^{T^*} \|b(\cdot,t)\|^2_{H^{-1}(B)} \, \mathrm{d}t, \quad A^- = \int_{-T^*}^0 \|b(\cdot,t)\|^2_{H^{-1}(B)} \, \mathrm{d}t. \tag{2.64}$$

Lemma 2.12. (Weak–L^1 estimates*) With the above notation, we have*

$$\|v_+\|_{L^{1,\infty}(B_{3/4}\times(-T^*,0))} \lesssim 1 + T^*(T^* + A^-) \tag{2.65}$$

and

$$\|v_-\|_{L^{1,\infty}(B_{3/4}\times(0,T^*))} \lesssim 1 + T^*(T^* + A^+). \tag{2.66}$$

Proof. By (2.59) and a weighted Poincaré inequality [29, Lemma 3, p. 120],

$$-\operatorname{sgn}(t) v_{\mathrm{avg}}(t) + \frac{1}{C_1} \int_{I_t} \left(|v - v_{\mathrm{avg}}|^2 \right)_{\mathrm{avg}} \mathrm{d}s \leq C_0 \int_{I_t} \left(1 + \|b(\cdot,t)\|^2_{H^{-1}(B)} \right) \mathrm{d}s, \tag{2.67}$$

where $C_0 > 0$ is the implied constant in (2.59). In the following, we focus on the case $t \in [-T^*, 0]$. We use (2.67) to obtain a sub/supersolution inequality corresponding to a quadratic ODE. First, we remove the forcing in the ODE by defining

$$p(x,t) := v(x,t) - C_0 \underbrace{\int_{I_t} \left(1 + \|b(\cdot,t)\|^2_{H^{-1}(B)}\right) \mathrm{d}s}_{\leq T^* + A^-} . \tag{2.68}$$

Then (2.67) becomes

$$p_{\mathrm{avg}}(t) + \frac{1}{C_1} \int_{I_t} \left(|p - p_{\mathrm{avg}}|^2\right)_{\mathrm{avg}} \mathrm{d}s \leq 0. \tag{2.69}$$

Let us introduce the super-level sets, whose measures η appear as a coefficient in the ODE:

$$\eta(\mu,t) := \left|\{x \in B_{3/4} : p(x,t) > \mu\}\right|, \quad \mu > 0. \tag{2.70}$$

Since $p_{\mathrm{avg}} \leq 0$, we have that $p(x,t) - p_{\mathrm{avg}}(t) > \mu - p_{\mathrm{avg}}(t) > 0$ whenever $p(x,t) > \mu$. Then

$$p_{\mathrm{avg}}(t) + \frac{1}{C_1 \mathrm{Vol}} \int_{I_t} \eta(\mu,s)(\mu - p_{\mathrm{avg}})^2 \,\mathrm{d}s \leq 0. \tag{2.71}$$

It is convenient to rephrase (2.71) in terms of a positive function evolving forward-in-time: $\bar{p}(t) = -p_{\mathrm{avg}}(-t)$ with $t \in [0,T^*]$. Then (2.71) becomes

$$\bar{p}(t) \geq \frac{1}{C_1 \mathrm{Vol}} \int_0^t \eta(\mu,|s|)(\mu + \bar{p}(s))^2 \,\mathrm{d}s. \tag{2.72}$$

The above inequality means that $\bar{p}$ is a supersolution of the quadratic ODE

$$\dot{q} = \frac{1}{C_1 \mathrm{Vol}} \times \eta(\mu,|t|)(\mu + q)^2 \tag{2.73}$$

with $q(0) = 0$. The above scalar ODE has a comparison principle. *A priori*, since (2.73) is quadratic, its solutions may quickly blow-up depending on the size of $\eta(\mu,\cdot)$ and μ. However, because $\bar{p}$ lies above the solution q, q does not blow up, and we obtain a bound for the density $\eta(\mu,\cdot)$ in the following way. After separating variables in (2.73), we obtain

$$\frac{1}{C_1 \mathrm{Vol}} \int_0^{T^*} \eta(\mu,|s|)\,\mathrm{d}s = \frac{1}{\mu} - \frac{1}{\mu + q(T^*)} \leq \frac{1}{\mu}, \tag{2.74}$$

since $q \geq 0$. That is,

$$\|p_+\|_{L^{1,\infty}(B_{3/4}\times(-T^*,0))} \lesssim 1. \tag{2.75}$$

Finally, since $\|\cdot\|_{L^{1,\infty}(B_{3/4}\times(-T^*,0))}$ is a quasi-norm and $v \leq p + C_0(T^* + A^-)$ pointwise due to (2.68), we have

$$\begin{aligned} \|v_+\|_{L^{1,\infty}(B_{3/4}\times(-T^*,0))} &\leq 2\|p_+\|_{L^{1,\infty}(B_{3/4}\times(-T^*,0))} \\ &\quad + 2C_0\|T^* + A^-\|_{L^{1,\infty}(B_{3/4}\times(-T^*,0))} \\ &\lesssim 1 + T^*(T^* + A^-). \end{aligned} \tag{2.76}$$

The proof for $t \in [0, T^*]$ is similar except that one uses sub-level sets in (2.70) with $\mu < 0$.

We now require the following lemma of Moser [31], which we quote almost directly, and in which we denote by $Q(\varrho)$, $\varrho > 0$ any family of domains satisfying $Q(\varrho) \subset Q(r)$ for $0 < \varrho < r$.

Lemma 2.13. *(Lemma 3 in [31]) Let* m, ζ, c_0, $1/2 \le \theta_0 < 1$ *be positive constants, and let* $w > 0$ *be a continuous function defined in a neighborhood of* $Q(1)$ *for which*

$$\sup_{Q(\varrho)} w^\gamma < \frac{c_0}{(r-\varrho)^m \operatorname{meas}(Q(1))} \iint_{Q(r)} w^\gamma \, \mathrm{d}t \, \mathrm{d}x \tag{2.77}$$

for all ϱ, r, γ *satisfying*

$$\frac{1}{2} \le \theta_0 \le \varrho < r \le 1, \quad 0 < \gamma < \zeta^{-1}. \tag{2.78}$$

Moreover, let

$$\operatorname{meas}\{(x,t) \in Q(1) : \log w > \mu\} < \frac{c_0 \zeta}{\mu} \operatorname{meas}(Q(1)) \tag{2.79}$$

for all $\mu > 0$. *Then there exists a constant function* $q = q(\theta_0, m, c_0)$ *such that*

$$\sup_{Q(\theta_0)} w < q^\zeta. \tag{2.80}$$

Proof of Theorem 2.2 (Harnack inequality). Recall that, without loss of generality, we may assume that θ is strictly positive by considering $\theta + \kappa$ and letting $\kappa \to 0^+$. We apply Lemma 2.13 to $w = \theta/\mathbf{K}$ with $Q(\varrho) = B_\varrho \times (-T^* + 2\ell(1-\varrho), 0)$ and $\theta_0 = 1/2$.[7] Indeed, the requirement (2.77) with $m = (n+2)\max(\alpha, 2)/2$ and $\zeta = 1/2$ follows directly from Theorem 2.1, and we recognize (2.79) as the weak L^1 estimate from Lemma 2.12. This gives

$$\sup_{B_{1/2} \times (-T^* + \ell, 0)} \frac{\theta}{\mathbf{K}} \lesssim 1. \tag{2.81}$$

Here, we suppress also the dependence on the time lag ℓ. Meanwhile, $v_- = \log_+(\mathbf{K}/\theta)$ is a subsolution. Hence,

$$\|v_-\|_{L^\infty(B_{1/2} \times (\ell, T^*))} \lesssim \|v_-\|_{L^{1,\infty}(B_{3/4} \times (0, T^*))}. \tag{2.82}$$

On the other hand,

$$\frac{\mathbf{K}}{\inf \theta} = \sup \frac{\mathbf{K}}{\theta} = \exp\left(\sup \log \frac{\mathbf{K}}{\theta}\right) \le \exp\left(\sup v_-\right) \lesssim \exp\left(\|v_-\|_{L^{1,\infty}}\right) \lesssim 1, \tag{2.83}$$

[7] Technically, to satisfy the conditions in Lemma 2.13, $w = \theta/\mathbf{K}$ should be extended arbitrarily to be continuous in a neighborhood of $Q(1)$.

where the inf and sup are taken on $B_{1/2} \times (\ell, T^*)$. Combining (2.81) and (2.83), we arrive at

$$\sup_{B_{1/2}\times(-T^*+\ell,0)} \theta \lesssim \mathbf{K} \lesssim \inf_{B_{1/2}\times(\ell,T^*)} \theta, \tag{2.84}$$

as desired. □

3. Bounded Total Speed

In this section, we prove the statements in Theorem 1.1 concerning the space $L^1_t L^\infty_x$.

Proposition 3.1. (Local boundedness) *Let* $T \in (-\infty, 0)$ *and* $\tau \in (T, 0)$. *Let* $b\ Q_{1,T} \to \mathbb{R}^n$ *be a smooth divergence-free drift satisfying*

$$\|b\|_{L^1_t L^\infty_x(Q_{1,T})} \le 1/8. \tag{3.1}$$

Let θ *be a non-negative Lipschitz subsolution on* $Q_{1,T}$. *Then, for all* $\gamma \in (0, 2]$, *we have*

$$\|\theta\|_{L^\infty(Q_{1/2,\tau})} \lesssim_\gamma \left(1 + \frac{1}{\tau - T}\right)^{\frac{n+2}{2\gamma}} \|\theta\|_{L^\gamma(Q_{1,T})}. \tag{3.2}$$

Under the assumption $b \in L^1_t L^\infty_x$, it is not evident how to absorb the boundary term $\int |\theta|^2 b \cdot \nabla\varphi\, \varphi\, dx$ in the standard energy estimate due to the presence of $\nabla\varphi$. We require a different strategy.

Proof. For smooth $\lambda : [T, 0] \to (0, +\infty)$, we define $x = \lambda y$ and

$$\tilde\theta(y, t) = \theta(\lambda y, t). \tag{3.3}$$

That is, $\tilde\theta$ is obtained by dynamically rescaling θ in space. The new PDE is

$$\partial_t\tilde\theta - \frac{1}{\lambda^2}\Delta_y\tilde\theta + \frac{1}{\lambda}\tilde b \cdot \nabla_y\tilde\theta \le 0 \tag{3.4}$$

where

$$\tilde b(y, t) = b(\lambda y, t) - \dot\lambda y. \tag{3.5}$$

Choose $\lambda(T) = 1$, $\dot\lambda = -2\|b(\cdot, t)\|_{L^\infty}$ when $t \in [T, 0]$. Clearly, $3/4 \le \lambda \le 1$. Our picture is that $\tilde\theta$ dynamically 'zooms in' on θ. In particular, using (3.1) and (3.5),

$$\tilde b(\cdot, t) \cdot \frac{y}{|y|} \ge -\|b(\cdot, t)\|_{L^\infty} + 2\|b(\cdot, t)\|_{L^\infty}|y| \ge 0 \text{ when } y \in B_1 \setminus B_{1/2}, \tag{3.6}$$

and

$$\operatorname{div} \tilde b = 2n\|b(\cdot, t)\|_{L^\infty} \ge 0. \tag{3.7}$$

We now demonstrate Caccioppoli's inequality in the new variables. Let $3/4 \leq \varrho < R \leq 1$. Let $\varphi \in C_0^\infty(B_R)$ be a radially symmetric and decreasing function satisfying $0 \leq \varphi \leq 1$ on $\mathbb{R}^n$, $\varphi \equiv 1$ on B_ϱ, and $|\nabla\varphi| \lesssim 1/(R-\varrho)$. Let $\Phi = \varphi^2$. We integrate Eq. (3.4) against $\tilde{\theta}\Phi$ on B_R. Then

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\tilde{\theta}|^2(y,t)\Phi\,\mathrm{d}y + \int \lambda^{-2}|\nabla\tilde{\theta}|^2\Phi\,\mathrm{d}y \\
&\quad \leq -\int \lambda^{-2}\tilde{\theta}\nabla\tilde{\theta}\cdot\nabla\Phi\,\mathrm{d}y + \frac{1}{2}\int \lambda^{-1}\underbrace{\tilde{b}\cdot\nabla\Phi}_{\leq 0 \text{ by } (3.6)}|\tilde{\theta}|^2\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int \lambda^{-1}\operatorname{div}\tilde{b}|\tilde{\theta}|^2\Phi\,\mathrm{d}y\,.
\end{aligned}
\tag{3.8}
$$

While $\operatorname{div}\tilde{b}$ has a disadvantageous sign, it acts as a potential in $L^1_t L^\infty_x$. Simple manipulations involving the integral form of Gronwall's inequality give the Caccioppoli inequality:

$$
\begin{aligned}
&\sup_{t\in(\tau,0)}\int_{B_\varrho}|\tilde{\theta}|^2(y,t)\,\mathrm{d}y + \iint_{Q_{\varrho,\tau}}|\nabla\tilde{\theta}|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \lesssim \left(\frac{1}{\tau-T} + \frac{1}{(R-\varrho)^2}\right)\iint_{Q_{R,T}\setminus Q_{\varrho,\tau}}|\tilde{\theta}|^2\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
\tag{3.9}
$$

The remainder of the proof proceeds as in Theorem 2.1 except in the (y,t) variables. Namely, we have the interpolation inequality as in Corollary 2.8, and $\tilde{\theta}^\beta$ is a subsolution of (3.4) whenever $\beta \geq 1$. Therefore, we may perform Moser's iteration verbatim. As in (2.44), the L^2 norm on the right-hand side may be replaced by the L^γ norm. Finally, undoing the transformation yields the inequality (3.2) in the (x,t) variables, since $(y,t) \in B_R \times \{t\}$ corresponds to $(x,t) \in B_{\lambda(t)R} \times \{t\}$.

The quantitative local boundedness property in Theorem 1.1 follows from applying Proposition 3.1 and its rescalings on finitely many small time intervals. In Remark 4.3, we justify that the constant depends on the 'profile' of b and not just its norm.

4. Counterexamples

4.1. Elliptic Counterexamples

Let $n \geq 3$. Our counterexamples will be axisymmetric in 'slab' domains $B_R \times (0,1)$, where $R > 0$ is arbitrary and B_R is a ball in $\mathbb{R}^{n-1}$. We use the notation $x = (x', z)$, where $x' \in \mathbb{R}^{n-1}$, $r = |x'|$, and $z \in (0,1)$. Let

$$\underline{\theta}(x) = u(r)z, \tag{4.1}$$

$$b(x) = V(r)e_z. \tag{4.2}$$

Since b is a shear flow in the e_z direction, it is divergence free. Then

$$-\Delta\underline{\theta} + b\cdot\nabla\underline{\theta} = -z\Delta_{x'}u + Vu. \tag{4.3}$$

We will a construct a subsolution $\underline{\theta}$ and supersolution $\bar{\theta}$ using the *steady Schrödinger equation*

$$-\Delta u + Vu = 0 \tag{4.4}$$

in dimension $n-1$, where additionally $u \geq 0$ and $V \leq 0$. The way to proceed is well known. We define

$$u = \log\log\frac{10R}{r}, \tag{4.5}$$

$$V = \frac{\Delta u}{u}, \tag{4.6}$$

for $r \leq R$. A simple calculation verifies that $0 \leq u \in H^1_{\mathrm{loc}}$, $\Delta u, V \leq 0$, and $\Delta u, V \in L^{(n-1)/2}_{\mathrm{loc}}$.[8] Therefore, $Vu = \Delta u \in L^1_{\mathrm{loc}}$, and the PDE (4.4) is satisfied in the sense of distributions. Using (4.3), we verify that $\underline{\theta}$ is a distributional subsolution:

$$-\Delta\underline{\theta} + b\cdot\nabla\underline{\theta} = -z\Delta_{x'}u + Vu = (1-z)Vu \leq 0 \text{ in } B_R\times(0,1), \tag{4.7}$$

with equality at $\{z = 1\}$. We also wish to control solutions from above. Since $\Delta u \leq 0$, we define

$$\bar{\theta}(x', z) = u(r). \tag{4.8}$$

Clearly, $\underline{\theta} \leq \bar{\theta}$, and $\bar{\theta}$ is a distributional supersolution:

$$-\Delta\bar{\theta} + b\cdot\nabla\bar{\theta} = -\Delta_{x'}u \geq 0 \text{ in } B_R\times(0,1). \tag{4.9}$$

We now construct smooth subsolutions and supersolutions approximating $\underline{\theta}$ and $\bar{\theta}$ according to the above procedure. Let φ be standard mollifier and

$$\varphi_\varepsilon = \frac{1}{\varepsilon^{n-1}}\varphi\left(\frac{\cdot}{\varepsilon}\right). \tag{4.10}$$

Define $u_\varepsilon = \varphi_\varepsilon * u$, $V_\varepsilon = \Delta u_\varepsilon/u_\varepsilon$, $b_\varepsilon = V_\varepsilon(r)e_z$, $\underline{\theta}_\varepsilon = zu_\varepsilon(r)$, and $\bar{\theta}_\varepsilon = u_\varepsilon(r)$. Then $(\underline{\theta}_\varepsilon)$ and $(\bar{\theta}_\varepsilon)$ trap a family (θ_ε) of smooth solutions to the PDEs

$$-\Delta\theta_\varepsilon + b_\varepsilon\cdot\nabla\theta_\varepsilon = 0 \text{ on } B_{R/2}\times(0,1) \tag{4.11}$$

when $\varepsilon \in (0, R/2)$. Moreover, we have the desired estimates

$$\sup_{\varepsilon\in(0,R/2)} \|\theta_\varepsilon\|_{L^p(B_{R/2}\times(0,1))} \leq \|\bar{\theta}\|_{L^p(B_R\times(0,1))} < +\infty, \quad p \in [1,+\infty), \tag{4.12}$$

[8] Since Schrödinger solutions with critical potentials V belong to L^p_{loc} for all $p < \infty$ (see Han and Lin [16], Theorem 4.4), it is natural to choose u with a log. The double log ensures that u has finite energy when $n = 3$. Notice also that $\Delta u = -(r\log r^{-1})^{-2}$ when $n = 3$.

$$\sup_{\varepsilon\in(0,R/2)} \|V_\varepsilon\|_{L^{\frac{n-1}{2}}(B_{R/2})} \lesssim \sup_{\varepsilon} \|\Delta u_\varepsilon\|_{L^{\frac{n-1}{2}}(B_{R/2})} \lesssim \|\Delta u\|_{L^{\frac{n-1}{2}}(B_R)} < +\infty, \tag{4.13}$$

and the singularity, as $\varepsilon \to 0^+$,

$$\sup_{B_{\frac{R}{4}}\times(\frac14,\frac34)} \theta_\varepsilon \geq \sup_{B_{\frac{R}{4}}\times(\frac14,\frac34)} \underline{\theta}_\varepsilon \to +\infty \text{ as } \varepsilon \to 0^+. \tag{4.14}$$

Remark 4.1. (Line singularity) The solutions constructed above are singular on the z-axis, as the maximum principle demands.

Remark 4.2. (Time-dependent examples) The above analysis of unbounded solutions for the steady Schrödinger equation with critical potential is readily adapted to the parabolic PDE $\partial_t u - \Delta u + Vu = f$ in $B_R \times (-T, 0) \subset \mathbb{R}^{n+1}$, $n \geq 2$, (i) with potential V belonging to $L^q_t L^p_x$, $2/q + n/p = 2$, $q > 1$, and zero force, or (ii) with force f belonging to the same space and zero potential. For example, one can define $u = \log\log(-t + r^2)$, $V = -(\partial_t u - \Delta u)/u$, and $f = 0$. The case $q = 1$ is an endpoint case in which solutions remain bounded. These examples are presumably well known, although we do not know a suitable reference.

4.2. Parabolic Counterexamples

Proof of borderline cases: $L^q_t L^p_x$, $\frac{2}{q} + \frac{n}{p} = 2$, $q > 1$. 1. A heat subsolution. Let

$$\Gamma(x,t) = \begin{cases} (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} & t > 0 \\ \delta_0(x) & t = 0 \\ 0 & t < 0 \end{cases} \tag{4.15}$$

be the heat kernel, where δ_0 is the Dirac mass at the origin. Let

$$E(x,t) = (\Gamma - c_n)_+, \tag{4.16}$$

where $c_n = (8\pi)^{-n/2}$. Then E is globally Lipschitz away from $t = 0$, and $E(\cdot, t)$ is supported in the ball $B_{R(t)}$, where

$$R(t)^2 = 2nt \log \frac{2}{t}, \quad t < 2, \tag{4.17}$$

and E vanishes in $t \geq 2$.

2. A steady, compactly supported drift. There exists a divergence-free vector field $U \in C^\infty_0(B_4)$ satisfying

$$U \equiv \vec{e}_1 \text{ when } |x| \leq 2. \tag{4.18}$$

Here is a construction: Let $\phi \in C^\infty_0(B_4)$ be a radially symmetric cut-off function such that $\phi \equiv 1$ on B_3. By applying Bogovskii's operator in the annulus $B_4\backslash\overline{B_2}$, see [14, Theorem III.3.3, p. 179], there exists $W \in C^\infty_0(B_4\backslash\overline{B_2})$ solving

$$\operatorname{div} W = -\operatorname{div}(\phi\vec{e}_1) \in C^\infty_0(B_4 \setminus \overline{B_2}). \tag{4.19}$$

Notably, the property of compact support is preserved. Finally, we define

$$U = \phi \vec{e}_1 + W. \tag{4.20}$$

3. Building blocks. Let $0 \leq S \in C_0^\infty(0,1)$ and $X\ \mathbb{R} \to \mathbb{R}^n$ be the solution of the ODE

$$\dot{X}(t) = S(t)\vec{e}_1, \quad X(0) = -10n\vec{e}_1. \tag{4.21}$$

Define

$$b_S(x,t) = S(t) U\left(\frac{x - X(t)}{R(t)}\right), \tag{4.22}$$

where $R(t)$ was defined in (4.17) above, and

$$E_S(x,t) = E(x - X(t), t). \tag{4.23}$$

Then E_S is a subsolution:

$$\begin{aligned} &(\partial_t - \Delta + b_S \cdot \nabla)\, E_S \\ &\quad = [(\partial_t - \Delta)E](x - X(t), t) + [(b_S - S(t)\vec{e}_1) \cdot \nabla E](x - X(t), t) \\ &\quad = [(\partial_t - \Delta)E](x - X(t), t) \leq 0 \qquad \text{on } \mathbb{R}^n \times (0,1)\,. \end{aligned} \tag{4.24}$$

If $[a, a'] \subset (0,1)$, $S \in C_0^\infty(a, a')$, and $\int S \,\mathrm{d}t \geq 20n$, then we have $E_S(\cdot, t)|_{B_3} \equiv 0$ when $t \leq a$ or $t \geq a'$. Additionally, $E_S(\cdot, \tilde{t}) = E(\cdot, \tilde{t})$ for some $\tilde{t} \in (a, a')$.

We also consider the solution Φ_S to the PDE:

$$\begin{aligned} (\partial_t - \Delta + b_S \cdot \nabla)\, \Phi_S = 0 \quad &\text{on } \mathbb{R}^{n+1} \setminus \{(X(0), 0)\} \\ \Phi_S|_{t=0} = \delta_{x = X(0)}&. \end{aligned} \tag{4.25}$$

For short times $|t| \ll 1$ and negative times is equal to the heat kernel $\Gamma(x - X(0), t)$. By the comparison principle,

$$E_S \leq \Phi_S. \tag{4.26}$$

We have the following measurements on the size of the drift:

$$\|b_S\|^q_{L^q_t L^p_x(\mathbb{R}^{n+1})} \lesssim \int_{\mathbb{R}} S(t)^q R(t)^{\frac{nq}{p}} \,\mathrm{d}t. \tag{4.27}$$

There $1 \leq p, q < +\infty$, and

$$\|b_S\|_{L^1_t L^\infty_x(\mathbb{R}^{n+1})} = \|U\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}} S(t) \,\mathrm{d}t. \tag{4.28}$$

4. Large displacement. For $S_k \in C_0^\infty(t_k, t'_k)$, $k \geq 1$, with $t_k = o_{k \to +\infty}(1)$ and $[t_k, t'_k] \subset (0,1)$ disjoint-in-k, we consider the drifts b_{S_k}. Let $M > 0$. We claim that it is possible to choose S_k satisfying

$$\|b_{S_k}\|_{L^1_t L^\infty_x(\mathbb{R}^{n+1})} = M \tag{4.29}$$

and

$$\left\| \sum_{k\geq 1} b_{S_k} \right\|_{L_t^q L_x^p(\mathbb{R}^{n+1})} < +\infty \tag{4.30}$$

for all $p, q \in [1, +\infty]$ satisfying $2/q + n/p = 2$ and $q > 1$. Indeed, consider

$$\bar{S}(t) = \left(t \log t^{-1} \log\log t^{-1}\right)^{-1} \tag{4.31}$$

when $t \leq c_0$ so that the above expression is well defined, and extended smoothly on $[c_0, 1]$. We ask also that $t_1 \leq c_0$. Since

$$\int_a^{a'} \bar{S}(t)dt = \log\log\log t^{-1}\Big|_a^{a'} \tag{4.32}$$

when $a' \leq c_0$, we have

$$\int_0^1 \bar{S}(t)\,\mathrm{d}t = +\infty, \tag{4.33}$$

whereas

$$\begin{aligned} \int_{t=0}^1 \bar{S}(t)^q R(t)^{\frac{nq}{p}}\,\mathrm{d}t &\leq O(1) + C_n \int_{t=0}^{c_0} (t\log t^{-1})^{-q+\frac{nq}{2p}} (\log\log t^{-1})^{-q}\,\mathrm{d}t \\ &\leq O(1) + C_n \int_{t=0}^{c_0} (t\log t^{-1})^{-1} (\log\log t^{-1})^{-q}\,\mathrm{d}t < +\infty \end{aligned} \tag{4.34}$$

when $q \in (1, +\infty)$. The case $q = +\infty$ is similar. We choose $S_k = \bar{S}(t)\varphi_k$ with suitable smooth cut-offs φ_k to complete the proof of the claim.

5. Unbounded solution. We choose $M = 20n$ and a suitable sequence of S_k as above. We reorder the building blocks we defined above so that the kth subsolution and kth drift are 'activated' on times $(1 - t_k', 1 - t_k)$. Define

$$b_k(\cdot, t) = b_{S_k}(\cdot, t - (1 - t_k') + t_k), \quad b = \sum_{k\geq 1} b_k \tag{4.35}$$

and, for size parameters $A_k \geq 0$,

$$E_k(\cdot, t) = E_{S_k}(\cdot, t - (1 - t_k') + t_k)\mathbf{1}_{(1-t_k', 1-t_k)}, \quad E = \sum_{k\geq 1} A_k E_k. \tag{4.36}$$

Then E is a subsolution of the PDE

$$(\partial_t - \Delta + b \cdot \nabla)E \leq 0 \quad \text{on } B_3 \times (-\infty, 1). \tag{4.37}$$

We further define

$$\Phi_k(\cdot, t) = \Phi_{S_k}(\cdot, t - (1 - t_k') + t_k), \quad \theta = \sum_{k\geq 1} A_k \Phi_k, \tag{4.38}$$

which is a solution of the PDE

$$(\partial_t - \Delta + b \cdot \nabla)\theta = 0 \quad \text{on } B_2 \times (-\infty, 1). \tag{4.39}$$

Since $E_k \leq \Phi_k$, we have that $E \leq \theta$ on $B_2 \times (-\infty, 1)$.

Additionally, we have

$$\sup_t A_k \|E_k(\cdot, t)\|_{L^\infty(B_1)} \geq A_k \|E_k(\cdot, 1 - \tilde{t}_k)\|_{L^\infty(B_1)} \gtrsim A_k {t'_k}^{-n/2}, \tag{4.40}$$

where $\tilde{t}_k \in (t_k, t'_k)$ satisfies $X_{S_k}(\tilde{t}_k) = 0$. Therefore, by the comparison principle and (4.40), we have

$$\limsup_{t \to 1_-} \|\theta(\cdot, t)\|_{L^\infty(B_1)} \gtrsim \limsup_{k \to +\infty} A_k {t'_k}^{-n/2}. \tag{4.41}$$

To control the solution from above, we use

$$\|\theta\|_{L_t^\infty L_x^1(\mathbb{R}^{n+1})} \leq \sum A_k. \tag{4.42}$$

Therefore, it is possible to choose $A_k \to 0$ as $k \to +\infty$ while keeping the lim sup in (4.41) infinite. Hence, by 'pruning' the sequence of A_k (meaning we pass to a subsequence, without relabeling), we can always ensure that $\|\theta\|_{L_t^\infty L_x^1(\mathbb{R}^{n+1})} < +\infty$. □

Remark 4.3. The sequence of solutions $\{\theta_k\}$ above demonstrates that the constant in the quantitative local boundedness property in Theorem 1.1 for drifts $b \in L_t^1 L_x^\infty$ depends on the 'profile' of b rather than just its norm.

Proof of non-borderline cases: $L_x^p L_t^q$, $\frac{3}{q} + \frac{n-1}{p} > 2$, $p \leq q$. This construction exploits rescaled copies of E and is, in a certain sense, self-similar.

1. Building blocks. Let $(t_k) \subset (0, 1)$ be an increasing sequence, with $t_k \to 1$ as $k \to +\infty$. Define $I_k = (t_k, t_{k+1})$, $R_k^2 = |I_k|$.

Let $0 \leq S \in C^\infty(0, 1)$ satisfying $\int_0^1 S(t)\,\mathrm{d}t = M$ with $M = 20n$. Define $X_k\ \mathbb{R} \to \mathbb{R}^n$ to be the solution of the ODE

$$\dot{X}_k(t - t_k) = \underbrace{\frac{1}{|I_k|} S\left(\frac{t}{|I_k|}\right)}_{=:S_k(t-t_k)} \vec{e}_1, \quad X_k(t_k) = -10n\vec{e}_1. \tag{4.43}$$

The 'total speed' has been normalized: $\int |\dot{X}_k|\,\mathrm{d}t = \int S\,\mathrm{d}t = M$. Define also

$$b_k(x, t) = S_k(t) U\left(\frac{x - X_k(t)}{R_k}\right) \tag{4.44}$$

and

$$E_k(x, t) = \frac{1}{R_k^n} E\left(\frac{x - X_k(t)}{R_k}, \frac{t - t_k}{|I_k|}\right). \tag{4.45}$$

Then E_k is a subsolution

$$(\partial_t - \Delta + b_k \cdot \nabla) E_k \le 0 \text{ on } \mathbb{R}^{n+1} \setminus \{(X_k(t_k), t_k)\}, \tag{4.46}$$

and satisfies many of the same properties as E_k in the previous construction, among which is

$$E_k(\cdot, \tilde{t_k}) = \frac{1}{R_k^n} E\left(\frac{\cdot}{R_k}, \tilde{t}\right) \tag{4.47}$$

for some $\tilde{t_k} \in (t_k, t_{k+1})$ and $\tilde{t} \in (0, 1)$. We define the solution θ_k to the PDE:

$$\begin{aligned} (\partial_t - \Delta + b_k \cdot \nabla) \theta_k = 0 \quad \text{on } \mathbb{R}^{n+1} \setminus \{(X_k(t_k), t_k)\} \\ \theta_k|_{t=t_k} = \delta_{x=X_k(t_k)}. \end{aligned} \tag{4.48}$$

For short times $|t - t_k| \ll_k 1$ and times $t < t_k$ that is equal to the heat kernel $\Gamma(x - X_k(t_k), t)$. The comparison principle implies

$$E_k \le \theta_k. \tag{4.49}$$

2. *Estimating the drift*. We now estimate the size of b_k. To begin, we estimate the $L_t^q L_x^p$ norms, $\frac{2}{q} + \frac{n}{p} > 2$. Using the scalings from (4.44), we have

$$\max |b_k| \le \|U\|_{L^\infty} \|S\|_{L^\infty} |I_k|^{-1} \tag{4.50}$$

and

$$\|b_k\|_{L_t^q L_x^p(\mathbb{R}^{n+1})} \lesssim \|U\|_{L^\infty} \|S\|_{L^\infty} |I_k|^{\frac{1}{q}-1} R_k^{\frac{n}{p}} \lesssim R_k^{\varepsilon(p,q)} = o_{k\to+\infty}(1), \tag{4.51}$$

since $|I_k| = R_k^2$. Next, we estimate the $L_{x'}^p L_{x_n}^\infty L_t^{\tilde{q}}$ norm, where $\frac{2}{\tilde{q}} + \frac{n-1}{p} > 2$. We are most interested when $\tilde{q} = +\infty$ and $p = \frac{n-1}{2}-$, but it is not more effort to estimate this. Importantly, we have

$$\operatorname{supp} b_k \subset B_{C_n R_k}^{\mathbb{R}^{n-1}} \times (-C_n, C_n) \times I_k. \tag{4.52}$$

Using this and (4.50), we have

$$\|b_k\|_{L_{x'}^p L_{x_n}^\infty L_t^{\tilde{q}}(\mathbb{R}^{n+1})} \lesssim \|U\|_{L^\infty} \|S\|_{L^\infty} R_k^{\frac{n-1}{p}} |I_k|^{\frac{1}{\tilde{q}}-1} \lesssim R_k^{\varepsilon(p,\tilde{q})} = o_{k\to+\infty}(1). \tag{4.53}$$

Interpolating between (4.51) and (4.53) with $(p, \tilde{q}) = (\frac{n-1}{2}-, \infty)$, we thus obtain

$$\|b_k\|_{L_x^p L_t^q(\mathbb{R}^{n+1})} = o_{k\to+\infty}(1) \tag{4.54}$$

when $\frac{3}{q} + \frac{n-1}{p} > 2$ and $p \le q$. After 'pruning' the sequence in k (meaning we pass to a subsequence, without relabeling), we have

$$\|b\|_{L_t^q L_x^p(\mathbb{R}^{n+1})} \le \sum \|b_k\|_{L_t^q L_x^p(\mathbb{R}^{n+1})} < +\infty, \quad \frac{2}{q} + \frac{n}{p} > 2 \tag{4.55}$$

and

$$\|b\|_{L^p_x L^q_t(\mathbb{R}^{n+1})} \leq \sum \|b_k\|_{L^p_x L^q_t(\mathbb{R}^{n+1})} < +\infty, \quad \frac{3}{q} + \frac{n-1}{p} > 2, \quad p \leq q. \tag{4.56}$$

3. Concluding. The remainder of the proof proceeds as before, with the notable difference that we do not need to reorder the blocks in time. To summarize, we have

$$\sup_t A_k \|E_k(\cdot, t)\|_{L^\infty(B_1)} \geq A_k \|E_k(\cdot, \tilde{t}_k)\|_{L^\infty(B_1)} \gtrsim A_k R_k{}^{-n}, \tag{4.57}$$

where $\tilde{t}_k \in (t_k, t_{k+1})$ satisfies $X_k(\tilde{t}_k) = 0$, and hence,

$$\limsup_{t \to 1_-} \|\theta(\cdot, t)\|_{L^\infty(B_1)} \gtrsim \limsup_{k \to +\infty} A_k R_k{}^{-n} \tag{4.58}$$

To control the solution from above, we again use (4.42) and choose $A_k \to 0$ as $k \to +\infty$ while maintaining that the right-hand side (4.58) is infinite. By again 'pruning' the sequence in k, we have $\|\theta\|_{L^\infty_t L^1_x(\mathbb{R}^{n+1})} < +\infty$. This completes the proof. □

Remark 4.4. (An open question) As mentioned in the introduction, we do not construct counterexamples in the endpoint cases $L^p_x L^q_t$, $\frac{3}{q} + \frac{n-1}{p} = 2$, except when $p = q = \frac{n+2}{2}$ or $(p, q) = (\frac{n-1}{2}, +\infty)$ (steady example constructed above). This seems to suggest, perhaps, that local boundedness should also fail on the line between these two points, but that the counterexamples may be more subtle. It would be interesting to construct these examples. Since each 'block' above is uniformly bounded in the desired spaces, we can say that, if local boundedness were to hold there, it must be depend on the 'profile' of b and not just its norm, as in Remark 4.3.

5. Upper Bounds on Fundamental Solutions

For the Gaussian-like upper bounds on fundamental solutions, we consider the following assumption, which makes sense for $b \in L^1_{\text{loc}}(\mathbb{R}^n \times [0, +\infty))$.

Assumption 5.1. ($\widetilde{\text{FBC}}$) There exist

- Parameters $\theta \in (0, 1]$, $q \in (\frac{1}{\theta}, +\infty]$, $\delta \in [-1, +\infty)$, and
- An upper bound $M_0 \geq 0$,

such that, for all

- Radii $R > 0$ and intervals $I_0 \subset \mathbb{R}_+$,

there exist

- An upper bound $0 \leq M \in L^q(I_0)$, with

$$\|M\|_{L^q(I_0)} \leq M_0\,, \tag{5.1}$$

and

- A measurable set $A = A(b, R, I_0) \subset (R/2, R)$, with $|A| \geq R/4$,

satisfying the following property:

- For all subintervals $I \subset I_0$ and Lipschitz functions $u \in W^{1,\infty}(B_R \times I)$,

$$\int_{B_A} |b \cdot n||u|^2(x,t)\,\mathrm{d}x \leq R^{-\delta} M(t)\|u(\cdot,t)\|^{2\theta}_{L^2(B_R)}(R^{-2}\|u(\cdot,t)\|^2_{L^2(B_R)} + \|\nabla u(\cdot,t)\|^2_{L^2(B_R)})^{1-\theta}, \quad \text{a.e. } t \in I\,. \tag{5.2}$$

Assumption 5.1 is invariant under $b \to -b$ (without changing the parameters) and time translation $b \to b(\cdot, \cdot + T)$ (after suitably shifting I_0); therefore, it will be applicable in the duality argument in Step 3 below.

Theorem 5.2. *If a divergence-free drift $b \in C_0^\infty(\mathbb{R}^n \times [0,+\infty))$ satisfies Assumption 5.1, then the fundamental solution $\Gamma = \Gamma(x,t;0,0)$ to the parabolic operator $L = \partial_t - \Delta + b \cdot \nabla$ with $\Gamma(\cdot,0;0,0) = \delta_0$ satisfies the following estimates:*

I. When $\theta < 1$, we have

$$\Gamma(x,t;0,0) \leq Ct^{-\frac{n}{2}} \max\left[\exp\left(-M_0^{-\frac{1}{1-\theta}} \frac{|x|^{1+\frac{\theta+\delta}{1-\theta}}}{Ct^{\frac{\theta-1/q}{1-\theta}}}\right), \exp\left(-\frac{|x|^2}{Ct}\right)\right], \tag{5.3}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, where $C = C(n,\theta,q) > 0$.

II. When $\theta = 1$, we have

$$\Gamma(x,t;0,0) \lesssim t^{-\frac{n}{2}} \exp\left[-\frac{1}{4Ct}\left(\frac{1}{4}|x| - CM_0|x|^{-\delta} t^{1-\frac{1}{q}}\right)^2\right] \tag{5.4}$$

for all $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, where $C = C(n,q) > 0$.

The pointwise upper bounds in Theorems 1.1 and 1.2 are a consequence of Theorem 5.2 and

Proposition 5.3. *Let $p, q, \beta, \gamma \in [1,+\infty]$, and $\kappa \in (0,+\infty]$. Assume $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times [0,+\infty))$.*

1. If

$$b \in L^q_t L^\beta_r L^\gamma_\sigma(\mathbb{R}^n \times \mathbb{R}_+), \quad \beta \geq \frac{n}{2}, \quad \zeta := \frac{2}{q} + \frac{1}{\beta} + \frac{n-1}{\gamma} < 2, \tag{5.5}$$

then b satisfies Assumption 5.1 with

$$\theta = \frac{2 - \zeta + 2/q}{2}, \quad \delta = 0, \quad M_0 = C\|b(\cdot,t)\|_{L^q_t L^\beta_r L^\gamma_\sigma(\mathbb{R}^n \times \mathbb{R}_+)}\,. \tag{5.6}$$

2. If

$$b \in L^\kappa_r L^q_t L^p_\sigma(\mathbb{R}^n \times \mathbb{R}_+), \quad q \geq \frac{n}{2}, \quad \zeta := \frac{3}{q} + \frac{n-1}{p} < 2, \tag{5.7}$$

then b satisfies Assumption 5.1 with

$$\theta = \frac{2 - \zeta + 2/q}{2}, \quad \delta = \frac{1}{\kappa} - \frac{1}{q}, \quad M_0 = C\|b\|_{L^\kappa_r L^q_t L^p_\sigma(\mathbb{R}^n \times \mathbb{R}_+)}\,. \tag{5.8}$$

Proof. This was verified in the proof of Proposition 2.3. For (5.5), we refer to (2.18–2.19). For (5.7), we refer to (2.24) and the discussion afterward.

As in Corollary 2.4, $b \in L^p_x L^q_t$ with $p \leq q$ is a special case of (5.7) with $\kappa = p$.

We record here two preliminary estimates. Define $\theta_q := \theta - 1/q$. First, we have

$$\int_0^t M(s)^{\frac{1}{\theta}}\, ds \leq M_0^{\frac{1}{\theta}} t^{\frac{\theta_q}{\theta}} . \tag{5.9}$$

Second, if we time-integrate (5.2) and apply Hölder's inequality with $M \in L^q$, we have

$$\begin{aligned} \iint_{B_A} |u|^2 |b \cdot n|(x,t)\, \mathrm{d}x \leq M_0 \|u\|^{2\theta_q}_{L^2(B_R \times I)} (\|u\|^2_{L^\infty_t L^2_x(B_R \times I)} \\ + R^{-2} \|u\|^2_{L^2(B_R \times I)} + \|\nabla u\|^2_{L^2(B_R \times I)})^{1-\theta_q} . \end{aligned} \tag{5.10}$$

If $\theta = 1$, then the $R^{-2}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}$ term can be removed from the right-hand side of (5.10).

An analogous calculation to (5.10) was contained in (2.21).

Proof of Theorem 5.2. Let $x_0 \in \mathbb{R}^n$ and $R := |x_0|$. We set $T = R^2$ when $\theta < 1$ and, otherwise, $T > 0$ is arbitrary. Let $I_0 = (0, T)$. We follow Davies' method [6], with some inspiration from [37]. Let $\psi = \psi(r)$ ($r = |x|$) be a bounded radial Lipschitz function, to be specified, such that $\psi = 0$ when $r < R/2$ and ψ is a constant when $r \geq R$. We record the property $|\nabla \psi| \leq \gamma$, where $\gamma > 0$ is to be optimized.

1. *Weighted energy estimates*. Let $f_0 \in C^\infty_0(\mathbb{R}^n)$ and u be the solution to the equation $Lu = 0$ in $\mathbb{R}^n \times \mathbb{R}_+$ with initial condition $u(\cdot, 0) = e^{-\psi} f_0$. For $t \geq 0$, denote

$$J(t) = \frac{1}{2} \int_{\mathbb{R}^n} e^{2\psi(y)} |u|^2(y,t)\, \mathrm{d}y. \tag{5.11}$$

For $t > 0$, we integrate by parts to compute that

$$\begin{aligned} \dot{J}(t) &= \int_{\mathbb{R}^n} e^{2\psi} u \partial_t u\, \mathrm{d}y = \int_{\mathbb{R}^n} e^{2\psi} u(\Delta u - b \cdot \nabla u)\, \mathrm{d}y \\ &= -\int_{\mathbb{R}^n} e^{2\psi} |\nabla u|^2\, dy - 2\int_{\mathbb{R}^n} e^{2\psi} u \nabla u \cdot \nabla \psi\, \mathrm{d}y - \int_{\mathbb{R}^n} e^{2\psi} b \cdot \nabla \frac{|u|^2}{2}\, \mathrm{d}y \\ &\leq -\frac{1}{2} \int_{\mathbb{R}^n} e^{2\psi} |\nabla u|^2\, \mathrm{d}y + C\gamma^2 J(t) + \int_{\mathbb{R}^n} e^{2\psi} b \cdot \nabla \psi |u|^2\, \mathrm{d}y . \end{aligned}$$

Let $f(\cdot, t) = e^{\psi} u(\cdot, t)$. Next, we integrate in time and use the elementary inequality

$$|\nabla f|^2 \leq 2|\nabla u|^2 e^{2\psi} + 4\gamma^2 |f|^2 \tag{5.12}$$

to obtain

$$\begin{aligned} J(t) &+ \frac{1}{4}\int_0^t \int_{\mathbb{R}^n} |\nabla f|^2 \,\mathrm{d}y \,\mathrm{d}s \\ &\leq J(0) + C\gamma^2 \int_0^t J(s)\, ds + \int_0^t \int_{B_R} b\cdot\nabla\psi |f|^2 \,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{5.13}$$

We choose $\psi'(r) = \gamma \mathbf{1}_A$, where $A = A(b, R, I_0)$ is the set of 'good slices' from Assumption 5.1. Then, for all $t \in (0, T]$, we have

$$\begin{aligned} \int_{B_R} b\cdot\nabla\psi|f|^2\, dy &= \gamma \int_{B_A} b\cdot n|f|^2 \,\mathrm{d}y \\ &\leq C(M(t)\gamma R^{-\delta})^{\frac{1}{\theta}} \|f(\cdot,t)\|^2_{L^2(B_R)} \\ &\quad + \frac{1}{100}\left[T^{-1}\|f(\cdot,t)\|^2_{L^2(B_R)} + \|\nabla f(\cdot,t)\|^2_{L^2(B_R)}\right], \end{aligned} \tag{5.14}$$

when $\theta < 1$. That is, we have applied Young's inequality in (5.2). When $\theta = 1$, Young's inequality is not necessary, and the terms with coefficient $1/100$ on the right-hand side of (5.14) are absent.

We now time-integrate (5.14) and absorb the last terms on its right-hand side into the left-hand side of (5.13). This yields

$$\sup_{s\in(0,t)} J(s) \leq 2J(0) + \int_0^t \left[C\gamma^2 + C(M(s)\gamma R^{-\delta})^{\frac{1}{\theta}}\right] J(s)\,\mathrm{d}s \tag{5.15}$$

for all $t \in (0, T]$. By Grönwall's inequality, we have

$$\begin{aligned} J(t) &\leq 2J(0)\exp\left[C\gamma^2 t + C(\gamma R^{-\delta})^{\frac{1}{\theta}} \int_0^t M(s)^{\frac{1}{\theta}}\,\mathrm{d}s\right] \\ &\overset{(5.9)}{\leq} 2J(0)\exp\left[C\gamma^2 t + C(M_0\gamma R^{-\delta})^{\frac{1}{\theta}} t^{\frac{\theta_q}{\theta}}\right]. \end{aligned} \tag{5.16}$$

2. *Global-in-space Moser iteration*. The goal of this step is to demonstrate that, for all $\tau \in (0, T]$, we have

$$\begin{aligned} &\|f\|_{L^\infty(\mathbb{R}^n\times(\tau/2,\tau))} \\ &\lesssim \left[\gamma^2\tau + (M_0\gamma R^{-\delta})^{\frac{1}{\theta_q}}\tau + 1\right]\tau^{-\frac{n+2}{4}}\|f\|_{L^2(\mathbb{R}^n\times(0,\tau))}. \end{aligned} \tag{5.17}$$

Without loss of generality, $u \geq 0$. Recall that u^p, $p \geq 1$, satisfies

$$\partial_t u^p - \Delta u^p + b\cdot\nabla u^p = -p(p-1)u^{p-2}|\nabla u|^2 \leq 0. \tag{5.18}$$

Let $0 \leq \eta \in C_0^\infty((0, T])$ with $\partial_t\eta \geq 0$. We multiply (5.18) by $u^p e^{2p\psi}\eta^2$ and perform an energy estimate analogous to Step 1. This yields

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x,t)|^{2p}\eta^2(t)\,\mathrm{d}x + \iint_{\mathbb{R}^n\times(0,t)} |\nabla|f|^p|^2\eta^2\,\mathrm{d}x\,\mathrm{d}s \\ &\lesssim \iint_{\mathbb{R}^n\times(0,t)} |f|^{2p}\eta\partial_t\eta\,\mathrm{d}x\,\mathrm{d}s + p\gamma^2 \iint_{\mathbb{R}^n\times(0,t)} |f|^{2p}\eta^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + p\gamma \iint_{B_A\times(0,T)} |b\cdot n||f|^{2p}\eta^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{5.19}$$

The last term above is estimated by applying (5.10), where we substitute $u = f^p\eta$ into the condition. After applying Young's inequality to split the product, we have

$$\begin{aligned}\sup_{t\in(0,T)}\int_{\mathbb{R}^n}|f(x,t)|^{2p}\eta^2(t)\,\mathrm{d}x+\iint_{\mathbb{R}^n\times(0,T)}|\nabla|f|^p|^2\eta^2\,\mathrm{d}x\,\mathrm{d}t\\ \lesssim\iint_{\mathbb{R}^n\times(0,T)}|f|^{2p}\eta\partial_t\eta\,\mathrm{d}x\,dt+\left[p\gamma^2+(pM_0\gamma R^{-\delta})^{\frac{1}{\theta_q}}\right]\\ \iint_{\mathbb{R}^n\times(0,T)}|f|^{2p}\eta^2\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}\tag{5.20}$$

We are now in a position to apply Moser's iteration. This is standard and similar to the proof of Theorem 2.1, so we omit it. The main difference is that the cut-off is only necessary in the time variable. This yields (5.17).

3. Duality. For $0 \le s < t \le T$, we define

$$P^{\psi}_{s\to t}f(x)=e^{\psi(x)}\int_{\mathbb{R}^n}\Gamma(x,t;y,s)e^{-\psi(y)}f(y)\,\mathrm{d}y.\tag{5.21}$$

We now combine the previous two steps. We can estimate

$$\|f\|_{L^2(\mathbb{R}^n\times(0,\tau))}\le\tau^{\frac12}\|f\|_{L^\infty_tL^2_x(\mathbb{R}^n\times(0,\tau))}\tag{5.22}$$

on the right-hand side of the L^∞ estimate (5.17) in Step 2. Of course, the right-hand side of (5.22) is further controlled by the energy estimate (5.16) in Step 1. Since $J(0) = \|f_0\|^2_{L^2}/2$, the above reasoning implies that, for all $t \in (0, T]$, we have

$$\begin{aligned}\|P^{\psi}_{0\to t}\|^2_{L^2\to L^\infty}\lesssim\left[\gamma^2t+(M_0\gamma R^{-\delta})^{\frac{1}{\theta_q}}t+1\right]^2t^{-\frac n2}\\ \times\exp\left[C\gamma^2t+C(M_0\gamma R^{-\delta})^{\frac1\theta}t^{\frac{\theta_q}{\theta}}\right].\end{aligned}\tag{5.23}$$

Since $X^ae^X \lesssim_a e^{2X}$ for any $a, X > 0$, we can incorporate two terms in the algebraic prefactor in (5.23) into the exponential term after increasing its growth rate:

$$\|P^{\psi}_{0\to t}\|^2_{L^2\to L^\infty}\lesssim t^{-\frac n2}\exp\left[C\gamma^2t+C(M_0\gamma R^{-\delta})^{\frac1\theta}t^{\frac{\theta_q}{\theta}}\right].\tag{5.24}$$

By duality, we also have that

$$\|P^{\psi}_{0\to t}\|^2_{L^1\to L^2}\lesssim t^{-\frac n2}\exp\left[C\gamma^2t+C(M_0\gamma R^{-\delta})^{\frac1\theta}t^{\frac{\theta_q}{\theta}}\right].\tag{5.25}$$

Therefore, after a translation in time, we can concatenate the estimates:

$$\begin{aligned}\|P^{\psi}_{0\to t}\|_{L^1\to L^\infty}\le\|P^{\psi}_{0\to t/2}\|_{L^1\to L^2}\|P^{\psi}_{t/2\to t}\|_{L^2\to L^\infty}\\ \lesssim t^{-\frac n2}\exp\left[C\gamma^2t+C(M_0\gamma R^{-\delta})^{\frac1\theta}t^{\frac{\theta_q}{\theta}}\right].\end{aligned}$$

In particular, we plug in the definition (5.21) and see that

$$\Gamma(x_0, t; 0, 0) \lesssim t^{-\frac{n}{2}} \exp\left[-\frac{\gamma|x_0|}{4} + C\gamma^2 t + C(M_0\gamma|x_0|^{-\delta})^{\frac{1}{\theta}} t^{\frac{\theta_q}{\theta}}\right], \quad (5.26)$$

since $\psi(0) = 0$ and $\psi(x_0) \geq \gamma|x_0|/4$.

4. Optimizing γ. The expression inside the exponential is

$$-\frac{1}{4}\gamma|x_0| + \underbrace{C\gamma^2 t}_{A} + \underbrace{C(M_0\gamma|x_0|^{-\delta})^{\frac{1}{\theta}} t^{\frac{\theta_q}{\theta}}}_{B}, \quad (5.27)$$

where we consider $C > 0$ to be fixed.

When $\theta = 1$, we can optimize γ explicitly via

$$2C\gamma t := \frac{1}{4}|x_0| - CM_0|x_0|^{-\delta} t^{1-\frac{1}{q}}, \quad (5.28)$$

and the resulting estimate is

$$\Gamma(x_0, t; 0, 0) \lesssim t^{-\frac{n}{2}} \exp\left[-\frac{1}{4Ct}\left(\frac{1}{4}|x_0| - CM_0|x_0|^{-\delta} t^{1-\frac{1}{q}}\right)^2\right] \quad (5.29)$$

for all $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$.[9]

From now on, suppose $\theta < 1$.

First, we consider scalings of γ in which $-\frac{1}{16}\gamma|x_0|$ overtakes B, namely,

$$(M_0\gamma|x_0|^{-\delta})^{\frac{1}{\theta}} t^{\frac{\theta_q}{\theta}} = \varepsilon\gamma|x_0|. \quad (5.30)$$

Then $\frac{1}{16}\gamma|x_0| \geq B$ when $\varepsilon \leq (C16)^{-1}$ and

$$\gamma^{1-\theta} = \varepsilon^{\theta}|x_0|^{\theta+\delta} M_0^{-1} t^{-\theta_q}. \quad (5.31)$$

With this scaling, we have that $B \geq A$ (or $C\varepsilon\gamma|x_0| \geq C\gamma^2 t$) when

$$\varepsilon^{2\theta-1}|x_0|^{2\theta+\delta-1} t^{1-\theta-\theta_q} \leq M_0. \quad (5.32)$$

In this region, under the additional assumption $t \leq |x_0|^2$, we have the exponential bound

$$\Gamma(x_0, t; 0, 0) \lesssim t^{-\frac{n}{2}} \exp\left(-\varepsilon^{\frac{\theta}{1-\theta}} M_0^{-\frac{1}{1-\theta}} \frac{|x_0|^{1+\frac{\theta+\delta}{1-\theta}}}{8t^{\frac{\theta_q}{1-\theta}}}\right). \quad (5.33)$$

Second, we consider scalings of γ in which $-\frac{1}{16}\gamma|x_0|$ overtakes A. Consider

$$\gamma^2 t = \varepsilon\gamma|x_0|. \quad (5.34)$$

[9] Recall that, when $\theta = 1$, we did not treat $t \geq |x_0|^2$ separately; the purpose of $t \leq |x_0|^2$ was only to allow certain terms of the form $R^{-2}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$ to be absorbed.

Then $\frac{1}{16}\gamma|x_0| \geq A$ when $\varepsilon \leq (C16)^{-1}$ and

$$\gamma = \varepsilon|x_0|t^{-1}\,. \tag{5.35}$$

With this scaling, we have that $A \geq B$ (or $C\varepsilon\gamma|x_0| \geq C(M_0\gamma|x_0|^{-\delta})^{\frac{1}{\theta}} t^{\frac{\theta_q}{\theta}}$) when

$$\varepsilon^{2\theta-1}|x_0|^{2\theta+\delta-1}t^{1-\theta-\theta_q} \geq M_0\,. \tag{5.36}$$

In this region, under the additional assumption $t \leq |x_0|^2$, we have the exponential bound

$$\Gamma(x_0, t; 0, 0) \lesssim t^{-\frac{n}{2}} \exp\left(-\varepsilon\frac{|x_0|^2}{8t}\right)\,. \tag{5.37}$$

We have demonstrated that, when $t \leq |x_0|^2$,

$$\Gamma(x_0, t; 0, 0) \lesssim t^{-\frac{n}{2}} \max\left[\exp\left(-\varepsilon^{\frac{\theta}{1-\theta}} M_0^{-\frac{1}{1-\theta}} \frac{|x_0|^{1+\frac{\theta+\delta}{1-\theta}}}{8t^{\frac{\theta_q}{1-\theta}}}\right), \exp\left(-\varepsilon\frac{|x_0|^2}{8t}\right)\right]\,. \tag{5.38}$$

Indeed, this is equivalent to (5.33) in the region (5.32) and (5.37) in the region (5.36) both holding. When $t \geq |x_0|^2$, the fundamental solution is controlled by the Nash estimate (1.27). Therefore, up to modifying the implicit constant in the symbol $\lesssim$, (5.38) remains true for arbitrary $x_0 \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. □

5.1. Examples

Let $l \in (1, +\infty]$. We consider power-law speeds

$$S(t) = \frac{M_0}{1 - {}^1/_l} t^{-\frac{1}{l}}\,, \tag{5.39}$$

which belong to the weak Lebesgue space $L_t^{l,\infty}$. For $b(x,t) = S(t)\vec{e}_n$, we have that the fundamental solution $\Gamma(x, t; 0, 0)$ is the translation of the heat kernel

$$\frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x - X(t)\vec{e}_n|^2}{4t}\right)\,, \tag{5.40}$$

where

$$X(t) = \int_0^t S(t)\,dt = M_0 t^{1-\frac{1}{l}}\,. \tag{5.41}$$

In particular, when $x = x_n\vec{e}_n$, the term inside the exponential is

$$-\frac{|x_n - M_0 t^{1-\frac{1}{l}}|^2}{4t}\,. \tag{5.42}$$

When $x_n \geq M_0 t^{1-\frac{1}{l}}$, this matches, up to prefactors, the upper bound in (1.14) in Theorem 1.1 for $L_t^l L_x^\infty$ drifts. In particular, we see that the power $1 - 1/q$ of t in the exponential in (1.14) cannot be increased.

We now generalize the above example. Namely, we identify an 'inner region' inside of which the upper bound of $Ct^{-n/2}$ cannot be improved.

Let $\mathbf{X}$ be a Banach space of distributional vector fields on $\mathbb{R}^n \times \mathbb{R}_+$. Let $\mathbf{Y}[\mathbf{X}] = \{b \in \mathbf{X} : \|b\|_{\mathbf{X}} \leq 1\} \cap \{b \in C_0^\infty(\mathbb{R}^n \times [0, +\infty)) : \operatorname{div} b = 0\}$. For given $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, we define

$$\Gamma_{\mathbf{X}}(x, t; 0, 0) := \sup_{b \in Y[X]} \Gamma[b](x, t; 0, 0), \tag{5.43}$$

where $\Gamma[b]$ is the fundamental solution to $\partial_t - \Delta + b \cdot \nabla$. That is, $\Gamma_{\mathbf{X}}$ is the optimal upper bound for fundamental solutions with divergence-free drifts $b \in C_0^\infty(\mathbb{R}^n \times [0, +\infty))$ belonging to the closed unit ball in $\mathbf{X}$.

Proposition 5.4. (Lower bounds on the optimal upper bound) *Let $p, q \in [1, +\infty]$. Consider*

$$\mathbf{X} = L_t^q L_x^p(\mathbb{R}^n \times \mathbb{R}_+) \text{ with } 1 \leq \zeta = \frac{2}{q} + \frac{n}{p} < 2 \tag{5.44}$$

or

$$\mathbf{X} = L_x^p L_t^q(\mathbb{R}^n \times \mathbb{R}_+) \text{ with } q \geq p \text{ and } 1 \leq \zeta = \frac{3}{q} + \frac{n-1}{p} < 2\,. \tag{5.45}$$

Let

$$l_0 = 2/\zeta > 1\,. \tag{5.46}$$

Then, for all $l > l_0$, there exists a constant $C = C(n, p, q, l_0, l) > 0$ such that

$$\Gamma_{\mathbf{X}}(x, t; 0, 0) \geq C^{-1} t^{-\frac{n}{2}}, \quad \forall t \in (0, 1], \ |x| \leq C^{-1} t^{1-\frac{1}{l}}\,. \tag{5.47}$$

Remark 5.5. When we substitute $|x| \leq t^{1-\frac{1}{l_0}}$ into the upper bounds in Theorems 1.1 and 1.2, we have that $\Gamma_{\mathbf{X}}(x, t; 0, 0) \lesssim t^{-\frac{n}{2}}$. Therefore, the lower bound (5.47) establishes the optimality of the upper bound in that region.

Proof of Proposition 5.4. By rotational invariance, it suffices to consider $x = x_n \vec{e}_n$ with $x_n \geq 0$. Let $l > l_0$. We use the subsolutions E_S with drift b_S we constructed in *3. Building blocks* in Sect. 4.2, except that now $X(0) = 0$ and $\dot{X} = S(t)\vec{e}_n$. We choose $S(t)$ to be the power law speed from (5.39) with $l_1 = (l + l_0)/2$ replacing l and $M_0 \ll 1$. We can approximate S by $S_k \in C_0^\infty(0, 1)$ to justify that $E_S(x, t) \leq \Gamma_{\mathbf{X}}(x, t; 0, 0)$ provided that $\|b_S\|_{\mathbf{X}} \leq 1$.

To estimate the $\mathbf{X}$ norm of b_S, we have

$$\begin{aligned} \|b_S\|^{q_1}_{L_t^{q_1} L_x^{p_1}(\mathbb{R}^n \times (0,1))} &\lesssim M_0^{q_1} \int_0^1 R(t)^{\frac{nq_1}{p_1}} S(t)_1^q \, dt \\ &\lesssim M_0^{q_1} \int_0^1 (t \log t^{-1})^{\frac{nq_1}{2p_1}} t^{-\frac{q_1}{l_1}} \, dt \lesssim M_0^{q_1} \end{aligned} \tag{5.48}$$

provided that $2/q_1 + n/p_1 > 2/l_1$, with an appropriate adjustment when $q_1 = +\infty$. Next, we estimate the $L_{x'}^{p_2} L_{x_n}^\infty L_t^{q_2}$ norm, where $2/q_2 + (n-1)/p_2 > 2/l_1$. We do this by estimating $b_k(x,t) = b_S(x,t)\mathbf{1}_{I_k}(t)$, $k \geq 1$, where $I_k = (2^{-k}, 2^{1-k})$. Define $R_k = R(2^{1-k})$. Then

$$\max |b_k| \lesssim M_0 2^{\frac{k}{l_1}} \|U\|_{L^\infty}, \tag{5.49}$$

$$\operatorname{supp} b_k \subset B^{\mathbb{R}^{n-1}}_{C_n R_k} \times (-C_n, C_n) \times I_k. \tag{5.50}$$

Hence, we have

$$\begin{aligned} \|b_S\|_{L^{p_2}_{x'} L^\infty_{x_n} L^{q_2}_t(\mathbb{R}^{n+1})} &\leq M_0 \|U\|_{L^\infty} \sum_{k\geq 1} 2^{\frac{k}{l_1}} R_k^{\frac{n-1}{p_2}} |I_k|^{\frac{1}{q_2}} \\ &\lesssim M_0 \sum_{k \geq 1} 2^{\frac{k}{l_1}} (k 2^{-k})^{\frac{n-1}{2p_2}} 2^{-\frac{k}{q_2}} \lesssim M_0, \end{aligned}$$

provided that $2/q_2 + (n-1)/p_2 > 2/l_1$, as desired.

Hence, we find that, for all M_0 sufficiently small, the drift b_S defined above satisfies $\|b_S\|_{\mathbf{X}} \leq 1$. Since $X(t) = M_0 t^{1-\frac{1}{l_1}}$, we have that $E_S(M_0 t^{1-\frac{1}{l_1}} \vec{e}_n, t) \gtrsim t^{-n/2}$ for $t \leq 1$. Finally, we use that $M_0 \ll 1$ was arbitrary. This completes the proof. □

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Declarations

Conflict of interest Since December 2023, DA has been co-organizing a conference with the handling editor, László Székelyhidi Jr. This manuscript was originally submitted in August 2021, when the handling editor was assigned, and revised in August 2022 and April 2023. We have no further conflicts of interest to report.

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