Received: 16 September 2021 | Revised: 8 April 2023 | Accepted: 6 June 2023

DOI: 10.1002/jgt.23005

ARTICLE

WILEY

Characterization of graphs of diameter 2 containing a homeomorphically irreducible spanning tree

Songling Shan[1] | **Shoichi Tsuchiya**[2]

[1]Department of Mathematics and Statistics, Auburn University, Auburn, Alabama, USA

[2]School of Network and Information, Senshu University, Kawasaki, Kanagawa, Japan

Correspondence
Shoichi Tsuchiya, School of Network and Information, Senshu University, Kawasaki, Kanagawa, Japan.
Email: s.tsuchiya@isc.senshu-u.ac.jp

Funding information
NSF, Grant/Award Number: DMS2153938; Senshu University 2022; Basic Science Research Projects from The Sumitomo Foundation; Japan Society for the Promotion of Science, Grant/Award Number: JP19K14584

Abstract

A spanning tree of a graph with no vertex of degree 2 is called a homeomorphically irreducible spanning tree (HIST) of the graph. In 1990, Albertson, Berman, Hutchinson, and Thomassen conjectured that every twin-free graph with diameter 2 contains a HIST. Recently, Ando disproved this conjecture and characterized twin-free graphs with diameter 2 that do contain a HIST. In this paper, we give a complete characterization of all graphs of diameter 2 that contain a HIST. This characterization gives alternative proofs for several known results.

KEYWORDS

diameter 2 graph, homeomorphically irreducible spanning tree, spanning tree

AMS CLASSIFICATION

05C10

1 | INTRODUCTION

We consider only simple graphs. Let G be a graph. For $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G and $d_G(v) := |N_G(v)|$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and the maximum degree of G, respectively. For $S \subset V(G)$, $G[S]$ is the subgraph of G induced on S, and $G - S := G[V(G) - S]$. For disjoint vertex sets $V_1, V_2 \subseteq V(G)$, let $E_G(V_1, V_2)$ be the set of all edges with one endvertex in V_1 and the other in V_2. When $V_1 = \{v\}$, we write $E_G(v, V_2)$ for $E_G(V_1, V_2)$. We use the standard notation K_n, P_n, and $K_{m,n}$ to denote the complete graph on n

This paper is dedicated to Prof. Katsuhiro Ota for his 60th birthday.

vertices, the path of order n, and the complete bipartite graph with one partition of size m and the other of size n, respectively. Two vertices x and y of a graph G are called *twins* if $N_G(x) = N_G(y)$. By the definition, x is not adjacent to y if they are twins. If G contains no twins, then G is *twin-free*.

A spanning tree of a graph with no vertex of degree 2 is called a *homeomorphically irreducible spanning tree* (*HIST*) of the graph. Let T be a HIST of a graph. In T, a vertex of degree 1 is called a *leaf* of T, and each vertex other than leaves is called a *stem* of T. For a graph G, a *blocking set* is a cutset of G consisting of vertices of degree 2. By the definition of a HIST, if G has a HIST, then G contains no blocking set.

In [1], Albertson, Berman, Hutchinson, and Thomassen proved that it is NP-complete to decide whether a given graph contains a HIST. On the other hand, there are sufficient conditions that guarantee the existence of a HIST in a graph.

Theorem A (Albertson, Berman, Hutchinson, and Thomassen [1]). *Let G be a graph of order $n \geq 4$. If each pair of vertices have a common neighbor, then G has a HIST.*

Theorem B (Furuya and Tsuchiya [3]). *Let G be a connected graph of order $n \geq 4$ with no induced P_4 (i.e., G is P_4-free). Then G has a HIST if and only if G is isomorphic to neither $K_{2,n-2}$ nor a graph obtained from K_4 by subdividing one edge.*

Theorem C (Ito and Tsuchiya [4]). *Let G be a graph of order $n \geq 8$. For any nonadjacent vertices x and y, if $d_G(x) + d_G(y) \geq n - 1$, then G has a HIST.*

Graphs satisfy conditions in Theorems A–C have diameter 2. In [1], Albertson, Berman, Hutchinson, and Thomassen conjectured that for a graph G of order $n \geq 10$ and diameter 2, if G is twin-free, then G contains a HIST. However, Ando [2] recently disproved this conjecture. In this paper, we give a complete characterization of all graphs of diameter 2 that contain a HIST. To state the result, we construct graphs described below of diameter 2 but contain no HIST.

Let $k \geq 1$ be an integer. For each $i \in \{1, \ldots, k\}$, let $A_i = K_{2,p_i}$ be a complete bipartite graph with bipartition X_i and Y_i such that $|X_i| = 2$ and $|Y_i| = p_i$, where $p_i \geq 1$ is an integer. Let $X_i = \{u_i, v_i\}$, and let K_1 and K_k be two complete graphs with $V(K_1) = \{x\}$ and $V(K_k) = \{y_1, \ldots, y_k\}$. Then a graph $A(p_1, \ldots, p_k)$ is obtained from $A_1, \ldots, A_k$, K_1 and K_k by identifying x with u_i and identifying y_i with v_i for each i. We let

$$\mathcal{A}_k = \{A(p_1, \ldots, p_k) | p_i \geq 1 \text{ for each } i\}$$

be the set of all such graphs $A(p_1, \ldots, p_k)$. In particular, if $p_i = 1$ (i.e., $A_i = P_3$ is a path) for each i, we denote $A(p_1, \ldots, p_k)$ by A_k^s, which is a graph in $\mathcal{A}_k$ with smallest order. See Figure 1 for an illustration of a graph in $\mathcal{A}_k$ and the graph A_k^s, respectively. Note that each graph in $\mathcal{A}_k$ contains no HIST as it has a blocking set, and each graph in $\mathcal{A}_k - \{A_k^s\}$ is a graph of diameter 2 with twins.

For $n \geq 6$, let B_n be a graph such that $V(B_n) = \{a, b_1, b_2, \ldots, b_{n-3}, c_1, c_2\}$ and

$$E(B_n) = \{ab_i : 1 \leq i \leq n-3\} \cup \{c_2 b_j : 3 \leq j \leq n-3\} \cup \{b_1 b_2, c_1 b_1, c_1 b_2, c_1 c_2\}.$$

It is clear that $B_n - \{b_1 b_2\} = A(2, n-5)$. See Figure 2 for a depiction of B_n.

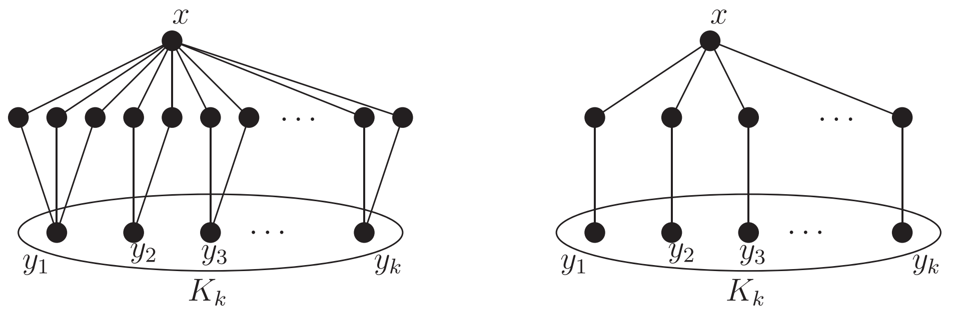

FIGURE 1 $A(3, 2, ..., 2) \in \mathcal{A}_k$ (left) and $A(1, 1, ..., 1) = A_k^s$ (right).

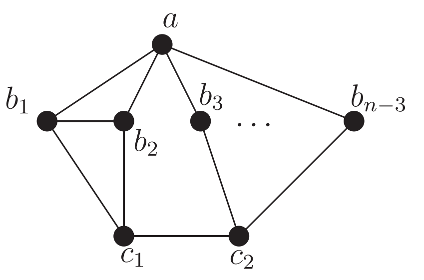

FIGURE 2 B_n.

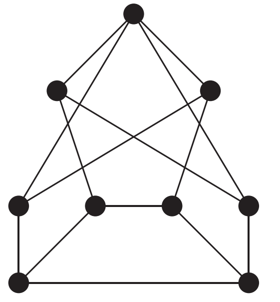

FIGURE 3 A twin-free graph of order 9 containing no homeomorphically irreducible spanning tree.

We now state Ando's result in disproving the conjecture of Albertson, Berman, Hutchinson, and Thomassen in [1] and our characterization of graphs of diameter 2 containing a HIST.

Theorem D (Ando [2]). *Let G be a graph of order $n \geq 10$ with diameter 2. Suppose that G is twin-free. Then G has a HIST if and only if G is not isomorphic to A_k^s for any $k \geq 1$.*

Theorem 1. *Let G be a graph of order $n \geq 10$ and diameter 2. Then G has a HIST if and only if $G \notin \mathcal{A}_k$ for any $k \geq 1$ and $G \neq B_n$.*

The condition $n \geq 10$ is best possible in both Theorems D and 1. The graph of order 9 given in Figure 3, constructed in [1], is twin-free, has no blocking set, is not isomorphic to B_9, but contains no HIST.

Theorem 1 gives alternative proofs of Theorems A–D if we assume the graphs are of order at least 10.

Proof of Theorem A. Since each pair of vertices of G have a common neighbor and so the diameter of G is 2. By Theorem 1, it suffices to show that $G \notin \mathcal{A}_k$ for any $k \geq 1$ and that $G \neq B_n$. Take a graph H in $\mathcal{A}_k$, let x' be a vertex in $N_H(x)$, where recall x is the vertex contained in K_1 when constructing the graph H. Then x and x' do not have a common neighbor in H. In B_n, a and b_1 do not have any common neighbor. □

Proof of Theorem B. Since G is a connected graph containing no induced P_4, the diameter of G is 2. Since all graphs in $\mathcal{A}_1$ are isomorphic to $K_{2,p}$ for some p, it suffices to show that $G \notin \mathcal{A}_k$ for any $k \geq 2$ and that $G \neq B_n$. For any $k \geq 2$, take a graph H in $\mathcal{A}_k$, and let x' be a common neighbor of x and y_1 in H, where recall y_1 is a vertex in K_k used to construct H. Then $xx'y_1y_2$ is an induced P_4. In B_n, $ab_1c_1c_2$ is an induced P_4. □

Proof of Theorem C. Since $d_G(x) + d_G(y) \geq n - 1$ for any nonadjacent vertices x and y, x and y have at least one common neighbor. Thus the diameter of G is 2. So, it suffices to show that $G \notin \mathcal{A}_k$ for any $k \geq 1$ and that $G \neq B_n$. Take a graph H in $\mathcal{A}_k$, let x' and x'' be vertices in $N_H(x)$. Then $d_H(x') + d_H(x'') = 4 < 9 \leq n - 1$. In B_n, $d_{B_n}(b_1) + d_{B_n}(b_3) = 5 < 9 \leq n - 1$. □

Proof of Theorem D. It suffices to show that $G \notin \mathcal{A}_k - \{A_k^s\}$ for any $k \geq 1$ and that $G \neq B_n$. Take a graph H in $\mathcal{A}_k - \{A_k^s\}$, and let $A_i = K_{2,p_i}$ be a complete bipartite subgraph of H with $p_i \geq 2$. Let x' and x'' be vertices of A_i from the partite set of size p_i. Then $N_H(x') = N_H(x'') = \{x, y_i\}$, and hence x' and x'' are twins. In B_n, there exists b_4 because $n \geq 10$. Then $N_{B_n}(b_3) = N_{B_n}(b_4) = \{a, c_2\}$, and hence b_3 and b_4 are twins. □

Theorem 1 is implied by the two theorems below, which we prove, respectively, in Sections 2 and 3.

Theorem 2. *Let G be a graph with diameter 2. Then G contains a blocking set if and only if $G \in \mathcal{A}_k$ for some integer $k \geq 1$.*

Theorem 3. *Let G be a graph of order $n \geq 10$ and diameter 2. Suppose that G contains no blocking set. Then G contains a HIST if and only if $G \neq B_n$.*

2 | PROOF OF THEOREM 2

Proof. If $G \in \mathcal{A}_k$ for some integer $k \geq 1$, then the vertices of G other than those corresponding to K_1 and K_k in the definition of $\mathcal{A}_k$ form a blocking set of G. Conversely, suppose that G is a diameter 2 graph containing a blocking set. We show that $G \in \mathcal{A}_k$ for some integer $k \geq 1$. □

Let B be a minimal blocking set of G and let $D_1, \ldots, D_p$ $(p \geq 2)$ be components of $G - B$.

Claim 2.1. For each $b \in B$ and each component D_i with $i \in \{1, ..., p\}$, b is adjacent to a vertex of D_i in G. Consequently, $p = 2$.

Proof. Suppose to the contrary that there exists $b \in B$ such that $N(b) \cap V(D_i) = \varnothing$ for some i. Then $B - \{b\}$ is a blocking set of G smaller than B, contradicting to B being minimal. Since each vertex in B has degree 2 in G, $p = 2$ as a consequence of the first part. □

Claim 2.2. For each $i \in \{1, 2\}$, every vertex in D_i is adjacent to at least one vertex from B in G.

Proof. Let $u \in V(D_i)$ be an arbitrary vertex. Take $v \in V(D_j)$ for $j \neq i$. Since the diameter of G is 2, u and v share a common neighbor in G. Consequently, u has a neighbor from B in G. □

Claim 2.3. For each $i \in \{1, 2\}$, $G[D_i]$ is a clique.

Proof. Suppose to the contrary that there exist $d, d' \in V(D_i)$ such that $dd' \notin E(D_i)$. By Claim 2.2, d is adjacent to a vertex $b \in B$ in G. Since $d_G(b) = 2$, it follows from Claim 2.1 that b is not adjacent to any vertices from $V(D_i) - \{d\}$ in G. Since $d'd \notin E(D_i)$ and so $d'd \notin E(G)$, d' has no common neighbor with b in G. This contradicts the assumption that the diameter of G is 2. □

Claim 2.4. There exists $i \in \{1, 2\}$ such that $G[D_i] = K_1$.

Proof. Suppose to the contrary that $G[D_i] \neq K_1$ for each $i \in \{1, 2\}$. As a consequence, we have $|V(D_1)| \geq 2$ and $|V(D_2)| \geq 2$. Let $d_1^1, d_2^1 \in V(D_1)$ be distinct. By Claim 2.2, for each $i \in \{1, 2\}$, d_i^1 is adjacent to at least one vertex from B in G. As each vertex from B has degree 2 in G, by Claim 2.1, each vertex from B is adjacent to exactly one vertex from D_1 in G. Thus d_1^1 and d_2^1 have no common neighbor from B in G. For each $i \in \{1, 2\}$, let

$$B_i = N_G(d_i^1) \cap B \quad \text{and} \quad B' = B - (B_1 \cup B_2).$$

Let $b_i \in B_i$ for each i. Since b_1 and b_2 have no common neighbor from D_1 and $b_1b_2 \notin E(G)$ in G by Claim 2.1, it follows that b_1 and b_2 have a common neighbor d_1^2 from D_2 in G because the diameter of G is 2. Since $|V(D_2)| \geq 2$, there exists a vertex $d_2^2 \in V(D_2) - \{d_1^2\}$. Since each vertex from B is adjacent to exactly one vertex from D_2 in G by Claim 2.1 and $d_1^2 \in N_G(b_1) \cap N_G(b_2)$, applying Claim 2.2, we know that d_2^2 is adjacent to a vertex $b' \in B - \{b_1, b_2\}$ in G. By symmetry, suppose $b' \in B_2 \cup B'$. Thus $b'd_1^1 \notin E(G)$ and so b' and b_1 have no common neighbor from D_1 in G. Since $d_2^2 b' \in E(G)$ and b' is adjacent to exactly one vertex from D_2 in G by Claim 2.1, it follows that b' is not adjacent to d_1^2 in G. Thus b' and b_1 have no common neighbor from D_2 in G. Furthermore, B is an independent set in G by Claim 2.1. Hence the distance between b_1 and b' is at least 3 in G, contradicting the assumption that the diameter of G is 2.

By Claim 2.4, we assume, without loss of generality, that $G[D_1] = K_1$. By Claim 2.1, each vertex from B is adjacent to exactly one vertex from D_2 in G. This together with Claims 2.2 and 2.3 implies that D_2 is a complete graph of order at most $|B|$. Now by Claims 2.1–2.4, we have $G \in \mathcal{A}_k$, where $k := |V(D_2)|$, completing the proof. □

3 | PROOF OF THEOREM 3

Theorem 3 is implied by the three lemmas below.

Lemma 4. *Let G be a graph of order $n \geq 10$, diameter 2, and containing no blocking set. Suppose that $\delta(G) \leq 2$. Then G has a HIST if and only if $G \neq B_n$.*

Lemma 5. *Let G be a graph of order $n \geq 10$ and diameter 2. If $\delta(G) = 3$, then G has a HIST.*

Lemma 6. *Let G be a graph of order $n \geq 10$ and diameter 2. If $\delta(G) \geq 4$, then G has a HIST.*

For a subtree T of a graph G, T is called an *extendable HIT* if T has no vertex of degree 2 and for any $v \in V(G) - V(T)$, v is adjacent to a stem of T. By the definition, the following statement is true.

Fact 3.1. A graph with an extendable HIT contains a HIST.

3.1 | Proof of Lemma 4

Proof. By tedious check, we can see that B_n has no HIST. Thus it suffices to show that G has a HIST if $G \neq B_n$. If G has a vertex of degree $n-1$, then G has a spanning star, which is a HIST of G. Thus we assume $\Delta(G) \leq n-2$.

Let v be a vertex of G such that $d_G(v) = \delta(G)$. If $d_G(v) = 1$, then the degree of the vertex adjacent to v is $n-1$ because the diameter of G is 2, contradicting $\Delta(G) \leq n-2$. Thus we have $d_G(v) = 2$. Let w_1 and w_2 be the vertices of G that are adjacent to v.

First we assume $w_1w_2 \in E(G)$. Suppose, by symmetry, $d_G(w_1) \geq d_G(w_2)$. Since $\Delta(G) \leq n-2$ and so $d_G(w_1) \leq n-2$, there exists $y \in V(G)\setminus\{v, w_1, w_2\}$ such that $w_1y \notin E(G)$. Then $w_2y \in E(G)$ as the distance between y and v is 2 in G. Since $d_G(w_1) \geq d_G(w_2)$, $n \geq 10$, and the diameter of G is 2, there exist distinct $x_1, x_2 \in N_G(w_1) - \{w_2, v\}$. Thus G has a double star T such that

$$V(T) = \{w_1, w_2, x_1, x_2, v, y\} \quad \text{and} \quad E(T) = \{w_1w_2, w_1x_1, w_1x_2, w_2v, w_2y\}.$$

Since each vertex of $V(G) - V(T)$ is adjacent to either w_1 or w_2 in G, T is an extendable HIT. Hence G has a HIST.

Next we assume $w_1w_2 \notin E(G)$. For each $i \in \{1, 2\}$, let

$$W_i = N_G(w_i) - N_G(w_{3-i}) \quad \text{and} \quad W_{12} = (N_G(w_1) \cap N_G(w_2)) - \{v\}.$$ □

Claim 3.2. If W_{12} has a vertex x of degree at least 3 in G, then G has a HIST.

Proof. Let $x \in W_{12}$ with $d_G(x) \geq 3$. Then x is adjacent to a vertex $y \in W_1 \cup W_2 \cup W_{12}$ in G. Let $W_i' = W_i - \{y\}$ for each $i \in \{1, 2\}$, and let $W_{12}' = W_{12} - \{y\}$. By symmetry, we assume $|W_1'| \geq |W_2'|$. Since $n \geq 10$, there exist distinct vertices $w, w' \in W_1' \cup W_{12}'$. If $W_2' \neq \emptyset$, then let $w'' \in W_2'$. We now construct a tree T as follows:

$$\begin{cases} V(T) = \{w_1, x, w_2, w, w', y, v, w''\} & \text{and} \\ E(T) = \{w_1x, w_2x, w_1w, w_1w', xy, w_2v, w_2w''\} & \text{if } W_2' \neq \emptyset; \\ V(T) = \{w_1, x, w, v, w_2, y\} & \text{and} \\ E(T) = \{w_1x, w_1w, w_1v, w_2x, xy\} & \text{if } W_2' = \emptyset. \end{cases}$$

In either case, T is an extendable HIT because each vertex in W_2' (resp., $W_1' \cup W_{12}'$) is adjacent in G to w_2 (resp., w_1). □

By Claim 3.2, we may assume that the degree of each vertex from W_{12} is 2 in G. Thus $E_G(W_i, W_{12}) = \emptyset$ for each $i \in \{1, 2\}$. If $W_i = \emptyset$ for each $i \in \{1, 2\}$, then $W_{12} \cup \{v\}$ is a blocking set of G. If $W_i = \emptyset$ for some $i \in \{1, 2\}$, then w_i has no common neighbor with each vertex of W_{3-i} in G, contradicting G being diameter 2. Thus for each $i \in \{1, 2\}$, $W_i \neq \emptyset$.

Claim 3.3. $|W_1| + |W_2| \geq 4$.

Proof. Suppose to the contrary that $|W_1| + |W_2| \leq 3$. If $|W_1| + |W_2| = 2$, then $|W_1| = |W_2| = 1$ as $W_i \neq \emptyset$ for each i. Since the vertex in W_1 is degree 2 in G, $W_1 \cup W_{12} \cup \{v\}$ is a blocking set of G, a contradiction. So we have $|W_1| + |W_2| = 3$. By symmetry, we assume that $|W_1| = 2$ and $|W_2| = 1$. Since $W_{12} \cup W_2 \cup \{v\}$ is not a blocking set of G, the vertex from W_2 has degree at least 3 in G. By Claim 3.2, the vertex from W_2 is adjacent in G to both of the vertices of W_1. If $G[W_1]$ contains no edges, then $W_1 \cup W_{12} \cup \{v\}$ is a blocking set of G. If $G[W_1]$ contains an edge, then $G = B_n$. In either case, we obtain a contradiction. □

Claim 3.4. There exist four distinct vertices $x_1 \in W_1$, $x_2 \in W_2$ and $y_1, y_2 \in W_1 \cup W_2$ such that

(1) $x_1x_2 \in E(G)$,
(2) $d_G(x_i) \geq 3$ for each i, and
(3) $x_1y_1, x_2y_2 \in E(G)$.

Proof. By Claim 3.2, we may assume that $E_G(W_i, W_{12}) = \emptyset$ for each $i \in \{1, 2\}$. Since G has diameter 2, $E_G(W_1, W_2) \neq \emptyset$. Since G contains no blocking set, G has two distinct vertices $a_1 \in W_1$ and $a_2 \in W_2$ satisfying Conditions (1) and (2). Since $d_G(a_i) \geq 3$, a_i is adjacent to a vertex $b_i \in (W_1 \cup W_2) - \{a_1, a_2\}$ in G for each i. If $b_1 \neq b_2$, then letting $x_i = a_i$ and $y_i = b_i$ gives the desired vertices. Hence we may assume that $b_1 = b_2$ and $d_G(a_i) = 3$ for each i. By symmetry, we assume $b_1 \in W_1$. By Claim 3.3, there exists a vertex $c \in W_i - \{a_1, a_2, b_1\}$ for some $i \in \{1, 2\}$. Since $d_G(a_{3-i}) = 3$, c is not adjacent to a_{3-i} in G. Since the diameter of G is 2, c and a_{3-i} have a common neighbor in G. Then c is adjacent to b_1 in G because $a_ic, w_{3-i}c \notin E(G)$. Then letting $x_1 = b_1$, $x_2 = a_2$, $y_1 = c$, and $y_2 = a_1$ gives four desired vertices. $\square$

By Claim 3.4, G has four distinct vertices $x_1 \in W_1$, $x_2 \in W_2$, and $y_1, y_2 \in W_1 \cup W_2$ satisfying Conditions (1)–(3). For each $i \in \{1, 2\}$, let $W_i' = W_i - \{x_1, x_2, y_1, y_2\}$. By symmetry, we assume $|W_1'| \geq |W_2'|$. Since $n \geq 10$, there are two distinct vertices $w, w' \in W_1' \cup W_{12}$. If $W_2' \neq \emptyset$, then let $w'' \in W_2'$. We now construct a tree T as follows:

$$\begin{cases} V(T) = \{w_1, x_1, x_2, w_2, w, w', y_1, y_2, v, w''\} & \text{and} \\ E(T) = \{w_1x_1, x_1x_2, x_2w_2, w_1w, w_1w', x_1y_1, x_2y_2, w_2v, w_2w''\} & \text{if } W_2' \neq \emptyset; \\ V(T) = \{w_1, x_1, x_2, w, v, y_1, y_2, w_2\} & \text{and} \\ E(T) = \{w_1x_1, x_1x_2, w_1w, w_1v, x_1y_1, x_2y_2, x_2w_2\} & \text{if } W_2' = \emptyset. \end{cases}$$

In either case, T is an extendable HIT because each vertex in W_2' (resp., $W_1' \cup W_{12}$) is adjacent in G to w_2 (resp., w_1).

3.2 | Proof of Lemmas 5 and 6

To prove Lemmas 5 and 6, we construct a spanning tree T of G as follows. Let v be a vertex of G such that $d_G(v) = \delta(G)$. We take a spanning tree T in G so that

- T contains all edges of $E_G(v, N_G(v))$,
- all vertices of T other than leaves are contained in $N_G(v) \cup \{v\}$ and
- the number of vertices of degree 2 in T is as small as possible.

Note that such T exists, as G has diameter 2 and so every vertex of $V(G) - (N_G(v) \cup \{v\})$ is adjacent in G to a vertex from $N_G(v)$. In T, we define subsets of $N_G(v)$ so that

- $N^1(v) := \{u_i^1 | d_T(u_i^1) \geq 4, u_i^1 \in N_G(v)\}$,
- $N^2(v) := \{u_i^2 | d_T(u_i^2) = 3, u_i^2 \in N_G(v)\}$,
- $N^3(v) := \{u_i^3 | d_T(u_i^3) = 2, u_i^3 \in N_G(v)\}$, and
- $N^4(v) := \{u_i^4 | d_T(u_i^4) = 1, u_i^4 \in N_G(v)\}$.

We define subsets of $V(G) - (N_G(v) \cup \{v\})$ so that

- $W_1 := \{w_i^1 | w_i^1 u \in E(T) \text{ for some } u \in N^1(v)\}$,
- $W_2 := \{w_i^2 | w_i^2 u \in E(T) \text{ for some } u \in N^2(v)\}$, and
- $W_3 := \{w_i^3 | w_i^3 u \in E(T) \text{ for some } u \in N^3(v)\}$.

See Figure 4 for a depiction of T and the sets defined above.

Claim 3.5. For each $w \in W_3$, w is not adjacent to any vertex from $N^1(v) \cup N^2(v)$ in G. Moreover, for any pair of vertices $w, w' \in W_3$, w and w' have no common neighbor from $N^3(v) \cup N^4(v)$ in G.

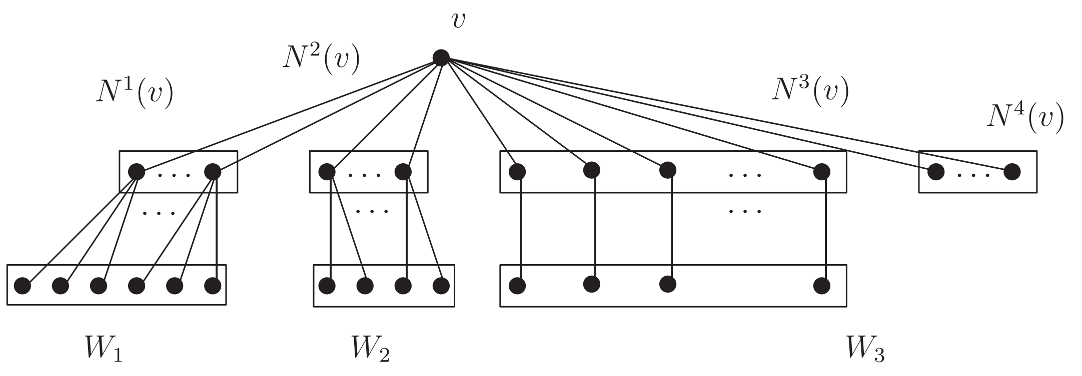

FIGURE 4 A spanning tree T.

Proof. Otherwise, we can take a spanning tree so that its number of vertices of degree 2 is smaller than that of T. □

Lemma 5. *Let G be a graph of order $n \geq 10$ and diameter 2. If $\delta(G) = 3$, then G has a HIST.*

Proof. We use the notation introduced above, and take T be a spanning tree of G with the described properties. □

Claim 3.6. For each $w \in W_3$, w is adjacent to at least two vertices from $N^4(v) \cup W_1 \cup W_2 \cup W_3$ in G.

Proof. Since $d_G(w) \geq d_G(v) = 3$, w is adjacent to at least three vertices in G. By the definition of T, w is adjacent to a vertex from $N^3(v)$ in G. By Claim 3.5, w is not adjacent to any vertex from $N^1(v) \cup N^2(v)$ in G. Thus the other two neighbors of w in G are contained in $N^4(v) \cup W_1 \cup W_2 \cup W_3$. □

If $W_3 = \emptyset$, then T is a HIST of G. So we suppose $W_3 \neq \emptyset$. By the definition of T, we have $|W_3| = |N^3(v)| \leq 3$. Since $n \geq 10$, $|W_3| \leq 3$ implies $N^1(v) \cup N^2(v) \neq \emptyset$. Thus $|W_3| \leq 2$.

First we assume $|W_3| = 2$. Let x_1 and x_2 be the two vertices in W_3. and for each $i \in \{1, 2\}$, let y_i be a vertex of $N^3(v)$ which is adjacent to x_i in T. Let y be the vertex in $N_G(v) - \{y_1, y_2\}$. Since $n \geq 10$, it follows from the definition of T that $\{y\} = N^1(v)$ and $N^4(v) \cup W_2 = \emptyset$.

Claim 3.7. If x_1 and x_2 have a common neighbor in G, then G has a HIST.

Proof. Let x' be a common neighbor of x_1 and x_2 in G. By adding x_1x', x_2x' to T and deleting x_1y_1, x_2y_2 from T, we obtain a HIST of G. □

By Claim 3.7, we suppose that x_1 and x_2 have no common neighbor in G. This implies that $x_1x_2 \in E(G)$ because the diameter of G is 2.

Claim 3.8. For each $i \in \{1, 2\}$, x_i is adjacent to a vertex from W_1 in G.

Proof. Since $d_G(x_i) \geq 3$, x_i is adjacent to a vertex other than x_{3-i} and y_i in G. By Claim 3.5, such a vertex is contained in W_1. □

By Claim 3.8, x_i is adjacent to a vertex $z_i \in W_1$ in G. By Claim 3.7, $z_1 \neq z_2$.

Claim 3.9. For each $i \in \{1, 2\}$, y_i is adjacent to y in G.

Proof. By the minimality of the number of degree 2 vertices of T, y_i is not adjacent to any vertex of W_1 in G for each $i \in \{1, 2\}$. Thus y_i and z_{3-i} have a common neighbor in G as G has diameter 2. Since $W_1 \cap N_G(y_i) = \varnothing$ and x_{3-i} is nonadjacent to z_i in G by Claim 3.7, y is the only possible common neighbor of y_i and z_{3-i}. Thus $yy_i \in E(G)$ for each i. □

By Claim 3.9, $yy_1, yy_2 \in E(G)$. Then we obtain a HIST of G by adding $yy_1, yy_2, x_1z_1, x_1x_2$ to T and deleting vy, vy_2, yz_1, x_2y_2 from T.

Next we assume $|W_3| = 1$. Let x_1 be the vertex in W_3, y_1 be the vertex in $N^3(v)$ which is adjacent to x_1 in T, and let y, y' be the two vertices in $N_G(v) - \{y_1\}$. Since $n \geq 10$, it follows from the definition of T that $W_1 \neq \varnothing$. By symmetry, we assume $y \in N^1(v)$.

Claim 3.10. If x_1 has a common neighbor with a vertex from W_1 in G, then G has a HIST.

Proof. Let z be a vertex in W_1 such that z and x_1 have a common neighbor x' in G. Let y'' be a vertex in $\{y, y'\}$ such that $y''z \in E(T)$. Note that $y'' \neq x'$ as x_1 is not adjacent to any vertex from $N^1(v) \cup N^2(v)$ in G by Claim 3.5. By adding x_1x', zx' to T and deleting $y''z, x_1y_1$ from T, we obtain a HIST of G. □

By Claim 3.10, we may assume that x_1 has no common neighbor with any vertex from W_1 in G. This implies that x_1 is adjacent to each vertex of W_1 in G because the diameter of G is 2. By this fact and Claim 3.10, we obtain the following claim.

Claim 3.11. If $G[W_1]$ contains an edge, then G has a HIST.

By Claim 3.11, we may assume that $G[W_1]$ contains no edge.

Claim 3.12. Let z be a vertex in W_1 such that $yz \in E(T)$. If $y'z \in E(G)$, then G has a HIST.

Proof. Suppose that there is a vertex $z' \in N_T(y) - \{v, z\}$. Then we obtain a HIST of G by adding $x_1z, x_1z', y'z$ to T and deleting $y_1v, y'v, yz'$ from T. □

Claim 3.13. If either $y' \in N^1(v)$ or $y' \in N^4(v)$, then G has a HIST.

Proof. Since $y' \in N^1(v) \cup N^4(v)$, it follows that $V(G) = \{v, y, y', y_1, x_1\} \cup W_1$. As W_1 is an independent set in G by Claim 3.11, and y_1 is not adjacent to any vertex of W_1 in G by the minimality of the number of degree 2 vertices of T, we see that each vertex in W_1 is adjacent to y and y' in G because $\delta(G) \geq 3$. Therefore G has a HIST by Claim 3.12. □

By Claim 3.13, we may assume $y' \in N^2(v)$. Let x_1' and x_2' be the two vertices in W_2 (i.e., $x_1'y', x_2'y' \in E(T)$). By the minimality of the number of degree 2 vertices of T, each vertex in W_1 is not adjacent to y_1 in G. By Claims 3.11 and 3.12, each vertex in W_1 is adjacent to x_i' for some $i \in \{1, 2\}$ in G because $\delta(G) \geq 3$.

Claim 3.14. If x_1' and x_2' have a common neighbor from W_1 in G, then G has a HIST.

Proof. Let z be a common neighbor of x_1' and x_2' from W_1. Then we obtain a HIST of G by adding $x_1z, x_1'z, x_2'z$ to T and deleting $x_1'y', x_2'y', x_1y_1$ from T. □

By Claim 3.14, we may assume that x_1' and x_2' have no common neighbor from W_1 in G.

Claim 3.15. If x_1 is adjacent to x_i' for some $i \in \{1, 2\}$ in G, then G has a HIST.

Proof. By symmetry, we assume $x_1x_1' \in E(G)$. By Claim 3.10, x_1' is adjacent to no vertex of W_1 in G. Thus x_2' is adjacent to all vertices of W_1 in G because every vertex of W_1 is adjacent to a vertex from $\{x_1', x_2'\}$ in G. Let z and z' be vertices in W_1. Then we obtain a HIST of G by adding $x_1x_1', x_1z, x_2'z, x_2'z'$ to T and deleting $vy_1, vy', x_1'y', yz'$ from T. □

By Claim 3.15, we may assume that x_1 is not adjacent to x_i' for any $i \in \{1, 2\}$ in G. Since the diameter of G is 2 and x_1 and y' are nonadjacent in G, x_1 and y' have a common neighbor, which is not contained in W_1 by Claim 3.12. Thus we have $y_1y' \in E(G)$. Recall that each vertex in W_1 is adjacent to x_i' for some $i \in \{1, 2\}$ in G. As $|W_1| \geq 3$, there exist $z, z' \in W_1$ such that $x_1'z, x_1'z' \in E(G)$ or $x_2'z, x_2'z' \in E(G)$. By symmetry, we assume $x_1'z, x_1'z' \in E(G)$. Then we obtain a HIST of G by adding $x_1z, x_1'z, x_1'z', y_1y'$ to T and deleting vy_1, vy', yz', x_1y_1 from T.

Lemma 6. *Let G be a graph of order $n \geq 10$ and diameter 2. If $\delta(G) \geq 4$, then G has a HIST.*

Proof. Again, we will use the notation introduced in the beginning of this subsection, and take T be a spanning tree of G with the described properties. □

Claim 3.16. For each $w \in W_3$, w is adjacent to exactly $d_G(w) - 1$ vertices from $N^4(v) \cup W_1 \cup W_2 \cup W_3$ in G.

Proof. The vertex w is only adjacent to exactly one vertex from $N^3(v)$ in G by the minimality of the number of vertices of degree 2 of T. By Claim 3.5, w is not adjacent to any vertex from $N^1(v) \cup N^2(v)$ in G. Thus the rest $d_G(w) - 1$ neighbors of w are contained in $N^4(v) \cup W_1 \cup W_2 \cup W_3$. □

If $W_3 = \emptyset$, then T is a HIST of G. So we suppose $W_3 \neq \emptyset$.

Claim 3.17. If $W_1 \cup W_2 = \emptyset$, then G has a HIST.

Proof. Let $W_3 = \{x_1, \ldots, x_k\}$ for some integer $k \geq 1$, and let $y_i \in N^3(v)$ such that $x_iy_i \in E(T)$ for each i. Since $d_G(x_1) \geq d_G(v)$, by Claims 3.5 and 3.16, x_1 is adjacent to all vertices from $N^4(v) \cup (W_3 - \{x_1\})$ in G. Since $d_G(y_1) \geq d_G(v)$, y_1 is adjacent to a vertex $z \in N_G(v) - \{y_1\}$ in G. If $k \neq 2$, then we take a spanning tree T' so that

$$E(T') = \left(E(T) \cup \{y_1 z\} \cup \bigcup_{i=2}^{k} \{x_1 x_i\}\right) - \left(\{vz\} \cup \bigcup_{i=2}^{k} \{x_i y_i\}\right).$$

If $k = 2$, then we have $d_G(v) \geq 7$ because $n \geq 10$. Let $z' \in N^4(v) - \{z\}$. We take a spanning tree T' so that

$$E(T') = (E(T) \cup \{y_1 z, x_1 z', x_1 x_2\}) - \{vz, vz', x_2 y_2\}.$$

In either case, T' is a HIST of G. $\square$

By Claim 3.17, we may assume $W_1 \cup W_2 \neq \emptyset$. Suppose that for some $k \geq 2$, there are vertices $a_1, \ldots, a_k \in W_3$ such that they have a common neighbor z in G. By Claims 3.5 and 3.16, $z \in (W_1 \cup W_2 \cup W_3) - \{a_1, \ldots, a_k\}$. Let $b_1, \ldots, b_k \in N^3(v)$ such that $a_i b_i \in E(T)$ $(1 \leq i \leq k)$. The *degree 2 vertex elimination* (D2VE) operation on z is a transformation of T by adding $a_i z$ and deleting $a_i b_i$ for each $i \in \{1, \ldots, k\}$. *A sequence of D2VE* is the following transformation of T:

Step (1) Set $S := W_1 \cup W_2$ and $S' := W_3$.
Step (2) Take $s \in S$ such that $|N(s) \cap S'|$ is maximum.
Step (3) If $|N(s) \cap S'| \leq 1$, then no transformation on s. Otherwise, go to the next step.
Step (4) Apply D2VE on s.
Step (5) Set $S := W_1 \cup W_2 \cup (N(s) \cap S')$.
Step (6) Set $S' := W_3 - S$.
Step (7) Return to Step (2).

Let $\hat{T}$ be a spanning tree of G obtained from T by a sequence of D2VE. By the definition, the number of vertices of degree 2 in $\hat{T}$ cannot be reduced further by applying the D2VE operation. In $\hat{T}$, we define subsets of $N_G(v)$ so that

- $\hat{N}^1(v) := N^1(v)$,
- $\hat{N}^2(v) := N^2(v)$,
- $\hat{N}^3(v) := \{u_i | u_i \in N^3(v), d_{\hat{T}}(u_i) = 2\}$,
- $\hat{N}'^3(v) := N^3(v) - \hat{N}^3(v)$, and
- $\hat{N}^4(v) := N^4(v)$.

Note that $\hat{N}'^3(v)$ were vertices of $N^3(v)$ that now become leaves of $\hat{T}$. We define subsets of $V(G) - (N_G(v) \cup \{v\})$ so that

- $\hat{W}_1 := W_1$,
- $\hat{W}_2 := W_2$,
- $\hat{W}_3 := \{w_i | w_i u \in E(\hat{T})$ for some $u \in \hat{N}^3(v)\}$, and
- $\hat{W}_3' := W_3 - \hat{W}_3$.

See Figure 5 for a depiction of $\hat{T}$ and the sets defined above.

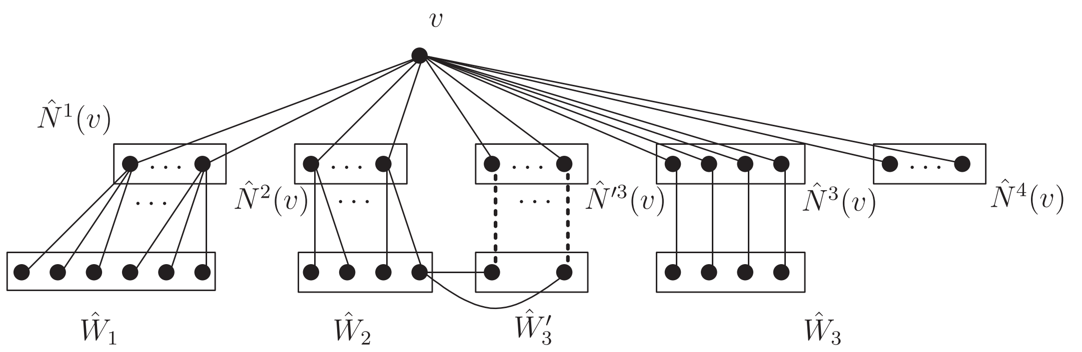

FIGURE 5 A spanning tree $\hat{T}$ (dot lines indicate edges of T deleted by the degree 2 vertex elimination operation).

Note that vertices of $\hat{W}_3'$ are adjacent to vertices of $\hat{W}_1 \cup \hat{W}_2 \cup \hat{W}_3'$ in $\hat{T}$. By the definition of $\hat{T}$, if $\hat{W}_3 = \varnothing$, then $\hat{T}$ is a HIST of G. So we suppose $\hat{W}_3 \neq \varnothing$. Note that each vertex in $\hat{W}_3$ is a leaf of $\hat{T}$. By Claim 3.16, we have the following claim.

Claim 3.18. For each $w \in \hat{W}_3$, w is adjacent to exactly $d_G(w) - 1$ vertices from $\hat{N}^4(v) \cup \hat{W}_1 \cup \hat{W}_2 \cup \hat{W}_3 \cup \hat{W}_3'$ in G.

By the choice of T and the D2VE procedure in getting $\hat{T}$ from T, we have the following claim.

Claim 3.19. For any pair of vertices $w, w' \in \hat{W}_3$, w and w' have no common neighbor from $\hat{N}^4(v) \cup \hat{W}_1 \cup \hat{W}_2 \cup \hat{W}_3'$ in G.

As the diameter of G is 2, by Claims 3.18 and 3.19, we know that any two vertices from $\hat{W}_3$ are adjacent in G or have a common neighbor from $\hat{W}_3$ in G. Thus the diameter of $G[\hat{W}_3]$ is 2. We consider below two cases regarding whether or not $|\hat{W}_3| = 2$.

Case 1. $|\hat{W}_3| \neq 2$.

Let $\hat{W}_3 = \{x_1, x_2, \ldots, x_k\}$ for some integer $k \neq 2$. Assume, without loss of generality, that the degree of x_1 in $G[\hat{W}_3]$ is maximum. Let $y_1, \ldots, y_k$ be vertices of $\hat{N}^3(v)$ such that $x_i y_i \in E(\hat{T})$. Since the diameter of $G[\hat{W}_3]$ is 2, if $k \geq 3$, then we can take a spanning tree H of $G[\hat{W}_3]$ so that

- H contains all edges of $E_G\left(x_1, N_{G[\hat{W}_3]}(x_1)\right)$,
- all vertices of H other than leaves are contained in $N_{G[\hat{W}_3]}(x_1) \cup \{x_1\}$, and
- the number of vertices of degree 2 in H is as small as possible.

When $k \geq 3$, by the maximality of $d_{G[\hat{W}_3]}(x_1)$, we have $d_{G[\hat{W}_3]}(x_1) = d_H(x_1) \geq 2$. For some integer $p \geq 0$, we assume that H has exactly p vertices from $V(H) \setminus \{x_1\}$ that are of degree 2 in H. We further assume, without loss of generality, that

$$d_H(x_i) = 2 \quad \text{for each } \; 2 \leq i \leq p+1 \; \text{(this expression is meaningful only if } \; p \geq 1\text{)}.$$

We modify $\hat{T}$ by adding $E(H)$ to $\hat{T}$ and deleting $\bigcup_{i=2}^{p+1}\{vy_i\} \cup (\bigcup_{i=p+2}^{k}\{x_i y_i\})$ from $\hat{T}$. Denote by $\hat{T}'$ the resulting spanning tree of G. If $k = 1$ (i.e., $|\hat{W}_3| = 1$), then let $\hat{T}' = \hat{T}$. We will now work on $\hat{T}'$ and modify it into a HIST of G. However, we will still use the sets $\hat{N}^1(v), \hat{N}^2(v), \hat{N}^3(v), \hat{N}'3(v), \hat{N}^4(v), \hat{W}_1, \hat{W}_2$ and $\hat{W}_3$, which recall are defined with respect to $\hat{T}$. By the definition, y_1 is a degree 2 vertex in $\hat{T}'$ (see Figure 6).

Claim 3.20. We have $d_{\hat{T}'}(v) \geq 4$. Thus y_1 is the only degree 2 vertex in $\hat{T}'$.

Proof. Suppose $k = 1$. Then we have $\hat{T}' = \hat{T}$. As $d_{\hat{T}}(v) = d_G(v) \geq \delta(G) \geq 4$, we have $d_{\hat{T}'}(v) \geq 4$. Since $\hat{W}_3 = \{x_1\}$, by the definition of $\hat{N}^3(v)$, y_1 is the only degree 2 vertex in $\hat{T}'$.

Thus we assume $k \geq 3$. In constructing $\hat{T}'$ from $\hat{T}$, the edges in $\bigcup_{i=2}^{p+1}\{vy_i\}$ were deleted. Thus $d_{\hat{T}'}(v) = d_{\hat{T}}(v) - p = d_G(v) - p$. When $|V(H)| = k \geq 3$, H has at least two leaves, thus $p \leq k - |\{x_1\}| - 2 = k - 3$. By Claim 3.17, we may assume $W_1 \cup W_2 \neq \emptyset$. Thus $k \leq d_G(v) - 1$. Therefore $d_{\hat{T}'}(v) = d_{\hat{T}}(v) - p = d_G(v) - p \geq 4$.

Note that the vertex x_1 has degree at least 3 in $\hat{T}'$. Thus in $\hat{T}'$, the set of possible vertices of degree 2 are contained in $\{y_1, \ldots, y_k\} \cup \{x_i \mid 2 \leq i \leq p + 1\}$. In constructing $\hat{T}'$ from $\hat{T}$, the edges in $\bigcup_{i=2}^{p+1}\{vy_i\} \cup (\bigcup_{i=p+2}^{k}\{x_i y_i\})$ were deleted. Thus every vertex from $\{y_2, \ldots, y_k\}$ has degree 1 in $\hat{T}'$. Each vertex x_i for $2 \leq i \leq p + 1$ (if exists) has degree 2 in H, and x_i is also adjacent to y_i in $\hat{T}'$. Thus x_i has degree 3 in $\hat{T}'$. Hence y_1 is the only degree 2 vertex in $\hat{T}'$. □

As $\hat{W}_1 \cup (\hat{W}_3 - \{x_1\}) \cup \hat{W}_3' = W_1 \cup (W_3 - \{x_1\})$, $\hat{W}_2 = W_2$, and $\hat{N}^2(v) = N^2(v)$, the minimality of the number of degree 2 vertices of T implies the following claim, which is similar to Claim 3.5.

Claim 3.21. The vertex y_1 is not adjacent to any vertex from $\hat{W}_1 \cup (\hat{W}_3 - \{x_1\}) \cup \hat{W}_3'$ in G. Moreover, for vertices $w, w' \in \hat{W}_2$ and $y \in \hat{N}^2(v)$ such that $wy, w'y \in E(\hat{T}')$, if $wy_1 \in E(G)$, then $w'y_1 \notin E(G)$.

Claim 3.22. If y_1 is adjacent to a vertex from $\hat{N}^1(v) \cup \hat{N}^2(v) \cup \hat{N}^4(v)$ in G, then G has a HIST.

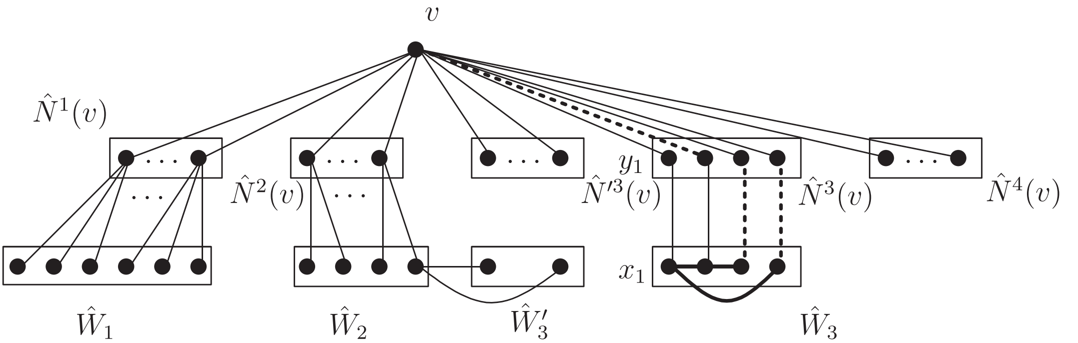

FIGURE 6 A spanning tree $\hat{T}'$ (dot lines indicate edges of $\hat{T}$ deleted by the transformation and strong lines indicate edges of H).

Proof. Suppose that $y_1y' \in E(G)$ for some $y' \in \hat{N}^1(v) \cup \hat{N}^2(v) \cup \hat{N}^4(v)$. Since $d_{\hat{T}'}(v) \geq 4$ by Claim 3.20, by adding y_1y' to $\hat{T}'$ and deleting vy' from $\hat{T}'$, we obtain a HIST of G. □

By Claim 3.22, we suppose that y_1 is not adjacent to any vertex from $\hat{N}^1(v) \cup \hat{N}^2(v) \cup \hat{N}^4(v)$ in G. Also y_1 is not adjacent to any vertex from $\hat{W}_1 \cup (\hat{W}_3 - \{x_1\}) \cup \hat{W}_3'$ in G by Claim 3.21. Thus the possible neighbors of y_1 in G are contained in $\hat{N}^3(v) \cup \hat{N}'^3(v) \cup \hat{W}_2 \cup \{x_1\}$. Suppose y_1 is adjacent to no vertex from $\hat{W}_2$ in G. Then y_1 is adjacent to $d_G(y_1) - 1 \geq d_G(v) - 1$ vertices from $\hat{N}^3(v) \cup \hat{N}'^3(v)$ in G. Since $|\hat{N}^3(v) \cup \hat{N}'^3(v)| = d_G(v) - |\hat{N}^1(v) \cup \hat{N}^2(v) \cup \hat{N}^4(v)| \leq d_G(v) - 1$ (note that $W_1 \cup W_2 \neq \emptyset$ by Claim 3.17, and hence $\hat{N}^1(v) \cup \hat{N}^2(v) \neq \emptyset$ because $\hat{N}^1(v) \cup \hat{N}^2(v) = N^1(v) \cup N^2(v)$), it follows that $|\hat{N}^3(v) \cup \hat{N}'^3(v)| = d_G(v) - 1 = d_G(y_1) - 1 \geq 3$ and so y_1 is adjacent to every vertex from $\hat{N}^3(v) \cup \hat{N}'^3(v)$ in G. We choose a vertex z so that

- $z \in \hat{N}'^3(v)$ if $\hat{N}'^3(v) \neq \emptyset$, and
- $z \in \hat{N}^3(v)$ such that $vz \in E(\hat{T}')$ if $\hat{N}'^3(v) = \emptyset$ (such z exists as H has leaves contained in $\hat{W}_3$ when $k = |\hat{W}_3| = |\hat{N}^3(v)| \geq 3$).

Then we obtain a HIST by adding y_1z to $\hat{T}'$ and deleting vz from $\hat{T}'$.

Claim 3.23. Suppose y_1 is adjacent to a vertex $w \in \hat{W}_2$ in G. Let $w' \in \hat{W}_2$ and $y \in \hat{N}^2(v)$ such that $yw, yw' \in E(\hat{T}')$ by the definition of $\hat{W}_2$. Then $y_1w' \notin E(G)$, and G has a HIST or $x_1w' \in E(G) - E(\hat{T}')$.

Proof. We have $y_1w' \notin E(G)$ by the second part of Claim 3.21. Thus we have

$$y_1w \in E(G) - E(\hat{T}'), \quad yw, yw' \in E(\hat{T}'), \quad \text{and} \quad y_1w' \notin E(G).$$

Since the diameter of G is 2, y_1 and w' have a common neighbor z in G.

By the first part of Claim 3.21, y_1 is not adjacent to any vertex from $\hat{W}_3 - \{x_1\}$ in G. By the minimality of the number of degree 2 vertices of T, w' is not adjacent to any vertex from $\hat{N}^3(v) \cup \hat{N}'3(v)$ in G (otherwise, if w' is adjacent to a vertex, say y^*, from $\hat{N}^3(v) \cup \hat{N}'3(v)$ in G, then in T we would delete yw and yw' and add y_1w and y^*w' to deduce the number of degree 2 vertices). As $N_G(y_1) \subset \hat{N}^3(v) \cup \hat{N}'^3(v) \cup \hat{W}_2 \cup \{x_1\}$, we thus have $z \in (\hat{W}_2 - \{w'\}) \cup \{x_1\}$.

If $z = w$, then we obtain a HIST of G by adding $ww'(= zw'), y_1w$ to $\hat{T}'$ and deleting vy, yw' from $\hat{T}'$. If $z \in \hat{W}_2 - \{w, w'\}$, then we obtain a HIST of G by adding $y_1w, y_1z, w'z$ to $\hat{T}'$ and deleting vy_1, yw, yw' from $\hat{T}'$ (in the resulting tree, the degree of y_1 is 3 and the degree of y is 1). Thus $z = x_1$ and so we have $x_1w' \in E(G) - E(\hat{T}')$ by the construction of $\hat{T}'$. □

We now suppose that y_1 is adjacent to a vertex $w_1 \in \hat{W}_2$ in G. By the definition of $\hat{W}_2$, there exist vertices $w_2 \in \hat{W}_2$ and $y \in \hat{N}^2(v)$ such that $yw_1, yw_2 \in E(\hat{T}')$. By Claim 3.23, we have

$y_1w_2 \notin E(G)$, and G has a HIST or $x_1w_2 \in E(G) - E(\hat{T}')$. So, we suppose that $x_1w_2 \in E(G) - E(\hat{T}')$.

If $d_{\hat{T}'}(x_1) \geq 3$, then we obtain a HIST of G by adding x_1w_2, y_1w_1 to $\hat{T}'$ and deleting yw_1, yw_2 from $\hat{T}'$. If $d_{\hat{T}'}(x_1) \leq 2$, since $d_{\hat{T}'}(x_1) \neq 2$, we must have $d_{\hat{T}'}(x_1) = 1$. This implies $V(H) = \{x_1\}$. Thus if y_1 is adjacent to a vertex $z \in \hat{N}'3(v)$ in G, then we obtain a HIST of G by adding y_1z to $\hat{T}'$ and deleting vz from $\hat{T}'$. Thus we suppose that y_1 is not adjacent to any vertex from $\hat{N}'3(v)$ in G. Note that when $V(H) = \{x_1\}$, we have $\hat{N}^3(v) = \{y_1\}$. As $\delta(G) \geq 4$, $N_G(y_1) \subset \hat{N}^3(v) \cup \hat{W}_2 \cup \{x_1\}$, and $y_1w_2 \notin E(G)$, it follows that y_1 is adjacent to at least two vertices of $\hat{W}_2 - \{w_1, w_2\}$ in G. Thus there exists $w_1' \in \hat{W}_2 - \{w_1, w_2\}$ such that $y_1w_1' \in E(G)$. Since $w_1' \in \hat{W}_2$ and $w_1' \notin \{w_1, w_2\}$, there exist vertices $w_2' \in \hat{W}_2$ and $y' \in \hat{N}^2(v) - \{y\}$ such that $y'w_1', y'w_2' \in E(\hat{T}')$. By Claim 3.23, we have $w_2'y_1 \notin E(G)$, and G has a HIST or $x_1w_2' \in E(G) - E(\hat{T}')$. So, we suppose that $x_1w_2' \in E(G) - E(\hat{T}')$.

Then we obtain a HIST of G by adding $x_1w_2, x_1w_2', y_1w_1, y_1w_1'$ to $\hat{T}'$ and deleting $yw_1, yw_2, y'w_1', y'w_2'$ from $\hat{T}'$.

Case 2. $|\hat{W}_3| = 2$.

Let $\hat{W}_3 = \{x_1, x_2\}$. Let y_1 and y_2 be the vertices in $\hat{N}^3(v)$ such that $x_1y_1, x_2y_2 \in E(\hat{T})$. By Claims 3.18 and 3.19, x_1 and x_2 have no common neighbor in G. Thus $x_1x_2 \in E(G)$ because the diameter of G is 2.

Claim 3.24. If $\hat{W}_1 = \emptyset$, then G has a HIST.

Proof. Note that $\hat{W}_1 = \emptyset$ implies $\hat{N}^1(v) = \emptyset$. By Claim 3.18, for each $i \in \{1, 2\}$, x_i is adjacent to exactly $d_G(x_i) - 1$ vertices from $\hat{N}^4(v) \cup \hat{W}_2 \cup \hat{W}_3 \cup \hat{W}'_3$ in G. Since x_1 and x_2 have no common neighbor in G, we have $|\hat{N}^4(v) \cup \hat{W}_2 \cup \hat{W}_3 \cup \hat{W}'_3| \geq d_G(x_1) - 1 + d_G(x_2) - 1 \geq 2d_G(v) - 2 = 2(|\hat{N}^2(v)| + |\hat{N}^3(v)| + |\hat{N}'^3(v)| + |\hat{N}^4(v)|) - 2$. As $|\hat{W}_2| = 2|\hat{N}^2(v)|$, $\hat{W}_3 = \{x_1, x_2\}$, and $|\hat{W}'_3| = |\hat{N}'^3(v)|$, it follows that $\hat{N}^4(v) \cup \hat{W}'_3 = \emptyset$. Thus $|\hat{N}^2(v)| \geq 2$ as $d_{\hat{T}}(v) \geq 4$. Since $d_G(x_i) \geq d_G(v)$, x_i is adjacent to at least $|\hat{N}^2(v)|$ vertices from $\hat{W}_2$ in G for each $i \in \{1, 2\}$. This, together with the facts that $|\hat{W}_2| = 2|\hat{N}^2(v)|$ and x_1 and x_2 have no common neighbor in G, implies that each vertex in $\hat{W}_2$ is adjacent to either x_1 or x_2 in G. For each i, let $\hat{W}_2(x_i) = N_G(x_i) \cap \hat{W}_2$. Note that we have $\hat{W}_2(x_1) \cup \hat{W}_2(x_1) = \hat{W}_2$ and $|\hat{W}_2(x_1)| = |\hat{W}_2(x_2)| = |\hat{N}^2(v)| \geq 2$. Since $d_G(y_1) \geq d_G(v) \geq 4$, $\hat{N}^1(v) \cup \hat{W}_1 \cup \hat{W}'_3 \cup \hat{N}^4(v) = \emptyset$, and $|\hat{W}_3| = 2$, it follows that y_1 is adjacent to a vertex $z \in \hat{N}^2(v) \cup \hat{W}_2$ in G.

If $z \in \hat{N}^2(v)$, then we obtain a HIST of G by adding x_1x_2, y_1z and $\{x_iw_i | w_i \in \hat{W}_2(x_i) \text{ for each } i = 1, 2\}$ to $\hat{T}$ and deleting x_2y_2, vz and $E_{\hat{T}}(\hat{W}_2, \hat{N}^2(v))$ from $\hat{T}$. Thus we suppose $z \notin \hat{N}^2(v)$ (i.e., $z \in \hat{W}_2$). If $z \in \hat{W}_2(x_1)$, then we obtain a HIST of G by adding x_1x_2, y_1z and $(\{x_iw_i | w_i \in \hat{W}_2(x_i) \text{ for each } i = 1, 2\} - \{x_1z\})$ to $\hat{T}$ and deleting x_2y_2 and $E_{\hat{T}}(\hat{W}_2, \hat{N}^2(v))$ from $\hat{T}$. If $z \in \hat{W}_2(x_2)$, then we obtain a HIST of G by adding x_1x_2, y_1z and $(\{x_iw_i | w_i \in \hat{W}_2(x_i) \text{ for each } i = 1, 2\} - \{x_2z\})$ to $\hat{T}$ and deleting vy_2 and $E_{\hat{T}}(\hat{W}_2, \hat{N}^2(v))$ from $\hat{T}$. □

By Claim 3.24, we may assume $\hat{W}_1 \neq \emptyset$. Thus $\hat{N}^1(v) \neq \emptyset$.

Claim 3.25. There exists a vertex $w \in \hat{W}_1$ which is adjacent to either x_1 or x_2 in G.

Proof. Suppose to the contrary that $(N_G(x_1) \cup N_G(x_2)) \cap \hat{W}_1 = \varnothing$. Then by the same argument as in Claim 3.24, we have $|\hat{N}^4(v) \cup \hat{W}_2 \cup \hat{W}_3 \cup \hat{W}_3'| \geq d_G(x_1) - 1 + d_G(x_2) - 1 \geq 2d_G(v) - 2 = 2(|\hat{N}^1(v)| + |\hat{N}^2(v)| + |\hat{N}^3(v)| + |\hat{N}'^3(v)| + |\hat{N}^4(v)|) - 2$. Thus $\hat{N}^1(v) \cup \hat{N}^4(v) \cup \hat{W}_3' = \varnothing$, contradicting the fact $\hat{N}^1(v) \neq \varnothing$. □

By Claim 3.25 and symmetry, we suppose that x_1 is adjacent to $w \in \hat{W}_1$ in G. Let y be the vertex in $\hat{N}^1(v)$ such that $wy \in E(\hat{T})$.

Claim 3.26. If w is adjacent to a vertex from $\hat{W}_1$ in G, then G has a HIST.

Proof. Suppose that w is adjacent to a vertex $w' \in \hat{W}_1 - \{w\}$ in G. Let y' be the vertex in $\hat{N}^1(v)$ such that $w'y' \in E(\hat{T})$. Then we obtain a HIST of G by adding ww', x_1x_2, x_1w to $\hat{T}$ and deleting $vy_1, x_2y_2, y'w'$ from $\hat{T}$. □

By Claim 3.26, we may assume that w is not adjacent to any vertex from $\hat{W}_1$ in G.

Claim 3.27. If y_1 is adjacent to a vertex from $N_G(v)$ in G, then G has a HIST.

Proof. Let $y' \in N_G(v) - \{y_1\}$ such that $y_1y' \in E(G)$. Then we obtain a HIST of G by adding y_1y', x_1x_2, x_1w to $\hat{T}$ and deleting vy', x_2y_2, yw from $\hat{T}$. □

By Claim 3.27, we suppose that y_1 is not adjacent to any vertex from $N_G(v)$ in G. By the minimality of the number of degree 2 vertices of T, y_1 is not adjacent to any vertex from $\hat{W}_1 \cup \hat{W}_3' \cup \{x_2\}$ in G. Thus $N_G(y_1) \subset \hat{W}_2 \cup \{v, x_1\}$ and so y_1 is adjacent to at least $|\hat{N}^1(v) \cup \hat{N}^2(v)| \geq 1$ vertices from $\hat{W}_2$ in G by $d_G(y_1) \geq d_G(v)$. Let $w_1 \in \hat{W}_2$ such that $y_1w_1 \in E(G)$. Then by the definition of $\hat{W}_2$, there exist $w_2 \in \hat{W}_2$ and $y' \in \hat{N}^2(v)$ such that $w_1y', w_2y' \in E(\hat{T})$. By the minimality of the number of degree 2 vertices of T, we have $y_1w_2 \notin E(G)$. Thus y_1 and w_2 have a common neighbor z in G because the diameter of G is 2. Since $N_G(y_1) \subseteq \hat{W}_2 \cup \{v, x_1\}$ and $w_2 \notin N_G(v)$, we have $z \in (\hat{W}_2 - \{w_2\}) \cup \{x_1\}$.

If $z = x_1$, then we obtain a HIST of G by adding y_1w_1, x_1x_2, x_1w_2 to $\hat{T}$ and deleting w_1y', w_2y', x_2y_2 from $\hat{T}$. If $z = w_1$, then we add $y_1w_1, w_1w_2, x_1x_2, x_1w$ to $\hat{T}$ and delete vy', w_2y', yw, x_2y_2 from $\hat{T}$ to get a HIST of G. Thus we assume $z \in \hat{W}_2 - \{w_1, w_2\}$. In this case, we obtain a HIST of G by adding $y_1w_1, y_1z, w_2z, x_1x_2, x_1w$ to $\hat{T}$ and deleting $vy_1, w_1y', w_2y', yw, x_2y_2$ from $\hat{T}$.

ACKNOWLEDGMENTS

This work was supported by NSF Grant number DMS2153938 (to Songling Shan), JSPS KAKENHI Grant number JP19K14584 (to Shoichi Tsuchiya), a grant for Basic Science Research Projects from The Sumitomo Foundation (to Shoichi Tsuchiya), and a research grant of Senshu University 2022 (to Shoichi Tsuchiya). The authors are very grateful to the two anonymous referees for their careful reading and valuable comments.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Songling Shan https://orcid.org/0000-0002-6384-2876

REFERENCES

1. M. O. Albertson, D. M. Berman, J. P. Hutchinson, and C. Thomassen, *Graphs with homeomorphically irreducible spanning trees*, J. Graph Theory. **14** (1990), 247–258.
2. K. Ando, *Homeomorphically irreducible spanning trees in graphs with diameter 2*, J. Graph Theory. to appear.
3. M. Furuya and S. Tsuchiya, *Forbidden subgraphs and existence of a spanning tree without small degree stems*, Discrete Math. **313** (2013), 2206–2212.
4. T. Ito and S. Tsuchiya, *Degree sum conditions for the existence of homeomorphically irreducible spanning trees*, J. Graph Theory. **99** (2022), no. 1, 162–170.

How to cite this article: S. Shan and S. Tsuchiya, *Characterization of graphs of diameter 2 containing a homeomorphically irreducible spanning tree*, J. Graph Theory. 2023;**104**:886–903. https://doi.org/10.1002/jgt.23005