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LONGTIME BEHAVIOR OF IMPULSIVE STOCHASTIC VEGETATION ECOSYSTEM WITH JUMP-DIFFUSION

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Abstract. We consider impulsive stochastic vegetation ecosystems with jump-diffusion, which have played a crucial role in the study of ecological protection. We are able to fully classify the longtime behavior of the underlying system. A threshold number is introduced so that its sign characterizes whether or not the vegetation goes extinct. In order to achieve this, we had to develop new analytical techniques to deal with jumps and impulses. The analysis is more subtle than in other population dynamics modeled by usual stochastic differential equations.

1. **Introduction.** Theoretically, ecological models play an important role in formulating and refining dynamic strategies to counter ecological degradation [9]. A common (deterministic) vegetation ecosystem [5] has the form of the following o-riginal differential equation (ODE)

$$\begin{aligned} dX(t) &= X(t)\Big[a_{12}Y(t) - a_{11}X(t) - \frac{b_1}{X(t)+1}\Big]dt, \\ dY(t) &= (R - a_{22}Y(t) - a_{21}X(t)Y(t))\,dt, \end{aligned} \tag{1.1}$$

where $X(t)$ is the vegetation biomass, $Y(t)$ is the soil water; and a_{12} is the maximum biomass growth rate, a_{11} is the ratio of the maximum biomass growth rate and the carrying capacity of biomass, and b_1 is the maximum consumption rate by herbivores or other factors. The parameters R, a_{22} and a_{21} are all positive real numbers representing the rainfall, the soil water loss rate and the consumption rate of water by biomass, respectively. Such above vegetation ecosystem is motivated from the study of environmental issues and ecological challenges like global warming, declining biodiversity, and land desertification. Which have intensified since the 1960s [8, 17] and references therein, and as a result, much attention has been devoted to the study of ecological protection [13] recently.

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On the other hand, deterministic models operate on the assumption that system parameters remain constant despite environmental fluctuations. From a biological perspective, this imposes limitation on the mathematical modeling of ecological systems because real-world population dynamics are inevitably influenced by environmental noise; see e.g., [16, 4] and references therein. From a different perspective, impulsive perturbations, as emphasized in [10, 6, 14], emerge across various domains including automatic control systems, computer networking, population models, neural networks, and economics. Sudden and intense changes take place abruptly in the form of impulses, presenting a modeling challenge when relying solely on continuous or discrete descriptions. Consequently, there is a substantial demand for research into impulsive systems.

The above needs motivate us to study the following impulsive stochastic vegetation ecosystem with jump-diffusion

$$\begin{aligned}
&dX(t) = X(t)\left[a_{12}Y(t) - a_{11}X(t) - \frac{b_1}{X(t)+b_2}\right]dt + \sigma_1 X(t)dW_1(t), \quad t\in[n,n+1)\\
&X(0)=x, \quad X(n) = (1+\rho)X(n-),\\
&dY(t) = (R - a_{22}Y(t) - a_{21}X(t)Y(t))\,dt + \sigma_2 Y(t)dW_2(t)\\
&\qquad\qquad + \int_{\mathbb{U}} Y(t-)h(u)\widetilde{N}(dt,du), \quad t\geq 0,\\
&Y(0)=y,
\end{aligned} \tag{1.2}$$

where $X(n-)=\lim_{s\to n^-}X(s)$, $Y(t-)=\lim_{s\to t^-}Y(s)$; $W_1(t)$, $W_2(t)$ are standard independent Brownian motions, and $N(du,dt)$ is a Poisson counting measure with characteristic measure ν on a measurable set $\mathbb{U}$ with $\nu(\mathbb{U})<\infty$, and $\widetilde{N}(dt,du) := N(dt,du) - \nu(du)dt$ is a Poisson martingale measure, which is independent of $W_1(t)$, $W_2(t)$.

In this paper, we first study fundamental properties of system (1.2), and then characterize fully its longtime behavior. Precisely, a threshold λ is introduced such that if $\lambda<0$ then $X(t)$ goes to 0 exponentially fast. In contrast, if $\lambda>0$, system (1.2) admits a unique invariant measure concentrating on $\{(u,v)\in\mathbb{R}^2 : u > 0, v>0\}$ and the transition probability converges to the invariant measure in total variation norm. A rate of convergence is also obtained. Compared with existing literature, the novelty and contribution of this work can be summarized as follows.

- From theoretical aspect, we provide a complete characterization for longtime behavior of a impulsive stochastic vegetation ecosystem with jump-diffusion for the first time, to be the best of the authors' current knowledge.
- From biological point of view, it provides helpful insights for ecologists as well as advances the study of ecological protection. The threshold λ, which fully forecast what will happen in the future, is computed explicitly from parameters in the model.
- The techniques developed in the paper to deal with jumps and impulses can be generalized to study other ecological and biological systems modeled by impulsive jump-diffusion.

The rest of paper is organized as follows. Section 2 presents our main results, which states basic properties of (1.2), introduces threshold λ, and provides a longtime characterization of (1.2). Section 3 is devoted to proofs, in which Section 3.1

proves fundamental properties, Section 3.2 is devoted to a proof of the permanence case while a proof of extinction is given in Section 3.3.

2. **Main results.** We denote by $\mathcal{L}$ the generator of the jump-diffusion process $(X(t), Y(t))$ from (1.2). We will also use $\mathbb{P}_{x,y,s}$, $\mathbb{E}_{x,y,s}$ to indicate the probability and expectation conditioned on $X(s) = x, Y(s) = y$. When $s = 0$, we simple use $\mathbb{P}_{x,y}, \mathbb{E}_{x,y}$ for $\mathbb{P}_{x,y,0}$ and $\mathbb{E}_{x,y,0}$, respectively. Denote $\mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2 : u, v \geq 0\}$, $\mathbb{R}^{2,\circ}_+ = \{(u,v) \in \mathbb{R}^2 : u, v > 0\}$.

The following assumption is held throughout the paper.

Assumption 2.1. *The following conditions hold:*

$$\int_{\mathbb{U}} h^2(u)d\nu(u) < \infty \text{ and } \int_{\mathbb{U}} (1 + h(u))^{-1} d\nu(u) < \infty.$$

The first theorem tells us that together with well-posedness, positivity, and Markov-Feller properties of (1.2), we can bound the moments of the process. Moreover, the solution process stays in compact sets with large probability if starting from a compact set.

Theorem 2.1. *We have the following claims.*

(i) For any initial value $(x, y) \in \mathbb{R}^2_+$, there exists a unique a global solution $(X(t), Y(t))$ to (1.2) such that $\mathbb{P}_{x,y}\{(X(t), Y(t)) \in \mathbb{R}^2_+,\ \forall t \geq 0\} = 1$. Moreover, $\mathbb{P}_{x,y}\{X(t) > 0,\ \forall t > 0\} = 1$ and $\mathbb{P}_{x,0}\{Y(t) = 0,\ \forall t \geq 0\} = 1$.

(ii) Let $p := \frac{a_{22}}{4\int_{\mathbb{U}} h^2(u)d\nu} \wedge \frac{a_{22}}{4(\sigma_1^2 \vee \sigma_2^2)} \wedge 0.1$ and $a_V := \frac{a_{21}}{a_{12}} \wedge 1$ and $V(x, y) = (a_V x + y)^{1+p}$. Then there exist constants K_1, K_2 and $k_2 > 0$ such that

$$\mathbb{E}_{x,y,s} Y^{1+p}(t) \leq \frac{2K_1}{a_{22}} + e^{-\frac{a_{22}}{2}(t-s)} y^{1+p},\ \forall (x, y) \in \mathbb{R}^2_+, t \geq s, \tag{2.1}$$

and

$$\mathbb{E}_{x,y,s} V(X(t), Y(t)) \leq K_2(1 + e^{-k_2(t-s)} V(x, y)),\ \forall (x, y) \in \mathbb{R}^2_+, t \geq s. \tag{2.2}$$

(iii) There exists $K_3 > 0$ such that

$$\mathbb{E}_{x,y,s}(1 + a_V X(t) + Y(t))^4 \leq e^{K_3(t-s)}(1 + a_V x + y)^2,\ \forall (x, y) \in \mathbb{R}^2_+, t \geq s. \tag{2.3}$$

(iv) For any $\varepsilon > 0, H > 0, T > 0$, there exists $\widetilde{K}(\varepsilon, H, T) > 0$ such that

$$\mathbb{P}_{x,y,s}\left\{X(t) + Y(t) \leq \widetilde{K}(\varepsilon, H, T),\ \forall s \leq t \leq T + s\right\} \geq 1 - \varepsilon \text{ given } x \vee y \leq H. \tag{2.4}$$

(v) $(X(t), Y(t))$ is a non-homogeneous Markov-Feller process.

We introduce the following threshold λ

$$\lambda = \ln(1 + \rho) + \frac{a_{12}R}{a_{22}} - \frac{b_1}{b_2} - \frac{\sigma_1^2}{2}, \tag{2.5}$$

and show that its sign fully characterize longtime behavior of (1.2).

If λ is positive, the system is persistence, that means, there exists uniquely an invariant probability measure μ^* of $\{(X(t), Y(t)), t \geq 0\}$ on $\mathbb{R}^{2,\circ}_+$. Moreover, the convergence and rate of convergence of the transition probability $P_t((x, y), \cdot)$ to μ^* is also obtained.

Theorem 2.2. *If $\lambda > 0$, then there exists uniquely an invariant probability measure μ^* of $\{(X(t), Y(t)), t \geq 0\}$ on $\mathbb{R}^{2,\circ}_+$. Moreover, the transition probability $P_t((x, y), \cdot)$ converges exponentially fast to μ^* in total variation norm.*

In contrast, if $\lambda < 0$, $X(t)$ goes extinct exponentially fast almost surely.

Theorem 2.3. *If $\lambda < 0$ then $\lim_{t\to\infty} \frac{\ln X(t)}{t} = \lambda < 0$ with probability 1.*

Remark 2.1. We make some remarks on an intuition of the threshold λ and Assumption 2.1 as follows.

- The definition of the threshold λ is inspired from a dynamical system theory point of view (the so-called Lyapunov exponent or stochastic growth rate). In particular, λ is actually defined as an approximation of $\lim_{t\to\infty} \frac{\ln X(t)}{t}$ when $X(t)$ is small. Precisely, let $\widetilde{Y}(t)$ be a positive solution to
$$d\widetilde{Y}(t) = \Big(R - a_{22}\widetilde{Y}(t)\Big)\,dt + \sigma_2\widetilde{Y}(t)dW_2(t) + \int_{\mathbb{U}} \widetilde{Y}(t-)h(u)\widetilde{N}(dt,du).$$
It is proved in [3] that the process $\{\widetilde{Y}(t)\}$ has a unique invariant measure, ν^* on $[0,\infty)$ satisfying
$$\int_{[0,\infty)} y\nu^*(dy) = \frac{R}{a_{22}}.$$
By the ergodicity of $\{\widetilde{Y}(t)\}$, we have
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{Y}(s)ds = \int_{[0,\infty)} y\nu^*(dy) = \frac{R}{a_{22}} \text{ a.s.}$$
Therefore, roughly speaking, when $X(t)$ is small and $Y(t)$ is close to $\widetilde{Y}(t)$,
$$\begin{aligned}\lim_{t\to\infty} \frac{\ln X(t)}{t} &= \limsup_{t\to\infty} \left(\frac{\ln X(t)}{t} + \frac{W_1(t)}{t} + \ln(1+\rho)\frac{[t]}{t} + \right)\\ &\quad + \lim_{t\to\infty} \frac{1}{t}\int_0^t \left\{(a_{12}Y(s) - a_{11}X(s) - \frac{b_1}{X(s)+b_2} - \frac{\sigma_1^2}{2}\right\} ds\\ &\approx \ln(1+\rho) + \limsup_{t\to\infty} \frac{1}{t}\int_0^t a_{12}\widetilde{Y}(s)ds - \frac{b_1}{b_2} - \frac{\sigma_1^2}{2}\\ &= \ln(1+\rho) + \frac{a_{12}R}{a_{22}} - \frac{b_1}{b_2} - \frac{\sigma_1^2}{2} := \lambda.\end{aligned} \tag{2.6}$$
- Assumption 2.1 guarantees the solution having finite $(1+p)^{th}$-order moment, for some small $p > 0$. Moreover, it also helps to ensure the existence and uniqueness of the invariant measure of the problem on the boundary, which is needed to define λ.

3. Proof of main results.

3.1. Proof of Theorem 2.1.

Proof of Theorem 2.1. Consider a system without impulses in each interval $t \in [n, n+1)$:
$$\begin{aligned}dX(t) &= X(t)\left[a_{12}Y(t) - a_{11}X(t) - \frac{b_1}{X(t)+b_2}\right]dt + \sigma_1 X(t)dW_1(t),\\ dY(t) &= (R - a_{22}Y(t) - a_{21}X(t)Y(t))\,dt + \sigma_2 Y(t)dW_2(t) + \int_{\mathbb{U}} Y(t-)h(u)\widetilde{N}(dt,du).\end{aligned} \tag{3.1}$$
The existence and uniqueness of continuous nonnegative solutions to (3.1), which can be proved by standard arguments, see e.g. [18], will imply the existence and

uniqueness of continuous nonnegative solutions to (1.2) on each interval $[n, n+1)$. Together with the impulses at $n \in \mathbb{Z}_+$, we can easily obtain part (i) of Theorem 2.1.

Next, to simplify the notation, we prove parts $(ii)-(v)$ only for $s=0$. The readers can see easily from the proofs that the statements of parts $(ii)-(v)$ hold for any $s>0$ with the constants being chosen independent of s.

Now, we prove part (ii). Consider $V(x,y)=(a_Vx+y)^{1+p}$. We have

$$\begin{aligned}
&\mathcal{L}V(x,y)\\
&=(1+p)(a_Vx+y)^p\Big[R-a_{22}y-(a_{21}-a_Va_{12})xy-a_Va_{11}x^2-\frac{a_Vb_1}{x+b_2}\Big]\\
&+p(1+p)(a_Vx+y)^{p-1}(a_V^2\sigma_1^2x^2+\sigma_2^2y^2)\\
&+\int_{\mathbb{U}}\big(a_Vx+y(1+h(u))^{1+p}-(a_Vx+y)^{1+p}-(1+p)(a_Vx+y)^pyh(u)\big)\,d\nu(u)\\
\leq& K_1-x^2-\frac{3a_{22}}{4}(1+p)(a_Vx+y)^{1+p}\\
&+(a_Vx+y)^{1+p}\int_{\mathbb{U}}\left(1+\frac{h(u)y}{a_Vx+y})^{1+p}-1-(1+p)\frac{h(u)y}{a_Vx+y}\right)d\nu(u),
\end{aligned} \tag{3.2}$$

for some finite constant $K_1>1$. From Taylor's expansion, we have

$$(1+z)^{1+p}\leq 1+(1+p)(1+z)^p+pz^2 \text{ for } z>0,$$

which leads to

$$\begin{aligned}
&\int_{\mathbb{U}}\left(1+\frac{h(u)y}{a_Vx+y})^{1+p}-1-(1+p)\frac{h(u)y}{a_Vx+y}\right)d\nu(u)\\
\leq& p\int_{\mathbb{U}}\left(\frac{h(u)y}{a_Vx+y}\right)^2d\nu(u)\\
\leq& p\int_{\mathbb{U}}h^2(u)d\nu\leq\frac{a_{22}}{4}.
\end{aligned} \tag{3.3}$$

Applying (3.3) to (3.2), we have

$$\mathcal{L}V(x,y)\leq K_1-x^2-\frac{a_{22}}{2}V(x,y). \tag{3.4}$$

In light of (3.4) and Dynkin's formula, one can easily implies for $n\leq s\leq t<n+1$ that

$$\begin{aligned}
&\mathbb{E}\left(e^{\frac{a_{22}}{2}(t\wedge\tau_{n,s}-s)}V(X(t\wedge\tau_{n,s}),Y(t\wedge\tau_{n,s})\right)\\
&\leq\mathbb{E}\left(V(X(s),Y(s)\right)+\mathbb{E}\int_s^t\left(e^{\frac{a_{22}}{2}(u-s)}K_1\right)du\\
&\leq\frac{2K_1}{a_{22}}e^{\frac{a_{22}}{2}(t-s)}+\mathbb{E}V(X(s),Y(s)),
\end{aligned} \tag{3.5}$$

if $\mathbb{E}V(X(s),Y(s))<\infty, n\in\mathbb{Z}_+$ where $\tau_{n,s}=\inf\{t\geq s: V(X(t),Y(t))\geq n\}$. Letting $n\to\infty$ in (3.5) and then dividing both sides by $e^{\frac{a_{22}}{2}(t-s)}$, we have

$$\mathbb{E}V(X(t),Y(t))\leq\frac{2K_1}{a_{22}}+e^{-\frac{a_{22}}{2}(t-s)}\mathbb{E}V(X(s),Y(s)) \tag{3.6}$$

if $n\leq s\leq t\leq n+1$ for some $n\in\mathbb{Z}_+$.

Similarly, we can obtain that

$$\mathbb{E}Y^{1+p}(t) \leq \frac{2K_1}{a_{22}} + e^{-\frac{a_{22}}{2}(t-s)}\mathbb{E}Y^{1+p}(s) \text{ if } n \leq s \leq t \leq n+1 \text{ for some } n \in \mathbb{Z}_+. \tag{3.7}$$

Applying (3.6) for $t \leq 1$, we have

$$\mathbb{E}_{x,y}V(X(1-), Y(1-)) \leq \frac{2K_1}{a_{22}} + e^{-\frac{a_{22}}{2}}V(x,y).$$

On the other hand, we also have from (3.4) that

$$\int_s^t \mathbb{E}X^2(u)du \leq \mathbb{E}\left(V(X(s), Y(s)) + K_1,\right.$$

which leads to $\mathbb{E}X^2(t_n) \leq \mathbb{E}\left(V(X(n), Y(n)) + K_1\right.$ for some $t_n \in [n, n+1)$. We have from Hölder's inequality and standard calculations that

$$\mathbb{E}X^{1+p}(t_n) \leq (\mathbb{E}(V(X(n), Y(n) + K_1)^{\frac{1}{1+p}} \leq C_1 + \frac{1}{2^{2+p}(1+\rho)^{1+p}}\mathbb{E}V(X(n), Y(n)), \tag{3.8}$$

for some C_1 depending only on K_1, p and ρ.

Combining (3.8) and (3.7) implies that

$$\begin{aligned}
&\mathbb{E}_{x,y}V(X((n+1)-), Y(n+1)-) \\
&\leq \frac{2K_1}{a_{22}} + V(X(t_n), Y(t_n) \\
&\leq \frac{2K_1}{a_{22}} + 2^{1+p}a_V^{1+p}\mathbb{E}_{x,y}X^{1+p}(t_n) + 2^{1+p}\mathbb{E}_{x,y}Y^{1+p}(t_n) \\
&\leq \frac{2K_1}{a_{22}} + 2^{1+p}C_1a_V^{1+p} + \frac{2^{1+p}a_V^{1+p}}{2^{2+p}(1+\rho)^{1+p}}\mathbb{E}_{x,y}V(X(n), Y(n)) + \frac{2^{2+p}K_1}{a_{22}} \\
&\quad + \frac{K_1 2^{2+p}}{a_{22}}e^{-\frac{a_{22}}{2}n}y^{1+p}.
\end{aligned} \tag{3.9}$$

Since $V(X((n+1)), Y(n+1)) \leq (1+\rho)^{1+p}V(X((n+1)-), Y(n+1)-)$ and $a_V \leq 1$, we have from (3.9) that

$$\mathbb{E}_{x,y}V(X((n+1)), Y(n+1)) \leq C_2 + \frac{K_1 2^{2+p}}{a_{22}}e^{-\frac{a_{22}}{2}n}y^{1+p} + \frac{1}{2}\mathbb{E}_{x,y}V(X(n), Y(n)),$$

for some constant C_2 independent of (x, y) and n. Applying this inequality recursively, we have

$$\mathbb{E}_{x,y}V(X(1), Y(1)) \leq C_2 + \frac{K_1 2^{2+p}}{a_{22}}y^{1+p} + \frac{1}{2}V(x,y),$$

and

$$\begin{aligned}
\mathbb{E}_{x,y}V(X(2), Y(2)) \leq& C_2 + \frac{K_1 2^{2+p}}{a_{22}}e^{-\frac{a_{22}}{2}}y^{1+p} + \frac{1}{2}\mathbb{E}_{x,y}V(X(1), Y(1)) \\
\leq& C_2(1+\frac{1}{2}) + \frac{K_1 2^{2+p}}{a_{22}}y^{1+p}\left(e^{-\frac{a_{22}}{2}} + \frac{1}{2}\right) + \frac{1}{4}V(x,y).
\end{aligned}$$

Continuing this way, we have

$$\begin{aligned}
&\mathbb{E}_{x,y}V(X(n+1),Y(n+1))\\
&\leq C_2\sum_{k=0}^{n}\frac{1}{2^k}+\frac{K_1 2^{2+p}}{a_{22}}y^{1+p}\sum_{k=0}^{n}\frac{1}{2^k}e^{-\frac{a_{22}(n+1-k)}{2}}+\frac{1}{2^{n+1}}V(x,y)\\
&\leq 2C_2+C_3\breve{\kappa}^n y^{1+p}+\frac{1}{2^{n+1}}V(x,y)\\
&\leq 2C_2+\left(C_3\breve{\kappa}^n+\frac{1}{2^{n+1}}\right)V(x,y),
\end{aligned}$$

for some constants $\breve{\kappa}\in(0,1)$, C_3 depending only on K_1, a_{22}. This together with (3.6) implies (2.3) when $s=0$. Similarly, we can obtain that

$$\mathbb{E}Y^{1+p}(t)\leq\frac{2K_1}{a_{22}}+e^{-\frac{a_{22}}{2}(t-s)}\mathbb{E}Y^{1+p}(s)\text{ for all }t\geq s. \tag{3.10}$$

Part (ii) is proved.

Similar to the estimate (3.4), we can show that

$$\mathcal{L}\check{V}(x,y)\leq K_3\check{V}(x,y)\text{ where }\check{V}(x,y)=(1+a_V x+y)^4,$$

for some constant K_3 independent of (x,y). Applying Dynkin's formula for $e^{-K_3t}\check{V}(X(t),Y(t))$, we can easily deduce that

$$\begin{aligned}
&\mathbb{E}e^{-K_3t}\check{V}(X(t),Y(t))\\
&\leq\mathbb{E}\check{V}(X(s),Y(s))+\mathbb{E}\int_s^t\left[-K_3e^{-K_3i}\check{V}(X(u),Y(u))+e^{-K_3u}\mathcal{L}\check{V}(X(u),Y(u))\right]du\\
&\leq\mathbb{E}\check{V}(X(s),Y(s))\text{ if }t\leq s,[t]=[s],
\end{aligned}$$

which leads to

$$\mathbb{E}_{x,y}\check{V}(X(t),Y(t))\leq e^{K_3(t-s)}\check{V}(X(s),Y(s)),\ \text{ if }t\leq s,[t]=[s].$$

Therefore, part (iii) is proved.

To prove (iv), we deduce from (3.6) that by Markov's property, we have

$$\begin{aligned}
&\mathbb{P}\{V(X(t),Y(t))\leq n\text{ for all }s\leq t\leq[s]+1\}\\
&\geq 1-\mathbb{P}\{\tau_{n,s}<[s]+1\}\\
&\geq 1-\frac{\mathbb{E}\left(e^{\frac{a_{22}}{2}(t\wedge\tau_{n,s}-s)}V(X(t\wedge\tau_{n,s}),Y(t\wedge\tau_{n,s}))\right)}{n}\\
&\geq 1-\frac{1}{n}\left(\frac{2K_1}{a_{22}}e^{\frac{a_{22}}{2}(t-s)}+\mathbb{E}V(X(s),Y(s))\right),
\end{aligned} \tag{3.11}$$

which together with (2.2) implies (2.4).

Finally, to proof the process $(X(t),Y(t))$ is a Markov Feller process can follow arguments from [12] with slight modification needed due to jumps and impulsive. □

3.2. **Proof of persistence.** This section is devoted to a proof of Theorem 2.2. Let $\delta_1>0$ such that

$$a_{11}x+\frac{b_1}{x+b_2}\leq\frac{b_1}{b_2}+\frac{\lambda}{8}\text{ for all }0\leq x\leq\delta_1;$$

and p be as in Theorem 2.1. Let $H^*>0$ be such that

$$a_{12}H^* - a_{11} - \frac{b_1}{b_2} - \frac{\sigma_1^2}{2} \geq \lambda, \tag{3.12}$$

and such that $K_{H^*} = \left(\frac{2K_1}{a_{22}} + H^*\right)^{\frac{1}{1+p}} < \frac{a_{22}\lambda}{4a_{12}}$. Now, we pick $T^* \in \mathbb{Z}_+$ such that

$$\frac{\ln(1+\rho)}{T^*} + K_{H^*}\frac{a_{12}}{a_{22T^*}} \leq \frac{\lambda}{8}, \tag{3.13}$$

and $\varepsilon^* \in (0,1)$ such that

$$\frac{a_{12}}{a_{22}}(a_{21}+a_{22})\varepsilon^* \leq \frac{\lambda}{8},\ a_{11}\varepsilon^* \leq \frac{\lambda}{8}. \tag{3.14}$$

Let $F^* = F(H^*,\varepsilon^*) > H^*$ be sufficiently large such that

$$\frac{1}{(F^*)^p}\left(\frac{2K_1}{a_{22}} + (H^*)^{1+p}\right) \leq \varepsilon^*.$$

To prove Theorem 2.2, we first need some Lemmas, and Propositions as follows.

Lemma 3.1. *For H^*, T^*, ε^* defined above, there exists $\delta_0 = \delta_0(T^*, H^*, \varepsilon^*) \in (0,1)$ such that*

$$\mathbb{E}_{x,y,s}X(t) \leq \varepsilon^* \text{ and } \mathbb{E}_{x,y,s}X(t)Y(t) \leq \varepsilon^* \text{ for } s \leq t \leq T^*,\ \text{ if } 0 < x < \delta_0, 0 \leq y \leq H^*. \tag{3.15}$$

Proof. In view of (2.4) (with T replaced by T^*), there exists $M = M(\varepsilon^*, H^*, T^*) > 0$ such that

$$\frac{1}{a_V^2}\left(\mathbb{P}_{x,y,s}\{\tau_{M,s} > t\}(a_V + H^*)^4 e^{K_3 T^*}\right)^{\frac{1}{2}} \leq \frac{\varepsilon^*}{2} \text{ if } 0 \leq x \leq 1, 0 \leq y \leq H^*,$$

where $\tau_{M,s} = \inf\{t \geq s : Y(t) \geq M\}$. We have

$$\begin{aligned} &X(t\wedge\tau_{M,s}) \\ &= X(s)(1+\rho)^{[t\wedge\tau_{M,s}]-[s]} e^{\int_s^{t\wedge\tau_{M,s}} a_{12}Y(u) - a_{11}X(u) - \frac{b_1}{X(u)+b_2}} \\ &\quad \cdot e^{\frac{\sigma_1^2}{2}(t\wedge\tau_{M,s}-s)+\sigma_1(W_1(t\wedge\tau_{M,s})-W_1(s))} \\ &\leq X(s)(1+\rho)^{T^*} e^{a_{12}MT^*} e^{\frac{\sigma_1^2}{2}(t\wedge\tau_{M,s}-s)+\sigma_1(W_1(t\wedge\tau_{M,s})-W_1(s))},\ s \leq t \leq T^*. \end{aligned} \tag{3.16}$$

Taking expectation both sides, we have

$$\begin{aligned} \mathbb{E}_{x,y,s}X(t\wedge\tau_{M,s}) \leq& x(1+\rho)^{T^*} e^{a_{12}MT^*}\mathbb{E}_{x,y,s} e^{\frac{\sigma_1^2}{2}(t\wedge\tau_{M,s}-s)+\sigma_1(W_1(t\wedge\tau_{M,s})-W_1(s))} \\ \leq& x(1+\rho)^{T^*} e^{a_{12}MT^*},\ s \leq t \leq T^*. \end{aligned} \tag{3.17}$$

As a result, one has

$$\begin{aligned} \mathbb{E}_{x,y,s}\left[\mathbf{1}_{\{\tau_{M,s}>t\}}X(t)Y(t)\right] \leq& \mathbb{E}_{x,y,s}X(t\wedge\tau_{M,s})Y(t\wedge\tau_{M,s}) \\ \leq& xM(1+\rho)^{T^*} e^{a_{12}MT^*}, s \leq t \leq T^*, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\mathbb{E}_{x,y,s}\left[\mathbf{1}_{\{\tau_{M,s}\le t\}}X(t)Y(t)\right] &\le \left(\mathbb{P}_{x,y,s}\{\tau_{M,s}<t\}\mathbb{E}_{x,y}[X(t)Y(t)]^2\right)^{\frac{1}{2}}\\ &\le \frac{1}{a_V^2}\left(\mathbb{P}_{x,y,s}\{\tau_{M,s}>t\}\mathbb{E}_{x,y}[a_VX(t)+Y(t)]^4\right)^{\frac{1}{2}}\\ &\le \frac{1}{a_V^2}\left(\mathbb{P}_{x,y,s}\{\tau_{M,s}>t\}(a_V+H^*)^4e^{-K_3T^*}\right)^{\frac{1}{2}}\\ &\le \frac{\varepsilon^*}{2}\ \text{if } x\le 1, y\le H^*.\end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19), we have

$$\mathbb{E}_{x,y,s}[X(t)Y(t)]\le xM(1+\rho)^{T^*}e^{a_{12}MT^*}+\frac{\varepsilon^*}{2},\ \forall s\le t\le T^*,\ \text{for } (x,y)\in[0,H^*]^2. \tag{3.20}$$

Similarly, we can have from (3.17), (2.3) and Holder's inequality that

$$\mathbb{E}_{x,y,s}[X(t)]\le x(1+\rho)^{T^*}e^{a_{12}MT^*}+\frac{\varepsilon^*}{2},\ s\le t\le T^*. \tag{3.21}$$

if $(x,y)\in[0,H^*]^2$. Picking $\delta_0=\frac{\varepsilon^*}{2}\left(M(1+\rho)^{T^*}e^{a_{12}MT^*}\right)^{-1}$, we obtain (3.15). □

Lemma 3.2. *Let δ_0 be as in Lemma 3.1. For any $T\le T^*$, we have*

$$\begin{aligned}&\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^{T}F^*\wedge Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\\ &\ge \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\frac{1}{a_{22}}\left(RT-K_{H^*}-(a_{21}+a_{22})\varepsilon^*T\right),\end{aligned} \tag{3.22}$$

where $\zeta=\inf\{t\ge 0: Y(t)\ge H^*\ \text{or}\ X(t)\ge\delta_0\}$.

Proof. Note that

$$\begin{aligned}Y(T)=&Y(T\wedge\zeta)+\int_{T\wedge\zeta}^{T}\left(R-a_{22}Y(t)-a_{21}X(t)Y(t)\right)dt\\ &+\int_{T\wedge\zeta}^{T}\sigma_2Y(t)dW_2(t)+\int_0^T\int_{\mathbb{U}}Y(t-)h(u)\widetilde{N}(dt,du).\end{aligned} \tag{3.23}$$

In view of Theorem 2.1, we can take the conditional expectation $\mathcal{F}_{T\wedge\zeta}$ both sides of the equation above to obtain

$$\mathbb{E}_{x,y}[Y(T)|\mathcal{F}_{T\wedge\zeta}]-Y(T\wedge\zeta)=\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^{T}\left(R-a_{22}Y(t)-a_{21}X(t)Y(t)\right)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]. \tag{3.24}$$

As a result,

$$\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^{T}Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\ge\frac{1}{a_{22}}\left(R[T-T\wedge\zeta]+Y(T\wedge\zeta)-\mathbb{E}_{x,y}[Y(T)|\mathcal{F}_{T\wedge\zeta}]\right). \tag{3.25}$$

It is noted that if $\zeta < T$ and $X(\zeta) < \delta_0 \le 1$ then $Y(\zeta) \ge H^*$. By the strong Markov property of $(X(t), Y(t))$ and (2.1), we have

$$\begin{aligned}\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}&\mathbb{E}_{x,y}[Y(T)|\mathcal{F}_{T\wedge\zeta}] - Y(T\wedge\zeta)\\ &\le \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y}[Y(T)|\mathcal{F}_{T\wedge\zeta}]\\ &= \mathbb{E}_{x,y}\left[\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y,\zeta}Y(T)\right]\\ &\le \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\left(\frac{2K_1}{a_{22}}+H^*\right)^{\frac{1}{1+p}},\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}&\mathbb{E}_{x,y}\left[\int_\zeta^T X(t)Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\\ &= \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^T X(t)Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]dt\\ &= \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\int_{T\wedge\zeta}^T \mathbb{E}_{x,y}\left[X(t)Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]dt\\ &\le \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\int_{T\wedge\zeta}^T \varepsilon^* dt \le \varepsilon^*(T-\zeta).\end{aligned} \tag{3.27}$$

Multiplying (3.25) with $\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}$ then using (3.26) and (3.27), we have

$$\begin{aligned}\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}&\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^T Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\\ &\ge \frac{\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}}{a_{22}}\left(R(T-\zeta)-\left(\frac{2K_1}{a_{22}}+H^*\right)^{\frac{1}{1+p}}-a_{21}\varepsilon(T-\zeta)\right).\end{aligned} \tag{3.28}$$

On the one hand, one has

$$\begin{aligned}\mathbb{E}_{x,y}&\left[\int_{T\wedge\zeta}^T \mathbf{1}_{\{Y(t)\ge F^*\}}Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\\ &\le \frac{1}{(F^*)^p}\int_{T\wedge\zeta}^T \mathbb{E}_{x,y}\left[Y^{1+p}(t)\Big|\mathcal{F}_{T\wedge\zeta}\right]dt\\ &\le \frac{1}{(F^*)^p}\left(\frac{2K_1}{a_{22}}+(H^*)^{1+p}\right)(T-\zeta) \le \varepsilon^*(T-\zeta).\end{aligned} \tag{3.29}$$

We deduce from (3.28) and (3.29) that

$$\begin{aligned}&\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^T F_H\wedge Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\\ &\ge \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^T Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\\ &- \mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}\mathbb{E}_{x,y}\left[\int_{T\wedge\zeta}^T \mathbf{1}_{\{Y(t)\ge F^*\}}Y(t)dt\Big|\mathcal{F}_{T\wedge\zeta}\right]\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}}{a_{22}}\left(R(T-\zeta)-\left(\frac{2K_1}{a_{22}}+H^*\right)^{\frac{1}{1+p}}-a_{21}\varepsilon^* T\right)+\varepsilon^*(T-\zeta)\\
&=\frac{\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}}}{a_{22}}\left(R(T-\zeta)-\left(\frac{2K_1}{a_{22}}+H^*\right)^{\frac{1}{1+p}}-(a_{21}+a_{22})\varepsilon^*(T-\zeta)\right).
\end{aligned}
\tag{3.30}
$$

Therefore, the proof is complete. □

Lemma 3.3. *There exist $c_1, c_2 > 0$ such that*

$$\mathbb{E}_{x,y,s}X^{-1}(t) \leq c_2 e^{c_1(t-s)}[x^{-1} \vee 1].$$

Proof. We prove this lemma for $s = 0$. It can be seen from the proof can be extended for any s with c_1, c_2 being independent of s. By Itô's formula,

$$
\begin{aligned}
dX^{-1}(t) =& \left(a_{11} + \frac{b_1}{X(t)(X(t)+b_2)} - \frac{a_{12}Y(t)}{X(t)} + \frac{\sigma_1^2}{X(t)}\right) dt - \frac{\sigma_1}{X(t)} dW_1(t)\\
\leq& a_{11} + \left(\frac{b_1}{b_2} + \sigma_1^2\right) X^{-1}(t)dt - \sigma_1 X^{-1}(t)dW_1(t), t \notin \mathbb{Z}_+.
\end{aligned}
$$

From this, we can easily obtain that

$$d\mathbb{E}_{x,y}X^{-1}(t) \leq a_{11} + \left(\frac{b_1}{b_2} + \sigma_1^2\right)\mathbb{E}_{x,y}X^{-1}(t)dt, 0 \leq t < 1.$$

This differential inequality implies that

$$\mathbb{E}_{x,y}X^{-1}(t) \leq \left(x^{-1} + a_{11}\left(\frac{b_1}{b_2}+\sigma_1^2\right)^{-1}\right)e^{\left(\frac{b_1}{b_2}+\sigma_1^2\right)t} - a_{11}\left(\frac{b_1}{b_2}+\sigma_1^2\right)^{-1}.$$

Thus,

$$
\begin{aligned}
\mathbb{E}_{x,y}X^{-1}(1) =& \frac{1}{1+\rho}\mathbb{E}_{x,y}X^{-1}(1-)\\
\leq& \left(x^{-1} + a_{11}\left(\frac{b_1}{b_2}+\sigma_1^2\right)^{-1}\right)e^{\left(\frac{b_1}{b_2}+\sigma_1^2\right)} - a_{11}\left(\frac{b_1}{b_2}+\sigma_1^2\right)^{-1}.
\end{aligned}
$$

Continuing this process, we have

$$\mathbb{E}_{x,y}X^{-1}(t) \leq \left(x^{-1} + a_{11}\left(\frac{b_1}{b_2}+\sigma_1^2\right)^{-1}\right)e^{\left(\frac{b_1}{b_2}+\sigma_1^2\right)t} - a_{11}\left(\frac{b_1}{b_2}+\sigma_1^2\right)^{-1}$$

for any $t \geq 0$.

The claim of the lemma follows obviously. □

Lemma 3.4. *For any $T \leq T^*$. Let $A_T := \{\zeta < T, X(\zeta) < \delta_0\} \cup \{\zeta \geq T\}$, and*

$$\Phi(T) := -\mathbf{1}_{A_T}\int_0^T \left[a_{12}(F^* \wedge Y(t)) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right]dt - \mathbf{1}_{A_T^c}\frac{\lambda T}{2}.$$

We have

$$\mathbb{E}_{x,y}\Phi(T) \leq -\frac{\lambda T}{2} + \ln(1+\rho)T,\ \forall (x,y) \in [0,H^*]^2, T \leq T^*.$$

Proof. By the definition of ζ, if $\zeta \geq T$ then $\Phi(T) \leq \frac{-\lambda T}{4}$. It is noted again that if $\zeta < T$ and $X(\zeta) < \delta_0 \leq 1$ then $Y(\zeta) \geq H^*$. Therefore, we have

$$\begin{aligned}\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}} \int_0^\zeta \left[a_{12}(F^*\wedge Y(t)) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right] dt \\ \geq \Big(a_{12}H^* - a_{11} - \frac{b_1}{b_2} - \frac{\sigma_1^2}{2}\Big)\zeta \geq \lambda\zeta.\end{aligned} \tag{3.31}$$

Moreover, by using (3.22), (3.15) and definition (2.5) of λ, one has

$$\begin{aligned}&\mathbb{E}_{x,y}\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}} \int_\zeta^T \left[a_{12}(F^*\wedge Y(t)) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right] dt \\ &\geq \mathbb{E}_{x,y}\bigg[1_{\{\zeta<T,X(\zeta)<\delta_0\}}\frac{a_{12}}{a_{22}}\left(R(T-\zeta) - (a_{21}+a_{22})\varepsilon(T-\zeta)\right) - K_{H^*}\frac{a_{12}}{a_{22}} \\ &\quad - \left(\frac{\sigma_1^2}{2} + \frac{b_1}{b_2}\right)(T-\zeta)\bigg] - \mathbb{E}_{x,y}\left[1_{\{\zeta<T,X(\zeta)<\delta_0\}}\int_\zeta^T a_{11}X(t)dt\right] \\ &\geq \mathbb{E}_{x,y}\left[1_{\{\zeta<T,X(\zeta)<\delta_0\}}\frac{3\lambda(T-\zeta)}{4} - \ln(1+\rho)(T-\zeta) - K_{H^*}\frac{a_{12}}{a_{22}}\right].\end{aligned} \tag{3.32}$$

From (3.31), (3.32) and definition of K_{H^*}, we have

$$\begin{aligned}&\mathbb{E}_{x,y}\mathbf{1}_{\{\zeta<T,X(\zeta)<\delta_0\}} \int_0^T \left[a_{12}(F^*\wedge Y(t)) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right] dt \\ &\geq \left(\frac{3\lambda T}{4} - \ln(1+\rho)T - K_{H^*}\frac{a_{12}}{a_{22}}\right)\mathbb{E}_{x,y}\left[1_{\{\zeta<T,X(\zeta)<\delta_0\}}\right] \\ &\geq \left(\frac{\lambda T}{2} - \ln(1+\rho)T\right)\mathbb{E}_{x,y}\left[1_{\{\zeta<T,X(\zeta)<\delta_0\}}\right].\end{aligned} \tag{3.33}$$

Since $a_{12}F^*\wedge Y(t) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2} \geq \frac{\lambda}{2}$ if $Y(t) \geq H^*$ and $X(t) \leq \delta$ (due to the definitions of F^*, H^*), we have

$$\begin{aligned}&\mathbb{E}_{x,y}\mathbf{1}_{\{\zeta\geq T\}} \int_0^T \left[a_{12}(F^*\wedge Y(t)) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right] dt \\ &\geq \frac{\lambda T}{2}\mathbb{E}_{x,y}[\mathbf{1}_{\{\zeta\geq T\}}].\end{aligned} \tag{3.34}$$

We have from (3.33), (3.34) and the definitions of $A_T, \Phi(t)$ that

$$\begin{aligned}&\mathbb{E}_{x,y}\Phi(T) \\ &= -\frac{\lambda T}{2}\mathbb{P}_{x,y}(A_T^c) - \mathbb{E}_{x,y}\mathbf{1}_{A_T}\int_0^T \left[a_{12}(F^*\wedge Y(t)) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right] dt \\ &\leq -\frac{\lambda T}{2} + \ln(1+\rho)T.\end{aligned} \tag{3.35}$$

The proof is complete. □

Lemma 3.5. *Let*

$$\Phi_1(T) = \Phi(T) - \mathbb{E}_{x,y}\Phi(T).$$

There exists $\check{K}^* = \check{K}^*(H^*, T^*) > 0$ *such that*

$$\mathbb{E}_{x,y}\Phi_1^2(T)e^{\theta\Phi_1(T)} \leq \check{K}^* \text{ for any } 0 \leq \theta \leq 1, x\vee y \leq H^*, T \leq T^*. \tag{3.36}$$

Proof. Note that for $\theta \in (0,1)$, $z^2 e^{\theta z} \le z^2$ if $z \le 0$ and $z^2 e^{\theta z} \le 4e^z$ if $z \ge 0$. As a result,

$$\begin{aligned}\mathbb{E}_{x,y}\Phi_1^2(T)e^{\theta\Phi_1(T)} &\le \mathbb{E}_{x,y}[\Phi_1(T)]^2 + 4\mathbb{E}_{x,y}e^{\Phi_1(T)} \\ &\le \mathbb{E}_{x,y}[\Phi(T)]^2 + 4\mathbb{E}_{x,y}e^{\Phi(T)}e^{-\mathbb{E}_{x,y}\Phi(T)}.\end{aligned}$$

Since $\Phi(T)$ is bounded below by $-F^*T$ so that $\mathbb{E}_{x,y}\Phi(T) \ge -F^*T$ it is easy to see that $\mathbb{E}_{x,y}([\Phi(T) - \mathbb{E}_{x,y}(T)] \wedge 0)^2 \le \mathbb{E}_{x,y}([\Phi(T)] \wedge 0)^2 \le (F^*T)^2$. On the other hand, one has

$$\begin{aligned}&\mathbb{E}_{x,y}\exp\left\{-\mathbf{1}_{A_T}\int_0^T\left[a_{12}(F^*\wedge Y(t)] - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right]\right\} \\ &\le \mathbb{E}_{x,y}\exp\left\{-\int_0^T\left[a_{12}(F^*\wedge Y(t)] - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right]\right\}\exp\{a_{12}F^*T\} \\ &\le \exp\{a_{12}F^*T\}\mathbb{E}_{x,y}\exp\left\{-\int_0^T\left[a_{12}(F^*\wedge Y(t)] - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right]\right\} \\ &\le \exp\{a_{12}F^*T\}\frac{\mathbb{E}_{x,y}X^{-1}(T)}{x^{-1}} \\ &\le \exp\{a_{12}F^*T\}c_2e^{c_1T} \text{ for } 0 < x \le 1.\end{aligned}$$

Moreover, because $-\Phi(T)$ is bounded above by F^*T, we have $e^{-\mathbb{E}_{x,y}\Phi(T)} \le e^{F^*T}$. Thus,

$$\mathbb{E}_{x,y}\Phi^2(T)e^{\theta\Phi(T)} \le (a_{12}F^*T)^2 + \exp\{a_{12}F^*T\}c_2e^{c_1T}.$$

Therefore, the proof is complete. □

Proposition 3.1. *With sufficiently small $\theta > 0$, there exists C^* independent of $(x,y) \in [0,\infty) \times [0,H^*]$ such that*

$$\mathbb{E}_{x,y}X^{-\theta T} \le x^{-\theta}e^{-\frac{\theta\lambda T}{4}} + C^*, \textit{ for } (x,y) \in [0,\infty)\times[0,H^*], T \le T^*.$$

Proof. Consider the function: $\phi(\theta) = \ln \mathbb{E}_{x,y}e^{\theta\Phi_1(T)}$ on $\theta \in [0, \frac{1}{2}]$. We have

$$\frac{d\phi}{d\theta} = \frac{\mathbb{E}_{x,y}\Phi_1(T)e^{\theta\Phi_1(T)}}{\mathbb{E}_{x,y}e^{\theta\Phi_1(T)}} \text{ and } \frac{d^2\phi}{d\theta^2} = \frac{\mathbb{E}_{x,y}\Phi_1^2(T)e^{\theta\Phi_1(T)} - \left(\mathbb{E}_{x,y}\Phi_1(T)e^{\Phi_1(T)}\right)^2}{\left(\mathbb{E}_{x,y}e^{\Phi_1(T)}\right)^2}.$$

Therefore, due to $e^x \le 1 + x + x^2e^x$ and Lemma 3.5, we have

$$\mathbb{E}_{x,y}e^{\theta\Phi_1(T)} = 1 + \theta\mathbb{E}_{x,y}\Phi_1(T) + \theta^2\mathbb{E}_{x,y}\Phi_1^2(T)e^{\theta\Phi_1(T)} \le 1 + \theta^2\check{K}^* \le e^{\theta^2\check{K}^*}.$$

On the other hand $e^{\theta\sigma_1W_1(T) - \frac{\theta^2\sigma_1^2}{2}T}$ is a martingale, so

$$\mathbb{E}_{x,y}e^{\theta\sigma_1W_1(T) - \frac{\theta^2\sigma_1^2}{2}T} = 1;$$

which implies

$$\mathbb{E}_{x,y}e^{\theta\sigma_1W_1T} \le e^{\frac{\theta^2\sigma_1^2}{2}T}.$$

By Holder's inequality, for $\theta \le \frac{1}{4}$,

$$\mathbb{E}_{x,y}e^{\theta\Phi_1T}e^{\theta\sigma_1W_1T} \le \left(\mathbb{E}_{x,y}e^{2\theta\Phi_1T}e^{2\theta\sigma_1W_1T}\right)^{\frac{1}{2}} \le e^{4\theta^2\check{K}^*}e^{2\theta^2\sigma_1^2T} \le e^{\frac{\theta\lambda T}{8}},$$

if we choose θ no larger than $\frac{\lambda T_1}{8(4\check{K}^* + 2\sigma_1^2T_2)}$.

Note that for $\omega \in A_T$, we have

$$\begin{aligned}
&X^{-\theta}(0) = x^{-\theta}(0)\\
&+\exp\left\{-\theta\int_0^T\left[a_{12}(Y(t) - a_{11}X(t) - \frac{b_1}{X(t)+b_2} - \frac{\sigma_1^2}{2}\right]\right\}\exp\left\{-\theta\sigma_1 W_1(T)\right\}\\
&\le X^{-\theta}(0)e^{\Phi(T)}\exp\left\{-\theta\sigma_1 W_1(T)\right\}\\
&= X^{-\theta}(0)e^{\mathbb{E}_{x,y}\Phi(T)}e^{\Phi_1(T)}\exp\left\{-\theta\sigma_1 W_1(T)\right\}.
\end{aligned} \tag{3.37}$$

As a result, by using Lemmas 3.4 and 3.5, we have

$$\begin{aligned}
\mathbb{E}_{x,y}\mathbf{1}_{A_T}X^{-\theta T} =&\mathbb{E}_{x,y}\left(x^{-\theta}(1+p)^{-\theta[T]}e^{\theta\mathbb{E}_{x,y}\Phi(T)}e^{\theta\Phi_1 T}e^{\theta\sigma_1 W_1 T}\right)\\
\le& x^{-\theta}\exp\left\{\theta\left(-\ln(1+p)[T] + (-\frac{\lambda T}{2} + \ln(1+p)T) + \frac{\lambda T}{8}\right)\right\}\\
\le& x^{-\theta}e^{-\frac{\theta\lambda T}{4}}.
\end{aligned} \tag{3.38}$$

It is noted that $A_T^c \subset \{\zeta \le T\}$. Thus,

$$\begin{aligned}
\mathbb{E}_{x,y}\mathbf{1}_{A_T^c}X^{-\theta T} \le&\mathbb{E}_{x,y}\mathbf{1}_{\{\zeta\le T\}}X^{-\theta T}\\
=&\mathbb{E}_{x,y}\mathbf{1}_{\{\zeta\le T\}}\mathbb{E}_{X(\zeta),Y(\zeta),\zeta}X^{-\theta T}\\
\le&(c_2e^{c_1T})^{\theta}\delta^{-\theta}.
\end{aligned} \tag{3.39}$$

In view of Lemma 3.3, there exists $C^* = C^*(\delta_0, T^*)$ such that

$$\mathbb{E}_{x,y}X^{-\theta T} \le C^* \text{ if } x \ge \delta_0, T \le T^*. \tag{3.40}$$

The proposition is proved by combining (3.38) and (3.39). □

Lemma 3.6. *For any compact subset $\mathcal{K} \in \mathbb{R}_+^{2,\circ}$, there exist a probability measure $\nu_{\mathcal{K}}$ and a constant $c_{\mathcal{K}} > 0$ such that*

$$\mathbb{P}_{x,y}\{(X(1), Y(1)) \in \cdot\} \ge c_{\mathcal{K}}\nu_{\mathcal{K}}(\cdot), \text{ for all } (x,y) \in \mathcal{K}.$$

Proof. Let $(\check{X}, \check{Y})$ be the solution to (3.1). Because the diffusion is nondegenerate, in view of [7, Lemma 3.6], there exists a probability measure $\nu_{\mathcal{K}}$ and a constant $c'_{\mathcal{K}} > 0$ such that

$$\mathbb{P}_{x,y}\{(\check{X}(1), \check{Y}(1)) \in \cdot\} \ge \check{c}_{\mathcal{K}}\nu_{\mathcal{K}}(\cdot), \text{ for all } (x,y) \in \mathcal{K}.$$

Because $(W_1(t), W_2(t))$ is independent of $\widetilde{N}$, and $\check{X}(1) = X(1), \check{Y}(1) = Y(1)$ if there is no jumps on $[0,1]$, we have

$$\mathbb{P}_{x,y}\{(X(1), Y(1)) \in \cdot\} \ge \mathbb{P}\left\{\int_{\mathbb{U}} N(1,du) = 0\right\}\mathbb{P}_{x,y}\{(\check{X}(1), \check{Y}(1)) \in \cdot\} \ge c_{\mathcal{K}}\nu_{\mathcal{K}}(\cdot)$$

for all $(x,y) \in \mathcal{K}$, where $c_{\mathcal{K}} = \mathbb{P}\left\{\int_{\mathbb{U}} N(1,du) = 0\right\}c'_{\mathcal{K}} > 0$. □

Lemma 3.7. *Let $U^*(x,y) = V(x,y) + x^{-\theta}$, where V is as in Theorem 2.1 and θ is as in Proposition 3.1. For any H, there exists $\check{c}_H > 0$ and a probability measure ν_H such that*

$$\mathbb{P}_{x,y}\{(X(T^*), Y(T^*)) \in \cdot\} \ge \check{c}_H\nu_H(\cdot), \text{ if } U^*(x,y) \le H.$$

Proof. By the variation of constants formula, see [1], we have

$$Y(t) = \psi(T)\left[Y(0) + R\int_0^T \psi^{-1}(t)dt\right] \geq \psi(T)\int_0^T \psi^{-1}(t)dt,$$

where

$$\psi(t) = e^{-\int_0^t (a_{22}+a_{21}X(s)+\frac{\sigma_2^2}{2}+\int_{\mathbb{U}}[\ln(h(u)+1)-h(u)du)ds+\sigma_2 W(t)+\int_0^t\int_{\mathbb{U}} h_2(u)\widetilde{N}(ds,du)}.$$

Since we can find $L_1 > 0$ such that

$$\mathbb{P}_{x,y}\left\{\sup_{t\in[0,1]}\left\{|X(t)|, |W(t)|, \left|\int_0^t\int_{\mathbb{U}} h(u)\widetilde{N}(ds,du)\right|\right\} \leq L_1\right\} \geq \frac{3}{4},$$

if $(x,y) \in (0,1]\times[0,H]$, there exists $L_2 > 0$ such that

$$\mathbb{P}_{x,y}\left\{\sup_{t\in[0,1]}\{|\psi(t)+\psi^{-1}(t)|\} \leq L_2\right\} \geq \frac{3}{4},\ \text{if } (x,y)\in(0,1]\times[0,H].$$

As a result,

$$\mathbb{P}_{x,y}\left\{Y(t) \geq \frac{1}{L_2^2}\right\} \geq \frac{3}{4},\ \text{if } U^*(x,y) \leq H. \tag{3.41}$$

In view of Lemma 3.3 and part (iii) of Theorem 2.1, there exists $L_3 > 0$ such that

$$\mathbb{P}_{x,y}\left\{a_V X(1) + Y(1) + X^{-1}(1) \geq L_3\right\} \leq \frac{1}{4},\ \text{if } U^*(x,y) \leq H. \tag{3.42}$$

Since the set $\left\{y \geq \frac{1}{L_2^2} \text{ and } a_V x + y + x^{-1}(1) \leq L_3\right\}$ is compact in $\mathbb{R}_+^{2,\circ}$, in view of Lemma 3.6, there exists a probability measure ν_H and a constant $\widetilde{c}_H > 0$ such that

$$\mathbb{P}_{x,y}\{(X(T^*-1), Y(T^*-1)) \in \cdot\} \geq \widetilde{c}_H\nu_H(\cdot),\ \text{if } y \geq \frac{1}{L_2^2} \text{ and } a_V x + y + x^{-1} \leq L_3. \tag{3.43}$$

By virtue of Markov properties of $\{X(n), Y(n)\}$, we have from (3.42) and (3.43) that

$$\mathbb{P}_{x,y}\{(X(T^*), Y(T^*)) \in \cdot\} \geq \frac{1}{2}\widetilde{c}_H\nu_H(\cdot),\ \text{if } U^*(x,y) \leq H. \tag{3.44}$$

The proof is complete. □

Proof of Theorem 2.2. Because of Proposition 3.1 and Theorem 2.1,

$$\mathbb{E}_{x,y}U^*(X(T^*), Y(T^*)) \leq \kappa U^*(x,y) + C^{**}, (x,y) \in \mathbb{R}_+^{2,\circ}, \tag{3.45}$$

for some $\kappa < 1$, where $U^*(x,y) = V(x,y) + x^{-\theta}$ as in Lemma 3.7. Because of Lemma 3.7 and (3.45), by applying [11], there exists an invariant probability measure μ^* of the Markov chain $\{(X(kT^*), Y(kT^*)), k \in \mathbb{Z}_+\}$ satisfying

$$\|P_{kT^*}((x,y),\cdot) - \mu^*\| \leq C_{U^*}(U^*(x,y))\kappa_{U^*}^k, \tag{3.46}$$

for some positive constants $C_{U^*} > 0, \kappa_{U^*} \in (0,1)$ independent of (x,y). On the other hand, (3.45) also implies the existence of invariant probability measures of the Markov process $\{(X(t), Y(t)), t \geq 0\}$. Since an invariant probability of $\{(X(t), Y(t)), t \geq 0\}$ is an invariant probability measure of $\{(X(kT^*), Y(kT^*)), k \in \mathbb{Z}_+\}$, we claim that μ^* is the unique invariant probability measure of $\{(X(t), Y(t)), t \geq 0\}$. Moreover, since the function $\|P_t((x,y),\cdot) - \mu^*\|$ is decreasing in t, we deduce from (3.46) that

$$\|P_t((x,y),\cdot) - \mu^*\| \leq C_{U^*}(U^*(x,y))\kappa_{U^*}^{t/T^*-1}, \tag{3.47}$$

which completes the proof. □

3.3. **Proof of extinction.**

Proof. By a comparison theorem, see e.g., [2], we have $Y(t) \le \widetilde{Y}(t)$ for any $t \ge 0$ with probability 1 if $Y(0) \le \widetilde{Y}(0)$, where $\widetilde{Y}(t)$ is a positive solution to

$$d\widetilde{Y}(t) = \Big(R - a_{22}\widetilde{Y}(t)\Big)\,dt + \sigma_2\widetilde{Y}(t)dW_2(t) + \int_{\mathbb{U}} \widetilde{Y}(t-)h(u)\widetilde{N}(dt, du).$$

Moreover, it is proved in [3] that the process $\{\widetilde{Y}(t)\}$ has a unique invariant measure, ν^* on $[0, \infty)$ satisfying

$$\int_{[0,\infty)} y\nu^*(dy) = \frac{R}{a_{22}}.$$

By the ergodicity of $\{\widetilde{Y}(t)\}$, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{Y}(s)ds = \int_{[0,\infty)} y\nu^*(dy) = \frac{R}{a_{22}} \text{ a.s.}$$

In Section 3.2, we presented methods and arguments to treat jumps and impulse. Combining these processes and the methods in [15] developed for diffusion, we can prove that

$$\limsup_{t\to\infty} \frac{\ln X(t)}{t} \le \frac{\lambda}{2} \text{ a.s.}$$

In particularly, $X(t)$ converges to 0 exponentially fast with probability 1. In the remaining, we will prove that $\limsup_{t\to\infty} \frac{\ln X(t)}{t}$ is exactly λ almost surely.

We have

$$\begin{aligned} d(\widetilde{Y}(t) - Y(t)) =& - a_{22}(\widetilde{Y}(t) - Y(t)) + a_{12}X(t)Y(t) + \sigma_2(\widetilde{Y}(t) - Y(t))dW_2(t) \\ &+ \int_{\mathbb{U}} h(u)(\widetilde{Y}(t) - Y(t))\widetilde{N}(dt, du). \end{aligned}$$

If $\widetilde{Y}(0) = Y(0)$, by the variation of constants formula, see e.e. [1], we have

$$\widetilde{Y}(t) - Y(t) = a_{22}\upsilon^{-1}(t)\int_0^t \upsilon(s)X(s)Y(s)ds,$$

where

$$\begin{aligned} &\upsilon(t) \\ &= \exp\left\{\left(a_{22} + \frac{\sigma_2^2}{2} - \int_{\mathbb{U}} [\ln(1 + h(u)) - h(u)]\nu(du)\right)t - \sigma_2 W_2(t) + \int_{\mathbb{U}} \widetilde{N}(t, du)\right\}. \end{aligned}$$

Since

$$\lim_{t\to\infty} \frac{\ln \upsilon(t)}{t} = \left(a_{22} + \frac{\sigma_2^2}{2} - \int_{\mathbb{U}} [\ln(1 + h(u)) - h(u)]\nu(du)\right) := \lambda_2 > 0,$$

and

$$\limsup_{t\to\infty} \frac{\ln Y(t)}{t} \le \limsup_{t\to\infty} \frac{\ln \widetilde{Y}(t)}{t} = 0,$$

and

$$\limsup_{t\to\infty} \frac{\ln X(t)}{t} \le \frac{\lambda}{2} < 0,$$

there exists $t_0 = t_0(\omega) > 0$ such that

$$\upsilon(t)X(t)Y(t) \le e^{(\lambda_2-\varepsilon)t}, t \ge 0,$$

for any $\varepsilon < \lambda$. As a result, with $\varepsilon < |\lambda| \wedge \lambda_2$, we have

$$\begin{aligned}\lim_{t\to\infty}[\widetilde{Y}(t) - Y(t)] \le& \frac{1}{a_{22}}\lim_{t\to\infty}\frac{\int_0^{t_0}\upsilon(s)X(s)Y(s)ds}{\upsilon(t)} + \frac{1}{a_{22}}\lim_{t\to\infty}\frac{\int_{t_0}^{t} e^{(\lambda_2-\varepsilon)s}ds}{\upsilon(t)}\\ \le& \frac{1}{a_{22}}\lim_{t\to\infty}\frac{\int_0^{t_0}\upsilon(s)X(s)Y(s)ds}{\upsilon(t)} + \frac{1}{a_{22}(\lambda_2-\varepsilon)}\lim_{t\to\infty}\frac{e^{(\lambda_2-\varepsilon)t}}{\upsilon(t)} = 0.\end{aligned}$$

Since $\lim_{t\to\infty}(X(t) + |\widetilde{Y}(t) - Y(t)|) = 0$, with probability 1, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln X(t)}{t} =& \limsup_{t\to\infty}\left(\frac{\ln X(t)}{t} + \frac{W_1(t)}{t} + \ln(1+\rho)\frac{[t]}{t} + \right)\\ &+ \lim_{t\to\infty}\frac{1}{t}\int_0^t\left\{(a_{12}Y(s) - a_{11}X(s) - \frac{b_1}{X(s)+b_2} - \frac{\sigma_1^2}{2}\right\}ds \qquad (3.48)\\ =& \ln(1+\rho) + \limsup_{t\to\infty}\frac{1}{t}\int_0^t a_{12}\widetilde{Y}(s)ds - \frac{b_1}{b_2} - \frac{\sigma_1^2}{2} = \lambda < 0 \text{ a.s.}\end{aligned}$$

The proof is complete.

□

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