Three Puzzles with Covariance and Supertranslation Invariance of Angular Momentum Flux and Their Solutions

Reza Javadinezhad* and Massimo Porrati†
Center for Cosmology and Particle Physics, Department of Physics, New York University, 726 Broadway, New York, New York 10003, USA

(Received 12 December 2023; revised 6 March 2024; accepted 26 March 2024; published 12 April 2024)

We describe and solve three puzzles arising in covariant and supertranslation-invariant formulas for the flux of angular momentum and other Lorentz charges in asymptotically flat spacetimes: (i) Supertranslation invariance and covariance imply invariance under spacetime translations; (ii) the flux depends on redundant auxiliary degrees of freedom that cannot be set to zero in all Lorentz frames without breaking Lorentz covariance; (iii) supertranslation-invariant Lorentz charges do not generate the transformations of the Bondi mass aspect implied by the isometries of the asymptotic metric. In this Letter, we solve the first two puzzles by presenting covariant formulas that unambiguously determine the auxiliary degrees of freedom and clarify the last puzzle by explaining the different roles played by covariant and canonical charges. Our construction makes explicit the choice of reference frame underpinning seemingly unambiguous results presented in the current literature.

DOI: 10.1103/PhysRevLett.132.151604

Introduction.—The problem of a proper definition of angular momentum and boost charges for asymptotically flat spacetimes in general relativity has existed since it was discovered in [1–4] that the metrics of spacetimes allowing for gravitational radiation enjoy an infinite-dimensional asymptotic symmetry algebra. This is the Bondi-Metzner-Sachs (BMS) algebra, which contains the Poincaré algebra as a proper *but not normal* subalgebra. The Poisson brackets of angular momentum with supertranslations, which are an infinite-dimensional Abelian subalgebra of BMS, do not vanish; consequently, the total angular momentum and the flux of angular momentum through null infinity can be changed by a gravitational wave of infinite wavelength. Since such a wave cannot be detected by any finite-size observer, the ambiguity due to BMS seems to preclude a meaningful definition of angular momentum and the other Lorentz charges in general relativity (GR). Problems inherent in defining angular momentum in GR were noticed by Penrose as early as 1964 [5].

The problem is even sharper in quantum gravity, because it implies that two states differing by a supertranslation have the same energy, so, in particular, the vacuum state is infinitely degenerate. This can be seen as follows: Denote schematically with S^a the generators of supertranslations and with $J^{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) the Lorentz generators; then the commutator $[S^a, J^{\mu\nu}]$ is nonzero, so two zero-energy states differing by a supertranslation have different angular momenta. Explicitly,

$$J^{ij}|0\rangle = 0 \Rightarrow J^{ij}[1 + c_a S^a]|0\rangle = [J^{ij}, S^a]|0\rangle \neq 0, i, \quad j = 1, 2, 3, \tag{1}$$

so $|0\rangle$ and $[1 + c_a S^a]|0\rangle$ are different (vacuum) states. In well-defined quantum field theories in flat spacetime and in holographic theories of quantum gravity in anti–de Sitter (AdS) space, instead, the vacuum is unique. The apparent nonuniqueness of vacuum in quantum field theories with spontaneously broken symmetries is resolved, because the physical Hilbert space decomposes into *superselection sectors* [6], while an infinity of states at zero energy are excluded in holographic theories in AdS, because the dual CFT has a unique vacuum. Besides, infinite vacuum degeneracy would violate generalizations of the Bekenstein bound [7] such as [8]. The previous observations suggest that the $J^{\mu\nu}$'s appearing in the BMS algebra may not be the correct operators to associate to angular momentum and boosts. Other, unambiguous quantities should be found.

Life is easy in Minkowski spacetime, because the Lorentz generators can be defined as $J^{\mu\nu} = \int_\Sigma n_\rho T^{\rho[\mu} x^{\nu]}$, where Σ is a complete Cauchy surface with timelike normal n_ρ. Unfortunately, this formula becomes ill defined in general relativity and must be substituted by an integral over a 2-surface at infinity, but this is precisely what introduces supertranslation ambiguities.

The problems we have outlined are not merely formal. In numerical general relativity, ambiguities due to supertranslations have been shown to affect the computation of waveforms [9,10] that are essential tools for detection and study of black hole mergers. The danger of supertranslation-dependent quantities is that they mix the effect of unobservable background noise due to very long waves to the signal due to a given physical process (gravitational scattering, black hole mergers, etc.). This danger manifest both at the level of formal definitions and in dealing with numerical simulations that by construction have finite resolution. Finding unambiguous conserved charges and fluxes in general relativity is, therefore, a serious problem, all the more urgent now after computational advances in general relativity (see, e.g., [11]) and observational discoveries [12].

In fact, the first direct detection of gravitational waves from black hole mergers [12] renewed interest in the quest for a better definition of symmetries and conserved charges in asymptotically flat 4D spacetime. Novel supertranslation-invariant definitions for angular momentum and angular momentum flux as well as for charges and fluxes of the other Lorentz algebra generators have recently appeared in the literature. To our knowledge, the first supertranslation-invariant formula for angular momentum was given in Ref. [13]. In fact, Ref. [13] defines the *Bondi charge* for angular momentum at any retarded time, so it also provides a definition of flux. References [14,15] give an independent definition of supertranslation-invariant Lorentz Bondi charges which, while agreeing with [13] at retarded time $u \to -\infty$, differ at finite u. References [16,17] provide formulas for supertranslation-invariant Lorentz charges and their fluxes based on a canonical formalism that applies equally well to the quantum theory.

The angular momentum flux given in [13,16,17] begins at $O(G^3)$ in a perturbative expansion in powers of the Newton constant G. This is in disagreement with explicit computation of mechanical angular momentum flux done by several groups with different methods. In particular, both [18] using techniques developed in [19] and [20–26] agree on the presence of a nonvanishing angular momentum flux at $O(G^2)$. The origin of this discrepancy was traced back in [27] to the difference between the supertranslation frame used in perturbative calculation of gravitational scattering and a "canonical" frame in which the Bondi angular momentum at $u = -\infty$ agrees [28] with the Arnowitt-Deser-Misner (ADM) definition [29]. Reference [30] employs the results of [27] to define, to all orders in G, a supertranslation-invariant angular momentum flux that agrees to $O(G^2)$ with perturbative calculations and is defined only in term of asymptotic metric data on $\mathcal{I}$.

All supertranslation-invariant formulas for the Lorentz charges flux depend on $C(u,\Theta)$, the "electric" component of the shear $C_{AB}(u,\Theta)$. These quantities are defined in the next section. The shear is independent of the first two harmonics of $C(u,\Theta)$, while the invariant flux depends on them. This fact requires an independent choice of the $l=0$, 1 harmonic components of $C(u,\Theta) \equiv \sum_{l=0}^{\infty} \times \sum_{-l\leq m\leq m} C_{l,m}(u) Y_{lm}(\Theta)$. The simplest choice would be to set them to zero *in all Lorentz frames*. Besides being aesthetically unpleasant, this choice is inconsistent with Lorentz covariance, because we will see that boosts mix the higher harmonics of C with the $l=0$, 1 ones. So, neither the Lorentz charges flux in [14,15] nor those in [30] are Lorentz covariant unless a new prescription is found. The effect of this noninvariance was explicitly verified in the case of two-particle scattering in [31], where it was shown that, while the charge defined in [30] agrees with perturbative calculations in the center of mass rest frame of the two particles, it does differs in the rest frame of one of the two particles.

In fact, all existing proposals for a supertranslation-invariant flux of Lorentz charges share an even more serious flaw: Any covariant formula for the flux (or the Lorentz charge) is a Lorentz tensor, and no such quantity can be supertranslation invariant without being also invariant under spacetime *translations*. The proof follows simply from the structure of the BMS algebra and is valid also for the quantum BMS algebra. So, whether we use the formulas of [13–16] or [30], one thing is clear: Whatever we are computing is an angular momentum *defined with respect to a particular choice of the origin of coordinates*.

In this Letter, we show that the need to define an "intrinsic" angular momentum flux depending on an independently prescribed origin of coordinates is not a problem but rather the feature that allows us to solve the puzzles due to the ambiguity in the choice of C_{lm}, $l=0$, 1, and to the lack of covariance. These are not mere technicalities but foundational problems that have so far precluded a covariant and unambiguous definition of angular momentum in general relativity and have also introduced redundant quantities, C_{lm}, $l=0$, 1, without apparent physical meaning. Our Letter instead shows that they do have a clear meaning; we will show that they define the origin of the spacetime coordinate system. Angular momentum depends on the point from which it is computed, but the quantities that determine that point are nowhere to be found among the asymptotic components of the metric. We will show that the "redundant" s- and p-wave components of the electric shear are precisely those "missing" quantities.

We begin by defining our notations and deriving the key result that supertranslation-invariant definitions of the flux must also be spacetime translation invariant. Next, we derive fully covariant formulas for the charges and fluxes defined in Refs. [14,15]. Key to the covariantization is the use of the transformation law of the boundary graviton under boosts, together with covariant and supertranslation-invariant definitions of the center of mass frame, which are used to determine the first two harmonics of the boundary graviton. Finally, we give a physical interpretation to the supertranslation-invariant charges we previously defined, show that differences among different prescriptions do not manifest themselves until $O(G^3)$, and show how current calculations found in the literature implicitly use the same choice of reference frame as our own.

We also solve the third puzzle by clarifying the difference between supertranslation-invariant charges and the generators of asymptotic symmetries.

Notations and transformation properties of covariant quantities.—The metric near future null infinity $\mathcal{I}^+$ in Bondi-Sachs coordinates is [1–4]

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + r^2\left(h_{AB} + \frac{C_{AB}}{r}\right)d\Theta^A d\Theta^B \\ & + D^A C_{AB} dud\Theta^B + \frac{2m}{r}du^2 + \frac{1}{16r^2}C_{AB}C^{AB}dudr \\ & + \frac{1}{r}\left(\frac{4}{3}(N_A + u\partial_A m) - \frac{1}{8}\partial_A(C_{BD}C^{BD})\right)dud\Theta^A \\ & + \frac{1}{4}h_{AB}C_{CD}C^{CD}d\Theta^A d\Theta^B + \cdots, \end{aligned} \tag{2}$$

where the mass aspect $m(u,\Theta)$ is a scalar, the angular momentum aspect $N_A(u,\Theta)$ is a vector, and the shear $C_{AB}(u,\Theta)$ is a symmetric traceless tensor. All these quantities are defined on the celestial sphere with coordinates Θ_A and round metric h_{AB} and also depend on the retarded time u. The dots in (2) denote subdominant terms in $1/r$. The coordinate system in Eq. (2) is invariant under the asymptotic symmetries $u \to u + f(\Theta)$, called supertranslations [2]. Energy, momentum, and Lorentz charges are defined in terms of m, N_A, the $l = 1$ spherical harmonics $\vec{Y}$, and the six conformal Killing vectors Y^A of the celestial sphere [2]:

$$\begin{aligned} E(u) &= \frac{1}{4\pi G}\int d^2\Theta\sqrt{h}m(u,\Theta), \\ \vec{P}(u) &= \frac{1}{4\pi G}\int d^2\Theta\sqrt{h}\vec{Y}m(u,\Theta), \\ J_Y(u) &= \frac{1}{8\pi G}\int d^2\Theta\sqrt{h}Y^A N_A(u,\Theta). \end{aligned} \tag{3}$$

By definition, the conformal Killing vectors obey $D_A Y_B + D_B Y_A = h_{AB} D_C Y^C$. For rotations $D_C Y^C = 0$, while for boosts $D_C Y^C$ is an $l = 1$ harmonic. In the latter case, we can write $Y_A = D_A \psi$ with ψ obeying $D_A D^A \psi = -2\psi$. When it is well defined, the total energy flux $\Delta E \equiv E(+\infty) - E(-\infty)$ is invariant under supertranslations. This is not true for the flux of the Lorentz charges; so, in particular, the angular momentum flux $\Delta J_Y \equiv J_Y(+\infty) - J_Y(-\infty)$ can be changed by a supertranslation.

The Lorentz Bondi charges at retarded time u defined in [15] are

$$\begin{aligned} J_Y^{CWWY}(u) &= J_Y(u) - j_Y[m(u), C(u)], \\ j_Y[m(u), C(u)] &= \frac{1}{8\pi G}\int d^2\Theta\sqrt{h}Y^A[3m(u,\Theta)D_A C(u,\Theta) + D_A m(u,\Theta)C(u,\Theta)] \\ &= \frac{1}{4\pi G}\int d^2\Theta\sqrt{h}m(\delta_Y^{-1/2}C) = -\frac{1}{4\pi G}\int d^2\Theta\sqrt{h}(\delta_Y^{3/2}m)C, \end{aligned} \tag{4}$$

where

$$\delta_Y^w F \equiv wD\cdot YF + Y\cdot DF. \tag{5}$$

The "electric shear" $C(u,\Theta)$ is defined by

$$D^A D^B C_{AB}(u,\Theta) = D^2(D^2+2)C(u,\Theta). \tag{6}$$

The operator $D^2(D^2+2)$ is diagonalized by spherical harmonics, on which $D^2(D^2+2)C_l = l(l^2-1)(l+2)C_l$, so the $l = 0$, 1 harmonics in C do not appear at all in the asymptotic metric. The constraint relating the change in the mass aspect to the change in the electric shear is

$$\Delta m(\Theta) = \frac{1}{4}D^2(D^2+2)\Delta C(\Theta) - \int_{-\infty}^{+\infty} duT_{uu}(u,\Theta), \tag{7}$$

$$\begin{aligned} \Delta C(\Theta) &= C(+\infty,\Theta) - C(-\infty,\Theta), \\ \Delta m(\Theta) &= m(+\infty,\Theta) - m(-\infty,\Theta). \end{aligned} \tag{8}$$

Here, $T_{uu} = \frac{1}{8}N_{AB}N^{AB} + \lim_{r\to\infty} r^2 T^M_{uu}$, with $T^M =$ matter stress-energy tensor and $N_{AB} \equiv \partial_u C_{AB}$ is the Bondi news. In gravitational scattering $N_{AB} = O(G^2)$ so, through $O(G^2)$, $\Delta m(\Theta) = \frac{1}{4}D^2(D^2+2)\Delta C(\Theta)$. This equation shows a further ambiguity in $C_{l=0,1}(u,\Theta)$: Their dependence on the retarded time u is completely arbitrary. They are undetermined even if the initial value for all harmonics of C is given at $u = -\infty$.

We now prove that a supertranslation-invariant tensor must also be spacetime translation invariant by computing the commutator

$$\delta_Y \delta_f \Delta J_Z^{CWWY} - \delta_f \delta_Y \Delta J_Z^{CWWY} = \delta_{[Y,f]} \Delta J_Z^{CWWY}, \tag{9}$$

with f a supertranslation and Y a Lorentz boost. The transformation of the flux is determined by Lorentz covariance to be $\delta_Y \Delta J_Z^{CWWY} = \Delta J_{[Y,Z]}^{CWWY}$, where $[Y,Z] \equiv \mathcal{L}_Y Z - \mathcal{L}_Z Y$ with $\mathcal{L}_W$ the Lie derivative along the vector W. Invariance under supertranslations, therefore, implies that the left-hand side of Eq. (9) vanishes. The right-hand side is the transformation of the flux under $[Y,f] \equiv \delta_Y^{-1/2} f$. A boost can be written as $Y^A = D^A \psi$ with ψ an $l=1$ harmonic obeying $D^2\psi = -2\psi$. So, for an $l=2$ supertranslation obeying $D^2 f = -6f$, we find

$$[Y,f] = -\frac{1}{2} f(D^2\psi) + \frac{1}{2} D^2(f\psi) + f\psi + 3f\psi. \tag{10}$$

The product $f\psi$ contains an $l=1$ harmonic on which $D^2(f\psi)|_{l=1} = -2(f\psi)|_{l=1}$; therefore, $[Y,f]|_{l=1} = 4(f\psi)|_{l=1} \neq 0$ and the boost of a supertranslation is a nonzero spacetime translation.

The flux $\Delta J_Y^{CWWY} = J_Y^{CWWY}(+\infty) - J_Y^{CWWY}(-\infty)$ is manifestly translation and supertranslation invariant if we define the transformation law of $C(u,\Theta)$ to be

$$C(u,\Theta) \to C'(u,\Theta) = C(u,\Theta) + f(\Theta) \tag{11}$$

for *any* function $f(\Theta)$. Its $l=0$, 1 harmonics are spacetime translations, so the $l=0$, 1 harmonics of $C(-\infty,\Theta)$ represent the choice of origin for the coordinate system used to define the angular momentum.

The upshot of this analysis is that any supertranslation invariant flux is necessarily an intrinsic flux, defined with respect to an origin of the coordinate system, which is equivalent to a choice of $C|_l$ for $l=0$, 1. *An independent, covariant prescription for the $l=0$, 1 components of the boundary graviton is necessary to define the flux.*

We cannot simply set $C|_{l\leq 1} = 0$ in all Lorentz frames, because this choice is inconsistent with Lorentz transformations. The problem is that the lth harmonic of the boundary graviton transforms under boosts exactly like a supertranslation; therefore, in parallel with Eq. (10), we find

$$\begin{aligned}\delta_Y^{-1/2} C &= -\frac{1}{2} D\cdot Y C + Y\cdot DC = \psi C + D\psi \cdot DC \\ &= \frac{1}{2}[D^2 + 4 + l(l+1)](\psi C_l).\end{aligned} \tag{12}$$

The same argument as given after Eq. (10) shows that $\delta_Y^{-1/2} C$ generically contains a nonvanishing $l=1$ harmonic.

Lorentz-covariant definitions of angular momentum and boost charges and fluxes.—We found in the previous section that an independent definition for $C(u)|_{l\leq 1}$ is necessary to completely define ΔJ_Y^{CWWY}. The prescription should maintain covariance and should not introduce an additional arbitrariness in the flux. Here, we propose a simple recipe, which is covariant by construction. We also present a small variation on the prescription that we later prove to coincide with the previous one to $O(G^2)$.

Let us consider the flux, computed in the initial center of mass rest frame (CMRF), which is defined by the condition $m_{1,m}^- \equiv \int d^2\Theta \sqrt{h} Y_{1m} m(-\infty,\Theta) = 0$. The definition of the frame is not complete, because the origin of the coordinate system can be translated arbitrarily in space and time. We remove this arbitrariness by requiring that in the CMRF the initial boost charge vanish. We denote by $\pm u$-dependent quantities evaluated at $\pm\infty$ and impose

$$J_{\bar{Y}}^- - j_{\bar{Y}}[m^-, C^-] = 0, \text{ for all } \bar{Y}^A = \text{boost} = D^A\psi,\ D^2\psi = -2\psi. \tag{13}$$

These are three conditions that uniquely determine the three components of $C|_{l=1}$. Explicitly, we use Eq. (4) and

$$\begin{aligned}\delta_Y^{3/2} m_l^- &= \frac{3}{2} D\cdot Y m_l^- + Y\cdot Dm_l^- \\ &= -3\psi m_l^- + D\psi\cdot Dm_l^- \\ &= \frac{1}{2}[D^2 - 4 + l(l+1)](\psi m_l^-),\\ (D^2+2)(\psi m_2^-)|_{l=1} &= 0,\end{aligned} \tag{14}$$

to obtain

$$\begin{aligned} j_{\bar{Y}}[m^-, C^-|_{l\leq 1}] &= j_{\bar{Y}}[m^-|_{l\leq 1}, C^-|_{l\leq 1}] \\ &= \frac{3m_0^-}{8\pi G}\int d^2\Theta\sqrt{h} D\psi\cdot DC^-|_{l\leq 1} \\ &= \frac{3m_0^-}{4\pi G}\int d^2\Theta\sqrt{h}\psi\cdot C^-|_{l\leq 1}.\end{aligned} \tag{15}$$

Equation (13) then reduces to

$$\frac{3m_0^-}{4\pi G} C_{1m}^- = J_{\bar{Y}^A}^- - j_{\bar{Y}^A}[m^-, C^-|_{l>1}], \quad \bar{Y}^A = D^A Y_{1-m}. \tag{16}$$

Under *both* supertranslations and spacetime translations and for any conformal Killing vector Y, $J_Y^- \to J_Y^- + (1/4\pi G)\int d^2\Theta\sqrt{h} m^- \delta_Y^{-1/2} f$, so $J_Y^- - j_Y[m^-, C^-]$ is by construction invariant. Our definition specifies the origin of the system of coordinates (up to a time translation). We are free to set $C|_{l=0}$ to whichever value we want, and we will choose $C|_{l=0} = 0$.

To extend our definition to any Lorentz frame with celestial sphere coordinates $\Theta = (\theta,\phi)$, we choose three conformal Killing vectors $\hat{Y}^A$, related to the pure boosts of the rest frame with coordinates $\bar{\Theta} = (\bar{\theta},\bar{\phi})$ by

$$\hat{Y}^A(\Theta) = \frac{\partial g^A}{\partial\bar{\Theta}^B}\bar{Y}^B(\bar{\Theta}), \qquad \bar{Y}_B = \frac{\partial\bar{\psi}}{\partial\bar{\Theta}^B}, \tag{17}$$

evaluated at $\Theta^A = g^A(\bar{\Theta})$. We fix the Lorentz transformation g^A from the CMRF to the Lorentz frame moving with velocity $\vec{\beta}$ by requiring that it is a pure boost along $\vec{\beta}$. Covariance is preserved by this choice, as it is most easily seen by writing the Lorentz charge as an antisymmetric matrix J, the boost from the CMRF as a pseudo-orthogonal matrix $B(\vec{\beta})$, and the constraint as

$$Jy = 0, \qquad y = B(\vec{\beta})\bar{y}, \qquad \bar{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

This definition is covariant, because any Lorentz transformation can be decomposed as $L = BR$, with B a pure boost and R a pure rotation. In the CMRF, $R\bar{y} = \bar{y}$ so $L\bar{y} = B\bar{y}$. Definition (18) is also unique, because Lorentz transformation L to a frame moving with velocity $\vec{\beta'}$ maps y into $y' = LB(\vec{\beta})\bar{y}$. This is not a pure boost from the CMRF, but the difference is a pure rotation of the CMRF, $B(-\vec{\beta'})LB(\vec{\beta}) = R$; hence, $LB(\vec{\beta})\bar{y} = B(\vec{\beta'})R\bar{y} = B(\vec{\beta'})\bar{y}$. So, a translation, supertranslation, and covariant definition of the Lorentz charges is

$$\mathfrak{J}^-_Y = J^-_Y - j_Y[m^-, C^-], \tag{19}$$

where the $l = 0$, 1 harmonics of the boundary graviton are given by solving the equation

$$J^-_{\hat{Y}} - j_{\hat{Y}}[m^-, C^-] = 0. \tag{20}$$

To define the flux, we have several possibilities. For all of them,

$$\mathfrak{J}^+_Y = J^+_Y - j_Y[m^+, C^+], \qquad \Delta\mathfrak{J}_Y = \mathfrak{J}^+_Y - \mathfrak{J}^-_Y. \tag{21}$$

The difference is in the choice of the equations for $C|_{l\leq 1}$. Two simple choices are (A) set $C^+|_{l\leq 1} = C^-|_{l\leq 1}$ and (B) solve $J^+_{\tilde{Y}} - j_{\tilde{Y}}[m^+, C^+] = 0$, where $\tilde{Y}^A$ is defined by Lorentz-transforming pure boosts defined in the *final* center of mass rest frame. We show in the next section that these two definitions coincide to $O(G^3)$.

Interpretation of the invariant charges and fluxes.—We proposed a Lorentz-covariant, supertranslation-invariant definition of the Lorentz Bondi charges at $u = \pm\infty$, which differs from [14,15] only in the condition used to fix the $l = 0$, 1 components of the boundary graviton. By construction, both $\mathfrak{J}^\pm_Y$ are invariant under supertranslations, so we can compute them by changing coordinates in the nonradiative far past and far future regions to set $C^\pm_{AB} = 0$:

$$\Delta\mathfrak{J}_Y = J_Y(+\infty)|_{C^+|_{l>1}=0} - J_Y(-\infty)|_{C^-|_{l>1}=0}. \tag{22}$$

So the covariant, supertranslation-invariant flux reduces to the difference of canonical Bondi charge at $+\infty$, computed in the frame where the angular metric at $u = +\infty$ is $h_{AB} + O(1/r^2)$, minus the canonical Bondi charge at $u = -\infty$, computed in the frame where the angular metric at $u = +\infty$ is also $h_{AB} + O(1/r^2)$. References [27,28] argue that $h_{AB} + O(1/r^2)$ is the frame where the Bondi charge reduces to the ADM charge. This identification suggests a very natural interpretation of $\Delta\mathfrak{J}_Y$: It is the canonical charge measured after a gravitational scattering in a "round metric" frame minus the initial canonical charge, also measured in a round metric frame. The reference frames are fixed by requiring that the initial metric is round before the scattering occurs and then after the scattering has occurred by requiring that the final metric is also round. This procedure "forgets" the initial frame fixing. The interpretation of the flux and its form is essentially the same as in [14,31–33] once the subtleties due to Lorentz covariance and covariant subtraction of the low-l harmonics in $C^\pm$ are properly taken into account using Eq. (20) for $C^-|_{l\leq 1}$ and either prescription (A) or (B) for $C^+|_{l\leq 1}$. The Lorentz-covariant prescription (20) can be used also to covariantize the flux defined in [16]. It differs from $\Delta\mathfrak{J}_Y$ by terms proportional to the gravitational memory $C^+ - C^-$. This computation was performed in the CMRF in [31,32], but the role of gravitational memory in the definition of angular momentum in general relativity was noticed long before, e.g., in [34]. We will expand on this question in a future publication [35].

The difference between prescriptions (A) and (B) can be seen most clearly for the flux of angular momentum $\Delta\vec{J}$. Prescription (A) computes the difference between initial and final angular momenta in the initial CMRF. So it includes a term due to the motion of the final CMRF, namely, $\Delta\vec{J} = \Delta\vec{J}^{\text{intrinsic}} + \vec{a} \times \Delta\vec{P}$, with $\vec{a}$ the displacement of the origin of the final CMRF with respect to the initial CMRF. Prescription (B) instead gives $\Delta\vec{J} = \Delta\vec{J}^{\text{intrinsic}}$. The difference between the two prescriptions amounts to a term proportional to $\Delta\vec{P}$, i.e., the change of the center of mass momentum due to gravitational radiation. Because of the constraint equation (7), the definition of momentum (3), and $T_{uu} = O(G^4)$, $\Delta\vec{P} = O(G^3)$ so prescriptions (A) and (B) agree to $O(G^2)$.

We defined the $l = 0$, 1 harmonics of C by requiring that the boost charges vanish in the initial CMRF. This is the same prescription used in, e.g., Ref. [25], which considered the scattering of two particles of initial 4-momenta p^μ_i, $i = 1, 2$, and impact parameters b_i obeying $p_i \cdot b_j = 0$ for all i, j. In the CMRF these constraints are satisfied by $p_1 = (E_1, p, 0, 0)$ and $p_2 = (E_2, -p, 0, 0)$, so $b_1 = (0, 0, y_1, z_1)$ and $b_2 = (0, 0, y_2, z_2)$, and the nonzero boost charges are $J^{02} = E_1 y_1 + E_2 y_2$ and $J^{03} = E_1 z_1 + E_2 z_2$. Setting $J^{02} = J^{03} = 0$, we reproduce the *particular* solution of the constraints chosen in [25]. Similar choices are done in other papers on gravitational scattering. They are natural because they set the origin of coordinates at the center of mass of the incoming particles.

Difference between two consistent definitions of Lorentz charges.—By construction, *any* definition of supertranslation-invariant Lorentz generators J^{inv}_Y, so, in particular, $J^{\text{inv}}_Y = \mathfrak{J}^-_Y$, makes them commute with the $l > 1$ harmonics of the mass aspect $m^-(\Theta)$:

$$[J^{\text{inv}}_Y, m^-(\Theta)|_{l>1}] = 0. \tag{23}$$

On the other hand, the transformation law of m^- can be found by performing a Lorentz transformation on the asymptotic metric (2). It is given by Eq. (5) with $w = 3/2$, from which it is obvious that neither boosts nor rotations vanish on $m_{l>1}$.

We must then conclude that, while the supertranslation-invariant Lorentz generators are useful quantum operators—in fact, essential for defining unambiguously the angular momentum of the vacuum as well as other quantum numbers—they should not be used to generate Lorentz transformations on the fields. In fact, the supertranslation-invariant Lorentz charges are elements of the *universal enveloping algebra* of an enlarged BMS algebra that includes logarithmic supertranslations. Their explicit form is given in Eq. (9.7) in Ref. [36].

In this Letter, we defined the covariantization of the charge J_Y^{CWWY}; in a future work, we will expand on the results presented here and discuss the covariantization and interpretation of other charges, such as those given in [16,30].

We thank Gabriele Veneziano, Massimiliano Riva, and Geoffrey Compère for useful discussions and one of the referees for pointing out to us Ref. [34]. M. P. is supported in part by National Science Foundation Grant No. PHY-2210349 and by the Leverhulme Trust through a Leverhulme Visiting Professorship at Imperial College, London.

[*]rj1154@nyu.edu
[†]mp9@nyu.edu

[1] H. Bondi, Gravitational waves in general relativity, Nature (London) **186**, 535 (1960).
[2] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems, Proc. R. Soc. A **269**, 21 (1962).
[3] R. Sachs, Asymptotic symmetries in gravitational theory, Phys. Rev. **128**, 2851 (1962).
[4] R. K. Sachs, Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times, Proc. R. Soc. A **270**, 103 (1962).
[5] R. Penrose, Conformal treatment of infinity, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 565, 10.1007/s10714-010-1110-5.
[6] F. Strocchi, Local and covariant gauge quantum field theories. Cluster property, superselection rules and the infrared problem, Phys. Rev. D **17**, 2010 (1978).
[7] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).
[8] R. Bousso, A. Shahbazi-Moghaddam, and M. Tomašević, Quantum information bound on the energy, Phys. Rev. D **100**, 126010 (2019).
[9] K. Mitman, N. Khera, D. A. B. Iozzo, L. C. Stein, M. Boyle, N. Deppe, L. E. Kidder, J. Moxon, H. P. Pfeiffer, M. A. Scheel *et al.*, Fixing the BMS frame of numerical relativity waveforms, Phys. Rev. D **104**, 024051 (2021).
[10] K. Mitman, L. C. Stein, M. Boyle, N. Deppe, F. Hébert, L. E. Kidder, J. Moxon, M. A. Scheel, S. A. Teukolsky, W. Throwe *et al.*, Fixing the BMS frame of numerical relativity waveforms with BMS charges, Phys. Rev. D **106**, 084029 (2022).
[11] U. Sperhake, The numerical relativity breakthrough for binary black holes, Classical Quantum Gravity **32**, 124011 (2015).
[12] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Binary black hole mergers in the first Advanced LIGO observing run, Phys. Rev. X **6**, 041015 (2016); Phys. Rev. X **8**, 039903(E) (2018).
[13] G. Compère, R. Oliveri, and A. Seraj, The Poincaré and BMS flux-balance laws with application to binary systems, J. High Energy Phys. 10 (2020) 116.
[14] P. N. Chen, M. T. Wang, Y. K. Wang, and S. T. Yau, Supertranslation invariance of angular momentum, Adv. Theor. Math. Phys. **25**, 777 (2021).
[15] P. N. Chen, M. T. Wang, Y. K. Wang, and S. T. Yau, BMS charges without supertranslation ambiguity, Commun. Math. Phys. **393**, 1411 (2022).
[16] R. Javadinezhad, U. Kol, and M. Porrati, Supertranslation-invariant dressed Lorentz charges, J. High Energy Phys. 04 (2022) 069.
[17] M. Porrati, Charges in the extended BMS algebra: Definitions and applications, SciPost Phys. Proc. **4**, 005 (2021).
[18] D. Bini and T. Damour, Gravitational radiation reaction along general orbits in the effective one-body formalism, Phys. Rev. D **86**, 124012 (2012).
[19] T. Damour, Radiative contribution to classical gravitational scattering at the third order in G, Phys. Rev. D **102**, 124008 (2020).
[20] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Radiation reaction from soft theorems, Phys. Lett. B **818**, 136379 (2021).
[21] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The eikonal operator at arbitrary velocities I: The soft-radiation limit, J. High Energy Phys. 07 (2022) 039.
[22] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, Radiative classical gravitational observables at $\mathcal{O}(G^3)$ from scattering amplitudes, J. High Energy Phys. 10 (2021) 148.
[23] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, The amplitude for classical gravitational scattering at third post-Minkowskian order, J. High Energy Phys. 08 (2021) 172.
[24] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, J. High Energy Phys. 10 (2021) 118.
[25] A. V. Manohar, A. K. Ridgway, and C. H. Shen, Radiated angular momentum and dissipative effects in classical scattering, Phys. Rev. Lett. **129**, 121601 (2022).
[26] P. Di Vecchia, C. Heissenberg, and R. Russo, Angular momentum of zero-frequency gravitons, J. High Energy Phys. 08 (2022) 172.
[27] G. Veneziano and G. A. Vilkovisky, Angular momentum loss in gravitational scattering, radiation reaction, and the Bondi gauge ambiguity, Phys. Lett. B **834**, 137419 (2022).

[28] A. Ashtekar and N. Khera, Unified treatment of null and spatial infinity III: Asymptotically Minkowski space-times, J. High Energy Phys. 02 (2024) 210; Unified treatment of null and spatial infinity IV: Angular momentum at null and spatial infinity, J. High Energy Phys. 01 (2024) 085.

[29] R. L. Arnowitt, S. Deser, and C. W. Misner, The dynamics of general relativity, Gen. Relativ. Gravit. **40**, 1997 (2008).

[30] R. Javadinezhad and M. Porrati, Supertranslation-invariant formula for the angular momentum flux in gravitational scattering, Phys. Rev. Lett. **130**, 011401 (2023).

[31] M. M. Riva, F. Vernizzi, and L. K. Wong, Angular momentum balance in gravitational two-body scattering: Flux, memory, and supertranslation invariance, Phys. Rev. D **108**, 104052 (2023).

[32] P. Mao, J. B. Wu, and X. Wu, Angular momentum and memory effect, Phys. Rev. D **107**, L101501 (2023).

[33] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The gravitational eikonal: From particle, string and brane collisions to black-hole encounters, arXiv:2306.16488.

[34] J. Winicour, Angular momentum in general relativity, in *General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein. Volume 2*, edited by A. Held (Plenum Press, New York, 1980), pp. 71–96.

[35] R. Javadinezhad and M. Porrati (to be published).

[36] O. Fuentealba, M. Henneaux, and C. Troessaert, Logarithmic supertranslations and supertranslation-invariant Lorentz charges, J. High Energy Phys. 02 (2023) 248.