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Viscosity Solutions of the Eikonal Equation on the Wasserstein Space

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Abstract
Dynamic programming equations for mean field control problems with a separable structure are Eikonal type equations on the Wasserstein space. Standard differentiation using linear derivatives yield a direct extension of the classical viscosity theory. We use Fourier representation of the Sobolev norms on the space of measures, together with the standard techniques from the finite dimensional theory to prove a comparison result among semi-continuous sub and super solutions, obtaining a unique characterization of the value function.

Keywords Mean Field Games · Wasserstein space · Viscosity Solutions · Eikonal Equation · Mean-field control

Mathematics Subject Classification 35Q89 · 35D40 · 49L25 · 60G99

1 Introduction

We consider the Hamilton-Jacobi equations related to mean-field control problems in which the state process X_t taking values in the d-dimensional Euclidean space $\mathbb{R}^d$ has the following simple dynamic structure,

$$\mathrm{d}X_u = \alpha_u \,\mathrm{d}u + \sigma \,\mathrm{d}W_u,$$

where α is the *control process* adapted to the information flow but unrestricted otherwise, positive square matrix σ is the diffusion coefficient, and W_t is a standard

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Brownian motion. The cost functional of these problems have a separable structure given by,

$$J(\alpha) := \int_t^T (\ell(u, \mathcal{L}(X_u)) + \frac{1}{2}\,\mathbb{E}|\alpha_u|^2)\,\mathrm{d}u + g(\mathcal{L}(X_T))\,,$$

where ℓ, g, are given functions, and $\mathcal{L}(X_u) \in \mathcal{P}(\mathbb{R}^d)$ is the law of the random variable X_u. Let $v(t, \mu)$ be the value function defined by,

$$v(t, \mu) := \inf_{\alpha.} J(\alpha), \qquad \mathcal{L}(X_t) = \mu.$$

By appropriately scaling time and space, we assume that σ is the identity matrix. Then, the corresponding dynamic programming equation is given by,

$$-\partial_t v(t, \mu) + H(\mu, \delta_\mu v(t, \mu)) = \ell(t, \mu), \tag{1.1}$$

where the function $\delta_\mu v(t, \mu)(\cdot)$ is the linear derivative of v with respect to μ as defined in Section 2 below, and for a twice differentiable function κ and a probability measure μ,

$$H(\mu, \kappa) = -\frac{1}{2}\mu(\Delta\kappa) + \frac{1}{2}\mu(|\nabla\kappa|^2), \tag{1.2}$$

and $\mu(f) = \int f(x)\mu(\mathrm{d}x)$.

Under natural assumptions on ℓ, g (cf. Assumption 3.1, below), dynamic programming holds and the value function is a viscosity solution of (1.1) using the standard notion of linear derivative. Many similar results of this type have already been proved in far greater generality. We refer the reader to our previous paper [1] for these types of results, and the relevant references therein.

Mean-field optimal control problems are part of the exciting general program of Lasry & Lions [2–4] as outlined by Lions during his College de France lectures [5]. Similar type of differential games were independently introduced by Huang, Malhamé, & Caines [6–8], and we refer the reader to the classical book of Carmona & Delarue [9], and to the lecture notes of Cardaliaguet [10] for detailed information and more references.

Our central goal is the characterization of the value function as the unique weak solution of (1.1). While the impressive paper of Cardaliaguet *et. al.* [11] provides regularity results for mean field games, it is well known that dynamic programming equations in general do not admit classical solutions, and we naturally consider the celebrated viscosity solutions of Crandall & Lions [12–15]. However, in infinite dimensions the Hamiltonian is often not defined when the derivative of the solution is not in the domain of corresponding unbounded operators, as explained in the excellent book of Gozzi & Swiech [16]. Thus, the original definition must be modified, and there are several alternatives. Among those we pursue the standard definition of a viscosity solution using the linear derivative on the convex set of probability measures, as we have done in our earlier paper [1].

Our main contribution Theorem 5.1 is a comparison result for the dynamic programming equation (1.1) among all semi-continuous sub and supersolutions. More general results in this direction have already been proved by Cosso *et. al.* [17], and more recently by Daudin, Seeger [18] and by Daudin, Jackson & Seeger [19]. However, we use a different and an alternate technique developed in [1] based on negative Sobolev norms and their Fourier representations, but without using the strong structure imposed on the controls in [1]. An important ingredient is the Lipschitz regularity in the negative Sobolev norms of the value of optimal control problems with smooth coefficients proved in Proposition 3.3. These estimates were first used in [1] in this context. In the separable structure that we consider, it is proved more generally by Daudin, Delarue & Jackson [20] using the theory of elliptic equations, and were then used in [19] to obtain a general comparison on the d-dimensional torus. We also leverage this Lipschitz regularity of the value functions and the techniques of [1] to prove the general comparison result Theorem 5.1 on the whole $\mathbb{R}^d$, under a weak regularity condition Assumption 3.1.

Properties of the solutions of Hamilton-Jacobi equations on the spaces of probability measures have been actively researched in the past two decades. A milestone in the these studies is the lifting introduced by Lions in [5]. This approach maps the problem to an $\mathbb{L}^2$ space and connects to the earlier results exploiting the Hilbert structure. and is further developed in several papers including [21, 22]. Additionally, the novel *Lions derivative* and its properties are explored in the book of Carmona & Delarue [9].

As mentioned earlier, [17] proves a very general comparison result by extending the deep techniques developed by Lions [23] to the Wasserstein space and covering essentially all convex Hamiltonians. Two recent papers [18, 19] also prove comparison results with techniques closer to ours. While an intriguing new definition together with the differentiable structure of the Wasserstein two metric is used in [18, 19] uses amalgam of deep techniques including the negative Sobolev norms and a change of variables introduced in [24] to prove several interesting results on the d-dimensional torus. Also a general Crandall-Ishii type result is proved in [24] using the negative Sobolev norms introduced in [1] and in this paper. Additionally, in another recent study [25] related to stochastic optimal transport, Bertucci introduces a highly original new definition of viscosity solutions and proves general comparison principles. An interesting approach developed by Gangbo & Swiech [16] and Marigonda & Quincampoix [26], and Jimenez *et.al.* [27] utilizes deep connections to geometry. Gangbo & Tudorascu [28] connects this method to Lions lifting. Cecchin and Delarue [9] uses Fourier approximations of the measures and exploits the semi-concavity, and provides an excellent overview of the problem. In our earlier work [1, 29], we have used the direct definition of the viscosity solutions and employed the classical techniques.

Alternatively, projections of these equations to finite-dimensional spaces yield approximate equations that can be directly analyzed by classical results [14]. A second-order problem studied in [30] provides a clear example of this approach as its projections exactly solve the projected finite dimensional equations. However, in general these projections are only approximate solutions, and clearly one has to effectively control the approximation error to obtain relevant results. This is achieved by Cosso *et.al.* [17] via the smooth variational principle together with Gaussian smoothed

Wasserstein metrics. Bayraktar *et.al.* [31] use a different approach, and Gangbo *et.al.* [32] studies the pure projection problem.

Other highly relevant studies include Wu & Zhang [33] for path-dependent equations, Conforti *et.al.* [34] for gradient flows, and Talbi *et.al.* [35, 36] for mean-field stopping problems. Additionally, Ambrosio & Feng [37], and Feng & Katsoulakis [38] study the closely connected Hamilton Jacobi equations on metric spaces.

The paper is organized as follow. General structure and notations are given in the next section. In Sect. 3 we briefly define the problem, and state the standing assumptions. Viscosity solutions are defined in Sect. 4, and the main comparison result Theorem 5.1 is stated and proved in Sect. 5. In the Appendices, we prove a technical lemma and outline the proof of the regularity result proved in [20].

2 Notations

The dimension of the ambient space is denoted by d, T is the finite horizon, $\mathcal{M}(\mathbb{R}^d)$ is the set of all bounded Radon measures, $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d$, and

$$\mathcal{P}_2(\mathbb{R}^d) := \{\, \mu \in \mathcal{P}(\mathbb{R}^d) : \int |x|^2 \, \mu(\mathrm{d}x) < \infty \,\}.$$

We write $\mathcal{M}, \mathcal{P}, \mathcal{P}_2$ when the ambient space is clear or redundant and endow them with the weak topology $\sigma(\mathcal{M}, \mathcal{C}_b(\mathbb{R}^d))$. Then, for a sequence of measures $\{\mu_k\}$ the weak convergence $\mu_k \rightharpoonup \mu$ means $\lim_{k\to\infty} \mu_k(f_k) = \mu(f)$ for every $f \in \mathcal{C}_b(\mathbb{R}^d)$.

We set $\mathcal{O} := (0, T) \times \mathcal{P}_2$ and endow $\overline{\mathcal{O}} := [0, T] \times \mathcal{P}_2$ with the product of Euclidean and the weak topology $\sigma(\mathcal{C}_b(\mathbb{R}^d), \mathcal{M})$. We utilize the local compactness of $\overline{\mathcal{O}}$. Indeed, set

$$\begin{aligned} \vartheta(\mu) &:= \mu(q) = \int q(x)\, \mu(\mathrm{d}x), \quad \mu \in \mathcal{P}_2, \\ q(x) &:= \sqrt{1 + |x|^2}, \quad x \in \mathbb{R}^d. \end{aligned} \tag{2.1}$$

Then, for any constant $c > 0$, the sublevel set $\{(t, \mu) \in \overline{\mathcal{O}} \ : \ \vartheta(\mu) \le c\,\}$ is weakly compact.

For metric spaces $E, F, \mathcal{C}(E \mapsto F)$ denotes the F-valued continuous functions on E. We write $\mathcal{C}(E)$ when $F = \mathbb{R}$ and $\mathcal{C}_b(E)$ for the bounded ones. For a positive integer n, $\mathcal{C}^n(\mathbb{R}^d)$ is the set of all n-times continuously differentiable, real-valued functions, and we set

$$\mathcal{C}_* := \mathcal{C}_*(\mathbb{R}^d) = \{\, f \in \mathcal{C}(\mathbb{R}^d) : \ |f(x)| \le c(1 + |x|^2), \text{ for some constant } c\}.$$

It is clear that $\int f \mathrm{d}\mu$ is well-defined for the pair $\mu \in \mathcal{P}_2$, $f \in \mathcal{C}_*$, and whenever defined we write $\mu(f)$ for the integral $\int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x)$. We also use the notation,

$$\mathcal{C}^2_* := \{\, f \in \mathcal{C}^2(\mathbb{R}^d) : \ f, \ |\nabla f|^2 \in \mathcal{C}_*, \ D^2 f \in \mathcal{C}_b \,\}. \tag{2.2}$$

Using the standard notion of linear derivative on the convex set $\mathcal{P}_2$, we say that $\varphi \in \mathcal{C}(\mathcal{P}_2)$ is *continuously differentiable* if there exists $\delta_\mu \varphi \in \mathcal{C}(\mathcal{P}_2 \mapsto \mathcal{C}_*)$ satisfying,

$$\varphi(\nu) = \varphi(\mu) + \int_0^1 (\nu - \mu)(\delta_\mu \varphi(\mu + \tau(\nu - \mu))\, \mathrm{d}\tau, \qquad \forall\, \mu, \nu \in \mathcal{P}_2.$$

Clearly, $\delta_\mu \varphi(\mu) \in \mathcal{C}_*$ has many representatives. However, when $\delta_\mu \varphi(\mu)$ is twice differentiable, then $\mu(\Delta \delta_\mu \varphi(\mu))$, and $\mu(h(\nabla \delta_\mu \varphi(\mu)))$ with any continuous function h and appropriate integrability are independent of this choice, see for instance [30][Appendix B]. For $\psi \in \mathcal{C}(\overline{\mathcal{O}})$ and $(t, \mu) \in \mathcal{O}$, $\partial_t \psi(t, \mu) \in \mathbb{R}$ is the time derivative evaluated at (t, μ), and $\delta_\mu \psi(t, \mu) \in \mathcal{C}_*$ is the derivative in the μ-variable.

We consider the Fourier basis given by,

$$e(x, \xi) := (2\pi)^{-\frac{d}{2}}\, e^{i \xi \cdot x}, \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d,$$

where $i = \sqrt{-1}$ and z^* is the complex conjugate of z. Then, for any $f \in \mathbb{L}^2(\mathbb{R}^d)$,

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} \mathfrak{F}(f)(\xi) e(x, \xi)\, \mathrm{d}\xi, \quad \text{where} \\ \mathfrak{F}(f)(\xi) &:= \int_{\mathbb{R}^d} f(x) e^*(x, \xi)\, \mathrm{d}x, \quad x, \xi \in \mathbb{R}^d. \end{aligned} \tag{2.3}$$

For $s \in \mathbb{R}$, $\mathcal{H}_s(\mathbb{R}^d)$ is the classical Sobolev space with fractional derivatives [37, 39]. Then,

$$\|f\|_s^2 := \|f\|^2_{\mathcal{H}_s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s\, |\mathfrak{F}(f)(\xi)|^2\, \mathrm{d}\xi.$$

Moreover, for $s > k + \frac{d}{2}$, $\mathcal{H}_s(\mathbb{R}^d)$ continuously embeds into $\mathcal{C}_b^k(\mathbb{R}^d)$. Therefore, for $s > \frac{d}{2}$, $\mathcal{M}(\mathbb{R}^d) \subset \mathcal{H}_{-s}(\mathbb{R}^d)$, and $\|\cdot\|_{-s}$ is well defined on $\mathcal{M}(\mathbb{R}^d)$. Then, for $\eta \in \mathcal{M}(\mathbb{R}^d)$,

$$\begin{aligned} \|\eta\|^2_{-s} &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s} |\mathfrak{F}(\eta)(\xi)|^2\, \mathrm{d}\xi, \quad \text{where} \\ \mathfrak{F}(\eta)(\xi) &= \int_{\mathbb{R}^d} e^*(x, \xi)\, \eta(\mathrm{d}x), \quad \xi \in \mathbb{R}^d. \end{aligned}$$

Moreover, by duality,

$$\|\eta\|_{-s} = \sup\{\, \eta(\psi) :\ \psi \in \mathcal{H}_s(\mathbb{R}^d),\ \|\psi\|_s \le 1\,\}.$$

We use the choice

$$n_* := n_*(d) = 3 + \lfloor \frac{d}{2} \rfloor, \qquad \varrho := \|\cdot\|_{-n_*}, \tag{2.4}$$

where $\lfloor a \rfloor$ is the integer part of a real number a. As $n_* > 2 + \frac{d}{2}$, $\mathcal{H}_{n_*}(\mathbb{R}^d) \subset \mathcal{C}_b^2(\mathbb{R}^d)$, and by Morrey's inequality there is a constant k_d depending only on the dimension such that (see for instance, [40][Chapter 4])

$$\|\kappa\|_{\mathcal{C}^2(\mathbb{R}^d)} \le k_d \|\kappa\|_{n^*}, \qquad \forall\, \kappa \in \mathcal{H}_{n^*}(\mathbb{R}^d). \tag{2.5}$$

3 McKean-Vlasov control

Let $v(t,\mu)$ be the value function of the *McKean-Vlasov optimal control* problem defined in the Introduction by using all square integrable, adapted controls. For more information, we refer the reader to Chapter 6 in [9] and [1, 41]. In particular, the recent paper of Daudin [41] outlines the connections between several formulations and proves the existence of optimal feedback controls.

Following is the only assumption of the paper. Recall that $\overline{\mathcal{O}} := [0,T] \times \mathcal{P}_2$ is endowed with the product of Euclidean and the weak topology $\sigma(\mathcal{P}_2, \mathcal{C}_b(\mathbb{R}^d))$.

Assumption 3.1 We assume that $\ell : \overline{\mathcal{O}} \mapsto \mathbb{R}$, $g : \mathcal{P}_2 \mapsto \mathbb{R}$ are bounded and continuous. We additionally assume that, there exists a sequence of smooth functions (ℓ_n, g_n) approximating (ℓ, g) uniformly, a constant $k_* > 0$, a modulus ω (i.e., $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function with $\omega(0) = 0$), and constants c_n, such that for each n, $t, s \in [0,T]$, and $\mu \in \mathcal{P}_2$,

$$\begin{aligned} &|\ell_n(t,\mu)| + |g_n(\mu)| \le k_*, \quad |\ell_n(t,\mu) - \ell_n(s,\mu)| \le k_*\, \omega(|t-s|), \\ &\quad \|\delta_\mu \ell_n(t,\mu)\|_{\mathcal{H}_{2n^*}(\mathbb{R}^d)} + \|\delta_\mu \ell_n(t,\mu)\|_{\mathcal{C}^{2n^*}(\mathbb{R}^d)} + \|\delta_\mu g_n(\mu)\|_{\mathcal{H}_{2n^*}(\mathbb{R}^d)} \\ &\quad + \|\delta_\mu g_n(\mu)\|_{\mathcal{C}^{2n^*}(\mathbb{R}^d)} \le c_n. \end{aligned}$$

Above assumption is satisfied by a large class of functions, and the choice $2n^*$ is arbitrary but does not decrease the generality. Below we provide a natural class of such functions. In fact, regularization techniques developed in [9] can be used to construct the approximating sequence directly under assumptions on (ℓ, g).

Example 3.2 Consider a function $\ell(\mu) = L(\mu(f))$ for some $L \in \mathcal{C}_b(\mathbb{R})$, $f \in \mathcal{C}_b(\mathbb{R}^d)$. Additionally, assume that L is Lipschitz, and f is square integrable. Then, by mollification one can construct smooth functions (L_n, f_n) approximating (L, f) uniformly, and satisfying $\|f_n\|_{\mathcal{H}_{2n^*}(\mathbb{R}^d)} + \|f_n\|_{\mathcal{C}^{2n^*}(\mathbb{R}^d)} \le c_n$,

$$\sup_n (\|L_n\|_{\mathcal{C}^1} + \|f_n\|_\infty) \le \|L\|_\infty + \|L'\|_\infty + \|f\|_\infty =: k_*.$$

Moreover, as $\delta_\mu \ell_n(t,\mu)(x) = L_n'(\mu(f_n)) f_n(x)$ for $x \in \mathbb{R}^d$,

$$\begin{aligned} &\|\delta_\mu \ell_n(t,\mu)\|_{\mathcal{H}_{2n^*}(\mathbb{R}^d)} \le k_*\, \|f_n\|_{\mathcal{H}_{2n^*}(\mathbb{R}^d)}, \\ &\|\delta_\mu \ell_n(t,\mu)\|_{\mathcal{C}^{2n^*}(\mathbb{R}^d)} \le k_*\, \|f_n\|_{\mathcal{C}^{2n^*}(\mathbb{R}^d)}. \end{aligned}$$

Thus, $\ell(\mu)$ satisfies the above assumptions. More generally, a natural class of functions for the above assumption is given by $\ell(t,\mu) = L(t, \mu(f_1(t,\cdot)), \dots, \mu(f_m(t,\cdot)))$ for some functions $L, f_1, \dots, f_m$ satisfying appropriate conditions.

Let (ℓ_n, g_n) be as in the Assumption 3.1, and v_n be the value function of the optimal control problem with running cost ℓ_n and terminal cost g_n, and same dynamics as in the original problem. The following regularity of v_n is essentially proved in [20][Proposition 3.2] improving a similar result proved in [1][Theorem 4.2].

Proposition 3.3 (Proposition 3.2 [20]) *Let ϱ be as in (2.4). Under the Assumption 3.1, there exists constants $\hat{c}_n$ such that*

$$|v_n(t,\mu) - v_n(t,\nu)| \leq \hat{c}_n\, \varrho(\mu - \nu), \qquad \forall\, t \in [0,T],\ \mu,\nu \in \mathcal{P}_2. \tag{3.1}$$

Proposition 3.2 in [20] proves exactly the above estimate but in the d-dimensional torus. However, their proof can be directly adopted to the current context with no changes. As the above estimate is used centrally in our proofs, for the convenience of the readers we provide an outline proof of the above result in the Appendix.

Corollary 3.4 *Under the Assumption 3.1, $v_n, v \in \mathcal{C}_b(\overline{\mathcal{O}})$, i.e., both v_n and v are bounded and are continuous in the product of Euclidean and weak topology $\sigma(\mathcal{C}_b(\mathbb{R}^d), \mathcal{P}_2)$.*

Proof The continuity of v_n in the time variable is straightforward [1]. The above Lipschitz continuity in ϱ and Lemma A.1 implies that $v_n \in \mathcal{C}_b(\overline{\mathcal{O}})$. The uniform convergence of (ℓ_n, g_n) to (ℓ, g) implies that v_n converges to v uniformly and therefore $v \in \mathcal{C}_b(\overline{\mathcal{O}})$ as well. $\square$

4 Viscosity Solutions

We start by defining the class of test functions used in the definition of the viscosity solutions.

Definition 4.1 A continuous function $\varphi \in \mathcal{C}(\overline{\mathcal{O}})$ is called a test function if there exists a version of $\delta_\mu \varphi$ such that the map

$$(t,\mu,x) \in \overline{\mathcal{O}} \times \mathbb{R}^d \mapsto \delta_\mu \psi(t,\mu)(x)$$

is continuous, and $\delta_\mu \varphi(t,\mu) \in \mathcal{C}^2_*$ for every $(t,\mu) \in \mathcal{O}$. Let $\mathcal{C}_s(\mathcal{O})$ be the set of all test functions.

We can now directly define the notion of viscosity solutions [12–15]. Recall that we endow $\overline{\mathcal{O}}$ with the product of Euclidian and weak topologies.

Definition 4.2 We say that an upper semicontinuous function $u : \overline{\mathcal{O}} \mapsto \mathbb{R}$ is a viscosity subsolution of (1.1) if for every test function $\varphi \in \mathcal{C}_s(\overline{\mathcal{O}})$ we have

$$-\partial_t \varphi(t_0,\mu_0) + H(\mu_0, \delta_\mu \varphi(t_0,\mu_0)) \leq \ell(t_0,\mu_0),$$

at every $(t_0, \mu_0) \in \mathcal{O}$ satisfying $(u - \varphi)(t_0, \mu_0) = \max_{\overline{\mathcal{O}}}(u - \varphi)$.

We say that a lower semicontinuous function $w : \overline{\mathcal{O}} \mapsto \mathbb{R}$ is a viscosity supersolution of (1.1) if for every test function $\varphi \in \mathcal{C}_s(\overline{\mathcal{O}})$ we have

$$-\partial_t \varphi(t_0, \mu_0) + H(\mu_0, \delta_\mu \varphi(t_0, \mu_0)) \geq \ell(t_0, \mu_0),$$

at every $(t_0, \mu_0) \in \mathcal{O}$ satisfying $(w - \varphi)(t_0, \mu_0) = \min_{\overline{\mathcal{O}}}(w - \varphi)$.

A function $v : \overline{\mathcal{O}} \mapsto \mathbb{R}$ is a viscosity solution if its lower semicontinuous envelope v_* is a supersolution and its upper semicontinuous envelope v^* is a subsolution.

Remark 4.3 In view of (2.2), if φ is a test function, then $\delta_\mu \varphi(t, \mu) \in \mathcal{C}^2$ with its derivatives satisfying $\delta_\mu \varphi(t, \mu)$, $|\nabla \delta_\mu \varphi(t, \mu)|^2 \in \mathcal{C}_*$, and $D^2 \delta_\mu \varphi(t, \mu) \in \mathcal{C}_b$. Note that these test functions are not necessarily bounded and may grow quadratically. As our analysis is in the Wasserstein space $\mathcal{P}_2$, this relaxation is natural, and is utilized in the comparison proof.

The following is standard and is proved in [1].

Corollary 4.4 *Under Assumption 3.1, the dynamic programming holds. Consequently, v is a viscosity solution of (1.1), and for each n, v_n is a viscosity solution of*

$$-\partial_t v(t, \mu) + H(\mu, \delta_\mu v(t, \mu)) = \ell_n(t, \mu), \qquad \textit{on } \ (0, T) \times \mathcal{P}_2.$$

5 Comparison

Our main result is the comparison for the Eikonal equation (1.1), and its proof is given later in this section. Recall that the state space is $\overline{\mathcal{O}} = [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, and we endow it with the product of Euclidean and weak topology $\sigma(\mathcal{P}_2, \mathcal{C}_b(\mathbb{R}^d))$.

Theorem 5.1 *Suppose that Assumption 3.1 holds, $u : \overline{\mathcal{O}} \mapsto \mathbb{R}$ is an upper semi-continuous, bounded viscosity sub-solution of (1.1), and $w : \overline{\mathcal{O}} \mapsto \mathbb{R}$ is a lower semi-continuous, bounded viscosity super-solution of (1.1). Further assume that $u(T, \cdot) \leq w(T, \cdot)$. Then, $u \leq w$ on $\overline{\mathcal{O}}$. In particular, the value function v is the unique continuous, bounded viscosity solution of the dynamic programming equation (1.1) and the terminal condition $v(T, \cdot) = g$.*

We start with a simple computation and estimates. Recall the test functions $\mathcal{C}_s(\overline{\mathcal{O}})$ of Definition 4.1, n_*, ϱ of (2.4), and the Fourier basis $e(x, \xi)$.

Lemma 5.2 *For $\eta \in \mathcal{M}(\mathbb{R}^d)$, set $\psi(\eta) := \frac{1}{2}\varrho^2(\eta)$. Then, for $\mu, \nu \in \mathcal{P}_2$,*

$$\kappa(x) := \delta_\mu \psi(\mu - \nu)(x) = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-n^*} \, \mathfrak{F}(\mu - \nu)(\xi) e(x, \xi) \, d\xi, \qquad x \in \mathbb{R}^d.$$

Moreover, $\|\kappa\|_{n^*} = \varrho(\mu - \nu)$.

Proof Fix $\mu, \nu \in \mathcal{P}_2$ and set $\eta = \mu - \nu$. A straightforward computation implies that $\kappa := \delta_\mu \psi(\eta)$ has the claimed form $\kappa(x) = \int g(\xi)\, e(x, \xi) \mathrm{d}\xi$, where $g(\cdot) := (1 + |\cdot|^2)^{-n^*} \mathfrak{F}(\eta)(\cdot)$. Then, in view of the inverse Fourier formula (2.3), we conclude that $\mathfrak{F}(\kappa) = g$.

Proof of Theorem 5.1 We complete the proof in several steps. Recall that the functions $q(\cdot)$ and $\vartheta(\mu) = \mu(q)$ are defined in (2.1), and ϑ is weakly lower-semicontinuous on $\mathcal{P}_2$, and any sublevel set $\{\mu \in \mathcal{P}_2 : \ \vartheta(\mu) \le c\}$ is compact. □

Step 1 (*Set-up*). Let u, w be as in the statement of the theorem. Towards a contraposition suppose that $\sup_{\overline{\mathcal{O}}}(u - w) > 0$. Let v be the value function. Then,

$$0 < \sup_{\overline{\mathcal{O}}}(u - w) \le \sup_{\overline{\mathcal{O}}}(u - v) + \sup_{\overline{\mathcal{O}}}(v - w).$$

Hence, either $\sup_{\overline{\mathcal{O}}}(u - v) > 0$, or $\sup_{\overline{\mathcal{O}}}(v - w) > 0$, or both must hold. We analyze the first case and this analysis can be followed *mutatis mutandis* to prove the other case.

For a small constant δ_*, set $\bar{u}(t, \mu) := u(t, \mu) - 2\delta_*(T - t + 1)$. We first fix δ_* satisfying $\sup_{\overline{\mathcal{O}}}(\bar{u} - v) > 0$. We then fix n sufficiently large so that

$$-\partial_t \bar{u}(t, \mu) + H(\mu, \delta_\mu \bar{u}(t, \mu)) \le \ell(t, \mu) - 2\delta_* \le \ell_n(t, \mu) - \delta_*, \tag{5.1}$$

and $\bar{u}(T, \cdot) \le g - 2\delta_* \le g_n$. In the remainder of the proof we fix δ_*, n as above. Next, set $l := \sup_{\overline{\mathcal{O}}}(\bar{u} - v_n)/3$.

Step 2 (*Doubling the variables*). Set $\mathcal{X} = \overline{\mathcal{O}} \times \overline{\mathcal{O}}$. For $\epsilon, \gamma > 0$, and $(t, \mu, s, \nu) \in \mathcal{X}$, define

$$\begin{aligned}\Psi_{\epsilon,\gamma}(t, \mu, s, \nu) := \bar{u}(t, \mu) - v_n(s, \nu) - \frac{1}{2\epsilon}((t - s)^2 + \varrho^2(\mu - \nu)) \\ -\gamma\vartheta(\mu) - \epsilon\vartheta(\nu).\end{aligned}$$

By the previous step, there is $(t_0, \mu_0) \in \overline{\mathcal{O}}$ such that

$$2l \le (\bar{u} - v_n)(t_0, \mu_0) = \Phi_{\epsilon,\gamma}(t_0, \mu_0, t_0, \mu_0) + \gamma\vartheta(\mu_0) + \epsilon\vartheta(\mu_0).$$

Then, for all $0 < \epsilon \le \gamma \le \gamma_* := l/(2\vartheta(\mu_0) + 1)$, $\max_{\mathcal{X}} \Phi_{\epsilon,\gamma} \ge l > 0$. In the remainder of this proof, we always assume that $\epsilon \le \gamma \le \gamma_*$.

Let (t_k, μ_k, s_k, ν_k) be a maximizing sequence of $\Phi_{\epsilon,\gamma}$. Since $\bar{u}, v_n$ are bounded,

$$\gamma\vartheta(\mu_k) + \epsilon\vartheta(\nu_k) \le \ (\|\bar{u}\|_\infty + \|v_n\|_\infty) =: c_*.$$

As the sub-level sets of ϑ are compact, the sequences μ_k, ν_k have limit points. Since additionally, v_m, ϱ are continuous, and $\bar{u}, -\ \vartheta$ are upper-semicontinuous, $\Phi_{\epsilon,\gamma}$ is also upper-semicontinuous, and these limit points achieve the maximum of $\Phi_{\epsilon,\gamma}$. Hence, there exists a quadruple $(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}, s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}) \in \mathcal{X}$ satisfying, $\Phi_{\epsilon,\gamma}(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}, s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}) = \max_{\mathcal{X}} \Phi_{\epsilon,\gamma} \geq l > 0$. Set

$$\eta_{\epsilon,\gamma} := \mu_{\epsilon,\gamma} - \nu_{\epsilon,\gamma}, \qquad \tau_{\epsilon,\gamma} := t_{\epsilon,\gamma} - s_{\epsilon,\gamma}$$

Then, we also have

$$\begin{aligned} \frac{1}{2\epsilon}(\tau_{\epsilon,\gamma}^2 + \varrho^2(\eta_{\epsilon,\gamma})) + \gamma\vartheta(\mu_{\epsilon,\gamma}) + \epsilon\vartheta(\nu_{\epsilon,\gamma}) &\leq \bar{u}(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) - v_n(s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}) \\ &\leq c_*. \end{aligned} \tag{5.2}$$

Step 3 (*Norm estimate*). We now use the Lipschitz estimate (3.1) of v_n to obtain a uniform bound for $\varrho(\eta_{\epsilon,\gamma})/\epsilon$. Note that n is already chosen and remains fixed throughout the proof. As $\Phi_{\epsilon,\gamma}(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}, s_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) \leq \Phi_{\epsilon,\gamma}(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}, s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma})$, we have

$$\begin{aligned} &u(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) - v_n(s_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) - \frac{1}{2\epsilon}\tau_{\epsilon,\gamma}^2 - \gamma\vartheta(\mu_{\epsilon,\gamma}) - \epsilon\vartheta(\mu_{\epsilon,\gamma}) \\ &\leq\ u(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) - v_n(s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}) \\ &\quad - \frac{1}{2\epsilon}(\tau_{\epsilon,\gamma}^2 + \varrho^2(\eta_{\epsilon,\gamma})) - \gamma\vartheta(\mu_{\epsilon,\gamma}) - \epsilon\vartheta(\nu_{\epsilon,\gamma}). \end{aligned}$$

Then, by Proposition 3.3 and (5.2),

$$\begin{aligned} \frac{1}{2\epsilon}\varrho^2(\eta_{\epsilon,\gamma}) &\leq v_n(s_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) - v_n(s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}) + \epsilon(\vartheta(\mu_{\epsilon,\gamma}) - \vartheta(\nu_{\epsilon,\gamma})) \\ &\leq \hat{c}_n\varrho(\eta_{\epsilon,\gamma}) + \epsilon\vartheta(\mu_{\epsilon,\gamma}) \leq \hat{c}_n\varrho(\eta_{\epsilon,\gamma}) + \epsilon\frac{c_*}{\gamma}. \end{aligned}$$

Therefore, there is a constant $\hat{c}$ depending only on $\hat{c}_n$, c_* such that for all $0 < \epsilon, \gamma \leq 1$,

$$\frac{\varrho(\eta_{\epsilon,\gamma})}{\epsilon} \leq \frac{\hat{c}}{\sqrt{\gamma}}. \tag{5.3}$$

Step 4 (*Letting ϵ to zero*). By (5.2), $\vartheta(\mu_{\epsilon,\gamma}) \leq c_*/\gamma$. Therefore, for each $\gamma \in (0, \delta_*]$ there are subsequences $\{(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma})\} \subset \overline{\mathcal{O}}$, $\{t_{\epsilon,\gamma}\} \subset [0, T]$, denoted by ϵ again, and limit points $(s_\gamma, \mu_\gamma) \in \overline{\mathcal{O}}$, $t_\gamma \in [0, T]$ such that as $\epsilon \downarrow 0$, $\mu_{\epsilon,\gamma} \rightharpoonup \mu_\gamma$, $t_{\epsilon,\gamma} \to t_\gamma$, and $s_{\epsilon,\gamma} \to s_\gamma$. By (5.2), it is clear that $t_\gamma = s_\gamma$, and $\lim_{k\to\infty} \varrho(\mu_{\epsilon,\gamma} - \nu_{\epsilon,\gamma}) = 0$. We now use Lemma A.1 to conclude that as $\epsilon \downarrow 0$, we also have $\nu_{\epsilon,\gamma} \rightharpoonup \mu_\gamma$.

As $\bar{u}(T, \cdot) \leq g_n = v_n(T, \cdot)$, if t_γ were to be equal to T, we would have

$$\begin{aligned} 0 < l &\leq \liminf_{k\to\infty} \Phi_{\epsilon,\gamma}(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}, s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}) \leq \liminf_{k\to\infty} [\bar{u}(t_{\epsilon,\gamma}, \mu_{\epsilon,\gamma}) - v_n(s_{\epsilon,\gamma}, \nu_{\epsilon,\gamma})] \\ &\leq \bar{u}(T, \mu_\gamma) - v_n(T, \mu_\gamma) \leq 0. \end{aligned}$$

Hence, $t_\gamma < T$ and consequently, both $t_{\epsilon,\gamma} < T$, and $s_{\epsilon,\gamma} < T$ for all sufficiently small $\epsilon > 0$.

Step 5 (*Viscosity property*). Set

$$\psi_{\epsilon,\gamma}(t,\mu) := \frac{1}{2\epsilon}((t-s_{\epsilon,\gamma})^2 + \varrho^2(\mu-\nu_{\epsilon,\gamma})) + \gamma\vartheta(\mu),$$
$$\phi_{\epsilon,\gamma}(s,\nu) := -\frac{1}{2\epsilon}((t_{\epsilon,\gamma}-s)^2 + \varrho^2(\mu_{\epsilon,\gamma}-\nu)) - \epsilon\vartheta(\nu).$$

By Lemma 5.2, both $\delta_\mu\psi_{\epsilon,\gamma}(t,\mu), \delta_\mu\phi_{\epsilon,\gamma}(t,\mu) \in \mathcal{C}_*^2$. Hence, $\psi_{\epsilon,\gamma}, \phi_{\epsilon,\gamma} \in \mathcal{C}_s(\overline{\mathcal{O}})$, i.e., they are smooth test functions in the sense of Definition 4.1. By using Lemma 5.2, we calculate that

$$\delta_\mu\psi_{\epsilon,\gamma}(t_{\epsilon,\gamma},\mu_{\epsilon,\gamma}) = \kappa_{\epsilon,\gamma} + \gamma q, \qquad \partial_\nu\phi_{\epsilon,\gamma}(s_{\epsilon,\gamma},\nu_{\epsilon,\gamma}) = \kappa_{\epsilon,\gamma} - \epsilon q,$$

where q is as in (2.1), and for $x \in \mathbb{R}^d$,

$$\kappa_{\epsilon,\gamma}(x) := \frac{1}{\epsilon}\int_{\mathbb{R}^d} (1+|\xi|^2)^{-n_*}\,\mathfrak{F}(\eta_{\epsilon,\gamma})(\xi)\,e(x,\xi)\,\mathrm{d}\xi \quad\Rightarrow\quad \|\kappa_{\epsilon,\gamma}\|_{n^*} = \frac{1}{\epsilon}\varrho(\eta_{\epsilon,\gamma}).$$

It is clear that, $\bar{u}(t,\mu) - \psi_{\epsilon,\gamma}(t,\mu)$ is maximized at $(t_{\epsilon,\gamma},\mu_{\epsilon,\gamma})$. Since $t_{\epsilon,\gamma} < T$, $\psi_{\epsilon,\gamma} \in \mathcal{C}_s(\overline{\mathcal{O}})$ and $\bar{u}$ is a viscosity subsolution of (5.1),

$$-\frac{t_{\epsilon,\gamma}-s_{\epsilon,\gamma}}{\epsilon} + H(\mu_{\epsilon,\gamma},\kappa_{\epsilon,\gamma}+\gamma q) \le \ell_n(t_{\epsilon,\gamma},\mu_{\epsilon,\gamma}) - \delta_*.$$

By the viscosity property of v_n, a similar argument implies that

$$-\frac{t_{\epsilon,\gamma}-s_{\epsilon,\gamma}}{\epsilon} + H(\nu_{\epsilon,\gamma},\kappa_{\epsilon,\gamma}-\epsilon q) \ge \ell_n(s_{\epsilon,\gamma},\nu_{\epsilon,\gamma}).$$

Step 6 (*Estimation*). We subtract the above inequalities to arrive at

$$\begin{aligned}0 < \delta_* \le& H(\nu_{\epsilon,\gamma},\kappa_{\epsilon,\gamma}-\epsilon q) - H(\mu_{\epsilon,\gamma},\kappa_{\epsilon,\gamma}+\gamma q) + \ell_n(t_{\epsilon,\gamma},\mu_{\epsilon,\gamma}) - \ell_n(s_{\epsilon,\gamma},\nu_{\epsilon,\gamma})\\ &=: \mathcal{I}_{\epsilon,\gamma} + \mathcal{J}_{\epsilon,\gamma} + \mathcal{K}_{\epsilon,\gamma},\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{\epsilon,\gamma} &:= \frac{1}{2}(\mu_{\epsilon,\gamma}(\Delta(\kappa_{\epsilon,\gamma}+\gamma q) - \nu_{\epsilon,\gamma}(\Delta(\kappa_{\epsilon,\gamma}-\epsilon q),\\ \mathcal{J}_{\epsilon,\gamma} &:= \frac{1}{2}(\nu_{\epsilon,\gamma}(|\nabla\kappa_{\epsilon,\gamma}-\epsilon q|^2) - \mu_{\epsilon,\gamma}(|\nabla(\kappa_{\epsilon,\gamma}+\gamma q)|^2)),\\ \mathcal{K}_{\epsilon,\gamma} &:= \ell_n(t_{\epsilon,\gamma},\mu_{\epsilon,\gamma}) - \ell_n(s_{\epsilon,\gamma},\nu_{\epsilon,\gamma}).\end{aligned}$$

By Assumption 3.1, $\mathcal{K}_{\epsilon,\gamma}$ converges to zero as $\epsilon \downarrow 0$. Moreover, since $\Delta q \le d$, for $\epsilon \le \gamma$,

$$\mathcal{I}_{\epsilon,\gamma} = -\frac{1}{2\epsilon}\int_{\mathbb{R}^d} \frac{|\xi|^2}{(1+|\xi|^2)^{n_*}}\,|\mathfrak{F}(\eta_{\epsilon,\gamma})(\xi)|^2\,\mathrm{d}\xi \;+\; \frac{1}{2}(\gamma\mu_{\epsilon,\gamma} + \epsilon\nu_{\epsilon,\gamma})(\Delta q) \le \gamma d.$$

Hence, $0 < \delta_* \leq \mathcal{J}_{\epsilon,\gamma} + \gamma d$.

Step 7 (*Estimation of* $\mathcal{J}_{\epsilon,\gamma}$). In view of Lemma 5.2, (2.5), and (5.3),

$$\|\kappa_{\epsilon,\gamma}\|_{\mathcal{C}^1(\mathbb{R}^d)} \leq k_d \|\kappa_{\epsilon,\gamma}\|_{n^*} = k_d \frac{\varrho(\eta_{\epsilon,\gamma})}{\epsilon} \leq \frac{k_d \hat{c}}{\sqrt{\gamma}}.$$

Since $\nabla q(x) = x/q(x)$, $|\nabla q| \leq 1$, and by algebra,

$$\nu_{\epsilon,\gamma}(|\nabla(\kappa_{\epsilon,\gamma} - \epsilon q)|^2 - |\nabla \kappa_{\epsilon,\gamma}|^2) = -\nu_{\epsilon,\gamma}(\nabla(2\kappa_{\epsilon,\gamma} - \epsilon q) \cdot \epsilon \nabla q) \leq \epsilon(\frac{2k_d\hat{c}}{\sqrt{\gamma}} + \epsilon).$$

Similarly,

$$\mu_{\epsilon,\gamma}(|\nabla \kappa_{\epsilon,\gamma}|^2 - |\nabla(\kappa_{\epsilon,\gamma} + \gamma q)|^2) = -\mu_{\epsilon,\gamma}(\nabla(2\kappa_{\epsilon,\gamma} + \gamma q) \cdot \gamma \nabla q) \leq 2k_d\hat{c}\sqrt{\gamma}.$$

Therefore,

$$\begin{aligned}
\mathcal{J}_{\epsilon,\gamma} &= \frac{1}{2}(\nu_{\epsilon,\gamma}(|\nabla \kappa_{\epsilon,\gamma} - \epsilon q|^2) - \mu_{\epsilon,\gamma}(|\nabla(\kappa_{\epsilon,\gamma} + \gamma q + \epsilon q)|^2)) \\
&\leq -\frac{1}{2}\eta_{\epsilon,\gamma}(|\nabla \kappa_{\epsilon,\gamma}|^2) + \bar{c}(\sqrt{\gamma} + \epsilon),
\end{aligned}$$

for some constant $\bar{c}$ independent of ϵ.

We have shown that as $\epsilon \downarrow 0$, $\mu_{\epsilon,\gamma}, \nu_{\epsilon,\gamma} \rightharpoonup \mu_\gamma$. In particular, $\mu_{\epsilon,\gamma}, \nu_{\epsilon,\gamma}$ are tight sequences and $\eta_{\epsilon,\gamma} \rightharpoonup 0$. Additionally, since $\|\kappa_{\epsilon,\gamma}\|_{\mathcal{C}^1(\mathbb{R}^d)}$ is uniformly bounded, on a subsequence $\kappa_{\epsilon,\gamma}$ is locally uniformly convergent. These imply that $\eta_{\epsilon,\gamma}(|\nabla \kappa_{\epsilon,\gamma}|^2)$ converges to zero as $\epsilon \downarrow 0$. Therefore,

$$\liminf_{k \to \infty} \ \mathcal{J}_{\epsilon,\gamma} \leq \bar{c}\sqrt{\gamma}.$$

Step 8 (*Conclusion*). By the previous steps, for every $\gamma > 0$ the following holds,

$$0 < \delta_* \leq \limsup_{k \to \infty} \ \mathcal{J}_{\epsilon,\gamma} + \gamma d \leq \bar{c}\sqrt{\gamma} + \gamma d.$$

Since $\delta_* > 0$, we obtain a contradiction by letting $\gamma \downarrow 0$. Hence, $\sup_{\overline{\mathcal{O}}}(u - w) \leq 0$. □

A Convergence of Measures in ϱ

For any $s > d/2$, any finite Borel measure is an element of the Sobolev space $\mathcal{H}_{-s}(\mathbb{R}^d)$. Hence, $\varrho = \|\cdot\|_{-n^*}$ is a metric on $\mathcal{P}_2(\mathbb{R}^d)$. Although $(\mathcal{P}_2, \varrho)$ is not complete, convergence in this space is equivalent to the weak convergence in the following sense.

Lemma A.1 *A sequence of probability measures μ_k converge to a probability measure μ in the weak topology $\sigma(\mathcal{M}(\mathbb{R}^d), \mathcal{C}_b(\mathbb{R}^d))$ if and only if $\lim_k \varrho(\mu_k - \mu) = 0$. Additionally, if a sequence of probability measures ν_k satisfies $\lim_k \varrho(\nu_k - \mu_k) = 0$, then ν_k to μ in the weak topology $\sigma(\mathcal{M}(\mathbb{R}^d), \mathcal{C}_b(\mathbb{R}^d))$ as well.*

Proof Suppose that μ_k weakly converges to μ. Then, we have $\lim_{k\to\infty} \mu_k(f) = \mu(f)$ for every $f \in \mathcal{C}_b(\mathbb{R}^d)$. Consequently, $\lim_k \mathfrak{F}(\mu_k - \mu)(\xi) = 0$ for every ξ, and dominated convergence implies $\lim_k \varrho(\mu_k - \mu) = 0$. Conversely, if $\lim_k \varrho(\mu_k - \mu) = 0$, then the dual characterization of ϱ implies that $\lim_k \nu_k(f) = \mu(f)$ for every $f \in \mathcal{H}_{n^*}(\mathbb{R}^d)$. As any $f \in \mathcal{C}_b(\mathbb{R}^d)$ is approximated uniformly by functions in $\mathcal{H}_{n^*}(\mathbb{R}^d)$, a direct approximation argument implies that $\lim_k \nu_k(f) = \mu(f)$ for every $f \in \mathcal{C}_b(\mathbb{R}^d)$. This proves the equivalence of weak convergence to the convergence in ϱ.

Moreover, $\limsup_k \varrho(\nu_k - \mu) \le \lim_k \varrho(\nu_k - \mu_k) + \lim_k \varrho(\mu_k - \mu) = 0$. Then, by the proved equivalence, we conclude that ν_k converges to μ in the weak topology $\sigma(\mathcal{M}(\mathbb{R}^d), \mathcal{C}_b(\mathbb{R}^d))$. ☐

B Proposition 3.3

Here, we outline the proof of Proposition 3.3 in several steps. We fix n and set

$$L(t,\mu,x) := \delta_\mu \ell_n(t,\mu)(x), \qquad G(\mu,x) := \delta_\mu g_n(\mu)(x), \qquad (t,\mu,x) \in \overline{\mathcal{O}} \times \mathbb{R}^d.$$

Step 1. (*Reformulation*). The optimal control problem is in fact a deterministic control problem which has an equivalent representation. Indeed, for a given initial condition $(t_0, \mu_0) \in \overline{\mathcal{O}}$, let $\mathcal{A}(t_0, \mu_0)$ be the set of all pairs (α, m) satisfying,

- $m : [t_0, T] \mapsto \mathcal{P}_2$ is continuous with $m(t_0, \cdot) = \mu_0$;
- $\alpha : [t_0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is Borel measurable and $\int |\alpha(t,x)|^2 m(t, \mathrm{d}x)\, \mathrm{d}t < \infty$;
- for any $\phi \in \mathcal{C}^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi(x) m(s, \mathrm{d}x) = \mu(\phi) + \int_{t_0}^{s} \int_{\mathbb{R}^d} \left(\frac{1}{2}\Delta\phi(x) + \alpha(t,x)\cdot\nabla\phi(x)\right) m(t,\mathrm{d}x)\, \mathrm{d}t.$$

The final condition simply states that $m(t,\cdot)$ is the law of a solution to the stochastic differential equation $dX_t = \alpha(t, X_t)\mathrm{d}t + \mathrm{d}W_t$.

Then, the value function has the following equivalent representation [41](Section 2),

$$v_n(t_0,\mu_0) = \inf_{(\alpha,m)\in\mathcal{A}(t_0,\mu_0)} \int_{t_0}^{T} \left[\ell_n(t, m(t,\cdot)) + \frac{1}{2}\int_{\mathbb{R}^d} |\alpha(t,x)|^2 m(t,\mathrm{d}x)\right] \mathrm{d}t + g_n(m(T,\cdot)).$$

Step 2. (*Smooth optimal feedback control*). By Pontryagin maximum principle (see Theorem 2.2 of [41] with constraint $\Psi \equiv 0$), for any initial condition (t_0, μ_0) there exists an optimal pair $(\alpha^*, m^*) \in \mathcal{A}(t_0, \mu_0)$. Moreover, $\alpha^*(t,x) = -\nabla u(t,x)$ where u is the solution of the following Eikonal equation,

$$-\partial_t u(t,x) - \frac{1}{2}\Delta u(t,x) + \frac{1}{2}|\nabla u(t,x)|^2 = \hat{L}(t,x) := L(t, m^*(t,\cdot), x),$$
$$(t,x) \in (0,T) \times \mathbb{R}^d,$$

with the final condition $u(T,x) = \hat{G}(x) := G(m^*(T,\cdot), x)$.

Recall that L, G have continuous and bounded derivatives of order $2n^*$. By standard elliptic regularity (see Lemma 3.1 [20]), the solution u of the above equation satisfies $u(t,\cdot) \in \mathcal{C}_b^{2n^*}(\mathbb{R}^d)$ with norms uniformly bounded in time. We may then rewrite the above equation as

$$-\partial_t u(t,x) - \frac{1}{2}\Delta u(t,x) + \frac{1}{2}A(t,x) \cdot \nabla u(t,x) = \hat{L}(t,x),$$

where $A(t,x) := \nabla u(t,x)$. We now know that $A(t,\cdot) \in \mathcal{C}_b^{2n^*-1}(\mathbb{R}^d)$. Also by hypothesis $\hat{L}(t,\cdot), \hat{G}$ are in $\mathcal{H}_{2n^*}(\mathbb{R}^d)$. As the above equation is linear with smooth coefficients, standard techniques imply that $u(t,\cdot) \in \mathcal{H}_{2n^*-1}(\mathbb{R}^d)$ with norms uniformly bounded in time. In particular, we conclude that there is a feedback optimal control α^* satisfying the estimate

$$\|\alpha^*(t,\cdot)\|_{\mathcal{C}^{2n^*-1}(\mathbb{R}^d)} + \|\alpha^*(t,\cdot)\|_{\mathcal{H}_{2n^*-1}(\mathbb{R}^d)} \leq C,$$

with a constant C depending only on the norms of $\hat{L}, \hat{G}$. In particular, C is independent of the initial condition (t_0, μ_0).
Step 3. (*Conclusion*). We now follow *mutadis mutandis* the proofs of Proposition 3.2 and Lemma 3.3 in [20], (that proves exactly the same result on the torus), to obtain the Lipschitz estimate (3.1). Alternatively, Section 7 of [1] also implies the Lipschitz continuity using the smoothness of the optimal feedback control.

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Declarations

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