Optimal estimates for transmission problems including relative conductivities with different signs

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A R T I C L E   I N F O

A B S T R A C T

We study the gradient and higher order derivative estimates for the transmission problem in the presence of closely located inclusions. We show that in two dimensions, when relative conductivities of circular inclusions have different signs, the gradient and higher order derivatives are bounded independent of ε, the distance between the inclusions. We also show that for general smooth strictly convex inclusions, when one inclusion is an insulator and the other one is a perfect conductor, the derivatives of any order is bounded independent of ε in any dimensions $n \geq 2$.

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1. Introduction and main result

In this paper, we first study the second-order elliptic equations in divergence form with discontinuous coefficients in two dimensions

$$L_{\varepsilon;r_1,r_2}u := D_i(a(x)D_iu) = D_if_i \quad \text{in} \ \ \mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is a bounded open subset of $\mathbb{R}^2$,

$$a(x) = k_1\chi_{\mathcal{B}_1} + k_2\chi_{\mathcal{B}_2} + k_0\chi_{\mathcal{B}_0},$$

$k_0 = 1$, $k_1, k_2, r_1, r_2 \in (0, \infty)$ are constants,

$$\mathcal{B}_1 := B_{r_1}(\varepsilon/2 + r_1, 0), \quad \mathcal{B}_2 := B_{r_2}(-\varepsilon/2 - r_2, 0), \quad \mathcal{B}_0 := \mathbb{R}^2 \setminus (\overline{\mathcal{B}_1 \cup \mathcal{B}_2}),$$

and χ is the indicator function. The equation models the conductivity problem in composite material. The gradient of the voltage potential u represents the electric field, and $a(x)$ is the conductivity which is a constant on each inclusion, and a different constant on the background matrix. It is significant from an engineering point of view to estimate the derivatives of the solutions.

In [6], Babuška et al. analyzed an analogous elliptic system, and numerically showed that, when the ellipticity constants are away from 0 and infinity, the gradient of solutions remains bounded independent of ε, the distance between inclusions. When $\varepsilon = 0$, Bonnetier and Vogelius [14] proved that $|Du|$ is bounded for a fixed $k = k_1 = k_2$ away from 0 and infinity. This result was extended by Li and Vogelius [32] to general second order elliptic equations in divergence form with piecewise Hölder coefficients and general shape of inclusions in any dimension. Furthermore, they established a stronger piecewise $C^{1,\alpha}$ control of u, which is independent of ε. Li and Nirenberg [31] further extended this global Lipschitz and piecewise $C^{1,\alpha}$ result to general second order elliptic systems in divergence form, including the linear system of elasticity. Some higher order derivative estimates in dimension $n = 2$ were obtained in [17,20,22].

On the other hand, if k_1, k_2 are allowed to be 0 or ∞, it was shown in [15,26,36] that the gradient of solutions generally becomes unbounded as $\varepsilon \to 0$. For the perfect conductivity problem ($k_1 = k_2 = \infty$), it has been proved that the generic blow-up rate of $|Du|$ is $\varepsilon^{-1/2}$ in two dimensions, $|\varepsilon \log \varepsilon|^{-1}$ in three dimensions, and ε^{-1} in dimensions greater than three; see [4,5,8,9,38,39]. These bounds were shown to be optimal and are independent of the shape of inclusions, as long as the inclusions are relatively strictly convex. Moreover, more detailed characterizations of the singular behavior of ∇u have been obtained. For further works on the perfect conductivity problem and closely related works, see e.g. [1–3,10–13,16,17,20,21,23–25,27–30,35] and the references therein.

For the insulated conductivity problem ($k_1 = k_2 = 0$), it was shown in [4,5] that the optimal blow-up rate is $\varepsilon^{-1/2}$ in two dimensions. The proof uses a harmonic conjugate

argument to link the insulated conductivity case to the perfect conductivity case, which fails in dimensions greater than two. For the higher dimensional case, Bao, Li, and Yin in [9] established an upper bound of order $\varepsilon^{-1/2}$. Yun in [40] proved the optimal blow-up rate on the shortest line segment connecting two spherical inclusions in three dimensions is $\varepsilon^{\frac{\sqrt{2}-2}{2}}$. Later, the upper bound $\varepsilon^{-1/2}$ was improved by Li and Yang in [34] to be $\varepsilon^{-1/2+\beta}$, for some $\beta > 0$. See also [33]. Weinkove in [37] used a Bernstein-type argument to obtain a more explicit upper bound of β in dimensions greater than three. Dong, Li, and Yang in recent works [18,19] identified the optimal blow-up rate. They proved the optimal gradient estimate for a class of inclusions including balls and "almost" optimal gradient estimate for general strictly convex inclusions. Unlike the perfect conductivity case, the optimal blow-up rate is related to the principal curvature of the inclusions.

Recently, among other results, Ji and Kang in [22] used some spectral properties of Neumann-Poincaré operators to study the problem (1.1) for the case when $0 < k_1 < 1$ and $k_2 > 1$, and proved that

$$|D^m u| \leq C \left(-\frac{(k_1+1)(k_2+1)}{(k_1-1)(k_2-1)} - 1 + \sqrt{\frac{2(r_1+r_2)\varepsilon}{r_1 r_2}} \right)^{-m+1}, \quad m = 1, 2, \ldots.$$

In particular, when $k_1 \to 0$ and $k_2 \to \infty$, this implies

$$|D^m u| \leq C\varepsilon^{-(m-1)/2}, \quad m = 1, 2, \ldots.$$

In the first part of this paper, we apply the Green function method developed in [17] to show that $|D^m u|$ is in fact bounded independent of ε for any $m = 1, 2, \ldots$. Compared to the method in [22], we use m-th order finite differences to estimate the m-th order derivatives more precisely.

Let

$$\alpha = \frac{k_1 - 1}{k_1 + 1}, \quad \beta = \frac{k_2 - 1}{k_2 + 1}, \quad \text{and} \quad \gamma = -\alpha\beta.$$

In particular, we are interested in the case when $k_1 \to 0, k_2 \to +\infty$ (or $\alpha \to -1, \beta \to 1$). Therefore, we may restrict $\gamma > \frac{1}{2}$. First we assume that $r_1 = r_2 = 1$.

Theorem 1.1. *Let $\varepsilon \in (0, 1/2)$ and $\mu \in (0, 1)$ be constants. Assume that u is a weak solution of (1.1) in $B_1 := B_1(0)$ with $r_1 = r_2 = 1$, $k_1 \in (0, 1)$, $k_2 \in (1, \infty)$, and $\gamma \in (1/2, 1)$. For any $m \in \mathbb{N}$, if f is piecewise $C^{2m-1,\mu}$ in B_1, and for some constant $C_m > 0$,*

$$\|u\|_{L^2(B_1)} \leq C_m, \quad \|f\|_{C^{2m-1,\mu}(B_1 \cap \mathcal{B}_j)} \leq C_m k_j, \quad j = 0, 1, 2,$$

then we have

$$|D^m u(x)| \leq C C_m \quad in \ \ B_{1/2}, \tag{1.2}$$

where $C > 0$ is a positive constant depending only on m and μ, and in particular is independent of ε, k_1, and k_2.

When u satisfies (1.1) in a domain that contains $\mathcal{B}_1$ and $\mathcal{B}_2$, we have more precise estimates as follows.

Theorem 1.2. Let $\varepsilon \in (0, 1/2)$ and $\mu \in (0, 1)$ be constants. Assume that $\mathcal{B}_1 \cup \mathcal{B}_2 \Subset \mathcal{D}_1 \Subset \mathcal{D}$ for some domain $\mathcal{D}_1$, u is a weak solution of (1.1) in $\mathcal{D}$ with $r_1 = r_2 = 1$, $0 < k_1 < 1$, $k_2 > 1$, and $\gamma \in (1/2, 1)$. For any $m \in \mathbb{N}$, if f is piecewise $C^{2m-1,\mu}$ in $\mathcal{D}$, and for some constant $C_m > 0$,

$$\|u\|_{L^2(\mathcal{D})} \leq C_m, \quad \|f\|_{C^{2m-1,\mu}(B_1 \cap B_j)} \leq C_m \min\{1, k_j\}, \quad j = 0, 1, 2,$$

then we have

$$|D^m u(x)| \leq \begin{cases} C C_m & in \ \ \mathcal{D}_1 \cap \mathcal{B}_0, \\ \dfrac{C C_m}{k_1 + 1} & in \ \ \mathcal{B}_1, \\ \dfrac{C C_m}{k_2 + 1} & in \ \ \mathcal{B}_2, \end{cases}$$

where $C > 0$ is a constant depending only on m, μ, $\mathcal{D}_1$, and $\mathcal{D}$.

For the general case when r_1 and r_2 are not necessarily equal to 1, we have the following theorem.

Theorem 1.3. Let $\varepsilon \in (0, 1/2)$, $\mu \in (0, 1)$ be constants, and $1/2 < r_1, r_2 < 10$. Then there exist domains $\mathcal{D}_1 \Subset \mathcal{D}$ that depend on r_1, r_2, such that if $\mathcal{B}_1 \cup \mathcal{B}_2 \Subset \mathcal{D}_1$, and u is a weak solution of (1.1) in $\mathcal{D}$ with $0 < k_1 < 1$, $k_2 > 1$, and $\gamma \in (1/2, 1)$, we have, for any $m \in \mathbb{N}$, if f is piecewise $C^{2m-1,\mu}$ in $\mathcal{D}$, and for some constant $C_m > 0$,

$$\|u\|_{L^2(\mathcal{D})} \leq C_m, \quad \|f\|_{C^{2m-1,\mu}(B_1 \cap B_j)} \leq C_m \min\{1, k_j\}, \quad j = 0, 1, 2,$$

then

$$|D^m u(x)| \leq \begin{cases} C C_m & in \ \ \mathcal{D}_1 \cap \mathcal{B}_0, \\ \dfrac{C C_m}{k_1 + 1} & in \ \ \mathcal{B}_1, \\ \dfrac{C C_m}{k_2 + 1} & in \ \ \mathcal{B}_2, \end{cases}$$

where $C > 0$ is a constant depending only on m, μ, r_1, and r_2.

In the final part of this paper, we partially answer a question raised by Kang in [23], where in the conclusion section, he mentioned that the extensions to general shape of inclusions and higher dimensions for the case $(k_1 - 1)(k_2 - 1) < 0$ are quite challenging. We prove the derivatives estimates for the extreme case when $k_1 = 0$, $k_2 = \infty$, and $f = 0$ for general strictly convex inclusions in dimensions $n \geq 2$.

The setting of this problem is as follows. Let $\mathcal{D} \subset \mathbb{R}^n$ be a bounded domain containing two smooth relatively strictly convex open sets $\mathcal{B}_1$ and $\mathcal{B}_2$ so that $\mathcal{B}_1 \cup \mathcal{B}_2 \Subset \mathcal{D}$, $\mathrm{dist}(\mathcal{B}_1 \cap \mathcal{B}_2) = \varepsilon$, and $\mathrm{dist}(\mathcal{B}_1 \cup \mathcal{B}_2, \mathcal{D}) > 0$. It is known that when $k_1 = 0$, $k_2 = \infty$, and $f = 0$, (1.1) is reduced to

$$
\begin{cases}
\Delta u = 0 & \text{in } \mathcal{D} \setminus \overline{(\mathcal{B}_1 \cup \mathcal{B}_2)}, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathcal{B}_1, \\
u = C \text{ (Constant)} & \text{on } \partial \mathcal{B}_2, \\
\int_{\partial \mathcal{B}_2} \frac{\partial u}{\partial \nu} = 0. &
\end{cases}
$$

We use the notation $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$. After choosing a coordinate system properly, we can assume that near the origin, the part of $\partial \mathcal{B}_1$ and $\partial \mathcal{B}_2$, denoted by Γ_+ and Γ_-, are respectively the graphs of two smooth functions in terms of x'. That is,

$$
\Gamma_+ = \left\{ x_n = \frac{\varepsilon}{2} + h_1(x'), \ |x'| < 1 \right\} \quad \text{and} \quad \Gamma_- = \left\{ x_n = -\frac{\varepsilon}{2} + h_2(x'), \ |x'| < 1 \right\},
$$

where h_1 and h_2 are sufficiently smooth functions satisfying

$$
h_1(x') > h_2(x') \quad \text{for } 0 < |x'| < 1,
$$

$$
h_1(0') = h_2(0') = 0, \quad D_{x'} h_1(0') = D_{x'} h_2(0') = 0, \quad D^2(h_1 - h_2)(0') > 0.
$$

For $0 < r \leq 1$, we denote

$$
\Omega_r := \left\{ (x', x_n) \in \mathcal{D} \setminus \overline{(\mathcal{B}_1 \cup \mathcal{B}_2)} \ \middle| \ -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), \ |x'| < r \right\}.
$$

We focus on the following problem near the origin:

$$
\begin{cases}
-D_i(a^{ij}(x) D_j u(x)) = 0 & \text{in } \Omega_1, \\
a^{ij}(x) D_j u(x) \nu_i = 0 & \text{on } \Gamma_+, \\
u = C \text{ (Constant)} & \text{on } \Gamma_-,
\end{cases}
\tag{1.3}
$$

where $(a^{ij}(x))$ satisfies, for some constants $\sigma \in (0, 1)$ and any $x \in \Omega_1, \xi \in \mathbb{R}^n$,

$$
\sigma |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j, \quad |a^{ij}(x)| \leq \frac{1}{\sigma}.
$$

Theorem 1.4. *Assume the above and let $u \in H^1(\Omega_1)$ be a weak solution of (1.3). For $m \in \mathbb{N}$, some constants $\alpha \in (0,1)$, and $C_{m,\alpha} > 0$, if*

$$\|a\|_{C^{m-1,\alpha}(\Omega_1)} + \|h_1\|_{C^{m,\alpha}(\{|x'|<1\})} + \|h_2\|_{C^{m,\alpha}(\{|x'|<1\})} \le C_{m,\alpha},$$

then there exist constants $\mu \in (0,1)$ and C, depending only on n, σ, m, α, and $C_{m,\alpha}$ such that

$$|D^m u(x)| \le C \mu^{\frac{1}{\sqrt{\varepsilon}+|x'|}} \|u\|_{L^2(\Omega_1)} \quad for \ \ x \in \Omega_{1/2}.$$

The rest of this paper is organized as follows. First, we review the Green function of the operator $L_{\varepsilon;1,1}$ constructed in [17], and derive some preliminary estimates in Section 2. In Section 3, we prove Theorem 1.1 with $m = 1$ to illustrate the main idea without getting into too much technicalities. Then we prove Theorem 1.1 with general $m \in \mathbb{N}$ in Section 4, and Theorem 1.2 in Section 5. In Section 6, we prove Theorem 1.3 by introducing a conformal map to reduce the problem to the case considered in Theorem 1.2. Finally, Theorem 1.4 is proved in Section 7.

2. Preliminary

In this section, we first review the Green function of the operator $L_{\varepsilon;1,1}$ constructed in [17], and then derive some preliminary estimates.

Let $\Phi_1(x), \Phi_2(x)$ denote the inversion maps of a point $x \in \mathbb{R}^2$ with respect to $\partial \mathcal{B}_1$ and $\partial \mathcal{B}_2$, respectively, that is

$$\Phi_1(x_1, x_2) := \left(\frac{x_1 - (1 + \varepsilon/2)}{(x_1 - 1 - \varepsilon/2)^2 + x_2^2} + 1 + \varepsilon/2, \frac{x_2}{(x_1 - 1 - \varepsilon/2)^2 + x_2^2} \right)$$

and

$$\Phi_2(x_1, x_2) := \left(\frac{x_1 + 1 + \varepsilon/2}{(x_1 + 1 + \varepsilon/2)^2 + x_2^2} - 1 - \varepsilon/2, \frac{x_2}{(x_1 + 1 + \varepsilon/2)^2 + x_2^2} \right).$$

The auxiliary function $\mathcal{G}(x, y)$ is given as follows:

(1) When $y \in \mathcal{B}_0$, $\mathcal{G}(x, y)$ equals

$$\frac{2}{k_1 + 1} \sum_{k=0}^{\infty} (\alpha\beta)^k \left(\log |(\Phi_1 \Phi_2)^k(x) - y| - \beta \log |(\Phi_2 \Phi_1)^k \Phi_2(x) - y| \right)$$

$$\text{for} \ \ x \in \overline{\mathcal{B}}_1;$$

$$\log |x - y| + \sum_{k=0}^{\infty} \left[(\alpha\beta)^k \left(\log |(\Phi_1 \Phi_2)^k(x) - y| + \log |(\Phi_2 \Phi_1)^k(x) - y| \right) \right.$$

$$- (\alpha\beta)^{k-1} \Big(\beta \log |(\Phi_2\Phi_1)^k \Phi_2(x) - y| + \alpha \log |(\Phi_1\Phi_2)^k \Phi_1(x) - y| \Big) \Big]$$
$$\text{for} \quad x \in \mathcal{B}_0;$$

$$\frac{2}{k_2 + 1} \sum_{l=0}^{\infty} (\alpha\beta)^k \Big(\log |(\Phi_2\Phi_1)^k(x) - y| - \alpha \log |(\Phi_1\Phi_2)^k \Phi_1(x) - y| \Big)$$
$$\text{for} \quad x \in \overline{\mathcal{B}}_2;$$

(2) When $y \in \mathcal{B}_1$, $\mathcal{G}(x, y)$ equals

$$\frac{1}{k_1} (\log |x - y| + \alpha \log |\Phi_1(x) - y|) - \frac{4\beta}{(k_1 + 1)^2} \sum_{k=0}^{\infty} (\alpha\beta)^k \log |(\Phi_2\Phi_1)^k \Phi_2(x) - y|$$
$$\text{for} \quad x \in \overline{\mathcal{B}}_1 \setminus \{(1 + \varepsilon/2, 0)\};$$

$$\frac{2}{k_1 + 1} \sum_{k=0}^{\infty} (\alpha\beta)^k \Big(\log |(\Phi_2\Phi_1)^k(x) - y| - \beta \log |(\Phi_2\Phi_1)^k \Phi_2(x) - y| \Big)$$
$$\text{for} \quad x \in \mathcal{B}_0;$$

$$\frac{4}{(k_1 + 1)(k_2 + 1)} \sum_{k=0}^{\infty} (\alpha\beta)^k \log |(\Phi_2\Phi_1)^k(x) - y|$$
$$\text{for} \quad x \in \overline{\mathcal{B}}_2;$$

(3) When $y \in \mathcal{B}_2$, $\mathcal{G}(x, y)$ equals

$$\frac{4}{(k_1 + 1)(k_2 + 1)} \sum_{k=0}^{\infty} (\alpha\beta)^k \log |(\Phi_1\Phi_2)^k(x) - y|$$
$$\text{for} \quad x \in \overline{\mathcal{B}}_1;$$

$$\frac{2}{k_2 + 1} \sum_{k=0}^{\infty} (\alpha\beta)^k \Big(\log |(\Phi_1\Phi_2)^k(x) - y| - \alpha \log |(\Phi_1\Phi_2)^k \Phi_1(x) - y| \Big)$$
$$\text{for} \quad x \in \mathcal{B}_0;$$

$$\frac{1}{k_2} (\log |x - y| + \beta \log |\Phi_2(x) - y|) - \frac{4\alpha}{(k_2 + 1)^2} \sum_{k=0}^{\infty} (\alpha\beta)^k \log |(\Phi_1\Phi_2)^k \Phi_1(x) - y|$$
$$\text{for} \quad x \in \overline{\mathcal{B}}_2 \setminus \{(-1 - \varepsilon/2, 0)\}.$$

Define

$$G(x, y) = \begin{cases} \mathcal{G}(x, y) & \text{for} \quad y \in \mathcal{B}_0, \\ \mathcal{G}(x, y) + \dfrac{\alpha}{1 - \alpha} \mathcal{G}(x, (1 + \varepsilon/2, 0)) & \text{for} \quad y \in \mathcal{B}_1, \\ \mathcal{G}(x, y) + \dfrac{\beta}{1 - \beta} \mathcal{G}(x, (-1 - \varepsilon/2, 0)) & \text{for} \quad y \in \mathcal{B}_2. \end{cases} \tag{2.1}$$

By [17, Proposition 2.3], G is a Green function of $L_{\varepsilon;1,1}$ in the sense that

$$a(x)\Delta_x G(x,y) = \delta(x-y) \quad \text{for} \ \ x \notin \partial \mathcal{B}_1 \cup \partial \mathcal{B}_2,$$

and $G(\cdot, y)$, $aD_\nu G(\cdot, y)$ are continuous across $\partial \mathcal{B}_1$ and $\partial \mathcal{B}_2$. Let $\eta \in C_0^\infty(B_{3/4})$ be a cutoff function such that $\eta = 1$ in $B_{1/2}$. Let $v = u\eta$, where u is a solution of (1.1). Then v satisfies

$$D_i(a(x)D_i v) = D_i \tilde{f}_i + \tilde{f}_3 \quad \text{in} \ \ \mathbb{R}^2,$$

where

$$\tilde{f}_i = f_i \eta + a u D_i \eta, \quad \tilde{f}_3 = -f_i D_i \eta + a D_i u D_i \eta. \tag{2.2}$$

We define

$$\tilde{u}(x) = -\int_{\mathcal{B}_1} D_{y_i} G(x,y) \tilde{f}_i(y)\, dy - \int_{\mathcal{B}_2} D_{y_i} G(x,y) \tilde{f}_i(y)\, dy$$

$$- \int_{\mathcal{B}_0} D_{y_i} G(x,y) \tilde{f}_i(y)\, dy + \int_{\mathcal{B}_1} G(x,y) \tilde{f}_3(y)\, dy$$

$$:= -w_1(x) - w_2(x) - w_0(x) + w_3(x). \tag{2.3}$$

We know from [17, pp. 1447] that $u = \tilde{u} + C_0$ for some constant C_0. We define for $j = 0, 1, 2$,

$$h_j(x) = \int_{\mathcal{B}_j} D_{y_i} \log|x-y| \tilde{f}_i(y)\, dy \tag{2.4}$$

and

$$g_j(x) = \int_{\mathcal{B}_j} \log|x-y| \tilde{f}_3(y)\, dy. \tag{2.5}$$

Since $\text{supp}(D\eta) \subset B_{3/4} \setminus B_{1/2}$, by [17, Lemma 3.2] and [20, Lemma 2.1], we have for $m \in \mathbb{N}$,

$$\|u\|_{C^{2m,\mu}(\mathcal{B}_j \cap \text{supp}(D\eta))} \le CC_m, \ \ \|\tilde{f}_i\|_{C^{2m-1,\mu}(\mathcal{B}_j)} \le CC_m k_j \ \text{for} \ \ i=1,2,3, \ \ j=0,1,2, \tag{2.6}$$

and

$$\|h_j\|_{C^{2m,\mu}(B_3 \cap \mathcal{B}_i)} + \|g_j\|_{C^{2m,\mu}(B_3 \cap \mathcal{B}_i)} \le CC_m k_j \quad \text{for} \ \ i,j = 0,1,2. \tag{2.7}$$

From now on, we identify a point $x = (x_1, x_2) \in \mathbb{R}^2$ with a complex number $z = x_1 + ix_2 \in \mathbb{C}$. We will derive some derivative estimates of the maps $(\Phi_2\Phi_1)^k(z)$ with respect to k and $z \in \mathcal{B}_2$. Estimates of $(\Phi_1\Phi_2)^k(z)$ with respect to k and z for $z \in \mathcal{B}_1$ will follow similarly.

We denote $a = 1 + \varepsilon/2$ for convenience, after a change of variable

$$2az - (2a^2 - 1) \to z,$$

we have

$$(\Phi_2\Phi_1)(z) = -1/z - 2(2a^2 - 1),$$

and the two fixed points of $\Phi_2\Phi_1$ are given by

$$\lambda_1 := -(2a^2 - 1) + 2a\sqrt{a^2 - 1} \sim -1 + 2\sqrt{\varepsilon},$$
$$\lambda_2 := -(2a^2 - 1) - 2a\sqrt{a^2 - 1} \sim -1 - 2\sqrt{\varepsilon}.$$

We denote $\psi = \Phi_2\Phi_1$ for simplicity. For $z \in \mathcal{B}_2$, we denote $r := |z - \lambda_2|$, and

$$I_k := (z - \lambda_2^{-1}) - |\lambda_2|^{-k}(z - \lambda_2) = (z - \lambda_2^{-1})(1 - |\lambda_2|^{-k}) + (\lambda_2 - \lambda_2^{-1})|\lambda_2|^{-k}.$$

It is easy to see that $|z - \lambda_2^{-1}| \sim r + \sqrt{\varepsilon}$, $\mathrm{Re}\,(z - \lambda_2^{-1}) < 0$, and $\lambda_2 - \lambda_2^{-1} \sim -\sqrt{\varepsilon}$. Hence

$$|I_k| \sim (r + \sqrt{\varepsilon})(1 - \lambda_2^{-k}) + \sqrt{\varepsilon}.$$

Following the iteration argument from Section 3 of [17], we have, for any $\alpha \in \mathbb{N}$,

$$\psi^k(z) = \lambda_2 + (\lambda_2^2 - 1)\lambda_2^{-2k-1}(z - \lambda_2)I_{2k}^{-1}, \tag{2.8}$$

$$D^\alpha(\psi^k(z)) = \frac{(\lambda_2 - \lambda_2^{-1})^2}{\lambda_2^{2k}}(-1)^{\alpha-1}\alpha!(1 - \lambda_2^{-2k})^{\alpha-1}I_{2k}^{-(\alpha+1)}, \tag{2.9}$$

and in particular, since $|\lambda_2 - \lambda_2^{-1}| \lesssim \sqrt{\varepsilon}$ and $|I_2| \gtrsim \sqrt{\varepsilon}$,

$$|D^\alpha(\psi(z))| \leq C, \tag{2.10}$$

where C is a positive constant depending only on α. By (2.8), for any $\beta \geq 1$,

$$D_k^\beta(\psi^k)(z) = D_k^\beta\Big[(\lambda_2^2 - 1)\lambda_2^{-2k-1}(z - \lambda_2)I_{2k}^{-1}\Big]$$
$$= \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}(\lambda_2^2 - 1)(z - \lambda_2)[D_k^{\beta_1}\lambda_2^{-2k-1}][D_k^{\beta_2}I_{2k}^{-1}].$$

Since

$$D_k \lambda_2^{-2k} = -2 \log |\lambda_2| \lambda_2^{-2k} \sim -\sqrt{\varepsilon} \lambda_2^{-2k}$$

and

$$D_k^{\beta_2} I_{2k}^{-1} = \sum_{j=0}^{\beta_2} C_j (\log |\lambda_2|)^{\beta_2} \lambda_2^{-2jk} (z - \lambda_2)^j I_{2k}^{-j-1},$$

where C_j is some constant independent of k and ε, we obtain

$$|D_k^{\beta}(\psi^k)(z)| \lesssim \frac{\varepsilon^{(1+\beta)/2}}{\lambda_2^{2k}} \sum_{j=0}^{\beta} \frac{r^{j+1}}{|I_{2k}|^{j+1}}. \qquad (2.11)$$

For any $\beta \geq 0$ and $\alpha \in \mathbb{N}$, by (2.9), we have

$$D_k^{\beta} D^{\alpha}(\psi^k(z)) = \sum_{\beta_1 + \beta_2 + \beta_3 = \beta} C_{\beta_1, \beta_2, \beta_3} (\lambda_2 - \lambda_2^{-1})^2 (-1)^{\alpha-1} \alpha! [D_k^{\beta_1} \lambda_2^{-2k}] \times$$

$$\times [D_k^{\beta_2} (1 - \lambda_2^{-2k})^{\alpha-1}] [D_k^{\beta_3} I_{2k}^{-(\alpha+1)}].$$

Observe that

$$\left| D_k^{\beta_2} (1 - \lambda_2^{-2k})^{\alpha-1} \right| = \left| \sum_{j=0}^{\min(\beta_2, \alpha-1)} C_j (1 - \lambda_2^{-2k})^{\alpha-j-1} \lambda_2^{-2jk} (\log |\lambda_2|)^{\beta_2} \right|$$

$$\lesssim \varepsilon^{\beta_2/2} \sum_{j=0}^{\min(\beta_2, \alpha-1)} |1 - \lambda_2^{-2k}|^{\alpha-j-1},$$

and

$$|D_k^{\beta_3} I_{2k}^{-(\alpha+1)}| = \left| \sum_{j=0}^{\beta_3} C_j (\log |\lambda_2|)^{\beta_3} \lambda_2^{-2jk} (z - \lambda_2)^j I_{2k}^{-(\alpha+j+1)} \right|$$

$$\lesssim \varepsilon^{\beta_3/2} \sum_{j=0}^{\beta_3} \frac{r^j}{|I_{2k}|^{\alpha+j+1}},$$

where C_j is some constant and can be different from line to line. Thus we have

$$|D_k^{\beta} D^{\alpha}(\psi^k(z))| \lesssim \frac{\varepsilon^{1+\beta/2}}{\lambda_2^{2k}} \sum_{\substack{\beta_2 + \beta_3 \leq \beta \\ \beta_2 \leq \alpha-1}} |1 - \lambda_2^{-2k}|^{\alpha-\beta_2-1} \frac{r^{\beta_3}}{|I_{2k}|^{\alpha+\beta_3+1}}. \qquad (2.12)$$

Note that β_2, β_3 in (2.12) might be different from the ones above.

3. First order derivative

In this section, we prove Theorem 1.1 when $m = 1$. Without loss of generality, we may assume that $C_1 = 1$.

Case 1: $x \in \mathcal{B}_0 \cap B_{1/2}$. In this case, $\psi^k(x) \in \mathcal{B}_2$ for $k \in \mathbb{N}$. By (2.1) and (2.3), we have

$$Dw_1(x) = \frac{2}{k_1 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(D[h_1(\psi^k(x))] - \beta D[h_1(\psi^k \Phi_2(x))] \Big).$$

We denote

$$\Theta_k := \gamma^k D[h_1(\psi^k(x))] = \gamma^k Dh_1[\psi^k(x)] D(\psi^{k-1})[\psi(x)] D\psi(x).$$

Then

$$\sum_{k=0}^{\infty} (-1)^k \Theta_k = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k (\Theta_k - \Theta_{k+1}) + \Theta_0 - \frac{1}{2}\Theta_1.$$

By (2.6) and (2.7), it is clear that $|\Theta_0 - \frac{1}{2}\Theta_1| \leq Ck_1$, where C is a positive constant independent of ε and γ. Therefore, the goal is to estimate $|\Theta_k - \Theta_{k+1}|$. By the mean value theorem,

$$\Theta_k - \Theta_{k+1} = D_k \Theta_k \Big|_{k=\bar{k}}$$

for some $\bar{k} \in (k, k+1)$. We claim that

$$\sum_{k=1}^{\infty} |\Theta_k - \Theta_{k+1}| \lesssim \sum_{k=1}^{\infty} |D_k \Theta_k| \lesssim k_1. \tag{3.1}$$

By the chain rule and the product rule, we have

$$
\begin{aligned}
D_k \Theta_k = &(\log \gamma) \gamma^k Dh_1[\psi^k(x)] D(\psi^{k-1})[\psi(x)] D\psi(x) \\
&+ \gamma^k D^2 h_1[\psi^k(x)] D_k(\psi^{k-1})[\psi(x)] D(\psi^{k-1})[\psi(x)] D\psi(x) \\
&+ \gamma^k Dh_1[\psi^k(x)] D_k D(\psi^{k-1})(x) D\psi(x) \\
= : &J_k^1 + J_k^2 + J_k^3.
\end{aligned}
$$

Recall that $|\lambda_2| > 1$ and $|\log \gamma| \sim 1 - \gamma$, by (2.7), (2.10), (2.11), and (2.12), we have

$$|J_k^1| \lesssim k_1 |\log \gamma| \gamma^k \frac{\varepsilon}{\lambda_2^{2k} |I_{2k}|^2},$$

$$|J_k^2| \lesssim k_1 \gamma^k \frac{\varepsilon}{\lambda_2^{2k}} \left(\frac{r}{|I_{2k}|} + \frac{r^2}{|I_{2k}|^2} \right) \frac{\varepsilon}{\lambda_2^{2k}} \frac{1}{|I_{2k}|^2} \lesssim k_1 \frac{\varepsilon^2}{\lambda_2^{2k}} \left(\frac{r}{|I_{2k}|^3} + \frac{r^2}{|I_{2k}|^4} \right),$$

$$|J_k^3| \lesssim k_1 \gamma^k \frac{\varepsilon^{3/2}}{\lambda_2^{2k}} \left(\frac{1}{|I_{2k}|^2} + \frac{r}{|I_{2k}|^3} \right).$$

When $k < \frac{1}{r+\sqrt{\varepsilon}}$, we have

$$|I_{2k}| \gtrsim \sqrt{\varepsilon}.$$

Therefore,

$$\left| \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} J_k^1 \right| \lesssim k_1 (1-\gamma) \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \gamma^k \frac{\varepsilon}{\lambda_2^{2k} |I_{2k}|^2}$$

$$\lesssim k_1 (1-\gamma) \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \gamma^k \lesssim k_1,$$

$$\left| \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} J_k^2 \right| \lesssim k_1 \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \frac{\varepsilon^2}{\lambda_2^{2k}} \left(\frac{r}{|I_{2k}|^3} + \frac{r^2}{|I_{2k}|^4} \right)$$

$$\lesssim k_1 \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \varepsilon^2 \left(\frac{r}{\varepsilon^{3/2}} + \frac{r^2}{\varepsilon^2} \right)$$

$$\lesssim k_1 \frac{\varepsilon^2}{r+\sqrt{\varepsilon}} \left(\frac{r}{\varepsilon^{3/2}} + \frac{r^2}{\varepsilon^2} \right) \lesssim k_1 (\sqrt{\varepsilon} + r),$$

and

$$\left| \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} J_k^3 \right| \lesssim k_1 \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \gamma^k \frac{\varepsilon^{3/2}}{\lambda_2^{2k}} \left(\frac{1}{|I_{2k}|^2} + \frac{r}{|I_{2k}|^3} \right)$$

$$\lesssim k_1 \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \varepsilon^{3/2} \left(\frac{1}{\varepsilon} + \frac{r}{\varepsilon^{3/2}} \right)$$

$$\lesssim k_1 \frac{\varepsilon^{3/2}}{r+\sqrt{\varepsilon}} \left(\frac{1}{\varepsilon} + \frac{r}{\varepsilon^{3/2}} \right) \lesssim k_1.$$

When $\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}$, we have

$$|I_{2k}| \gtrsim (r+\sqrt{\varepsilon}) k \sqrt{\varepsilon}.$$

Therefore,

$$\left| \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} J_k^1 \right| \lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} |\log \gamma| \gamma^k \frac{\varepsilon}{\lambda_2^{2k} |I_{2k}|^2}$$

$$\lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} |\log \gamma| \gamma^k \frac{\varepsilon}{\lambda_2^{2k} (r+\sqrt{\varepsilon})^2 k^2 \varepsilon}$$

$$\lesssim k_1 \gamma^{\frac{1}{r+\sqrt{\varepsilon}}} |\log\gamma| \frac{1}{(r+\sqrt{\varepsilon})^2} \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \frac{1}{k^2}$$

$$\lesssim k_1 e^{\frac{\log\gamma}{r+\sqrt{\varepsilon}}} \frac{|\log\gamma|}{r+\sqrt{\varepsilon}} \lesssim k_1,$$

where we used the fact that $|e^{-x}x| \leq C$ for any $x > 0$,

$$\left| \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} J_k^2 \right| \lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \gamma^k \frac{\varepsilon^2}{\lambda_2^{2k}} \left(\frac{r}{|I_{2k}|^3} + \frac{r^2}{|I_{2k}|^4} \right)$$

$$\lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \varepsilon^2 \left(\frac{r}{(r+\sqrt{\varepsilon})^3 k^3 \varepsilon^{3/2}} + \frac{r^2}{(r+\sqrt{\varepsilon})^4 k^4 \varepsilon^2} \right)$$

$$\lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \left(\frac{1}{(r+\sqrt{\varepsilon})k^3} + \frac{1}{(r+\sqrt{\varepsilon})^2 k^4} \right)$$

$$\lesssim k_1 (r+\sqrt{\varepsilon}),$$

and

$$\left| \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} J_k^3 \right| \lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \gamma^k \frac{\varepsilon^{3/2}}{\lambda_2^{2k}} \left(\frac{1}{|I_{2k}|^2} + \frac{r}{|I_{2k}|^3} \right)$$

$$\lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \varepsilon^{3/2} \left(\frac{1}{(r+\sqrt{\varepsilon})^2 k^2 \varepsilon} + \frac{r}{(r+\sqrt{\varepsilon})^3 k^3 \varepsilon^{3/2}} \right)$$

$$\lesssim k_1 \sum_{\frac{1}{r+\sqrt{\varepsilon}} \leq k < \frac{1}{\sqrt{\varepsilon}}} \left(\frac{1}{(r+\sqrt{\varepsilon})k^2} + \frac{1}{(r+\sqrt{\varepsilon})^2 k^3} \right) \lesssim k_1.$$

Finally, when $k \geq \frac{1}{\sqrt{\varepsilon}}$, we have

$$|I_{2k}| \gtrsim r + \sqrt{\varepsilon}.$$

Therefore,

$$\left| \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} J_k^1 \right| \lesssim k_1 \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} |\log\gamma| \gamma^k \frac{\varepsilon}{\lambda_2^{2k} |I_{2k}|^2}$$

$$\lesssim k_1 (1-\gamma) \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} \gamma^k \lesssim k_1 \gamma^{\frac{1}{\sqrt{\varepsilon}}},$$

$$\left| \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} J_k^2 \right| \lesssim k_1 \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} \gamma^k \frac{\varepsilon^2}{\lambda_2^{2k}} \left(\frac{r}{|I_{2k}|^3} + \frac{r^2}{|I_{2k}|^4} \right)$$

$$\lesssim k_1 \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} \frac{\gamma^k}{\lambda_2^{2k}} \varepsilon^2 \left(\frac{r}{(r + \sqrt{\varepsilon})^3} + \frac{r^2}{(r + \sqrt{\varepsilon})^4} \right)$$

$$\lesssim k_1 \frac{\varepsilon(\gamma/\lambda_2^2)^{\frac{1}{\sqrt{\varepsilon}}}}{1 - \gamma/\lambda_2^2} \lesssim k_1 \sqrt{\varepsilon}(\gamma/\lambda_2^2)^{\frac{1}{\sqrt{\varepsilon}}},$$

where we used $1 - \gamma/\lambda_2^2 \gtrsim \sqrt{\varepsilon}$, and

$$\left| \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} J_k^3 \right| \lesssim k_1 \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} \gamma^k \frac{\varepsilon^{3/2}}{\lambda_2^{2k}} \left(\frac{1}{|I_{2k}|^2} + \frac{r}{|I_{2k}|^3} \right)$$

$$\lesssim k_1 \sum_{k \geq \frac{1}{\sqrt{\varepsilon}}} \frac{\gamma^k}{\lambda_2^{2k}} \varepsilon^{3/2} \left(\frac{1}{(r + \sqrt{\varepsilon})^2} + \frac{r}{(r + \sqrt{\varepsilon})^3} \right)$$

$$\lesssim k_1 \frac{\sqrt{\varepsilon}(\gamma/\lambda_2^2)^{\frac{1}{\sqrt{\varepsilon}}}}{1 - \gamma/\lambda_2^2} \lesssim k_1 (\gamma/\lambda_2^2)^{\frac{1}{\sqrt{\varepsilon}}}.$$

Therefore, (3.1) follows from the estimates above, and hence

$$\left| \sum_{k=0}^{\infty} (-1)^k \Theta_k \right| \leq C k_1,$$

where C is a positive constant independent of ε and γ.

By the same argument, we can estimate

$$\left| \sum_{k=0}^{\infty} (-1)^k \gamma^k D[h_1(\psi^k \Phi_2(x))] \right| \leq C k_1.$$

Therefore,

$$|Dw_1(x)| \leq \frac{C k_1}{k_1 + 1} \quad \text{for} \ x \in \mathcal{B}_0 \cap B_{1/2}.$$

By (2.1) and (2.3) again, we have, for $x \in \mathcal{B}_0 \cap B_{1/2}$,

$$Dw_2(x) = \frac{2}{k_2 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(D[h_2((\Phi_1 \Phi_2)^k(x))] - \alpha D[h_2((\Phi_1 \Phi_2)^k \Phi_1(x))] \Big),$$

$$Dw_0(x) = Dh_0(x) + \sum_{k=0}^{\infty} \Big[(-1)^k \gamma^k \Big(D[h_0((\Phi_1 \Phi_2)^k(x))] + D[h_0((\Phi_2 \Phi_1)^k(x))] \Big)$$

$$- (-1)^{k-1} \gamma^{k-1} \Big(\beta D[h_0((\Phi_2 \Phi_1)^{k-1} \Phi_2(x))] + \alpha D[h_0((\Phi_1 \Phi_2)^{k-1} \Phi_1(x))] \Big) \Big].$$

Therefore, in the same way we can estimate

$$|Dw_2(x)| \le \frac{Ck_2}{k_2+1} \quad \text{and} \quad |Dw_0(x)| \le C \quad \text{for } x \in \mathcal{B}_0 \cap B_{1/2},$$

where C is a positive constant independent of ε, k_1, and k_2. To estimate w_3, since $\text{supp}(\tilde{f}_3) \subset B_1$, we can write

$$
\begin{aligned}
w_3(x) &= \int\limits_{\mathcal{B}_1} G(x,y)\tilde{f}_3(y)\,dy + \int\limits_{\mathcal{B}_2} G(x,y)\tilde{f}_3(y)\,dy + \int\limits_{\mathcal{B}_0} G(x,y)\tilde{f}_3(y)\,dy \\
&= \frac{2}{k_1+1}\sum_{k=0}^{\infty}(-1)^k\gamma^k\Big(g_1((\Phi_2\Phi_1)^k(x)) - \beta g_1((\Phi_2\Phi_1)^k\Phi_2(x))\Big) \\
&\quad + \alpha\sum_{k=0}^{\infty}(-1)^k\gamma^k\Big(\log|(\Phi_2\Phi_1)^k(x) - (a,0)| \\
&\quad - \beta\log|(\Phi_2\Phi_1)^k\Phi_2(x) - (a,0)|\Big)\int\limits_{\mathcal{B}_1}\tilde{f}_3(y)\,dy \\
&\quad + \frac{2}{k_2+1}\sum_{k=0}^{\infty}(-1)^k\gamma^k\Big(g_2((\Phi_1\Phi_2)^k(x)) - \alpha g_2((\Phi_1\Phi_2)^k\Phi_1(x))\Big) \\
&\quad + \beta\sum_{k=0}^{\infty}(-1)^k\gamma^k\Big(\log|(\Phi_1\Phi_2)^k(x) + (a,0)| \\
&\quad - \alpha\log|(\Phi_1\Phi_2)^k\Phi_1(x) + (a,0)|\Big)\int\limits_{\mathcal{B}_2}\tilde{f}_3(y)\,dy \\
&\quad + g_0(x) + \sum_{k=0}^{\infty}\Big[(-1)^k\gamma^k\Big(g_0((\Phi_1\Phi_2)^k(x)) + g_0((\Phi_2\Phi_1)^k(x))\Big) \\
&\quad - (-1)^{k-1}\gamma^{k-1}\Big(\beta g_0((\Phi_2\Phi_1)^{k-1}\Phi_2(x)) + \alpha g_0((\Phi_1\Phi_2)^{k-1}\Phi_1(x))\Big)\Big].
\end{aligned}
$$

Note that for any $x \in \mathcal{B}_0 \cap B_{1/2}$,

$$
\begin{aligned}
|(\Phi_2\Phi_1)^k(x) - (a,0)| &\ge 1, & |(\Phi_2\Phi_1)^k\Phi_2(x) - (a,0)| &\ge 1, \\
|(\Phi_1\Phi_2)^k(x) + (a,0)| &\ge 1, & |(\Phi_1\Phi_2)^k\Phi_1(x) + (a,0)| &\ge 1.
\end{aligned}
$$

By (2.2) and (1.1),

$$\int\limits_{B_1}\tilde{f}_3 = \int\limits_{B_1} -f_i D_i\eta + a(x)D_i u D_i\eta = 0.$$

Therefore, by (2.6),

$$\left|\int\limits_{\mathcal{B}_2}\tilde{f}_3\right| = \left|\int\limits_{\mathcal{B}_1\cup\mathcal{B}_0} -f_i D_i\eta + a(x)D_i u D_i\eta\right| \le CC_m,$$

where C is some positive constant independent of ε, k_1, and k_2. Then we can estimate Dw_3 similarly as above to obtain

$$|Dw_3(x)| \le C \quad \text{for} \quad x \in \mathcal{B}_0 \cap B_{1/2}.$$

Therefore,

$$|Du(x)| \le C \quad \text{for} \quad x \in \mathcal{B}_0 \cap B_{1/2},$$

where C is a positive constant independent of ε, k_1, and k_2.

Case 2: $x \in \mathcal{B}_1 \cap B_{1/2}$. By (2.1) and (2.3), we have

$$w_1(x) = \frac{1}{k_1} h_1(x) + \frac{\alpha}{k_1} h_1(\Phi_1(x)) - \frac{4\beta}{(k_1+1)^2} \sum_{k=0}^{\infty} (-1)^k \gamma^k h_1((\Phi_2\Phi_1)^k \Phi_2(x)),$$

$$w_2(x) = \frac{4}{(k_1+1)(k_2+1)} \sum_{k=0}^{\infty} (-1)^k \gamma^k h_2((\Phi_1\Phi_2)^k(x)),$$

$$w_0(x) = \frac{2}{k_1+1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(h_0((\Phi_1\Phi_2)^k(x)) - \beta h_0((\Phi_2\Phi_1)^k \Phi_2(x)) \Big),$$

and

$$
\begin{aligned}
w_3(x) &= \int_{\mathcal{B}_1} G(x,y)\tilde{f}_3(y)\,dy + \int_{\mathcal{B}_2} G(x,y)\tilde{f}_3(y)\,dy + \int_{\mathcal{B}_0} G(x,y)\tilde{f}_3(y)\,dy \\
&= \frac{1}{k_1} g_1(x) + \frac{\alpha}{k_1} g_1(\Phi_1(x)) - \frac{4\beta}{(k_1+1)^2} \sum_{k=0}^{\infty} (-1)^k \gamma^k g_1((\Phi_2\Phi_1)^k \Phi_2(x)) \\
&\quad + \frac{\alpha}{k_1} \log|x-(a,0)| \int_{\mathcal{B}_1} \tilde{f}_3(y)\,dy \\
&\quad - \frac{2\alpha\beta}{k_1+1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \log|(\Phi_2\Phi_1)^k \Phi_2(x)-(a,0)| \int_{\mathcal{B}_1} \tilde{f}_3(y)\,dy \\
&\quad + \frac{4}{(k_1+1)(k_2+1)} \sum_{k=0}^{\infty} (-1)^k \gamma^k g_2((\Phi_1\Phi_2)^k(x)) \\
&\quad + \frac{2\beta}{k_1+1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \log|(\Phi_1\Phi_2)^k(x)+(a,0)| \int_{\mathcal{B}_2} \tilde{f}_3(y)\,dy \\
&\quad + \frac{2}{k_1+1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(g_0((\Phi_1\Phi_2)^k(x)) - \beta g_0((\Phi_2\Phi_1)^k \Phi_2(x)) \Big),
\end{aligned}
$$

where we used $\log|x-(a,0)| = -\log|\Phi_1(x)-(a,0)|$ for $x \in \mathcal{B}_1 \cap B_{1/2}$. Note that

$$|x - (a,0)| \geq 1/2, \quad |(\Phi_2\Phi_1)^k\Phi_2(x) - (a,0)| \geq 1, \quad |(\Phi_1\Phi_2)^k(x) + (a,0)| \geq 1.$$

We can estimate $Dw_i, i = 0, 1, 2, 3$, as above to obtain

$$|Dw_1(x)| \leq C, \quad |Dw_2(x)| \leq \frac{Ck_2}{(k_1 + 1)(k_2 + 1)},$$

$$|Dw_0(x)| \leq \frac{C}{k_1 + 1}, \quad |Dw_3(x)| \leq C.$$

Therefore,

$$|Du(x)| \leq C \quad \text{for} \quad x \in \mathcal{B}_1 \cap B_{1/2},$$

where C is a positive constant independent of ε, k_1, and k_2.

Case 3: $x \in \mathcal{B}_2 \cap B_{1/2}$. By (2.1) and (2.3) again, we have

$$w_1(x) = \frac{4}{(k_1 + 1)(k_2 + 1)} \sum_{k=0}^{\infty} (-1)^k \gamma^k h_1((\Phi_2\Phi_1)^k(x)),$$

$$w_2(x) = \frac{1}{k_2} h_2(x) + \frac{\beta}{k_2} h_2(\Phi_2(x)) - \frac{4\alpha}{(k_2 + 1)^2} \sum_{k=0}^{\infty} (-1)^k \gamma^k h_2((\Phi_1\Phi_2)^k\Phi_1(x)),$$

$$w_0(x) = \frac{2}{k_2 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(h_0((\Phi_2\Phi_1)^k(x)) - \alpha h_0((\Phi_1\Phi_2)^k\Phi_1(x)) \Big),$$

and

$$w_3(x) = \int_{\mathcal{B}_1} G(x,y)\tilde{f}_3(y)\,dy + \int_{\mathcal{B}_2} G(x,y)\tilde{f}_3(y)\,dy + \int_{\mathcal{B}_0} G(x,y)\tilde{f}_3(y)\,dy$$

$$= \frac{4}{(k_1 + 1)(k_2 + 1)} \sum_{k=0}^{\infty} (-1)^k \gamma^k g_1((\Phi_2\Phi_1)^k(x))$$

$$+ \frac{2\alpha}{k_2 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \log |(\Phi_2\Phi_1)^k(x) - (a,0)| \int_{\mathcal{B}_1} \tilde{f}_3(y)\,dy$$

$$+ \frac{1}{k_2} g_2(x) + \frac{\beta}{k_2} g_2(\Phi_2(x)) - \frac{4\alpha}{(k_2 + 1)^2} \sum_{k=0}^{\infty} (-1)^k \gamma^k g_2((\Phi_1\Phi_2)^k\Phi_1(x))$$

$$+ \frac{\beta}{k_2} \log |x + (a,0)| \int_{\mathcal{B}_2} \tilde{f}_3(y)\,dy$$

$$- \frac{2\alpha\beta}{k_2 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \log |(\Phi_1\Phi_2)^k\Phi_1(x) + (a,0)| \int_{\mathcal{B}_2} \tilde{f}_3(y)\,dy$$

$$+ \frac{2}{k_2 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(g_0((\Phi_2 \Phi_1)^k(x)) - \alpha g_0((\Phi_1 \Phi_2)^k \Phi_1(x)) \Big),$$

where we used $\log |x + (a, 0)| = -\log |\Phi_2(x) + (a, 0)|$ for $x \in \mathcal{B}_2 \cap B_{1/2}$. Note that

$$|x + (a, 0)| \geq 1/2, \quad |(\Phi_1 \Phi_2)^k \Phi_1(x) + (a, 0)| \geq 1, \quad |(\Phi_2 \Phi_1)^k(x) - (a, 0)| \geq 1.$$

We can estimate $Dw_i, i = 0, 1, 2, 3$, as above to obtain

$$|Dw_1(x)| \leq \frac{Ck_1}{(k_1 + 1)(k_2 + 1)}, \quad Dw_2(x)| \leq C,$$

$$|Dw_0(x)| \leq \frac{C}{k_2 + 1}, \quad |Dw_3(x)| \leq C.$$

Therefore,

$$|Du(x)| \leq C \quad \text{for} \ \ x \in \mathcal{B}_2 \cap B_{1/2},$$

where C is a positive constant independent of ε, k_1 and k_2. Hence (1.2) is proved for $m = 1$.

4. Higher order derivatives

In this section, we prove Theorem 1.1 for $m \geq 2$. The idea is essentially the same as the case when $m = 1$. Without loss of generality, we assume that $C_m = 1$.

For $x \in \mathcal{B}_0 \cap B_{1/2}$, by (2.1) and (2.3), we have

$$D^m w_1(x) = \frac{2}{k_1 + 1} \sum_{k=0}^{\infty} (-1)^k \gamma^k \Big(D^m[h_1(\psi^k(x))] - \beta D^m[h_1(\psi^k \Phi_2(x))] \Big).$$

We denote

$$\Theta_{m,k} := \gamma^k D^m[h_1(\psi^k(x))].$$

Since $\sum_{i=0}^{m} \binom{m}{i} = 2^m$, we have

$$\sum_{k=0}^{\infty} (-1)^k \Theta_{m,k} = \frac{1}{2^m} \sum_{k=1}^{\infty} (-1)^k \left[\sum_{i=0}^{m} (-1)^i \binom{m}{i} \Theta_{m,k+i} \right]$$

$$+ \Theta_{m,0} + \frac{1}{2^m} \sum_{i=1}^{m} \left[(-1)^i \Theta_{m,i} \sum_{j=i}^{m} \binom{m}{j} \right]. \tag{4.1}$$

The second line of (4.1) is clearly bounded independent of ε and γ. The main goal is to estimate the first term on the right-hand side of (4.1). By the mean value theorem,

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Theta_{m,k+i}=D_k^m\Theta_{m,k}\Big|_{k=\bar{k}}$$

for some $\bar{k}\in(k,k+m)$. We claim that

$$\left|\sum_{k=1}^{\infty}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\Theta_{m,k+i}\right|\lesssim\sum_{k=1}^{\infty}|D_k^m\Theta_{m,k}|\lesssim k_1. \qquad (4.2)$$

By the chain rule and the product rule,

$$D^m[h_1(\psi^k(x))]=\sum_{n=1}^{m}\left(D^n h_1(\psi^k(x))\sum_{\sum_{j=1}^{n}\alpha_{n,j}=m}\prod_{j=1}^{n}D^{\alpha_{n,j}}(\psi^k)(x)\right),$$

where $\alpha_{n,j}\geq1$. For each n and $\{\alpha_{n,j}\}_{j=1}^{n}$ satisfying $\sum_{j=1}^{n}\alpha_{n,j}=m$,

$$D^{\alpha_{n,j}}(\psi^k)(x)=\sum_{\tilde{\alpha}_{n,j}=1}^{\alpha_{n,j}}\left(D^{\tilde{\alpha}_{n,j}}\psi^{k-1}[\psi(x)]\sum_{\sum_{p=1}^{\tilde{\alpha}_{n,j}}\alpha_{n,j,p}=\alpha_{n,j}}\prod_{p=1}^{\tilde{\alpha}_{n,j}}D^{\alpha_{n,j,p}}\psi(x)\right),$$

where $\alpha_{n,j,p}\geq1$. Then by (2.10),

$$\left|D_k^m\gamma^k D^m[h_1(\psi^k(x))]\right|$$

$$\lesssim\sum_{\substack{\tau+\beta_0=0\\\tau,\beta_0\geq0}}^{m}\sum_{n=1}^{m}\left(\gamma^k|\log\gamma|^\tau\big|D_k^{\beta_0}[D^n h_1(\psi^k(x))]\big|\times\right.$$

$$\left.\times\sum_{\sum_{j=1}^{n}\beta_j=m-\tau-\beta_0}\prod_{j=1}^{n}\sum_{\tilde{\alpha}_{n,j}=1}^{\alpha_{n,j}}\big|D_k^{\beta_j}D^{\tilde{\alpha}_{n,j}}(\psi^{k-1})[\psi(x)]\big|\right). \qquad (4.3)$$

When $\beta_0\geq1$, by the chain rule and the product rule,

$$D_k^{\beta_0}[D^n h_1(\psi^k(x))]=\sum_{s=1}^{\beta_0}\left(D^{n+s}h_1(\psi^k(x))\sum_{\sum_{i=1}^{s}\gamma_{s,i}=\beta_0}\prod_{i=1}^{s}D_k^{\gamma_{s,i}}\psi^{k-1}[\psi(x)]\right),$$

where $\gamma_{s,i}\geq1$. For each s and $\{\gamma_{s,i}\}_{i=1}^{s}$ satisfying $\gamma_{s,i}\geq1$ and $\sum_{i=1}^{s}\gamma_{s,i}=\beta_0$, by (2.11)

$$\left|\prod_{i=1}^{s}D_k^{\gamma_{s,i}}\psi^{k-1}[\psi(x)]\right|\lesssim\prod_{i=1}^{s}\left(\frac{\varepsilon^{(1+\gamma_{s,i})/2}}{\lambda_2^{2k}}\sum_{j=0}^{\gamma_{s,i}}\frac{r^{j+1}}{|I_{2k}|^{j+1}}\right)$$

$$\lesssim\frac{\varepsilon^{(s+\beta_0)/2}}{\lambda_2^{2k}}\sum_{j=0}^{\beta_0}\frac{r^{j+s}}{|I_{2k}|^{j+s}}. \qquad (4.4)$$

To prove the claim, we consider the following two cases:

Case 1: $\beta_0 \geq 1$. For each n, by (4.4), (2.7), and (2.12), the right-hand side of (4.3) can be estimated by (up to a positive constant independent of k)

$$
k_1 \gamma^k |\log \gamma|^\tau \sum_{s=1}^{\beta_0} \left(\frac{\varepsilon^{(s+\beta_0)/2}}{\lambda_2^{2k}} \sum_{j=0}^{\beta_0} \frac{r^{j+s}}{|I_{2k}|^{j+s}} \right) \frac{\varepsilon^{n+(m-\tau-\beta_0)/2}}{\lambda_2^{2k}} \times
$$

$$
\times \sum_{\substack{0 \leq \tilde{\beta}_2 + \tilde{\beta}_3 \leq m-\tau-\beta_0 \\ 0 \leq \tilde{\beta}_2 \leq \tilde{m}-n}} |1 - \lambda_2^{-2k}|^{\tilde{m}-n-\tilde{\beta}_2} \frac{r^{\tilde{\beta}_3}}{|I_{2k}|^{\tilde{m}+n+\tilde{\beta}_3}}
$$

$$
\lesssim k_1 \sum_{s=1}^{\beta_0} \gamma^k |\log \gamma|^\tau \frac{\varepsilon^{n+(m-\tau+s)/2}}{\lambda_2^{2k}} \times
$$

$$
\times \sum_{\substack{0 \leq \tilde{\beta}_2 + \tilde{\beta}_3 \leq m-\tau-\beta_0 \\ 0 \leq \tilde{\beta}_2 \leq \tilde{m}-n \\ 0 \leq j \leq \beta_0}} |1 - \lambda_2^{-2k}|^{\tilde{m}-n-\tilde{\beta}_2} \frac{r^{\tilde{\beta}_3+j+s}}{|I_{2k}|^{\tilde{m}+n+\tilde{\beta}_3+j+s}},
$$

where $n \leq \tilde{m} \leq m$. For every $s, \tau, j, \tilde{\beta}_2,$ and $\tilde{\beta}_3$ that satisfy $1 \leq s \leq \beta_0$, $0 \leq j \leq \beta_0$, $0 \leq \tilde{\beta}_2 \leq \tilde{m}-n$, $\tau \geq 0$, and $0 \leq \tilde{\beta}_2 + \tilde{\beta}_3 \leq m-\tau-\beta_0$, we consider the following three cases:

When $k < \frac{1}{r+\sqrt{\varepsilon}}$, we have

$$
|1 - \lambda_2^{-2k}| \lesssim k\sqrt{\varepsilon} \quad \text{and} \quad |I_{2k}| \gtrsim \sqrt{\varepsilon}.
$$

Since

$$
\sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \left(\frac{\gamma}{\lambda_2^2} \right)^k k^{\tilde{m}-n-\tilde{\beta}_2}
$$

$$
\lesssim \min \left\{ \frac{1}{(1-\gamma/\lambda_2^2)^{\tilde{m}-n-\tilde{\beta}_2+1}}, \frac{1}{(r+\sqrt{\varepsilon})^{\tilde{m}-n-\tilde{\beta}_2+1}} \right\}
$$

$$
\lesssim \frac{1}{(1-\gamma+r+\sqrt{\varepsilon})^{\tilde{m}-n-\tilde{\beta}_2+1}},
$$

we have

$$
\sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \gamma^k |\log \gamma|^\tau \frac{\varepsilon^{n+(m-\tau+s)/2}}{\lambda_2^{2k}} |1 - \lambda_2^{-2k}|^{\tilde{m}-n-\tilde{\beta}_2} \frac{r^{\tilde{\beta}_3+j+s}}{|I_{2k}|^{\tilde{m}+n+\tilde{\beta}_3+j+s}}
$$

$$
\lesssim \sum_{k < \frac{1}{r+\sqrt{\varepsilon}}} \left(\frac{\gamma}{\lambda_2^2} \right)^k k^{\tilde{m}-n-\tilde{\beta}_2} |\log \gamma|^\tau \varepsilon^{(m+\tilde{m}+n-\tau-\tilde{\beta}_2+s)/2} \frac{r^{\tilde{\beta}_3+j+s}}{\varepsilon^{(\tilde{m}+n+\tilde{\beta}_3+j+s)/2}}
$$

$$\lesssim \frac{1}{(1-\gamma+r+\sqrt{\varepsilon})^{\tilde{m}-n-\tilde{\beta}_2+1}}(1-\gamma)^{\tau}\varepsilon^{(m-\tau-\tilde{\beta}_2-\tilde{\beta}_3-j)/2}r^{\tilde{\beta}_3+j+s}$$

$$\lesssim \frac{1}{(1-\gamma+r+\sqrt{\varepsilon})^{\tilde{m}-n-\tilde{\beta}_2+1}}(1-\gamma+r+\sqrt{\varepsilon})^{m-\tilde{\beta}_2+s}\lesssim 1,$$

as $\tilde{m}\leq m$ and $n-1+s\geq 1>0$. When $\frac{1}{r+\sqrt{\varepsilon}}\leq k<\frac{1}{\sqrt{\varepsilon}}$, we have

$$|1-\lambda_2^{-2k}|\lesssim k\sqrt{\varepsilon} \quad \text{and} \quad |I_{2k}|\gtrsim (r+\sqrt{\varepsilon})k\sqrt{\varepsilon}.$$

Therefore,

$$\sum_{\frac{1}{r+\sqrt{\varepsilon}}\leq k<\frac{1}{\sqrt{\varepsilon}}}\gamma^k|\log\gamma|^{\tau}\frac{\varepsilon^{n+(m-\tau+s)/2}}{\lambda_2^{2k}}|1-\lambda_2^{-2k}|^{\tilde{m}-n-\tilde{\beta}_2}\frac{r^{\tilde{\beta}_3+j+s}}{|I_{2k}|^{\tilde{m}+n+\tilde{\beta}_3+j+s}}$$

$$\lesssim \sum_{\frac{1}{r+\sqrt{\varepsilon}}\leq k<\frac{1}{\sqrt{\varepsilon}}}\gamma^{\frac{1}{r+\sqrt{\varepsilon}}}k^{\tilde{m}-n-\tilde{\beta}_2}|\log\gamma|^{\tau}\varepsilon^{(m+\tilde{m}+n-\tau-\tilde{\beta}_2+s)/2}\frac{r^{\tilde{\beta}_3+j+s}}{[(r+\sqrt{\varepsilon})k\sqrt{\varepsilon}]^{\tilde{m}+n+\tilde{\beta}_3+j+s}}$$

$$\lesssim \gamma^{\frac{1}{r+\sqrt{\varepsilon}}}|\log\gamma|^{\tau}\varepsilon^{(m-\tau-\tilde{\beta}_2-\tilde{\beta}_3-j)/2}\frac{1}{(r+\sqrt{\varepsilon})^{\tilde{m}+n}}\sum_{\frac{1}{r+\sqrt{\varepsilon}}\leq k<\frac{1}{\sqrt{\varepsilon}}}\frac{1}{k^{2n+\tilde{\beta}_2+\tilde{\beta}_3+s+j}}$$

$$\lesssim e^{\frac{\log\gamma}{r+\sqrt{\varepsilon}}}|\log\gamma|^{\tau}\varepsilon^{(m-\tau-\tilde{\beta}_2-\tilde{\beta}_3-j)/2}\frac{1}{(r+\sqrt{\varepsilon})^{\tilde{m}+n}}(r+\sqrt{\varepsilon})^{2n+\tilde{\beta}_2+\tilde{\beta}_3+s+j-1}$$

$$\lesssim e^{\frac{\log\gamma}{r+\sqrt{\varepsilon}}}\left|\frac{\log\gamma}{r+\sqrt{\varepsilon}}\right|^{\tau}(r+\sqrt{\varepsilon})^{m-\tilde{m}+n+s-1}\lesssim 1,$$

where we used the fact that $|e^{-x}x^{\tau}|\leq C_{\tau}$ for any $x>0$, $m-\tau-\tilde{\beta}_2-\tilde{\beta}_3-j\geq 0$, $\tilde{m}\leq m$, and $n+s-1\geq 1>0$.

When $k\geq\frac{1}{\sqrt{\varepsilon}}$, we have

$$|1-\lambda_2^{-2k}|\lesssim 1 \quad \text{and} \quad |I_{2k}|\gtrsim r+\sqrt{\varepsilon}.$$

Therefore,

$$\sum_{k\geq\frac{1}{\sqrt{\varepsilon}}}\gamma^k|\log\gamma|^{\tau}\frac{\varepsilon^{n+(m-\tau+s)/2}}{\lambda_2^{2k}}|1-\lambda_2^{-2k}|^{\tilde{m}-n-\tilde{\beta}_2}\frac{r^{\tilde{\beta}_3+j+s}}{|I_{2k}|^{\tilde{m}+n+\tilde{\beta}_3+j+s}}$$

$$\lesssim \sum_{k\geq\frac{1}{\sqrt{\varepsilon}}}\left(\frac{\gamma}{\lambda_2^2}\right)^k|\log\gamma|^{\tau}\varepsilon^{n+(m-\tau+s)/2}\frac{1}{(r+\sqrt{\varepsilon})^{m+n}}$$

$$\lesssim \frac{\gamma^{\frac{1}{\sqrt{\varepsilon}}}}{1-\gamma/\lambda_2^2}\left|\frac{\log\gamma}{\sqrt{\varepsilon}}\right|^{\tau}\varepsilon^{(n+s)/2}$$

$$\lesssim e^{\frac{\log\gamma}{\sqrt{\varepsilon}}}\left|\frac{\log\gamma}{\sqrt{\varepsilon}}\right|^{\tau}\varepsilon^{(n+s-1)/2}\lesssim 1,$$

where we used the fact that $|e^{-x}x^\tau| \le C_\tau$ for any $x > 0$, $\tilde{m} - n - \tilde{\beta}_2 \ge 0$, $n + s - 1 \ge 1 > 0$, and $1 - \gamma/\lambda_2^2 \gtrsim \sqrt{\varepsilon}$. Therefore, (4.2) follows from the estimates above.

Case 2: $\beta_0 = 0$. For each n, by (2.12), the right-hand side of (4.3) can be estimated by (up to a positive constant independent of k)

$$k_1 \gamma^k |\log \gamma|^\tau \frac{\varepsilon^{n+(m-\tau)/2}}{\lambda_2^{2k}} \sum_{\substack{0 \le \tilde{\beta}_2 + \tilde{\beta}_3 \le m - \tau \\ 0 \le \tilde{\beta}_2 \le \tilde{m} - n}} |1 - \lambda_2^{-2k}|^{\tilde{m} - n - \tilde{\beta}_2} \frac{r^{\tilde{\beta}_3}}{|I_{2k}|^{\tilde{m} + n + \tilde{\beta}_3}},$$

where $n \le \tilde{m} \le m$. Then (4.2) follows by repeating the exact same argument as in Case 1 with $s = 0$ and $j = 0$.

Using the same argument but replacing $\psi^{k-1}(\psi(x))$ with $\psi^k \Phi_2(x)$, we have

$$\left| \sum_{k=0}^\infty (-1)^k \gamma^k D^m [h_1(\psi^k \Phi_2(x))] \right| \le C k_1,$$

where C is a positive constant independent of ε and γ. Therefore,

$$|D^m w_1(x)| \le \frac{C k_1}{k_1 + 1} \quad \text{for } x \in \mathcal{B}_0 \cap B_{1/2}.$$

As in Section 3, we can estimate $|D^m w_1(x)|$, $|D^m w_2(x)|$, $|D^m w_3(x)|$, and $|D^m w_0(x)|$ in all three regions. Therefore, (1.2) is proved.

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The proof is similar to that of Theorem 1.1. Without loss of generality, we assume $C_m = 1$. Take a domain $\mathcal{D}_2$ such that $\mathcal{D}_1 \Subset \mathcal{D}_2 \Subset \mathcal{D}$, and take a cutoff function $\eta \in C_0^\infty(\mathcal{D}_2)$ such that $\eta = 1$ on $\mathcal{D}_1$. Then $v := u\eta$ satisfies

$$D_i(a(x)D_i v) = D_i \tilde{f}_i + \tilde{f}_3 \quad \text{in } \mathbb{R}^2,$$

where

$$\tilde{f}_i = f_i \eta + u D_i \eta, \quad \tilde{f}_3 = -f_i D_i \eta + D_i u D_i \eta.$$

For $i = 0, 1, 2, 3$, $j = 1, 2, 3$, we define $\tilde{u}$, w_i, h_j, and g_j as in (2.3), (2.4), and (2.5). Instead of (2.6) and (2.7), we have

$$\|\tilde{f}_i\|_{C^{2m-1,\mu}(\mathcal{B}_j)} \le C \min\{1, k_j\} \quad \text{for } i = 1, 2, 3, \ j = 0, 1, 2,$$

and

$$\|h_j\|_{C^{2m,\mu}(B_3 \cap \mathcal{B}_i)} + \|g_j\|_{C^{2m,\mu}(B_3 \cap \mathcal{B}_i)} \le C \min\{1, k_j\} \quad \text{for } i, j = 0, 1, 2. \tag{5.1}$$

As in the proof of Theorem 1.1, for $x \in \mathcal{B}_0 \cap \mathcal{D}_1$, we have, by (5.1),

$$|D^m w_1(x)| \leq \frac{Ck_1}{k_1 + 1}, \quad |D^m w_2(x)| \leq \frac{C}{k_2 + 1}, \quad \text{and} \quad |D^m w_0(x)| + |D^m w_3(x)| \leq C.$$

For $x \in \mathcal{B}_1$, we have

$$|D^m w_1(x)| \leq C + \frac{Ck_1}{(k_1 + 1)^2},$$

$$|D^m w_2(x)| \leq \frac{C}{(k_1 + 1)(k_2 + 1)},$$

$$|D^m w_0(x)| \leq \frac{C}{k_1 + 1}.$$

To estimate $|D^m w_3(x)|$, we note that

$$\frac{\alpha}{k_1} g_1(\Phi_1(x)) + \frac{\alpha}{k_1} \log |x - (a,0)| \int_{\mathcal{B}_1} \tilde{f}_3(y) \, dy$$

$$= \frac{\alpha}{k_1} \int_{\mathcal{B}_1} (\log |\Phi_1(x) - y| + \log |x - (a,0)|) \tilde{f}_3(y) \, dy,$$

and for fixed $y \in \mathcal{B}_1$, $x = (a,0)$ is a removable singular point for $\log |\Phi_1(x) - y| + \log |x - (a,0)|$. Therefore $\log |\Phi_1(x) - y| + \log |x - (a,0)|$ is harmonic in $\mathcal{B}_1$, and

$$\left| D^m \left(\frac{\alpha}{k_1} g_1(\Phi_1(x)) + \frac{\alpha}{k_1} \log |x - (a,0)| \int_{\mathcal{B}_1} \tilde{f}_3(y) \, dy \right) \right| \leq C.$$

The rest of the terms can be estimated similarly as before, and hence we obtain

$$|D^m w_3(x)| \leq C,$$

which yields

$$|D\tilde{u}(x)| \leq \frac{C}{k_1 + 1} \quad \text{for} \quad x \in \mathcal{B}_1.$$

Finally, for $x \in \mathcal{B}_2$, we have

$$|D^m w_1(x)| \leq \frac{Ck_1}{(k_1 + 1)(k_2 + 1)},$$

$$|D^m w_2(x)| \leq \frac{C}{k_2} + \frac{C}{(k_2 + 1)^2},$$

$$|D^m w_0(x)| \leq \frac{C}{k_2 + 1},$$

$$|D^m w_3(x)| \leq \frac{Ck_1}{(k_1 + 1)(k_2 + 1)} + \frac{Ck_1}{k_2 + 1} + \frac{C}{k_2} + \frac{C}{(k_2 + 1)^2} + \frac{C}{k_2 + 1},$$

which yields

$$|D\tilde{u}(x)| \leq \frac{C}{k_2 + 1} \quad \text{for} \quad x \in \mathcal{B}_2.$$

Theorem 1.2 is proved.

6. Proof of Theorem 1.3

When $r_1 = r_2$, Theorem 1.3 follows from Theorem 1.2 after scaling. When $r_1 \neq r_2$, we will find a conformal map $T : \mathbb{C} \to \mathbb{C}$ that maps $\mathcal{B}_1$ and $\mathcal{B}_2$ to circles of the same radius. Without loss of generality, we may assume $r_2 > r_1$.

Let $T(z) = \frac{1}{z - z_0}$, where $z_0 \in \mathbb{C}$ and $z_0 = z_1 + z_2 i$. It is well known that if $z_0 \notin \overline{\mathcal{B}_1 \cup \mathcal{B}_2}$, T maps $\mathcal{B}_1 \cup \mathcal{B}_2$ to two disks. After a direct computation, we know that T maps $\mathcal{B}_1$ to the disk of center $\frac{\bar{z}_0 - \varepsilon/2 - r_1}{r_1^2 - |z_0 - \varepsilon/2 - r_1|^2}$, radius $\frac{r_1}{|r_1^2 - |z_0 - \varepsilon/2 - r_1|^2|}$, and maps $\mathcal{B}_2$ to the disk of center $\frac{\bar{z}_0 + \varepsilon/2 + r_2}{r_2^2 - |z_0 + \varepsilon/2 + r_2|^2}$, radius $\frac{r_2}{|r_2^2 - |z_0 + \varepsilon/2 + r_2|^2|}$. We only need to find $z_0 = z_1 + z_2 i$ such that

$$r_1(r_2^2 - |z_0 + \varepsilon/2 + r_2|^2) = r_2(r_1^2 - |z_0 - \varepsilon/2 - r_1|^2).$$

This is equivalent to

$$\left(z_1 - \left(\frac{\varepsilon}{2}\frac{r_1 + r_2}{r_2 - r_1} + \frac{2r_1 r_2}{r_2 - r_1}\right)\right)^2 + z_2^2$$
$$= \left(\frac{2r_1 r_2}{r_2 - r_1}\right)^2 + \frac{2\varepsilon r_1 r_2(r_1 + r_2)}{(r_2 - r_1)^2} + \frac{\varepsilon^2}{4}\left[\left(\frac{r_1 + r_2}{r_2 - r_1}\right)^2 - 1\right].$$

We take

$$z_1 = \sqrt{\left(\frac{2r_1 r_2}{r_2 - r_1}\right)^2 + \frac{2\varepsilon r_1 r_2(r_1 + r_2)}{(r_2 - r_1)^2} + \frac{\varepsilon^2}{4}\left[\left(\frac{r_1 + r_2}{r_2 - r_1}\right)^2 - 1\right]}$$
$$+ \frac{\varepsilon}{2}\frac{r_1 + r_2}{r_2 - r_1} + \frac{2r_1 r_2}{r_2 - r_1},$$
$$z_2 = 0.$$

It is easy to see that $z_1 > \varepsilon/2 + 4r_1$. Therefore $z_0 \notin \overline{\mathcal{B}_1 \cup \mathcal{B}_2}$, and we can choose domains $\mathcal{D}_1$ and $\mathcal{D}$ such that $\mathcal{B}_1 \cup \mathcal{B}_2 \Subset \mathcal{D}_1 \Subset \mathcal{D}$, and $z_0 \notin \overline{\mathcal{D}}$. Hence T is smooth in $\mathcal{D}$ and is the desired conformal map.

7. The extreme case

In this section, we prove Theorem 1.4. The proof essentially follows that of [7, Theorem 1.1], with some modifications.

By considering $u - C$ instead of u, we may assume the constant $C = 0$ in (1.3). For any $0 < t < s < 1$, let $\eta \in C_c^\infty(\Omega_s)$ be a cutoff function such that $\eta = 1$ in Ω_t and $|D\eta| \le C(s - t)^{-1}$. Multiplying $u\eta^2$ on both sides of (1.3) and integrating by parts, we have

$$\int_{\Omega_1} a^{ij} D_j u D_i u \eta^2 + 2 a^{ij} D_j u D_i \eta u \eta = 0.$$

By Young's inequality,

$$\int_{\Omega_t} |Du|^2 \le \frac{C}{(s - t)^2} \int_{\Omega_s \setminus \Omega_t} u^2.$$

Since $u = 0$ on Γ_-, by the Poincaré inequality in the x_n direction, we have

$$\int_{\Omega_s \setminus \Omega_t} u^2 \le C(\varepsilon + s^2)^2 \int_{\Omega_s \setminus \Omega_t} |Du|^2.$$

Therefore,

$$\int_{\Omega_t} |Du|^2 \le C_0 \left(\frac{\varepsilon + s^2}{s - t}\right)^2 \int_{\Omega_s \setminus \Omega_t} |Du|^2. \tag{7.1}$$

Let $t_0 = r \in (\sqrt{\varepsilon}, 1/2)$ and $t_j = (1 - jr)r$ for $j \in \mathbb{N}$. Taking $s = t_j, t = t_{j+1}$ in (7.1), we have

$$\int_{\Omega_{t_{j+1}}} |Du|^2 \le 4C_0 \int_{\Omega_{t_j} \setminus \Omega_{t_{j+1}}} |Du|^2.$$

Adding both sides by $4C_0 \int_{\Omega_{t_{j+1}}} |Du|^2$ and dividing both sides by $1 + 4C_0$, we have

$$\int_{\Omega_{t_{j+1}}} |Du|^2 \le \frac{4C_0}{1 + 4C_0} \int_{\Omega_{t_j}} |Du|^2.$$

Let $k = \lfloor \frac{1}{2r} \rfloor$ and iterate the above inequality k times. We have

$$\int_{\Omega_{r/2}} |Du|^2 \le \left(\frac{4C_0}{1 + 4C_0}\right)^k \int_{\Omega_r} |Du|^2 \le C\mu^{\frac{1}{r}} \int_{\Omega_1} |u|^2, \tag{7.2}$$

where $\mu \in (0, 1)$ and C are constants depending only on n, σ, $\|h_1\|_{C^2}$, and $\|h_2\|_{C^2}$. For any $\bar{x} = (\bar{x}', \bar{x}_n) \in \Omega_{1/2}$, let $R = \varepsilon + h_1(\bar{x}') - h_2(\bar{x}')$. We make the change of variables by setting

$$
\begin{cases}
y' = x' - \bar{x}', \\
y_n = 2R \left(\dfrac{x_n - h_2(x') + \varepsilon/2}{\varepsilon + h_1(x') - h_2(x')} - \dfrac{1}{2} \right)
\end{cases}
$$

for $x = (x', x_n) \in \{|x' - \bar{x}'| < R, -\varepsilon/2 + h_2(x') < x_n < \varepsilon/2 + h_1(x')\}$. This change of variables maps the domain above to Q_R, where

$$
Q_s := \{(y', y_n) \in \mathbb{R}^n \mid |y'| < s, |y_n| < s\}
$$

for $s > 0$. Let

$$
(b^{ij}(y)) = \frac{(\partial_x y)\big(a^{ij}(x(y))\big)(\partial_x y)^t}{\det(\partial_x y)},
$$

$\tilde{b}^{ij}(y) = b^{ij}(Ry)$, and $\tilde{u}(y) = u(Rx)$. Then $\tilde{u}$ satisfies

$$
\begin{cases}
-\partial_i(\tilde{b}^{ij}(y)\partial_j \tilde{u}(y)) = 0 & \text{in } Q_1, \\
\tilde{b}^{nj}(y)\partial_j \tilde{u}(y) = 0 & \text{on } \{y_n = 1\}, \\
\tilde{u}(y) = 0 & \text{on } \{y_n = -1\}.
\end{cases}
$$

It is straightforward to verify that

$$
\frac{I}{C} \leq \tilde{b} \leq CI \quad \text{and} \quad \|\tilde{b}\|_{C^{m-1,\alpha}(Q_1)} \leq C,
$$

where C is a positive constant depending only on n, σ, m, α, and $C_{m,\alpha}$. Then by the Schauder estimate, we have

$$
\max_{-1 \leq y_n \leq 1} |D^m \tilde{u}(0', y_n)| \leq C \|D\tilde{u}\|_{L^2(Q_1)},
$$

which gives, after reversing the change of variables,

$$
|D^m u(\bar{x})| \leq C R^{1-m-n/2} \|Du\|_{L^2(\Omega_{|\bar{x}'|+R})} \quad \forall \bar{x} \in \Omega_{1/2}, \tag{7.3}
$$

where C is a positive constant depending only on n, σ, m, α, and $C_{m,\alpha}$. Then by using (7.2) and (7.3) with $r = 2 \max(\sqrt{\varepsilon}, |x'| + R)$, we conclude the proof.

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