

Antimagic orientation of forests

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ABSTRACT

An antimagic labeling of a digraph D with n vertices and m arcs is a bijection from the set of arcs of D to $\{1, 2, \dots, m\}$ such that all n oriented vertex-sums are pairwise distinct, where the oriented vertex-sum of a vertex is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. A graph G admits an antimagic orientation if G has an orientation D such that D has an antimagic labeling. Hefetz, Mütze and Schwartz conjectured every connected graph admits an antimagic orientation. In this paper, we support this conjecture by proving that any forest obtained from a given forest with at most one isolated vertex by subdividing each edge at least once admits an antimagic orientation.

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1. Introduction

Graphs considered in this paper are simple, and digraphs in consideration are the orientations of simple graphs. If there is no confusion, we use the same notation G to denote a graph or an underlying graph of a digraph. Also, we use the edges to denote the arcs of a digraph if we do not care about the direction of the edges. Let m be a positive integer and G be a graph with m edges. We use $V(G)$, $E(G)$ to denote the vertex set and edge set of G , respectively. A *labeling* of a graph G with m edges is a bijection from $E(G)$ to the set $\{1, 2, \dots, m\}$. A labeling of G is *antimagic* if for every two distinct vertices $u, v \in V(G)$, the sum of labels on the edges incident with u differs from that of v . A graph is *antimagic* if it admits an antimagic labeling.

Hartsfield and Ringel introduced antimagic labeling in 1990 and they conjectured that every connected graph other than K_2 is antimagic [10], which is called the Antimagic Labeling Conjecture. This conjecture has received intensive attention, but remains open in general, even for trees, as a relaxed version of the conjecture posed in the same paper. Some partial results on the Antimagic Labeling Conjecture can be found in [1–6,8,11,13,14,16,18,19,25].

Hefetz, Mütze and Schwartz [12] introduced a variation of antimagic labelings, i.e., antimagic labelings on directed graphs. An *antimagic labeling* of a directed graph with m arcs is a bijection from the set of arcs to the set $\{1, 2, \dots, m\}$ such that any two oriented vertex-sums are distinct, where an *oriented vertex-sum* of a vertex is the sum of labels of all arcs entering that vertex minus the sum of labels of all arcs leaving it. Given a graph G , we say G admits an *antimagic orientation* if G has an orientation D such that D is antimagic. Regarding antimagic orientation of graphs, Hefetz, Mütze and Schwartz [12] proposed the following conjecture.

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Conjecture 1.1 ([12]). *Every connected graph admits an antimagic orientation.*

For Conjecture 1.1, Hefetz, Mütze and Schwartz [12] showed that every orientation of a graph with order n and minimum degree at least $c \log n$ is antimagic, where c is an absolute constant. Particularly, they showed that every orientation of stars (other than $K_{1,2}$), wheels, and complete graphs (other than K_3) is antimagic. The conjecture is also verified for regular graphs [12,15,23], biregular bipartite graphs [21], Halin graphs [26], graphs with large maximum degree [24], graphs with a large independence set [22], lobsters [9], caterpillars [17] and subdivided caterpillars [7]. Recently, the first author showed that Conjecture 1.1 is true for all bipartite graphs with no vertex of degree two or zero, and for all graphs with minimum degree at least 33 [20].

Every antimagic bipartite graph G with partite sets X and Y admits an antimagic orientation, as we can direct all edges from X to Y and apply any of the antimagic labelings of G . Thus, by the result in [14] with a minor error corrected by Liang, Wong and Zhu [16] on antimagic labelings of trees, we know that every tree with at most one vertex of degree two admits an antimagic orientation, and any tree obtained from a tree with no vertex of degree two by subdividing every edge exactly once admits an antimagic orientation. These two results, together with the result of the first author [20] that every bipartite graph with no vertex of degree two or zero admits an antimagic orientation, suggest that it is hard to find an antimagic orientation if a graph has many vertices of degree two. In this paper, we overcome this issue for forests and obtain the results below.

Theorem 1.2. *Let $F = (V, E)$ be a forest with at most one isolated vertex. If the set of vertices of degree distinct from two is independent, then F admits an antimagic orientation.*

The following result is a consequence of Theorem 1.2.

Corollary 1.3. *Let F be obtained from any forest with at most one isolated vertex by subdividing each edge at least once. Then F admits an antimagic orientation.*

2. Notation and preliminary lemmas

Let G be a graph. For $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G , and $d_G(v) = |N_G(v)|$ is the degree of v in G . For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. If $F \subseteq E(G)$, then $G - F$ is obtained from G by deleting all the edges of F . For $F \subseteq E(\overline{G})$, $G + F$ is obtained from G by adding all the edges from F to G . We also use $G - F$ and $G + F$ if G is a digraph. For two disjoint subsets $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of edges in G with one endvertex in X and the other one in Y and let $e_G(X, Y) = |E_G(X, Y)|$. A *matching* M in G is a set of independent edges, and we use $V(M)$ to denote the set of vertices saturated by M , here the vertices *saturated* by M are the vertices belonging to the edges of M . If G is bipartite with two partite sets X and Y , we denote G by $G[X, Y]$ to emphasis the bipartitions. For any two integers a and b with $a \leq b$, let $[a, b] = \{i \in \mathbb{Z} \mid a \leq i \leq b\}$. If $a = 1$ and $b \geq 1$, we write $[1, b]$ as $[b]$ for simplicity.

Let G be a graph and D be an orientation of G . We denote by $A(D)$ the set of arcs of D . For a labeling τ on $A(D)$ and a vertex $v \in V(D)$, we use $s_{(D, \tau)}(v)$ to denote the oriented vertex-sum at v in D with respect to τ , which is the sum of labels on all arcs entering v minus the sum of labels on all arcs leaving v in D . For simplicity, we write $s_{(D, \tau)}(v)$ as $s(v)$ if D and τ are understood.

We will use the lemma below to partition an integer set such that all the sums of the elements from each subset are congruent to zero modulo an integer.

Lemma 2.1. *Let a and t be integers with $a \geq 0$ and $t \geq 1$, and let $r_1 + r_2 + \dots + r_t$ be a partition of a positive integer k , where $r_i \geq 2$ for each $i \in [t]$. Define $A = \left[1, \left\lfloor \frac{k}{2} \right\rfloor\right] \cup \left[\left\lfloor \frac{k}{2} \right\rfloor + a + 1, k + a\right]$. Then A can be partitioned into subsets $A_1, A_2, \dots, A_t$ such that for every $i \in [t]$, $|A_i| = r_i$ and*

$$\begin{cases} \sum_{x \in A_i} x \equiv 0 \pmod{k + a + 1}, & \text{if } k \text{ is even;} \\ \sum_{x \in A_i} x \equiv 0 \pmod{k + a}, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The case for even k is Corollary 2.2 (i) from [26]. Thus we assume k is odd. Since $k = r_1 + r_2 + \dots + r_t$ and $r_i \geq 2$ for each $i \in [t]$, there exists an odd $r_{i_0} \geq 3$ for some $i_0 \in [t]$. Let $r'_{i_0} = r_{i_0} - 1$ and $r'_i = r_i$ for $i \in [t] \setminus \{i_0\}$. Since $k - 1$ is even and $k - 1 = r'_1 + r'_2 + \dots + r'_t$, applying the case when k is even, the set $A \setminus \{k + a\}$ can be partitioned into pairwise disjoint subsets $B_1, B_2, \dots, B_t$ such that for every $i \in [t]$,

$$|B_i| = r'_i \quad \text{and} \quad \sum_{x \in B_i} x \equiv 0 \pmod{k + a}.$$

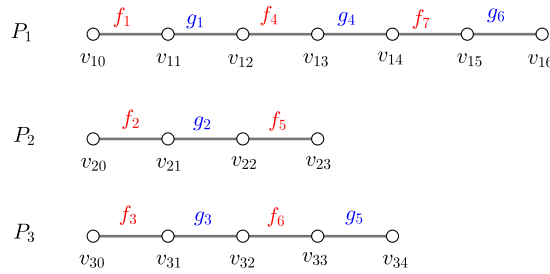


Fig. 1. Illustration of the orderings of edges from three directed paths.

Define

$$A_i = \begin{cases} B_i, & \text{if } i \in [t] \setminus \{i_0\}; \\ B_i \cup \{k+a\}, & \text{if } i = i_0. \end{cases}$$

It is clear that $|A_i| = r_i$ and $\sum_{x \in A_i} x \equiv 0 \pmod{k+a}$. $\square$

This lemma below tells us how to label paths in a forest, which will be used in the proof of Theorem 1.2.

Lemma 2.2. Let $n \geq 1$ be an integer and $\vec{P}_1, \vec{P}_2, \dots, \vec{P}_n$ be n internally disjoint directed paths. For each $i \in [n]$, let $\vec{P}_i = v_{i0}v_{i1} \dots v_{i\ell_i}$, where $\ell_i \geq 1$ is the length of $\vec{P}_i$. Suppose $\sum_{1 \leq i \leq n} \ell_i = \ell$, and the set of first edges $v_{i0}v_{i1}$ of each $\vec{P}_i$ is labeled by a mapping σ such that $\sigma(v_{i0}v_{i1}) = i$ for every $i \in [n]$. Then there exists a bijection $\tau : \bigcup_{1 \leq i \leq n} A(\vec{P}_i) \rightarrow [\ell]$ satisfying the following properties:

- (i) $\tau(v_{i0}v_{i1}) = \sigma(v_{i0}v_{i1})$ for each $i \in [n]$; and
- (ii) $s(u) \neq s(v)$ for any two distinct vertices u and v such that u is internal of some $\vec{P}_i$ and v is internal of some $\vec{P}_j$ with $i, j \in [n]$; and
- (iii) for any vertex v that is internal of some $\vec{P}_i$ with $i \in [n]$, we have $|s(v)| \in [\ell - 1]$ and $s(v) \leq \ell - n - 1$ if $s(v) > 0$.

Proof. Let $i \in [n]$, $j \in [\ell_i]$, and $e_{ij} = v_{i(j-1)}v_{ij}$. For two distinct edges e_{ij} and e_{st} with j and t having the same parity, we write $e_{ij} < e_{st}$ if either $j < t$ or $j = t$ and $i < s$. Note that if $e_{ij} < e_{st}$, then $e_{i(j+1)} < e_{s(t+1)}$. By this definition, all arcs from $\bigcup_{1 \leq i \leq n} A(\vec{P}_i)$ are ordered into two disjoint linear orderings:

$$L_1 : e_{11} < e_{21} < \dots < e_{n1} < \dots,$$

$$L_2 : e_{i_12} < e_{i_22} < \dots < e_{i_k2} < \dots,$$

where $i_1, i_2, \dots, i_k$ with $i_1 < i_2 < \dots < i_k$ are indices such that $\vec{P}_{i_j}$ has length at least two for each $j \in [k]$. To simplify the labels assignment, we assume that the linear ordering L_1 contains ℓ_o edges and the linear ordering L_2 contains ℓ_e edges for some $\ell_o, \ell_e \in [\ell]$, and we denote the i -th edge of L_1 by f_i , and the j -th edge of L_2 by g_j for each $i \in [\ell_o]$ and $j \in [\ell_e]$:

$$L_1 : f_1 < f_2 < \dots < f_{\ell_o},$$

$$L_2 : g_1 < g_2 < \dots < g_{\ell_e}.$$

See Fig. 1 for an illustration of such orderings.

Define $\tau : \bigcup_{1 \leq i \leq n} A(\vec{P}_i) \rightarrow [\ell]$ such that

$$\begin{cases} \tau(f_i) = i & \text{for } i \in [\ell_o]; & \text{(a)} \\ \tau(g_i) = \ell - i + 1 & \text{for } i \in [\ell_e]. & \text{(b)} \end{cases} \quad (1)$$

The mapping τ is a bijection. We show below that τ satisfies properties (i) to (iii). By the definition of τ in (1)(a), we have $\tau(v_{i0}v_{i1}) = i = \sigma(v_{i0}v_{i1})$. Thus, Lemma 2.2 (i) is true.

Since $\vec{P}_i$ is a directed path from v_{i0} to $v_{i\ell_i}$ for any $i \in [n]$, for each $j \in [\ell_i - 1]$, we have $s(v_{ij}) = \tau(e_{ij}) - \tau(e_{i(j+1)})$. Therefore, by the definition of τ , $s(v_{ij}) > 0$ if j is even and $s(v_{ij}) < 0$ if j is odd.

We now show Lemma 2.2 (ii). Let u and v be two distinct vertices such that u is internal of some $\vec{P}_i$ and v is internal of some $\vec{P}_j$ with $i, j \in [n]$. Assume $u = v_{is}$ and $v = v_{jt}$ for some $s \in [1, \ell_i - 1]$ and $t \in [1, \ell_j - 1]$. If s and t have different parities, then one of $s(u)$ and $s(v)$ is positive and another is negative, and so $s(u) \neq s(v)$. Thus, we assume s and t have

the same parity and assume without loss of generality that $e_{is} < e_{jt}$. The assumption $e_{is} < e_{jt}$ further implies that $e_{i(s+1)} < e_{j(t+1)}$. Then, as $\tau(e_{is}) > \tau(e_{jt})$ and $\tau(e_{i(s+1)}) < \tau(e_{j(t+1)})$ when s is even, and $\tau(e_{is}) < \tau(e_{jt})$ and $\tau(e_{i(s+1)}) > \tau(e_{j(t+1)})$ when s is odd, we have

$$|s(u)| = |s(v_{is})| = |\tau(e_{is}) - \tau(e_{i(s+1)})| > |\tau(e_{jt}) - \tau(e_{j(t+1)})| = |s(v_{jt})| = |s(v)|.$$

This proves (ii).

We lastly show Lemma 2.2 (iii). Let v be an internal vertex of some $\vec{P}_i$ with $i \in [n]$. Since the oriented vertex-sum at v is the difference of the two labels on the edges incident to v , and the two labels are distinct numbers from $[\ell]$, we have $|s(v)| \in [\ell - 1]$. Recall that $s(v) > 0$ if $v = v_{ij}$ for some even $j \in [2, \ell_i - 1]$. Since all the labels from $[n]$ are assigned to the set of the first edges of the n paths and $j + 1 \geq 3$ is odd, we have $\tau(e_{i(j+1)}) \geq n + 1$. Thus $s(v) = \tau(e_{ij}) - \tau(e_{i(j+1)}) \leq \ell - (n + 1) = \ell - n - 1$, proving the second part of (iii). $\square$

3. Proof of Theorem 1.2

Let $m = e(F)$. We may assume $m > 0$, as otherwise, the statement is vacuously true. If F has an isolated vertex, then its oriented vertex-sum is zero. We will define next an orientation D of F and an antimagic labeling τ of D such that each oriented vertex-sum at a vertex of degree at least one is non-zero. Thus, in the following, we focus only on the nontrivial components of F . Let

$$X = \{v \in V \mid d_F(v) = 2\} \quad \text{and} \quad Y = V \setminus X.$$

Since F has leaves, we have $Y \neq \emptyset$. Furthermore, Y is independent by the condition on F . Since $Y \neq \emptyset$ and $m > 0$, we have $X \neq \emptyset$. Because $F[X]$ is a subgraph of F and all vertices in X have degree two in F , it follows that $F[X]$ is a forest with each component being a path. Let $P_1, P_2, \dots, P_s$ be all the paths of $F[X]$ for some integer $s \geq 1$, and for every $i \in [s]$, let

$$P_i = v_{i0}v_{i1}v_{i2} \dots v_{i\ell_i} \quad \text{for some integer } \ell_i \geq 0.$$

As $d_{P_i}(v_{i0}) = d_{P_i}(v_{i\ell_i}) \leq 1$ and $d_F(v_{i0}) = d_F(v_{i\ell_i}) = 2$, it follows that each one of the vertices v_{i0} and $v_{i\ell_i}$ is adjacent to a vertex from Y . Let v'_{i0} and $v'_{i\ell_i}$ be the vertices from Y such that

$$v_{i0}v'_{i0}, v_{i\ell_i}v'_{i\ell_i} \in E_F(X, Y).$$

It is possible to have $v_{i0} = v_{i\ell_i}$, but v'_{i0} and $v'_{i\ell_i}$ are two distinct vertices as F is simple and contains no cycle.

Claim 3.1. *There exists a matching $M \subseteq E_F(X, Y)$ saturating exactly one of the endvertices of P_i for each $i \in [s]$ and saturating all vertices of degree at least three from Y .*

Proof. We construct a new bipartite graph H based on F . For each P_i we contract it into a vertex w_i . Let $V(H) = \{w_1, \dots, w_s\} \cup Y$ and $E(H) = E_F(X, Y)$. As the neighbor of v_{i0} from Y is distinct with that of $v_{i\ell_i}$, H is a forest such that $d_H(w_i) = 2$ for each $i \in [s]$. It suffices to show that H has a matching saturating $\{w_1, \dots, w_s\}$ and all vertices of Y that have degree at least three in H . We claim first that H has a matching M_1 saturating $\{w_1, \dots, w_s\}$ and a matching M_2 saturating all vertices of Y that have degree at least three in H . Suppose M_1 does not exist. Then by Hall's Theorem, there is $A \subseteq \{w_1, \dots, w_s\}$ such that $|N_H(A)| < |A|$. This in turn implies that $e_H(A, N_H(A)) = 2|A| > |A| + |N_H(A)|$. Thus $H[A, N_H(A)]$ contains a cycle, which also corresponds to a cycle of F , a contradiction. Let H_2 be a resulting graph of deleting the edges of M_1 in H . For every component of H_2 , note that every vertex from X_1 has degree one in H_2 , and every vertex from Y has degree at least one in H_2 . By Hall's theorem, there exists a matching, denoted by M_2 , that saturates all vertices of degree at least three in $V(H) \cap Y$.

Let $X_1 = \{w_1, \dots, w_s\}$ and $Y_1 = V(M_2) \cap Y$. We claim next that we can construct a matching M that saturates both X_1 and Y_1 . Let D be a component of the graph induced by the edges of M_1 and M_2 . By the constructions of M_1 and M_2 , it holds that $M_1 \cap M_2 = \emptyset$. Therefore, D can only be a path. Because M_1 saturates all vertices of X_1 , by the definition of D , D can not be a path of even length and start and so end at two vertices from X_1 . If D is a path of even length and start and so end at two vertices from Y_1 , we let $M_D = E(D) \cap M_2$ be a matching of D . If D is a path of odd length, then D has a matching M_D saturating $V(D)$. In both cases, by the construction, M_D saturates all vertices in $(X_1 \cap V(D)) \cup (Y_1 \cap V(D))$. Let M be the union of matchings M_D for all the components D . By the construction, M is a matching with the desired property. $\square$

By Claim 3.1, we let $M \subseteq E_F(X, Y)$ be a matching saturating exactly one of the end-vertices of P_i for each $i \in [s]$ and all vertices of degree at least three from Y that exists by Claim 3.1. By the choice of M , we have $|M| = s$. Assume

$$Y = \{y_1, y_2, \dots, y_{n_2}\},$$

for some integer $n_2 \geq 2$ ($|Y| \geq 2$ as F has at least two leaves). Note that $n_2 \geq |M|$ as $M \subseteq E_F(X, Y)$. Since M saturates all the vertices having degree at least three from Y , each vertex $y_i \notin V(M)$ has degree one. Denote by e_i the only edge incident to such y_i . Let $M^* = M \cup \{e_i \mid y_i \notin V(M) \text{ for } i \in [n_2]\}$, $H = F[X, Y] - M^*$, and n_1 be the number of vertices from Y that have degree at least three in F . Note that $|M^*| = n_2$. As Y contains no vertex of degree 2, Y has $n_2 - n_1$ vertices of degree one in F . By renaming vertices of Y , assume

$$\begin{cases} d_F(y_i) \geq 3 & \text{for } i \in [n_1]; \\ d_F(y_i) = 1 & \text{for } i \in [n_1 + 1, n_2]. \end{cases}$$

Because $d_H(y_i) = d_F(y_i) - 1 \geq 2$ for $i \in [n_1]$, it holds that either $e(H) = 0$ or $e(H) \geq 2$. As $e_F(X, Y) = 2s$ and $|M| = s$, we have

$$h := e(H) = e_F(X, Y) - |M^*| = 2s - n_2.$$

In the remainder, we find an orientation D of F and an antimagic labeling τ of D in four steps.

Step 1 Orient and label H : direct each edge from Y to X . For each $i \in [n_1]$, let A_i be the set of all edges incident to y_i in H . Clearly, $|A_1| + |A_2| + \dots + |A_{n_1}| = h$. Since each y_i with $i \in [n_1]$ has degree at least 2 in H , we have $|A_i| \geq 2$. By applying Lemma 2.1 with $t = n_1$, $a = m - s - h$, $k := h$ and $r_i = |A_i|$ for each $i \in [n_1]$, the set

$$A = \left[1, \left\lfloor \frac{h}{2} \right\rfloor\right] \cup \left[\left\lfloor \frac{h}{2} \right\rfloor + a + 1, h + a\right] = \left[1, \left\lfloor \frac{h}{2} \right\rfloor\right] \cup \left[m - s - \left\lfloor \frac{h}{2} \right\rfloor + 1, m - s\right]$$

can be partitioned into $R_1, R_2, \dots, R_{n_1}$ such that for each $i \in [n_1]$, $|R_i| = |A_i|$ and $\sum_{r \in R_i} r \equiv 0 \pmod{m - s + 1}$ if h is even, and $\sum_{r \in R_i} r \equiv 0 \pmod{m - s}$ if h is odd. Label edges in A_i by integers from R_i arbitrarily such that distinct edges receive distinct labels.

Let D_1 be the orientation of H defined above and σ_1 be the labeling of H defined in Step 1. Then for every $i \in [n_1]$, we have

$$s_{(D_1, \sigma_1)}(y_i) = \begin{cases} -a_i(m - s + 1), & \text{if } h \text{ is even;} \\ -a_i(m - s), & \text{if } h \text{ is odd,} \end{cases} \quad (2)$$

for some positive integer a_i .

Step 2 Orient and label edges in $M^* \setminus M$: direct each edge from Y to X . Note that $|M^* \setminus M| = n_2 - s$. We assign arbitrarily the labels in $[m - \lceil h/2 \rceil - n_2 + 1, m - \lceil h/2 \rceil - s]$ to edges in $M^* \setminus M$ such that distinct edges receive distinct labels. We let D_2 be the orientation of $H + M^* \setminus M$ given through Steps 1 and 2 and σ_2 be the labeling of D_2 obtained through Steps 1 and 2. The set of labels used so far on D_2 is

$$\left[1, \left\lfloor \frac{h}{2} \right\rfloor\right] \cup \left[m - n_2 - \left\lfloor \frac{h}{2} \right\rfloor + 1, m - s\right].$$

By renaming the vertices of each path P_i , we may assume that the endvertex $v_{i\ell_i}$ of P_i is saturated by M . Thus the edge $v'_{i0}v_{i0} \in E_F(X, Y) \setminus M$. By permuting the s paths $P_1, \dots, P_s$, we further assume that

$$\begin{cases} \sigma_2(v'_{i0}v_{i0}) = i, & i \in \left[1, \left\lfloor \frac{h}{2} \right\rfloor\right]; & \text{(a)} \\ m - s + \lfloor h/2 \rfloor - i + 1, & i \in \left[\left\lfloor \frac{h}{2} \right\rfloor + 1, s\right]. & \text{(b)} \end{cases} \quad (3)$$

Notice that $2s = e_F(X, Y) = e(H) + |M^*| = h + n_2$.

Step 3 Orient and label edges of $F[X]$: direct each P_i from v_{i0} to $v_{i\ell_i}$ for $i \in [s]$, and denote the orientation by $\vec{P_i}$. Let g be the number of paths P_i with length at least one for all $i \in \left[\left\lfloor \frac{h}{2} \right\rfloor + 1, s\right]$. Denote these g paths by $P_{r_1}, P_{r_2}, \dots, P_{r_g}$, where $\left\lfloor \frac{h}{2} \right\rfloor + 1 \leq r_1 < r_2 < \dots < r_g \leq s$. Define a bijection σ_3^* from $\{v_{r_j0}v_{r_j1} \mid j \in [g]\}$ to $[\left\lfloor \frac{h}{2} \right\rfloor + 1, \left\lfloor \frac{h}{2} \right\rfloor + g]$ such that

$$\sigma_3^*(v_{r_j0}v_{r_j1}) = \lfloor h/2 \rfloor + j \quad \text{for } j \in [g]. \quad (4)$$

Denote by σ_3' the combination of the labeling σ_2 and σ_3^* on $D_3' := D_2 + \{v_{r_j0}v_{r_j1} \mid j \in [g]\}$. Since $s_{(D_3', \sigma_3')}(v_{r_j0}) = \sigma_2(v'_{r_j0}v_{r_j0}) - \sigma_3^*(v_{r_j0}v_{r_j1})$ for $j \in [g]$, by (3)(b) and (4), we have

$$\begin{aligned} m - s - (\lfloor h/2 \rfloor + 1) &\geq s_{(D_3', \sigma_3')}(v_{r_10}) > s_{(D_3', \sigma_3')}(v_{r_20}) > \dots > s_{(D_3', \sigma_3')}(v_{r_g0}) \\ &\geq m - 2s + \lfloor h/2 \rfloor + 1 - (\lfloor h/2 \rfloor + g) \\ &= m - 2s - g + 1 > 0, \end{aligned} \quad (5)$$

as $m \geq e_F(X, Y) + e(F[X]) \geq 2s + g$.

For $i \in \left[\left\lfloor \frac{h}{2} \right\rfloor\right]$, let $R_i = v'_{i0} v_{i0} P_i$ and $\vec{R}_i$ be the directed path from v'_{i0} to $v_{i\ell_i}$. For $i \in \left[\left\lfloor \frac{h}{2} \right\rfloor + 1, \left\lfloor \frac{h}{2} \right\rfloor + g\right]$, let $\vec{R}_i = \vec{P}_{r_j}$, where $j = i - \left\lfloor \frac{h}{2} \right\rfloor \in [g]$. For every $i \in [h/2 + g]$, we have

$$e(R_i) \geq 1 \quad \text{and} \quad \sum_{1 \leq j \leq [h/2] + g} e(R_j) = m - |M^*| - h + [h/2] = m - n_2 - [h/2].$$

We apply Lemma 2.2 on $\vec{R}_1, \dots, \vec{R}_{[h/2] + g}$ with

$$n := [h/2] + g, \quad \ell := m - n_2 - [h/2], \quad \text{and} \quad \sigma := \sigma'_3$$

to get a labeling for those arcs of the n directed paths.

Denote by D_3 the orientation of $F - M$ obtained through Steps 1 to 3, and let σ_3 be the labeling of D_3 obtained through the three steps. Then for any two distinct vertices $u, v \in X \setminus \{v_{i\ell_i}, v_{r_j0}, v_{r_j\ell_{r_j}} \mid i \in [h/2], j \in [h/2 + 1, s]\}$,

where the set consists of all internal vertices from the paths $\vec{R}_1, \dots, \vec{R}_n$, by Lemma 2.2, we have

- (i) $s_{(D_3, \sigma_3)}(u) \neq s_{(D_3, \sigma_3)}(v)$; and
- (ii) $|s_{(D_3, \sigma_3)}(v)| \in [1, m - n_2 - [h/2] - 1]$; and
- (iii) if $s_{(D_3, \sigma_3)}(v) > 0$, then $s_{(D_3, \sigma_3)}(v) \leq m - n_2 - g - h - 1$.

Step 4 Orient and label edges of M : direct each edge in M from Y to X . Recall that $M = \{v'_{i\ell_i} v_{i\ell_i} \mid i \in [s]\}$. Without loss of generality, assume

$$s_{(D_3, \sigma_3)}(v_{i_1\ell_{i_1}}) \leq s_{(D_3, \sigma_3)}(v_{i_2\ell_{i_2}}) \leq \dots \leq s_{(D_3, \sigma_3)}(v_{i_s\ell_{i_s}}).$$

Let σ_4 be a bijection from M to $[m - s + 1, m]$ such that

$$\sigma_4(v'_{i_j\ell_{i_j}} v_{i_j\ell_{i_j}}) = m - s + i$$

for every $i_j \in [s]$.

Denote by D the orientation of F obtained through Steps 1 to 4, and let τ be the labeling of D obtained through the four steps. Then by Step 4, we have

$$\begin{cases} s_{(D, \tau)}(v_{i\ell_i}) = s_{(D_3, \sigma_3)}(v_{i\ell_i}) + \sigma_4(v'_{i\ell_i} v_{i\ell_i}) \geq m - s + 2 & \text{for every } i \in [s]. & \text{(a)} \\ s_{(D, \tau)}(v_{i_1\ell_{i_1}}) < s_{(D, \tau)}(v_{i_2\ell_{i_2}}) < \dots < s_{(D, \tau)}(v_{i_s\ell_{i_s}}) & \text{for every } i_j \in [s]. & \text{(b)} \end{cases} \quad (6)$$

Since M saturates all vertices of degree at least three from Y by Claim 3.1, every vertex from $V(M^* \setminus M) \cap Y$ is a degree one vertex of F . By Step 4 and Step 2,

$$\begin{cases} s_{(D, \tau)}(y_i) \leq -(m - s + 1), & y_i \in V(M) \cap Y; & \text{(a)} \\ -s_{(D, \tau)}(y_i) \in [m - [h/2] - n_2 + 1, m - [h/2] - s], & y_i \in V(M^* \setminus M) \cap Y. & \text{(b)} \end{cases} \quad (7)$$

Next, we show that τ is an antimagic orientation of D . Note first that τ is a bijection from $A(D)$ to $[m]$ as the new labels used in each step are all distinct and they all together form the set $[m]$, and the labelings defined in each step are all bijections. Thus we show that for any two distinct vertices $u, v \in V(D)$, $s_{(D, \tau)}(u) \neq s_{(D, \tau)}(v)$. Let

$$X_1 = \{v_{i\ell_i} \mid i \in [s]\}, \quad X_2 = \{v_{r_j0} \mid j \in [g]\}, \quad X_3 = X \setminus (X_1 \cup X_2).$$

Note that $X_1 = X \cap V(M)$ and X_3 is the set of internal vertices from the paths $\vec{R}_1, \dots, \vec{R}_n$ defined in Step 3. We have the following cases to consider:

- (1) $u, v \in X_1$, $u, v \in X_2$, $u, v \in X_3$, or $u, v \in Y$;
- (2) $u \in X_1$ and $v \in V(D) \setminus X_1$;
- (3) $u \in Y$ and $v \in X_2 \cup X_3$;
- (4) $u \in X_2$ and $v \in X_3$.

Case (1): If $u, v \in X_1$, we get $s_{(D, \tau)}(u) \neq s_{(D, \tau)}(v)$ by (6)(b). If $u, v \in X_2$, we have $s_{(D, \tau)}(u) \neq s_{(D, \tau)}(v)$ by (5). If $u, v \in X_3$, we have $s_{(D, \tau)}(u) \neq s_{(D, \tau)}(v)$ by Property (i) of σ_3 defined in Step 3. Thus we assume $u, v \in Y$. If $u, v \in Y \setminus V(M)$, then u and v are leaves of F by the choice of M and so $s_{(D, \tau)}(u) \neq s_{(D, \tau)}(v)$. If $u \in Y \setminus V(M)$ and $v \in Y \cap V(M)$, then $s_{(D, \tau)}(u) > s_{(D, \tau)}(v)$ by (7)(a) and (7)(b). Hence we assume $u, v \in Y \cap V(M)$. Then by (2) and (7)(a), we have

$$s_{(D,\tau)}(u) - s_{(D,\tau)}(v) \equiv \begin{cases} b \pmod{m-s+1}, & \text{if } h \text{ is even;} \\ b \pmod{m-s}, & \text{if } h \text{ is odd,} \end{cases}$$

where b is an integer satisfying $b \in [-(s-1), s-1]$ and $b \neq 0$. Since $m \geq 2s$, we have $b < m-s$. Thus $s_{(D,\tau)}(u) \neq s_{(D,\tau)}(v)$.

Case (2): Let $u \in X_1$ and $v \in V(D) \setminus X_1$. By (6)(a), we have $s_{(D,\tau)}(u) \geq m-s+2$. If $v \in Y$, we have $s_{(D,\tau)}(v) < 0$. Thus we assume $v \in X_2 \cup X_3$. By (5) and Property (ii) of σ_3 defined in Step 3, we have $s_{(D,\tau)}(v) \leq \max\{m-s-(\lfloor h/2 \rfloor + 1), m-n_2 - \lfloor h/2 \rfloor - 1\} < m-s$, as $n_2 \geq s$. Thus $s_{(D,\tau)}(u) \neq s_{(D,\tau)}(v)$.

Case (3): Let $u \in Y$ and $v \in X_2 \cup X_3$. Note that $s_{(D,\tau)}(u) \neq s_{(D,\tau)}(v)$ if $v \in X_2$, as $s_{(D,\tau)}(u) < 0$ and $s_{(D,\tau)}(v) > 0$. Thus we assume $v \in X_3$. By (7)(a) and (7)(b), we have $|s_{(D,\tau)}(u)| \geq m-n_2 - \lfloor h/2 \rfloor + 1$. By Property (ii) of σ_3 defined in Step 3, we have $|s_{(D,\tau)}(v)| \leq m-n_2 - \lfloor h/2 \rfloor - 1 < |s_{(D,\tau)}(u)|$. Thus $s_{(D,\tau)}(u) \neq s_{(D,\tau)}(v)$.

Case (4): Let $u \in X_2$ and $v \in X_3$. By (5), we have $s_{(D,\tau)}(u) \geq m+1-2s-g-h > 0$. Thus we only consider $v \in X_3$ such that $s_{(D,\tau)}(v) > 0$. By Property (iii) of σ_3 defined in Step 3, we have $s_{(D,\tau)}(v) \leq m-n_2-g-h-1 = m-2s-g-1 < m+1-2s-g$, where recall $2s = h+n_2$. Thus $s_{(D,\tau)}(u) \neq s_{(D,\tau)}(v)$.

The proof is complete. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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