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Hybrid stochastic SIS epidemic models with vaccination: Stability of the disease-free state and applications

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ABSTRACT

In this paper, we consider a stochastic SIS epidemic model with vaccination in random switching environment. The system is formulated as a hybrid stochastic differential equation. We provide a threshold number that characterizes completely its longtime behavior. It turns out that if the threshold is negative, the number of the infected class converges to zero or the extinction happens. The rate of convergence is also obtained. In contrast, if the threshold is positive, the infection is endemic. We are able to obtain an algebraic formula for the threshold, which helps us to study some strategies for controlling the disease such as: (i) determining the minimum vaccination rate needed to keep the population from the disease and (ii) determining the strategy with minimum cost of vaccination and treatment. To illustrate the results, a number of mathematical simulations and numerical examples are also presented.

1. Introduction

It has been observed in human history that infectious diseases have been making a significant negative impact on the population's health, economics, and social life. Because of that, much attention has been devoted to modeling the dynamics of epidemic systems, analyzing their dynamic behaviors, and predicting what may happen in the future. Introduced by Kermack and McKendrick [1,2], the compartmental model now is a very popular technique to model the dynamics of epidemic systems. Depending on the type of disease, we will divide the population into different compartments. In particular, for diseases with permanent immunity we often use the model with three compartments that includes susceptible group, infectious group, and recovered group. This model is the so-called SIR epidemic model. However, there are some infections that do not confer any long-lasting immunity, for example, common colds, influenza, etc. Upon recovery, the infections do not give a permanent immunization and individuals return to the susceptible compartment again. This is the so-called SIS epidemic model, which has been recognized as one of the most important models in epidemiology and mathematical biology; see e.g., [3–8] and the references therein. To control diseases, together with improving treatments, vaccination is also researched, produced, and used. In recent years, many authors have modeled and analyzed the SIS epidemic models allowing vaccination; see e.g., [9–15] and the references therein.

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A SIS epidemic model with vaccination introduced by Li and Ma [10] has the form

$$\begin{cases} \dfrac{S(t)}{dt} = (1-q)A - \beta S(t)I(t) - (p+\mu)S(t) + \gamma I(t) + \varepsilon V(t), \\ \dfrac{I(t)}{dt} = \beta S(t)I(t) - (\mu+\gamma+\alpha)I(t), \\ \dfrac{V(t)}{dt} = qA + pS(t) - (\mu+\varepsilon)V(t), \end{cases} \tag{1.1}$$

where $S(t)$ denotes the number of susceptible group containing individuals who are susceptible to an infection, $I(t)$ stands for the number of the infection group and $V(t)$ is the number of members who are immune to an infection as the result of vaccination; and A is an input of new members into the population, q is a fraction of vaccinated for newborns, μ is the nature death rate of the population, β is the transmission coefficient between compartments S and I, p is the proportional coefficient of vaccinated for the susceptible, γ is the recovery rate of I, ε is the rate of losing their immunity for vaccinated individuals, α is the disease-related death rate. All these constants are assumed to be positive.

It has also been well-recognized that epidemic models are inevitably subjected to environmental white noise and may be perturbed by color noise which can cause the system to switch from one environmental regime to another; see e.g., [16,17] and references therein. Precisely, to take account these kinds of randomness into consideration, instead of considering systems (1.1), we consider the following system (see also [14])

$$\begin{cases} dS(t) = & \Big[(1-q(\Lambda(t)))\,A(\Lambda(t)) - \beta(\Lambda(t))S(t)I(t) - (p(\Lambda(t)) + \mu(\Lambda(t)))\,S(t) \\ & + \gamma(\Lambda(t))I(t) + \varepsilon(\Lambda(t))V(t)) \Big]\, dt + \sigma_1(\Lambda(t))S(t)dW_1(t), \\ dI(t) = & \Big(\beta(\Lambda(t))I(t)S(t) - [\mu(\Lambda(t)) + \gamma(\Lambda(t)) + \alpha(\Lambda(t))]\, I(t)\Big)dt \\ & + \sigma_2(\Lambda(t))I(t)dW_2(t), \\ dV(t) = & \Big[q(\Lambda(t))A(\Lambda(t)) + p(\Lambda(t))S(t) - (\mu(\Lambda(t)) + \varepsilon(\Lambda(t)))\,V(t)\Big]dt \\ & + \sigma_3(\Lambda(t))V(t)dW_3(t), \\ S(0) = & s \ge 0, \quad I(0) = i \ge 0, \quad V(0) = v \ge 0, \quad \Lambda(0) = k \in \mathcal{M}, \end{cases} \tag{1.2}$$

where, $\Lambda(t)$ is a Markov chain with state space $\mathcal{M} = \{1, \dots, m_0\}$, and $W_i(t)$ are Brownian motions that are used to model color noise (which cause the system to switch from one environmental regime to another) and white noises, respectively; see e.g., [16,17].

In this paper, we will analyze system (1.2) and provide a complete characterization of longtime properties of the system. Our main purpose is that: a threshold λ is introduced such that its sign will characterize completely the longtime behavior of the underlying system. In particular, we prove that if λ is negative, the disease will be eradicated; the rate is also obtained. In contrast, if λ is positive, the infection is endemic. We also show that the system has an invariant probability measure, and the transition probability of the solution process converges to the invariant measure. Our method to determine the thresholds is to look at the dynamics of the system near the boundary, and that can be generalized to deal with numerous other epidemic models. The algebraic representation of this threshold is introduced to make it very easy to be computed and facilitate further study of the epidemic model. A number of mathematical simulations and numerical examples are also provided to illustrate our theoretical results.

The contributions and novelties in this work can be summarized as follows.

- In the literature, although system (1.2) has been studied, for example in [14] and references therein, a complete characterization has still not been treated. In particular,
  all existing results only provided sufficient conditions for the persistence and extinction of the disease and the conditions left a sizable gap. By characterizing these properties with a threshold level, we provide sufficient and necessary conditions for both persistence and extinction except for a critical case.
- We also introduce novel and systematic approaches. Our methodology (using the Lyapunov exponents from dynamical systems theory to define the thresholds) is general and systematic that can be generalized for other models. Although a similar approach has been developed in [18,19], those results are not applicable here and some new techniques are required. In particular, in SIS models, the term $\gamma I(t)$ in the first equations (due to the short immunity making recovered individuals susceptible again) leads significant challenges in understanding when the extinction happens. To overcome that, we need to consider auxiliary perturbed systems to understand how small changes in infected groups affect to the whole systems.
- Being able to obtain an algebraic formula for the threshold λ, we can determine the minimum rate of vaccination needed to keep the disease-free state of the system. In case the disease persists even with full vaccination, we are able to reduce the problem of minimizing the long-term average cost of vaccination and treatment into an elementary optimization problem. That will facilitate the decision making process of controlling a disease.

The rest of the paper is organized as follows. Section 2 introduces the threshold λ and proves the main results. Section 3 answers the practical question: What is the rate of vaccination needed to keep the disease from persistence. Moreover, we consider the optimal strategies for minimizing the cost of vaccination and treatments when the disease persists. Section 4.1 is devoted to discussion. A number of simulated examples are showcased in Section 4.2 to illustrate our main findings.

2. A complete characterization of longtime behavior

We begin with introduce some notations: $\mathbb{R}^2_+ = \{(s,i) \in \mathbb{R}^2 : s \geq 0, i \geq 0\}$, $\mathbb{R}^3_+ = \{(s,i,v) \in \mathbb{R}^3 : s \geq 0, i \geq 0, v \geq 0\}$, $\mathbb{R}^{2,\circ}_+ = \{(s,i) \in \mathbb{R}^2 : s > 0, i > 0\}$, $\mathbb{R}^{3,\circ}_+ = \{(s,i,v) \in \mathbb{R}^3 : s > 0, i > 0, v > 0\}$, $\mathbb{R}^{2,*}_+ = \{(s,i) \in \mathbb{R}^2 : s \geq 0, i > 0\}$, and $\mathbb{R}^{3,*}_+ = \{(s,i,v) \in \mathbb{R}^3 : s \geq 0, i > 0, v \geq 0\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. $W_1(t)$, $W_2(t)$, $W_3(t)$ are independent Brownian motions, and $\Lambda(t)$ is a Markov chain with finite state space $\mathcal{M}$. We also assume that $\Lambda(t)$ has the (irreducible) generator $Q = (q_{kl})_{m_0\times m_0}$ and an ergodic measure π and is independent of $W_1(t)$, $W_2(t)$, $W_3(t)$ such that

$$\begin{aligned}&\mathbb{P}\{\Lambda(t+\Delta) = j | \Lambda(t) = i, \Lambda(u), u \leq t\} = q_{ij}\Delta + o(\Delta) \text{ if } i \neq j \quad \text{and}\\ &\mathbb{P}\{\Lambda(t+\Delta) = i | \Lambda(t) = i, \Lambda(u), u \leq t\} = 1 + q_{ii}\Delta + o(\Delta).\end{aligned} \tag{2.1}$$

Let $\mathbb{E}_{s,i,v,k}$ and $\mathbb{P}_{s,i,v,k}$ denote the expectation and probability associated with the initial data $S(0) = s$, $V(0) = v$, $I(0) = i$, $\Lambda(0) = k$, respectively. We first establish the existence, uniqueness and basic properties of the solution process of (1.2).

Theorem 2.1. *Starting at $(s,i,v,k) \in \mathbb{R}^3_+ \times \mathcal{M}$, there is uniquely a global nonnegative solution $(S(t), I(t), V(t))$ to (1.2) satisfying $\mathbb{P}_{s,i,v,k}\{S(t) > 0, V(t) > 0, I(t) \geq 0, \ \forall t > 0\} = 1$. Moreover, $\mathbb{P}_{s,0,v,k}\{I(t) = 0, \ \forall t \geq 0\} = 1$, $\mathbb{P}_{s,0,v,k}\{S(t) > 0, \ V(t) > 0, \ \forall t \geq 0\} = 1$ and $\mathbb{P}_{s,i,v,k}\{S(t), I(t), V(t) \in \mathbb{R}^{3,\circ}_+, \ \forall t > 0\} = 1$, if $i > 0$. Moreover, the joint-process $(S(t), I(t), V(t), \Lambda(t))$ is a Markov–Feller process.*

Proof. The proof is standard and similar to [20, Theorem 2.1] or [21, Theorem 2.1], and therefore is omitted here. $\square$

We continue to provide moment boundedness and tightness (boundedness in probability) of the solution.

Lemma 2.1. *For any $q > 0$ sufficiently small, there exist $C_q > 0$ and $D_q > 0$ such that*

$$\mathbb{E}_{s,i,v,k}\,(1 + S(t) + V(t) + I(t))^{1+q} \leq \frac{C_q}{D_q} + \frac{(1+s+v+i)^{1+q}}{e^{D_q t}}, \quad \forall t \geq 0. \tag{2.2}$$

As a result, for any $H > 0$, $\varepsilon > 0$, $T > 0$, there is a constant $M_{H,\varepsilon,T} > 0$ such that

$$\mathbb{P}_{s,i,v,k}\left\{\sup_{t\in[0,T]}\{S(t) + V(t) + I(t)\} \geq M_{H,\varepsilon,T}\right\} \leq \varepsilon, \quad \forall (s,i,v,k) \in [0,H]^3 \times \mathcal{M}. \tag{2.3}$$

Proof. Denote by $\mathcal{L}$ the operator associated with the solution process of (1.2). It is well-known that for a function V of (s,i,v,k) that is twice differentiable with respect to (s,i,v), we have

$$\begin{aligned}\mathcal{L}V(s,i,v,k) =& V_s(s,i,v,k)\big((1-q(k))A - \beta(k)si - (p(k)+\mu(k))s + \varepsilon(k)v + \gamma(k)i\big)\\ &+ V_i(s,i,v,k)\big(\beta(k)si - (\mu(k)+\alpha(k)+\gamma(k))i\big)\\ &+ V_v(s,i,v,k)\big(q(k)A(k) + p(k)s - (\mu(k)+\varepsilon(k))v\big)\\ &+ \frac{1}{2}V_{ss}(s,i,k)\sigma_1^2(k)s^2 + \frac{1}{2}V_{ii}(s,i,v,k)\sigma_2^2(k)i^2 + \frac{1}{2}V_{vv}(s,i,v,k)\sigma_3^2(k)i^2\\ &+ \sum_{l\in\mathcal{M}} q_{kl}V_n(s,i,v,l).\end{aligned}$$

Thus, by direct computations, there exists $C_q, D_q > 0$ for sufficiently small $q > 0$ satisfying

$$\mathcal{L}(1+s+v+i)^{1+q} \leq -D_q(1+s+i+v)^{1+q} + C_q.$$

Therefore, as an application of Itô's formula, one has (2.2) and (2.3); see e.g., [22] or [23] for this well-known argument. $\square$

2.1. A complete characterization of longtime behavior

In this section, we will define a threshold λ that will characterize completely the longtime behavior of system (1.2).
For each $\theta \geq \theta_m := -\min_{k\in\mathcal{M}}\{\frac{(1-q(k))A(k)}{2\gamma(k)}\}$, consider a perturbed system of (1.2) when $I(t)$ is small:

$$\begin{cases} d\widetilde{S}^\theta(t) = & \Big[(1-q(\Lambda(t)))\,A(\Lambda(t)) - (p(\Lambda(t)) + \mu(\Lambda(t)))\,\widetilde{S}^\theta(t)\\ & +\gamma(\Lambda(t))\theta + \varepsilon(\Lambda(t))\widetilde{V}^\theta(t)\Big]\,dt + \sigma_1(\alpha(t))\widetilde{S}^\theta(t)dW_1(t),\\ d\widetilde{V}^\theta(t) = & \Big[q(\Lambda(t))A(\Lambda(t)) + p(\Lambda(t))\widetilde{S}^\theta(t) - (\mu(\Lambda(t)) + \varepsilon(\Lambda(t)))\,\widetilde{V}^\theta(t)\Big]dt\\ & +\sigma_3(\Lambda(t))\widetilde{V}^\theta(t)dW_2(t).\end{cases} \tag{2.4}$$

Similar to Theorem 2.1, we can easily show that for each initial value $(s,v,k) \in \mathbb{R}^2_+ \times \mathcal{M}$, there exists a global solution $(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t))$ to (2.4). Moreover, by a similar argument as in Lemma 2.1, we can show the solution satisfies that $(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t)) \in \mathbb{R}^{2,\circ}_+$ for all $t > 0$ almost surely and that

$$\mathbb{E}_{s,v,k}\left(\widetilde{S}^\theta(t) + \widetilde{V}^\theta(t)\right)^{1+q} \leq \frac{(1+s+v)^{1+q}}{e^{\widetilde{D}_q t}} + \frac{\widetilde{C}_q}{\widetilde{D}_q}, \quad \forall t \geq 0, \tag{2.5}$$

for some $q > 0$, $\widetilde{C}_q, \widetilde{D}_q > 0$, which can be taken uniformly for any $\theta \in [\theta_m, 1]$.

Since the diffusion in (2.4) in nondegenerate on $\mathbb{R}^{2,\circ}_+ \times \mathcal{M}$, there exists uniquely an invariant probability distribution $\boldsymbol{\nu}_\theta$ of $(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t), \alpha(t))$ on $\mathbb{R}^{2,\circ}_+ \times \mathcal{M}$. Moreover, it is noted that π is the unique invariant measure of $\Lambda(t)$. Therefore, π is the marginal distribution of the third component in the joint distribution $\boldsymbol{\nu}_\theta$. Due to (2.5), we have

$$\sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} (s+v)^{1+q} \boldsymbol{\nu}_\theta(ds, dv, k) \le \frac{\widetilde{C}_q}{\widetilde{D}_q} < \infty.$$

Thus, we can well define the following λ_θ which is an approximated growth rate of $I(t)$ as its density is small:

$$\begin{aligned}\lambda_\theta :=& \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \left(\beta(k)s - (\mu(k)+\gamma(k)+\alpha(k)) - \frac{\sigma_2^2(k)}{2}\right) \boldsymbol{\nu}_\theta(ds, dv, k)\\ =& \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \beta(k)s\boldsymbol{\nu}_\theta(ds, dv, k) - \sum_{k\in\mathcal{M}} \left(\mu(k)+\alpha(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2}\right)\pi_k.\end{aligned}$$

The equality above follows from the fact that π is the marginal distribution of the third component in $\boldsymbol{\nu}_\theta$. In particular, when $\theta = 0$, we denote

$$\lambda := \lambda_0 = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \left(\beta(k)s - (\mu(k)+\alpha(k)+\gamma(k)) - \frac{\sigma_2^2(k)}{2}\right) \boldsymbol{\nu}_0(ds, dv, k). \tag{2.6}$$

This λ will be shown to be the threshold that characterizes the longtime behavior of the disease. Note that, when $\theta = 0$, (2.4) is the solution to (1.2) with $I(t) \equiv 0$. Thus, we can consider $\boldsymbol{\nu}_0$ as the unique invariant probability measure of the solution process $(S(t), I(t), V(t), \Lambda(t))$ on the boundary $[0,\infty)\times\{0\}\times[0,\infty)\times\mathcal{M}$ (by embedding $[0,\infty)\times[0,\infty)\times\mathcal{M}$ to $[0,\infty)\times\{0\}\times[0,\infty)\times\mathcal{M}$).

Algebraic representation of λ. To proceed, we provide an algebraic representation for the proposed threshold λ. This representation is very useful from a computational point of view since we can compute λ by solving a system of linear equations.

Lemma 2.2. *Let* $\left(c_1(1), c_1(2), \dots, c_1(m_0), c_2(1), c_2(2, \dots, c_2(m_0))\right)^\top$ *be the unique solution to the linear system:*

$$\begin{cases} \beta(k) - (p(k)+\mu(k))\, c_1(k) + p(k)c_2(k) + \sum_{l\in\mathcal{M}} \gamma_{kl} c_1(l) = 0,\\ \varepsilon(k)c_1(k) - (\mu(k)+\varepsilon(k))\, c_2(k) + \sum_{l\in\mathcal{M}} \gamma_{kl} c_2(l) = 0, \quad k = 1,2,\dots,m_0.\end{cases} \tag{2.7}$$

We have:

$$\lambda_\theta = \sum_{k\in\mathcal{M}} \left[c_1(k)[(1-q(k))A(k)+\gamma(k)\theta] - \left(\mu(k)+\alpha(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2}\right)\right]\pi_k.$$

Proof. The system (2.7) can be written in the following form

$$AC = \beta, \tag{2.8}$$

where $C \in \mathbb{R}^{2n}$, $\beta = \left(\beta(1), \beta(2), \dots, \beta(m_0), 0, 0, \dots, 0\right)^\top$, and

$$A = \begin{bmatrix} \mu(1)+p(1)-\gamma_{11} & \dots & -\gamma_{1m_0} & -p(1) & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ -\gamma_{m_0 1} & \dots & \mu(m_0)+p(m_0)-\gamma_{m_0m_0} & 0 & \dots & -p(m_0)\\ -\varepsilon(1) & \dots & 0 & \mu(1)+\varepsilon(1)-\gamma_{11} & \dots & -\gamma_{1m_0}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \dots & -\varepsilon(m_0) & -\gamma_{m_01} & \dots & \mu(m_0)+\varepsilon(m_0)-\gamma_{m_0m_0}\end{bmatrix}.$$

Since $-\gamma_{ii} = \sum_{j\neq i}\gamma_{ij}$, it is obvious that matrix $A = [a_{ij}]_{2m_0\times 2m_0}$ is diagonally dominant, i.e., $|a_{ii}| \ge \sum_{j\neq i, j=1}^{2m_0} |a_{ij}|$ for any $i = 1,\dots,m_0$. It is well-known that a diagonally dominant matrix is non-singular. Thus, there exists a unique solution C to $AC = \beta$.

Denote by $\mathcal{L}_\theta$ the operator associated with the solution process of (2.4). It is well known that if we let $U(s,v,k) = c_1(k)s + c_2(k)v$ then

$$\begin{aligned}\mathcal{L}_\theta U(s,v,k) =& c_1(k)\left[(1-q(k))A(k)+\gamma(k)\theta - (p(k)+\mu(k))\,s + \varepsilon(k)v\right]\\ &+ c_2(k)\left[q(k)A(k) + p(k)s - (\mu(k)+\varepsilon(k))\,v\right]\\ &+ \Big(\sum_l \gamma_{kl}c_1(l)\Big)s + \Big(\sum_l \gamma_{kl}c_2(l)\Big)v\\ =& c_1(k)\left[(1-q(k))A(k)+\gamma(k)\theta\right] - \beta(k)s\\ =& h_\theta(k) - \beta(k)s,\end{aligned}$$

where

$$h_\theta(k) = c_1(k)\left[(1-q(k))A(k)+\gamma(k)\theta\right]. \tag{2.9}$$

Then, by Dynkin's formula (see e.g., [24]) we have

$$\begin{aligned}
\mathbb{E}_{s,v,k} & \quad U(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t), \Lambda(t)) - U(\widetilde{S}^\theta(0), \widetilde{V}^\theta(0), \Lambda(0)) \\
= & \quad \mathbb{E}_{s,v,k} \int_0^t \mathcal{L}_\theta U(\widetilde{S}^\theta(u), \widetilde{V}^\theta(u), \Lambda(u))du \\
= & \quad \mathbb{E}_{s,v,k} \int_0^t h(\Lambda(u))du - \mathbb{E}_{s,v,k} \int_0^t \beta(\Lambda(u))\widetilde{S}^\theta(u)du.
\end{aligned} \tag{2.10}$$

Due to ergodicity of $\Lambda(t)$ and $\widetilde{S}^\theta(t)$, we have

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{s,v,k} \int_0^t h(\Lambda(u))du = \sum_{k\in\mathcal{M}} h_\theta(k)\pi_k, \tag{2.11}$$

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{s,v,k} \int_0^t \beta(\Lambda(u))\widetilde{S}^\theta(u)du = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \beta(k)s\nu(ds,dv). \tag{2.12}$$

Moreover, because of (2.5), one has

$$\lim_{t\to\infty} \frac{\mathbb{E}_{s,v,k}U(\widetilde{S}^\theta(t), \widetilde{V}^\theta(t), \Lambda(t)) - U(\widetilde{S}^\theta(0), \widetilde{V}^\theta(0), \Lambda(0))}{t} = 0. \tag{2.13}$$

Plugging (2.11), (2.12), and (2.13) in (2.10), we have

$$\sum_{k\in\mathcal{M}} h(k)\pi_k = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} \beta(k)s\boldsymbol{\nu}_\theta(ds,dv,k),$$

which implies

$$\lambda_\theta = \sum_{k\in\mathcal{M}} h(k)\pi_k - \sum_{k\in\mathcal{M}} \left(\mu(k) + \alpha(k) + \gamma(k) + \frac{\sigma_2^2(k)}{2}\right)\pi_k.$$

The proof is complete because of (2.9) □

Now, we state our main results in this section.

Theorem 2.2. *Suppose $\lambda > 0$. Then there the transition probability converges to a unique invariant probability measure $\boldsymbol{\nu}^*$ in total variation with an exponential rate.*

Theorem 2.3. *If $\lambda < 0$, $I(t)$ converges to 0 exponentially fast, or*

$$\mathbb{P}_{s,i,v,k}\left\{\lim_{t\to\infty} \frac{\ln I(t)}{t} = \lambda\right\} = 1. \tag{2.14}$$

Remark 2.1. The following are some remarks:

- The sufficient conditions for persistence and extinction in our results are also almost necessary conditions. The only case that left untreated is when $\lambda = 0$. If $\gamma(k) \equiv 0$ and $\lambda = 0$, the disease is not persistent in either time average or space average sense, that is,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds = 0 \text{ a.s, and } \lim_{t\to\infty} \mathbb{E}_{s,i,v,k}I(t) = 0.$$

  This claim can be derived from the method introduced in [25]. However, if $\gamma(k) > 0$ for some k, it is not easy to treat the case $\lambda = 0$.
- In this work, we treat with linear incidence rate βSI for simplifying notation to introduce our new methods and ideas. It can be easily applied to other types of the incidence rate functions to improve and generalize many existing results, e.g., [14,26–28]. Moreover, the method in [14] cannot work for nonlinear incidence rate. However, our method can be applied to a general model with nonlinear incidence rate; see Section 4.1. From computational point of view, when the incidence rate is nonlinear, we might not have an algebraic representation for λ, however it can be approximated by a numerical scheme.

2.2. Proof of Theorem 2.2

Lemma 2.3. *There exists $c_{-1} > 0$ such that*

$$\mathbb{E}_{s,i,v,k}I^{-1}(t) \le i\exp\{c_{-1}t\}, \quad \forall t \ge 0, (s,i,v) \in \mathbb{R}^{3,*}_+.$$

Proof. We have

$$\begin{aligned}
dI^{-1}(t) = & \left[\mu(\Lambda(t)) + \gamma(\Lambda(t)) + \alpha(\Lambda(t)) - \beta(\Lambda(t))S(t) - \frac{\sigma_2^2(\Lambda(t))}{2}\right] I^{-1}(t)dt - \sigma_2(\Lambda(t))I^{-1}(t)dW_2(t) \\
\le & c_{-1}I^{-1}(t)dt - \sigma_2(\Lambda(t))I^{-1}(t)dW_2(t),
\end{aligned}$$

where $c_{-1} = \max\{0, \mu(k)+\gamma(k)+\alpha(k)-\sigma_2^2(k)/2\}$. From this estimate, we can easily imply the existence of $\mathbb{E}_{s,i,v,k} I^{-1}(t)$, see [29]. Then taking expectation both sides, we obtain that

$$d\mathbb{E}_{s,i,v,k} I^{-1}(t) \le c_{-1}\mathbb{E}_{s,i,v,k} I^{-1}(t).$$

The Lemma follows easily this differential inequality. □

In this subsection, we assume $\lambda > 0$. Let $\theta_1 < 0$ small enough such that $\lambda_{\theta_1} \ge \frac{7\lambda}{8} > 0$. Let $(\widetilde{S}^{\theta_1}, \widetilde{V}^{\theta_1})$ be the solution to (2.4) with $\theta = \theta_1$ and $\widetilde{S}^{\theta_1}(0) = \widetilde{V}^{\theta_1}(0) = 0$. Note from the ergodicity of $(\widetilde{S}^{\theta_1}, \widetilde{V}^{\theta_1})$ that

$$\lim_{T\to\infty} \mathbb{E}_k \frac{1}{T}\int_0^T \left(\beta(\Lambda(t))\widetilde{S}^{\theta_1}(t)dt - \big(\mu(\Lambda(u))+\gamma(\Lambda(u))+\alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2}\right)du = \lambda_{\theta_1},$$

there exists $T > 0$ such that

$$\mathbb{E}_k \frac{1}{T}\int_0^T \left(\beta(\Lambda(t))\widetilde{S}^{\theta_1}(t)dt - \big(\mu(\Lambda(u))+\gamma(\Lambda(u))+\alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2}\right)du \ge \frac{3\lambda}{4}. \tag{2.15}$$

Lemma 2.4. *There exists $\varsigma \in (0, \frac{1}{4})$, $\kappa \in (0,1)$ and $C_I > 0$ such that*

$$\mathbb{E}_{s,i,v,k} I^{-\varsigma}(T) \le i^{-\varsigma}\kappa + C_I \text{ for all } (s,i,v,k) \in \mathbb{R}_+^{3,*} \times \mathcal{M}. \tag{2.16}$$

Proof. Define

$$\xi = \inf\{t \ge 0 : I(t) \ge |\theta_1|\}$$

and

$$\begin{aligned}\Phi(t) = &-\int_0^T \left(\beta(\Lambda(t))\widetilde{S}^{\theta_1}(t)dt - \big(\mu(\Lambda(u))+\gamma(\Lambda(u))+\alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2}\right)du \\ &+ \mathbb{E}_k\int_0^T \left(\beta(\Lambda(t))\widetilde{S}^{\theta_1}(t)dt - \big(\mu(\Lambda(u))+\gamma(\Lambda(u))+\alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2}\right)du.\end{aligned}$$

Due to the boundedness (2.5), there exists $C_T > 0$ such that

$$\mathbb{E}_k[\Phi(T)]^2 \le C_T,$$

and

$$\Phi(T) \le \int_0^T \big(\mu(\Lambda(u))+\gamma(\Lambda(u))+\alpha(\Lambda(u)) - \frac{\sigma_2^2(\Lambda(u))}{2}\big)du + \mathbb{E}_k\int_0^T \beta(\Lambda(t))\widetilde{S}^{\theta_1}(t)dt \le C_T.$$

Note that for $\varsigma \in (0, \frac{1}{2})$, $z^2e^{\varsigma z} \le z^2$ if $z \le 0$ $z^2e^{\varsigma z} \le 4e^z$ if $z \ge 0$. As a result,

$$\mathbb{E}_{x,y}\Phi^2(T)e^{\varsigma\Phi(T)} \le \mathbb{E}_{x,y}[\Phi(T)]^2 + 4\mathbb{E}_{x,y}e^{\Phi_1(T)} \le C_T + 4e^{C_T}. \tag{2.17}$$

We deduce from the inequality $e^z \le 1 + z + z^2e^z$ that

$$\mathbb{E}_x e^{\varsigma\Phi(T)} \le 1 + \varsigma\mathbb{E}_k\Phi(T) + \varsigma^2\mathbb{E}_k[\Phi^2(T)e^{\varsigma\Phi(T)}] \le 1 + \varsigma^2(C_T + 4e^{C_T}) \le e^{\varsigma^2(C_T+4e^{C_T})} := e^{\varsigma^2\overline{C}_T}.$$

Since

$$\exp\left\{\varsigma\int_0^t \sigma_2(\Lambda(u))dW_2(u) - \varsigma^2\int_0^t \frac{\sigma_2(\Lambda(u))}{2}du\right\}$$

is a martingale, we have

$$\mathbb{E}_k \exp\left\{\int_0^t \sigma_2(\Lambda(u))dW_2(u) - \int_0^t \frac{\sigma_2(\Lambda(u))}{2}du\right\} = 1,$$

which implies

$$\mathbb{E}_k \exp\left\{\int_0^t \sigma_2(\Lambda(u))dW_2(u)\right\} \le e^{\varsigma^2\frac{\sigma_2^M t}{2}}, \text{ where } \sigma_2^M = \max_{k\in\mathcal{M}}\{\sigma_2(k)\}$$

$$\begin{aligned}\mathbb{E}_k e^{\varsigma\Phi(t)}e^{\varsigma\int_0^t \sigma_2(\Lambda(u))dW_2(u)} &\le \left[\mathbb{E}_k e^{2\varsigma\Phi(t)}\mathbb{E}_k e^{2\varsigma\int_0^t \sigma_2(\Lambda(u))dW_2(u)}\right]^{\frac{1}{2}} \\ &\le \left[e^{4\varsigma^2(\overline{C}_T)}e^{2\sigma_2^M T\varsigma^2}\right]^{\frac{1}{2}} \\ &\le e^{\check{C}_T\varsigma^2},\end{aligned} \tag{2.18}$$

where $\check{C}_T = 2\overline{C}_T + \sigma_2^M T$.

Because $S(t) \geq \widetilde{S}^{\theta_1}(t)$ for $t \leq \zeta$, due to a comparison theorem, we deduce that

$$\begin{aligned}\frac{I^{-\varsigma}(T)}{I^{-\varsigma}(0)} =& \exp\left\{ \varsigma \int_0^t \left(\beta(\Lambda(u))S(u) - \big(\mu(\Lambda(u)) + \gamma(\Lambda(u)) + \alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2}\right) du \right.\\ &\left. + \varsigma \int_0^t \sigma_2(\alpha(u)) dW_2(u) \right\}\\ =& \exp\left\{ \mathbb{E}_k \int_0^T \left(\beta(\Lambda(t))\widetilde{S}^{\theta_1}(t)dt - \big(\mu(\Lambda(u)) + \gamma(\Lambda(u)) + \alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2}\right) du \right\}\\ &\times \exp\{\varsigma \Phi(T)\} \exp\left\{ \varsigma \int_0^t \sigma_2(\alpha(u)) dW_2(u)\right\}\\ \leq& \exp\{-\frac{3\lambda}{4}T\} \exp\{\varsigma\Phi(T)\} \exp\left\{ \varsigma \int_0^T \sigma_2(\alpha(u)) dW_2(u)\right\}.\end{aligned} \tag{2.19}$$

As a result,

$$\begin{aligned}\mathbb{E}_{s,i,v,k}\left[\mathbf{1}_{\{\zeta > T\}} \frac{I^{-\varsigma}(T)}{I^{-\varsigma}(0)}\right] \leq& \exp\{-\frac{3\lambda}{4}T\}\mathbb{E}_{s,i,v,k}\left\{ \mathbf{1}_{\{\zeta>T\}} \exp\{\varsigma\Phi(T)\} \exp\left\{ \varsigma \int_0^T \sigma_2(\alpha(u)) dW_2(u)\right\}\right\}\\ \leq& \exp\{-\varsigma\frac{3\lambda}{4}T\}\mathbb{E}_{s,i,v,k}\left\{ \exp\{\varsigma\Phi(T)\} \exp\left\{ \varsigma \int_0^T \sigma_2(\alpha(u)) dW_2(u)\right\}\right\}\\ \leq& \exp\{-\varsigma\frac{3\lambda}{4}T\} \exp\{\check{C}_T\varsigma^2\},\end{aligned} \tag{2.20}$$

where the last inequality is due to (2.18). Moreover, because of the strong Markov's property, we have from Lemma 2.3 that

$$\begin{aligned}\mathbb{E}_{s,i,v,k}\left[\mathbf{1}_{\{\zeta \leq T\}} I^{-\varsigma}(T)\right] =& \mathbb{E}_{s,i,v,k}\left[\mathbf{1}_{\zeta \leq T} \mathbb{E}_{S(\zeta),I(\zeta),V(\zeta),\Lambda(\zeta)} I^{-\varsigma}(T)\right]\\ \leq& \mathbb{E}_{s,i,v,k}\left[\mathbf{1}_{\{\zeta \leq T\}} |\theta_1|^{-\varsigma} \exp\{\varsigma c_{-1}(T-\zeta)\}\right]\\ \leq& |\theta_1|^{-\varsigma} \exp\{\varsigma c_{-1} T\}.\end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21), we have

$$\mathbb{E}_{s,i,v,k} I^{-\varsigma(T)} \leq e^{-\frac{\varsigma\lambda T}{2}} i^{-\varsigma} + |\theta_1|^{-\varsigma} \exp\{\varsigma c_{-1} T\}.$$

if $\varsigma \leq \frac{\lambda T}{4\check{C}_T}$, □

Lemma 2.5. *For any $H > 1$, there exists a compact subset $\mathcal{K} \subset \mathbb{R}_+^{3,\circ}$ such that*

$$\mathbb{P}_{s,i,v,k}\{(S(T-1), I(T-1), V(T-1)) \in \mathcal{K}\} \geq \frac{1}{2}, \text{ if } 0 \leq s, v \leq H, H^{-1} \leq i \leq H.$$

Proof. By the variation of constants formula, see [29], we have

$$S(t) = \psi(t)\left(S(0) + \int_0^t \psi^{-1}(u)(1 - q(\Lambda(u))A(\Lambda(u)) + \varepsilon(\Lambda(u))V(u))du\right), \tag{2.22}$$

where

$$\psi(t) = \exp\left\{ \int_0^t -\beta(\Lambda(u))I(u) - p(\Lambda(u)) - \mu(\Lambda(u)) - \frac{\sigma_1^2(\Lambda(u))}{2} du + \int_0^t \sigma_1(\Lambda(u))dW_1(u)\right\}.$$

Due to (2.3), there exists $\check{K} > 0$ such that

$$\mathbb{P}_{s,i,v,k}\left\{ \sup_{t\in[0,T]} \left\{ S(t), I(t), V(t), \left| \int_0^t \sigma_1(\Lambda(u))dW_1(u)\right|\right\} \leq \check{K}\right\} \geq \frac{7}{8} \text{ if } 0 \leq s, v \leq H, H^{-1} \leq i \leq H.$$

We can easily deduce from (2.22) the existence of $\check{k}_1, \check{K}_1 > 0$ such that

$$\mathbb{P}_{s,i,v,k}\left\{ \check{k}_1 \leq S(t) \leq \check{K}_1, t \in [0,T]\right\} \geq \frac{7}{8}.$$

Similarly, we have

$$\mathbb{P}_{s,i,v,k}\left\{ \check{k}_2 \leq S(t) \leq \check{K}_2, t \in [0,T]\right\} \geq \frac{7}{8},$$

$$\mathbb{P}_{s,i,v,k}\left\{ \check{k}_3 \leq S(t) \leq \check{K}_3, t \in [0,T]\right\} \geq \frac{7}{8},$$

for some positive constants $\check{k}_i, \check{K}_i, i = 2, 3$. Combining three estimates above we complete the proof. □

Proof of Theorem 2.2. In view of Lemma 2.4 and (2.2), we have

$$\mathbb{E}_{s,i,v,k}U(S(T),I(T),V(T)) \le \kappa U(s,i,v)+C^*,\ \text{for } (s,i,v)\in\mathbb{R}_+^{3,*}, \tag{2.23}$$

where $U(s,i,v)=(1+s+i+v)^{1+q}+i^{-\varsigma}$ with ς satisfying Lemma 2.4 and $\kappa\in(0,1)$ and $C^*>0$ are constants independent of (s,i,v,k).

On the other hand, because the diffusion is nondegenerate, in view of [19, Lemma 3.6], there exists a probability measure $\nu_{\mathcal{K}}$ on $\mathbb{R}_+^{3,\circ}\times\mathcal{M}$ and a constant $\check{c}_{\mathcal{K}}$ such that

$$\mathbb{P}_{s,i,v,k}\{(S(T),I(T),V(T),\Lambda(T))\in\cdot\}\ge \check{c}_{\mathcal{K}}\nu_{\mathcal{K}}(\cdot),\ \text{for all } (x,y)\in\mathcal{K}. \tag{2.24}$$

By Markov's property of $(S(t),I(t),V(t),\Lambda(t))$, we have from 2.5 and 2.24 that

$$\mathbb{P}_{s,i,v,k}\{(S(T),I(T),V(T))\in\cdot\}\ge \frac{\check{c}_H}{2}\nu_H(\cdot),\ \text{if } U(s,i,v)\le H, \tag{2.25}$$

for some probability measure ν_H and a constant $\check{c}_H$. It is well know that, (see e.g., [30] or [31])) with (2.23) and (2.25), there exists a probability measure μ^* on $\mathbb{R}_+^{3,*}\times\mathcal{M}$ and a constant $C_U>0,\kappa_U\in(0,1)$ such that

$$\|P_{nT}((s,i,v,k),\cdot)-\mu^*(\cdot)\|\le C_U(U(s,i,v))\kappa_U^n. \tag{2.26}$$

Moreover, we deduce from (2.23), 2.1, and (2.16) that

$$\limsup_{t\to\infty}\mathbb{E}_{s,i,v,k}U(S(t),I(t),V(t))<\infty,$$

which shows the existence of invariant probability measures of the Markov process $\{(S(t),I(t),V(t),\Lambda(t)),t\ge 0\}$. Since an invariant probability of $\{(S(t),I(t),V(t),\Lambda(t)),t\ge 0\}$ is also an invariant probability measure of its skeleton $\{(S(nT),I(nT),V(nT),\Lambda(nT)),n\in\mathbb{Z}_+\}$, μ^* must be the unique invariant probability measure of $\{(S(t),I(t),V(t),\Lambda(t)),t\ge 0\}$. Moreover, since the function $\|P_t((s,i,v,k),\cdot)-\mu^*(\cdot)\|$ is decreasing in t, we deduce from (2.26) that

$$\|P_t((s,i,v,k),\cdot)-\mu^*(\cdot)\|\le C_U(U(s,i,v))\kappa_U^{t/T-1}, \tag{2.27}$$

which completes the proof. □

2.3. Proof of Theorem 2.3

By Lemma 2.2, it is readily seen that λ_θ is continuous in θ. Therefore, there exists θ_0 such that

$$\lambda_{\theta_0}<\frac{\lambda_0}{2}=\frac{\lambda}{2}. \tag{2.28}$$

It is noted that we are considering the case $\lambda<0$, so $\lambda_{\theta_0}<0$.

Lemma 2.6. *Let θ_0 be as in (2.28). For any $\varepsilon>0$, $H>0$, there is a constant $\theta_1>0$ such that*

$$\mathbb{P}_{s,i,v,k}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t}=\lambda<0\right\}\ge 1-\varepsilon,\ \forall (s,i,v,k)\in[0,H]\times(0,\theta_1]\times[0,H]\times\mathcal{M}. \tag{2.29}$$

Proof. Let

$$g(s,k)=\beta(k)s-(\mu(k)+\alpha(k)+\gamma(k))-\frac{\sigma_2^2(k)}{2}. \tag{2.30}$$

We obtain from the ergodicity of $(\widetilde{S}^{\theta_0}(t),\widetilde{V}^{\theta_0}(t),\Lambda(t))$ that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t g(\widetilde{S}^{\theta_0}(u),\alpha(u))du=\sum_{k\in\mathcal{M}}\int_{\mathbb{R}_+^2} g(s,k)\boldsymbol{\nu}_{\theta_0}(ds,dv,k)=\lambda_{\theta_0},\ \text{a.s.} \tag{2.31}$$

Therefore, for any $\varepsilon>0$, there exists a $T_1=T_1(H,\varepsilon)>0$ such that $\mathbb{P}_{H,H,k}(\Omega_1)\ge 1-\frac{\varepsilon}{4}$, where

$$\Omega_1=\left\{\omega:\frac{1}{t}\int_0^t g(\widetilde{S}^{\theta_0}_{H,H,k}(u),\alpha(u))du\le\frac{|\lambda_{\theta_0}|}{4}+\lambda_{\theta_0},\quad\forall t\ge T_1\right\}. \tag{2.32}$$

Here, the initial value of $(\widetilde{S}^{\theta_0}(t),\widetilde{V}^{\theta_0}(t),\Lambda(t))$ is indicated in the subscript of $(\widetilde{S}^{\theta_0}_{H,H,k}(t),\widetilde{V}^{\theta_0}_{H,H,k}(t))$. Because of comparison theorem (see e.g., [24]), we have $S_{s,i,v,k}(u)\le\widetilde{S}^{\theta}_{H,H,k}(u)$, $V_{s,i,v,k}(u)\le\widetilde{V}^{\theta}_{H,H,i}(u)$, $\forall 0\le u\le\widetilde{\tau}$ almost surely if (s,i,v,k) satisfies $s\le H,v\le H$. On the other hand, the strong law of large numbers for martingales leads to that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sigma_2(\Lambda(u))dW_2(u)=0,\ \text{a.s.} \tag{2.33}$$

This deduces that there exists $T_2(\varepsilon)>0$ such that $\mathbb{P}(\Omega_2)\ge 1-\frac{\varepsilon}{4}$, where

$$\Omega_2=\left\{\omega\in\Omega:\frac{1}{t}\left|\int_0^t\sigma_2(\Lambda(u))dW_2(u)\right|\le\frac{|\lambda_{\theta_0}|}{4},\quad\text{for all } t\ge T_2\right\}. \tag{2.34}$$

Let $T = \max\{T_1, T_2\}$. As a consequence of (2.3), there is $M > 0$ satisfying that $\mathbb{P}(\Omega_3) \geq 1 - \frac{\varepsilon}{4}$, where

$$\Omega_3 = \left\{ \omega \in \Omega : \int_0^T \beta(\Lambda(u)) \widetilde{S}^{\theta_0}_{H,H,k}(u) du \leq \frac{M}{2} \right\}. \tag{2.35}$$

Moreover, Doob's inequality allows us to pick $M > 0$ sufficiently large satisfying

$$\mathbb{P}\left(\Omega_4 := \left\{ \omega \in \Omega : \left| \int_0^t \sigma_i(\Lambda(u)) W_i(u) du \right| \leq \frac{M}{2} \right\} \right) \geq 1 - \frac{\varepsilon}{4}, \forall t \in [0, T], i = 1, 2. \tag{2.36}$$

Let $\theta_1 \in \left(0, \theta_0 e^{-M}\right)$. The second equation of (1.2), together with (2.35) and (2.36) implies that for all $\omega \in \cap_{j=1}^4 \Omega_j$, we have

$$\begin{aligned} I(t) =& I(0) \exp\left\{ \int_0^t \left(\beta(\Lambda(u)) S(u) - \big(\mu(\Lambda(u)) + \gamma(\Lambda(u)) + \alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2} \right) du \right. \\ &\left. + \int_0^t \sigma_2(\alpha(u)) dW_2(u) \right\} \\ \leq& I(0) \exp\left\{ \int_0^t \beta(\Lambda(u)) S(u) du + \int_0^t \sigma_2(\Lambda(u)) dW_2(u) \right\} \\ \leq& \theta_1 e^M < \theta_0 \ \forall t \in [0, T], \text{ given } I(0) \leq \theta_1. \end{aligned} \tag{2.37}$$

Define the stopping time

$$\widetilde{\zeta} := \inf\left\{ t \geq 0 : I(t) \geq \theta_0 \right\}. \tag{2.38}$$

Because of (2.37), for $\omega \in \Omega_3 \cap \Omega_4$ one has $\widetilde{\zeta} > T$. As a consequence of comparison theorem (see e.g., [24]), we have $S(t) \leq \widetilde{S}^{\theta_0}(t)$ for any $t \leq \widetilde{\zeta}$ given that they have the same initial value. Thus, from (2.37), if $t \leq \widetilde{\zeta}$, one has

$$\begin{aligned} I(t) \leq& I(0) \exp\left\{ \int_0^t \left(\beta(\Lambda(u)) \widetilde{S}^{\theta_0}_{H,H,k}(u) - \big(\beta(\Lambda(u)) + \gamma(\Lambda(u)) + \alpha(\Lambda(u))\big) - \frac{\sigma_2^2(\Lambda(u))}{2} \right) du \right. \\ &\left. + \int_0^t \sigma_2(\Lambda(u)) dW_2(u) \right\}. \end{aligned} \tag{2.39}$$

Combining (2.32), (2.34), (2.37), and (2.39) we claim that $\widetilde{\zeta} > T$ and

$$\begin{aligned} I_z(t) &\leq I(0) \exp\left\{ \int_0^t g(\widetilde{S}^{\theta_0}_{H,H,k}(u), \Lambda(u)) du + \int_0^t \sigma_2(\Lambda(u)) dW_2(u) \right\} \\ &\leq I(0) \exp\left\{ \frac{\lambda_{\theta_0} t}{2} \right\} < I(0) < \theta_0, \text{ for any } t \in [T, \widetilde{\zeta}), \end{aligned}$$

for any $\omega \in \bigcap_{j=1}^4 \Omega_j$ and $I(0) = i \leq \theta_1$. As a result, we have $\widetilde{\zeta} = \infty$ if $\omega \in \bigcap_{j=1}^4 \Omega_j$, $I(0) \leq \theta_1$. Since $\widetilde{\zeta} = \infty$, we have

$$I(t) \leq I(0) \exp\left\{ \frac{\lambda_{\theta_0} t}{2} \right\}, \forall t \geq T, \text{ given that } \omega \in \bigcap_{j=1}^4 \Omega_j, I(0) = i \leq \theta_1.$$

This clearly implies that $\lim_{t\to\infty} I(t) = 0 \ \forall \omega \in \bigcap_{j=1}^4 \Omega_j$, provided $I(0) = i \leq \theta_1$.

Next, we define the randomized occupation measure

$$\widetilde{\Pi}^t_{s,i,v,k}(\cdot) := \frac{1}{t} \int_0^t \mathbf{1}_{\{(S(u), I(u), V(u), \Lambda(u)) \in \cdot\}} du, \quad t > 0.$$

In the above, the subscript in $\widetilde{\Pi}^t_{s,i,v,k}(\cdot)$ indicates the initial condition. Because $S_{s,i,v,k}(t) \leq \widetilde{S}^\theta_{H,H,k}(t), V_{s,i,v,k}(t) \leq \widetilde{V}^\theta_{H,H,k}(t), u \geq 0$ and $\lim_{t\to\infty} I_{s,i,v,k}(t) = 0$ for $\omega \in \cap_{j=1}^4 \Omega_j$, $(s, i, v, k) \in [0, H] \times (0, \theta_1] \times [0, H] \times \mathcal{M}$, we claim that the family of measures $\{\widetilde{\Pi}^t_{s,i,v,k}(\cdot; \omega), t > 0, \omega \in \cap_{j=1}^4 \Omega_j\}$ is tight in the space $\mathbb{R}^3_+ \times \mathcal{M}$ and any of its weak limit $\widetilde{\boldsymbol{\nu}}$ as $t \to \infty$ satisfying $\boldsymbol{\nu}([0, \infty) \times \{0\} \times [0, \infty) \times \mathcal{M}) = 1$. Moreover, we can also claim with probability 1 that $\boldsymbol{\nu}$ is invariant with respect to the process $(S(t), I(t), V(t), \Lambda(t))$ on $\mathbb{R}^3_+ \times \mathcal{M}$. We can refer these well-known claims to [19,32]. On the other hand, we have showed that $\boldsymbol{\nu}_0$, through an embedding, is the only invariant measure on $[0, \infty) \times \{0\} \times [0, \infty) \times \mathcal{M}$. Therefore, for almost every $\omega \in \cap_{j=1}^4 \Omega_j$, we have the weak convergence of $\widetilde{\Pi}^t_{s,i,v,k}(\cdot)$ to $\boldsymbol{\nu}_0$ as $t \to \infty$. The weak convergence implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(S(u), \Lambda(u)) du = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} g(s, k) \boldsymbol{\nu}_0(ds, dv, k) = \lambda, \tag{2.40}$$

for almost every $\omega \in \cap_{j=1}^4 \Omega_j$. It is noted that the limit (2.40) is valid because of the weak convergence and the uniform integrability

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \int_0^t (S(u))^{1+\widetilde{p}} du &\leq \lim_{t\to\infty} \frac{1}{t} \int_0^t (\widetilde{S}^{\theta_0}(u))^{1+\widetilde{p}} du \\ &= \sum_{k\in\mathcal{M}} \int_{\mathbb{R}^2_+} s^{1+\widetilde{p}} \boldsymbol{\nu}_{\theta_0}(ds, dv, k) < \infty \text{ for some } \widetilde{p} > 0; \end{aligned}$$

see e.g., [19, Lemma 5.6]. By (2.30) and (2.37), one has

$$\frac{\ln I(t)}{t} = \frac{1}{t}\int_0^t g(S(u),\Lambda(u))du + \frac{\ln I(0)}{t} + \frac{1}{t}\int_0^t \sigma_2(\Lambda(u))dW_2(u). \tag{2.41}$$

Therefore, letting $t\to\infty$ in (2.41) and because of (2.33) and (2.40), we have that for almost every $\omega\in\cap_{j=1}^4\Omega_j$,

$$\lim_{t\to\infty}\frac{\ln I(t)}{t} = \lambda.$$

As a result, by noticing that $\mathbb{P}(\cap_{j=1}^4\Omega_j)\ge 1-\varepsilon$, the Lemma is proved. □

Proof of Theorem 2.3. Because of Lemma 2.6, the solution process $\{(S(\cdot),I(\cdot),V(\cdot))\}$ is transient in $\mathbb{R}_+^{3,\circ}$. This fact leads to that the process has no invariant probability measure in $\mathbb{R}_+^{3,\circ}$. As a result, $\boldsymbol{\nu}_0$ is the sole invariant probability measure of $(S(t),I(t),V(t),\Lambda(t))$ in $[0,\infty)\times\{0\}\times[0,\infty)\times\mathcal{M}$. Now, pick $H>0$ satisfying $\boldsymbol{\nu}_0(\{s\in(0,H)\})>1-\varepsilon$. Because the process $(S(t),I(t),V(t),\Lambda(t))$ is tight (due to (2.2)), for any initial value $(s,i,v,k)\in\mathbb{R}_+^3\times\mathcal{M}$, the family of occupation measures

$$\left\{\Pi^t_{s,i,v,k}(\cdot) := \frac{1}{t}\int_0^t \mathbb{P}\{(S(w),I(w),V(w),\Lambda(w))\in\cdot\}\,dw, t\ge 1\right\}$$

is tight in the whole space $\mathbb{R}_+^3\times\mathcal{M}$. Because of the invariability of the weak-limits of $\Pi^t_{s,i,v,k}$ as $t\to\infty$, we deduce that $\Pi^t_{s,i,v,k}$ weakly converges to $\boldsymbol{\nu}_0$ as $t\to\infty$. Sequentially, for any $\delta>0$, there is a constant $\hat{T}>0$ satisfying that

$$\Pi^{\hat{T}}_{s,i,v,k}((0,H)\times(0,\delta)\times(0,H)\times\mathcal{M})>1-\varepsilon,$$

which can be rewritten as

$$\frac{1}{\hat{T}}\int_0^{\hat{T}}\mathbb{P}_{s,i,v,k}\{0<S(t),V(t)<H, 0<I(t)<\delta\}dt>1-\varepsilon.$$

Therefore,

$$\mathbb{P}_{s,i,v,k}\{\hat{\zeta}\le\hat{T}\}>1-\varepsilon,$$

where

$$\hat{\zeta} = \inf\{t\ge 0 : (S(t),I(t),V(t))\in(0,H)\times(0,\delta)\times(0,H)\}.$$

We obtain from the strong Markov property and Lemma 2.6 that

$$\mathbb{P}_{s,i,v,k}\left\{\lim_{t\to\infty}\frac{\ln I(t)}{t}=\lambda\right\}\ge 1-\varepsilon,\ \forall(s,i,v,k)\in\mathbb{R}_+^{3,*}\times\mathcal{M}.$$

Letting $\varepsilon\to 0$ we obtain (2.14). The proof of Theorem 2.2 is complete. □

3. Some control policies

3.1. Determining the herd immunity

In (1.2), we let $q(k)$ and $p(k)$ be switching-dependent so that the model can be as general as possible. However, they are the immunization rates and human-controlled so they might not depend on k in practice. We will derive a formula for λ when $q(k)\equiv q$ and $p(k)\equiv q$. The formula will help us determine the immunization rates needed to maintain the disease-free state.

Let $\mathbf{c}_i=(c_i(1),\dots,c_i(m_0))^\top, i=1,2,$

$$\mathbf{B}=\begin{bmatrix}\mu(1)-\gamma_{11} & \cdots & -\gamma_{1m_0}\\ \vdots & \vdots & \vdots\\ -\gamma_{m_01} & \cdots & \mu(m_0)-\gamma_{m_0m_0}\end{bmatrix},$$

$\mathbf{D}=\operatorname{diag}(\varepsilon(1),\dots,\varepsilon(m_0))$ and $\mathbf{I}_{m_0}$ be the identity matrix on $\mathbb{R}_{m_0\times m_0}$. Then, consider the equation

$$\begin{cases}(\mathbf{B}+p\mathbf{I}_{m_0})\mathbf{c}_1-p\mathbf{c}_2= & \beta,\\ -\mathbf{D}\mathbf{c}_1+(\mathbf{B}+\mathbf{D})\mathbf{c}_2= & 0.\end{cases}$$

Substituting $\mathbf{c}_2=(\mathbf{B}+\mathbf{D})^{-1}\mathbf{D}\mathbf{c}_1=(\mathbf{I}_{m_0}-(\mathbf{B}+\mathbf{D})^{-1}\mathbf{B})\mathbf{c}_1$, which is from the second equation, into the first equation, we have

$$(\mathbf{B}+p(\mathbf{B}+\mathbf{D})^{-1}\mathbf{B})\mathbf{c}_1=\boldsymbol{\beta},$$

or

$$(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})(\mathbf{B}+\mathbf{D})^{-1}\mathbf{B}\mathbf{c}_1=\boldsymbol{\beta},$$

which leads to

$$\mathbf{c}_1 = \mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta}.$$

In view of Lemma 2.2, we have

$$\lambda = \sum_{k\in\mathcal{M}} \left[c_1(k)(1-q)A(k) - \left(\mu(k)+\alpha(k)+\gamma(k)+\frac{\sigma_2^2(k)}{2}\right)\right]\pi_k = (1-q)\pi\mathbf{c}_1\circ\mathbf{a} - \pi\mathbf{f},$$

where $\mathbf{c}_1 = \mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta}$, $\mathbf{a} = (A(1),\dots,A(m_0))^\top$, $\circ$ is the Hadamard product and

$$\mathbf{f} = \left(\mu(1)+\alpha(1)+\gamma(1)+\frac{\sigma_2^2(1)}{2},\dots,\mu(m_0)+\alpha(m_0)+\gamma(m_0)+\frac{\sigma_2^2(m_0)}{2}\right)^\top.$$

In what follows, for a matrix (or a vector) M, by writing $M>0$ we mean all entries of M are positive.

Lemma 3.1. *$\mathbf{c}_1 = \mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta}$ is decreasing in p, that is if $p_1 \ge p_2$ then $\mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p_1\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta} - \mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p_2\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta} > 0$.*

Proof. Neumann's series expansion of resolvents give

$$(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1} = (\mathbf{B}+\mathbf{D}+p_0\mathbf{I}_{m_0})^{-1} + \sum_{n=1}^{\infty}(p_0-p)^n(\mathbf{B}+\mathbf{D}+p_0\mathbf{I}_{m_0})^{-n-1}, \text{ for } |p-p_0| \text{ sufficiently small},$$

which leads to

$$\frac{d(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}}{dp}(p_0) = -(\mathbf{B}+\mathbf{D}+p_0\mathbf{I}_{m_0})^{-2}.$$

As a result,

$$\frac{d\mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta}}{dp} = -\mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-2}\beta.$$

Note that $\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0}$ is a diagonally dominant matrix. Thus, it is well-known that $(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}$ is a matrix of all positive entries and so is $(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-2}$. As a result,

$$\widetilde{\boldsymbol{\beta}} := (\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-2}\boldsymbol{\beta} > 0.$$

So

$$\mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-2}\beta = \mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})\widetilde{\boldsymbol{\beta}} = \widetilde{\boldsymbol{\beta}} + \mathbf{B}^{-1}\mathbf{D}\widetilde{\boldsymbol{\beta}}.$$

Since $\mathbf{B}^{-1} > 0$ and $\widetilde{\boldsymbol{\beta}} > 0$, $\mathbf{D}\widetilde{\boldsymbol{\beta}} > 0$, we deduce that

$$-\frac{d\mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta}}{dp} = \widetilde{\boldsymbol{\beta}} + \mathbf{B}^{-1}\mathbf{D}\widetilde{\boldsymbol{\beta}} > 0,$$

which implies $\mathbf{c}_1$ is a decreasing vector-valued function of p. □

From Lemma 3.1, in order to maintain the disease-free state of the system, the vaccination rate p must be smaller greater than p_0, which is the solution to

$$\lambda(p_0) = (1-q)\pi\mathbf{B}^{-1}(\mathbf{B}+\mathbf{D})(\mathbf{B}+\mathbf{D}+p_0\mathbf{I}_{m_0})^{-1}\boldsymbol{\beta}\circ\mathbf{a} - \pi\mathbf{f} = 0. \tag{3.1}$$

3.2. Optimal strategies for controlling the disease

Suppose that p^* is the maximum vaccination rate of susceptible group that can be achieved. Note that Eq. (3.1) does not always have a solution on $[0,p^*]$. As a result, we may want to minimize the cost of vaccination and treatment. Suppose the coefficients are not switching-dependent, we consider the following system:

$$\begin{cases} dS(t) = & \Big[(1-q)A - \beta S(t)I(t) - (p+\mu)S(t), \\ & +(\gamma+r)I(t) + \varepsilon V(t)\Big]dt + \sigma_1 S(t)dW_1(t), \\ dI(t) = & \Big(\beta I(t)S(t) - [\mu+\gamma+\alpha-r]I(t)\Big)dt + \sigma_2 I(t)dW_2(t), \\ dV(t) = & \Big[qA + pS(t) - (\mu+\varepsilon)V(t)\Big]dt + \sigma_3 V(t)dW_3(t), \\ S(0) = & s\ge 0, \quad I(0)=i\ge 0, \quad V(0)=v\ge 0, \end{cases} \tag{3.2}$$

where r is the rate of recovering due to treatment. It is reasonable to assume that $r\le\alpha$, because α is the extra rate of death due to the disease.

Using the definition (2.6) of λ, we obtain that for this model,

$$\lambda = \int_{\mathbb{R}_+^2} \beta s \widetilde{\mu}(ds, dv) - (\mu + \alpha + \gamma - r) - \frac{\sigma_2^2}{2},$$

where $\widetilde{\mu}$ is the invariant measure of process $(\widetilde{S}(t), \widetilde{V}(t))$, which is the solution to the following system obtained by considering (3.2) on the boundary with $I(t) = 0$

$$\begin{cases} d\widetilde{S}(t) = & \left[(1-q)A - (p+\mu)\widetilde{S}(t) + \varepsilon \widetilde{V}(t)\right]dt + \sigma_1 \widetilde{S}(t) dW_1(t), \\ d\widetilde{V}(t) = & \left[qA + p\widetilde{S}(t) - (\mu+\varepsilon)\widetilde{V}(t)\right]dt + \sigma_3 \widetilde{V}(t) dW_3(t). \end{cases} \tag{3.3}$$

Because of [19, Lemma 2.1], we can obtain that $M_{\widetilde{S}} := \int_{\mathbb{R}_+^2} s\widetilde{\mu}(ds, dv)$ and $M_{\widetilde{V}} := \int_{\mathbb{R}_+^2} v\widetilde{\mu}(ds, dv)$ is the solution to

$$\begin{cases} (1-q)A - (p+\mu) M_{\widetilde{S}} + \varepsilon M_{\widetilde{V}} = 0 \\ qA + pM_{\widetilde{S}} - (\mu+\varepsilon) M_{\widetilde{V}} = 0. \end{cases}$$

Therefore, we get $M_{\widetilde{S}} = \frac{A((1-q)\mu+\varepsilon)}{\mu(p+\mu+\varepsilon)}$, so $\lambda = \beta\frac{A((1-q)\mu+\varepsilon)}{\mu(p+\mu+\varepsilon)} - (\mu+\gamma+\alpha-r) - \frac{\sigma_2^2}{2}$. We assume that $\lambda > 0$ for all $0 \le p \le p^*, 0 \le r \le \alpha$, that is

$$\beta\frac{A((1-q)\mu+\varepsilon)}{\mu(p^*+\mu+\varepsilon)} - (\mu+\gamma) - \frac{\sigma_2^2}{2} > 0. \tag{3.4}$$

Under this assumption, the disease persists even with treatment and vaccination. Suppose that c_1, c_2, and c_3 are the per capital costs of vaccination, treatment, and infection on a short period of time Δt. Then the total cost in the period of time $[0, T]$ is

$$\int_0^T (c_1 pS(t) + c_2 rI(t) + c_3 I(t))dt.$$

In the long run, assuming (3.4) holds, we wish to do the following optimization problem:

$$\text{Minimize } \lim_{T\to\infty} \mathbb{E}_{s,i,v} \frac{1}{T}\int_0^T (c_1 pS(t) + c_2 rI(t) + c_3 I(t))dt, \quad \text{subject to } 0 \le p \le p^*, 0 \le r \le \alpha. \tag{3.5}$$

Due to the ergodicity of the process, the following limits exist.

$$x_1 = \lim_{T\to\infty} \mathbb{E}_{s,i,v} \frac{1}{T}\int_0^T S(t)dt,$$

$$x_2 = \lim_{T\to\infty} \mathbb{E}_{s,i,v} \frac{1}{T}\int_0^T I(t)dt,$$

$$x_3 = \lim_{T\to\infty} \mathbb{E}_{s,i,v} \frac{1}{T}\int_0^T V(t)dt,$$

$$x_4 = \lim_{T\to\infty} \mathbb{E}_{s,i,v} \frac{1}{T}\int_0^T S(t)I(t)dt.$$

From the first equation of (3.2), we have

$$\begin{aligned} \frac{\mathbb{E}_{s,i,v} S(T)}{T} = & \frac{s}{T} + (1-q)A - \beta\mathbb{E}_{s,i,v}\frac{1}{T}\int_0^T S(t)I(t)dt - (p+\mu)\mathbb{E}_{s,i,v}\frac{1}{T}\int_0^T S(t)dt \\ & + (\gamma+r)\mathbb{E}_{s,i,v}\frac{1}{T}\int_0^T I(t)dt + \varepsilon\mathbb{E}_{s,i,v}\frac{1}{T}\int_0^T V(t)dt. \end{aligned}$$

Letting $T \to \infty$, we have

$$(1-q)A - \beta x_4 - (p+\mu)x_1 + \gamma x_2 + \varepsilon x_3 = 0. \tag{3.6}$$

Similarly, we have from the other equations of (3.2) that

$$\beta x_4 - (\mu+\gamma+\alpha-r) = 0, \tag{3.7}$$

and

$$qA + px_1 - (\mu+\varepsilon)x_3 = 0. \tag{3.8}$$

Moreover, we have

$$\frac{\ln I(T)}{T} = \frac{\ln I(0)}{t} + \beta\frac{1}{T}\int_0^T S(t)dt - (\mu+\gamma+\alpha+\frac{\sigma_2^2}{2} - r) + \frac{\sigma_2 W_2(t)}{t}.$$

Letting $T \to \infty$, noting that $\lim_{T\to\infty} \frac{1}{T}\int_0^T S(t)dt = x_1$ (due to the ergodicity of (3.2)), we have

$$\lim_{t\to\infty} \frac{\ln I(T)}{T} = \beta x_1 - (\mu + \gamma + \alpha + \frac{\sigma_2^2}{2} - r).$$

Since the process $(S(t), I(t), V(t))$ is recurrent on $\mathbb{R}_+^{3,\circ}$, we must have

$$\beta x_1 - (\mu + \gamma + \alpha + \frac{\sigma_2^2}{2} - r) = 0. \tag{3.9}$$

From (3.6), (3.7),(3.8) and (3.9), we have

$$x_1 = \frac{(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2} - r)}{\beta},$$

and

$$(\gamma + r)x_2 = (\mu + \gamma + \alpha - r) - A(\mu + \varepsilon + \mu q) + \left(\mu + p - \frac{p\varepsilon}{\mu + \varepsilon}\right) \frac{(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2} - r)}{\beta}.$$

Then (3.5) becomes an elementary optimization problem:

$$\text{Minimizing } f(p, r), \quad \text{subject to } 0 \le p \le p^*, 0 \le r \le \alpha, \tag{3.10}$$

where

$$\begin{aligned} f(p, r) = &\frac{c_1 p(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2} - r)}{\beta} \\ &+ \frac{c_2 r + c_3}{\gamma + r}\left((\mu + \gamma + \alpha - r) - A(\mu + \varepsilon + \mu q) + \left(\mu + p - \frac{p\varepsilon}{\mu + \varepsilon}\right) \frac{(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2} - r)}{\beta}\right). \end{aligned}$$

Remark 3.1. Controlling diseases often involves enhancing vaccination rates for susceptible groups and implementing treatment methods, typically represented by parameters p and r (as outlined in (3.2)). Ideally, maximizing the improvement in rates p and r would be optimal for disease control. However, practical constraints such as budgetary limitations and economic factors limit these rates in real-world applications. We consider the scenario that the disease persists and find optimal rates, which optimize the costs that are due to vaccination and treatment and that penalize infection, as given in (3.5). The longtime cost function in (3.5) is intractable. However, we are able to transform the problem to an elementary optimization problem (3.10).

4. Discussion and numerical examples

4.1. Discussion

Possible generalization. Although, in this paper, we consider the linear incidence rate function βSI in the dynamics of disease transmission, it can be seen that our approach and proofs of main results do not depend on this specific formulation. In particular, we can replace the linear incidence rate function $\beta(\Lambda(t))S(t)I(t)$ in (1.2) by a general function $I(t)f(\Lambda(t), S(t), I(t))$ and consider

$$\begin{cases} dS(t) = & \Big[(1 - q(\Lambda(t)))\, A(\Lambda(t)) - I(t)f(\Lambda(t), S(t), I(t)) - (p(\Lambda(t)) + \mu(\Lambda(t)))\, S(t) \\ & + \gamma(\Lambda(t))I(t) + \varepsilon(\Lambda(t))V(t) \Big]\, dt + \sigma_1(\Lambda(t))S(t)dW_1(t), \\ dI(t) = & \Big(I(t)f(\Lambda(t), S(t), I(t)) - [\mu(\Lambda(t)) + \gamma(\Lambda(t)) + \alpha(\Lambda(t))]\, I(t) \Big) dt \\ & + \sigma_2(\Lambda(t))I(t)dW_2(t), \\ dV(t) = & \Big[q(\Lambda(t))A(\Lambda(t)) + p(\Lambda(t))S(t) - (\mu(\Lambda(t)) + \varepsilon(\Lambda(t)))\, V(t) \Big] dt \\ & + \sigma_3(\Lambda(t))V(t)dW_2(t), \\ S(0) = & s \ge 0, \quad I(0) = i \ge 0, \quad V(0) = v \ge 0, \quad \Lambda(0) = k \in \mathcal{M}. \end{cases} \tag{4.1}$$

In this case, by the same approach and similar argument we have developed in Section 2, we can define

$$\lambda_f = \sum_{k\in\mathcal{M}} \int_{\mathbb{R}_+^2} \left(f(k, s, 0) - (\mu(k) + \alpha(k) + \gamma(k)) - \frac{\sigma_2^2(k)}{2} \right) \boldsymbol{v}_0(ds, dv, k),$$

where $\boldsymbol{v}_0$ is the invariant measure of the solution process to (2.4) with $\theta = 0$, and prove that this λ_f can be used to characterize completely longtime behaviors of (4.1).

Theorem 4.1. *Assume that*

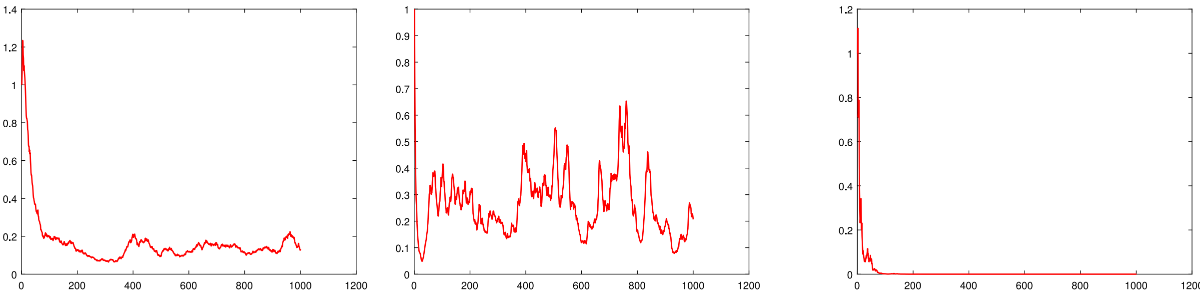

Fig. 1. Sample paths of $I(t)$. From left to right: in fixed state system with $\Lambda(t) = 1$, in fixed state system with $\Lambda(t) = 2$, and in the switched system, respectively in Example 4.1.

- *$f(k, s, i)$ is locally Lipschitz, $f(k, 0, i) = 0$, $f(k, s, i) \le K(1 + s)$ for some finite K, and*
- *$f(k, s, i)$ is uniformly continuous at $i = 0$, i.e., $\lim_{i\to 0} \sup |f(k, s, i) - f(k, s, 0)| = 0$.*

We have:

1. *If $\lambda_f > 0$, the transition probability converges to an invariant measure with an exponential rate.*
2. *If $\lambda_f < 0$, $\lim_{t\to\infty} \frac{\ln I(t)}{t} = \lambda_f$ with probability 1.*

Remark 4.1. Common incidence rate functions used in epidemic models are: the bilinear functional response βSI, the nonlinear functional response $\frac{\beta SI^l}{1+m_2 I^h}$, the Beddington–DeAngelis functional response $\frac{\beta SI}{1+m_1 S+m_2 I}$, the Holling type II functional response $\frac{\beta SI}{m_1+S}$, and their variants; see e.g., [16,26–28,33] and references therein. We note that all these common incidence rate functions satisfy conditions in Theorem 4.1. *Recall that the incidence rate in our setting is $If(S, I)$ rather than $f(S, I)$.*

In Theorem 4.1, if we assume further that

$$\sum_{k\in\mathcal{M}} \left(\lim_{s\to\infty} f(k, s, 0) - (\mu(k) + \alpha(k) + \gamma(k)) - \frac{\sigma_2^2(k)}{2} \right) \pi(k) > 0,$$

then we can prove the convergence has an exponential rate.

4.2. Numerical examples

In this section, we focus on some numerical examples for model (1.2). As in Lemma 2.2, we obtain an algebraic representation for λ in this model, which is an interesting finding, allowing us to easily evaluate this threshold. The value λ is computed simply by solving a system of linear equations together with basic multiplication and addition. As a result, we can have more insight about how switching changes the dynamics of the system. Moreover, it will be shown in following examples that extinction or persistence of two fixed systems can be reversed by switching.

Example 4.1. Consider system (1.2) with two random switching states: $\mathcal{M} = \{1, 2\}$. The other parameters are $q_1 = 0.8$, $q_2 = 0.4$, $A(1) = 2.5$, $A(2) = 1$, $\beta_1 = 4$, $\beta_2 = 8$, $\mu(1) = 1$, $\mu(2) = 0.5$, $p(1) = 1$, $p(2) = 0.3$, $\gamma_1 = 0.2$, $\gamma_2 = 3$, $\varepsilon(1) = 0.2$, $\varepsilon(2) = 0.5$, $\sigma_1(1) = 0.2, \sigma_1(2) = 1, \alpha(1) = 0.1, \alpha(2) = 3, \sigma_2(1) = 0.2, \sigma_2(2) = 2, \sigma_3(1) = 0.2, \sigma_3(2) = 1$.

In this example, the thresholds for the system without switching in state 1 and state 2 are $\lambda_1 = 0.1216$ and $\lambda_2 = 1.0524$. As a result, without switching, the disease will persist in either state. However, with switching rates $\gamma_{12} = \gamma_{21} = 30$, we have $\lambda = -0.3804$, which implies that the disease will go away. This example shows that the random switching can make persistence become extinction, see Fig. 1.

The algebraic representation of λ in Lemma 2.2 allows us to examine in details λ as a function of the parameters. In this example, with $\gamma_{12} = \gamma_{21} = y$ and $q(1) = q(2) = x$ while the other parameters receive values as above, we have Fig. 2 for λ as a function of x, y and Fig. 3 for λ as a function of x or y given some fixed values of the other.

Example 4.2. Consider system (1.2) with two random switching states: $\mathcal{M} = \{1, 2\}$. The other parameters are $q_1 = q_2 = 0.8, A(1) = 1.2, A(2) = 5, \beta(1) = 5, \beta(2) = 1.0, \mu(1) = \mu(2) = 0.8, p(1) = p(2) = 0.5, \gamma(1) = \gamma(2) = 2, \varepsilon(1) = 0.2, \varepsilon(2) = 0.5, \sigma_1(1) = \sigma_1(2) = 0.1, \alpha(1) = \alpha(2) = 0.11, \sigma_2(1) = \sigma_2(2) = \sigma_3(1) = \sigma_3(2) = 0.1$.

In this example, the thresholds for two fixed system in state 1 and state 2 are $\lambda_1 = -1.115$ and $\lambda_2 = -0.6233$. As a result, without switching, the disease will die out in either state. However, with switching rates $\gamma_{12} = \gamma_{21} = 20$, we have $\lambda = 1.0261$, which implies the disease persists, see Fig. 4.

With $\gamma_{12} = \gamma_{21} = y$ and $q(1) = q(2) = x$ while the other parameters receive values as above, we have Fig. 5 for λ as a function of x, y and Fig. 6 for λ as a function of x or y given some fixed values of the other.

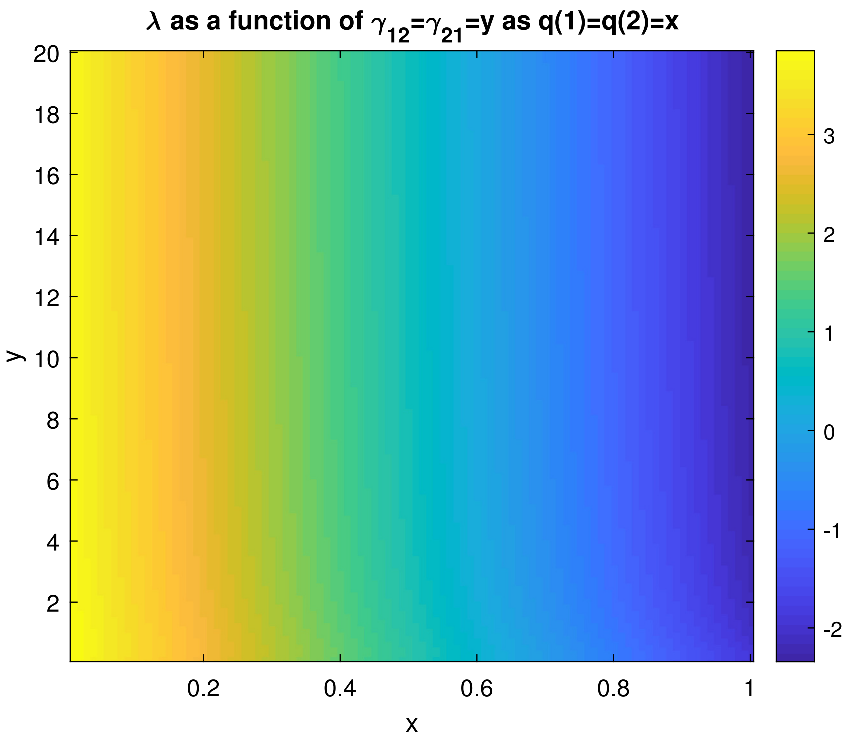

Fig. 2. λ as a function of $q(1) = q(2) = x$ and $\gamma_{12} = \gamma_{21} = y$.

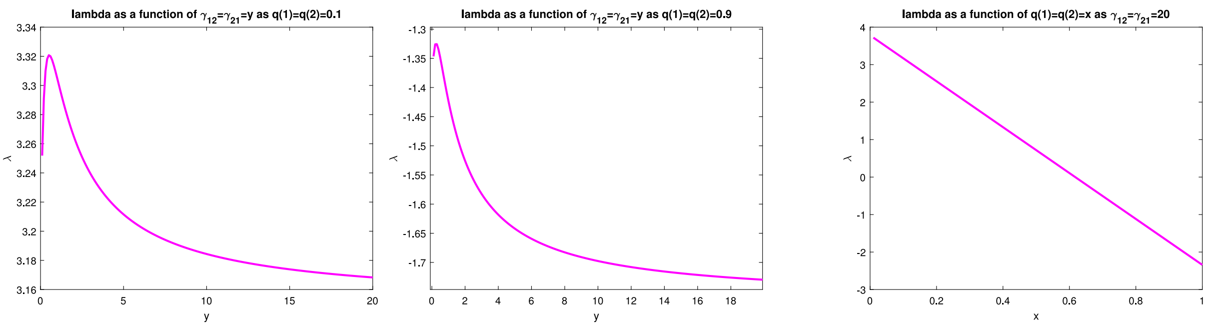

Fig. 3. From left to right: the graphs of $\lambda(y)$ with $x = 0.1$; $\lambda(y)$ with $x = 0.9$ and $\lambda(x)$ with $y = 1$ respectively.

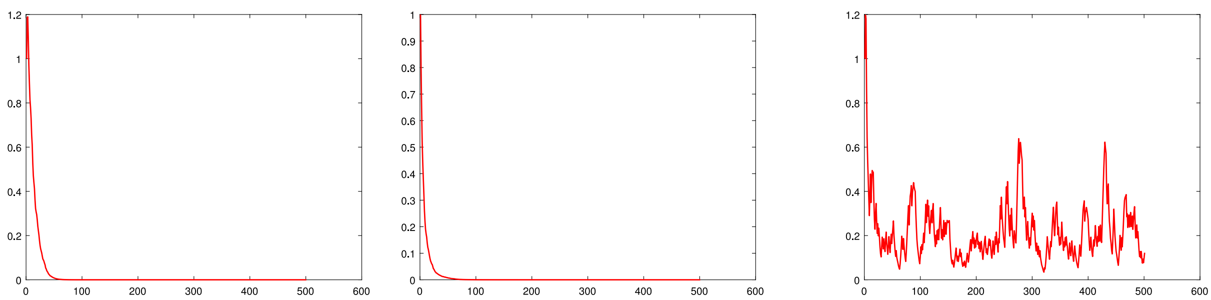

Fig. 4. Sample paths of $I(t)$. From left to right: in fixed state system with $\Lambda(t) = 1$, in fixed state system with $\Lambda(t) = 2$, and in the switched system, respectively in Example 4.2.

Example 4.3. Consider system (1.2) with two random switching states: $\mathcal{M} = \{1, 2\}$, $p(1) = p(2) = p$ and parameters $q(1) = q(2) = 0$, $\Lambda(1) = \Lambda(2) = 1$, $\beta(1) = 3, \beta(2) = 3.5, \mu(1) = 1, \mu(2) = 2, \gamma(1) = \gamma(2) = 0.1, \varepsilon(1) = \varepsilon(2) = 0.1, \sigma_2(1) = \sigma_2(2) = \sqrt{0.2}, \alpha(1) = \alpha(2) =$

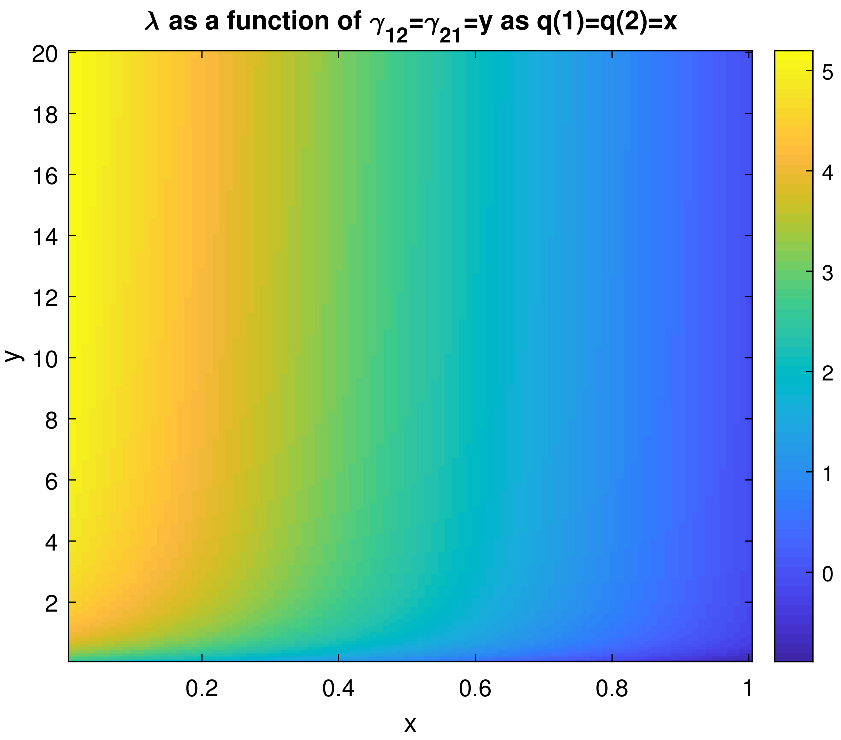

Fig. 5. λ as a function of $q(1) = q(2) = x$ and $\gamma_{12} = \gamma_{21} = y$.

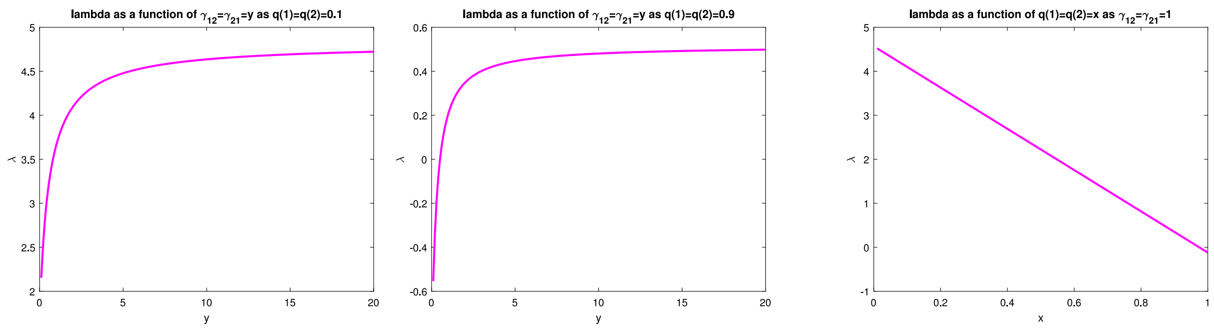

Fig. 6. From left to right: the graphs of $\lambda(y)$ with $x = 0.1$; $\lambda(y)$ with $x = 0.9$ and $\lambda(x)$ with $y = 1$ respectively.

$0.1, \sigma_1(1) = \sigma_1(2) = \sigma_3(1) = \sigma_3(2) = 0.1$. The equation $\lambda = 0$ leads to

$$3600p^2 + 11770p - 4959 = 0,$$

which has a unique root $p_0 = 0.3777$ in $(0, 1)$. Thus, the vaccination rate p of susceptible group must be greater than 0.3777 in order to maintain the disease-free state of the system.

Biological interpretation. Our theoretical findings indicate that the long-term dynamics of the SIS epidemic model with vaccination are entirely determined by the threshold number λ, derived from parameters outlined in (2.6). In epidemic models, because our target is to end the pandemic or to control disease, the extinction stage (corresponding with $\lambda < 0$) is preferable. Our results rigorously affirm the significant roles played by the disease-transmission rate β, recovery rate γ, and disease-related death rate α in pandemic control. Additionally, our findings suggest that the uncertainty associated with the evolution of the infected group (described by σ_2) can have positive effects.

Furthermore, above numerical examples demonstrate that random switching environments can make extinction to be persistence and vice versa. Although the dependence of λ on vaccinated rate of newborns q is not too explicit from (2.6) in the general setting, our numerical examples (see Figs. 2 and 5) reveal that the vaccinated rate q has a significant impact. This emphasizes the importance of developing vaccines and vaccinating newborns in disease control efforts.

CRediT authorship contribution statement

Nguyen T. Hieu: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Dang H. Nguyen:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Nhu N. Nguyen:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Tran D. Tuong:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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