Holography for the trace anomaly action

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(Received 6 July 2023; accepted 12 September 2023; published 9 October 2023)

A recently proposed effective action for the trace anomaly describes a tensor-scalar theory that is weakly coupled up to a certain high energy scale, where it becomes strongly interacting. Its ultraviolet completion is obtained by coupling to gravity a quantum field theory in which conformal invariance is spontaneously broken. In this paper, we show that if the field theory that gives rise to the trace anomaly is a large N_c conformal field theory, then the trace anomaly action has a completion above the strong scale in a holographic Randall-Sundrum two-brane theory, with the radion as a low energy remnant of the spontaneously broken conformal symmetry. Furthermore, we note that the subleading N_c terms can be derived by adding localized fields to the UV brane, so that the theory remains weakly coupled. The subleading terms are also obtained by introducing the Weyl squared terms in the 5D bulk. These, however, exhibit strongly coupled behavior at the respective sub-Planckian energy scales.

DOI: 10.1103/PhysRevD.108.085006

I. INTRODUCTION AND SUMMARY

A local diffeomorphism-invariant action capturing the gravitational trace anomaly [1] has been know thanks to the works of Riegert, and Efim Fradkin and Tseytlin [2,3]; it is referred to as the local Riegert action, albeit the works [2,3] are practically simultaneous, with Riegert's derivation being more general. The current understanding of the Riegert action as an effective action for the trace anomaly was achieved by Komargodski and Schwimmer [4] [see also an $SO(2,4)/ISO(1,3)$ coset construction leading to this action in [5]].[1]

However, general relativity (GR) coupled to the Riegert action is a strongly coupled theory at an arbitrary low energy scale (see [6,7], and references therein). One of us proposed in [7] to resolve this problem by augmenting the classical GR action so that the amended theory, together with the Riegert action, is weakly coupled all the way up to a certain high energy scale $\bar{M}$, which could presumably be in the interval, $M_0 \sim 10^5$ GeV $\ll \bar{M} \ll M_{\rm Pl} = 1/\sqrt{8\pi G} \sim 10^{18}$ GeV. The augmented action [7], without the Riegert term, reads:

$$S_{R-\bar{R}} = M^2 \int d^4x\sqrt{-g}R - \bar{M}^2 \int d^4x\sqrt{-\bar{g}}\bar{R}, \quad (1)$$

where $\bar{R} \equiv R(\bar{g})$, $M = M_{\rm Pl}/\sqrt{2} \gg \bar{M}$, and the two metric tensors are related as

$$g_{\mu\nu} = e^{2\tau}\bar{g}_{\mu\nu}, \quad (2)$$

where τ is a scalar field. Consequently, the total effective action that captures classical gravitational physics and the trace anomaly equation reads as follows [7]:

$$S_{\rm eff} = S_{R-\bar{R}} + S_A(\tau, \bar{g}), \quad (3)$$

where S_A—denoting the local Riegert action—has the form [2–4]

$$S_A = -2a \int d^4x\sqrt{-\bar{g}}(\tau\bar{E} - 4\bar{G}^{\mu\nu}\bar{\nabla}_\mu\tau\bar{\nabla}_\nu\tau - 4(\bar{\nabla}^2\tau)(\bar{\nabla}\tau)^2 - 2(\bar{\nabla}\tau)^4) + 2c' \int d^4x\sqrt{-\bar{g}}\tau\bar{W}^2, \quad (4)$$

[1]After deriving the local but seemingly noncovariant anomaly action, Riegert went on to rewrite it as a manifestly diff-invariant, but nonlocal action [2]. This rewriting introduced a fourth-order derivative into the action via a nonlocal field redefinition used in Eq. (19) of [2]. The resulting action in Eqs. (24) and (25) of [2] has a ghost because of the fourth-order derivative term. This ghost, even if projected out classically, will still lead to unphysical quantum instabilities. For that reason, the anomaly actions containing fourth-order derivatives will not be considered here. In this work we only use Riegert's local action, presented in Eqs. (6) and (8) of [2], which is diffeomorphism invariant [4], and has no four-derivative kinetic terms.

with the covariant derivative $\overline{\nabla}$, the Euler (Gauss-Bonnet) invariant, $\bar{E} = \bar{R}^2_{\mu\nu\alpha\beta} - 4\bar{R}^2_{\mu\nu} + \bar{R}^2$, and the Weyl tensor squared, $\bar{W}^2 = \bar{R}^2_{\mu\nu\alpha\beta} - 2\bar{R}^2_{\mu\nu} + \bar{R}^2/3$, all made out of the metric $\bar{g}$.

The action (3), with $c' = 0$, was introduced earlier in [8] in a different, entirely classical context without reference to the quantum trace anomaly, but as a scalar-tensor action maintaining two derivative conformally invariant equations of motion, in spite of the action itself not being conformally invariant. This action belongs to a more general class of Horndeski's scalar-tensor theories, which maintain second order equations of motion [9].

It is evident that Eq. (3) should be regarded as defining an effective low energy action. Since $\bar{M} \ll M_{\rm Pl}$, all Planck-scale suppressed higher dimensional terms can be ignored in this action; it describes a weakly coupled theory up to the scale $\bar{M}$, where it becomes strongly coupled [7].

Recall that the GR action breaks the scale invariance explicitly, with $M_{\rm Pl}$ being the breaking scale.[2] The parameter $\bar{M}$, on the other hand, can be viewed as a scale where the conformal symmetry is spontaneously broken [5,7]. The low energy remnant of the spontaneous breaking is τ. Thus, the action (3), when viewed as a functional of g and τ, contains terms for explicit, spontaneous, and anomalous breaking of the conformal symmetry. That being said, the respective high energy theory above $\bar{M}$ is likely to contain more terms and degrees of freedom. The goal of this paper is to discuss the UV completion of the above action at the scale $\bar{M}$.

The Riegert action is in fact part of the universal action describing the dynamics of a conformal field theory (CFT) with spontaneously broken conformal invariance below the breaking scale [4]. The field τ is identified with the dilaton, that is the Nambu-Goldstone boson of spontaneously broken conformal invariance, and the second term in (1) contains its kinetic term.

When a generic CFT is coupled to gravity many relevant operators—such as scalar masses—are induced, which make the theory gapped and lift the flat directions in the scalar potential. The flat directions are the very reason why conformal invariance is broken spontaneously instead of being broken explicitly, so spontaneous conformal symmetry breaking is nongeneric in quantum field theory. An exception to this scenario is represented by theories that come from certain highly supersymmetric compactifications of string theory.

Another interesting example is large-N_c CFTs, which have a holographically dual description in 5D. Here we will study the UV completion of the action (3) in the case when a quantum field theory (QFT) that gives rise to the Riegert action is a large N_c CFT that has a holographic dual. There are simplifications in this case that help our analysis. Indeed, the general trace anomaly equation, $\langle T^\mu_\mu \rangle = -aE + c'W^2$, simplifies in the large N_c limit of a CFT where $a = c'$

$$\langle T^\mu_\mu \rangle = a(W^2 - E) = \frac{N_c^2}{32\pi^2}\left(R^2_{\mu\nu} - \frac{1}{3}R^2\right). \quad (5)$$

Henningson and Skenderis have shown in [10] that the term, $R^2_{\mu\nu} - \frac{1}{3}R^2$, generically emerges in equations of motion via 5D AdS holography, where they derived the trace anomaly equation for a large N_c CFT (5). One would therefore expect that the corresponding 4D effective action for the trace anomaly could also be derived from the 5D AdS theory. We will show how this expectation is indeed fulfilled.

After writing 5D gravity in the formalism that we will use throughout the paper (see, Sec. II) we will derive the Riegert action as an effective action in Sec. III. The 4D action would appear nonlocal if only the metric field were to be used. However, it can be rewritten as a local diff-invariant action by integrating into it the field τ. Since τ is also the dilaton, we need a 5D model for spontaneously broken conformal invariance coupled to gravity. This requirement can be realized by the two-brane Randall-Sundrum model [11] (RS1), which is indeed holographic to a CFT coupled to gravity [12]. Importantly, the position of the IR brane in RS1, which is not stabilized, is precisely the massless dilaton, as shown in [13].

In Sec. IV we will recover the result of [13–15] that shows how the $R - \bar{R}$ term naturally emerges in the holographic picture of the RS1 model. The main result of Sec. IV is to show that the RS1 action also generates the Riegert term with $a = c'$, together with certain specific conformally-invariant dimension-4 terms. Unlike the Riegert terms, the latter are nonuniversal and model-dependent. In Sec. V we show that our action satisfies the Ward identities of spontaneously broken conformal invariance. In Sec. VI we will study deviations from the $a = c'$ limit. In Sec. VII we will discuss some open questions.

II. THE 5D THEORY

First of all we summarize the 5D action and equations in the formalism developed by Shiromizu *et al.* [16]. The bulk action is

$$S_{\rm Bulk} = M_5^3 \int d^4x dz \sqrt{\hat{g}} N(\hat{R} + \hat{K}^2 - \hat{K}^2_{\alpha\beta} - 2\Lambda_5), \quad (6)$$

where N is the lapse in the 5th dimension, and N_μ is the respective shift (see Ref. [17] for more details). The 5D cosmological constant will be chosen to be negative,

[2]To be more precise, the scale transformations here refer to the transformations formed by the dilatations together with simultaneous global diffeomorphisms, such that the coordinates do not transform under the combined transformations, but the metric tensor transforms as, $g \to e^{2\lambda} g$, with a constant λ.

$-2\Lambda_5 = 12/L^2$, where L is the radius-curvature of a 5D AdS spacetime. The extrinsic curvature of 4D spacetime is defined as follows

$$\hat{K}_{\mu\nu} = \frac{1}{2N}(\partial_z \hat{g}_{\mu\nu} - \hat{\nabla}_\mu N_\nu - \hat{\nabla}_\nu N_\mu). \tag{7}$$

Variation of the bulk action (6) with respect to the lapse N gives

$$\hat{R} - 2\Lambda_5 = \hat{K}^2 - \hat{K}^2_{\alpha\beta}, \tag{8}$$

and the equation obtained by variation of the action with respect to N_β reads

$$\hat{\nabla}^\alpha \hat{K}_{\alpha\beta} = \hat{\nabla}_\beta \hat{K}. \tag{9}$$

As long as the above equation is satisfied, we can substitute $N_\mu = 0$ in the bulk $\{\mu\nu\}$ equation, which then reads as follows:

$$\begin{aligned}\hat{G}_{\mu\nu} + \Lambda_5 \hat{g}_{\mu\nu} = {}& \frac{1}{2}\hat{g}_{\mu\nu}(\hat{K}^2 - \hat{K}^2_{\alpha\beta}) + 2(\hat{K}^\rho_\mu \hat{K}_{\rho\nu} - \hat{K}\hat{K}_{\mu\nu}) \\ & + \frac{\hat{\nabla}_\mu \hat{\nabla}_\nu N - \hat{g}_{\mu\nu}\hat{\nabla}^2 N}{N} \\ & - \hat{g}_{\mu\alpha}\hat{g}_{\nu\beta}\frac{\partial_z(\sqrt{-\hat{g}}(\hat{K}\hat{g}^{\alpha\beta} - \hat{K}^{\alpha\beta}))}{N\sqrt{-\hat{g}}}. \end{aligned} \tag{10}$$

We will use these equations below, together with the appropriate boundary conditions, to determine an effective 4D theory.

In the next section we will consider only one brane, together with the corresponding boundary conditions specified there. We will show that by integrating out the bulk in the single-brane RS model one gets at low energies 4D GR plus the Riegert action, with $a = c'$.

III. THE RIEGERT ACTION FROM AdS$_5$

After taking into account the equations of motion, (8) and (9), we can set $N_\mu = 0$, $N = A(z) = L/(z+L)$ for $z \geq 0$, and consider the metric:

$$ds^2 = \hat{g}_{\mu\nu}(x,z)dx^\mu dx^\nu + A^2(z)dz^2. \tag{11}$$

We will follow Kanno and Soda [18] to integrate out the 5D bulk. This is done via a classical nonlinear order-by-order expansion of the 5D equations of motion in powers of RL^2, where R is a 4D curvature experienced by a 4D observer on the positive tension brane in the single-brane RS model. The corresponding expansion of the metric is parametrized as follows:

$$\hat{g}_{\mu\nu}(x,z) = A^2(z)\Big(g_{\mu\nu}(x) + g^{(1)}_{\mu\nu}(x,z) + g^{(2)}_{\mu\nu}(x,z) + \ldots\Big), \tag{12}$$

where $g^{(j)}_{\mu\nu}(x, z=0) = 0$, for $j = 1,2,3,\ldots$ Using the above expansion one can find the corresponding power series expression for the extrinsic curvature

$$\hat{K}^\mu_\nu = \sum_{n=0}^{\infty} K^{(n)\mu}_\nu. \tag{13}$$

The expression for $K^{(2)\mu}_\nu$ depends of an unknown tensor that cannot be written as a variation of a local tensor made out of the metric and its derivatives [18]. On the other hand, traces of $K^{(n)\mu}_\nu$ can be determined unambiguously in terms of local quantities [18]. We use this observation and come up with a scheme that only utilizes the traces of the extrinsic curvatures evaluated at $z = 0$:

$$\begin{aligned} & K^{(0)\mu}_\mu\Big|_{z=0} = -\frac{4}{L}, \qquad K^{(1)\mu}_\mu\Big|_{z=0} = -\frac{L}{6}R(g), \\ & K^{(2)\mu}_\mu\Big|_{z=0} = -\frac{L^3}{24}\left(R^2_{\mu\nu} - \frac{1}{3}R^2\right). \end{aligned} \tag{14}$$

Taking the trace of the junction condition at $z = 0^+$

$$T_4 \hat{g}_{\mu\nu} - 2M_5^3(\hat{K}_{\mu\nu} - \hat{K}g_{\mu\nu}) = 0, \tag{15}$$

one gets

$$4T_4 + 6M_5^3 \hat{K} = 0. \tag{16}$$

Using the RS fine tuning condition

$$T_4 = 6M_5^3/L, \tag{17}$$

and noticing that $\hat{R}(\hat{g})|_{z=0^+} = R(g)$, we can rewrite the trace equation (16) as follows:

$$M^2 R = -\frac{M_5^3 L^3}{4}\left(R^2_{\mu\nu} - \frac{1}{3}R^2\right) \equiv -a(W^2 - E), \tag{18}$$

where $M^2 = M_5^3 L$ is the 4D Planck mass squared divided by 2, and $a \equiv \frac{M_5^3 L^3}{8} \sim {N_c}^2$. The above is the trace anomaly equation. As noted earlier, the emergence of the 4D trace anomaly equation from a 5D AdS bulk was first shown in [10].

Our goal is to derive the action that gives rise to Eq. (18). To achieve this we can follow the method used by Riegert [2]. We multiply both sides of the Eq. (18) by $\sqrt{-g}$, and perform the following field redefinition, $g_{\mu\nu}(x) = e^{2\tau}\bar{g}_{\mu\nu}$; this gives

$$M^2\sqrt{-\bar{g}}e^{2\tau}(\bar{R}(\bar{g})-6(\bar{\nabla}^2\tau)-6\bar{\nabla}^2\tau)=-2a\sqrt{-\bar{g}}\left(-\frac{\bar{E}}{2}+\frac{\bar{W}^2}{2}-4\bar{R}^{\mu\nu}(\bar{\nabla}_\mu\bar{\nabla}_\nu\tau-\bar{\nabla}_\mu\tau\bar{\nabla}_\nu\tau)+2\bar{R}\bar{\nabla}^2\tau\right)$$
$$-2a\sqrt{-\bar{g}}(-4(\bar{\nabla}^2\tau)^2+4(\bar{\nabla}_\mu\bar{\nabla}_\nu\tau)^2-4(\bar{\nabla}^2\tau)(\bar{\nabla}\tau)^2-8(\bar{\nabla}^\alpha\bar{\nabla}^\beta\tau)(\bar{\nabla}_\alpha\tau)\bar{\nabla}_\alpha\tau). \quad (19)$$

We now look for an action, as a functional of τ and $\bar{g}$ and their derivatives, that can be varied with respect to τ to give rise to (19). This action reads

$$S=M^2\int d^4x\sqrt{-\bar{g}}e^{2\tau}(\bar{R}(\bar{g})+6(\bar{\nabla}\tau)^2)+I(\bar{g})$$
$$-2a\int d^4x\sqrt{-\bar{g}}(\tau\bar{E}-4\bar{G}^{\mu\nu}\bar{\nabla}_\mu\tau\bar{\nabla}_\nu\tau-4(\bar{\nabla}^2\tau)(\bar{\nabla}\tau)^2-2(\bar{\nabla}\tau)^4)+2a\int d^4x\sqrt{-\bar{g}}\tau\bar{W}^2. \quad (20)$$

The second line is the Riegert action with $a=c'=M_5^3L^3/8$, consistent with a large N_c CFT. The first term in the first line is the Einstein-Hilbert term written in the Jordan frame. The second term in the first line, denoted by $I(\bar{g})$, is a functional that does not depend on τ, but can depend on $\bar{g}$ and its derivatives. In a sense, $I(\bar{g})$ is an "integration functional" which cannot be determined by the above procedure. If one assumes $I(\bar{g})=0$, as it was done by Riegert [2], then the theory is strongly coupled at arbitrarily low energies [6,7]. One way to avoid this problem is to postulate that, $I(\bar{g})=-\bar{M}^2\int d^4x\sqrt{-\bar{g}}\bar{R}$, based on symmetry and anomaly considerations, as was done in [7].

From the 5D perspective the strong coupling issue stems from the fact that the gravitational Kaluza-Klein modes are infinitely strongly coupled at the nonlinear level near the AdS horizon in the single-brane RS model [19]. This issue gets resolved by introducing a second brane which cuts off the AdS horizon in the RS model. Interestingly, the expression for $I(\bar{g})$—precisely with the negative sign—naturally emerges in the holographic picture as soon as one introduces the second brane in the RS model [11]; this leads to a radion field, which can then be identified with the τ field introduced above. All this is discussed in gradually increasing detail in the subsequent sections.

IV. DERIVATION OF $I(\bar{g})$

The expression for $I(\bar{g})$ could be obtained from the comprehensive work of Kanno and Soda [20]; it is for sake of presentation that we will give a different derivation of $I(\bar{g})$ here.

First we use the bulk equation (8) in the bulk action (6) to get a partially "on-shell" action

$$S_{\rm Bulk}|=2M_5^3\int d^4xdz\sqrt{\hat{g}}N(\hat{R}-2\Lambda_5). \quad (21)$$

To introduce a radion field we parametrize the metric as follows:

$$ds^2=\hat{g}_{\mu\nu}(x,z)dx^\mu dx^\nu+N^2(x,z)dz^2, \quad (22)$$

and adopt new notations, as well as more general expansion for the metric than the one used in the previous section:

$$\hat{g}_{\mu\nu}(x,z)\equiv\Omega^2(x,z)g_{\mu\nu}(x,z)$$
$$\equiv\Omega^2(x,z)(g_{\mu\nu}(x)+\delta g_{\mu\nu}(x,z)), \quad (23)$$

where $\delta g_{\mu\nu}(x,z=0)=0$. The metric $g_{\mu\nu}(x)$ denotes the RS zero mode, while $\delta g_{\mu\nu}$ encodes subleading curvature corrections due to the bulk. Using (22) and (23) one can find the corresponding expression for the extrinsic curvature

$$\hat{K}_{\alpha\beta}=\frac{\partial_z\Omega^2}{2N\Omega^2}\hat{g}_{\alpha\beta}(x,z)+\frac{\Omega^2}{2N}\partial_z\delta g_{\alpha\beta}. \quad (24)$$

Note that the first term on the right-hand side contains the leading and subleading pieces in $\hat{g}$. Substituting the latter expression into (9), and focusing on the leading order, we get the following relation between N and Ω

$$N(x,z)=\frac{U}{2}\frac{\partial_z\Omega^2}{\Omega^2}, \quad (25)$$

where U is an integration constant. By substituting (25) into the bulk equation (8) one can determine the integration constant from the obtained relation

$$-2\Lambda_5=\frac{12}{U^2}=\frac{12}{L^2}. \quad (26)$$

We can now use (25) and (26) in (21) to obtain

$$S_{\rm Bulk}|=M_5^3L\int d^4x\int dz\sqrt{-g}(\partial_z(\Omega^2)R(g)$$
$$+6\partial_z(\nabla\Omega)^2-(\partial_z\Omega^4)\Lambda_5). \quad (27)$$

It is clear that the z integration can be done explicitly and that the result of that integration depends only on the boundary values of Ω. We choose the following boundary conditions

$$\Omega(x, z=0) = 1,$$
$$\Omega(x, z=\chi(x)) = \Phi(x) = \frac{L}{z_{\rm IR}+L} e^{-\tau(x)}, \qquad (28)$$

where we parametrized the x-dependent boundary, $\chi(x)$, by the field $\Phi(x)$. Using the junction conditions at both branes to integrate (27) with respect to z from the UV, $z=0$, to the IR, $z=\chi(x)$, we obtain

$$M^2 \int d^4x\sqrt{-g}(R - \Phi^2 R - 6(\nabla\Phi)^2)$$
$$= M^2 \int d^4x(\sqrt{-g}R - \epsilon^2\sqrt{-\bar{g}}\bar{R}). \qquad (29)$$

Here $\epsilon = L/(z_{\rm IR}+L) \equiv \bar{M}/M$, and hence we find that $I(\bar{g}) = -\bar{M}^2 \int d^4x\sqrt{-\bar{g}}\bar{R}$.

Note that we require $\bar{M} \ll M$. This hierarchy can be achieved if the average distance between the two branes is much greater than the radius curvature of AdS_5, $z_{\rm IR} \gg L$. This implies that the mass scale of the lightest Kaluza-Klein (KK) modes, $\sim z_{\rm IR}^{-1}$, is below the scale of the curvature of AdS_5. Moreover, we find that, $\bar{M} \sim N_c/z_{\rm IR}$, which is much higher than the KK mass scale, $z_{\rm IR}^{-1}$, and, by construction, is much smaller than the 4D Planck scale, $M_{\rm Pl} \sim M \sim N_c/L$.

The action (29) was found in [13–15] by dimensional reduction of the 5D RS1 action. We will also derive it in the next sections by a dimensional reduction that solves the radial Hamiltonian and momentum constraints Eqs. (8) and (9). By taking care of the constraints we can also compute the first order correction to the action, which depends on $\delta g_{\mu\nu}$. This can be done systematically using the formalism of Ref. [20] and by selecting appropriate integration constants in their equations. When considering the first correction to the lowest-order results—i.e., $\delta g_{\mu\nu} = g^{(1)}_{\mu\nu} \equiv h_{\mu\nu}$ in the expansion (23)—we can simplify the derivation and explicitly perform integrals in z that are left implicit in [18,20]. We can also find at the same time the Riegert action and the next to leading Weyl-invariant terms.

A. Solving the equations for the extrinsic curvature $K_{\mu\nu}$

We are interested in an effective field theory in which the expansion parameter is $z_{\rm IR}$ times gradients of the fields. The first term in the expansion is Eq. (29) while the second is obtained by expanding $g_{\mu\nu}(x,y) = g_{\mu\nu}(x) + h_{\mu\nu}(x,y)$ in the action (6) up to the fourth order in $z_{\rm IR}\partial_\mu$. It is also convenient to define a new radial coordinate y as $z = z_{\rm IR}y - L$, and since $0 \le z \le z_{\rm IR}$ in this section, the integration range for y is $\frac{L}{z_{\rm IR}} \le y \le 1 + \frac{L}{z_{\rm IR}}$. In the rest of this section and in Sec. VI we also use a different definition of Ω, which is $(z+L)/L$ times the Ω used in the previous section. With this new definition the exact anti–de Sitter metric $ds^2 = \frac{L^2}{(L+z)^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2)$ corresponds to $\Omega = 1$. We find

$$S_{\rm Bulk} = \frac{c}{z_{\rm IR}^2}\int_{L/z_{\rm IR}}^{1+L/z_{\rm IR}} \frac{dy}{y^3}\int d^4x\sqrt{-g}\Big[N\Omega^2(\nabla_\mu\nabla_\nu h^{\mu\nu} - \nabla^2 h - h^{\mu\nu}G_{\mu\nu}) - 6h^{\mu\nu}\nabla_\mu(N\Omega)\nabla_\nu\Omega + 3h\nabla_\mu(N\Omega)\nabla^\mu\Omega$$
$$+ \frac{\Omega^4}{4Nz_{\rm IR}^2}\left(\frac{dh}{dy}\frac{dh}{dy} - \frac{dh^{\mu\nu}}{dy}\frac{dh_{\mu\nu}}{dy}\right)\Big]. \qquad (30)$$

Here, $c = M_5^3L^3$, $h^{\mu\nu} \equiv g^{\mu\rho}g^{\nu\sigma}h_{\rho\sigma}$, $h \equiv g^{\mu\nu}h_{\mu\nu}$, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, and $K_{\mu\nu} = \frac{1}{2N}\frac{d}{dy}h_{\mu\nu}$, since we set $N_\mu = 0$. The equations of motion for $h_{\mu\nu}$ are

$$\frac{1}{z_{\rm IR}^2}\frac{d}{dy}\left[\frac{\Omega^4}{2Ny^3}\left(\frac{dh_{\mu\nu}}{dy} - g_{\mu\nu}\frac{dh}{dy}\right)\right] = \frac{1}{y^3}(g_{\mu\nu}\nabla^2(N\Omega^2) - \nabla_\mu\nabla_\nu(N\Omega^2) + G_{\mu\nu}\Omega^2$$
$$+ 3\nabla_\mu(N\Omega)\nabla_\nu\Omega + 3\nabla_\nu(N\Omega)\nabla_\mu\Omega - 3g_{\mu\nu}\nabla_\lambda(N\Omega)\nabla^\lambda\Omega). \qquad (31)$$

Using the identities

$$g^{\mu\nu}\nabla_\mu(N\Omega)\nabla_\nu\Omega = -\frac{1}{2}y^3\frac{d}{dy}(y^{-2}g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega), \qquad N\Omega^2 = -\frac{1}{2}y^3\frac{d}{dy}(y^{-2}\Omega^2), \qquad (32)$$

we find that the left-hand side (lhs) of (31) is a total derivative so we can write

$$\frac{\Omega^4}{z_{\rm IR}^2 N y^3}\left(\frac{dh_{\mu\nu}}{dy} - g_{\mu\nu}\frac{dh}{dy}\right) = \frac{1}{y^2}\Omega^2 X_{\mu\nu} + C_{\mu\nu},$$
$$X_{\mu\nu} \equiv \Omega^{-2}(\nabla_\mu\nabla_\nu\Omega^2 - g_{\mu\nu}\nabla^2\Omega^2 - G_{\mu\nu}\Omega^2 + - 6\nabla_\mu\Omega\nabla_\nu\Omega + 3g_{\mu\nu}\nabla_\lambda\Omega\nabla^\lambda\Omega). \quad (33)$$

The y-independent integration function $C_{\mu\nu}$ can be constrained by demanding that (33) solves the momentum constraint (9). Using the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ and the identity

$$(\nabla_\mu\nabla_\nu\nabla^\mu - \nabla_\nu\nabla^2)S = R_{\mu\nu}\nabla^\mu S, \quad (34)$$

which is valid for any scalar S, we find

$$\nabla^\mu\left[\frac{\Omega^4}{z_{\rm IR}^2 N y^3}\left(\frac{dh_{\mu\nu}}{dy} - g_{\mu\nu}\frac{dh}{dy}\right)\right] = \frac{1}{y^2}\partial_\mu\Omega(\Omega R - 6\nabla^2\Omega) + \nabla^\mu C_{\mu\nu}. \quad (35)$$

This solves the constraint if $\nabla^\mu C_{\mu\nu} = 0$.

Furthermore, it is convenient to rewrite $X_{\mu\nu}$ in terms of a new variable σ defined as $\Omega = \exp(\sigma)$. We find:

$$X_{\mu\nu} = -2\partial_\mu\sigma\partial_\nu\sigma + 2\nabla_\mu\partial_\nu\sigma - G_{\mu\nu} - g_{\mu\nu}\partial_\lambda\sigma\nabla^\lambda\sigma - 2g_{\mu\nu}\nabla^2\sigma. \quad (36)$$

Another useful identity is $X_{\mu\nu} = -G_{\mu\nu}(\Omega^2 g) = -G_{\mu\nu}(e^{2\sigma}g)$.

B. The boundary condition at $y = 1$

We need to find the boundary conditions for $K_{\mu\nu}$ at $y = 1 + L/z_{\rm IR} \approx 1$ (recall that $L/z_{\rm IR} \ll 1$). The variation of Eq. (30) contains the boundary term

$$\frac{c}{z_{\rm IR}^2}\int d^4x\sqrt{-g}\frac{\Omega^4}{2L^2N}\left(\frac{dh_{\mu\nu}}{dy} - g_{\mu\nu}\frac{dh}{dy}\right)\delta h^{\mu\nu}|_{y=1} = -c\int d^4x\sqrt{-g}[e^{-2\tau}G_{\mu\nu}(e^{-2\tau}g) - C_{\mu\nu}]\delta h^{\mu\nu}|_{y=1}, \quad (37)$$

where $\tau \equiv -\sigma|_{y=1}$. Note that there is no boundary term at $y = L/z_{\rm IR}$ because there we impose the Dirichlet boundary conditions $\delta h_{\mu\nu} = 0$. We can decompose, $\delta h_{\mu\nu} = \delta h^{TT}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu - 2g_{\mu\nu}\omega$, and notice that the gradient and trace term in the variation can be canceled by varying ψ in the action of the zero modes (29) according to $\delta\psi = \xi^m\nabla_\mu\psi + \omega\psi$. So, to make the action stationary we have to impose only that the transverse-traceless part of $K_{\mu\nu}$ vanish: $K^{TT}_{\mu\nu} = 0$. This identifies $C_{\mu\nu}$ as $C_{\mu\nu} = [e^{-2\tau}G_{\mu\nu}(e^{-2\tau}g)]^{TT}$.

C. The effective action to $O(z^0_{\rm IR})$

Let us substitute the solution of the $h_{\mu\nu}$ equations of motion into the action (6). Since we chose free boundary conditions at $y = 1$, and the Dirichlet boundary conditions at $y = L/z_{\rm IR}$, we can discard the boundary contributions and get the action $S_{\rm eff} + S_1$, with $S_{\rm eff}$ given in Eq. (29) and

$$S_1 = -\frac{c}{{z_{\rm IR}}^2}\int_{L/z_{\rm IR}}^1\frac{dy}{y^3}\int d^4x\sqrt{-g}\frac{\Omega^4}{4Nz_{\rm IR}^2}\left(\frac{dh}{dy}\frac{dh}{dy} - \frac{dh^{\mu\nu}}{dy}\frac{dh_{\mu\nu}}{dy}\right)$$
$$= \frac{c}{4}\int_{L/z_{\rm IR}}^1 dyy^3\int d^4x\sqrt{-g}N\Omega^{-4}\left[\left(-\frac{1}{y^2}\Omega^2G_{\mu\nu}(\gamma)) + C_{\mu\nu}\right)^2 - \frac{1}{3}\left(\frac{1}{y^2}\Omega^2 g^{\mu\nu}R_{\mu\nu}(\gamma) + C\right)^2\right]. \quad (38)$$

Here, $\gamma_{\mu\nu}(x,y) \equiv \Omega^2(x,y)g_{\mu\nu}(x)$. A key identity that follows from Eq. (36) is

$$\frac{d}{dy}G_{\mu\nu}(\gamma) = -2\mathcal{D}_\mu\mathcal{D}_\nu\frac{d\sigma}{dy} + 2\gamma_{\mu\nu}\mathcal{D}^2\frac{d\sigma}{dy}, \quad (39)$$

where $\mathcal{D}_\mu$ denotes the covariant derivative with respect to the metric $\gamma_{\mu\nu}(x,y)$. Expanding Eq. (38) in powers of $C_{\mu\nu}$ we get three terms. Using integration by part in y we rewrite the first term as

$$\frac{c}{4}\int_{L/z_{\rm IR}}^1\frac{dy}{y}\int d^4x\sqrt{-g}N\left(g^{\mu\rho}g^{\nu\sigma}G_{\mu\nu}(\gamma)G_{\rho\sigma}(\gamma) - \frac{1}{3}[g^{\mu\nu}G_{\mu\nu}(\gamma)]^2\right)$$
$$= -\frac{c}{4}\int_{L/z_{\rm IR}}^1 dy\int d^4x\sqrt{-g}\left(\frac{d}{dy}(\sigma - \log y)\right)\left(g^{\mu\rho}g^{\nu\sigma}G_{\mu\nu}(\gamma)G_{\rho\sigma}(\gamma) - \frac{1}{3}[g^{\mu\nu}G_{\mu\nu}(\gamma)]^2\right)$$
$$= -\frac{c}{4}\int d^4x\sqrt{-g}\left[-\tau\left(G^2_{\mu\nu}(e^{-2\tau}g) - \frac{1}{3}G^2(e^{-2\tau}g)\right) + \log(L/z_{\rm IR})\left(G^2_{\mu\nu}(g) - \frac{1}{3}G^2(g)\right)\right]$$
$$- c\int_{L/z_{\rm IR}}^1 dy\int d^4x\sqrt{-\gamma}(\sigma - \log y)\left(\mathcal{D}_\mu\mathcal{D}_\nu\frac{d\sigma}{dy}\right)G_{\rho\sigma}(\gamma)\gamma^{\mu\rho}\gamma^{\nu\sigma}. \quad (40)$$

Notice that we define $G^2_{\mu\nu} = G_{\mu\nu}G_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$ etc. An integration by parts in $\mathcal{D}_\mu$ transforms the last term into

$$\begin{aligned} & c\int_{L/z_{\rm IR}}^1 dy \int d^4x\sqrt{-g}\partial_\mu\sigma\left(\partial_\nu \frac{d\sigma}{dy}\right)G_{\rho\sigma}(\gamma)g^{\mu\rho}g^{\nu\sigma} \\ & = \frac{c}{2}\int_{L/z_{\rm IR}}^1 dy\int d^4x\sqrt{-g}\left(\frac{d}{dy}\partial_\mu\sigma\partial_\nu\sigma\right)G_{\rho\sigma}(\gamma)g^{\mu\rho}g^{\nu\sigma} \\ & = \frac{c}{2}\int d^4x\sqrt{-g}\partial_\mu\tau\partial_\nu\tau G^{\mu\nu}(e^{-2\tau}g) + c\int_0^1 dy\int d^4x\sqrt{-\gamma}\gamma^{\mu\rho}\gamma^{\nu\sigma}\partial_\mu\sigma\partial_\nu\sigma(\mathcal{D}_\rho\mathcal{D}_\sigma - \gamma_{\rho\sigma}\mathcal{D}^2)\frac{d\sigma}{dy}. \end{aligned} \tag{41}$$

We set $L/z_{\rm IR} = 0$ in the converging integrals and used the Bianchi identity $\mathcal{D}^\mu G_{\mu\nu}(\gamma) = 0$. To deal with the last integral in y we use

$$\begin{aligned} \mathcal{D}_\mu V_\nu + \mathcal{D}_\nu V_\mu &= \gamma^{\rho\sigma}V_\rho\partial_\sigma\gamma_{\mu\nu} + \partial_\mu(\gamma^{\sigma\rho}V_\sigma)\gamma_{\rho\nu} + \partial_\nu(\gamma^{\sigma\rho}V_\sigma)\gamma_{\rho\mu} \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu + \Omega^{-2}V^\lambda(\partial_\lambda\Omega^2)g_{\mu\nu} + \Omega^2\partial_\mu(\Omega^{-2})V_\nu + \Omega^2\partial_\nu(\Omega^{-2})V_\mu \\ &= \Omega^2[\nabla_\mu(\Omega^{-2}V_\nu) + \nabla_\nu(\Omega^{-2}V_\mu) - V^\lambda(\partial_\lambda\Omega^{-2})g_{\mu\nu}], \end{aligned} \tag{42}$$

where indices are raised and lowered with the metric $g_{\mu\nu}$. For $V_\mu = \partial_\mu(d\sigma/dy)$ Eq. (42) gives

$$(\mathcal{D}_{(\mu}\partial_{\nu)} - \gamma_{\mu\nu}\gamma^{\rho\sigma}\mathcal{D}_\rho\partial_\sigma)\frac{d\sigma}{dy} = \frac{d}{dy}\left[(\nabla_\mu\partial_\nu - g_{\mu\nu}\nabla^2)\sigma - \partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g_{\mu\nu}\partial_\rho\sigma\nabla^\rho\sigma\right]. \tag{43}$$

Inserting this identity in (41) we find that the integral in y is a total derivative

$$\begin{aligned} & c\int_0^1 dy\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}\partial_\rho\sigma\partial_\sigma\sigma\frac{d}{dy}\left[(\nabla_\mu\partial_\nu - g_{\mu\nu}\nabla^2)\sigma - \partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g_{\mu\nu}\partial_\lambda\sigma\nabla^\lambda\sigma\right]. \\ & = c\int d^4x\sqrt{-g}\left(-\partial_\mu\tau\partial_\nu\tau\nabla^\mu\nabla^\nu\tau - \frac{3}{4}\partial_\mu\tau\nabla^\mu\tau\partial_\nu\tau\nabla^\nu\tau\right). \end{aligned} \tag{44}$$

The other terms in (38) are treated analogously. The cross term is

$$\begin{aligned} & -\frac{c}{4}\int_{L/z_{\rm IR}}^1 dy\int d^4x\sqrt{-g}\left(\frac{d}{dy}(y^2\Omega^{-2})\right)\left(G_{\mu\nu}(\gamma)C^{\mu\nu} - \frac{1}{3}G(\gamma)C\right) \\ & = -\frac{c}{4}\int d^4x\sqrt{-g}e^{2\tau}\left(G_{\mu\nu}(e^{-2\tau}g)C^{\mu\nu} - \frac{1}{3}G(e^{-2\tau}g)C\right) - \frac{c}{2}\int_{L/z_{\rm IR}}^1 dy\int d^4x\sqrt{-g}(y^2\Omega^{-2})\left(\mathcal{D}_\mu\mathcal{D}_\nu\frac{d\sigma}{dy}\right)C^{\mu\nu}. \end{aligned} \tag{45}$$

Here too the indices are raised and lowered with the metric $g_{\mu\nu}$ and the last term vanishes because $C_{\mu\nu}$ is transverse with respect to the covariant derivative ∇_μ and the traceless part of Eq. (42) is $\Omega^{-2}\mathcal{D}_{(\mu}V_{n)_T} = \nabla_{(\mu}\Omega^{-2}V_{\nu)_T}$.

The last term is

$$c\int_{L/z_{\rm IR}}^1 dy\int d^4x\sqrt{-g}\left(\frac{d}{dy}(y^4\Omega^{-4})\right)\left(C^2_{\mu\nu} - \frac{1}{3}C^2\right) = c\int d^4x\sqrt{-g}e^{4\tau}\left(C^2_{\mu\nu} - \frac{1}{3}C^2\right). \tag{46}$$

Notice that (45) and (46) give only Weyl-invariant terms so the anomaly arises only from (40)

$$\begin{aligned} S_{\rm Anomaly} = \frac{c}{4}\int d^4x\sqrt{-g}\Big[&-\tau\left(-G^2_{\mu\nu}(e^{-2\tau}g) + \frac{1}{3}G^2(e^{-2\tau}g)\right) + 2\partial_\mu\tau\partial_\nu\tau G^{\mu\nu}(e^{2\tau}g) + \\ &- 4\partial_\mu\tau\partial_\nu\tau\nabla^\mu\nabla^\nu\tau - 3\partial_\mu\tau\nabla^\mu\tau\partial_\nu\tau\nabla^\nu\tau\Big]. \end{aligned} \tag{47}$$

This is the same as Eq. (4) with $a = c'$ once the identities

$$\int d^4x \partial_\mu \tau \partial_\nu \tau \nabla^\mu \nabla^\nu \tau = -\frac{1}{2}\int d^4x (\nabla\tau)^2 \nabla^2 \tau,$$
$$\int d^4x (\nabla\tau)^2 \nabla^2\tau = \int d^4x (\bar{\nabla}\tau)^2 [\bar{\nabla}^2\tau + 2(\bar{\nabla}\tau)^2] \quad (48)$$

are taken into account.

V. HOLOGRAPHY AND WARD IDENTITIES

The action (6) computed on shell at a fixed value of the metric at $z = 0$ has a dual holographic interpretation as the effective action obtained from integrating out the CFT degrees of freedom [12]. Because it is a 5D gravitational action, it is invariant under the infinitesimal coordinate change $z = w + \omega(q)(w + L)$, $x^\mu = q^\mu + F^\mu(q, w)$, $F^\mu(q, 0) = 0$. When $(w + L)\partial_\mu \omega(q) + g_{\mu\nu}\partial_w F^\nu(q, w) = 0$ the metric still has the form (11) and (23) and its boundary value transforms to $g_{\mu\nu}(x) = (1 - 2\omega(x))g_{\mu\nu}(x)$. To first order in ω, the limits of integration in (6), $0 \le z \le z_{\rm IR}$, change to $-\omega(x)L \le w \le z_{\rm IR}(x) - \omega(x)(z_{\rm IR}(x) + L)$. General coordinate invariance thus gives the equation

$$S[(1 - 2\omega(x))g_{\mu\nu}(x), -\omega(x)L, z_{\rm IR}(x) - \omega(x)(z_{\rm IR}(x) + L)] = S[g_{\mu\nu}(x), 0, z_{\rm IR}(x)] + O(\omega^2). \quad (49)$$

This action diverges in the limit $L \to 0$. It can be written as [10]

$$S[g_{\mu\nu}, L, z_{\rm IR}] = S_D[g_{\mu\nu\nu}, L] + S_F[g_{\mu\nu}, z_{\rm IR}] + O(L). \quad (50)$$

The divergent part depends only on the behavior of the metric near the $z = 0$ boundary so it does not depend on $z_{\rm IR}(x)$ [10,13,21]. The finite part does not depend on L and the rest vanishes in the limit $L \to 0$. The first variation of S with respect to ω vanishes thanks to Eq. (49) so, up to terms that vanish in the $L \to 0$ limit we find

$$0 = \delta_\omega S_D + \int d^4x \sqrt{-g}\left[\delta_\omega g_{\mu\nu}(x)\frac{\delta S_F}{\delta g_{\mu\nu}(x)} + \delta_\omega z_{\rm IR}(x)\frac{\delta S_F}{\delta z_{\rm IR}(x)}\right]. \quad (51)$$

Reference [10] finds $\delta S_D/\delta\omega(x) = -a(E - W^2)$ [see Eq. (18)]. Notice that

$$\delta_\omega S_D = \int d^4x \sqrt{-g}\left[\delta_\omega g_{\mu\nu}(x)\frac{\delta S_D}{\delta g_{\mu\nu}(x)} + \delta_\omega L(x)\frac{\delta S_D}{\delta L(x)}\right]. \quad (52)$$

The second term is a variation with respect to L. In RS1 L is the position of the UV brane, which is kept fixed. Keeping L fixed and introducing a bare Einstein-Hilbert term on the UV brane explicitly breaks Weyl invariance. The terms that break Weyl invariance in S_D are a 4D cosmological constant $\propto L^{-4}\int d^4x\sqrt{-g}$ and an Einstein-Hilbert term $\propto L^{-2}\int d^4x\sqrt{-g}R(g)$. The holographic interpretation of these terms is that they are induced by the CFT loops. The induced 4D cosmological constant is canceled by the brane tension while the bare and induced Newton constants give the 4D Newton constant $G = 2/M^2$. So, S satisfies the equation

$$\begin{aligned}\int d^4x\sqrt{-g}\left[\delta_\omega g_{\mu\nu}(x)\frac{\delta S}{\delta g_{\mu\nu}(x)} + \delta_\omega z_{\rm IR}(x)\frac{\delta S}{\delta z_{\rm IR}(x)}\right] &= -\int d^4x\sqrt{-g}\left[\delta_\omega L(x)\frac{\delta S_D}{\delta L(x)}\right] + O(L)\\ &= \int d^4x\sqrt{-g}\omega(x)a(E - W^2) + \int d^4x\sqrt{-g}\delta_\omega g_{\mu\nu}(x)\frac{\delta S_D}{\delta g_{\mu\nu}(x)} + O(L).\\ &= \int d^4x\sqrt{-g}\omega(x)a(E - W^2) - \frac{2}{16\pi G}\int d^4x\sqrt{-g}\omega(x)R(g) + O(L).\end{aligned} \quad (53)$$

This is indeed the Ward identity for a spontaneously broken CFT coupled to gravity.

VI. DERIVATION OF THE SUBLEADING TERMS

A. Subleading W^2 contributions to the anomaly

The general anomaly is the sum of the term $R_{\mu\nu}^2 - \frac{1}{3}R^2$ and the square of the Weyl tensor. The additional 5D term that gives rise to this anomaly is [22–24]

$$S_W = \alpha k\int_L^{L+z_{\rm IR}} dz \int d^4x\sqrt{G}C^M_{NPQ}C_M^{NPQ}, \quad (54)$$

where $C^M_{NPQ}(G)$ is the Weyl tensor in 5D, $\alpha \ll c$, and for convenience in this section we will use the z coordinate that is shifted by L with respect to the one used in the previous sections, so the z integration range is, $L \le z \le L + z_{\rm IR} \simeq z_{\rm IR}$. Its key property is that it is Weyl Invariant: $C^M_{NPQ}(\omega^2 G) = C^M_{NPQ}(G)$. We only need to evaluate this integral on the zero-mode metric because the contribution of $h_{\mu\nu}$ terms is $O(z_{\rm IR}^2\partial_\mu^2)$. By defining Ω as in Sec. IV A we can write this metric as

$$ds^2 = \frac{\Omega^2 L^2}{z^2}(\Omega^{-2}(1 - z\partial_z\Omega\Omega^{-1})^2 dz^2 + g_{\mu\nu}dx^\mu dx^\nu),\quad \text{with}\quad \frac{dg_{\mu\nu}}{dz} = 0. \tag{55}$$

The coordinate change $T = z\Omega^{-1}$, $(\Omega^{-1} - z\partial_z\Omega\Omega^{-2})dz = dT + T\partial_\mu\Omega\Omega^{-1}dx^m$ transforms metric (55) into

$$\begin{aligned}ds^2 &= \frac{L^2}{T^2}d\tilde{s}^2\\ &= \frac{L^2}{T^2}[dT^2 + 2T\partial_\mu\sigma dx^m dT\\ &\quad + (g_{\mu\nu}(x) + T^2\partial_\mu\sigma\partial_\nu\sigma)dx^m dx^\nu],\qquad \sigma = \log\Omega.\end{aligned} \tag{56}$$

Since the Weyl tensor is invariant under conformal rescaling, we can compute it on the metric $d\tilde{s}^2$. Moreover, since $T \lesssim z_{\rm IR}$, when we compute curvature tensor in the effective 4D theory we expand in powers of $z_{\rm IR}\partial_\mu$. So, in considering the most relevant terms in the expansion, we can neglect all terms proportional to T in the metric *except* for those where derivatives of T cancel out the T dependence. In the Riemann tensor there is only one such term:

$$\begin{aligned}R_{T\mu T\nu} &= -\partial_T K_{\mu\nu}\\ &= -\frac{1}{2}\partial_T(2T\partial_\mu\sigma\partial_\nu\sigma - 2T\nabla_\mu\partial_\nu\sigma)\\ &= -\partial_\mu\sigma\partial_\nu\sigma + \nabla_\mu\partial_\nu\sigma.\end{aligned} \tag{57}$$

Hence $R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}(g)$ and

$$\begin{aligned}R_{\mu\nu} &= R_{\mu\nu}(g) - \partial_\mu\sigma\partial_\nu\sigma + \nabla_\mu\partial_\nu\sigma,\\ R_{TT} &= -\partial_\mu\sigma\nabla^\mu\sigma + \nabla^2\sigma,\\ R &= R(g) - 2\partial_\mu\sigma\nabla^\mu\sigma + 2\nabla^2\sigma.\end{aligned} \tag{58}$$

A short calculations gives

$$C^M_{NPQ}C_M^{NPQ} = 4R_{T\mu T\nu}R^{T\mu T\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{4}{3}R_{\mu\nu}R^{\mu\nu} - \frac{4}{3}R_{TT}R^{TT} + \frac{1}{6}R^2, \tag{59}$$

with indices $\mu, \nu, \cdots$ raised and lowered with the metric $g_{\mu\nu}$. Substituting Eqs. (57) and (58) into (59) a tedious but straightforward calculation gives

$$\sqrt{g}C^M_{NPQ}C_M^{NPQ} = \sqrt{g}\left[R_{\mu\nu\rho\sigma}(g)R^{\mu\nu\rho\sigma}(g) - 2R_{\mu\nu}(g)R^{\mu\nu}(g) + \frac{1}{3}R^2(g)\right] + \sqrt{\gamma}\left[\frac{2}{3}R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{6}R^2(\gamma)\right], \tag{60}$$

where all the terms in the second bracket are defined with respect to the metric $\gamma_{\mu\nu} = e^{2\sigma}g_{\mu\nu}$, which is also used there to raise and lower indices. Notice that the first term is proportional to the Weyl anomaly term $W^2 = (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2)$. We can therefore recall the definition $\tau = -\sigma|_{y=1}$ and rewrite S_W as

$$\begin{aligned}S_W &= \alpha\int d^4x\int_L^{z_{\rm IR}e^\tau}\frac{dT}{T}\left[\sqrt{g}W^2 + \sqrt{\gamma}\left(\frac{2}{3}R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{6}R^2(\gamma)\right)\right]\\ &= 64\alpha\int d^4x\sqrt{g}\tau W^2 + \alpha\int d^4x\int_L^{z_{\rm IR}e^\tau}\frac{dT}{T}\sqrt{\gamma}\left(\frac{2}{3}R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{6}R^2(\gamma)\right).\end{aligned} \tag{61}$$

The first term is the Weyl tensor contribution to the anomaly. To compute the last term we transform back from T to z using $dT = (\Omega^{-1} - z\Omega^{-2}\partial_z\Omega)dz - z\Omega^{-2}\partial_\mu\Omega dx^\mu$. Setting $z = z_{\rm IR}y$, the second term in (61) becomes

$$\alpha\int_{L/z_{\rm IR}}^1\frac{dy}{y}\int d^4x\sqrt{\gamma}N\left(\frac{2}{3}\gamma^{\mu\rho}\gamma^{\nu\sigma}G_{\mu\nu}(\gamma)G^{\rho\sigma}(\gamma) - \frac{1}{6}[\gamma^{\mu\nu}G_{\mu\nu}(\gamma)]^2\right). \tag{62}$$

This action vanishes when $G_{\mu\nu}(\gamma) = 0$ so it can be canceled up by a redefinition of the metric $h_{\mu\nu}$ in Eq. (30). Concretely, we set

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \Delta h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{z_{\rm IR}^2 y^2}{\Omega^2 N}[VG_{\mu\nu}(\gamma) + B\gamma_{\mu\nu}G(\gamma)], \tag{63}$$

where $\bar{h}_{\mu\nu}$ solves Eq. (33). Substituting in (30) the linear term in $\Delta h_{\mu\nu}$ vanishes and, discarding terms of higher order in z_{IR} we get the additional $O(\Delta h_{\mu\nu})^2$ term

$$\frac{c}{{z_{\mathrm{IR}}}^4}\int_{L/z_{\mathrm{IR}}}^1 \frac{dy}{y^3}\int d^4x\sqrt{-g}\frac{\Omega^4}{4N}\left(\frac{d\Delta h}{dy}\frac{d\Delta h}{dy}-\frac{d\Delta h^{\mu\nu}}{dy}\frac{d\Delta h_{\mu\nu}}{dy}\right)$$
$$= c\int_{L/z_{\mathrm{IR}}}^1 \frac{dy}{y^3}\int d^4x\sqrt{-\gamma}[V^2G_{\mu\nu}(\gamma)^2 - (V^2+6VB+12B^2)G(\gamma)^2]. \tag{64}$$

Hence $V=\pm\frac{\alpha}{c}\sqrt{6}/3$, $B=\mp\frac{\alpha}{c}\sqrt{6}/12$ so we need $\alpha>0$. When translated into the coefficients of the general anomaly, $\langle T^\mu_\mu\rangle = c'W^2 - aE$, it means $c'>a$. Presumably the other sign could be dealt with by studying the 5D Euler density (Lovelock) action as in Ref. [24].

We note that the departure from the $a=c'$ can also be achieved by introducing the bulk 5D Gauss-Bonnet term [23]. In this case, similarly to the case of the 5D Weyl-tensor square considered above, one would be introducing in the bulk another scale below M_5 at which the 5D theory would become strongly coupled (or else these 5D terms would have to be suppressed by M_5, in which case such terms cannot be differentiated from generic terms emerging from quantum gravity at M_5.)

In the next section we will discuss a framework in a weakly coupled theory that enables to relax the condition $a=c'$.

B. Subleading terms from a boundary-localized QFT

There is another way for the theory to depart from the $a=c'$ limit. Suppose there is a 4D boundary QFT localized on a positive tension brane. This QFT is coupled to the metric $\hat{g}_{\mu\nu}(x,z=0)=g_{\mu\nu}(x)$. Let us integrate out the boundary QFT in the path integral, and use dimensional regularization to deal with the divergences. This would make the brane world-volume to become $D=4-2\delta$ dimensional, with $\delta\to 0$ to be taken after all the divergences are subtracted. The result of this calculation—with the massless limit taken at the very end—has long been known (see, Duff's work in [1] and an overview in [25]), it is proportional to

$$\Gamma(2-D/2)\int d^Dx\sqrt{-g}(a_bE(g)-c_b(R^2_{\mu\nu\alpha\beta}-2R^2_{\mu\nu}+R^2/3)), \tag{65}$$

where Γ is the Euler gamma function, E is the Euler (Gauss-Bonnet) invariant, $E=R^2_{\mu\nu\alpha\beta}-4R^2_{\mu\nu}+R^2$, a_b and c_b are calculable coefficients, and in general $a_b\neq c_b$. The combination of the curvature square invariants proportional to c_b combines into the Weyl tensor squared only after the $\delta\to 0$ limit is taken. Since the term (65) is localized on the UV brane, it will have to be added to the holographic 4D action. This would shift the holographic theory away from the $a=c'$ limit, to $a_{\mathrm{tot}}\neq c_{\mathrm{tot}}$, where $a_{\mathrm{tot}}=a+a_b$ and $c_{\mathrm{tot}}=c'+c_b$. For simplicity, we will focus on the E term in (65), while requiring a QFT that has a_b not equal to c_b.

As we mentioned earlier, it is known that the trace of the variation of (65) with respect to g gives the right trace anomaly equation after taking the limit $\delta\to 0$. If so, then (65) should also contain the Riegert action in the $\delta\to 0$ limit. This can be shown under the assumption of validity of the analytic continuation to negative values of δ, and by representing the interval of the $4+n$ dimensional theory as,

$$d_{4+n}s = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\tau(x)}d_n^2z, \tag{66}$$

where $n=-2\delta$. Then, taking the limit $n=-2\delta\to 0$, the divergent coefficient proportional to $1/\delta$ coming from the Γ function is canceled by the terms proportional to δ coming from E and the resulting finite term is exactly the Riegert term written in terms of the metric g and the scalar τ [26] (see, also [6] and references therein)

$$\int d^4x\sqrt{-g}(\tau E + 4G^{\mu\nu}\nabla_\mu\tau\nabla_\nu\tau - 4(\nabla^2\tau)(\nabla\tau)^2 + 2(\nabla\tau)^4). \tag{67}$$

Furthermore, using $g=e^{2\tau}\bar{g}$ in (67), one recovers the a-terms of (4).

To summarize, with the help of a localized QFT on the positive tension RS brane one obtains a weakly coupled completion for the trace anomaly action with $a_{\mathrm{tot}}\neq c_{\mathrm{tot}}$.

VII. DISCUSSION

The location of the IR brane in the 5D theory considered in the present work is not stabilized. One could introduce a potential to stabilize its location in the 5th dimension, and stabilize the radion via the Goldberger-Wise (GW) mechanism [15]; however, this would in general alter the ability of the resulting theory to correctly recover the 4D trace anomaly equation. That said, if the scales in the GW potential are much smaller than $\bar{M}$, the resulting equation could give a good approximation to the trace anomaly equation [4]. Without the GW mechanism, the vacuum expectation value of the Φ field is a modulus. This VEV sets the value of the scale $\bar{M}$, which is not dynamically determined. As we discussed, $\bar{M}$ has to be significantly below the Planck scale for the trace anomaly effective field theory to be distinguished from other higher dimensional terms, which are suppressed by the Planck scale. Furthermore, $1/\bar{M}$ serves as a constant determining the self-interactions of the radion, which in the 4D holographic theory can be regarded as a dilaton of spontaneously broken conformal symmetry.

What are the couplings of the matter fields to the metric and dilaton? The answer depends on where the matter fields are assumed to be placed in 5D. If they are localized on the IR brane, as they are in the RS1 scenario for the sake of solving the hierarchy problem, then matter would coupe to the metric times Ω^4. Because of the presence of a long-range radion such a theory would be ruled out observationally in our case. In the holographic approach adopted in this work it is more natural to assume that matter—that is the weakly coupled matter that we have added to the strongly coupled CFT—is localized on the UV brane, in which case it would couple to the metric g. Based on symmetry considerations matter was coupled in [7] to the metric $\hat{g} = g(1 - \Phi^2/M^2)$. Since $M = M_{\rm Pl}/\sqrt{2} \gg \bar{M}$, the difference between coupling to g and $\hat{g}$ is small.

In either case, there is a fifth force produced by the long-range dilaton. If the matter couples to g the coupling to the dilaton appears at the linear level and its strength is proportional to $(\bar{M}/M)^2$, while in the case when matter couples to $\hat{g}$ the coupling to dilaton only emerges at the nonlinear level and its strength is proportional to $a(\bar{M}/M)^2$, as shown by Tsujikawa [27]–who has recently found a black hole solution in the effective trace anomaly action and explicitly calculated the corrections to the Schwarzschild geometry proportional to $a(\bar{M}/M)^2$. Comparison of these predictions with observational data will likely impose strong bounds on the value of the scale $\bar{M}$.

While the 5D construction serves the point of identifying a completion of the 4D effective theory above its strong scale $\bar{M}$, the 4D effective theory should still be a more convenient tool for practical calculations: in general, it is significantly easier to work with 4D differential equations (in many symmetric cases being ordinary differential equations) rather than to work with 5D partial differential equations.

ACKNOWLEDGMENTS

We'd like to thank Justin Khoury, Shinji Tsujikawa, and Giorgi Tukhashvili for useful discussions. M. P. is supported in part by the Leverhulme Trust through a Leverhulme Visiting Professorship. This work is supported in part by NSF Grant No. PHY-2210349.

[1] D. M. Capper and M. J. Duff, Conformal anomalies and the renormalizability problem in quantum gravity, Phys. Lett. A **53**, 361 (1975); M. J. Duff, Observations on conformal anomalies, Nucl. Phys. **B125**, 334 (1977); S. M. Christensen and M. J. Duff, Axial and conformal anomalies for arbitrary spin in gravity and supergravity, Phys. Lett. **76B**, 571 (1978).
[2] R. J. Riegert, A nonlocal action for the trace anomaly, Phys. Lett. **134B**, 56 (1984).
[3] E. S. Fradkin and A. A. Tseytlin, Conformal anomaly in Weyl theory and anomaly free superconformal theories, Phys. Lett. **134B**, 187 (1984).
[4] Z. Komargodski and A. Schwimmer, On renormalization group flows in four dimensions, J. High Energy Phys. 12 (2011) 099.
[5] G. Gabadadze and G. Tukhashvili, Conformal/Poincaré Coset, cosmology, and descendants of Lovelock terms, Phys. Rev. D **102**, 024054 (2020).
[6] J. Bonifacio, K. Hinterbichler, and L. A. Johnson, Amplitudes and 4D Gauss-Bonnet theory, Phys. Rev. D **102**, 024029 (2020).
[7] G. Gabadadze, A new gravitational action for the trace anomaly, Phys. Lett. B **843**, 138031 (2023).
[8] P. G. S. Fernandes, Gravity with a generalized conformal scalar field: Theory and solutions, Phys. Rev. D **103**, 104065 (2021).
[9] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. **10**, 363 (1974).
[10] M. Henningson and K. Skenderis, The holographic Weyl anomaly, J. High Energy Phys. 07 (1998) 023.
[11] L. Randall and R. Sundrum, A Large Mass Hierarchy from a Small Extra Dimension, Phys. Rev. Lett. **83**, 3370 (1999).
[12] N. Arkani-Hamed, M. Porrati, and L. Randall, Holography and phenomenology, J. High Energy Phys. 08 (2001) 017.
[13] R. Rattazzi and A. Zaffaroni, Comments on the holographic picture of the Randall-Sundrum model, J. High Energy Phys. 04 (2001) 021.
[14] C. Csaki, M. Graesser, L. Randall, and J. Terning, Cosmology of brane models with radion stabilization, Phys. Rev. D **62**, 045015 (2000).
[15] W. D. Goldberger and M. B. Wise, Phenomenology of a stabilized modulus, Phys. Lett. B **475**, 275 (2000).
[16] T. Shiromizu, K. i. Maeda, and M. Sasaki, The Einstein equation on the 3-brane world, Phys. Rev. D **62**, 024012 (2000).
[17] C. Deffayet and J. Mourad, Multigravity from a discrete extra dimension, Phys. Lett. B **589**, 48 (2004).
[18] S. Kanno and J. Soda, Brane world effective action at low-energies and AdS/CFT, Phys. Rev. D **66**, 043526 (2002).
[19] L. Randall and R. Sundrum, An Alternative to Compactification, Phys. Rev. Lett. **83**, 4690 (1999).
[20] S. Kanno and J. Soda, Radion and holographic brane gravity, Phys. Rev. D **66**, 083506 (2002).
[21] S. de Haro, S. N. Solodukhin, and K. Skenderis, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence, Commun. Math. Phys. **217**, 595 (2001).

[22] M. Blau, K. S. Narain, and E. Gava, On subleading contributions to the AdS/CFT trace anomaly, J. High Energy Phys. 09 (1999) 018.
[23] S. Nojiri and S. D. Odintsov, On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence, Int. J. Mod. Phys. A **15**, 413 (2000).
[24] M. Banados, A. Schwimmer, and S. Theisen, Chern-Simons gravity and holographic anomalies, J. High Energy Phys. 05 (2004) 039.
[25] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1994).
[26] H. Lu and Y. Pang, Horndeski gravity as $D \to 4$ limit of Gauss-Bonnet, Phys. Lett. B **809**, 135717 (2020).
[27] S. Tsujikawa, Black holes in a new gravitational theory with trace anomalies, Phys. Lett. B **843**, 138054 (2023).