

Existence of 2-factors in tough graphs without forbidden subgraphs

Elizabeth Grimm^a, Anna Johnsen^b, Songling Shan^{a,*}

^a Department of Mathematics, Illinois State University, Normal, IL 61790, United States of America

^b Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States of America

ARTICLE INFO

Article history:

Received 12 January 2022

Received in revised form 8 June 2023

Accepted 11 June 2023

Available online 28 June 2023

Keywords:

2-factor

Toughness

Forbidden subgraphs

ABSTRACT

For a given graph R , a graph G is R -free if G does not contain R as an induced subgraph. It is known that every 2-tough graph with at least three vertices has a 2-factor. In graphs with restricted structures, it was shown that every $2K_2$ -free $3/2$ -tough graph with at least three vertices has a 2-factor, and the toughness bound $3/2$ is best possible. In viewing $2K_2$, the disjoint union of two edges, as a linear forest, in this paper, for any linear forest R on 5, 6, or 7 vertices, we find the sharp toughness bound t such that every t -tough R -free graph on at least three vertices has a 2-factor.

© 2023 Elsevier B.V. All rights reserved.

1. Introduction

Let G be a simple, undirected graph and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G , respectively. We denote the set of neighbors of a vertex $x \in V(G)$ by $N_G(x)$. The closed neighborhood of a vertex x in G , denoted by $N_G[x]$, is the set $\{x\} \cup N_G(x)$. For any subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S , $G - S$ denotes the subgraph $G[V(G) \setminus S]$, and $N_G(S) = \bigcup_{v \in S} N_G(v)$. Given disjoint subsets S and T of $V(G)$, we denote by $E_G(S, T)$ the set of edges which have one endvertex in S and the other endvertex in T , and let $e_G(S, T) = |E_G(S, T)|$. If $S = \{s\}$ is a singleton, we write $e_G(s, T)$ for $e_G(\{s\}, T)$. If $H \subseteq G$ is a subgraph of G , and $T \subseteq V(G)$ with $T \cap V(H) = \emptyset$, we write $E_G(H, T)$ and $e_G(H, T)$ for notational simplicity.

For a given graph R , we say that G is R -free if there does not exist an induced copy of R in G . For integers a and b with $a \geq 0$ and $b \geq 1$, we denote by aP_b the graph consisting of a disjoint copies of the path P_b . When $a = 1$, $1P_b$ is just P_b , and when $a = 0$, $0P_b$ is the null graph. For two integers p and q , let $[p, q] = \{i \in \mathbb{Z} : p \leq i \leq q\}$.

Denote by $c(G)$ the number of components of G . Let $t \geq 0$ be a real number. We say a graph G is t -tough if for each cutset S of G we have $t \cdot c(G - S) \leq |S|$. The toughness of a graph G , denoted by $\tau(G)$, is the maximum value of t for which G is t -tough if G is non-complete and is defined to be ∞ if G is complete.

For an integer $k \geq 1$, a k -regular spanning subgraph is a k -factor of G . It is well known, according to a theorem by Enomoto, Jackson, Katerinis, and Saito [3] from 1995, that every k -tough graph with at least three vertices has a k -factor if $k|V(G)|$ is even and $|V(G)| \geq k + 1$. In terms of a sharp toughness bound, particular research interest has been taken for 2-factors for graphs with restricted structures. For example, it was shown that every $3/2$ -tough 5-chordal graph (graphs with no induced cycle of length at least 6) on at least three vertices has a 2-factor [1]. As $2K_2$ -free graphs are 5-chordal, as a consequence, every $3/2$ -tough $2K_2$ -free graph on at least three vertices has a 2-factor [5]. The toughness bound $3/2$ is best possible in both results.

* Corresponding author.

E-mail addresses: evgrimm@ilstu.edu (E. Grimm), ajohnsen2@gsu.edu (A. Johnsen), sshan12@ilstu.edu (S. Shan).

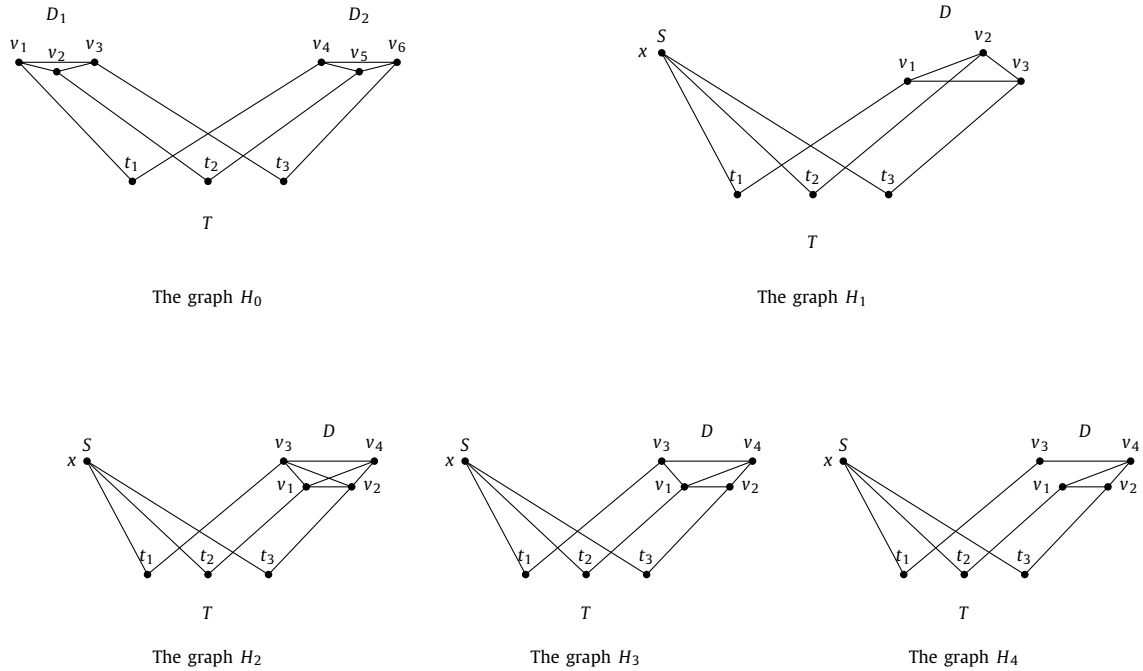


Fig. 1. The five exceptional graphs for Theorem 1(1), where $S = \{x\}$ and $T = \{t_1, t_2, t_3\}$.

A *linear forest* is a graph consisting of disjoint paths. In viewing $2K_2$ as a linear forest on 4 vertices and the result by Ota and Sanka [5] that every $3/2$ -tough $2K_2$ -free graph on at least three vertices has a 2-factor, we investigate the existence of 2-factors in R -free graphs when R is a linear forest on 5, 6, or 7 vertices. These graphs R are listed below, where the unions are vertex disjoint unions.

1. $P_5 \quad P_4 \cup P_1 \quad P_3 \cup P_2 \quad P_3 \cup 2P_1 \quad 2P_2 \cup P_1 \quad P_2 \cup 3P_1 \quad 5P_1$;
2. $P_6 \quad P_5 \cup P_1 \quad P_4 \cup P_2 \quad P_4 \cup 2P_1 \quad 2P_3 \quad P_3 \cup P_2 \cup P_1 \quad P_3 \cup 3P_1 \quad 3P_2 \quad 2P_2 \cup 2P_1 \quad P_2 \cup 4P_1 \quad 6P_1$;
3. $P_7 \quad P_6 \cup P_1 \quad P_5 \cup P_2 \quad P_5 \cup 2P_1 \quad P_4 \cup P_3 \quad P_4 \cup P_2 \cup P_1 \quad P_4 \cup 3P_1 \quad 2P_3 \cup P_1 \quad P_3 \cup 2P_2 \quad P_3 \cup P_2 \cup 2P_1 \quad P_3 \cup 4P_1 \quad 3P_2 \cup P_1 \quad 2P_2 \cup 3P_1 \quad P_2 \cup 5P_1 \quad 7P_1$.

Our main results are the following.

Theorem 1. Let $t > 0$ be a real number, R be any linear forest on 5 vertices, and G be a t -tough R -free graph on at least 3 vertices. Then G has a 2-factor provided that

- (1) $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$ unless

- (a) $R = P_2 \cup 3P_1$, and $G \cong H_0$ or G contains H_i as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [1, 4]$, where H_i , S and T are defined in Fig. 1.
- (b) $R = P_3 \cup 2P_1$, and G contains H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$.

- (2) $R = 5P_1$ and $t > 1$.

- (3) $R \in \{P_5, P_3 \cup P_2, 2P_2 \cup P_1\}$ and $t = 3/2$.

Theorem 2. Let $t > 0$ be a real number, R be any linear forest on 6 vertices, and G be a t -tough R -free graph on at least 3 vertices. Then G has a 2-factor provided that

- (1) $R \in \{P_4 \cup 2P_1, P_3 \cup 3P_1, P_2 \cup 4P_1, 6P_1\}$ and $t > 1$ unless $R = 6P_1$ and G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, where H_5 , S and T are defined in Fig. 2.
- (2) $R \in \{P_6, P_5 \cup P_1, P_4 \cup P_2, 2P_3, P_3 \cup P_2 \cup P_1, 3P_2, 2P_2 \cup 2P_1\}$ and $t = 3/2$.

Theorem 3. Let $t > 0$ be a real number, R be any linear forest on 7 vertices, and G be a t -tough R -free graph on at least 3 vertices. Then G has a 2-factor provided that

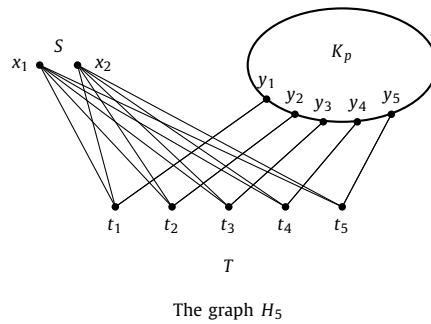


Fig. 2. The exceptional graph for Theorem 2(1), where $S = \{x_1, x_2\}$, $T = \{t_1, \dots, t_5\}$, and $p = 5$.

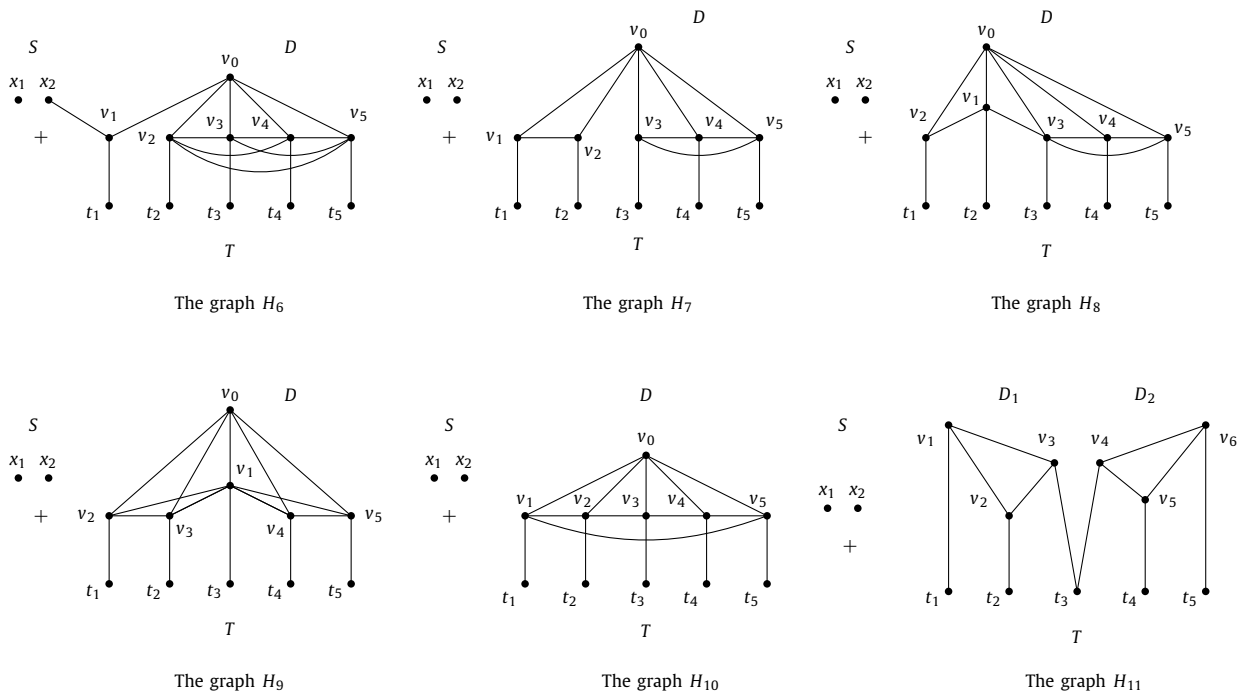


Fig. 3. The six exceptional graphs for Theorem 3(1)(b), where $S = \{x_1, x_2\}$, $T = \{t_1, t_2, t_3, t_4, t_5\}$, and “+” represents the join of $H_i[S]$ and $H_i[T]$, $i \in [6, 11]$.

(1) $R \in \{P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1\}$ and $t > 1$ unless

- (a) when $R \neq P_4 \cup 3P_1$, G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, where H_5 , S and T are defined in Fig. 2.
- (b) $R = P_2 \cup 5P_1$ and G contains one of H_6, \dots, H_{11} as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S]) \cup E(G[V(G) \setminus (T \cup S)])$, where H_i , S and T are defined in Fig. 3 for each $i \in [6, 11]$.

(2) $R = 7P_1$ and $t > \frac{7}{6}$ unless G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$.

(3) $R \in \{P_7, P_6 \cup P_1, P_5 \cup P_2, P_5 \cup 2P_1, P_4 \cup P_3, P_4 \cup P_2 \cup P_1, 2P_3 \cup P_1, P_3 \cup 2P_2, P_3 \cup P_2 \cup 2P_1, 3P_2 \cup P_1, 2P_2 \cup 3P_1\}$ and $t = 3/2$.

Remark 4 (Examples demonstrating sharp toughness bounds). The toughness bounds in Theorems 1 to 3 are all sharp.

- (1) Theorem 1(1) when $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$. The graph showing that the toughness 1 is best possible is the complete bipartite $K_{n-1, n}$ for any integer $n \geq 2$. The graph $K_{n, n-1}$ is $(P_2 \cup P_1)$ -free and so is R -free, with $\lim_{n \rightarrow \infty} \tau(K_{n, n-1}) = \lim_{n \rightarrow \infty} \frac{n-1}{n} = 1$, but contains no 2-factor.
- (2) Theorem 1(2), Theorem 2(1) and Theorem 3(1) and $t > 1$. The graph showing that the toughness is best possible is the graph H_{12} , which is constructed as follows: let $p \geq 3$, K_p be a complete graph, and $y_1, y_2, y_3 \in V(K_p)$ be distinct,

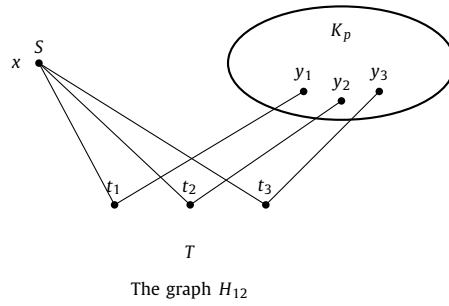


Fig. 4. Sharpness example for Theorem 1(2), Theorem 2(1) and Theorem 3(1), where $S = \{x\}$ and $T = \{t_1, t_2, t_3\}$.

- $S = \{x\}$, and $T = \{t_1, t_2, t_3\}$, then H_{12} is obtained from K_p , S and T by adding edges $t_i x$ and $t_i y_i$ for each $i \in [1, 3]$. See Fig. 4 for a depiction. By inspection, the graph is $5P_1$ -free and $(P_4 \cup 2P_1)$ -free. So the graph is R -free for any $R \in \{5P_1, P_4 \cup 2P_1, P_3 \cup 3P_1, P_2 \cup 4P_1, 6P_1, P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1\}$. For any given $p \geq 3$, the graph H_{12} does not contain a 2-factor, as any 2-factor has to contain the edges $t_1 x, t_2 x$ and $t_3 x$. We will show $\tau(H_{12}) = 1$ in the last section.
- (3) For Theorem 1(3), Theorem 2(2) and Theorem 3(3) and $t = \frac{3}{2}$: note that all the graphs R in these cases contain $2K_2$ as an induced subgraph. Chvátal [2] constructed a sequence $\{G_k\}_{k=1}^{\infty}$ of split graphs (graphs whose vertex set can be partitioned into a clique and an independent set) having no 2-factors and $\tau(G_k) = \frac{3k}{2k+1}$ for each positive integer k . As the class of $2K_2$ -free graphs is a superclass of split graphs, $\frac{3}{2}$ -tough is the best possible toughness bound for a $2K_2$ -free graph to have a 2-factor.
- (4) Theorem 3(2) and $t > \frac{7}{6}$. The graph showing that the toughness is best possible is the graph H_5 with $p \geq 6$, which is constructed as follows: let $p \geq 5$, K_p be a complete graph, and $y_1, y_2, y_3, y_4, y_5 \in V(K_p)$ be distinct, $S = \{x_1, x_2\}$, and $T = \{t_1, t_2, t_3, t_4, t_5\}$. Then H_5 is obtained from K_p , S and T by adding edges $t_i x_j$ and $t_i y_i$ for each $i \in [1, 5]$ and each $j \in [1, 2]$. See Fig. 2 for a depiction. By inspection, the graph is $7P_1$ -free. For any given $p \geq 5$, the graph H_5 does not contain a 2-factor, as any 2-factor has to contain at least three edges from one of x_1 and x_2 to at least three vertices of T . We will show $\tau(H_5) = \frac{7}{6}$ when $p \geq 6$ in the last section.

To supplement Theorems 1 to 3, we show that the exceptional graphs in Figs. 1 to 3 satisfy the corresponding conditions below.

Theorem 5. *The following statements hold.*

- (1) *The graph H_i is $(P_2 \cup 3P_1)$ -free, contains no 2-factor, and $\tau(H_i) = 1$ for each $i \in [0, 4]$, the graph H_1 is also $(P_3 \cup 2P_1)$ -free.*
 (2) *The graph H_i is $(P_2 \cup 5P_1)$ -free and contains no 2-factor for each $i \in [5, 11]$, H_5 with $p = 5$ is $(P_3 \cup 4P_1)$ -free and $6P_1$ -free. Furthermore, $\tau(H_5) = \frac{6}{5}$ when $p = 5$ and $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 11]$.*

We have explained that H_5 and H_{12} are R -free for the corresponding linear forests R and contain no 2-factor in Remark 4(2) and (4). The Theorem below is to verify the toughness of the graphs H_5 with $p \geq 6$ and H_{12} .

Theorem 6. *The following statements hold.*

- (1) $\tau(H_5) = \frac{7}{6}$ when $p \geq 6$;
 (2) $\tau(H_{12}) = 1$.

The remainder of this paper is organized as follows. In section 2, we introduce more notation and preliminary results on proving existence of 2-factors in graphs. In section 3, we prove Theorems 1-3. Theorems 5 and 6 are proved in the last section.

2. Preliminaries

One of the main proof ingredients of Theorems 1 to 3 is to apply Tutte's 2-factor Theorem. We start with some notation. Let S and T be disjoint subsets of vertices of a graph G , and D be a component of $G - (S \cup T)$. The component D is said to be an *odd component* (resp. *even component*) of $G - (S \cup T)$ if $e_G(D, T) \equiv 1 \pmod{2}$ (resp. $e_G(D, T) \equiv 0 \pmod{2}$). Let $h(S, T)$ be the number of all odd components of $G - (S \cup T)$. Define

$$\delta(S, T) = 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h(S, T).$$

It is easy to see that $\delta(S, T) \equiv 0 \pmod{2}$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We use the following criterion for the existence of a 2-factor, which is a restricted form of Tutte's f -factor Theorem.

Lemma 7 (Tutte [6]). *A graph G has a 2-factor if and only if $\delta(S, T) \geq 0$ for every $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.*

An ordered pair (S, T) , consisting of disjoint subsets of vertices S and T in a graph G , is called a *barrier* if $\delta(S, T) \leq -2$. By Lemma 7, if G does not have a 2-factor, then G has a barrier. In [4], a *biased barrier* of G is defined as a barrier (S, T) of G such that among all the barriers of G ,

- (1) $|S|$ is maximum; and
- (2) subject to (1), $|T|$ is minimum.

The following properties of a biased barrier were derived in [4].

Lemma 8. *Let G be a graph without a 2-factor, and let (S, T) be a biased barrier of G . Then each of the following holds.*

- (1) *The set T is independent in G .*
- (2) *If D is an even component with respect to (S, T) , then $e_G(T, D) = 0$.*
- (3) *If D is an odd component with respect to (S, T) , then for any $y \in T$, $e_G(y, D) \leq 1$.*
- (4) *If D is an odd component with respect to (S, T) , then for any $x \in V(D)$, $e_G(x, T) \leq 1$.*

Let G be a graph without a 2-factor and (S, T) be a barrier of G . For an integer $k \geq 0$, let \mathcal{C}_k denote the set of components D of $G - (S \cup T)$ such that $e_G(D, T) = k$. The following result was proved as a claim in [4] but we include a short proof here for self-completeness.

Lemma 9. *Let G be a graph without a 2-factor, and let (S, T) be a biased barrier of G . Then $|T| \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| + 1$.*

Proof. Let $U = V(G) \setminus S$. Since (S, T) is a barrier,

$$\begin{aligned} \delta(S, T) &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h(S, T) \\ &= 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - \sum_{k \geq 0} |\mathcal{C}_{2k+1}| \leq -2. \end{aligned}$$

By Lemma 8(1) and Lemma 8(2),

$$\sum_{y \in T} d_{G-S}(y) = \sum_{y \in T} e_G(y, U) = e_G(T, U) = \sum_{k \geq 0} (2k+1)|\mathcal{C}_{2k+1}|.$$

Therefore, we have

$$-2 \geq 2|S| - 2|T| + \sum_{k \geq 0} (2k+1)|\mathcal{C}_{2k+1}| - \sum_{k \geq 0} |\mathcal{C}_{2k+1}|,$$

which yields $|T| \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| + 1$. \square

We use the following lemmas in our proof.

Lemma 10. *Let $t \geq 1$, G be a t -tough graph on at least three vertices containing no 2-factor, and (S, T) be a biased barrier of G . Then the following statements hold.*

- (1) *If $\mathcal{C}_1 \neq \emptyset$, then $|S| + 1 \geq 2t$. Consequently, $S = \emptyset$ implies $\mathcal{C}_1 = \emptyset$, and $|S| = 1$ implies $\mathcal{C}_1 = \emptyset$ when $t > 1$.*
- (2) $\bigcup_{k \geq 1} \mathcal{C}_{2k+1} \neq \emptyset$.

Proof. By Lemma 9, $T \neq \emptyset$. Since G is 1-tough and thus is 2-connected, $d_G(y) \geq 2$ for every $y \in T$. This together with Lemma 8(1)-(3) implies $|S| + \sum_{k \geq 0} |\mathcal{C}_{2k+1}| \geq 2$.

For the first part of (1), suppose to the contrary that $|S| + 1 < 2t$. Let $D \in \mathcal{C}_1$ and $y \in V(T)$ be adjacent in G to some vertex $v \in V(D)$. As $|S| + \sum_{k \geq 0} |\mathcal{C}_{2k+1}| \geq 2$ and $|T| \geq |S| + 1$ by Lemma 9, we have $c(G - (S \cup \{y\})) \geq 2$ regardless of whether or not $S = \emptyset$. But $c(G - (S \cup \{y\})) \geq 2$ implies $\tau(G) < 2t/2 = t$, contradicting G being t -tough. The second part of (1) is a consequence of the first part.

For (2), suppose to the contrary that $\bigcup_{k \geq 1} \mathcal{C}_{2k+1} = \emptyset$. By Lemma 10(1), $|S| + |C_1| \geq 2$ implies $|S| \geq 1$. Consequently, $|T| \geq 2$ by Lemma 9. As every component of $G - (S \cup T)$ in \mathcal{C}_1 is connected to exactly one vertex of T , S is a cutset of G with $c(G - S) \geq |T|$. However, $|T| \geq |S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| + 1 = |S| + 1$, implying $\tau(G) < 1$, a contradiction. \square

A path P connecting two vertices u and v is called a (u, v) -path, and we write uPv or vPu in order to specify the two endvertices of P . Let uPv and xQy be two disjoint paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx . Let G be a graph without a 2-factor, and let (S, T) be a barrier of G . For $y \in T$, define

$$h(y) = |\{D : D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1} \text{ and } e_G(y, D) \geq 1\}|.$$

Lemma 11. *Let G be a graph without a 2-factor, and let (S, T) be a biased barrier of G . Then the following statements hold.*

- (1) *If $|\bigcup_{k \geq 1} \mathcal{C}_{2k+1}| \geq 1$, then G contains an induced $P_4 \cup aP_1$, where $a = |T| - 2$.*
- (2) *If there exists $y_0 \in T$ with $h(y_0) \geq 2$, then for some integer $b \geq 7$, G contains an induced $P_b \cup aP_1$, where $a = |T| - 3$. Furthermore, an induced $P_b \cup aP_1$ can be taken such that the vertices in aP_1 are from T and the path P_b has the form $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$, where $y_0, y_1, y_2 \in T$ and $x_1^*P_1x_1$ and $x_2^*P_2x_2$ are respectively contained in two distinct components from $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$ such that $e_G(x, T) = 0$ for every internal vertex x from P_1 and P_2 .*

Proof. Lemma 8(1), (3) and (4) will be applied frequently in the arguments sometimes without mentioning it.

Let $D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}$. The existence of D implies $|T| \geq 3$ and $|V(D)| \geq 3$ by Lemma 8(3) and (4). We claim that for a fixed vertex $x_1 \in V(D)$ such that $e_G(x_1, T) = 1$, there exists distinct $x_2 \in V(D)$ and an induced (x_1, x_2) -path P in D with the following two properties: (a) $e_G(x_2, T) = 1$, and (b) $e_G(x, T) = 0$ for every $x \in V(P) \setminus \{x_1, x_2\}$. Note that the vertex x_1 exists by Lemma 8(4). Let $y_1 \in T$ be the vertex such that $e_G(x_1, T) = e_G(x_1, y_1) = 1$ and $W = N_G(T \setminus \{y_1\}) \cap V(D)$. Since $|N_G(T) \cap V(D)| \geq 3$ and $|N_G(y_1) \cap V(D)| = 1$, we have $W \neq \emptyset$. By Lemma 8(4), $x_1 \notin W$. Now in D , we find a shortest path P connecting x_1 and some vertex from W , say x_2 . Then x_2 and P satisfy properties (a) and (b), respectively. Let $y_2 \in T$ such that $e_G(x_2, T) = e_G(x_2, y_2) = 1$. The vertex y_2 uniquely exists by the choice x_2 and Lemma 8(4). By Lemma 8(1) and (4), and the choice of P , we know that $y_1x_1Px_2y_2$ and $T \setminus \{y_1, y_2\}$ together contain an induced $P_4 \cup aP_1$. This proves (1).

We now prove (2). By Lemma 8(3), the existence of y_0 implies $|\bigcup_{k \geq 1} \mathcal{C}_{2k+1}| \geq 2$, which in turn gives $|T| \geq 3$ by Lemma 8(3) again. We let $D_1, D_2 \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}$ be distinct such that $e_G(y_0, D_1) = 1$ and $e_G(y_0, D_2) = 1$. Let $x_i \in D_i$ such that $e_G(y_0, D_i) = e_G(y_0, x_i) = 1$. By the argument in the first paragraph above, we can find $x_i^* \in V(D_i) \setminus \{x_i\}$ and an (x_i, x_i^*) -path P_i in D_i for each $i \in \{1, 2\}$. By the choice of P_i and Lemma 8(4), there are unique $y_1, y_2 \in T \setminus \{y_0\}$ such that $x_i^*y_i \in E(G)$. If $y_1 \neq y_2$, by the choice of P_1 and P_2 and Lemma 8(1) and (4), we know that $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ and $T \setminus \{y_0, y_1, y_2\}$ together contain an induced $P_b \cup aP_1$ for some integer $b \geq 7$. Thus we assume $y_1 = y_2$. Then the vertex y_1 can also play the role of y_0 . Let $W = N_G(T \setminus \{y_0, y_1\}) \cap V(D_2)$. By Lemma 8(4), $x_2, x_2^* \notin W$. Now in D_2 , we find a shortest path P_2^* connecting some vertex of $\{x_2, x_2^*\}$ and some vertex from W , say z . Let the neighbor of z in G from T be z_1 . If P_2^* is an (x_2, z) -path, then $y_1x_1^*P_1x_1y_0x_2P_2^*z$ and $T \setminus \{y_0, y_1, z_1\}$ together contain an induced $P_b \cup aP_1$. If P_2^* is an (x_2^*, z) -path, then $y_0x_1P_1x_1^*y_1x_2^*P_2^*z$ and $T \setminus \{y_0, y_1, z_1\}$ together contain an induced $P_b \cup aP_1$. The second part of (2) is clear by the construction above. \square

Let G be a non-complete graph. A cutset S of $V(G)$ is a *toughset* of G if $\frac{|S|}{c(G-S)} = \tau(G)$.

Lemma 12. *If G is a connected graph and S is a toughset of G , then for every $x \in S$, x is adjacent in G to vertices from at least two components of $G - S$.*

Proof. Assume to the contrary that there exists $x \in S$ such that x is adjacent in G to vertices from at most one component of $G - S$. Then $c(G - (S \setminus \{x\})) = c(G - S) \geq 2$ and

$$\frac{|S \setminus \{x\}|}{c(G - (S \setminus \{x\}))} < \frac{|S|}{c(G - S)} = \tau(G),$$

contradicting G being $\tau(G)$ -tough. \square

3. Proof of Theorems 1, 2, and 3

Let R be any linear forest on at most 7 vertices. If G is R -free, then G is also R^* -free for any supergraph R^* that contains R as an induced subgraph. To prove Theorems 1 to 3, we will show that the corresponding statements hold for a supergraph R^* of R , where R^* contains R as an induced subgraph. This strategy simplifies the cases of possibilities of R . Let us first list the supergraphs that we will use.

- (1) $P_4 \cup 3P_1$ is a supergraph of the following graphs: $P_4 \cup 2P_1, P_3 \cup 3P_1$, and $P_2 \cup 4P_1$;

- (2) $6P_1$ is a supergraph of $5P_1$;
 (3) $P_3 \cup 2P_2$ is a supergraph of $3P_2$;
 (4) $P_7 \cup 2P_1$ is a supergraph of the following graphs:

- (a) $P_5, P_3 \cup P_2, 2P_2 \cup P_1$;
 (b) $P_6, P_5 \cup P_1, P_4 \cup P_2, 2P_3, P_3 \cup P_2 \cup P_1, 2P_2 \cup 2P_1$;
 (c) $P_7, P_6 \cup P_1, P_5 \cup 2P_1, P_4 \cup P_2 \cup P_1, 2P_3 \cup P_1, P_3 \cup P_2 \cup 2P_1, 2P_2 \cup 3P_1$.

Those supergraphs above together with the graphs R listed below cover all the 33 R graphs described in Theorems 1 to 3. Theorems 1 to 3 are then consequences of the theorem below.

Theorem 13. Let $t > 0$ be a real number, R be a linear forest, and G be a t -tough R -free graph on at least 3 vertices. Then G has a 2-factor provided that

- (1) $R \in \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ and $t = 1$ unless
 (a) $R = P_2 \cup 3P_1$, and $G \cong H_0$ or G contains H_i as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [1, 4]$, where H_i, S and T are defined in Fig. 1.
 (b) $R = P_3 \cup 2P_1$, and G contains H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$.
 (2) $R \in \{P_4 \cup 3P_1, P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1\}$ and $t > 1$ unless
 (a) when $R \neq P_4 \cup 3P_1$, G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$, where H_5, S and T are defined in Fig. 2.
 (b) $R = P_2 \cup 5P_1$ and G contains one of H_6, \dots, H_{11} as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S]) \cup E(G[V(G) \setminus (T \cup S)])$, where H_i, S and T are defined in Fig. 3 for each $i \in [6, 11]$.
 (3) $R = 7P_1$ and $t > \frac{7}{6}$ unless G contains H_5 with $p = 5$ as a spanning subgraph such that $E(G) \setminus E(H_5) \subseteq E_G(S, V(G) \setminus (T \cup S)) \cup E(G[S])$.
 (4) $R \in \{P_7 \cup 2P_1, P_5 \cup P_2, P_4 \cup P_3, P_3 \cup 2P_2, 3P_2 \cup P_1\}$ and $t = 3/2$.

Proof. Assume by contradiction that G does not have a 2-factor. By Lemma 7, G has a barrier. We choose (S, T) to be a biased barrier. Thus (S, T) and G satisfy all the properties listed in Lemma 8. These properties will be used frequently even without further mentioning sometimes. By Lemma 9,

$$|T| \geq |S| + \sum_{k \geq 1} k|C_{2k+1}| + 1. \quad (1)$$

Since $t \geq 1$, by Lemma 10(2), we know that

$$\bigcup_{k \geq 1} C_{2k+1} \neq \emptyset. \quad (2)$$

This implies $|T| \geq 3$ and so G contains an induced $P_4 \cup P_1$ by Lemma 11(1). Thus we assume $R \neq P_4 \cup P_1$ in the rest of the proof.

Claim 1. $R \notin \{P_3 \cup 2P_1, P_2 \cup 3P_1\}$ unless G falls under one of the exceptional cases as in (a) and (b) of Theorem 13(1).

Proof. Assume instead that $R \in \{P_3 \cup 2P_1, P_2 \cup 3P_1\}$. Thus $t = 1$. We may assume that G does not fall under any of the exceptional cases as in (a) and (b) of Theorem 13(1).

It must be the case that $|T| = 3$, as otherwise G contains an induced $P_4 \cup 2P_1$ by Lemma 11(1), and so contains an induced R . By Equation (1), we have $|\bigcup_{k \geq 1} C_{2k+1}| + |S| \leq 2$. By Lemma 10(1), we have that $C_1 = \emptyset$ if $S = \emptyset$. Since G is 1-tough and so $\delta(G) \geq 2$, Lemma 8(1)-(3) implies that $|\bigcup_{k \geq 1} C_{2k+1}| + |S| = 2$. By (2), we have the two cases below.

CASE 1: $|\bigcup_{k \geq 1} C_{2k+1}| = 2$ and $S = \emptyset$.

Let $D_1, D_2 \in \bigcup_{k \geq 1} C_{2k+1}$ be the two odd components of $G - (S \cup T)$. Since $|T| = 3$, Lemma 8(3) implies that $e_G(D_i, T) = 3$ for each $i \in [1, 2]$. Let $y \in T$ and $x \in V(D_1)$ such that $xy \in E(G)$. We let x_1 be a neighbor of x from D_1 . Then yx_1 is an induced P_3 by Lemma 8(3). Let $y_1 \in T \setminus \{y\}$ such that $y_1x_1 \notin E(G)$, which is possible as $|T| = 3$ and $e_G(x_1, T) \leq 1$ by Lemma 8(4). We now let $x_2 \in V(D_2)$ such that $e_G(x_2, \{y, y_1\}) = 0$, which is again possible as $|N_G(T) \cap V(D_2)| = 3$ and each vertex of D_2 is adjacent in G to at most one vertex of T . However, yx_1, y_1 and x_2 together form an induced copy of $P_3 \cup 2P_1$. Therefore, we assume $R = P_2 \cup 3P_1$.

We first claim that $|V(D_i)| = 3$ for each $i \in [1, 2]$. Otherwise, say $|V(D_2)| \geq 4$. Let $y \in T$ and $x \in V(D_1)$ such that $xy \in E(G)$. Take $x_1 \in V(D_2)$ such that $e_G(x_1, T) = 0$, which exists as $|N_G(T) \cap V(D_2)| = 3$. Then xy, x_1 and $T \setminus \{y\}$ together form an induced copy of $P_2 \cup 3P_1$, giving a contradiction. We next claim that $D_i = K_3$ for each $i \in [1, 2]$. Otherwise, say $D_1 \neq K_3$. As D_1 is connected, it follows that $D_1 = P_3$. If also $D_2 \neq K_3$ and so $D_2 = P_3$, then deleting the two vertices of degree 2 from both D_1 and D_2 gives three components (note that each vertex of T is adjacent in G to one vertex of D_1 and one vertex of D_2), showing that $\tau(G) \leq 2/3 < 1$. Thus $D_2 = K_3$. We let $x_1, x_2 \in V(D_1)$ be nonadjacent, $y_1, y_2 \in T$ such that $e_G(x_i, y_i) = 1$ for each $i \in [1, 2]$, and $z_1, z_2 \in V(D_2)$ such that $e_G(y_i, z_i) = 1$ for each $i \in [1, 2]$. Let $y \in T \setminus \{y_1, y_2\}$. Then $z_1 z_2, y, x_1$ and x_2 together form an induced copy of $P_2 \cup 3P_1$, giving a contradiction.

Thus $|V(D_i)| = 3$ and $D_i = K_3$ for each $i \in [1, 2]$. However, this implies that $G \cong H_0$.

CASE 2: $|\bigcup_{k \geq 1} C_{2k+1}| = 1$ and $|S| = 1$.

Since G is 1-tough and so is 2-connected, we have $C_{2k} = \emptyset$ for any $k \geq 0$: there is no edge between T and any even component of $G - (S \cup T)$ by Lemma 8(2), and $|S| = 1$. Let $D \in \bigcup_{k \geq 1} C_{2k+1}$ be the odd component of $G - (S \cup T)$. Assume first that $R = P_3 \cup 2P_1$. Then we have $|V(D)| = 3$. Otherwise, $|V(D)| \geq 4$. Let $x \in V(D)$ such that $e_G(x, T) = 0$ and P be a shortest path of D from x to a vertex, say $x_1 \in V(D) \cap N_G(T)$. Let $y \in T$ such that $e_G(x_1, y) = 1$. Then xPx_1y and $T \setminus \{y\}$ contain an induced copy of R , a contradiction.

Since G does not contain H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$, it follows that $D \neq K_3$. As D is connected, it follows that $D = P_3$. Now deleting the vertex in S together with the degree 2 vertex of D produces three components, showing $\tau(G) \leq 2/3 < 1$.

Therefore, we assume now that $R = P_2 \cup 3P_1$. Since G does not contain H_1 as a spanning subgraph such that $E(G) \setminus E(H_1) \subseteq E_G(S, V(G) \setminus (T \cup S))$, the argument for the case $R = P_3 \cup 2P_1$ above implies that $|V(D)| \geq 4$. We claim that $|V(D)| = 4$. If $|V(D)| \geq 5$, we let $x_1, x_2 \in V(D) \setminus N_G(T)$ be any two distinct vertices. If $x_1 x_2 \in E(G)$, then $x_1 x_2$ together with T form an induced copy of R , a contradiction. Thus $V(D) \setminus N_G(T)$ is an independent set in G . However, $c(G - (S \cup (N_G(T) \cap V(D)))) = |T| + |V(D) \setminus N_G(T)| \geq 5$, implying $\tau(G) \leq 4/5 < 1$.

Thus $|V(D)| = 4$. Let $x \in V(D)$ such that $e_G(x, T) = 0$. Since G does not contain H_i as a spanning subgraph such that $E(G) \setminus E(H_i) \subseteq E_G(S, V(G) \setminus (T \cup S))$ for each $i \in [2, 4]$, it follows that either $d_D(x) \leq 2$ or $d_D(x) = 3$ and $D = K_{1,3}$. If $d_D(x) = 3$, then as $D = K_{1,3}$, we have $c(G - (S \cup \{x\})) = 3$, implying $\tau(G) \leq 2/3 < 1$. Thus $d_D(x) \leq 2$. Let $V(D) = \{x, x_1, x_2, x_3\}$ and assume $xx_1 \notin E(D)$. Then $c(G - (S \cup \{x_2, x_3\})) = 4$, implying $\tau(G) \leq 3/4 < 1$. The proof of Case 2 is complete. \square

Thus by Claim 1 and the fact that $R \neq P_4 \cup P_1$, we can assume $R \notin \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1\}$ from this point on. Therefore we have $t > 1$. This implies that G is 3-connected and so $\delta(G) \geq 3$. Thus $|S| + |\bigcup_{k \geq 0} C_{2k+1}| \geq 3$ by Lemma 8(1)-(4).

Claim 2. $|T| \geq 5$.

Proof. Equation (2) implies $|T| \geq 3$. Assume to the contrary that $|T| \leq 4$. We consider the following two cases.

CASE 1: $|T| = 3$.

Since $|S| + |\bigcup_{k \geq 0} C_{2k+1}| \geq 3$, we already have a contradiction to Equation (1) if $C_1 = \emptyset$. Thus $C_1 \neq \emptyset$, which gives $|S| \geq 2$ by Lemma 10(1). However, we again get a contradiction to Equation (1) as $\bigcup_{k \geq 1} C_{2k+1} \neq \emptyset$ by Equation (2).

CASE 2: $|T| = 4$.

By Lemma 8(3), we know that $C_{2k+1} = \emptyset$ for any $k \geq 2$. First assume $|S| \leq 1$. Then $C_1 = \emptyset$ by Lemma 10(1). By Lemma 8, there are at least $3|T| = 12$ edges going from T to vertices in S and components in $\bigcup_{k \geq 1} C_{2k+1}$. As $C_{2k+1} = \emptyset$ for any $k \geq 2$, it follows that $|C_3| \geq 4$ if $|S| = 0$ and $|C_3| \geq 3$ if $|S| = 1$, contradicting Equation (1).

Next, assume $|S| \geq 2$. By Equations (1) and (2), we have $|S| = 2$. Let D be the single component in C_3 . Define W_D to be a set of 2 vertices in D which are all adjacent in G to some vertex from T . Then $S \cup W_D$ is a cutset in G such that $|S \cup W_D| = 4$ and $c(G - (S \cup W_D)) \geq |T| = 4$, contradicting $\tau(G) \geq t > 1$. \square

By Claim 2 and Lemma 11(1), we see that G contains an induced $R = P_4 \cup 3P_1$. Thus we may assume $R \notin \{P_4 \cup P_1, P_3 \cup 2P_1, P_2 \cup 3P_1, P_4 \cup 3P_1\}$ from this point on.

Claim 3. $R \notin \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$ unless G falls under the exceptional cases as in (a) and (b) of Theorem 13(2) or as in Theorem 13(3).

Proof. We may assume that G does not fall under any of the exceptional cases as in (a) and (b) of Theorem 13(2) and as in Theorem 13(3). Thus we show that $R \notin \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$.

Assume to the contrary that $R \in \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$. By Lemma 11(1), G contains an induced $P_4 \cup aP_1$, where $a = |T| - 2$. If $a \geq 5$, then each of $P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1$, and $7P_1$ is an induced subgraph of $P_4 \cup aP_1$, a contradiction.

Thus $a \leq 4$ and so $|T| \leq 6$. As $|T| \leq 6$, we have that $\bigcup_{k \geq 2} \mathcal{C}_{2k+1} = \emptyset$ by Lemma 8(3). By Claim 2, $|T| \geq 5$. Thus, we have two cases.

CASE 1: $|T| = 5$.

As $|T| = 5$, we have $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 3$. We consider two cases regarding whether or not $|\mathcal{C}_3 \cup \mathcal{C}_5| \geq 2$.

CASE 1.1: $|\mathcal{C}_3 \cup \mathcal{C}_5| = 1$.

Let $D \in \mathcal{C}_{2k+1} \subseteq \mathcal{C}_3 \cup \mathcal{C}_5$ be the only odd component of $G - (S \cup T)$. Note that k is either one or two. By Equation (1), $5 \geq |S| + k + 1$, so $|S| \leq 4 - k$.

We first claim that $\mathcal{C}_{2\ell} = \emptyset$ for any $\ell \geq 0$. Suppose to the contrary that $\mathcal{C}_{2\ell} \neq \emptyset$ for some $\ell \geq 0$. Then $\ell = 0$ by Lemma 8(2). Let W_D be a set of $2k$ vertices (which exist by Lemma 8(4)) from D which are adjacent in G to vertices from T . Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k}{|T| + 1} \leq \frac{4 + k}{6} \leq \frac{6}{6} = 1,$$

contradicting $t > 1$.

We then claim $k = 2$. For otherwise, $k = 1$. Let W_D be a set of $2k$ vertices from D which are adjacent in G to vertices from T . Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k}{5} \leq \frac{4 + k}{5} = \frac{5}{5} = 1,$$

contradicting $t > 1$. Thus $k = 2$.

From $k = 2$, we get $|S| \leq 2$. We lastly claim $|S| = 2$. For otherwise, $|S| \leq 1$. Let W_D be a set of $2k$ vertices from D which are adjacent in G to vertices from T . Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k}{5} \leq \frac{1 + 2k}{5} = \frac{5}{5} = 1,$$

contradicting $t > 1$.

Since $|S| = 2$ and D is the only odd component of $G - (S \cup T)$, we know that every vertex of T is adjacent in G to every vertex from S by $\delta(G) \geq 3$. We now consider two subcases to finish the proof of Case 1.1.

CASE 1.1.1: $|V(D)| \geq 6$.

For $R = P_3 \cup 4P_1$, let $x \in V(D)$ such that $e_G(x, T) = 0$. Let P be a shortest path in D from x to a vertex, say x^* from $N_G(T) \cap V(D)$. Let $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then xPx^*y^* and $T \setminus \{y^*\}$ contain $P_3 \cup 4P_1$ as an induced subgraph. We consider next that $R = 6P_1$. Then T and the vertex of D that is not adjacent in G to any vertex from T form an induced $6P_1$, giving a contradiction. For $R = 7P_1$, let W_D be the set of $2k + 1$ vertices (which exist by Lemma 8(4)) from D which are adjacent in G to vertices from T . Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 2k + 1}{|T| + 1} \leq \frac{4 + k + 1}{6} = \frac{7}{6},$$

giving a contradiction to $t > 7/6$.

Lastly, we consider $R = P_2 \cup 5P_1$. For any $x \in V(D)$ such that $e_G(x, T) = 0$, it must be the case that x is adjacent in G to every vertex from $N_G(T) \cap V(D)$. Otherwise, let $x^* \in N_G(T) \cap V(D)$ such that $xx^* \notin E(G)$. Let $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then x^*y^* and $(T \setminus \{y^*\}) \cup \{x\}$ contain $P_2 \cup 5P_1$ as an induced subgraph. Furthermore, if $|V(D)| - |N_G(T) \cap V(D)| \geq 2$, then $V(D) \setminus (N_G(T) \cap V(D))$ is an independent set in G . Otherwise, an edge with both endvertices from $V(D) \setminus (N_G(T) \cap V(D))$ together with T induces $P_2 \cup 5P_1$. Thus if $|V(D)| \geq 7$, let W_D be the set of $2k + 1$ vertices (which exist by Lemma 8(4)) from D which are adjacent in G to vertices from T . Then $S \cup W_D$ forms a cutset and we have

$$t \leq \frac{|S| + 5}{|T| + 2} \leq \frac{7}{7},$$

giving a contradiction to $t > 1$. Thus $|V(D)| = 6$. Let $x \in V(D)$ be the vertex such that $e_G(x, T) = 0$. Then it must be the case that $D - x$ has at most two components. Otherwise, we have $t \leq \frac{|S \cup \{x\}|}{3} = 1$.

Assume first that $c(D - x) = 2$. Let D_1 and D_2 be the two components of $D - x$, and assume further that $|V(D_1)| \leq |V(D_2)|$. Then as $|V(D - x)| = 5$, we have two possibilities: either $|V(D_1)| = 1$ and $|V(D_2)| = 4$ or $|V(D_1)| = 2$ and $|V(D_2)| = 3$. Since $\delta(G) \geq 3$, if $|V(D_1)| = 1$, then the vertex from D_1 must be adjacent in G to at least one vertex from S . When $|V(D_2)| = 4$ and $D_2 \neq K_4$, then D_2 has a cutset W of size 2 such that $c(D_2 - W) = 2$. Then $S \cup W \cup \{x\}$ is a cutset of G such that $c(G - (S \cup W \cup \{x\})) = 5$, showing $t \leq 1$. Thus $D_2 = K_4$. However, this shows that G contains H_6 as a spanning subgraph. When $|V(D_2)| = 3$ and $D_2 \neq K_3$, then D_2 has a cutvertex x^* . Then $S \cup \{x, x^*\}$ is a cutset of G such that $c(G - (S \cup \{x, x^*\})) = 4$, showing that $t \leq \frac{4}{4} = 1$. Thus $D_2 = K_3$; however, this shows that G contains H_7 as a spanning subgraph.

Assume then that $c(D - x) = 1$. Let $D^* = D - x$. If $\delta(D^*) \geq 3$, then D^* is Hamiltonian and so G contains H_{10} as a spanning subgraph. Thus we assume $\delta(D^*) \leq 2$.

Assume first that D^* has a cutvertex x^* . Then $c(D^* - x^*) = 2$: as if $c(D^* - x^*) \geq 3$, then $c(G - (S \cup \{x, x^*\})) \geq 4$, implying $t \leq 1$. Let D_1^* and D_2^* be the two components of $D^* - x^*$, and assume further that $|V(D_1^*)| \leq |V(D_2^*)|$. Then as $|V(D^* - x^*)| = 4$, we have two possibilities: either $|V(D_1^*)| = 1$ and $|V(D_2^*)| = 3$ or $|V(D_1^*)| = 2$ and $|V(D_2^*)| = 2$. When $|V(D_2^*)| = 3$ and $D_2^* \neq K_3$, then D_2^* has a cutvertex x^{**} . Then $S \cup \{x, x^*, x^{**}\}$ is a cutset of G such that $c(G - (S \cup \{x, x^*, x^{**}\})) = 5$, showing that $t \leq 1$. Thus $D_2^* = K_3$. The vertex x^* is a cutvertex of D^* and so is adjacent in D^* to a vertex of D_1^* and a vertex of D_2^* . However, this shows that G contains H_8 as a spanning subgraph. When $|V(D_2^*)| = 2$, as G does not contain H_8 or H_9 as a spanning subgraph, x^* is adjacent in G to exactly one vertex, say x_1^* , of D_1^* and to exactly one vertex, say x_2^* , of D_2^* . Then $S \cup \{x, x_1^*, x_2^*\}$ is a cutset of G whose removal produces 5 components, showing $\tau(G) \leq 1$.

Assume then that D^* is 2-connected. As $\delta(D^*) \leq 2$, D^* has a minimum cutset W of size 2. If $c(D^* - W) = 3$, then we have $c(G - (S \cup W \cup \{x\})) = 5$, showing $t \leq 1$. Thus $c(D^* - W) = 2$. Then by analyzing the connection in D^* between W and the two components of $D^* - W$, we see that D^* contains C_5 as a spanning subgraph, showing that G contains H_{10} as a spanning subgraph.

CASE 1.1.2: $|V(D)| = 5$.

Since H_5 is an exceptional graph for each $R \in \{P_3 \cup 4P_1, P_2 \cup 5P_1, 6P_1, 7P_1\}$, we assume that G does not contain H_5 as a spanning subgraph. Thus $D \neq K_5$. As $D \neq K_5$, D has a cutset W_D of size at most 3 such that each component of $D - W_D$ is a single vertex. Then

$$t \leq \frac{|S| + |W_D|}{|T|} \leq \frac{4 - 2 + 3}{5} = 1,$$

a contradiction.

CASE 1.2: $|C_3 \cup C_5| \geq 2$.

By Equation (1), we have

$$4 \geq |S| + \sum_{k \geq 1} k|C_{2k+1}|.$$

So one of the following holds:

1. $S = \emptyset$ and either $|C_5| = 2$ and $|C_3| = 0$, $|C_5| = 1$ and $|C_3| \leq 2$, or $|C_5| = 0$ and $|C_3| \leq 4$. In this case, $C_1 = \emptyset$ by Lemma 10(1). Thus by Lemma 8(3), we have $e_G(T, V(G) \setminus T) \leq 12 < 3|T| = 15$.
2. $|S| = 1$ and either $|C_5| = 1$ and $|C_3| = 1$, or $|C_5| = 0$ and $|C_3| \leq 3$. In this case, again $C_1 = \emptyset$ by Lemma 10(1). This implies there are a maximum of 14 edges incident to vertices in T , a contradiction.
3. $|S| = 2$ and either $|C_5| = 1$ and $|C_3| = 0$, or $|C_5| = 0$ and $|C_3| = 2$.

If $|C_5| = 1$ and $|C_3| = 0$, then we let $D \in C_5$ be the only odd component of $G - (S \cup T)$. If $|C_5| = 0$ and $|C_3| = 2$, let $C_3 = \{D_1, D_2\}$. Note that $|V(D_i)| \geq 3$ by Lemma 8(4) for each $i \in [1, 2]$.

We first claim that $C_{2\ell} = \emptyset$ for any $\ell \geq 0$. Suppose to the contrary that $C_{2\ell} \neq \emptyset$ for some $\ell \geq 0$. Then $\ell = 0$ by Lemma 8(2). Let W be a set of 4 vertices from either D or consisting of 2 vertices from D_1 and 2 vertices from D_2 that are all adjacent in G to vertices from T .

$$t \leq \frac{|S| + 4}{|T| + 1} \leq \frac{6}{6} = 1,$$

contradicting $t > 1$.

When $|C_5| = 1$ and $|C_3| = 0$, we also know that every vertex of T is adjacent in G to every vertex from S by $\delta(G) \geq 3$. Then $D \neq K_5$ since G does not contain H_5 as a spanning subgraph. As $D \neq K_5$, D has a cutset W_D of size at most 3 such that each component of $D - W_D$ is a single vertex. Then

$$t \leq \frac{|S| + |W_D|}{|T|} \leq \frac{4 - 2 + 3}{5} = 1,$$

a contradiction.

Thus we assume that $|C_5| = 0$ and $|C_3| = 2$. Since $|T| = 5$, there exists $y_0 \in T$ such that $e_G(y_0, D_i) = 1$ for each $i \in [1, 2]$. If $R = P_3 \cup 4P_1$, then T together with the two neighbors of y_0 from $V(D_1) \cup V(D_2)$ induce R . If $R = 6P_1$, then $T \setminus \{y_0\}$ together with the two neighbors of y_0 from $V(D_1) \cup V(D_2)$ gives an induced $6P_1$. If $R = 7P_1$, let $W_{D_i} \subseteq V(D_i) \setminus N_G(y_0)$ be the two vertices of D_i that are adjacent in G to vertices from T . Then $c(G - (S \cup W_{D_1} \cup W_{D_2} \cup \{y_0\})) = |T| - 1 + 2 = 6$. Thus $t \leq \frac{2+2+2+1}{6} = \frac{7}{6}$, contradicting $t > \frac{7}{6}$. Lastly, assume $R = P_2 \cup 5P_1$. If one of D_i has at least 4 vertices, say $|V(D_2)| \geq 4$, then let $x \in V(D_2)$ such that $e_G(x, T) = 0$, $x^* \in V(D_1)$ and $y^* \in T$ such that $e_G(x^*, y^*) = 1$. Then x^*y^* and $(T \setminus \{y^*\}) \cup \{x\}$ induce $P_2 \cup 5P_1$. Thus $|V(D_1)| = |V(D_2)| = 3$. If one of D_i , say $D_2 \neq K_3$, then D_2 has a cutvertex x . Let W be the set of any two vertices of D_1 . Then $S \cup W \cup \{x\}$ is a cutset of G such that $c(G - (S \cup W \cup \{x\})) = 5$, showing that $t \leq \frac{5}{5} = 1$. Thus $D_1 = D_2 = K_3$. However, this shows that G contains H_{11} as a spanning subgraph.

CASE 2: $|T| = 6$.

In this case, by Lemma 11(1), G has an induced $P_4 \cup 4P_1$, which contains each of $P_3 \cup 4P_1$, $P_2 \cup 5P_1$ and $6P_1$ as an induced subgraph. So we assume $R = 7P_1$ in this case and thus $t > \frac{7}{6}$.

Recall for $y \in T$, $h(y) = |\{D : D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1} \text{ and } e_G(y, D) \geq 1\}|$. If there exists $y_0 \in T$ such that $h(y_0) \geq 2$, we let x_1, x_2 be the two neighbors of y_0 from the two corresponding components in $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$, respectively. Then $T \setminus \{y_0\}$ together with $\{x_1, x_2\}$ induces $7P_1$. Thus $h(y) \leq 1$ for each $y \in T$. This, together with $|T| = 6$, implies that we have either $|\mathcal{C}_3| \in \{1, 2\}$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$ or $|\mathcal{C}_5| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $1 \leq k \neq 2$.

If $|\mathcal{C}_3| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, then $|S| \leq 4$ by Equation (1). Let W be a set of two vertices from the component in \mathcal{C}_3 that are adjacent in G to vertices from T . Then $c(G - (S \cup W)) \geq 6$, indicating $t \leq \frac{4+2}{6} < \frac{7}{6}$. For the other two cases, we have $|S| \leq 3$. If $|\mathcal{C}_3| = 2$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $k \geq 2$, let W be a set of four vertices, with two from one component in \mathcal{C}_3 and the other two from the other component in \mathcal{C}_3 , which are adjacent in G to vertices from T . If $|\mathcal{C}_5| = 1$ and $\mathcal{C}_{2k+1} = \emptyset$ for any $1 \leq k \neq 2$, let W be a set of four vertices from the component in \mathcal{C}_5 that are adjacent in G to vertices from T . Then we have $c(G - (S \cup W)) \geq 6$, indicating that $t \leq \frac{3+4}{6} = \frac{7}{6}$. \square

By Claim 3, we now assume that $R \in \{P_7 \cup 2P_1, P_5 \cup P_2, P_4 \cup P_3, P_3 \cup 2P_2, 3P_2 \cup P_1\}$ and $t = 3/2$.

Claim 4. *There exists $y \in T$ with $h(y) \geq 2$.*

Proof. Assume to the contrary that for every $y \in T$, we have $h(y) \leq 1$. Define the following partition of T :

$$T_0 = \{y \in T : e_G(y, D) = 0 \text{ for all } D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}\},$$

$$T_1 = \{y \in T : e_G(y, D) = 1 \text{ for some } D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}\}.$$

Note that $|T_1| = \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}|$ by Lemma 8(3) and (4). For each $D \in \mathcal{C}_{2k+1}$ for some $k \geq 1$, we let W_D be a set of $2k$ vertices that each has in G a neighbor from T . As each $D - W_D$ is connected to exactly one vertex from T and each component from \mathcal{C}_1 is connected to exactly one vertex from T , it follows that

$$W = S \cup \bigcup_{D \in \bigcup_{k \geq 1} \mathcal{C}_{2k+1}} W_D$$

satisfies $c(G - W) \geq |T| \geq 5$, where $|T| \geq 5$ is by Claim 2.

By the toughness of G , we have

$$\begin{aligned} |S| + \sum_{k \geq 1} 2k|\mathcal{C}_{2k+1}| &= |W| \geq t|T| = t(|T_0| + |T_1|) \\ &= t \left(|T_0| + \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}| \right) \\ &\geq \frac{3}{2} \left(|T_0| + \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}| \right). \end{aligned}$$

Thus

$$|S| + \sum_{k \geq 1} k|\mathcal{C}_{2k+1}| \geq 3|T_0|/2 + \sum_{k \geq 1} (2k+3/2)|\mathcal{C}_{2k+1}| > |T_0| + \sum_{k \geq 1} (2k+1)|\mathcal{C}_{2k+1}| = |T|,$$

contradicting Equation (1). \square

By Claim 4, there exists $y \in T$ such that $h(y) \geq 2$. Then as $|T| \geq 5$, by Lemma 11(2), G contains an induced $P_7 \cup 2P_1$. Thus we assume that $R \neq P_7 \cup 2P_1$. We assume first that $|\bigcup_{k \geq 1} \mathcal{C}_{2k+1}| \geq 3$ and let D_1, D_2, D_3 be three distinct odd components from $\bigcup_{k \geq 1} \mathcal{C}_{2k+1}$. Let $y_0 \in T$ such that $h(y_0) \geq 2$. We assume, without loss of generality, that $e_G(y_0, D_1) = e_G(y_0, D_2) = 1$. By Lemma 11(2), G contains an induced $P_b \cup aP_1$, where $b \geq 7$ and $a = |T| - 3$, and the graph $P_b \cup aP_1$ can be chosen such that the vertices in aP_1 are from T and the path P_b has the form $y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$, where $y_0, y_1, y_2 \in T$ and $x_1^* P_1 x_1$ and $x_2^* P_2 x_2$ are respectively contained in D_1 and D_2 such that $e_G(x, T) = 0$ for every internal vertex x from P_1 and P_2 . If one of y_1 and y_2 , say y_1 has a neighbor z_1 from $V(D_3)$, then $z_1 y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ and $T \setminus \{y_0, y_1, y_2\}$ induce $P_8 \cup 2P_1$, which contains each of $P_5 \cup P_2$, $P_4 \cup P_3$, and $3P_2 \cup P_1$ as an induced subgraph. Let $z_2 \in V(D_3)$ be a neighbor of z_1 . Then $z_2 z_1 y_1 x_1^* P_1 x_1 y_0 x_2 P_2 x_2^* y_2$ contains an induced $P_3 \cup 2P_2$ whether $e_G(z_2, \{y_0, y_2\}) = 0$ or 1. Thus we

assume $e_G(y_i, D_3) = 0$ for each $i \in [1, 2]$ and so we can find $y_3 \in T \setminus \{y_0, y_1, y_2\}$ and $z \in V(D_3)$ such that $y_3z \in E(G)$. Then $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ and zy_3 contains an induced $P_7 \cup P_2$, which contains each of $P_5 \cup P_2$, $P_3 \cup 2P_2$ and $3P_2 \cup P_1$ as an induced subgraph. We are only left to consider $R = P_4 \cup P_3$. As $e_G(y_i, D_3) = 0$ for each $i \in [1, 2]$, we can find distinct $y_3, y_4 \in T \setminus \{y_0, y_1, y_2\}$ and distinct $z_1, z_2 \in V(D_3)$ such that $y_3z_1, y_4z_2 \in E(G)$. We let P be a shortest path in D_3 connecting z_1 and z_2 . If $e_G(y_0, V(P)) = 0$, then $y_3z_1Pz_2y_4$ and $y_1x_1^*P_1x_1y_0x_2P_2x_2^*y_2$ contains an induced $P_4 \cup P_3$. Thus $e_G(y_0, V(P)) = 1$. Then $y_3z_1Pz_2y_4$ and $y_1x_1^*P_1x_1$ together contain an induced $P_4 \cup P_3$.

Thus $|\bigcup_{k \geq 1} C_{2k+1}| = 2$. Let $D_1, D_2 \in \bigcup_{k \geq 1} C_{2k+1}$ be the two components. Define the following partition of T :

$$\begin{aligned} T_0 &= \{y \in T : e_G(y, D_1) = e_G(y, D_2) = 0\}, \\ T_{11} &= \{y \in T : e_G(y, D_1) = 1 \text{ and } e_G(y, D_2) = 0\}, \\ T_{12} &= \{y \in T : e_G(y, D_1) = 0 \text{ and } e_G(y, D_2) = 1\}, \\ T_2 &= \{y \in T : e_G(y, D_1) = e_G(y, D_2) = 1\}. \end{aligned}$$

By Claim 4, we have $T_2 \neq \emptyset$. Now define the following vertex sets:

$$W_1 = N_G(T_{11}) \cap V(D_1), \quad \text{and} \quad W_2 = N_G(T_{12}) \cap V(D_2).$$

We have that $|W_1| = |T_{11}|$ and $|W_2| = |T_{12}|$. Now let $W = S \cup W_1 \cup W_2$. Then W is a cutset of G with $c(G - W) \geq |T_0| + |T_{11}| + |T_{12}| + 1$ since $T_2 \neq \emptyset$. By toughness, $|W| \geq \frac{3}{2}(|T_0| + |T_{11}| + |T_{12}| + 1)$. Since $|W| = |S| + |W_1| + |W_2| = |S| + |T_{11}| + |T_{12}|$, we have $|S| + |T_{11}| + |T_{12}| \geq \frac{3}{2}|T_0| + \frac{3}{2}|T_{11}| + \frac{3}{2}|T_{12}| + \frac{3}{2}$. This implies

$$|S| \geq \frac{3}{2}|T_0| + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| + 1.$$

Thus,

$$|S| + k_1 + k_2 \geq \frac{3}{2}|T_0| + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}| + 1 + k_1 + k_2, \quad (3)$$

where $2k_i + 1 = e_G(T, D_i)$ for each $i \in [1, 2]$.

On the other hand, we have

$$\begin{aligned} |T| &= |T_0| + (2k_1 + 1 + 2k_2 + 1 - |T_2|) \\ &= |T_0| + (2k_1 + 2k_2 + 2) - \frac{1}{2}(2k_1 + 1 + 2k_2 + 1 - |T_{11}| - |T_{12}|) \\ &= |T_0| + k_1 + k_2 + 1 + \frac{1}{2}|T_{11}| + \frac{1}{2}|T_{12}|. \end{aligned}$$

Using the size of T and (3), we get $|S| + k_1 + k_2 \geq |T|$, showing a contradiction to Equation (1).

The proof of Theorem 13 is now finished. \square

4. Proof of Theorems 5 and 6

Recall that for a graph G , $\alpha(G)$, the independence number of G , is the size of a largest independent set in G .

Proof of Theorem 5. For each $i \in [0, 11]$, H_i does not contain a 2-factor by Lemma 7. Thus to finish proving Theorem 5, we are only left to show the three claims below.

Claim 5. The graph H_i is $(P_2 \cup 3P_1)$ -free, H_1 is $(P_3 \cup 2P_1)$ -free, and $\tau(H_i) = 1$ for each $i \in [0, 4]$.

Proof. We first show that H_i is $(P_2 \cup 3P_1)$ -free for each $i \in [0, 4]$. We only show this for H_0 , as the proofs for H_i for $i \in [1, 4]$ are similar. In H_0 , there are two types of edges xy : $x, y \in V(D_j)$ or $x \in V(D_j)$ and $y \in V(T)$, where $j \in [1, 2]$. Without loss of generality first consider the edge $v_1v_2 \in E(D_1)$ and the subgraph $F_1 = H_0 - (N_{H_0}[v_1] \cup N_{H_0}[v_2])$. We see $\alpha(F_1) = 2$. Now, without loss of generality, consider the edge v_1t_1 and the subgraph $F_2 = H_0 - (N_{H_0}[v_1] \cup N_{H_0}[t_1])$. We see $\alpha(F_2) = 2$. In either case, $P_2 \cup 3P_1$ cannot exist as an induced subgraph in H_0 . Thus H_0 is $(P_2 \cup 3P_1)$ -free.

Then we show that H_1 is $(P_3 \cup 2P_1)$ -free. Three types of induced paths abc of length 3 exist: $a \in S, b \in T, c \in V(D)$, $a \in T, b, c \in V(D)$, or $a \in T, b = x, c \in T$. Without loss of generality, consider the path xt_1v_1 and the subgraph $F_1 = H_1 - (N_{H_1}[x] \cup N_{H_1}[t_1] \cup N_{H_1}[v_1])$. We see that F_1 is a null graph. Then, without loss of generality, consider the path $t_1v_1v_2$ and the subgraph $F_2 = H_1 - (N_{H_1}[t_1] \cup N_{H_1}[v_1] \cup N_{H_1}[v_2])$. We see $|V(F_2)| = 1$. Now, without loss of generality, consider the path t_1xt_2 and the subgraph $F_2 = H_1 - (N_{H_1}[t_1] \cup N_{H_1}[x] \cup N_{H_1}[t_2])$. We see $|V(F_2)| = 1$. In either case, $P_3 \cup 2P_1$ cannot exist as an induced subgraph in H_1 . Thus H_1 is $(P_3 \cup 2P_1)$ -free.

Let $i \in [0, 4]$. As $\delta(H_i) = 2$, $\tau(H_i) \leq 1$. It suffices to show $\tau(H_i) \geq 1$. Since H_i is 2-connected, we show that $c(H_i - W) \leq |W|$ for any $W \subseteq V(H_i)$ such that $|W| \geq 2$. If $|W| = 2$, by considering all the possible formations of W , we have $c(H_i - W) \leq |W|$. Thus we assume $|W| \geq 3$.

Assume by contradiction that there exists $W \subseteq V(H_i)$ with $|W| \geq 3$ and $c(H_i - W) \geq |W| + 1 \geq 4$. The size of a largest independent set of each H_0, H_2, H_3 , and H_4 is 4, and of H_1 is 3. Since $c(H_i - W)$ is bounded above by the size of a largest independent set of H_i , we already obtain a contradiction if $i = 1$ or $|W| \geq 4$. So we assume $i \in \{0, 2, 3, 4\}$ and $|W| = 3$.

As $c(H_i - W) \geq 4$, for the graph H_0 , we must have $\{v_1, v_2, v_3\} \cap W \neq \emptyset$ and $\{v_4, v_5, v_6\} \cap W \neq \emptyset$. As $|W| = 3$, we have either $W \cap T = \emptyset$ or $|W \cap T| = 1$. In either case, by checking all the possible formations of W , we get $c(H_0 - W) \leq 2$, contradicting the choice of W .

As $c(H_i - W) \geq 4$, for each $i \in [2, 4]$, we must have $x \in W$. Thus $t_j \notin W$ for $j \in [1, 3]$, as otherwise, $c(H_i - (W \setminus \{t_j\})) \geq 4$, contradicting the argument previously that $c(H_i - W^*) \leq 2$ for any $W^* \subseteq V(H_i)$ and $|W^*| \leq 2$. As $|W| = 3$, we then have $|W \cap \{v_1, v_2, v_3, v_4\}| = 2$. However, $c(H_i - W) \leq 3$ for $W = \{x, v_k, v_\ell\}$ for all distinct $k, \ell \in [1, 4]$. We again get a contradiction to the choice of W . \square

Claim 6. The graph H_5 with $p = 5$ is $(P_3 \cup 4P_1)$ -free, $(P_2 \cup 5P_1)$ -free, and $6P_1$ -free with $\tau(H_5) = \frac{6}{5}$.

Proof. Let $p = 5$ and D be the odd component of $H_5 - (S \cup T)$. Note that $D = K_p = K_5$.

We first show that H_5 is $(P_3 \cup 4P_1)$ -free. There are three types of induced paths xyz of length 3 in H_5 : $x \in S, y \in T, z \in V(D)$ or $x \in T, y, z \in V(D)$ or $x, z \in T, y \in S$. Without loss of generality, consider the path $x_1 t_1 y_1$ and the subgraph $F_1 = H_5 - (N_{H_5}[x_1] \cup N_{H_5}[t_1] \cup N_{H_5}[y_1])$. We see that F_1 is a null graph. Now consider the path $t_1 y_1 y_2$ and the subgraph $F_2 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[y_1] \cup N_{H_5}[y_2])$. We see $\alpha(F_2) = 3$. Finally consider the path $t_1 x_1 t_2$ and the subgraph $F_3 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[x_1] \cup N_{H_5}[t_2])$. We see $\alpha(F_3) = 3$. In any case, an induced copy of $P_3 \cup 4P_1$ cannot exist in H_5 . Thus H_5 is $(P_3 \cup 4P_1)$ -free.

We then show that H_5 is $(P_2 \cup 5P_1)$ -free. There are three types of edges xy in H_5 : $x \in S, y \in T$ or $x \in T, y \in V(D)$ or $x, y \in V(D)$. Without loss of generality, consider the edge $x_1 t_1$ and the subgraph $F_1 = H_5 - (N_{H_5}[x_1] \cup N_{H_5}[t_1])$. We see $|V(F_1)| = 4$. Now consider the edge $t_1 y_1$ and the subgraph $F_2 = H_5 - (N_{H_5}[t_1] \cup N_{H_5}[y_1])$. We see $|V(F_2)| = 4$. Finally, consider the edge $y_1 y_2$ and the subgraph $F_3 = H_5 - (N_{H_5}[y_1] \cup N_{H_5}[y_2])$. We see $\alpha(F_3) = 3$. In any case, no induced copy of $P_2 \cup 5P_1$ can exist in H_5 . Thus H_5 is $(P_2 \cup 5P_1)$ -free.

We lastly show that H_5 is $6P_1$ -free. There are three types of vertices x in H_5 : $x \in S, x \in T$, or $x \in V(D)$. Without loss of generality, consider the vertex x_1 and the subgraph $F_1 = H_5 - N_{H_5}[x_1]$. We see $\alpha(F_1) = 1$. Now consider the vertex t_1 and the subgraph $F_2 = H_5 - N_{H_5}[t_1]$. We see $\alpha(F_2) = 4$. Finally, consider the vertex y_1 and the subgraph $F_3 = H_5 - N_{H_5}[y_1]$. We see $\alpha(F_3) = 4$. In any case, no induced copy of $6P_1$ can exist in H_5 . Thus H_5 is $6P_1$ -free.

We now show that $\tau(H_5) = \frac{6}{5}$. Let W be a toughset of H_5 . Then $S \subseteq W$. Otherwise, by the structure of H_5 , we have $c(H_5 - W) \leq 3$ and $|W| \geq 5$. As $S \subseteq W$ and the only neighbor of each vertex of T in $H_5 - S$ is contained in a clique of H_5 , we have $T \cap W = \emptyset$. Since $c(H_5 - W) \geq 2$, it follows that $W \cap V(D) \neq \emptyset$. Then $c(H_5 - W) = |W \cap V(D)|$ if $|W \cap V(D)| \leq 3$ or $|W \cap V(D)| = 5$, and $c(H_5 - W) = |W \cap V(D)| + 1$ if $|W \cap V(D)| = 4$. The smallest ratio of $\frac{|W|}{c(H_5 - W)}$ is $\frac{6}{5}$, which happens when $|W \cap V(D)| = 4$. \square

Claim 7. The graph H_i is $(P_2 \cup 5P_1)$ -free with $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 11]$.

Proof. We show first that each H_i is $(P_2 \cup 5P_1)$ -free. We do this only for the graph H_6 , as the proofs for the rest graphs are similar. For any edge $ab \in E(H_6)$, we see $\alpha(H_6 - (N_{H_6}[a] \cup N_{H_6}[b])) \leq 4$. Thus no induced copy of $(P_2 \cup 5P_1)$ can exist in H_6 . Thus H_6 is $(P_2 \cup 5P_1)$ -free.

We next show that $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 10]$. We have $c(H_i - (S \cup \{v_1, \dots, v_5\})) = 6$, implying $\tau(H_i) \leq \frac{7}{6}$. Suppose $\tau(H_i) < \frac{7}{6}$. Let W be a toughset of H_i . As each H_i is 3-connected, we have $|W| \geq 3$. Thus $c(H_i - W) \geq 3$. We have that either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_i - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one which contains $S \setminus W$. Since $c(H_i - W) \geq 3$, by the structure of H_i , it follows that we have either $T \subseteq W$ or $\{v_1, \dots, v_5\} \subseteq W$. In either case, we have $c(H_i - W) \leq 3$, implying $\frac{|W|}{c(H_i - W)} \geq \frac{5}{3} > \frac{7}{6}$, a contradiction. So $S \subseteq W$. By Lemma 12, $t_j \notin W$ for all $j \in [1, 5]$. Thus each $t_j \in V(H_i - W)$. Now either $v_0 \in W$ or $v_0 \notin W$. Suppose $v_0 \in W$, then we cannot have all $v_j \in W$ without violating Lemma 12. In this case, the minimum ratio $\frac{|W|}{c(H_i - W)}$ occurs when $|W \cap \{v_1, v_2, v_3, v_4, v_5\}| = 3$. This implies $\frac{|W|}{c(H_i - W)} \geq \frac{6}{5} > \frac{7}{6}$, a contradiction. Thus $v_0 \notin W$ and we must have $v_0 \in V(H_i - W)$. This implies $\{v_1, \dots, v_5\} \subseteq W$ and $\frac{|W|}{c(H_i - W)} = \frac{7}{6}$, a contradiction. Thus $\tau(H_i) = \frac{7}{6}$ for each $i \in [6, 10]$.

Lastly we show $\tau(H_{11}) = \frac{7}{6}$. We have $c(H_{11} - (S \cup \{v_1, v_2, t_3, v_5, v_6\})) = 6$, implying $\tau(H_{11}) \leq \frac{7}{6}$. Suppose $\tau(H_{11}) < \frac{7}{6}$. Let W be a tough set of H_{11} . As H_{11} is 3-connected, we have $|W| \geq 3$. Thus $c(H_{11} - W) \geq 3$. We have that either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_{11} - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one which contains $S \setminus W$. Since $c(H_{11} - W) \geq 3$, by the structure of H_{11} , it follows that $|W| \geq 5$ and $c(H_{11} - W) \leq 4$. This implies $\frac{|W|}{c(H_{11} - W)} \geq \frac{5}{4} > \frac{7}{6}$, a contradiction. So $S \subseteq W$. By Lemma 12, $t_i \notin W$ for $i \in \{1, 2, 4, 5\}$. Thus $t_i \in V(H_{11} - W)$ for $i \in \{1, 2, 4, 5\}$ and we must have $W \cap \{v_1, v_2, v_3, v_4, v_5, v_6, t_3\} \neq \emptyset$. If $t_3 \notin W$, then $\frac{|W|}{c(H_{11} - W)} \geq \frac{6}{5} > \frac{7}{6}$,

a contradiction. Thus $t_3 \in W$. Then v_3 and v_4 are respectively in two distinct components of $H_{11} - W$ by Lemma 12. Thus $W \cap \{v_1, v_2, v_5, v_6\} \neq \emptyset$ as $c(H_{11} - W) \geq 3$. Furthermore, we have $c(H_{11} - W) = |W \cap \{v_1, v_2, v_5, v_6\}| + 2$. The smallest ratio of $\frac{|W|}{c(H_{11}-W)}$ is $\frac{7}{6}$, which happens when $\{v_1, v_2, v_5, v_6\} \subseteq W$. Again we get a contradiction to the assumption that $\tau(H_{11}) < \frac{7}{6}$. Thus $\tau(H_{11}) = \frac{7}{6}$. \square

The proof of Theorem 5 is complete. \square

Proof of Theorem 6. Let $p \geq 6$ and D be the odd component of $H_5 - (S \cup T)$. Note that $D = K_p$. Since $c(H_5 - (S \cup \{y_1, \dots, y_5\})) = 6$, we have $\tau(H_5) \leq \frac{7}{6}$. We show $\tau(H_5) \geq \frac{7}{6}$. Let W be a toughset of H_5 . Then either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_5 - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one containing $S \setminus W$. Since $c(H_5 - W) \geq 2$, by the structure of H_5 , it follows that we have either $T \subseteq W$ or $\{y_1, \dots, y_5\} \subseteq W$. In either case, we have $c(H_5 - W) \leq 3$, implying $\frac{|W|}{c(H_5-W)} \geq \frac{5}{3} > \frac{7}{6}$. Now suppose $S \subseteq W$. By Lemma 12, $t_i \notin W$ for all i . Thus each $t_i \in V(H_5 - W)$. Furthermore, $c(H_5 - W) = |W \cap V(D)| + 1$. Since W is a cutset of G , we have $|W \cap V(D)| \geq 2$. The smallest ratio of $\frac{|W|}{c(H_5-W)}$ is $\frac{7}{6}$, which happens when $|W \cap V(D)| = 5$.

For the graph H_{12} , we have $\tau(H_{12}) \leq 1$ by $\delta(H_{12}) \leq 2$. We show $\tau(H_{12}) \geq 1$. Let W be a toughset of H_{12} . Then either $S \subseteq W$ or $S \not\subseteq W$. Suppose the latter. Then we have $S \cap V(H_{12} - W) \neq \emptyset$. Then all vertices in $T \setminus W$ are contained in the same component as the one containing $S \setminus W$. Since $c(H_{12} - W) \geq 2$, by the structure of H_{12} , it follows that we have either $T \subseteq W$ or $\{y_1, y_2, y_3\} \subseteq W$. In either case, we have $c(H_{12} - W) \leq 2$, implying $\frac{|W|}{c(H_{12}-W)} \geq \frac{3}{2} > 1$. Now suppose $S \subseteq W$. By Lemma 12, $t_i \notin W$ for all i . Thus each $t_i \in V(H_{12} - W)$. This implies $|\{y_1, y_2, y_3\} \cap W| = 2$ or 3 . In either case we see $\frac{|W|}{c(H_{12}-W)} = 1$. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

The authors are very grateful to the anonymous referees for their careful reading and valuable comments. Songling Shan was supported by NSF grant DMS-2153938.

References

- [1] D. Bauer, G.Y. Katona, D. Kratsch, H.J. Veldman, Chordality and 2-factors in tough graphs, *Discrete Appl. Math.* 99 (2000) 323–329.
- [2] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* 5 (1973) 215–228.
- [3] H. Enomoto, B. Jackson, P. Katerinis, A. Saito, Toughness and the existence of k -factors, *J. Graph Theory* 9 (1) (1985) 87–95.
- [4] J. Kanno, S. Shan, Vizing's 2-factor conjecture involving toughness and maximum degree conditions, *Electron. J. Comb.* 26 (2) (2019) P2.17.
- [5] K. Ota, M. Sanka, Hamiltonian cycles in 2-tough $2K_2$ -free graphs, *J. Graph Theory* 101 (4) (2022) 769–781.
- [6] W.T. Tutte, A short proof of the factor theorem for finite graphs, *Can. J. Math.* 6 (1954) 347–352.