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Integrating Products of Quadratic Forms

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Abstract

We prove that if $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ are quadratic forms in variables $x_1, \ldots, x_n$ such that each q_k depends on at most r variables and each q_k has common variables with at most r other forms, then the average value of the product $(1 + q_1) \cdots (1 + q_m)$ with respect to the standard Gaussian measure in $\mathbb{R}^n$ can be approximated within relative error $\epsilon > 0$ in quasi-polynomial $n^{O(1)} m^{O(\ln m - \ln \epsilon)}$ time, provided $|q_k(x)| \leq \gamma \|x\|^2/r$ for some absolute constant $\gamma > 0$ and $k = 1, \ldots, m$. The integral in question is viewed as the independence polynomial of an auxiliary weighted graph and then the method of polynomial interpolation is applied. When q_k are interpreted as pairwise squared distances for configurations of points in Euclidean space, the average can be interpreted as the partition function of systems of particles with mollified logarithmic potentials. We sketch possible applications to testing the feasibility of systems of real quadratic equations and to computing permanents of positive definite Hermitian matrices.

Keywords Quadratic equations · Algorithm · Interpolation method · Integration · Permanent

Mathematics Subject Classification 14Q30 · 65H14 · 68Q25 · 68W25 · 90C23 · 15A15

Dedicated to the memory of Eli Goodman.

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1 Introduction and Main Results

Integration of high degree multivariate polynomials is computationally difficult and no efficient algorithms are known except in few special cases, when the polynomials have a rather simple algebraic structure (for example, close to a power of a linear form on a simplex, cf. [3]), or have some very nice analytic properties (slowly varying or, most notably, log-concave, cf. [14]). Since a general n-variate polynomial p of degree d is defined by $\binom{n+d}{d}$ parameters (for example, coefficients), the problem becomes interesting for large n and d only if p has some special structure (such as the product of low-degree polynomials), which allows us to define p using much fewer parameters.

In this paper, we integrate products of quadratic forms with respect to the Gaussian measure in $\mathbb{R}^n$. We relate the problem to partition functions of mollified logarithmic potentials, to testing the feasibility of systems of real quadratic equations and to computing permanents of positive definite Hermitian matrices.

Our algorithms are deterministic and based on the method of polynomial interpolation, which has been recently applied to a variety of partition functions in combinatorial (discrete) problems, cf. [6]. In a continuous setting, the method was applied to computing partition functions arising in quantum models [7, 12].

1.1 Quadratic Forms on $\mathbb{R}^n$

We consider Euclidean space $\mathbb{R}^n$ endowed with the standard inner product

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n,$$

for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, and corresponding Euclidean norm

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + \cdots + x_n^2}$$

for $x = (x_1, \ldots, x_n)$. Let $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ be quadratic forms defined by

$$q_k(x) = \frac{\langle Q_k x, x\rangle}{2} \qquad \text{for } k = 1, \ldots, m, \tag{1.1.1}$$

where $Q_1, \ldots, Q_m$ are $n \times n$ real symmetric matrices. Our first result concerns computing the integral

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + q_1(x)) \cdots (1 + q_m(x)) e^{-\|x\|^2/2}\, dx. \tag{1.1.2}$$

The idea of the interpolation method is to consider (1.1.2) as a one-parameter perturbation a much simpler integral, in our case, of

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\|x\|^2/2}\, dx = 1. \tag{1.1.3}$$

For the method to work, one should show that there are no zeros in the vicinity of a path in the complex plane which connects (1.1.2) and (1.1.3). We prove the following result.

Theorem 1.1 *There is an absolute constant* $\gamma > 0$ (*one can choose* $\gamma = 0.151$) *such that the following holds. Let* $q_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \ldots, m$, *be quadratic forms. Then*

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + \omega q_1(x)) \cdots (1 + \omega q_m(x)) e^{-\|x\|^2/2} \, dx \neq 0$$

for all $\omega \in \mathbb{C}$ *such that* $|\omega| \leq \gamma$, *provided*

$$|q_k(x)| \leq \frac{\|x\|^2}{\max\{m, n\}} \quad \text{for } k = 1, \ldots, m.$$

By interpolation, for any constant $0 < \gamma' < \gamma$, fixed in advance, we obtain an algorithm which, given quadratic forms $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$, computes (1.1.2) within relative error $0 < \epsilon < 1$ in quasi-polynomial $n^{O(1)} m^{O(\ln m - \ln \epsilon)}$ time provided

$$|q_k(x)| \leq \frac{\gamma' \|x\|^2}{\max\{m, n\}} \quad \text{for } k = 1, \ldots, m. \tag{1.2.1}$$

Note that by Theorem 1.1 and (1.1.3), the value of (1.1.2) is positive, as long as (1.2.1) holds. Some remarks are in order.

First, we note that the integrand in (1.1.2) can vary wildly. Indeed, for large n the bulk of the standard Gaussian measure in $\mathbb{R}^n$ is concentrated in the vicinity of the sphere $\|x\| = \sqrt{n}$. More precisely, if γ_n is the standard Gaussian measure in $\mathbb{R}^n$ with density $(2\pi)^{-n/2} e^{-\|x\|^2/2}$ then

$$\begin{aligned} \gamma_n \left\{ x \in \mathbb{R}^n : \|x\|^2 \geq \frac{n}{1 - \epsilon} \right\} &\leq e^{-\epsilon^2 n/4} \quad \text{and} \\ \gamma_n \left\{ x \in \mathbb{R}^n : \|x\|^2 \leq (1 - \epsilon) n \right\} &\leq e^{-\epsilon^2 n/4} \quad \text{for all } \ 0 < \epsilon < 1, \end{aligned} \tag{1.2.2}$$

see for example, [5, Sect. V.5]. Assuming that $m = n$, we can choose $q_k(x) \sim \|x\|^2/n$ so that (1.2.1) is satisfied. Then, in the vicinity of the sphere $\|x\| = \sqrt{n}$, the product $(1 + q_1(x)) \cdots (1 + q_n(x))$ in (1.1.2) varies within an exponential in n factor, and is not at all well-concentrated.

Second, if the quadratic forms $q_1, \ldots, q_m$ exhibit a simpler combinatorics, we can improve the bounds accordingly. We prove the following result.

Theorem 1.2 *There is an absolute constant* $\gamma > 0$ (*one can choose* $\gamma = 0.151$) *such that the following holds. Let* $q_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \ldots, m$, *be quadratic forms. Suppose further that each form depends on not more than* r *variables among* $x_1, \ldots, x_n$ *and that each form has common variables with not more than* r *other forms. Then*

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + \omega q_1(x)) \cdots (1 + \omega q_m(x)) e^{-\|x\|^2/2} \, dx \neq 0$$

for all $w \in \mathbb{C}$ *such that* $|\omega| \leq \gamma$, *provided*

$$|q_k(x)| \leq \frac{\|x\|^2}{r} \quad \textit{for} \ \ k = 1, \ldots, m.$$

By interpolation, for any constant $0 < \gamma' < \gamma$, fixed in advance, we obtain an algorithm which, given quadratic forms $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ as in Theorem 1.2, computes (1.1.2) within relative error $0 < \epsilon < 1$ in quasi-polynomial $n^{O(1)} m^{\ln m - \ln \epsilon}$ time provided

$$|q_k(x)| \leq \frac{\gamma' \|x\|^2}{r} \quad \text{for} \quad k = 1, \ldots, m. \tag{1.3.1}$$

We prove Theorems 1.1 and 1.2 in Sect. 3 and describe the algorithm for computing (1.1.2) in Sect. 4. In Sect. 2, we discuss connections with systems of particles with mollified logarithmic potentials, possible applications to testing the feasibility of systems of multivariate real quadratic equations and to computing permanents of positive definite Hermitian matrices.

2 Connections and Possible Applications

2.1 Partition Functions of Mollified Logarithmic Potentials

Let $n = ds$ and let us interpret $\mathbb{R}^n = \mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d$ as the space of all ordered s-tuples $(v_1, \ldots, v_s)$ of points $v_i \in \mathbb{R}^d$. Hence the distance between v_i and v_j is $\|v_i - v_j\|$.

Let us fix some set E of m pairs $\{i, j\}$ of indices $1 \leq i < j \leq s$ and suppose that the energy of a set of points $(v_1, \ldots, v_s)$ is defined by

$$-\sum_{\{i,j\} \in E} \ln\left(1 + \alpha \|v_i - v_j\|^2\right) + \frac{1}{2} \sum_{i=1}^{n} \|v_i\|^2, \tag{2.1.1}$$

where $\alpha > 0$ is a parameter. The first sum in (2.1.1) indicates that there a repulsive force between any pair $\{v_i, v_j\}$ with $\{i, j\} \in E$ (so that the energy decreases if the distance between v_i and v_j increases), while the second sum indicates that there is a force pushing the points towards 0 (so that the energy decreases when each v_i approaches 0). When $\alpha = 0$, the repulsive force disappears altogether, and when $\alpha \to +\infty$, the repulsive force behaves as a Coulomb's force with logarithmic potential, since

$$\lim_{\alpha \to +\infty} \ln\left(1 + \alpha \|v_i - v_j\|^2\right) - \ln \alpha = 2 \ln \|v_i - v_j\|.$$

Thus the integral

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \prod_{\{i,j\} \in E} \left(1 + \alpha \|v_i - v_j\|^2\right) e^{-(\|v_1\|^2 + \cdots + \|v_s\|^2)/2} \, dx, \tag{2.1.2}$$

which is a particular case of (1.1.2), can be interpreted as the partition function of points with “mollified” or “damped” logarithmic potentials. One can think of (2.1.2) as the partition function for particles with genuine logarithmic potentials, provided each particle is confined to its own copy of $\mathbb{R}^d$ among a family of parallel d-dimensional affine subspaces in some higher-dimensional Euclidean space.

The integral (2.1.2) can be considered as a ramification of classical Selberg-type integrals for logarithmic potentials:

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \prod_{1 \le i < j \le n} |x_i - x_j|^{2\gamma} e^{-(x_1^2+\ldots+x_n^2)/2}\, dx_1 \ldots dx_n \prod_{j=1}^{n} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}, \tag{2.1.3}$$

see for example, [16, Chap. 17]. The integral (2.1.3) corresponds to points in $\mathbb{R}^1$ and a similar integral is computed explicitly for points in $\mathbb{R}^2$ (and $\gamma = 1$), see [16, Sect. 17.11]. For higher dimensions d no explicit formulas appear to be known.

In contrast, we compute integrals (2.1.2) approximately for certain values of α, but we allow arbitrary dimensions and can choose an arbitrary set of pairs of interacting points (and we can even choose different α's for different pairs of points). Theorem 1.2 can be interpreted as the absence of phase transition in the Lee–Yang sense [17], if α is sufficiently small. For example, if the set E consists of all $\binom{s}{2}$ pairs $\{i, j\}$, Theorem 1.2 implies that there is no phase transition and the integral (2.1.2) can be efficiently approximated if

$$\alpha < \frac{\beta}{\max\{d, s\}}$$

for some absolute constant $\beta > 0$.

2.2 Applications to Systems of Quadratic Equations

Every system of real polynomial equations can be reduced to a system of quadratic equations, as one can successively reduce the degree by introducing new variables via substitutions of the type $z := xy$. A system of quadratic equations can be solved in polynomial time when the number of equations is fixed in advance, [4, 11], but as the number of equations grows, the problem becomes computationally hard. Here we are interested in the systems of equations of the type

$$q_k(x) = 1 \quad \text{for} \quad k = 1, \ldots, m, \tag{2.2.1}$$

where $q_k : \mathbb{R}^n \to \mathbb{R}$ are positive semidefinite quadratic forms. Such systems naturally arise in problems of distance geometry, where we are interested to find out if there are configurations of points in $\mathbb{R}^d$ with prescribed distances between some pairs of points and in which case q_k are scaled squared distances between points, see [8, 13] and Sect. 2.1. Besides, finding if a system of homogeneous quadratic equations has a non-trivial solution

$$q_k(x) = 0 \qquad \text{for} \quad k = 1, \ldots, m \ \text{ and } \ \|x\| = 1, \tag{2.2.2}$$

can be reduced to (2.2.1) with positive definite forms q_k by adding $\|x\|^2$ to the appropriately scaled equations in (2.2.2). Suppose that

$$\sum_{k=1}^{m} q_k(x) = \frac{\|x\|^2}{2} \tag{2.2.3}$$

in (2.2.1). By itself, the condition (2.2.3) is not particularly restrictive: if the sum in the left-hand side of (2.2.3) is positive definite, it can be brought to the right-hand side by an invertible linear transformation of x.

Let us choose an $\alpha > 0$ such that the scaled forms αq_k satisfy (1.3.1), so that the integral

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + \alpha q_1(x)) \cdots (1 + \alpha q_m(x)) e^{-\|x\|^2/2} \, dx \tag{2.2.4}$$

can be efficiently approximated. We would like to argue that the value of the integral (2.2.4) can provide a reasonable certificate which allows one to distinguish systems (2.2.1) with many "near solutions" x from the systems that are far from having a solution.

We observe that the system (2.2.1) has a solution if and only if the system

$$q_k(x) = t \qquad \text{for} \quad k = 1, \ldots, m, \tag{2.2.5}$$

has a solution $x \in \mathbb{R}^n$ for any $t > 0$. Let us find $0 < \beta < 1$ such that

$$2m \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) = \frac{n}{1 - \beta}. \tag{2.2.6}$$

Indeed, the equation (2.2.6) always has a (necessarily unique) solution $0 < \beta < 1$, since for $\beta \approx 0$ the left-hand side is bigger than the right-hand side, while for $\beta \approx 1$ the right-hand side is bigger than the left-hand side. Because of (2.2.3), we can rewrite (2.2.4) as

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-(1-\beta)\|x\|^2/2} \prod_{k=1}^{m} (1 + \alpha q_k(x)) e^{-\beta q_k(x)} \, dx. \tag{2.2.7}$$

We observe that if $\alpha > \beta$ then the maximum value of $(1 + \alpha t)e^{-\beta t}$ for $t \geq 0$ is attained at

$$t = \frac{1}{\beta} - \frac{1}{\alpha} > 0 \tag{2.2.8}$$

and is equal to

$$\frac{\alpha}{\beta} \exp\left\{\frac{\beta}{\alpha} - 1\right\} > 1,$$

and hence the maximum value of the product of the m factors in (2.2.7) is

$$\left(\frac{\alpha}{\beta} \exp\left\{\frac{\beta}{\alpha} - 1\right\}\right)^m$$

and attained if and only if the system (2.2.1) and hence (2.2.5) has a solution x. Also, if x is a solution to (2.2.5), by (2.2.3), (2.2.6), and (2.2.8), we have

$$\|x\|^2 = 2tm = \frac{n}{1-\beta}.$$

The Gaussian probability measure in $\mathbb{R}^n$ with density

$$\frac{(1-\beta)^{n/2}}{(2\pi)^{n/2}} e^{-(1-\beta)\|x\|^2/2},$$

is concentrated in the vicinity of the sphere $\|x\|^2 = n/(1-\beta)$, cf. (1.2.2). Therefore, if for the system (2.2.1) there are sufficiently many "near solutions" x, we should have the value of the integral (2.2.4) sufficiently close to

$$\left(\frac{\alpha}{\beta} \exp\left\{\frac{\beta}{\alpha} - 1\right\}\right)^m (1-\beta)^{-n/2},$$

while if the system (2.2.1) is far from having a solution, the value of the integral will be essentially smaller.

2.3 Connection to Permanents

Let μ_n be the standard Gaussian measure in $\mathbb{C}^n$ with density

$$\frac{e^{-\|z\|^2}}{\pi^n} \quad \text{where} \quad \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 \quad \text{for} \quad z = (z_1, \ldots, z_n).$$

Let $f_1, \ldots, f_m; g_1, \ldots, g_m : \mathbb{C}^n \to \mathbb{C}$ be linear functions and let us define an $m \times m$ complex matrix $A = (a_{ij})$ by

$$\begin{gathered} a_{ij} = \int_{\mathbb{C}^n} f_i(z) \overline{g_j(z)} \, d\mu_n(z) = \sum_{k=1}^n f_{ik} \overline{g_{jk}} \quad \text{where} \\ f_i(z) = \sum_{k=1}^n f_{ik} z_k \quad \text{and} \quad g_j(z) = \sum_{k=1}^n g_{jk} z_k \quad \text{for} \quad z = (z_1, \ldots, z_n). \end{gathered} \tag{2.3.1}$$

Equivalently, $A = FG^*$ where $F = (f_{ik})$ and $G = (g_{jk})$ are $m \times n$ matrices. Wick's formula states that

$$\int_{\mathbb{C}^n} f_1(z) \cdots f_m(z) \overline{g_1(z) \cdots g_m(z)} \, d\mu_n(z) = \operatorname{per} A, \tag{2.3.2}$$

where the permanent of A is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_m} \prod_{i=1}^{m} a_{i\sigma(i)},$$

and S_m is the symmetric group of all $m!$ permutations of the set $\{1, \ldots, m\}$, see for example, [6, Sect. 3.1.4]. It follows then that

$$\int_{\mathbb{C}^n} (1 + f_1(z)\overline{g_1(z)}) \cdots (1 + f_m(z)\overline{g_m(z)}) \, d\mu_n = \operatorname{per}(I + A), \tag{2.3.3}$$

where I is the $n \times n$ identity matrix. Indeed, using (2.3.2), we can write the left-hand side of (2.3.3) as

$$\sum_{S \subset \{1,\ldots,m\}} \int_{\mathbb{C}^n} \prod_{i \in S} f_i(z)\overline{g_i(z)} \, d\mu_n = \sum_{S \subset \{1,\ldots,m\}} \operatorname{per} A_S,$$

where A_S is the $|S| \times |S|$ submatrix of A consisting of the entries a_{ij} with $i, j \in S$.

Suppose now that $f_i = g_i \neq 0$ for $i = 1, \ldots, m$. Identifying $\mathbb{C}^n = \mathbb{R}^{2n}$, we observe that $f_i\overline{g_i} = |f_i|^2$ is a quadratic form of real rank 2 on $\mathbb{R}^{2n}$ and that, up to a rescaling of the Gaussian measure, (2.3.3) is the integral of the type (1.1.2), where each q_i is a positive semidefinite form of rank 2. Hence the integral (1.1.2) for general positive semidefinite forms q_i can be viewed as the expectation of $\operatorname{per}(I + A)$ over some distribution on the set of positive semidefinite matrices A, obtained as follows: for $i = 1, \ldots, m$, we sample random vectors $f_i = (f_{i1}, \ldots, f_{in})$ independently from some distributions in $\mathbb{C}^n$ and define the $m \times m$ matrix $A = (a_{ij})$ by

$$a_{ij} = \sum_{k=1}^{n} f_{ik}\overline{f_{jk}}.$$

Equivalently, $A = FF^*$. We note that any positive definite Hermitian matrix B with the smallest eigenvalue at least 1 can be written as $B = I + A$, where A is positive semidefinite, and that computing (approximating) permanents of positive definite matrices was of some interest recently [1, 2], in particular in connection with quantum optics.

2.4 More of Related Integrals

Let $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ be quadratic forms and let $k_1, \ldots, k_m$ be positive integers. It is (implicitly) shown in [4] that the integral

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} q_1^{k_1}(x) \cdots q_m^{k_m}(x) e^{-\|x\|^2/2}\, dx \tag{2.4.1}$$

can be computed in $(n(k_1 + \cdots + k_m))^{O(m)}$ time exactly. In particular, if m is fixed in advance, the complexity of the algorithm is polynomial in n and $k_1 + \cdots + k_m$. Since it will be useful for us later, we describe the algorithm here. Let $Q_1, \ldots, Q_m$ be the matrices of $q_1, \ldots, q_m$ defined by (1.1.1) and let us consider the function

$$G(z) = \det{}^{-1/2}\left(I - \sum_{k=1}^{m} z_i Q_i\right) \quad \text{for} \quad z = (z_1, \ldots, z_m)$$

of m complex variables $z_1, \ldots, z_m$ in the vicinity of $z = 0$. We show in Lemma 3.1 below that in the vicinity of $z = 0$, we have the Taylor series expansion

$$G(z) = \sum_{k_1, \ldots, k_m \geq 0} c_{k_1, \ldots, k_m} z_1^{k_1} \cdots z_m^{k_m}, \tag{2.4.2}$$

where the coefficient $c_{k_1, \ldots, k_m}$ is the integral (2.4.1) divided by $k_1! \cdots k_m!$ To compute (2.4.1), we write

$$\det\left(I - \sum_{k=1}^{m} z_k Q_k\right) = 1 - p(z_1, \ldots, z_m),$$

where $p(0, \ldots, 0) = 0$, explicitly as a polynomial of degree n by standard multivariate interpolation in $n^{O(m)}$ time, and then use the standard expansion

$$(1 - p)^{-1/2} = 1 + \sum_{s=1}^{\infty} \frac{(2s-1)!!}{s!\, 2^s} p^s$$

to extract the coefficient $c_{k_1, \ldots, k_m}$ from (2.4.2) in $(n(k_1 + \cdots + k_m))^{O(m)}$ time. It follows that the integral (1.1.2) can be computed in polynomial time exactly, provided the number of distinct forms among $q_1, \ldots, q_m$ is fixed in advance.

Finally, we consider connections to permanents. Let $F = (f_{ik})$ be an $m \times n$ complex matrix and let us consider the Hermitian form $q : \mathbb{C}^n \to \mathbb{R}$,

$$q(z) = \sum_{i=1}^{m} \left| \sum_{k=1}^{n} f_{ik} z_k \right|^2 \quad \text{for} \quad z = (z_1, \ldots, z_n).$$

Equivalently, $q(z) = \|Fz\|^2$. For a multiset

$$I = \{i_1, \dots, i_n\}, \qquad 1 \le i_1 \le \dots \le i_n \le m,$$

of indices, let F_I be the $n \times n$ matrix consisting of the rows of F indexed by $i_1, \dots, i_n$ (some rows may be repeated and the order of rows in F_I is not important), let α_i be the number of occurrences of i in I and let

$$\alpha(I) = \prod_{i=1}^{m} \frac{1}{\alpha_i!}.$$

It follows from Wick's formula (2.3.2) that

$$\frac{1}{n!} \int_{\mathbb{C}^n} q^n(z)\, d\mu_n = \sum_I \alpha(I) \operatorname{per}(F_I F_I^*), \tag{2.4.3}$$

where μ_n is the standard Gaussian measure in $\mathbb{C}^n$ and the sum in the right hand side is taken over all multisets I of n indices. Identifying $\mathbb{C}^n = \mathbb{R}^{2n}$, we observe that the sum in the right hand side can be computed exactly in polynomial $(mn)^{O(1)}$ time. A similar looking sum

$$\sum_I \alpha(I) |\operatorname{per} F_I|^2 \tag{2.4.4}$$

is considered in [1] in connection with boson based quantum computers and, apparently, is much harder to compute by classical (as opposed to quantum) means. We note the inequality

$$|\operatorname{per} F_I|^2 \le \operatorname{per}(F_I F_I^*)$$

from [15], from which it follows that (2.4.3) is an efficiently computable upper bound for (2.4.4).

3 Proofs of Theorems 1.1 and 1.2

Choosing $r = \max\{m, n\}$, we obtain Theorem 1.1 as a particular case of Theorem 1.2. Hence we prove Theorem 1.2 only. For a real symmetric $n \times n$ matrix Q we denote

$$\|Q\| = \max_{\|x\|=1} \|Qx\|$$

its operator norm. We start with a simple formula, cf. also [4].

Lemma 3.1 *Let $q_1, \dots, q_m : \mathbb{R}^n \to \mathbb{R}$ be quadratic forms,*

$$q_k(x) = \frac{\langle Q_k x, x \rangle}{2},$$

$k = 1, \ldots, m$, where $Q_1, \ldots, Q_m$ are $n \times n$ real symmetric matrices such that

$$\sum_{k=1}^{m} \|Q_k\| < 1.$$

Then

$$\det{}^{-1/2}\left(I - \sum_{k=1}^{m} z_k Q_k\right) \tag{3.1.1}$$
$$= \sum_{k_1, \ldots, k_m \geq 0} \frac{z_1^{k_1} \cdots z_m^{k_m}}{k_1! \cdots k_m!} \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} q_1^{k_1}(x) \cdots q_m^{k_m}(x) e^{-\|x\|^2/2} \, dx,$$

for all $z_1, \ldots, z_m \in \mathbb{C}$ such that

$$|z_1|, \ldots, |z_m| \leq 1. \tag{3.1.2}$$

Here we take the principal branch of $\det^{-1/2}$ in the left-hand side of (3.1.1), which is equal to 1 when $z_1 = \ldots = z_m = 0$. The series in the right hand side converges absolutely and uniformly on the polydisc (3.1.2).

Proof For $z = (z_1, \ldots, z_m)$, let

$$Q_z = I - \sum_{k=1}^{m} z_k Q_k$$

and let

$$q_z(x) = \frac{\langle Q_z x, x \rangle}{2} = \frac{\|x\|^2}{2} - \sum_{k=1}^{m} z_k q_k(x).$$

If $z_1, \ldots, z_m$ are real and satisfy (3.1.2), then $q_z \colon \mathbb{R}^n \to \mathbb{R}$ is a positive definite quadratic form, and, as is well known,

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-q_z(x)} \, dx = \frac{1}{\sqrt{\det Q_z}}.$$

Since both sides of the above identity are analytic in the domain (3.1.2), we obtain

$$\det{}^{-1/2}\left(I - \sum_{k=1}^{m} z_k Q_k\right) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{-\frac{\|x\|^2}{2} + \sum_{k=1}^{m} z_k q_k(x)\right\} dx.$$

Since

$$\exp \sum_{k=1}^{m} z_k q_k(x) = \prod_{k=1}^{m} \exp\{z_k q_k(x)\} = \prod_{k=1}^{m} \sum_{j=0}^{\infty} \frac{z_k^j}{j!} q_k^j(x),$$

the proof follows. $\square$

Next, we extract the integral (1.1.2) from the generating function of Lemma 3.1. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and let

$$\mathbb{T}^m = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{m \text{ times}}$$

be the m-dimensional torus endowed with the uniform (Haar) probability measure $\mu = \mu_1 \times \cdots \times \mu_m$, where μ_k is the uniform probability measure on the k-th copy of $\mathbb{S}^1$. If $s \in \mathbb{Z}^m$, $s = (s_1, \ldots, s_m)$, then for the Laurent monomial

$$\mathbf{z}^s = z_1^{s_1} \cdots z_m^{s_m},$$

we have

$$\int_{\mathbb{T}^m} \mathbf{z}^s \, d\mu = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

Lemma 3.2 *Let $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ be quadratic forms,*

$$q_k(x) = \frac{\langle Q_k x, x \rangle}{2},$$

$k = 1, \ldots, m$, where $Q_1, \ldots, Q_m$ are $n \times n$ real symmetric matrices such that

$$\sum_{k=1}^{m} \|Q_k\| < 1.$$

Then for every $\omega \in \mathbb{C}$ such that $|\omega| < 1$ we have

$$\begin{aligned} &\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + \omega q_1(x)) \cdots (1 + \omega q_m(x)) e^{-\|x\|^2/2} \, dx \\ &\quad = \int_{\mathbb{T}^m} \prod_{k=1}^{m} (1 + \omega z_k^{-1}) \prod_{(k_1, \ldots, k_s)} \left(1 + \frac{\operatorname{trace}(Q_{k_1} \cdots Q_{k_s})}{2s} z_{k_1} \cdots z_{k_s}\right) d\mu, \end{aligned}$$

where the second product is taken over all non-empty ordered tuples $(k_1, \ldots, k_s)$ of distinct indices from $\{1, \ldots, m\}$.

Proof From Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+\omega q_1(x)) \cdots (1+\omega q_m(x)) e^{-\|x\|^2/2}\, dx \\
&\qquad = \int_{\mathbb{T}^m} \prod_{k=1}^{m} (1+\omega z_k^{-1}) \det{}^{-1/2}\left(I - \sum_{k=1}^{m} z_k Q_k\right) d\mu.
\end{aligned} \tag{3.2.1}$$

Next, we write

$$\begin{aligned}
&\det{}^{-1/2}\left(I - \sum_{k=1}^{m} z_k Q_k\right) = \exp\left\{-\frac{1}{2} \ln\det\left(I - \sum_{k=1}^{m} z_k Q_k\right)\right\} \\
&= \exp\left\{-\frac{1}{2} \operatorname{trace} \ln\left(I - \sum_{k=1}^{m} z_k Q_k\right)\right\} = \exp\left\{\frac{1}{2}\sum_{s=1}^{\infty} \frac{1}{s} \operatorname{trace}\left(\sum_{k=1}^{m} z_k Q_k\right)^s\right\} \\
&= \exp \sum_{s=1}^{\infty} \frac{1}{2s} \sum_{1 \le k_1, \ldots, k_s \le m} \operatorname{trace}(Q_{k_1} \cdots Q_{k_s}) z_{k_1} \cdots z_{k_s} \\
&= \prod_{s=1}^{\infty} \prod_{1 \le k_1, \ldots, k_s \le m} \exp\left\{\frac{\operatorname{trace}(Q_{k_1} \cdots Q_{k_s})}{2s} z_{k_1} \cdots z_{k_s}\right\},
\end{aligned}$$

where the product converges absolutely and uniformly on $\mathbb{T}^m$. We expand each of the exponential functions into the Taylor series and observe that only square-free monomials in $z_1, \ldots, z_m$ contribute to the integral (3.2.1), from which it follows that

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+\omega q_1(x)) \cdots (1+\omega q_m(x)) e^{-\|x\|^2/2}\, dx \\
&\quad = \int_{\mathbb{T}^m} \prod_{k=1}^{m} (1+\omega z_k^{-1}) \prod_{(k_1, \ldots, k_s)} \left(1 + \frac{\operatorname{trace}(Q_{k_1} \cdots Q_{k_s})}{2s} z_{k_1} \cdots z_{k_s}\right) d\mu,
\end{aligned}$$

where the second product is taken over all non-empty ordered tuples of distinct indices $k_1, \ldots, k_s \in \{1, \ldots, m\}$. $\square$

Our next goal is to write the integral in Lemma 3.2 as the value of the independence polynomial of an appropriate (large) graph.

3.1 Independent Sets in Weighted Graphs

Let $G = (V, E)$ be a finite undirected graph with set V of vertices, set E of edges and without loops or multiple edges. A set $S \subset V$ of vertices is called *independent*, if no two vertices from S span an edge of G. We agree that $S = \emptyset$ is an independent set. Let $w\colon V \to \mathbb{C}$ be a function assigning to each vertex a complex *weight* $w(v)$.

We define the *independence polynomial* of G by

$$\mathrm{Ind}_G(w) = \sum_{\substack{S \subset V \\ S \text{ independent}}} \prod_{v \in S} w(v).$$

Hence $\mathrm{Ind}_G(w)$ is a multivariate polynomial in complex variables $w(v)$ with constant term 1, corresponding to $S = \emptyset$.

Corollary 3.3 *Let $q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ be quadratic forms,*

$$q_k(x) = \frac{\langle Q_k x, x \rangle}{2},$$

$k = 1, \ldots, m$, where Q_k are real symmetric $n \times n$ matrices and let $\omega \in \mathbb{C}$ be a complex number. We define a weighted graph $G = (V, E; w)$ as follows. The vertices of G are all non-empty ordered tuples $(k_1, \ldots, k_s)$ of distinct indices $k_1, \ldots, k_s \in \{1, \ldots, m\}$ and two vertices span an edge of G if they have at least one common index k, in arbitrary positions. We define the weight of the vertex $(k_1, \ldots, k_s)$ by

$$\frac{\omega^s}{2s} \operatorname{trace}(Q_{k_1} \cdots Q_{k_s}).$$

Then

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + \omega q_1(x)) \cdots (1 + \omega q_k(x)) e^{-\|x\|^2/2} \, dx = \mathrm{Ind}_G(w). \quad (3.4.1)$$

Proof From Lemma 3.2 it follows that (3.4.1) holds provided $|\omega|$ and $\|Q_k\|$ for $k = 1, \ldots, m$ are small enough. Since both sides of (3.4.1) are polynomials in $Q_1, \ldots, Q_k$ and ω, the proof follows. □

The following criterion provides a sufficient condition for $\mathrm{Ind}_G(w) \neq 0$ for an arbitrary weighted graph G. The result is known as the Dobrushin criterion and also as the Kotecký–Preiss condition for the cluster expansion, see, e.g., [10, Chap. 5].

Lemma 3.4 *Given a graph $G = (V, E)$ and a vertex $v \in V$, we define its neighborhood $\mathcal{N}_v \subset V$ by*

$$\mathcal{N}_v = \{v\} \cup \{u \in V : \{u, v\} \in E\}.$$

Let $w : V \to \mathbb{C}$ be an assignment of complex weights to the vertices of G. Suppose that there is a function $\rho : V \to \mathbb{R}_+$ with positive real values such that for every vertex $v \in V$, we have

$$\sum_{u \in \mathcal{N}_v} |w(u)| e^{\rho(u)} \leq \rho(v).$$

Then

$$\mathrm{Ind}_G(w) \neq 0.$$

Proof See, for example, [9, Sect. 5.2] for a concise exposition. □

Now we are ready to prove Theorem 1.2.

3.2 Proof of Theorem 1.2

Let $Q_1, \ldots, Q_m$ be the matrices of the quadratic forms $q_1, \ldots, q_m$, so that

$$q_k(x) = \frac{\langle Q_k x, x\rangle}{2} \quad \text{and} \quad \|Q_k\| \leq \frac{2}{r},$$

$k = 1, \ldots, m$. Since each quadratic form q_k depends of at most r variables, we have rank $Q_k \leq r$ for $k = 1, \ldots, m$. In particular,

$$|\mathrm{trace}\,(Q_{k_1} \cdots Q_{k_s})| \leq r\left(\frac{2}{r}\right)^s = \frac{2^s}{r^{s-1}}. \tag{3.6.1}$$

Since each quadratic form q_k has a common variable with at most r other forms, we have:

$$\begin{aligned}&\text{For every } k \text{ there are at most } r \text{ indices}\\ &j \neq k \text{ such that } Q_k Q_j \neq 0 \text{ or } Q_j Q_k \neq 0.\end{aligned} \tag{3.6.2}$$

Let $\omega \in \mathbb{C}$ be a complex number satisfying

$$|\omega| \leq \gamma = \frac{e^{-1/2}}{4} \approx 0.1516326649. \tag{3.6.3}$$

Given $Q_1, \ldots, Q_k$ and ω, we construct a weighted graph $G = (V, E; w)$ as in Corollary 3.3. Our goal is to prove that $\mathrm{Ind}_G(w) \neq 0$, for which we use Lemma 3.4.

We say that the *level* of a vertex $v = (k_1, \ldots, k_s)$ is s for $s = 1, \ldots, m$. Thus for the weight of v, we have

$$w(v) = \frac{\omega^s}{2s}\,\mathrm{trace}\,(Q_{k_1} \cdots Q_{k_s}).$$

Combining (3.6.1) and (3.6.3), we conclude that for a vertex of level s, we have

$$|w(v)| \leq \frac{1}{s 2^{s+1} r^{s-1}} e^{-s/2}. \tag{3.6.4}$$

We observe that there are at most str^{t-1} vertices u of level t with $w(u) \neq 0$ that are neighbors of a given vertex v (for $t = s$, we count v as its own neighbor). Indeed, there are at most s ways to choose a common index k, after which there are at most t positions to place k in u. By (3.6.2), we conclude that there are at most str^{t-1} vertices $u \in \mathcal{N}_v$ of level t with $w(u) \neq 0$. Choosing $\rho(v) = s/2$ for a vertex of level s and using (3.6.4), we conclude that for a vertex v of level s, we have

$$\sum_{u \in \mathcal{N}_v} |w(u)| e^{\rho(u)} \leq \sum_{t=1}^{m} \frac{e^{-t/2}}{t 2^{t+1} r^{t-1}} str^{t-1} e^{t/2} = s \sum_{t=1}^{m} \frac{1}{2^{t+1}} < \frac{s}{2} = \rho(v),$$

and the proof follows by Corollary 3.3 and Lemma 3.4.

4 Approximating the Integral

The interpolation method is based on the following simple observation.

Lemma 4.1 *Let $p\colon \mathbb{C} \to \mathbb{C}$ be a polynomial,*

$$p(z) = \sum_{s=0}^{m} c_s z^s,$$

and $\beta > 1$ be a real number such that $p(z) \neq 0$ provided $|z| < \beta$. Let us choose a branch of $f(z) = \ln p(z)$ for $|z| < \beta$ and let

$$T_k(z) = f(0) + \sum_{s=1}^{k} \frac{f^{(s)}(0)}{s!} z^s$$

be the Taylor polynomial of degree k of f computed at $z = 0$.

(1) *We have*

$$|f(1) - T_k(1)| \leq \frac{m}{(k+1)\beta^k(\beta - 1)}.$$

(2) *We have $f(0) = \ln p(0)$, while the numbers $f^{(s)}(0)$ satisfy the system of linear equations*

$$p^{(s)}(0) = \sum_{j=0}^{s-1} \binom{s-1}{j} p^{(j)}(0) f^{(s-j)}(0) \quad \text{for} \quad s = 1, \ldots, k,$$

with a $k \times k$ invertible triangular matrix of coefficients.

Proof See, for example, [6, Sect. 2.2]. $\square$

As follows from part (1) of Lemma 4.1, if $\beta > 1$ is fixed in advance, to estimate the value of $f(1)$ within additive error $0 < \epsilon < 1$ (in which case we say that we estimate the value of $p(1) = e^{f(1)}$ within relative error ϵ), it suffices to compute the numbers $f^{(s)}(0)$ for $s = 0, \ldots, k$ with $k = O(\ln m - \ln \epsilon)$, where the implied constant in the "O" notation depends only on β. As follows from part (2), the numbers $f^{(s)}(0)$ in turn can be computed from the coefficients $c_s = p^{(s)}(0)/s!$ for $s = 0, \ldots, k$ in $O(k^2)$ time by solving a triangular system of linear equations with an invertible matrix of coefficients (the diagonal entries are $p(0) \neq 0$).

A similar to Lemma 4.1 result holds if $p(z) \neq 0$ in an arbitrary, fixed in advance, connected open set $U \subset \mathbb{C}$ such that $\{0, 1\} \subset U$, see [6, Sect. 2.2] (in Lemma 4.1, the neighborhood U is the disc of radius β).

4.1 Computing the Integrals

Let us fix a constant

$$0 < \gamma' < \gamma,$$

where γ is the constant of Theorem 1.2 (so one can choose $\gamma' = 0.15$). Let $q_1, \ldots, q_m \colon \mathbb{R}^n \to \mathbb{R}$ be quadratic forms, defined by their matrices $Q_1, \ldots, Q_m$ as in (1.1.1), such that each form depends on not more than r variables among $x_1, \ldots, x_n$ and each form has common variables with not more than r other forms. Suppose that the bound (1.3.1) holds. We define a univariate polynomial $p \colon \mathbb{C} \to \mathbb{C}$ by

$$p(z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + zq_1(x)) \cdots (1 + zq_m(x)) e^{-\|x\|^2/2}\, dx.$$

Hence $\deg p \leq m$ and by Theorem 1.2 we have

$$p(z) \neq 0 \quad \text{provided } |z| < \beta \text{ where } \beta = \frac{\gamma}{\gamma'} > 1.$$

In view of Lemma 4.1, to approximate

$$p(1) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + q_1(x)) \cdots (1 + q_m(x)) e^{-\|x\|^2/2}\, dx \tag{4.2.1}$$

within relative error $0 < \epsilon < 1$, it suffices to compute $p(0) = 1$ and $p^{(s)}(0)$ for $s = O(\ln m - \ln \epsilon)$, where the implied constant in the "O" notation is absolute.

One way to proceed is to notice that

$$p^{(s)}(0) = s! \sum_{1 \leq i_1 < \ldots < i_s \leq m} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} q_{i_1}(x) \cdots q_{i_s}(x) e^{-\|x\|^2/2}\, dx$$

and compute each of the $\binom{m}{s}$ integrals in $(nm)^{O(s)}$ time using the algorithm of Sect. 2.4. Another possibility is to notice that from Corollary 3.3, we have

$$p^{(s)}(0) = s! \sum_{\substack{(k_{11},\dots,k_{1s_1}),\dots,\\(k_{j1},\dots,k_{js_j})\\ s_1+\dots+s_j=s}} \frac{\operatorname{trace}(Q_{k_{11}}\cdots Q_{1s_1})}{2s_1}\cdots\frac{\operatorname{trace}(Q_{k_{j1}}\cdots Q_{k_{js}})}{2s_j},$$

where the sum is taken over all unordered collections of pairwise disjoint ordered tuples $(k_{11},\dots,k_{1s_1}),\dots,(k_{j1},\dots,k_{js_j})$ of distinct indices k_{ij} from the set $\{1,\dots,m\}$, with the total number s of chosen indices. A crude upper bound for the number of such collections is $(2m)^s$: writing all the indices k_{ij} as a row, we have at most $2m$ choices for each index k_{ij}, including the choice on whether the index remains in the current tuple or starts a new one. Given that $s = O(\ln m - \ln \epsilon)$ and that computing the traces of the products of $n \times n$ matrices can be done in $(ns)^{O(1)}$ time, we obtain an algorithm approximating the integral in quasi-polynomial $n^{O(1)}m^{O(\ln m - \ln \epsilon)}$ time.

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