An Ore-type condition for hamiltonicity in tough graphs and the extremal examples

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Abstract

Let G be a t-tough graph on $n \geqslant 3$ vertices for some $t > 0$. It was shown by Bauer et al. in 1995 that if the minimum degree of G is greater than $\frac{n}{t+1} - 1$, then G is hamiltonian. In terms of Ore-type hamiltonicity conditions, the problem was only studied when t is between 1 and 2, and recently the second author proved a general result. The result states that if the degree sum of any two nonadjacent vertices of G is greater than $\frac{2n}{t+1} + t - 2$, then G is hamiltonian. It was conjectured in the same paper that the "$+t$" in the bound $\frac{2n}{t+1} + t - 2$ can be removed. Here we confirm the conjecture. The result generalizes the result by Bauer, Broersma, van den Heuvel, and Veldman. Furthermore, we characterize all t-tough graphs G on $n \geqslant 3$ vertices for which $\sigma_2(G) = \frac{2n}{t+1} - 2$ but G is non-hamiltonian.

Keywords. Ore-type condition; toughness; hamiltonian cycle.

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1 Introduction

We consider only finite simple graphs. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let $v \in V(G)$, $S \subseteq V(G)$, and $H \subseteq G$. Then $N_G(v)$ denotes the set of neighbors of v in G, $d_G(v) := |N_G(v)|$ is the degree of v in G, and $\delta(G) := \min\{d_G(v) : v \in V(G)\}$ is the minimum degree of G. Define $\deg_G(v, H) = |N_G(v) \cap V(H)|$, $N_G(S) = (\bigcup_{x \in S} N_G(x)) \setminus S$, and we write $N_G(H)$ for $N_G(V(H))$. Let $N_H(v) = N_G(v) \cap V(H)$ and $N_H(S) = N_G(S) \cap V(H)$. Again, we write $N_H(R)$ for $N_H(V(R))$ for any subgraph R of G. We use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G) \setminus S$, respectively. For notational simplicity we

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write $G-x$ for $G-\{x\}$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in G with one endvertex in V_1 and the other endvertex in V_2. For two integers a and b, let $[a, b] = \{i \in \mathbb{Z} : a \leqslant i \leqslant b\}$.

Throughout this paper, if not specified, we will assume t to be a nonnegative real number. The number of components of a graph G is denoted by $c(G)$. The graph G is said to be *t-tough* if $|S| \geqslant t \cdot c(G-S)$ for each $S \subseteq V(G)$ with $c(G-S) \geqslant 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or is ∞ if G is complete. This concept was introduced by Chvátal [7] in 1973. It is easy to see that if G has a hamiltonian cycle then G is 1-tough. Conversely, Chvátal [7] conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed t-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so t_0 must be at least $\frac{9}{4}$ if Chvátal's toughness conjecture is true.

Chvátal's toughness conjecture has been verified for certain classes of graphs including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs [2]. The classes also include $2K_2$-free graphs [6, 15, 13], and R-free graphs for $R \in \{P_2 \cup P_3, P_3 \cup 2P_1, P_2 \cup kP_1\}$ [16, 9, 17, 12, 19], where $k \geqslant 4$ is an integer. In general, the conjecture is still wide open. In finding hamiltonian cycles in graphs, sufficient conditions such as Dirac-type and Ore-type conditions are the most classic ones.

Theorem 1.1 (Dirac's Theorem [8])**.** *If G is a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant \frac{n}{2}$, then G is hamiltonian.*

Define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G)$ and they are nonadjacent$\}$ if G is noncomplete, and define $\sigma_2(G) = \infty$ otherwise. Ore's Theorem, as a generalization of Dirac's Theorem, is stated below.

Theorem 1.2 (Ore's Theorem [11])**.** *If G is a graph on $n \geqslant 3$ vertices with $\sigma_2(G) \geqslant n$, then G is hamiltonian.*

Analogous to Dirac's Theorem, Bauer, Broersma, van den Heuvel, and Veldman [4] proved the following result by incorporating the toughness of the graph.

Theorem 1.3 (Bauer et al. [4])**.** *Let G be a t-tough graph on $n \geqslant 3$ vertices. If $\delta(G) > \frac{n}{t+1} - 1$, then G is hamiltonian.*

A natural question here is whether we can find an Ore-type condition involving the toughness of G that generalizes Theorem 1.3. Various theorems were proved prior to Theorem 1.3 by only taking $\tau(G)$ between 1 and 2 [10, 3, 5]. Let G be a t-tough graph on $n \geqslant 3$ vertices. The author showed in [14] that if $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, then G is hamiltonian. It was also conjectured in [14] that $\sigma_2(G) > \frac{2n}{t+1} - 2$ is the right bound. In

this paper, we confirm the conjecture. For any odd integer $n \geqslant 3$, the complete bipartite graph $G := K_{\frac{n-1}{2},\frac{n+1}{2}}$ is $\frac{n-1}{n+1}$-tough and satisfies $\sigma_2(G) = n-1 = \frac{2n}{1+\frac{n-1}{n+1}} - 2$. However, G is not hamiltonian. Thus, the degree sum condition that $\sigma_2(G) > \frac{2n}{t+1} - 2$ is best possible for a t-tough graph on at least three vertices to be hamiltonian. In fact, for any odd integers $n \geqslant 3$, any graph from the family $\mathcal{H} = \{H_{\frac{n-1}{2}} + \overline{K}_{\frac{n+1}{2}} : H_{\frac{n-1}{2}}$ is any graph on $\frac{n-1}{2}$ vertices$\}$ is an extremal graph, where "+" represents the join of two graphs. We also show that $\mathcal{H}$ is the only family of extremal graphs.

Theorem 1. *Let G be a t-tough graph on $n \geqslant 3$ vertices. Then the following statements hold.*

(a) If $\sigma_2(G) > \frac{2n}{t+1} - 2$, then G is hamiltonian.

(b) If $\sigma_2(G) = \frac{2n}{t+1} - 2$ and G is not hamiltonian, then $G \in \mathcal{H}$.

The remainder of this paper is organized as follows: in Section 2, we introduce some notation and preliminary results, and in Section 3, we prove Theorem 1.

2 Preliminary results

Let G be a graph and λ be a positive integer. Following [18], a cycle C of G is a *D_λ-cycle* if every component of $G - V(C)$ has order less than λ. Clearly, a D_1-cycle is just a hamiltonian cycle. We denote by $c_\lambda(G)$ the number of components of G with order at least λ, and write $c_1(G)$ just as $c(G)$. Two subgraphs H_1 and H_2 of G are *remote* if they are disjoint and there is no edge of G joining a vertex of H_1 with a vertex of H_2. For a subgraph H of G, let $d_G(H) = |N_G(H)|$ be the degree of H in G. We denote by $\delta_\lambda(G)$ the minimum degree of a connected subgraph of order λ in G. Again $\delta_1(G)$ is just $\delta(G)$.

Lemma 2.1 ([16])**.** *Let $t > 0$ and G be a non-complete n-vertex t-tough graph. Then $|W| \leqslant \frac{n}{t+1}$ for every independent set W in G.*

Denote by $\overrightarrow{C}$ an orientation of C. We assume that the orientation is clockwise throughout the rest of this paper. For $x \in V(C)$, denote the immediate successor of x on $\overrightarrow{C}$ by x^+ and the immediate predecessor of x on $\overrightarrow{C}$ by x^-. We use $N_C^+(x)$ to denote the set of immediate successors for vertices from $N_C(x)$. For $u, v \in V(C)$, $u\overrightarrow{C}v$ denotes the segment of $\overrightarrow{C}$ starting at u, following $\overrightarrow{C}$ in the orientation, and ending at v. Likewise, $u\overleftarrow{C}v$ is the opposite segment of $\overrightarrow{C}$ with endpoints as u and v. Let $\mathrm{dist}_{\overrightarrow{C}}(u, v)$ denote the length of the path $u\overrightarrow{C}v$. For any vertex $u \in V(C)$ and any positive integer k, define

$$L_u^+(k) = \{v \in V(C) \,:\, \mathrm{dist}_{\overrightarrow{C}}(u, v) \in [1, k]\}$$

to be the set of k consecutive successors of u. Hereafter, all cycles under consideration are oriented, and we will not distinguish between the notation C and $\overrightarrow{C}$.

The following lemma provides a way of extending a cycle C provided that the vertices outside C have many neighbors on C. The proof follows from Lemma 2.1 and is very similar to the proof of Lemma 10 in [16]: if we assume instead that C cannot be extended by including x, then $N_C^+(x) \cup \{x\}$ is an independent set in G.

Lemma 2.2. *Let $t > 0$ and G be an n-vertex t-tough graph, and let C be a non-hamiltonian cycle of G. If $x \in V(G) \setminus V(C)$ satisfies $\deg_G(x, C) > \frac{n}{t+1} - 1$, then G has a cycle C' such that $V(C') = V(C) \cup \{x\}$.*

A path P connecting two vertices u and v is called a (u, v)-*path*, and we write uPv or vPu in order to specify the two endvertices of P. Let uPv and xQy be two paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx.

For an integer $\lambda \geqslant 1$, if a graph G contains a $D_{\lambda+1}$-cycle C but no D_λ-cycle, then $V(G) \setminus V(C) \neq \emptyset$. Furthermore, $G - V(C)$ has a component of order λ. The result below with $d_G(H)$ replaced by $\delta_\lambda(G)$ and H replaced by any component of $G - V(C)$ with order λ was proved in [4, Corollary 7(a)].

Lemma 2.3 ([14])**.** *Let G be a t-tough 2-connected graph of order n. Suppose G has a D_{s+1}-cycle but no D_s-cycle for some integer $s \geqslant 1$. Let C be a D_{s+1}-cycle of G such that C minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p, q \in [1, s]$ with $p > q$. Then $n \geqslant (t + |V(H)|)(d_G(H) + 1)$ for any component H of $G - V(C)$.*

The lemma below is the key to get rid of the "$+t$" in the lower bound $\frac{2n}{t+1} + t - 2$ on $\sigma_2(G)$ for guaranteeing the existence of a hamiltonian cycle [14].

Lemma 2.4. *Let G be a t-tough 2-connected graph of order n. Suppose that G has a $D_{\lambda+1}$-cycle but no D_λ-cycle for some integer $\lambda \geqslant 1$. Let C be a cycle of G. Then $G - V(C)$ has a component H with order at least λ such that $\deg_G(x, C) \leqslant \frac{n}{t+1} - \lambda$ for some $x \in V(H)$.*

Proof. Since G has no D_λ-cycle, it is clear that $G - V(C)$ has a component of order at least λ. We suppose to the contrary that for each component H with order at least λ of $G - V(C)$ and each $x \in V(H)$, we have $\deg_G(x, C) > \frac{n}{t+1} - \lambda$. Among all cycles C' of G that satisfy the two conditions below, we may assume that C is one that minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p \geqslant \lambda$ and any q with $q < p$.

(1) Each component of $G - V(C)$ either has order at most $\lambda - 1$, or

(2) the component H has order at least λ such that for each $x \in V(H)$, we have $\deg_G(x, C) > \frac{n}{t+1} - \lambda$.

We take a component H with order at least λ and assume that $N_C(H)$ has size k for some integer $k \geqslant 2$, and that the k neighbors are $v_1, \ldots, v_k$ and appear in the same order along $\overrightarrow{C}$. Note that $k > \frac{n}{t+1} - \lambda$ by our assumption. For each $i \in [1,k]$, and each $v \in V(v_i^+ \overrightarrow{C} v_{i+1}^-)$, where $v_{k+1} := v_1$, we let $\mathcal{C}(v)$ be the set of components of $G - V(C)$ that have a vertex joining to v by an edge in G. As $N_C(H) \cap V(v_i^+ \overrightarrow{C} v_{i+1}^-) = \emptyset$, we have $H \notin \mathcal{C}(v)$. Let $w_i^* \in V(v_i^+ \overrightarrow{C} v_{i+1}^-)$ be the vertex with $\mathrm{dist}_{\overrightarrow{C}}(v_i, w_i^*)$ minimum such that

$$\sum_{D \in \bigcup\limits_{v \in V(v_i^+ \overrightarrow{C} w_i^*)} \mathcal{C}(v)} |V(D)| + |V(v_i^+ \overrightarrow{C} w_i^*)| \geqslant \lambda.$$

If such a vertex w_i^* exists, let $L_{v_i}^*(\lambda)$ be the union of the vertex set $V(v_i^+ \overrightarrow{C} w_i^*)$ and all those vertex sets of graphs in $\bigcup\limits_{v \in V(v_i^+ \overrightarrow{C} w_i^*)} \mathcal{C}(v)$; if such a vertex w_i^* does not exist, let $L_{v_i}^*(\lambda) = L_{v_i}^+(\lambda)$. Note that when w_i^* exists, by its definition, $w_i^* \in V(v_i^+ \overrightarrow{C} v_{i+1}^-)$. Thus $V(v_i^+ \overrightarrow{C} w_i^*) \cap V(v_j^+ \overrightarrow{C} w_j^*) = \emptyset$ if both w_i^* and w_j^* exist for distinct $i, j \in [1,k]$. If w_i^* exists, for any $r \in [1, \mathrm{dist}_{\overrightarrow{C}}(v_i, w_i^*)]$, let v_i^r be the vertex from $v_i^+ \overrightarrow{C} w_i^*$ such that $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_i^r) = r$. Then define

$$L_{v_i}^*(\lambda, r) = \left(\bigcup_{D \in \bigcup\limits_{v \in V(v_i^+ \overrightarrow{C} v_i^r)} \mathcal{C}(v)} V(D) \right) \cup V(v_i^+ \overrightarrow{C} v_i^r).$$

We will show that we can make the following assumptions:

(a) If for some $i \in [1,k]$, it holds that $L_{v_i}^*(\lambda) = L_{v_i}^+(\lambda)$, then $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_j) \geqslant \lambda + 1$ for any $j \in [1,k]$ with $j \neq i$. Thus the vertex w_i^* exists for each $i \in [1,k]$.

(b) $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are pairwise remote for any distinct $i, j \in [1,k]$.

With Assumptions (a) and (b), we can reach a contradiction as follows: note that $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are remote for any distinct $i, j \in [1,k]$ and H and $G[L_{v_i}^*(\lambda)]$ are remote for any $i \in [1,k]$. Let $S = V(G) \setminus \Big((\bigcup_{i=1}^k L_{v_i}^*(\lambda)) \cup V(H) \Big)$. Then $|S| \leqslant n - (k+1)\lambda$ and $c(G - S) = k + 1$. As G is t-tough, we get

$$n - (k+1)\lambda \geqslant |S| \geqslant t \cdot c(G - S) = t(k+1),$$

giving $k \leqslant \frac{n}{t+\lambda} - 1$. Since $n \geqslant (\lambda + t)(2t+1)$ by Lemma 2.3 (G has a $D_{\lambda+1}$-cycle C' such that $G - V(C')$ has a component H' of order λ, and $d_G(H') \geqslant 2t$ by G being t-tough),

we get

$$\frac{n}{t+1}-\lambda-\left(\frac{n}{t+\lambda}-1\right) \;=\; \frac{(\lambda-1)(n-(t+1)(t+\lambda))}{(t+1)(t+\lambda)} \geqslant 0,$$

and so $k \leqslant \frac{n}{t+\lambda}-1 \leqslant \frac{n}{t+1}-\lambda$. This gives a contradiction to $k > \frac{n}{t+1}-\lambda$. Thus we are only left to show Assumptions (a) and (b). We show that if any one of the assumptions is violated, then we can decrease $c_p(G-V(C))$ for some $p \geqslant \lambda$.

For Assumption (a), if $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$ for some $i \in [1,k]$ but $\mathrm{dist}_{\overrightarrow{C}}(v_i,v_j) \leqslant \lambda$ for some $v_j \in N_C(H)$ with $j \neq i$, then there must exist two consecutive indices $i,j \in [1,k]$ such that $\mathrm{dist}_{\overrightarrow{C}}(v_i,v_j) \leqslant \lambda$. Thus we may just assume $j = i+1$, where the index is taken modulo k. Let $v^*_i, v^*_{i+1} \in V(H)$ such that $v_iv^*_i, v_{i+1}v^*_{i+1} \in E(G)$, and let P be a (v^*_i, v^*_{i+1})-path in H. Let $C_1 = v_i\overleftarrow{C}v_{i+1}v^*_{i+1}Pv^*_iv_i$.

Note that every component of $G-V(C)$ not having any vertex joining to a vertex from $v_i^+\overrightarrow{C}v_{i+1}^-$ in G is still a component of $G-V(C_1)$. Those components automatically satisfy Conditions (1) and (2) as listed in the beginning of this proof. Vertices in $v_i^+\overrightarrow{C}v_{i+1}^-$ are contained in a distinct component of $G-V(C_1)$, and the component has order at most $\lambda-1$ by the assumption that $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$ and $\mathrm{dist}_{\overrightarrow{C}}(v_i,v_{i+1}) \leqslant \lambda$. Finally, as any vertex from each component of $H-V(v^*_{i+1}Pv^*_i)$ is not adjacent in G to any vertex from $v_i^+\overrightarrow{C}v_{i+1}^-$, we know that components of $H-V(v^*_{i+1}Pv^*_i)$ are components of $G-V(C_1)$, and that $\deg_G(w,C_1) > \frac{n}{t+1}-\lambda$ for any $w \in V(H-V(v^*_{i+1}Pv^*_i))$. Hence each component of $G-V(C_1)$ either has order at most $\lambda-1$ or is a component of order at least λ such that each vertex from the component has in G more than $\frac{n}{t+1}-\lambda$ neighbors on C_1. However, $c_{|V(H)|}(G-V(C_1)) < c_{|V(H)|}(G-V(C))$ and $c_q(G-V(C_1)) = c_q(G-V(C))$ for any $q > |V(H)|$, contradicting the choice of C. Therefore we have Assumption (a), which implies that the vertex w^*_i exists for each $i \in [1,k]$.

For Assumption (b), suppose it is false. Then there exist distinct $i,j \in [1,k]$ such that $G[L^*_{v_i}(\lambda)]$ and $G[L^*_{v_j}(\lambda)]$ are not remote. By the definition of remote subgraphs, we have either $L^*_{v_i}(\lambda) \cap L^*_{v_j}(\lambda) \neq \emptyset$ or $L^*_{v_i}(\lambda) \cap L^*_{v_j}(\lambda) = \emptyset$ but $E_G(L^*_{v_i}(\lambda), L^*_{v_j}(\lambda)) \neq \emptyset$. In order to achieve a contradiction, we first show the following general claim, call it Claim $(*)$.

Claim $(*)$: For any $r \in [1, \mathrm{dist}_{\overrightarrow{C}}(v_i,w^*_i)]$ and $s \in [1, \mathrm{dist}_{\overrightarrow{C}}(v_j,w^*_j)]$, if $L^*_{v_i}(\lambda,r) \cap L^*_{v_j}(\lambda,s) = \emptyset$, then $E_G(L^*_{v_i}(\lambda,r), L^*_{v_j}(\lambda,s)) = \emptyset$.

Suppose otherwise that $E_G(L^*_{v_i}(\lambda,r), L^*_{v_j}(\lambda,s)) \neq \emptyset$. Since $L^*_{v_i}(\lambda,r) \cap L^*_{v_j}(\lambda,s) = \emptyset$, $E_G(L^*_{v_i}(\lambda,r), L^*_{v_j}(\lambda,s)) \neq \emptyset$ implies that there exist $y \in V(v_i^+\overrightarrow{C}w^*_i) \cap L^*_{v_i}(\lambda,r)$ and $z \in V(v_j^+\overrightarrow{C}w^*_j) \cap L^*_{v_j}(\lambda,s)$ such that $yz \in E(G)$. We choose $y \in V(v_i^+\overrightarrow{C}w^*_i) \cap L^*_{v_i}(\lambda,r)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_i,y)$ minimum and $z \in V(v_j^+\overrightarrow{C}w^*_j) \cap L^*_{v_j}(\lambda,s)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_j,z)$ minimum such that $yz \in E(G)$. By the choice of y and z, we have $E_G(V(v_i^+\overrightarrow{C}y^-), V(v_j^+\overrightarrow{C}z^-)) = \emptyset$.

Let $v_i^*, v_j^* \in V(H)$ such that $v_iv_i^*, v_jv_j^* \in E(G)$, P be a (v_i^*, v_j^*)-path in H, and let $C_1 = v_i\overleftarrow{C}zy\overrightarrow{C}v_jv_j^*Pv_i^*v_i$. Note that no vertex of H is adjacent in G to any vertex of $v_i^+\overrightarrow{C}y^-$ or $v_j^+\overrightarrow{C}z^-$ by the fact that $V(v_i^+\overrightarrow{C}y^-) \subseteq V(v_i^+\overrightarrow{C}w_i^*)$ and $V(v_j^+\overrightarrow{C}z^-) \subseteq V(v_j^+\overrightarrow{C}w_j^*)$ and Assumption (a). By the assumption that $L_{v_i}^*(r) \cap L_{v_j}^*(s) = \emptyset$ and the definitions of $L_{v_i}^*(\lambda)$ and $L_{v_j}^*(\lambda)$, we know that $v_i^+\overrightarrow{C}y^-$ and $v_j^+\overrightarrow{C}z^-$ are respectively contained in distinct components of $G - V(C_1)$ that each have order at most $\lambda - 1$. By the same reasoning as in proving Assumption (a), we know that each component of $G - V(C_1)$ has order at most $\lambda - 1$ or is a component such that each vertex from the component has in G more than $\frac{n}{t+1} - \lambda$ neighbors on C_1. However, $c_{|V(H)|}(G - V(C_1)) < c_{|V(H)|}(G - V(C))$ and $c_q(G - V(C_1)) = c_q(G - V(C))$ for any $q > |V(H)|$, contradicting the choice of C. Thus Claim $(*)$ holds.

Now let us get back to proving Assumption (b) by contradiction. Assume first that $L_{v_i}^*(\lambda) \cap L_{v_j}^*(\lambda) \neq \emptyset$. Then there exist $v \in V(v_i^+\overrightarrow{C}w_i^*)$ and $u \in V(v_j^+\overrightarrow{C}w_j^*)$ such that $\mathcal{C}(v) \cap \mathcal{C}(u) \neq \emptyset$, we then further choose v closest to v_i and u closest to v_j along $\overrightarrow{C}$ such that $\mathcal{C}(v) \cap \mathcal{C}(u) \neq \emptyset$. Thus for any $w_i \in V(v_i^+\overrightarrow{C}v^-)$ and any $w_j \in V(v_j^+\overrightarrow{C}u^-)$, it holds that $\mathcal{C}(w_i) \cap \mathcal{C}(w_j) = \emptyset$. Let $D \in \mathcal{C}(v) \cap \mathcal{C}(u)$ and $v', u' \in V(D)$ such that $vv', uu' \in E(G)$, and P' be a (v', u')-path of D. Let $v_i^*, v_j^* \in V(H)$ such that $v_iv_i^*, v_jv_j^* \in E(G)$, and let P be a (v_i^*, v_j^*)-path in H. Then $C_1 = v_iv_i^*Pv_j^*v_j\overleftarrow{C}vv'P'u'u\overrightarrow{C}v_i$ is a cycle. Since each of $V(v_i^+\overrightarrow{C}v^-)$ and $V(v_j^+\overrightarrow{C}u^-)$ contains at most $\lambda - 1$ vertices and they are proper subsets of $V(v_i^+\overrightarrow{C}w_i^*)$ and $V(v_j^+\overrightarrow{C}w_j^*)$ respectively, by Assumption (a) above, we have $N_C(H) \cap (V(v_i^+\overrightarrow{C}v^-) \cup V(v_j^+\overrightarrow{C}u^-)) = \emptyset$. By the choices of v and u that for any $w_i \in V(v_i^+\overrightarrow{C}v^-)$ and any $w_j \in V(v_j^+\overrightarrow{C}u^-)$, it holds that $\mathcal{C}(w_i) \cap \mathcal{C}(w_j) = \emptyset$, Claim $(*)$ implies that the components of $G - V(C_1)$ that respectively contain $v_i^+\overrightarrow{C}v^-$ and $v_j^+\overrightarrow{C}u^-$ are disjoint. Since $V(v_i^+\overrightarrow{C}v^-)$ is a proper subset of $V(v_i^+\overrightarrow{C}w_i^*)$ and $V(v_j^+\overrightarrow{C}u^-)$ is a proper subset of $V(v_j^+\overrightarrow{C}w_j^*)$, it follows by the definitions of $L_{v_i}^*(\lambda)$ and $L_{v_j}^*(\lambda)$ that the components of $G - V(C_1)$ that respectively contain $v_i^+\overrightarrow{C}v^-$ and $v_j^+\overrightarrow{C}u^-$ have order at most $\lambda - 1$. By the same reasoning as in proving Assumption (a), we know that each component of $G - V(C_1)$ has order at most $\lambda - 1$ or is a component such that each vertex from the component has in G more than $\frac{n}{t+1} - \lambda$ neighbors on C_1. However, $c_{|V(H)|}(G - V(C_1)) < c_{|V(H)|}(G - V(C))$ and $c_q(G - V(C_1)) = c_q(G - V(C))$ for any $q > |V(H)|$, contradicting the choice of C. Thus we must have $L_{v_i}^*(\lambda) \cap L_{v_j}^*(\lambda) = \emptyset$. Applying Claim $(*)$ again with $r = \text{dist}_{\overrightarrow{C}}(v_i, w_i^*)$ and $s = \text{dist}_{\overrightarrow{C}}(v_j, w_j^*)$, we have $E_G(L_{v_i}^*(\lambda), L_{v_j}^*(\lambda)) = \emptyset$. Therefore, $G[L_{v_i}^*(\lambda)]$ and $G[L_{v_j}^*(\lambda)]$ are remote, contradicting our assumption. Thus Assumption (b) holds. □

3 Proof of Theorem 1

Let G be a t-tough graph on $n \geqslant 3$ vertices such that $\sigma_2(G) \geqslant \frac{2n}{t+1} - 2$. We may assume that G is not a complete graph. Thus G is $\lceil 2t \rceil$-connected as it is t-tough. Suppose to the contrary that G is not hamiltonian.

Claim 1. *We may assume that G is 2-connected.*

Proof. Since $t > 0$, G is connected. Assume to the contrary that G has a cut vertex x. By considering the degree sum of two vertices respectively from two components of $G-x$, we know that $\sigma_2(G) \leqslant n-1$. On the other hand, G has a cut vertex implies $t \leqslant \frac{1}{2}$ and so $\sigma_2(G) \geqslant \frac{2n}{t+1} - 2 \geqslant \frac{4n}{3} - 2$. If $\sigma_2(G) > \frac{4n}{3} - 2$, then we get a contradiction to $\sigma_2(G) \leqslant n-1$ as $n \geqslant 3$. Thus we assume $\sigma_2(G) = \frac{4n}{3} - 2$, which contradicts $\sigma_2(G) \leqslant n-1$ if $n \geqslant 4$. Thus $n = 3$ and so $G = P_3$, but this implies $G \in \mathcal{H}$. □

Since G is 2-connected, Lemma 2.3 implies

$$n \geqslant (t+1)(\lceil 2t \rceil + 1).$$

Also as G is 2-connected, G contains cycles. Let $\lambda \geqslant 0$ be the integer such that G admits no D_λ-cycle but a $D_{\lambda+1}$-cycle. Then we choose C to be a longest $D_{\lambda+1}$-cycle that minimizes $c_p(G-V(C))$ prior to minimizing $c_q(G-V(C))$ for any $p, q \in [1, \lambda]$ with $p > q$. As G is not hamiltonian, we have $\lambda \geqslant 1$. Thus $V(G) \setminus V(C) \neq \emptyset$. Since C is not a D_λ-cycle but a $D_{\lambda+1}$-cycle, $G - V(C)$ has a component H of order λ. Let

$$W = N_C(H) \quad \text{and} \quad \omega = |W|.$$

Since G is a connected t-tough graph, it follows that $\omega \geqslant \lceil 2t \rceil$. On the other hand, Lemma 2.3 implies that $\omega \leqslant \frac{n}{t+\lambda} - 1$.

Claim 2.

$$\begin{cases} \lambda + \omega < \dfrac{n}{t+1} & \text{if } \lambda \geqslant 2, \\ \lambda + \omega \leqslant \dfrac{n}{t+1} & \text{if } \lambda = 1. \end{cases}$$

Proof. If $\lambda = 1$, then the assertion holds by $\omega \leqslant \frac{n}{t+\lambda} - 1$. Thus we assume $\lambda \geqslant 2$ and assume to the contrary that $\lambda + \omega \geqslant \frac{n}{t+1}$. Then we have $n \leqslant (\lambda+\omega)(t+1)$. By Lemma 2.3, we have $n \geqslant (\lambda + t)(\omega + 1)$. Thus we have

$$(\lambda + t)(\omega + 1) \leqslant (\lambda + \omega)(t + 1),$$

which implies $\lambda\omega + \lambda + t\omega + t \leqslant \lambda t + \lambda + t\omega + \omega$ and so $(\lambda - 1)\omega \leqslant (\lambda - 1)t$. Since $\lambda \geqslant 2$, we get $\omega \leqslant t$, a contradiction to $\omega \geqslant 2t$. Note that the argument above for $\lambda \geqslant 2$ holds for all components of $G - V(C)$ as Lemma 2.3 holds for all components of $G - V(C)$. □

Claim 3. *If $\sigma_2(G) \geqslant \frac{2n}{t+1} - 2$, then H is the only component of $G - V(C)$.*

Proof. Suppose $H^* \neq H$ is another component of $G - V(C)$. Then we have $d_G(x) + d_G(y) \geqslant \sigma_2(G)$ for any $x \in V(H)$ and $y \in V(H^*)$. Since $d_G(x) \leqslant \lambda + \omega - 1$ and $d_G(y) \leqslant |V(H^*)| + |N_C(H^*)| - 1$, Claim 2 implies that $|V(H^*)| + |N_C(H^*)| > \sigma_2(G) - (\frac{n}{t+1} - 1) + 1 \geqslant \frac{n}{t+1}$ if $\lambda \geqslant 2$. Repeating exactly the same argument for $|V(H^*)| + |N_C(H^*)|$ as in the proof of Claim 2 leads to a contradiction.

Thus we assume $\lambda = 1$. We get the same contradiction as above if $\sigma_2(G) > \frac{2n}{t+1} - 2$ or $\lambda + \omega < \frac{n}{t+1}$. Thus we have $\sigma_2(G) = \frac{2n}{t+1} - 2$ and $\omega = \frac{n}{t+1} - 1$ by Claim 2. Then H and H^* each contains only one vertex, say x and y, respectively. We first claim that the vertex y is adjacent in G to at most one vertex from W^+. For otherwise, suppose there are distinct $u, v \in W^+$ such that $yu, yv \in E(G)$. Then $C^* = u^-\overleftarrow{C}vyu\overrightarrow{C}v^-xu^-$ is a $D_{\lambda+1}$-cycle of G with $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$. This contradicts the choice of C.

We then claim that the set W^+ is an independent set in G. For otherwise, suppose there are distinct $u, v \in W^+$ such that $uv \in E(G)$. Then $C^* = u^-\overleftarrow{C}vu\overrightarrow{C}v^-xu^-$ is a $D_{\lambda+1}$-cycle of G with $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$. This contradicts the choice of C.

Now let $S = V(G) \setminus (W^+ \cup V(H) \cup V(H^*))$. Then $c(G - S) \geqslant \omega + 1$. However

$$\frac{|S|}{c(G-S)} \leqslant \frac{n - \omega - 2}{\omega + 1} = \frac{\frac{tn}{t+1} - 1}{\frac{n}{t+1}} < t,$$

a contradiction.

Therefore, H is the only component of $G - V(C)$. □

Since H is the only component of $G - V(C)$, every vertex $v \in V(C) \setminus W$ is only adjacent in G to vertices on C. As vertices from $V(C) \setminus W$ are nonadjacent in G with vertices from H, we have

$$\deg_G(v, C) \geqslant \sigma_2(G) - (\omega + \lambda - 1) \quad \text{for any } v \in V(C) \setminus W. \tag{1}$$

We construct the vertex sets L_u^+ for each $u \in W$ as follows:

$$L_u^+ = \begin{cases} \{v \in V(C) : \operatorname{dist}_{\overrightarrow{C}}(u, v) < \dfrac{n}{t+1} - \omega + 1\} & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2; \\ \{v \in V(C) : \operatorname{dist}_{\overrightarrow{C}}(u, v) \leqslant \dfrac{n}{t+1} - \omega + 1\} & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2. \end{cases}$$

Claim 4. *The following statements hold.*

(a) If $\sigma_2(G) = \frac{2n}{t+1} - 2$, then for any two distinct vertices $u, v \in W$, we have $\operatorname{dist}_{\overrightarrow{C}}(u, v) \geqslant \frac{n}{t+1} - \omega + 1$ and $E_G(L_u^+, L_v^+) = \emptyset$.

(b) If $\sigma_2(G) > \frac{2n}{t+1} - 2$, then for any two distinct vertices $u, v \in W$, we have $\operatorname{dist}_{\overrightarrow{C}}(u, v) > \frac{n}{t+1} - \omega + 1$ and $E_G(L_u^+, L_v^+) = \emptyset$.

Proof. We only show Claim 4(a), as the proof for Claim 4(b) follows the same argument by just using the strict inequality. Let $u^* \in N_H(u), v^* \in N_H(v)$ and P be a (u^*, v^*)-path of H. For the first part of the statement, it suffices to show that when we arrange the vertices of W along $\overrightarrow{C}$, for any two consecutive vertices u and v from the arrangement, we have $\mathrm{dist}_{\overrightarrow{C}}(u, v) \geqslant \frac{n}{t+1} - \omega + 1$. Note that $V(u^+\overrightarrow{C}v^-) \cap W = \emptyset$ for such pairs of u and v. Assume to the contrary that there are distinct $u, v \in W$ with $V(u^+\overrightarrow{C}v^-) \cap W = \emptyset$ and $\mathrm{dist}_{\overrightarrow{C}}(u, v) < \frac{n}{t+1} - \omega + 1$. Let $C^* = u\overleftarrow{C}vv^*Pu^*u$. Since H has order λ and $V(u^+\overrightarrow{C}v^-) \cap W = \emptyset$, $H - V(P)$ is a union of components of $G - V(C^*)$ that each is of order at most $\lambda - 1$ and $u^+\overrightarrow{C}v^-$ is a component of $G - V(C^*)$ of order less than $\frac{n}{t+1} - \omega$ but at least λ (G has no D_λ-cycle). By (**1**), for each vertex $x \in V(u^+\overrightarrow{C}v^-)$, $\deg_G(x, C^*) > \sigma_2(G) - (\omega + \lambda - 1) - (\frac{n}{t+1} - \omega - 1) = \frac{n}{t+1} - \lambda$. This shows a contradiction to Lemma 2.4.

For the second part of the statement, we assume to the contrary that $E_G(L_u^+, L_v^+) \neq \emptyset$. Applying the first part, we know that $\mathrm{dist}_{\overrightarrow{C}}(u, v) \geqslant \frac{n}{t+1} - \omega + 1$ and $\mathrm{dist}_{\overrightarrow{C}}(v, u) \geqslant \frac{n}{t+1} - \omega + 1$ (exchanging the role of u and v). Thus $L_u^+ \cap L_v^+ = \emptyset$. We choose $x \in L_u^+$ with $\mathrm{dist}_{\overrightarrow{C}}(u, x)$ minimum and $y \in L_v^+$ with $\mathrm{dist}_{\overrightarrow{C}}(v, y)$ minimum such that $xy \in E(G)$. By this choice of x and y, it follows that $E_G(V(u^+\overrightarrow{C}x^-), V(v^+\overrightarrow{C}y^-)) = \emptyset$. Let $C^* = u\overleftarrow{C}y\overrightarrow{x}\overrightarrow{C}vv^*Pu^*u$. Since H is of order λ and no vertex of H is adjacent in G to any vertex of $u^+\overrightarrow{C}x^-$ or $v^+\overrightarrow{C}y^-$ by the first part of the statement, $H - V(P)$ is a union of components of $G - V(C^*)$ that each have order at most $\lambda - 1$. Also $u^+\overrightarrow{C}x^-$ and $v^+\overrightarrow{C}y^-$ are components of $G - V(C^*)$ that each have order less than $\frac{n}{t+1} - \omega$ but at least one of them has order at least λ.

Since $E_G(V(u^+\overrightarrow{C}x^-), V(v^+\overrightarrow{C}y^-)) = \emptyset$, by (**1**), for each vertex $w \in V(u^+\overrightarrow{C}x^-) \cup V(v^+\overrightarrow{C}y^-)$, $\deg_G(w, C^*) > \frac{n}{t+1} - \lambda$. This shows a contradiction to Lemma 2.4. □

By Claim 4, $G[L_u^+]$ and $G[L_v^+]$ are remote for any two distinct $u, v \in W$. Furthermore, H is remote with $G[L_u^+]$ for any $u \in W$. In addition, we have $|L_u^+| \geqslant \frac{n}{t+1} - \omega$ if $\sigma_2(G) = \frac{2n}{t+1} - 2$, and $|L_u^+| > \frac{n}{t+1} - \omega$ if $\sigma_2(G) > \frac{2n}{t+1} - 2$. Let $S = V(G) \setminus \big((\bigcup_{u \in W} L_u^+) \cup V(H)\big)$. Then $c(G - S) = \omega + 1$ and

$$\begin{cases} |S| < n - \omega\left(\dfrac{n}{t+1} - \omega\right) - \lambda & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\ |S| \leqslant n - \omega\left(\dfrac{n}{t+1} - \omega\right) - \lambda & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2. \end{cases}$$

As G is t-tough and so $|S| \geqslant tc(G-S) = t(\omega+1)$, we get

$$\begin{cases} n > \omega\left(\dfrac{n}{t+1} - \omega + t\right) + \lambda + t & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\ n \geqslant \omega\left(\dfrac{n}{t+1} - \omega + t\right) + \lambda + t & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2. \end{cases}$$

Claim 5. *It holds that* $\sigma_2(G) = \frac{2n}{t+1} - 2$, $\lambda = 1$, *and* $\omega = \frac{n}{t+1} - 1$.

Proof. Note that we have $\omega \leqslant \frac{n}{t+1} - \lambda \leqslant \frac{n}{t+1} - 1$ by Claim 2. Suppose to the contrary that $\sigma_2(G) > \frac{2n}{t+1} - 2$, $\lambda \geqslant 2$, or $\omega < \frac{n}{t+1} - 1$. Now we have

$$n \geqslant \omega\left(\frac{n}{t+1} - \omega + t\right) + \lambda + t,$$

implying

$$\left(\frac{\omega}{t+1} - 1\right) n \leqslant \omega(\omega - t) - \lambda - t. \tag{2}$$

The inequality (**2**) cannot achieve equality when $\sigma_2(G) > \frac{2n}{t+1} - 2$, since we have $n > \omega\left(\frac{n}{t+1} - \omega + t\right) + \lambda + t$ under the current assumptions. If $\omega < t+1$, then we have $\omega < 2$ because $2t \leqslant \omega < t+1$ implies $t < 1$, a contradiction to Claim 1. Thus we have $\omega \geqslant t+1$, implying $\frac{\omega}{t+1} - 1 \geqslant 0$. Then by Claim 2, we have

$$\left(\frac{\omega}{t+1} - 1\right) n \geqslant \left(\frac{\omega}{t+1} - 1\right)(\omega + \lambda)(t+1). \tag{3}$$

Note that if $\lambda \geqslant 2$ or $\omega < \frac{n}{t+1} - 1$, then the inequality (**3**) cannot achieve the equality. By the assumption for the contrary, at least one of the inequalities (**2**) or (**3**) cannot achieve the equality. Therefore, combining (**2**) and (**3**), we get

$$\omega(\omega - t) - \lambda - t > \left(\frac{\omega}{t+1} - 1\right)(\omega + \lambda)(t+1),$$

which implies

$$\begin{aligned} \omega^2 - \omega t - \lambda - t &> \omega(\omega + \lambda) - (\omega + \lambda)(t+1) \\ &= \omega^2 + \omega\lambda - \omega t - \omega - \lambda t - \lambda. \end{aligned}$$

This gives $(\lambda - 1)t > (\lambda - 1)\omega$, leading to $0 < 0$ or $\omega < t$, a contradiction. □

By Claim 5, Theorem 1(a) holds. In the rest of the proof, we show Theorem 1(b). Recall that H consists of a single vertex, and this assumption will be used in the remainder of the proof. Let

$$W^* = W^+ \cup V(H).$$

Since $u^+ \in L_u^+$ for each $u \in W$, Claim 4 implies that W^* is an independent set in G.

Claim 6. *Every vertex in $V(G) \setminus W^*$ is adjacent in G to at least two vertices from W^*.*

Proof. Suppose to the contrary that there exists $x \in V(G) \setminus W^*$ such that x is adjacent in G to at most one vertex from W^*. Let $S = V(G) \setminus (W^* \cup \{x\})$. Then $c(G-S) \geqslant \omega+1$. However

$$\frac{|S|}{c(G-S)} \leqslant \frac{n-\omega-2}{\omega+1} = \frac{\frac{tn}{t+1}-1}{\frac{n}{t+1}} < t,$$

a contradiction. □

Claim 7. *For every $v \in W^+$, we have $\deg_G(v, C) = \frac{n}{t+1} - 1$ and v is not adjacent in G to any two consecutive vertices on C.*

Proof. Since $\sigma_2(G) = \frac{2n}{t+1} - 2$, we have $\deg_G(v, C) \geqslant \frac{n}{t+1} - 1$ for every $v \in W^+$. As W^* is an independent set in G, $v^+ \notin W^*$. By Claim 6, v^+ is adjacent in G to another vertex u from W^*. If $\{u\} = V(H)$, then $C^* = v^-\overleftarrow{C}v^+uv^-$ is a $D_{\lambda+1}$-cycle of G with v being the only component of $G - V(C^*)$. Assume instead that $u \in W^+$. Let $V(H) = \{x\}$. Then $C^* = v^+u\overrightarrow{C}v^-xu^-\overleftarrow{C}v^+$ is a $D_{\lambda+1}$-cycle of G with v being the only component of $G - V(C^*)$.

Again, since G has no D_λ-cycle and v is the only vertex of G outside $V(C)$, it follows that v is not adjacent in G to any two consecutive vertices on C^*. Furthermore, we must have $\deg_G(v, C^*) = \frac{n}{t+1} - 1$. As if $\deg_G(v, C^*) > \frac{n}{t+1} - 1$, then $\{v\} \cup N_C(v)^+$ is an independent set of G with more than $\frac{n}{t+1}$ vertices, a contradiction to Lemma 2.1. The claim follows as $\deg_G(v, C) = \deg_G(v, C^*)$ and two neighbors of v that are consecutive on C will also be consecutive on C^* as v is not adjacent in G to any vertex from W^+. □

Our goal is to show that $N_C(W^+) = N_C(H)$. To do so, we investigate how vertices in $N_C(W^+)$ are located along $\overrightarrow{C}$. We start with some definitions. A *chord* of C is an edge uv with $u, v \in V(C)$ and $uv \notin E(C)$. Two chords ux and vy of C that do not share any endvertices are *crossing* if the four vertices u, x, v, y appear along $\overrightarrow{C}$ in the order u, v, x, y or u, y, x, v. For two distinct vertices $x, y \in N_C(W^+)$, we say x and y form a *crossing* if there exist distinct vertices $u, v \in W^+$ such that ux and vy are crossing chords of C.

Claim 8. *For any two distinct $x, y \in N_C(W^+)$ with $xy \in E(C)$, it follows that x and y do not form any crossing.*

Proof. Suppose to the contrary that for some distinct $x, y \in N_C(W^+)$ with $xy \in E(C)$, the two vertices x and y form a crossing. Let $u, v \in W^+$ such that $xu, yv \in E(G)$. Assume, without loss of generality, that the four vertices u, v, x, y appear in the order u, v, x, y along $\overrightarrow{C}$. Let $V(H) = \{w\}$. Then $ux\overleftarrow{C}vy\overrightarrow{C}u^-wv^-\overleftarrow{C}u$ is a hamiltonian cycle of G, a contradiction to our assumption that G is not hamiltonian. □

Claim 9. *For any vertex $v \in W^+$ and any two distinct $x, y \in N_C(v)$, $x\overrightarrow{C}y$ contains a vertex from W^+.*

Proof. By Claim 7, $x\overrightarrow{C}y$ has at least three vertices. Suppose to the contrary that $x\overrightarrow{C}y$ contains no vertex from W^+. We furthermore choose x and y so that $x\overrightarrow{C}y$ contains no other vertex from $N_C(v) \setminus \{x, y\}$. Assume that the three vertices v, x, y appear in the order v, x, y along $\overrightarrow{C}$. By Claim 6, each internal vertex of $x\overrightarrow{C}y$ is adjacent in G to a vertex from W^+. Then by our selection of x and y, we know that each internal vertex of $x\overrightarrow{C}y$ is adjacent in G to a vertex from $W^+ \setminus \{v\}$. Applying Claim 8, x^+ does not form a crossing with x, and so x^+ forms a crossing with y. Similarly, x^{++} does not form a crossing with x^+, and so forms a crossing with y. Continuing this argument for all the internal vertices of $x^{++}\overrightarrow{C}y$, we know that y^- forms a crossing with y, a contradiction to Claim 8. □

We assume that the ω neighbors of the vertex from $V(H)$ on C are $v_1, \ldots, v_\omega$ and they appear in the same order along $\overrightarrow{C}$. For each $i \in [1, \omega]$, let $I_i = V(v_i\overrightarrow{C}v_{i+1}) \setminus \{v_i\}$, where $v_{\omega+1} := v_1$.

Claim 10. *For every $v \in W^+$, it holds that $N_C(v) = W$.*

Proof. Since $x\overrightarrow{C}y$ contains a vertex from W^+ for any two distinct $x, y \in N_C(v)$ by Claim 9, it follows that no I_i can contain more than one vertex from $N_C(v)$. Since $\deg_G(v, C) = \omega = |W^+|$ by Claim 7 and $\{I_1, \ldots, I_\omega\}$ is a partition of $V(C)$, the Pigeonhole Principle implies that each I_i contains exactly one vertex from $N_C(v)$.

Assume to the contrary that $N_C(v) \neq W$. Let $i \in [1, \omega]$ be the index such that $\mathrm{dist}_{\overrightarrow{C}}(v, v_i)$ is largest and $vv_i \notin E(G)$. Note that the index i exists since $v^- \in W$ and $vv^- \in E(G)$. In particular, every vertex $u \in W \cap V(v_i^+\overrightarrow{C}v)$ is adjacent to v by the choice of i. Let z be the vertex in $N_C(v) \cap I_{i-1}$. We prove the four subclaims below. Let $V(H) = \{x\}$ in the rest arguments.

Claim A: $z = v_i^-$.

Proof of Claim A. Suppose otherwise that $z \neq v_i^-$. Then by Claim 6, z^+ is adjacent in G to at least two vertices from W^+. By Claim 8, $N_C(z^+) \cap W^+ \subseteq V(v_i^+\overrightarrow{C}v)$. Thus z^+ is adjacent in G to a vertex from $W^+ \cap V(v_i^+\overrightarrow{C}v^-)$ as z is the only neighbor of v from I_{i-1} in G. By repeating this procedure for all the vertices from $V(z^{++}\overrightarrow{C}v_i^-)$ iteratively, we conclude that v_i^- is adjacent in G to a vertex $u \in W^+ \cap V(v_i^+\overrightarrow{C}v^-)$. As $v_i^+v_i \in E(G)$ and $v_iv_i^- \in E(C)$, Claim 7 implies that v_i^+ is not adjacent in G to v_i^-. Thus we have $u \notin \{v_i^+, v\}$. However, since $u^-v \in E(G)$ by our choice of the index i, the cycle $xv^-\overleftarrow{C}uv_i^-\overleftarrow{C}vu^-\overleftarrow{C}v_ix$ is longer than C, a contradiction. Thus z must be v_i^-. □

Claim B: $v_{i+1} = v^-$.

Proof of Claim B. Suppose that $v_{i+1} \neq v^-$. Considering v^+_{i+1} in the place of v and applying Claim A to it, v^+_{i+1} must be adjacent to v_i or v_i^- (if $v^+_{i+1}v_i \notin E(G)$, then i is the index such that $\mathrm{dist}_{\overrightarrow{C}}(v^+_{i+1}, v_i)$ is largest and $v^+_{i+1}v_i \notin E(G)$). If $v^+_{i+1}v_i^- \in E(G)$, then the cycle $xv^-\overleftarrow{C}v^+_{i+1}v_i^-\overleftarrow{C}vv_{i+1}\overleftarrow{C}v_ix$ is longer than C, a contradiction. Thus we have $v^+_{i+1}v_i \in E(G)$. We consider the vertex v^+. Since W^- is independent in G and v is adjacent to $v_i^- \in W^-$, we have $v \notin W^-$. Thus $v^+ \notin W$. Then by Claim 6, v^+ is adjacent in G to a vertex $u \in W^+ \setminus \{v\}$. However, the cycle

$$\begin{cases} xv_i\overrightarrow{C}vv_i^-\overleftarrow{C}uv^+\overrightarrow{C}u^-x & \text{if } u \in V(v^+\overrightarrow{C}v^+_{i-1}),\\ xv_iv^+_{i+1}\overrightarrow{C}vv_i^-\overleftarrow{C}v^+v_i^+\overrightarrow{C}v_{i+1}x & \text{if } u = v_i^+,\\ xv^-\overleftarrow{C}uv^+\overrightarrow{C}v_i^-vu^-\overleftarrow{C}v_ix & \text{if } u \in V(v^+_{i+1}\overrightarrow{C}v^-), \end{cases}$$

is longer than C, a contradiction. □

Claim C: $\omega \geqslant 4$.

Proof of Claim C. By Claim 1, we have $\omega \geqslant 2$. Suppose that $\omega \in [2,3]$. First, suppose $\omega = 2$. Since $v \in W^+$ is adjacent to a vertex in W^- and W^+ is independent in G, we have $W^- \setminus W^+ \neq \emptyset$. Also a vertex $u \in W^- \setminus W^+$ is adjacent to all vertices in W^+ by Claim 6. Then $u^+ \in W$ and so $u^{++} \in W^+$ is adjacent to u^+ and u, contrary to Claim 7. Next, suppose $\omega = 3$. We let, without loss of generality, $v = v_1^+$. Then Claim B implies $v_1^+v_3^- \in E(G)$. Note that since W^+ is independent in G, v_3^- must not be v_2^+. We also have $v_2^+v_1 \notin E(G)$, as otherwise $v_1v_2^+\overrightarrow{C}v_3^-v_1^+\overrightarrow{C}v_2xv_3\overrightarrow{C}v_1$ is a cycle longer than C. Applying Claim A to v_2^+, we get $v_2^+v_1^- \in E(G)$. Similarly, $v_3^+v_2 \notin E(G)$, as otherwise $v_3^+v_2\overleftarrow{C}v_1xv_3\overleftarrow{C}v_2^+v_1^-\overleftarrow{C}v_3^+$ is a cycle longer than C. Applying Claim A to v_3^+, we get $v_3^+v_2^- \in E(G)$. Then as the degrees of all vertices from W^+ are of degree 3 in G, Claims 3, 6, and 7 imply that the graph G is isomorphic to the Petersen graph. However, $3 = \omega = \frac{10}{t+1} - 1$ implies that G is $\frac{3}{2}$-tough, contradicting that the toughness of the Petersen graph is at most $\frac{4}{3}$ (in the Petersen graph, deleting two independent vertices from one 5-cycle and another two independent vertices that are non-neighbors of the first two deleted vertices from the second disjoint 5-cycle gives three components). Thus we have $\omega \geqslant 4$. □

Claim D: For every $j \in [1,\omega]$, $|I_j| \neq 3$.

Proof of Claim D. Suppose that $|I_j| = 3$ for some $j \in [1,\omega]$. Then we have $v_j^+v_{j+1}^- \in E(C)$, which implies $N_C(v_j^+) \neq W$. Applying Claim A to v_j^+, we get $v_j^+v_{j-1}^- \in E(G)$. By symmetry of the orientation of C, we have $v_{j+1}^-v_{j+2}^+ \in E(G)$. Also we have $\omega \geqslant 4$ by Claim C, which implies $v_{j-1} \in V(v_{j+2}^+\overrightarrow{C}v_j^-)$. Then the cycle $xv_{j-1}\overrightarrow{C}v_j^+v_{j-1}^-\overleftarrow{C}v_{j+2}^+v_{j+1}^-\overrightarrow{C}v_{j+2}x$ is longer than C, a contradiction. □

We now show a contradiction. The vertex v^+ must not be in W since $vv_i^- \in E(G)$ by Claim A and W^- is independent in G. Thus v^+ is adjacent in G to a vertex $u \in W^+ \setminus \{v\}$ by Claim 6. If $u \neq v_i^+$, then the cycle $xv_i\overrightarrow{C}vv_i^-\overleftarrow{C}uv^+\overrightarrow{C}u^-x$ is longer than C, a contradiction. Thus we have $u = v_i^+$. We consider the cycle $C^* = v^+\overrightarrow{C}v_ixv^-\overleftarrow{C}v_i^+v^+$ in G. Note that we have $V(C^*) = V(G) \setminus \{v\}$. Then since the length of C^* is equal to the length of C, we can apply Claim D to C^*. However, v^-, x, v_i, v_i^- are four consecutive vertices on C^* appearing in the order v^-, x, v_i, v_i^- and $v^-, v_i^- \in N_{C^*}(v)$, showing that C^* does not satisfy Claim D, a contradiction. This completes the proof of Claim 10. □

Claim 10 implies that $N_C(W^*) = W$. Thus every vertex from W^* is adjacent in G to every vertex from W. Therefore $t \leqslant \tau(G) \leqslant \frac{|W|}{|W^*|}$ as W^* is an independent set in G. Consequently, $|W| \geqslant t|W^*| = \frac{tn}{t+1}$ and so $W = V(G) \setminus W^*$ by noticing $|W^*| = \frac{n}{t+1}$. Thus G contains a spanning complete bipartite graph between W^* and W. On the other hand, since $|W^+| = |W| = \frac{n}{t+1} - 1$ and $V(G) = W^* \cup W = (W^+ \cup V(H)) \cup W$, we know that $2(\frac{n}{t+1} - 1) + 1 = n$ and so $t = \frac{n-1}{n+1}$. Thus $|W| = \frac{n-1}{2}$ and $|W^*| = \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Therefore, $G \in \mathcal{H}$. The proof of Theorem 1 is now complete. □

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