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Off-shell form factor in $\mathcal{N}$=4 sYM at three loops

A.V. Belitsky,[a] **L.V. Bork**[b,c] **and V.A. Smirnov**[d,e]

[a]*Department of Physics, Arizona State University, Tempe, AZ 85287-1504, U.S.A.*
[b]*Institute for Theoretical and Experimental Physics, 117218 Moscow, Russia*
[c]*The Center for Fundamental and Applied Research, 127030 Moscow, Russia*
[d]*Skobeltsyn Institute of Nuclear Physics, Moscow State University, 119992 Moscow, Russia*
[e]*Moscow Center for Fundamental and Applied Mathematics, 119992 Moscow, Russia*

E-mail: andrei.belitsky@asu.edu, borkleonid@gmail.com, smirnov@theory.sinp.msu.ru

Abstract: In this paper we provide a detailed account of our calculation, briefly reported in arXiv:2209.09263, of a two-particle form factor of the lowest components of the stress-tensor multiplet in $\mathcal{N} = 4$ sYM theory on its Coulomb branch, which is interpreted as an off-shell kinematical regime. We demonstrate that up to three-loop order, both its infrared-divergent as well as finite parts do exponentiate in the Sudakov regime, with the coefficient accompanying the double logarithm being determined by the octagon anomalous dimension Γ_{oct}. We also observe that up to this order in 't Hooft coupling the logarithm of the Sudakov form factor is identical to twice the logarithm of the null octagon, which was introduced within the context of integrability-based computation of four point correlators with infinitely large R-charges. The null octagon is known in a closed form for all values of the 't Hooft coupling constant and kinematical parameters. We conjecture that the relation between the former and the off-shell Sudakov form factor holds to all loop orders.

Keywords: Higher-Order Perturbative Calculations, Scattering Amplitudes, Supersymmetric Gauge Theory, Integrable Field Theories

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Contents

1 Introduction

The past three decades have witnessed enormous progress in unravelling the structure of S-matrices in gauge theories in various space-time dimensions. One playground where this was achieved in a spectacular manner is the maximally supersymmetric Yang-Mills theory in four space-time dimensions, or $\mathcal{N} = 4$ sYM, for short. The structure of perturbative amplitudes in the latter is qualitatively similar to the one of parton-level scattering in Quantum Chromodynamics,— the theory of strong interaction —, and it was explicitly employed in aiding cutting-edge multi-loop QCD analyses starting from their $\mathcal{N} = 4$ sYM counterparts.

A number of analytical frameworks were devised, tested and employed to obtain a plethora of multi-loop and sometimes all-order scattering amplitudes, with unitarity-based [1], bootstrap [2], duality [3–5] and integrability [6] methods, to name just a few. The same or similar strategies can be applied to other (dimensionally regularized) quantities in $\mathcal{N} = 4$ sYM such as form factors and correlation functions. None of these results would be feasible with standard Feynman diagrammatic methods, which require calculational power far beyond current computational capabilities.

All-order results for scattering amplitudes [7, 8] allowed one to explicitly verify theorems regarding infrared (IR) behavior and factorization properties of massless amplitudes in gauge theories [9–11]. They were found in perfect agreement with general considerations stating that IR divergent parts of the color ordered amplitudes have to be given by products

JHEP11(2023)111

of two-particle Sudakov form factors [7, 10, 11]. The latter were in turn governed by the ubiquitous cusp anomalous dimension $\Gamma_{\rm cusp}$ [12, 13].

These perturbative studies are not of purely academic interest though, considering the unphysical nature of the model in question, but rather, as we briefly touched upon at the very beginning, are of phenomenological relevance as well. For many observables, $\mathcal{N}=4$ results represent the "most complicated" portion of the ones in QCD and thus can be used to facilitate tedious calculations. This is known under the name of the principle of maximal transcendentality [14].

In our present work, a two-particle form factor of a two-scalar field operator in the stress-tensor multiplet will take center stage. Starting with a precocious two-loop analysis in ref. [15], this on-shell observable in the massless $\mathcal{N}=4$ sYM perturbative expansion is currently known up to four-loop accuracy [16–19].

In the bulk of the analyses alluded to above, one dealt with the $\mathcal{N}=4$ sYM theory with the exact SU(N) gauge symmetry. Much less attention has been paid to its phase where the latter is broken by non-zero vacuum expectation values (VEVs) of scalar fields present in the model. This setup is referred in the literature as to the Coulomb branch [20]. One of the motivations to even address the theory with the spontaneously broken gauge symmetry is that so induced particle masses can be regarded as an IR regulator [21–23], — an alternative to the conventional dimensional regularization. As innocent as it may look, the use of the former leaves certain other (space-time) symmetries intact, such as the dual conformal symmetry [4, 24], violated otherwise [21]. Indeed, in $\mathcal{N}=4$ sYM with unbroken gauge symmetry, amplitudes and form factors can be made well-defined only in $D=4-2\varepsilon$ space-time dimensions and they become singular as $\varepsilon\to 0$ due to copious emissions of massless states and their presence in quantum loops, hence IR divergent. However, in the case of the spontaneously broken gauge symmetry, massive particles will play the role of an IR regulator and the limit $\varepsilon\to 0$ can safely be taken from the get-go such that the theory will safely reside in four space-time dimensions. IR divergences will now manifest themselves as logarithms of the particle mass m as $m\to 0$.

The $\mathcal{N}=4$ sYM on the Coulomb branch is also intricately connected to sYM theories in higher dimensions. In this correspondence, massless higher dimensional momenta of particles can be interpreted as massive four dimensional ones [20, 21, 25]. As a consequence, amplitudes and form factors with massive states in $\mathcal{N}=4$ sYM on the Coulomb branch are expected to be equivalent to their counterparts in $\mathcal{N}=1$ sYM in $D=10$ dimensions with loop momenta restricted to the $D=4$ space-time subspace [20, 21, 25]. A choice of VEVs can be enforced in such a manner that all external particles are truly massless in four dimensions, while nonvanishing masses emerge only for loop states propagating roughly around diagram perimeters. In this massive setup, IR factorization properties of scattering amplitudes and Sudakov form factors were briefly discussed in ref. [26]. Results obtained there were in line with general expectations about the structure of IR divergences in gauge theories. In particular, the leading IR behavior of form factors and amplitudes was controlled by the very same cusp anomalous dimension $\Gamma_{\rm cusp}$, as in the massless case.

Recently a duality was suggested [25], which relates correlation functions of half-BPS operators with infinitely-large R-charges to scattering amplitudes of massive particles, or

W-bosons, in planar $\mathcal{N} = 4$ sYM on the Coulomb branch in the regime when all states which propagate in internal loops are massless. Consistency and gauge invariance of such kinematical regime was advocated by the authors of [25] using the above massless/massive correspondence between sYM theories in various dimensions. This kinematical regime mimics a naive off-shell generalization of a purely massless scattering with unbroken gauge symmetries, so hereafter we will refer to it as *off-shell* to distinguish it from the one where massive particles are also present in quantum loops, such as discussed in refs. [21, 26]. Integrability allowed the authors of ref. [25] to obtain a closed-form all-loop expression for the scattering amplitude of four W-bosons starting from the four-point correlation function of very heavy half-BPS operators [27–29]. This conjecture was supported by a comparative analysis of the ten-dimensional null limit of the correlator's integrand with the D-dimensional integrands of four-point amplitudes up to four [30] and five [31] loops.

This conjecture allowed the authors of [25] to probe the IR behavior of the off-shell four point amplitude to all orders of perturbative series and to reveal quite an unexpected result: it turned out that the IR divergences in this case are *not* controlled by the Γ_{cusp}, as previously expected, but rather by a different function of the coupling, the so-called octagon anomalous dimension Γ_{oct}, which has made its debut in the four-dimensional null limit of the aforementioned four-point large R-charge correlator [27–29]. Further studies performed in ref. [32], this time involving a five-leg off-shell amplitude, supported these results and solidified the role of Γ_{oct} as the off-shell counterpart of Γ_{cusp}. These observations immediately raise the question about the structure of Sudakov form factors and IR factorization properties of amplitudes in the off-shell kinematical regime. This is a very important and nontrivial endeavor since these recent findings regarding IR behavior of the off-shell scattering amplitudes are in tension with what was expected previously in $\mathcal{N} = 4$ sYM [4] as well as in other gauge theories such as QCD [33, 34].

The aim of the current paper is to report details of a three-loop computation of the off-shell two-particle form factor and its Sudakov limit in planar $\mathcal{N} = 4$ sYM. This was first announced in a short note in ref. [35]. Based on this analysis, we confirm that the IR behavior of the off-shell Sudakov form factor in $\mathcal{N} = 4$ sYM is indeed controlled by Γ_{oct} rather than Γ_{cusp}. Moreover, we conjecture a closed-form all-order expression for the finite part of the off-shell Sudakov form factor as well: it is found to be proportional to the non-logarithmic "hard" function of the so-called null octagon $\mathbb{O}_0$, which in turn was introduced within the context of integrability based computation of the four point correlation functions with infinitely-large R-charges [28, 29].

Our subsequent consideration is organized as follows. In section 2 we provide a lightening overview of salient facts about IR properties of amplitudes and form factors in $\mathcal{N} = 4$ sYM at different points of the Coulomb branch, its origin and beyond. In section 3, we briefly recall the structure of the Sudakov form factor on the Coulomb branch up to two loop order, which was previously known in the literature. We further discuss certain assumptions made in our off-shell calculation. In section 4, we present detailed analysis and explicit results for the three-loop off-shell form factor and its double logarithmic limit. Based on this computation we also make an all-order conjecture for the off-shell Sudakov form factor. In section 5, we summarize our observations regarding IR factorization properties of Coulomb branch amplitudes in the off-shell kinematical regime and compare them with statements made in earlier literature. Finally, we summarize.

2 IR structure of amplitudes and form factors

The IR structure of amplitudes and form factors in gauge theories are closely related to each other. Therefore, before diving into the subject of form factors, it is instructive to recall key facts regarding the IR behavior of massless scattering amplitudes in $\mathcal{N}=4$ sYM at the origin of the moduli space. This situation is very well understood.

2.1 Origin of moduli space

As in any four-dimensional gauge theories with massless particles, scattering amplitudes in $\mathcal{N}=4$ sYM possess IR divergences. One can tame them with conventional dimensional regularization (or its supersymmetric version, dimensional reduction) by considering the theory in $D=4-2\varepsilon$ space-time instead. The IR singularities then manifest themselves as poles in the ε regulator. For a planar color-ordered n-particle amplitude A_n with arbitrary helicity content, it is convenient to define the ratio $M_n = A_n/A_n^{\text{tree}}$. One can then expect the following form of M_n to hold at all-loop orders [7, 9–11]:

$$\log M_n = -\frac{1}{4}\sum_{i=1}^{n}\sum_{\ell=1}^{\infty} g^{2\ell}\left[\frac{\Gamma^{(\ell)}_{\text{cusp}}}{(\ell\varepsilon)^2}+\frac{G^{(\ell)}}{(\ell\varepsilon)}\right]\left(\frac{\mu^2}{s_{ii+1}}\right)^{\ell\varepsilon} + \mathcal{F}_n(\{p_i\},a) + O(\varepsilon)\,, \tag{2.1}$$

where the perturbative series is furnished in terms of the D-dimensional 't Hooft coupling $g^2 = g^2_{\text{YM}}N(4\pi e^{-\gamma_{\text{E}}})^{\varepsilon}/(4\pi)^2$. The generalized Mandelstam invariants $s_{ii+1}=(p_i+p_{i+1})^2$ are built from particles' momenta p_i $(i=1,\dots,n)$, and μ is a mass parameter of the dimensional regularization. Last but far from being the least, $\Gamma^{(\ell)}_{\text{cusp}}$ and $G^{(\ell)}$ are some numerical transcendental coefficients. The function $\mathcal{F}_n$ depends on the helicity configuration of the amplitude A_n, kinematical invariants as well as the coupling constant, however, it depends neither on the parameter ε nor the scale μ. The $1/\varepsilon^2$-pole structure originates from the overlap of soft and collinear divergences, where each of them manifests itself individually as $1/\varepsilon$.

We see that IR divergences of the ratio M_n factorize and exponentiate. Their structure is universal for all amplitudes (i.e., independent of particular helicity configurations) and is controlled by two sets of coefficients: $\Gamma^{(\ell)}_{\text{cusp}}$ and $G^{(\ell)}$. These coefficients in turn define two functions of the 't Hooft coupling, the cusp $\Gamma_{\text{cusp}}(g)$ and collinear $G(g)$ anomalous dimensions. The first few terms in their perturbative expansion are

$$\begin{aligned}\Gamma_{\text{cusp}}(g) &= \sum_{\ell=1}^{\infty}\Gamma^{(\ell)}_{\text{cusp}} g^{2\ell} = 4g^2 - 8\zeta_2^4 + 88\zeta_4 g^6 + \dots\,,\\ G(g) &= \sum_{\ell=1}^{\infty} G^{(\ell)} g^{2\ell} = -4\zeta_3 g^4 + \left(32\zeta_5 + \frac{80}{3}\zeta_2\zeta_3\right) g^6 + \dots\,.\end{aligned} \tag{2.2}$$

The above formulas exhibit standard folklore that the leading IR behavior (i.e. $1/\varepsilon^2$ poles) of planar amplitudes in $\mathcal{N}=4$ sYM at the origin of moduli space is controlled by $\Gamma_{\text{cusp}}(g)$.

Not only the cusp anomalous dimension, but the whole IR divergent part of the amplitude can be independently defined in terms of matrix elements of some local operators. Indeed one can show that the divergent part of the amplitude is given by the product of Sudakov form factors F_2, which are determined here as matrix elements of an operator from the $\mathcal{N}=4$ sYM stress-tensor supermultiplet and a pair of on-shell states. The lowest component of this supermultiplet is given by the operator $\mathcal{O}=\mathrm{tr}(\phi_{12}\phi_{12})$, built out of two scalar fields $\phi_{AB}=\phi^a_{AB}t_a$ from the $\mathcal{N}=4$ sYM Lagrangian in the **6** representation of $\mathrm{SU}(4)_\mathrm{R}$ with t_a being the $\mathrm{SU}(N)$ generators in the fundamental representation. So one can define F_2 as:

$$F_2=\langle 0|(\hat{a}^\dagger)^a_{12,p_1}(\hat{a}^\dagger)^a_{12,p_2}\mathcal{O}|0\rangle/\langle 0|(\hat{a}^\dagger)^a_{12,p_1}(\hat{a}^\dagger)^a_{12,p_2}\mathcal{O}|0\rangle_{\mathrm{tree}}\,. \tag{2.3}$$

Here $(\hat{a}^\dagger)^a_{12,p_i}$ is a creation operator of the on-shell scalar ϕ^a_{12} with momentum p_i, $p_i^2=0$. One can demonstrate that F_2 will in fact be identical to all other operators from the stress-tensor supermultiplet and all possible pairs of particles from the on-shell $\mathcal{N}=4$ sYM supermultiplet [36, 37]. Information about operator type and particle helicities is encoded in $\langle 0|(\hat{a}^\dagger)^a_{12,p_1}(\hat{a}^\dagger)^a_{12,p_2}\mathcal{O}|0\rangle_{\mathrm{tree}}$, which we factor out. Then one can rewrite (2.1) as:

$$\log M_n=\frac{1}{2}\sum_{i=1}^{n}\log F_2\left(\frac{\mu^2}{s_{ii+1}},g,\varepsilon\right)+\mathcal{F}_n(\{p_i\},g)-n\,c(g)+O(\varepsilon), \tag{2.4}$$

with:

$$\log F_2\left(\frac{\mu^2}{q^2},g,\varepsilon\right)=-\frac{1}{2}\sum_{\ell=1}^{\infty}g^{2\ell}\left[\frac{\Gamma^{(\ell)}_{\mathrm{cusp}}}{(\ell\varepsilon)^2}+\frac{G^{(\ell)}}{(\ell\varepsilon)}+c^{(\ell)}\right]\left(\frac{\mu^2}{q^2}\right)^{\ell\varepsilon}+O(\varepsilon). \tag{2.5}$$

Here $q=p_1+p_2$ is the off-shell momentum, $q^2\neq 0$, carried by the operator in question and $c(g)=\sum_\ell c^{(\ell)}g^{2\ell}$, where $c^{(\ell)}$ are some (potentially) scheme-dependent constants. These factorization theorems for F_2 and M_n were tested by multiple explicit computations of M_n for some values of n and F_2 and are in perfect agreement with each other, see, e.g., [7, 8, 15, 38]. They are also supported by general theoretical arguments which map IR behavior of amplitudes to UV behavior of cusped Wilson lines [12, 13]. Standard renormalization group (RG) machinery can be applied to tackle their UV behavior [33, 34]. For example, the all-order structure of F_2 (2.5) is a result of such analyses [7, 10, 33]. For illustrative purposes, scalar Feynman integrals contributing to M_4 and F_2 in the first two orders of perturbative series are displayed in figure 1. It is also worth to mention explicitly that because the operator insertion in definition of Sudakov form factor is color singlet, even in the planar limit there will be contributions of non-planar graphs to the Sudakov form factor in contrast to the planar nature of amplitude case. This fact makes relations (2.4) between amplitudes and form factor especially nontrivial.

In gauge theories with less or without supersymmetry, such as QCD, IR factorization relations similar to (2.4) will also hold, but their explicit structure will be more involved since one will have to take the running of coupling into consideration [10, 11, 33, 34].

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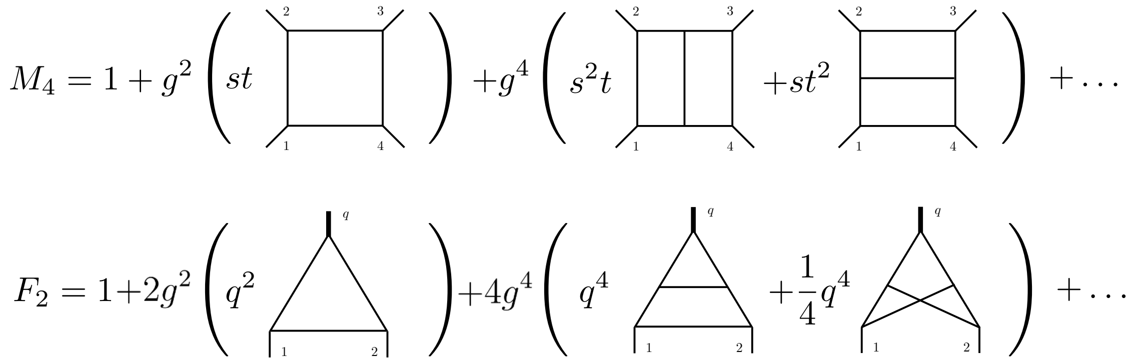

Figure 1. Two-loop expansion of the four-leg amplitude (top panel) and the two-leg form factor (bottom panel) in terms of scalar integrals.

JHEP11(2023)111

2.2 Coulomb branch

As we already mentioned in the introductory section, in addition to the coupling constant $g_{\rm YM}$ and the number of colors N, the $\mathcal{N}=4$ sYM with SU(N) gauge group possesses another set of free adjustable parameters,— the VEVs of the six real scalar fields ϕ_{AB} of the theory, aka moduli. In principle, there are no restrictions on their values and one can consider the theory at any point in its moduli space. So what about the above amplitude story away from its origin? As was advocated in [25], it is convenient to use the aforementioned D-dimensional framework together with observations that the integrands of $n=4$ and $n=5$ of planar amplitudes (or rather the ratio functions M_n), have universal structure shared among sYM theories [39–41]. Then by imposing kinematical constraints on the D-dimensional integrand, one can obtain integrands in $\mathcal{N}=4$ sYM at a nontrivial position in its moduli space away from its origin.

Ref. [25] found that it is useful to invoke the D-dimensional dual coordinates X_i to impose the aforementioned kinematical constraints. These are related to particles' momenta as $p_i = X_{ii+1} \equiv X_i - X_{i+1}$. Namely, all loop integrations $d^D X_l$ are accompanied by the constraint $\delta^{D-4}(X_\ell)$. This effectively decomposes all propagator denominators $X^2_{i\ell}$ into the four- and extra-dimensional components, $X^2_{i\ell} = x^2_{i\ell} + y^2_i$, where y^2_i must be identified in turn with particle masses $y^2_i \equiv m^2_i$ generated by a specific pattern of gauge symmetry breaking. Momenta of external particles encoded in $p^2_i = X^2_{ii+1} = 0$ should also be decomposed accordingly, $X^2_{ii+1} = x^2_{ii+1} + y^2_{ii+1}$, where once again $D > 4$ part is regarded as their mass $y^2_{ii+1} = m^2_{ii+1}$, see figure 2. The specifics of the VEV choice is not relevant for our discussion,— as long as it is possible —, so we sweep under the rug these irrelevant details about concrete patterns of gauge symmetry breaking, structure of the R-symmetry group after the latter took place etc. All this information is contained in $A_n^{\rm tree}$ amplitude, which we factor out anyway.

As an illustration, let us consider the contribution of the one-loop box integral from figure 1:

$$\int \frac{d^D X_\ell \, \delta^{D-4}(X_\ell)\, X^2_{13} X^2_{24}}{X^2_{1\ell} X^2_{2\ell} X^2_{3\ell} X^2_{4\ell}} \mapsto \int \frac{d^4 x_\ell \, X^2_{13} X^2_{24}}{(x^2_{1\ell}+y^2_1)(x^2_{2\ell}+y^2_2)(x^2_{3\ell}+y^2_3)(x^2_{4\ell}+y^2_4)}. \tag{2.6}$$

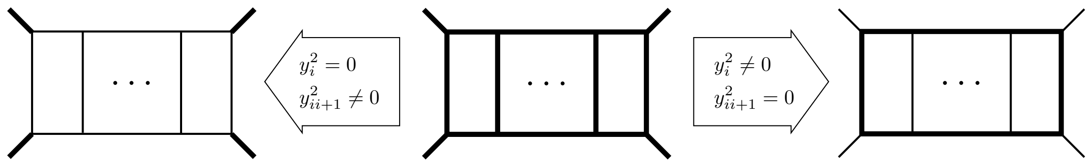

Figure 2. Various kinematical regimes discussed in the text stemming from a D-dimensional progenitor.

There exist two essentially different possibilities. One can consider the situation where all $m_{ii+1}^2 = 0$. This will correspond to the scattering of massless particles via massive states propagating in loops, such as in the four-photon scattering amplitude in QED. Here $y_i^2 = m_i^2$ will play the role of the IR regulator, and to investigate the IR properties of the amplitude one must evaluate loop integrals in $D = 4$ and then approach the limit $m_i^2 \equiv m^2 \to 0$. For eq. (2.6), this will yield

$$\int \frac{d^4 x_\ell \, x_{13}^2 x_{24}^2}{(x_{1\ell}^2 + m^2)(x_{2\ell}^2 + m^2)(x_{3\ell}^2 + m^2)(x_{4\ell}^2 + m^2)} , \tag{2.7}$$

where everything is four dimensional and $x_{ii+1}^2 = p_i^2 = 0$. In particular, these last conditions imply that $x_{13}^2 x_{24}^2$ is now equal to $s_{12}s_{23}$.

Another interesting option is to consider the opposite limit and put all $y_i^2 = m_i^2 = 0$ first. Then, the external masses m_{ii+1}^2 instead will play the role of an IR regulator. Once again, it is implemented in such a manner that all integrals are evaluated in $D = 4$ and then the limit $m_{ii+1}^2 \equiv m^2 \to 0$ is taken. Physically, this situation corresponds to the scattering of massive W-bosons in the limit where we neglect all masses of states propagating in quantum loops. For our one-loop box (2.6), this limit provides:

$$\int d^4 x_\ell \frac{x_{13}^2 x_{24}^2}{x_{1\ell}^2 x_{2\ell}^2 x_{3\ell}^2 x_{4\ell}^2} , \tag{2.8}$$

where every Lorentz invariant is four-dimensional and $x_{ii+1}^2 = p_i^2 = -m^2 \to 0$. This integral can be re-expressed in terms of the Davydychev-Usyukina one loop box function [42, 43]. Since the scalar integral families as well as their accompanying numerical coefficients will be identical to the on-shell case for $n = 4, 5$, this kinematical regime will be in fact identical to the naive off-shell regularization of purely massless results. This was briefly discussed in the earlier literature, see, e.g., [4]. At that time such an apparently naive off-shell continuation was obscured by potential problems with gauge invariance and, thus, overall consistency of such a procedure. Relations to higher dimensional sYM theories were not (widely) known or explored back then.

Let us consider the situation with the scattering of massless external particles via massive virtual states first. Using the aforementioned prescription, explicit $n = 4$ three loop and $n = 5$ two loop computations were performed[1] [21–23]. Results of these computations

[1]At two-loop level, they were also verified by explicit Feynman diagram calculations [21].

allow one to conjecture the following IR factorization formula for the ratio function M_n:

$$\log M_n = -\frac{1}{8}\sum_{i=1}^{n}\left[\Gamma_{\text{cusp}}(g)\log^2\left(\frac{m^2}{s_{ii+1}}\right) + 2\widetilde{G}(g)\log\left(\frac{m^2}{s_{ii+1}}\right)\right] + \mathcal{F}_n(\{p_i\},g) + O(m^2)\,. \tag{2.9}$$

Here $\widetilde{G}(g)$ may potentially be different from the pure massless case (2.1). Based on the results of refs. [21–23], one can also expect that the hard function $\mathcal{F}_n$ here is identical to the purely massless case. We see that this situation is essentially equivalent to the massless case (i.e., the theory at the origin of its moduli space) with the replacement of $1/\varepsilon$ poles with $\log m^2$. Leading IR logarithms are still controlled by $\Gamma_{\text{cusp}}(g)$, which is in line with the folklore that $\Gamma_{\text{cusp}}(g)$ is "the ultimate IR anomalous dimension" and all IR limits of the theory should be controlled by $\Gamma_{\text{cusp}}(g)$ of that theory. It is also curious to mention for eq. (2.9) to hold, it is sufficient to retain nonvanishing m_i^2 only in propagators which form a closed frame around graph sites. All other masses can be considered strictly set to zero, see, e.g., the right-hand side of figure 2 for the $n = 4$ amplitude.

The opposite situation of the massive particle scattering via massless virtual states revealed, however, a different picture. Based on the three loop $n = 4$ and two loop $n = 5$ computations [25, 32], one can conjecture the following IR factorization formula:

$$\log M_n = -\frac{1}{4}\sum_{i=1}^{n}\Gamma_{\text{oct}}(g)\log^2\left(\frac{m^2}{s_{ii+1}}\right) + \widetilde{\mathcal{F}}_n(\{p_i\},g) + O(m)\,, \tag{2.10}$$

with $\Gamma_{\text{oct}}(g)$ being a different function of 't Hooft coupling compared to $\Gamma_{\text{cusp}}(g)$:

$$\Gamma_{\text{oct}}(g) = 4g^2 - 16\zeta_2 g^4 + 256\zeta_4 g^6 + \dots\,, \tag{2.11}$$

and (potentially) different $\widetilde{\mathcal{F}}_n$ compared to pure massless case. More accurately for $n = 4, 5$ examples the kinematical dependence of $\widetilde{\mathcal{F}}_n$ and $\mathcal{F}_n$ was identical, but the dependence on coupling constant g was different, which can be captured by $\Gamma_{\text{oct}} \mapsto \Gamma_{\text{cusp}}$ replacement.

This unexpected result immediately raises the question about the IR factorization properties of amplitudes in the off-shell kinematical regime, i.e., can their IR divergent parts be captured by the product of the off-shell Sudakov form factors? The ultimate goal of this article is to shed light on these questions and we will address them in details in the next sections.

3 Form factors in $\mathcal{N} = 4$ sYM

Before discussing the off-shell regime for production of external (massive) particles by the operator $\mathcal{O}$ from the vacuum, let us briefly discuss the opposite situation for scattering of massless particles via massive virtual states.

3.1 Coulomb branch: massive internal lines

For simplicity,we will choose a single mass $y_i^2 = m^2 \ll 1$ for all i and then take the limit $m^2 \to 0$. In the pure massless case with the unbroken gauge symmetry, the Sudakov form

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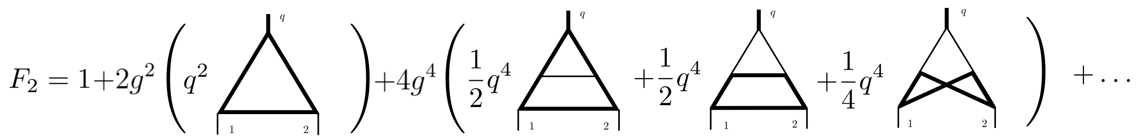

Figure 3. Scalar integrals determining the second order perturbative expansion of the on-shell form factor regularized by massive internal lines.

factor F_2 reads up to two loops

$$F_2 = 1 + g^2 F_2^{(1)} + g^4 F_2^{(2)} + \dots , \tag{3.1}$$

in terms of the following set of scalar integrals, shown graphically in figure 1,

$$\begin{aligned} F_2^{(1)} &= 2Q^2\, T_{1,1}^{\text{on-shell}}, \\ F_2^{(2)} &= 4Q^4 \left[T_{2,1}^{\text{on-shell}} + \frac{1}{4}\, T_{2,2}^{\text{on-shell}} \right], \end{aligned} \tag{3.2}$$

where we introduced Euclidean momentum transfer $Q^2 = -q^2 > 0$. These results for $p_i^2 = 0$ were (re)derived by different methods by multiple authors [15, 16, 44]. As in the amplitude case, one can argue that the integrands in these expansions are identical among sYM theories in all D's [16]. So it is tempting to use the massless/massive prescription of [25] to obtain scalar integral representation for form factors as well starting from eq. (3.2). One faces an immediate obstacle, however, due to the presence of non-planar[2] graphs, e.g., $T_{2,2}^{\text{on-shell}}$ is the first example of such scalar integral. A general consensus is that there is no well-defined way to introduce dual coordinates for such integrals. Despite this fact, one can still insert masses in massless propagators in non-planar $D = 4$ dimensional integrals "by hand" akin to how it is done for planar integrals, assuming that there is a way to choose D-dimensional momenta to replicate these mass insertions. Hereafter, we will adopt exactly this hands-on approach for obtaining candidates integrals for massive integrands from purely massless ones.

In current case, there are two possible ways to insert massive propagators to form closed frames in the $T_{2,1}^{\text{on-shell}}$ integral, which will lead to finite four-dimensional results. It is natural to consider both of them with equal apportions, i.e., $\frac{1}{2}$ coefficients, see figure 3. The choice of the massive frame in the nonplanar integral $T_{2,2}^{\text{on-shell}}$, which make this integral finite in $D = 4$, is unique. This results in the following conjectured expansion for the on-shell form factor with massive internal loops:

$$\begin{aligned} F_2^{(1)} &= 2Q^2\, T_{1,1}^{\text{massive}}, \\ F_2^{(2)} &= 4Q^4 \left[\frac{1}{2} T_{2,1}^{\text{massive},a} + \frac{1}{2} T_{2,1}^{\text{massive},b} + \frac{1}{4} T_{2,2}^{\text{massive}} \right], \end{aligned} \tag{3.3}$$

[2]These graphs are endowed nevertheless by leading color structures due to the fact that the operator vertex $\mathcal{O}$ is a unity matrix in the SU(N) group space contrary to other external legs which are in its adjoint representation.

where, as was explained earlier, $T_{i,j}^{\text{massive}}$ are scalar integrals corresponding to the very same graphs as $T_{i,j}^{\text{on-shell}}$ in (3.2), but massive instead of massless propagators, see figure 3. This decomposition can in fact be derived [26] in the standard perturbation theory in $\mathcal{N}=4$ sYM where a specific pattern of spontaneous gauge symmetry breaking is chosen, see refs. [21–23] for details. This provides a solid endorsement for our approach of uplifting massless contributions off of their mass shell.

The small-mass expansion for all $T_{i,j}^{\text{massive}}$ integrals leads to the following result [26]:

$$\log F_2\left(\frac{-m^2}{Q^2}, g\right) = -\frac{1}{4}\Gamma_{\text{cusp}}(g)\log^2\left(\frac{-m^2}{Q^2}\right) - \widetilde{G}(g)\log\left(\frac{-m^2}{Q^2}\right) + \widetilde{c}(g) + O(m^2)\,. \quad (3.4)$$

Here as before $\Gamma_{\text{cusp}}(a)$ and $\widetilde{G}(a)$ identical to those of (2.2) at this order of perturbative series, i.e., two loops.

One can expect that the structure of (2.9) will hold to all orders in g with the same $\Gamma_{\text{cusp}}(g)$ but (potentially) different $\widetilde{G}(g)$ and $\widetilde{c}(g)$ compared to eq. (2.5). This means that the factorization formula on this massive Coulomb branch will be identical to the one of the massless case (2.4). The structure of the Sudakov form factor itself is also very similar to purely massless case with bold replacements of all poles $1/\varepsilon$ by logarithms $\log m^2$.

The Sudakov form factor satisfies an evolution equation [4, 33, 34], which in the massless case is given by

$$\left(\frac{\partial}{\partial \log \mu^2}\right)^2 \log F_2 = -\frac{1}{2}\Gamma_{\text{cusp}}(g)\,. \quad (3.5)$$

We see from (3.4) that in the case of the massive kinematical regime, this evolution equation will be intact provided on replaces $\log \mu^2 \mapsto \log m^2$. At the time of its derivation, this result was completely in line with general expectations that all IR physics in gauge theories under consideration must be controlled by Γ_{cusp} of that theory and that Γ_{cusp} is the "ultimate anomalous dimension" of IR physics. The appearance of Γ_{cusp} in the form factors and amplitudes in this massive kinematical regime was considered for granted and was interpreted as the consequence of scheme independence of Γ_{cusp} [21].

3.2 Coulomb branch: off-shell regime

Let us now return to the main observable of our interest, i.e., the Sudakov form factor in the off-shell kinematical regime. Based on the two-loop purely massless result (3.3) and making use of the D-dimensional approach described in the previous sections, we can conjecture that the perturbative expansion of the Sudakov form factor of W-bosons

$$F_2 = 1 + g^2 F_2^{(1)} + g^4 F_2^{(2)} + g^6 F_2^{(3)} + \dots\,, \quad (3.6)$$

is determined up to two loop order by the following scalar integrals

$$\begin{aligned} F_2^{(1)} &= 2Q^2 T_{1,1}\,, \\ F_2^{(2)} &= 4Q^4\left[T_{2,1} + \frac{1}{4}T_{2,2}\right]\,, \end{aligned} \quad (3.7)$$

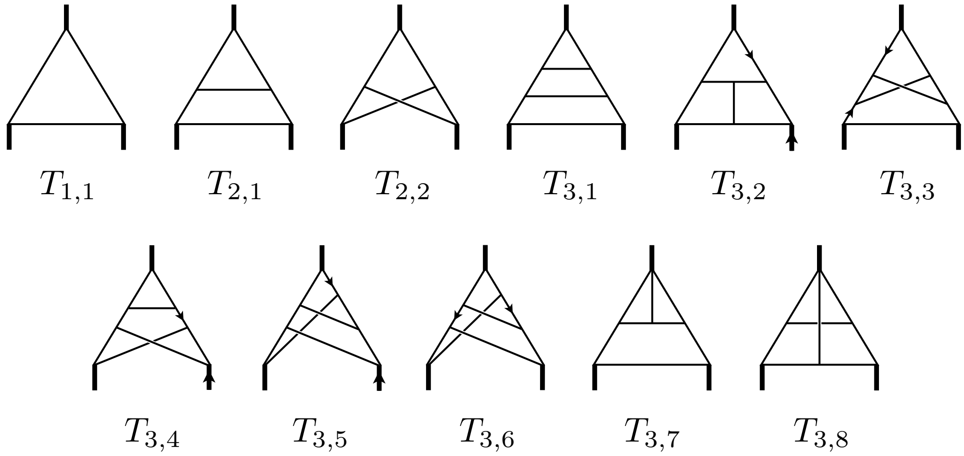

Figure 4. Scalar integrals contributing to the three-loop off-shell form factor. Arrows on the lines corresponds to the presence of the numerator $(p_a + p_b)^2$, where p_a and p_b are the momenta flowing through corresponding lines.

with very same graphs representation as in (3.2), but now with momenta of external particles being p_i massive, or off-shell. On the other hand, all internal lines (propagators) are massless, see figure 4. Note that we currently choose $m^2 = -p_i^2$ in contrast to the previous section, where m was the mass parameter in the massive propagators $1/(k^2 - m^2)$.

In the following, we will use the notation[3] $t \equiv m^2/Q^2 > 0$ for the dimensionless ratio entering all integrals. The one- and two-loop integrals $T_{1,1}$ and $T_{2,1}$, $T_{2,2}$ can be expressed in terms of the Davydychev-Usyukina ℓ-loop box ladder functions $\Phi_\ell(x,y)$ [42, 43, 45] as follows

$$Q^2 T_{1,1} = -\Phi_1(t,t)\,, \qquad Q^4 T_{2,1} = \Phi_2(t,t)\,, \tag{3.8}$$

$$Q^4 T_{2,2} = [\Phi_1(t,t)]^2\,, \tag{3.9}$$

where we grouped them together according to their (non)planarity. In turn, the Davydychev-Usyukina functions $\Phi_\ell(x,y)$ can be solely written in terms of (poly)logarithms,

$$\Phi_\ell(x,y) = -\sum_{j=\ell}^{2\ell} \frac{j!(-1)^j \log^{2\ell-j}\left(\frac{y}{x}\right)}{\ell!(j-\ell)!(2\ell-j)!} \frac{\mathrm{Li}_j\big(-(\rho x)^{-1}\big) - \mathrm{Li}_j\big(-(\rho y)^{+1}\big)}{\lambda}\,, \tag{3.10}$$

with ρ and λ being functions of x and y,

$$\lambda(x,y) = [(1-x-y)^2 - 4xy]^{1/2}\,, \qquad \rho(x,y) = 2[1-x-y-\lambda(x,y)]^{-1}. \tag{3.11}$$

It is worth to realize that eq. (3.9) is highly non-trivial since it re-expresses the nonplanar integral $T_{2,2}$ in terms of the planar $T_{1,1}$. It is well-known that all planar scalar integrals,[4]

[3]We hope that there will be no confusion with a Mandelstam invariant for the four-point amplitude.

[4]The integral $T_{2,1}$ can be obtained from the double box integral using a limiting procedure from refs. [42, 43]. The same is applicable to other planar integrals in eq. (3.6), at least up to three loops.

JHEP11(2023)111

including $T_{2,1}$ above, are invariant[5] with respect to the so-called dual conformal symmetry, i.e., conformal boosts in the momentum space [4], which is a harbinger of integrability of $\mathcal{N} = 4$ sYM. We will see in the next section that relations akin (3.9) between non-planar and planar integrals is likely to be a general pattern of the Sudakov form factor in $\mathcal{N} = 4$ in the off-shell kinematical regime.

For the first two ℓ's, i.e., $\ell = 1, 2$, the small-t expansion of the Davydychev-Usyukina functions immediately reads

$$\begin{aligned}\Phi_1(t,t) &= \log^2 t + 2\zeta_2 + O(m^2),\\ \Phi_2(t,t) &= \frac{\log^4 t}{4} + 3\zeta_2 \log^2 t + \frac{21\zeta_4}{2} + O(m^2)\,.\end{aligned} \tag{3.12}$$

Let us direct the reader's attention to the fact that contrary to the case of the dimensional regularization, used in the purely massless cases, there is no analog of $\varepsilon \times 1/\varepsilon$-interference between different orders in coupling constant g, and relation like eq. (3.12) are sufficient to completely determine $\log F_2$ up to terms $O(m^2)$ and three loop accuracy. At this point, let us quote the expansion for $\Phi_3(t,t)$ which will be useful in what follows

$$\Phi_3(t,t) = \frac{\log^6 t}{36} + \frac{5\zeta_2}{6}\log^4 t + \frac{35\zeta_4}{2}\log^2 t + \frac{155\zeta_6}{4} + O(m^2)\,. \tag{3.13}$$

Substituting these in $\log F_2$ and expanding, in turn, $\log F_2$ in powers of g up to two-loop accuracy, we obtain [32]:

$$\log F_2\,(t,g) = \left[-2g^2 + 8\zeta_2 g^4 + \dots\right]\log^2 t + \left[-4\zeta_2 g^2 + 32\zeta_4 g^4 + \dots\right] + O(m^2)\,. \tag{3.14}$$

The structure of this result is in line with general expectations about exponentiation of IR logarithms, however, there are some important differences. Indeed we see that IR logarithms exponentiate but the coefficient accompanying $\log^2 m^2$ is different from $-\Gamma_{\rm cusp}(g)$ divided by 4. Note also that the analogue of the collinear anomalous dimensions $G(g)$, that is $\widetilde{G}(g)$, is completely missing in this case, compared to (2.9). A naked eye inspection of the coefficient accompanying the double logarithm as well as the finite piece [32], allows one to verify that these are in agreement with the leading two terms of the perturbative expansion of the null octagon encoded by the two functions of 't Hooft coupling, which are known exactly in terms of elementary functions [28, 29, 46]

$$\Gamma_{\rm oct}(g) = \frac{2}{\pi^2}\log\left[\cosh(2\pi g)\right] = -4g^2 + 16\zeta_2 g^4 + \dots\,, \tag{3.15}$$

$$D(g) = \frac{1}{4}\log\left[\frac{\sinh(4\pi g)}{4\pi g}\right] = 4\zeta_2 g^2 - 32\zeta_4 g^4 + \dots\,. \tag{3.16}$$

The above off-shell kinematical regime was discussed in early literature, see, e.g., [4, 33]. It was predicted there that the coefficient before $\log^2 m^2$ should merely be given by $-\Gamma_{\rm cusp}$, that is, twice larger compared to the purely massless case (2.5) and (6.1). At one loop this is indeed the case, and the origin of the doubling is very well understood [47–50]. The

[5] Of course, these have to be accompanied by certain prefactors built out from external particle momenta to render them dimensionless.

latter is a consequence of an additional integration domain over the loop momentum in $T_{1,1}$ integral, dubbed the ultra-soft regime, compared to the on-shell case. This domain provides leading contribution on par with the soft-collinear regions intrinsic to both on- and off-shell integrals. However, as we observe, at higher loop orders this simple doubling relation is no longer true. Thus, a calculation of the three loop correction is highly desirable to clarify this clash and to support the claim that the coefficient of $\log^2 m^2$ is indeed proportional to $\Gamma_{\rm oct}$ rather that $\Gamma_{\rm cusp}$. This is what we are set to demonstrate in the next section.

4 Off-shell form factor at three loops and beyond

In this section, we present the main result of the current work.

4.1 Integral basis at three loops

The massless form factor at three loop order was first evaluated in [16]. We employ their basis of scalar integrals and cast this contribution $F_2^{(3)}$ to eq. (3.6) into the form

$$F_2^{(3)} = 8Q^4 \left[Q^2 T_{3,1} - \frac{1}{4} T_{3,2} + \frac{1}{2} T_{3,3} + \frac{1}{2} T_{3,4} - \frac{1}{2} T_{3,5} - \frac{1}{2} T_{3,6} - \frac{1}{2} T_{3,7} + \frac{1}{4} T_{3,8} \right], \tag{4.1}$$

see figure 4 for graphical representation of individual $T_{3,i}$'s. Our goal is to evaluate $T_{3,i}$. More precisely, we are interested in their small-m behavior up to $O(m^2)$.

Before we dive into technicalities, let us make a general comment about the task at hand. Of course, the most straightforward approach to the calculation would be to compute corresponding parametric integrals directly in four dimensions since all $T_{3,i}$'s are finite. This can immediately be accomplished with existing software packages such as, for instance, `HyperInt` [51, 52]. However taking into account the presence of integrals with irreducible numerators,[6] e.g., $T_{3,3}$, `HyperInt` is not an optimal code to tackle them. We found that it is more efficient to use other approaches, in particular, the method of differential equations and the method of regions. These two independent techniques were used in parallel to cross-check correctness of devised expressions. Both of these require, however, lifting considerations away from four dimensions to $D = 4 - 2\varepsilon$. In the former case, it is needed to render the system of master integrals complete, while in the latter, separation of integrals into various asymptotic regions makes them individually divergent and recover a finite expression only in their sum. We will discuss them in turn.

4.2 Evaluating $T_{3,i}$ integrals with differential equations

The first approach that we used is based on solving differential equations [55–57] for these integrals at general values of Q^2 and m^2, with $p_1^2 = p_2^2 = -m^2$. In this manner, we obtained three-loop corrections in exact kinematics, i.e., not just the limit $m \to 0$. In fact we adopted

[6] We tested `HyperInt` for some of the integrals without numerators in eq. (4.1). They turned out to be linearly reducible and expressible in terms of Goncharov polylogarithms [53] for a general value of m. They were further expanded at small m using `HyperInt` functions. Thus obtained expressions were confirmed by other methods that we employed for all $T_{3,i}$, with or without irreducible numerators. One can expect that all $T_{3,i}$ integrals are also linearly reducible [54].

JHEP11(2023)111

an earlier analysis from ref. [58] where these integrals were calculated at the symmetrical point, $p_1^2 = p_2^2 = q^2 = -1$. The homogeneity of $T_{3,i}$ in the two kinematical invariants allows us to factor out their mass dimension in terms of Q^2 and thus focus solely on their dependence on the dimensionless ratio $t = m^2/Q^2$.

The first step on the way to find the so-called canonical form of the differential equations [57]

$$\partial \boldsymbol{J} = \varepsilon \boldsymbol{A} \cdot \boldsymbol{J} \,, \tag{4.2}$$

is to perform an IBP reduction for all $T_{3,i}$ in order to reveal the initial set of master integrals $\boldsymbol{I}$ involved in their representation. This can be achieved by any software available for this purpose. We relied on `FIRE6` [59] and `LiteRed` [60]. For the resulting basis of preliminary master integrals, we then constructed a set of differential equation in the t-variable. To transform the resulting differential equations to a Fuchsian (aka `dLog`) and ε-form, it is convenient to introduce a new variable by $t = -(z-1)^2/z$. This choice rationalizes potential square roots in the matrix $\boldsymbol{A}$. Then the singular points $t = \{1, 0, \infty, 4\}$ of the latter are transformed into $z = \{\exp(\frac{i\pi}{3}), 1, 0, -1\}$. Let us emphasize that the appearance of the sixth root of unity $\sigma = \exp(\frac{i\pi}{3})$ in this problem is a feature that shows up for the first time at the three-loop order.

Making use of the public codes `CANONICA` [61] and `Libra` [62, 63], we transformed the initial system of differential equations for the integrals $\boldsymbol{I} = \boldsymbol{T} \cdot \boldsymbol{J}$ first to the `dLog` and ultimately to the ε-form, with the $\boldsymbol{A}$ matrix having only Fuchsian singularities

$$\boldsymbol{A} = \frac{\boldsymbol{R}_0}{z} + \frac{\boldsymbol{R}_1}{z-1} + \frac{\boldsymbol{R}_{-1}}{z+1} + \frac{\boldsymbol{R}_\sigma}{z-\sigma} + \frac{\boldsymbol{R}_{\sigma^*}}{z-\sigma^*} \,. \tag{4.3}$$

The main advantage of the canonical form of the resulting differential equations is that their general solution

$$\boldsymbol{J} = P \exp\left(\varepsilon \int dz \boldsymbol{A}\right) \cdot \boldsymbol{J}_0 \,, \tag{4.4}$$

when expanded in the power series in ε can easily be integrated order-by-order up to the desired weight-six contributions. The results are naturally expressed in terms of multiple Goncharov polylogarithms [53] or, presently, via harmonic polylogarithms (HPL) [64]. The integration constants $\boldsymbol{J}_0$ were fixed with the help of boundary conditions in the limit where one of the two external momenta was considered large in the Euclidean sense. In this limit well-known graph-theoretical methods can be used to perform systematic expansions. These prescriptions are implemented in the computer codes `EXP` [65, 66] and `MINCER` [67, 68], which we relied on.

As a result of our analysis, we deduced the analytical form for all integrals $T_{3,i}$ required for three-loop off-shell form factor. For example, we obtained for $T_{3,7}$

$$\begin{aligned} Q^4 T_{3,7} = &-\frac{40\,\mathrm{H}_6}{(t-1)^3(t+1)t^2} + \frac{4\pi^4\,\mathrm{H}_{0,0}}{9(t-1)^3(t+1)t^2} + \frac{20\pi^2\,\mathrm{H}_{0,0,0,0}}{3(t-1)^3(t+1)t^2} \\ &+ \frac{20\pi i\,\mathrm{H}_{0,0,0,0,0}}{(t-1)^3(t+1)t^2} - \frac{20\,\mathrm{H}_{0,0,0,0,0,0}}{(t-1)^3(t+1)t^2} + \frac{8\pi^6}{189(t-1)^3(t+1)t^2} \,, \end{aligned} \tag{4.5}$$

where we used the conventional nomenclature

$$\mathrm{H}_{a_1,\dots,a_n} \equiv \mathrm{HPL}_{a_1,\dots,a_n}(t)\,. \tag{4.6}$$

The rest of integrals are relegated to the `Mathematica` notebook `3loopTs.nb` in the supplementary material attached to this paper. Eventually, these were expanded in the limit $t \to 0$. Since all contributions are expressed in terms of HPL, this is a rather straightforward task and can be systematically performed by means of the so-called shuffle relations. For instance, for $T_{3,7}$ we find[7]

$$Q^4 T_{3,7} = \frac{\log^6 t}{36} + \frac{5\zeta_2}{6}\log^4 t + \frac{35\zeta_4}{2}\log^2 t + \frac{155\zeta_6}{4} + O(m^2)\,. \tag{4.7}$$

It is curious to observe that this turns out to be the small-t expansion of the Davydychev-Usyukina function $\Phi_3(t,t)$ in disguise. Explicit form of these series for all other integrals $T_{3,i}$ can be found in appendix A.

4.3 Evaluating $T_{3,i}$ integrals with expansion by regions

In order to cross check our findings, we relied on yet another method to evaluate the small-t expansion of the three-loop integrals. It is based on a strategy of the expansion by regions [69] (see also [70, 71]). It was originally introduced to tackle threshold expansions of Feynman integrals [69] and later generalized to any limit. The limit under consideration in the present work, i.e., $t \to 0$, is intrinsic to Minkowski space-time and cannot be formulated in Euclidean kinematics.[8] The essence of the expansion by region consists in classification of loop-momentum integrands with regard to their scaling in a small parameter involved, i.e., t for the case at hand. The contributions of these so-called regions are then evaluated according to the instructions formulated in ref. [69] by extending loop integrals to the *entire* infinite space without any kinematical restrictions. Setting to zero all emerging scaleless integrals, one obtains desired asymptotic expansion by summing up non-vanishing contributions of the regions.

Remarkably, this fairly dubious procedure formulated in the momentum space works extremely well in practice. However, insisting on the momentum-space language it proves rather difficult to reveal all regions in a given limit following the decomposition of loop momenta in terms of hard, collinear, soft and ultrasoft. It is the use of the Feynman parametric representation, see eq. (4.8) below, which allowed one to alleviate this drawback and provided the possibility to develop a systematic algorithm [77–79] and, moreover, to implement it in the computer code `asy` [78, 79]. Within this algorithm, relevant regions correspond to facets (i.e., faces of maximal dimension) of a Newton polytope connected with two Symanzik polynomials in the Feynman parametric representation. In fact, the expansion by regions can be applied with the use of the code `asy` to any parametric integrals over[9]

[7] If the appropriate branch of HPL functions is chosen.

[8] Limits of Feynman integrals typical to Euclidean space, such as a off-shell large momentum expansion, receive support from the Wilson operator product expansion. The latter can be formulated in a graph-theoretical language as in refs. [72–75], see [70, 76] for comprehensive reviews.

[9] For other domains, one should first map it to R_+^N and then proceed with `asy`. An example of its application to integrals, which are not Feynman integrals, can be found in [80] where the initial integration domain was a multidimensional unit cube.

R_+^N of products of polynomials raised to powers linearly depending on the regularization parameter ε. Expansion by regions has up to now the status of experimental mathematics. However, to date there are no known examples where it fails. Let us refer to [81, 82] for discussions of possible ways to prove this strategy.

For Feynman integrals with integrands determined by a product of N propagators $1/(p^2 - m^2 + i0)^{a_i}$, the corresponding Feynman parametric representation is an integral over a projective R_+^N,

$$I_{a_1,\ldots,a_N} = \frac{\left(i\pi^{D/2}\right)^L \Gamma(a - LD/2)}{\prod_i \Gamma(a_i)} \int_0^\infty \prod_{i=1}^N x_i^{a_i-1} \mathrm{d}x_i \delta\left(\sum x_i - 1\right) U^{a-(L+1)D/2} F^{LD/2-a}\,, \tag{4.8}$$

where $a = \sum a_i$, L is the number of loops and $F = U \sum m_i^2 x_i - V$. The functions U and V are the two Symanzik polynomials given by the well-known formulas with summations over graph's trees and 2-trees, respectively (see, e.g., [71]). To apply `asy` to (4.8), it proves more convenient to exploit it as a part of the `FIESTA5` distribution package [83, 84]. The advantage of its use is that the command `UF` will generate U and V automatically as well. In fact, the folklore Cheng-Wu theorem allows one to choose the sum over Feynman parameters in the argument of the delta-function to be over any nonempty subset of indices. For example, one can take $\delta\left(x_{i_0} - 1\right)$ for a conveniently chosen i_0. In circumstances when some indices a_i are negative, i.e., it's a numerator, the corresponding parametric integral is obtained by a limiting procedure, with a result which has no integration over the corresponding parameter and involves extra polynomials in the integrand. So, in eq. (4.8) we implied that all propagator indices a_i are positive and, if some indices are negative `FIESTA5` immediately yields corresponding expressions as well.

The heuristic formulation of the expansion by regions in the momentum space alluded to above can be repeated to the letter for parametric integrals as well. Let us emphasize, however, that the notion of a region here literally implies certain scaling of integration variables with powers of the small parameter in the problem. Our goal is then to select scalings that generate, after a subsequent expansion, non-zero contributions and we relegate this task to the code `asy`. The latter yields an output given as a set of N-dimensional vectors $\boldsymbol{r}_j = \{(r_j)_1, \ldots, (r_j)_N\}$.

In our case of a single small parameter t, the contribution of a given region $\boldsymbol{r}_j$ is obtained from the original integral by adopting the following three steps

$$\begin{aligned}
&\text{(i) rescaling variables as:} && x_i \to t^{(r_j)_i} x_i\,, \quad i = 1, \ldots, N\,, \\
&\text{(ii) multiplying integrands by:} && t^{\sum_{i=1}^N (r_j)_i}\,, \\
&\text{(iii) expanding integrands as:} && t \to 0\,.
\end{aligned} \tag{4.9}$$

To see how this works, let us consider an example, say, the three-loop integrals $T_{3,6}$ which is in fact the most complicated case. The "propagators" defining it read

$$\begin{aligned}
\texttt{Props} = \{ &-k_1^2, -k_2^2, -k_3^2, -(k_1+p_1)^2, -(k_1+k_2+p_1)^2, -(k_1+k_2+k_3+p_1)^2, \\
&-(k_1+k_2+k_3-p_2)^2, -(k_2+k_3-p_2)^2, -(k_2-p_2)^2, -(k_1+p_1+p_2)^2\}\,,
\end{aligned} \tag{4.10}$$

where k_i $(i = 1, 2, 3)$ are the loop momenta. The tenth invariant is in fact a numerator such that our integral $T_{3,6}$ possesses the indices $\{a_1, \ldots, a_9, a_{10}\} = \{1, \ldots, 1, -1\}$. The Feynman integral in question is finite in four dimensions. However, when expanded in the limit $t \to 0$ different regions being integrated over the entire R_+^N space inevitably induce divergences. These need to be regularized to get finite results. Dimensional regularization comes to the rescue as the most optimal choice and it mends singularities to becomes poles in ε. Ultimately, cancellations of the latter becomes then a very powerful check of the correctness of the expansion procedure.

After running `asy` with the help of the `FIESTA`'s command `SDExpandAsy`, we obtain information about all contributing regions. For the case at hand there are 35 of them

$$
\begin{aligned}
\boldsymbol{r} = \{&\{1,1,1,0,1,1,1,0,1,0\}, \{0,0,0,0,0,0,0,0,0,0\}, \{0,1,0,1,2,2,0,0,0,0\}, \\
&\{1,1,0,1,2,2,1,0,0,0\}, \{0,0,1,1,1,2,0,0,1,0\}, \{0,1,1,0,1,1,0,0,1,0\}, \\
&\{0,1,0,0,1,1,0,0,0,0\}, \{1,1,0,0,1,1,1,0,0,0\}, \{0,1,1,1,2,2,1,1,1,0\}, \\
&\{0,1,1,0,1,1,1,1,1,0\}, \{0,1,1,1,2,2,0,0,1,0\}, \{1,0,0,1,1,1,1,0,0,0\}, \\
&\{1,0,1,0,0,1,1,0,1,0\}, \{1,0,1,1,1,2,1,0,1,0\}, \{0,0,0,0,1,1,1,0,0,0\}, \\
&\{0,0,0,1,1,1,0,0,0,0\}, \{1,1,1,0,1,1,2,1,1,0\}, \{1,0,1,1,1,1,2,2,1,0\}, \\
&\{1,0,2,1,1,2,1,1,1,0\}, \{1,0,1,1,1,2,2,1,1,0\}, \{0,0,1,0,0,1,1,1,1,0\}, \\
&\{0,0,1,0,1,1,1,1,0,0\}, \{1,0,0,0,0,0,1,0,0,0\}, \{0,0,1,1,1,2,1,1,1,0\}, \\
&\{0,0,1,0,0,1,0,0,1,0\}, \{0,0,1,0,1,1,0,0,0,0\}, \{0,0,0,1,1,1,1,1,1,0\}, \\
&\{1,0,1,0,0,1,2,1,1,0\}, \{1,0,0,0,0,0,2,1,1,0\}, \{1,0,1,1,1,1,1,1,0,0\}, \\
&\{1,1,1,1,2,2,1,0,1,0\}, \{1,0,2,1,1,2,2,2,1,0\}, \{1,1,1,1,2,2,1,1,0,0\}, \\
&\{0,0,0,0,0,0,1,1,1,0\}, \{1,0,0,1,1,1,2,1,1,0\}\}\,,
\end{aligned} \tag{4.11}
$$

and they scale, respectively, as the following function of t

$$
\begin{aligned}
\Big\{&t^{-5\varepsilon}, t^{0}, t^{-6\varepsilon}, t^{-5\varepsilon}, t^{-6\varepsilon}, t^{-2\varepsilon}, t^{-3\varepsilon}, t^{-6\varepsilon}, t^{-4\varepsilon}, t^{-\varepsilon}, t^{-5\varepsilon}, t^{-2\varepsilon}, \\
&t^{-6\varepsilon}, t^{-5\varepsilon}, t^{-3\varepsilon}, t^{-3\varepsilon}, t^{-4\varepsilon}, t^{-4\varepsilon}, t^{-4\varepsilon}, t^{-4\varepsilon}, t^{-2\varepsilon}, t^{-2\varepsilon}, t^{-3\varepsilon}, t^{-5\varepsilon}, \\
&t^{-3\varepsilon}, t^{-3\varepsilon}, t^{-6\varepsilon}, t^{-5\varepsilon}, t^{-6\varepsilon}, t^{-\varepsilon}, t^{-4\varepsilon}, t^{-3\varepsilon}, t^{-4\varepsilon}, t^{-3\varepsilon}, t^{-5\varepsilon}\Big\}\,.
\end{aligned} \tag{4.12}
$$

It becomes immediately obvious that the momentum-space language would make the goal of identification of loop-momentum scalings quite a tedious task. While it is more or less clear that the t^0 behavior is associated with the hard-hard-hard region in the three loop momenta, $t^{-\varepsilon}$ is associated with regions where one of the loop momenta is collinear (to p_i) and the other two are hard, however, a proper identification of other regions in the momentum-space formalism looks quite challenging. Nevertheless, such an analysis will definitely be beneficial since it would result in a factorization theorem for the off-shell Sudakov form factor akin to the on-shell case [33, 34].

The analytical mode of the command `SDExpandAsy` provides explicit parametric integrals corresponding to the above 35 contributing regions. To systematically evaluate these in the ε-expansion, we rely on the method of the Mellin-Barnes (MB) representation (see, e.g.,

JHEP11(2023)111

ref. [71]). The latter is based on the simple formula

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_{\mathcal{C}} \frac{dz}{2\pi i} \frac{B^z}{A^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z)\,, \tag{4.13}$$

which allows one to partition a complicated polynomial in terms of its two 'simpler' components A and B. In this equation, the contour $\mathcal{C}$ runs from $-i\infty$ to $+i\infty$ in the complex plane and the poles of $\Gamma(\ldots+z)$ are to its left while the ones of $\Gamma(\ldots-z)$ are to its right with these left/right poles corresponding to infrared/ultraviolet singularities of the original integral. This formula is usually applied repeatedly enough number of times to a given parametric integral in order to transform it into a multiple MB integral. Of course, one attempts to arrive at as simple final representation as possible with fewer complex integrations. This procedure was recently automated with the code `MBcreate.m` [85].

Once a reasonable MB representation for a given parametric integral is obtained, the next step is to resolve integrand's singularities in ε. The goal here is to represent a given complex integral as a linear combination of MB integrals whose ε-expansion can be performed *under* the integral sign. There are two public codes `MB.m` and `MBresolve.m` [86, 87], where this algorithm is implemented. They are based on integrations strategies developed in [88, 89]. We relied on `MBresolve.m`.

The next step is to evaluate emerging MB integrals emerging as the coefficients in the Laurent expansion in ε. Here the command `DoAllBarnes` from Kosower's[10] `barnesroutines.m` automatically applies the first and the second Barnes lemmas (and their corollaries) and thereby performs some integrations in terms of the Euler gamma functions. After all these possibilities were exhausted and if some one- and two-fold MB integrals are still left, one can turn to numerical evaluations with high accuracy and then apply the `PSLQ` algorithm [90] to obtain analytic results given a basis of numbers,— typically values of Riemann zeta function,— entering the final result is known.

The strategy formulated above was successful in our calculation: we obtained asymptotics for all three-loop integrals $T_{3,i}$ and confirmed their complete agreement with the results determined via differential equations. Further making use of the numerical mode of `FIESTA`, these findings were also verified numerically.

5 Sudakov form factor: three loops and beyond

Having calculated individual contributions in the previous section, we can neatly combine them in linear combinations which arise in the three-loop expression (4.1) and observe massive cancellations of odd powers of the logarithm $\log t$ such that the small-m expansions for the $T_{3,i}$ integrals

$$\begin{aligned} Q^6 T_{3,1} &= -\Phi_3 + O(m^2), \\ Q^4 T_{3,2} &= \Phi_3 + O(m^2), \\ Q^4 [T_{3,3} - T_{3,5}] &= \frac{1}{2}\left[\Phi_3 - \Phi_1\Phi_2\right] + O(m^2), \end{aligned}$$

[10]All required MB tools can be downloaded from `bitbucket.org/feynmanIntegrals/mb/src/master/`.

$$
\begin{aligned}
Q^4[T_{3,4}-T_{3,6}] &= -\Phi_1\Phi_2+O(m^2),\\
Q^4T_{3,7} &= \Phi_3+O(m^2),\\
Q^4T_{3,8} &= \Phi_1\Phi_2+O(m^2)\,,
\end{aligned}
\tag{5.1}
$$

is determined exclusively by the Davydychev-Usyukina functions $\Phi_\ell \equiv \Phi_\ell(t,t)$. We remind the reader that indeed the small-m expansion of Φ_ℓ possesses even powers of $\log t$ one, as can be observed from the explicit expressions for $\Phi_{1,2}$ and Φ_3 previously quoted in eqs. (3.12) and (3.13), respectively. It is worth pointing out that making use of the symmetry properties of the Davydychev-Usyukina functions with respect to their two arguments, the above relations for $T_{3,2}$ and $T_{3,7}$ are similar to the so-called "magic identities" of ref. [91]. Other relations, however, cannot be obtained in this manner and are therefore unique in this regard.

Substituting (5.1) in $F_2^{(3)}$ and expanding $\log F_2$ in powers of g we found that, up to the three-loop order, $\log F_2$ equals to:

$$
\log F_2\,(t,g) = -\frac{1}{2}\Gamma_{\rm oct}(g)\log^2 t - D(g) + \mathcal{O}(m^2),
\tag{5.2}
$$

with the functions $\Gamma_{\rm oct}(g)$ and $D(g)$ of the coupling quoted earlier in eqs. (3.15) and (3.16), respectively. This is exactly the logarithm of the null octagon $\mathbb{O}_0(z,\bar z)$ [27, 28] multiplied by the factor of 2 and expanded up to $O(g^6)$,

$$
\log \mathbb{O}_0(z,\bar z) = -\frac{1}{4}\Gamma_{\rm oct}(g)\log^2\left(\frac{\bar z}{z}\right) - g^2\log(z\bar z) - \frac{1}{2}D(g)\,,
\tag{5.3}
$$

with $z = 1/\bar z = \sqrt{t}$. It appears natural to us to conjecture that this relation holds at any order of perturbation theory as well, i.e.,

$$
\log F_2 = 2\log \mathbb{O}_0 + O(m^2),
\tag{5.4}
$$

and thus conjecturing that the off-shell Sudakov form factor is given by the null octagon function $\mathbb{O}_0$ to all orders of the perturbation theory. This is the main result of our work.

Let us make a few comments regarding our claim. The above formula (5.2) is expected to hold in the *planar* limit, i.e., for the leading color structure. While the log-squared asymptotic behavior proportional to $\Gamma_{\rm oct}(g)$ was in fact anticipated in light of an explicit four loop computation of the four-point amplitude in ref. [25], the closed form of the D-piece is solely based on our three-loop analysis. This is the main (uncertain) ingredient of our all-loop conjecture. On a more technical level, we observe that all integrals (individual integrals or their linear combinations) in (4.1) can be represented as linear combinations of products of Davydychev-Usyukina function $\Phi_\ell(t,t)$ at least up to $O(m^2)$ terms. This observation is reminiscent of the results of [25] where all scalar integrals contributing to four point amplitude in off-shell kinematical regime where expressed in terms of linear combinations of products of $\Phi_\ell(x,y)$ functions up to four loops. The reason why such relations between integrals exist can be traced back to the integrability of underlying problem. The four point amplitude in off-shell kinematical regime is expected to be given by $\mathbb{O}_0(z,\bar z)$, which can be written in closed form, for instance as a perturbative series [27, 92, 93],

$$\mathbb{O}_0 = \det(1 - \mathbb{K}_0)\,, \qquad (\mathbb{K}_0)_{nm} = \sum_{\ell=n+m-1}^{\infty} (-g^2)^{\ell} C_{nm}^{(\ell)} \Phi_\ell(z,\bar{z}), \tag{5.5}$$

where $C_{nm}^{(\ell)}$ are explicitly known coefficients

$$C_{nm}^{(\ell)} = \frac{-(2m-1)[2\ell]![\ell-1]!\ell!}{[\ell-(n+m-1)]![\ell+(n+m-1)]![\ell-|n-m|]![\ell+|n-m|]!}\,. \tag{5.6}$$

The relations (5.1) together with (3.8) and (3.9) can be considered as subtle hints that the Sudakov problem (for the two-particle form factor, in the current case, or generally even multi-leg ones) in $\mathcal{N}=4$ sYM on the Coulomb branch in the off-shell kinematical regime can potentially be solved using integrability. Another hint pointing to this conclusion is that eqs. (5.1), (3.8) and (3.9) express nonplanar integrals in terms of planar one. The latter in turn possess well defined dual conformal symmetry properties, which being a part of a larger symmetry group,— the so-called Yangian symmetry [94],— is intrinsic to integrable systems, see, e.g., [95]. If the integrability of the off-shell Sudakov problem will hold in a fashion similar to the on-shell case [96] then it would not be probably too surprising that $\Gamma_{\rm oct}(g)$ and $D(g)$ appeared in tandem.

6 Conclusions

As a conclusion, let us make several comments regarding IR properties of the off-shell Sudakov form factor and amplitudes which we observed as a result of our analysis. The most obvious one is that in the off-shell regime the Sudakov logarithms indeed exponentiate but it is $\Gamma_{\rm oct}$ rather than $\Gamma_{\rm cusp}$ that governs the rate of its decay. As was already anticipated in [25], the $\Gamma_{\rm cusp}$ is not the archetypal IR anomalous dimension and IR behavior of amplitudes and form factors is far more involved than previously expected. Note also that in the off-shell case there are no $\log^{2n+1}$ terms and hence there is no analogue of the collinear anomalous dimension, at least, up to the three loop accuracy. In terms of evolution equation, the result (5.2) can be rewritten as:

$$\left(\frac{\partial}{\partial \log m^2}\right)^2 \log F_2 = -\Gamma_{\rm oct}(g)\,. \tag{6.1}$$

This evolution equation is obviously different from what was conjectured earlier in the literature, e.g., [4], which involved $\Gamma_{\rm cusp}$ instead.

Let us point out that factorization properties for four- and five-leg amplitudes for W-boson scattering still hold in this off-shell kinematical regime such that IR-sensitive parts are still driven by the product of Sudakov form factors identical to (2.4), but now with F_2 given by eq. (5.2). This demonstrates self-consistency of these considerations in such kinematical regime. A natural extension of these findings would be to study the structure of the Sudakov form factor in a similar kinematical regime in QCD and other four-dimensional gauge theories. Regarding QCD, our computation can be considered as a determination of its “most transcendental” part in the planar limit.

JHEP11(2023)111

In spite of the fact that the form of the evolution equations differs depending on the value of the external particles' off-shellness, (3.5) vs. (6.1), there is a chance that they are in fact next of kin. The two anomalous dimensions can be found as solutions to the so-called flux-tube equations, given in ref. [97] for $\Gamma_{\rm cusp}$ and [28] for $\Gamma_{\rm oct}$. The two can in fact be combined into a more general equation by introducing a deformation parameter [98]. Thus, it appears that the latter encodes a very subtle, anomalous effect of the non-commutativity of $p_i^2 \to 0$ and $\varepsilon \to 0$ limits. At the moment we have no adequate understanding of this fact.

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A Small off-shellness expansion of $T_{3,i}$

In this appendix, we present a list of the small-m expansions for all $T_{3,i}$ integrals given in terms of HPLs in the Mathematica notebook `3loopTs.nb` in the supplementary material attached to this paper. They read

$$Q^6 T_{3,1} = -\frac{1}{36}\log^6 t - \frac{5\zeta_2}{6}\log^4 t - \frac{35\zeta_4}{2}\log^2 t - \frac{155\zeta_6}{4}, \tag{A.1}$$

$$Q^4 T_{3,2} = \frac{1}{36}\log^6 t + \frac{5\zeta_2}{6}\log^4 t + \frac{35\zeta_4}{2}\log^2 t + \frac{155\zeta_6}{4}, \tag{A.2}$$

$$Q^4 T_{3,3} = -\frac{1}{36}\log^6 t + \frac{\zeta_2}{3}\log^4 t + \frac{2\zeta_3}{3}\log^3 t + \frac{27\zeta_4}{2}\log^2 t + (4\zeta_2\zeta_3 - 20\zeta_5)\log t + 32\zeta_6 - 4\zeta_3^2, \tag{A.3}$$

$$Q^4 T_{3,4} = -\frac{1}{36}\log^6 t - \frac{5\zeta_2}{6}\log^4 t - \frac{4\zeta_3}{3}\log^3 t + \frac{\zeta_4}{2}\log^2 t - 20\zeta_5 \log t - \frac{27\zeta_6}{4} - 16\zeta_3^2, \tag{A.4}$$

$$Q^4 T_{3,5} = \frac{1}{12}\log^6 t + \frac{5\zeta_2}{3}\log^4 t + \frac{2\zeta_3}{3}\log^3 t + \frac{35\zeta_4}{2}\log^2 t + (4\zeta_2\zeta_3 - 20\zeta_5)\log t + 31\zeta_6 - 4\zeta_3^2, \tag{A.5}$$

$$Q^4 T_{3,6} = \frac{2}{9}\log^6 t + \frac{8\zeta_2}{3}\log^4 t - \frac{4\zeta_3}{3}\log^3 t + 26\zeta_4 \log^2 t - 20\zeta_5 \log t + 30\zeta_6 - 16\zeta_3^2, \tag{A.6}$$

$$Q^4 T_{3,7} = \frac{1}{36}\log^6 t + \frac{5\zeta_2}{6}\log^4 t + \frac{35\zeta_4}{2}\log^2 t + \frac{155\zeta_6}{4}\,, \tag{A.7}$$

$$Q^4 T_{3,8} = \frac{1}{4}\log^6 t + \frac{7\zeta_2}{2}\log^4 t + \frac{51\zeta_4}{2}\log^2 t + \frac{147\zeta_6}{4}\,, \tag{A.8}$$

and valid up to $O(m^2)$.

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JHEP11(2023)111