

# MODE SOLUTIONS TO THE WAVE EQUATION ON A ROTATING COSMIC STRING BACKGROUND

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ABSTRACT. A static rotating cosmic string metric is singular along a timelike line and fails to be globally hyperbolic; these features make it difficult to solve the wave equation by conventional energy methods. Working on a single angular mode at a time, we use microlocal methods to construct forward parametrices for wave and Klein–Gordon equations on such backgrounds.

## 1. INTRODUCTION

In this note we construct a semi-global forward parametrix for mode solutions to the wave equation on a rotating cosmic string background. Cosmic strings, introduced by Kibble [7], are solutions to the Einstein equations that have topological defects along one-dimensional (“string”) structures. They may or may not be a feature of real cosmology [10]. The simplest cosmic string solutions, corresponding to a single, nonrotating string in equilibrium, may be viewed either as singular at  $(x_1, x_2) = 0$ ,  $(x_3, t) \in \mathbb{R}$  in  $3 + 1$  dimensions or, reducing along an axis of symmetry, as singular at  $(x_1, x_2) = 0$ ,  $t \in \mathbb{R}$  in  $2 + 1$  dimensions. The latter solutions are simply static metrics whose spatial slices are flat 2d cones. *Rotating* cosmic string solutions, by contrast, have a singularity with an authentically Lorentzian character, given in the static setting [5, Equation 4.17] by the metric (in cylindrical coordinates in  $\mathbb{R}_t \times \mathbb{R}^2$ )

$$(1) \quad g = (dr^2 + r^2 d\varphi^2) - (dt^2 - 2A dt d\varphi + A^2 d\varphi^2).$$

These are solutions, introduced by Deser–Jackiw–’t Hooft [5], to the Einstein equations corresponding to a one-dimensional rotating source with zero mass but with nonzero angular momentum; here  $A = -4GJ$  where  $G$  is the gravitational constant and  $J$  the angular momentum. Owing to their flatness, which can be seen *locally* by a change of coordinates reducing to Minkowski space, these metrics are manifestly

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singular solutions to the Einstein equations with vanishing cosmological constant. Among the interesting features of the rotating cosmic string metric (dubbed a “cos-mon” in [5]) are the singularity at  $r = 0$  and the causality violation entailed by the existence of closed timelike curves such as

$$(t_0, r_0, \varphi = s) : s \in [0, 2\pi]$$

for  $r_0 < A$ .

One might suppose that such a serious causality violation as exhibited by the metric (1) should be disastrous for the well-posedness of the wave equation on such a background, and certainly it does cause great difficulties for conventional energy methods. (Note that the conserved energy associated to  $t$ -translation invariance ceases to be positive when  $r < A$ .) A number of positive results on the behavior of the wave equation on causality-violating spacetimes similar to the one we study have been obtained in the work of Bachelot [1], however.<sup>1</sup> Motivated by [1], in this note we pursue the question of existence of *forward* solutions to the wave equation, where we specify a compactly supported inhomogeneity, and try to solve in forward time. The wave operator in this spacetime is given by

$$\left(1 - \frac{A^2}{r^2}\right)\partial_t^2 - \frac{1}{r^2}(r\partial_r)^2 - \frac{2A}{r^2}\partial_\varphi\partial_t - \frac{1}{r^2}\partial_\varphi^2.$$

Here we specialize to a single angular mode solution  $e^{ik\varphi}u$ , which leaves us with the 1+1 dimensional operator

$$\begin{aligned}\square_k &= \left(1 - \frac{A^2}{r^2}\right)\partial_t^2 - \frac{1}{r^2}(r\partial_r)^2 - \frac{2Aik}{r^2}\partial_t + \frac{k^2}{r^2} \\ &= -\frac{1}{r^2}(A\partial_t + ik)^2 + \partial_t^2 - \partial_r^2 - \frac{1}{r}\partial_r.\end{aligned}$$

The operator  $\square_k$  thus changes type from hyperbolic in  $r > A$  to elliptic in  $r < A$ . Near this interface, the equation is in fact of *Tricomi type*, with the added difficulty of a singularity at  $r = 0$ . We seek a forward solution operator modulo smoothing terms. One might hope for better: a solution to  $\square_k u = \delta_q$  *supported* only in the forward light cone emanating from a point  $q$  in the hyperbolic region, but this is ruled out by Lemma 11 below (cf. [1, Theorem 3.5]): the support of the solution must extend throughout the elliptic region.

Thus we rely on microlocal methods, in the spirit of the work of Payne [9] on equations of Tricomi type, which yield results on singularities of solutions without actually constraining their supports. *Our main result (fully stated in Section 2 below) is the existence of a forward parametrix for the equation  $(\square_k + m^2)u = f$  :* in

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<sup>1</sup>Our metric essentially fits into the framework of *Papapetrou metrics* considered by Bachelot, but the singularity at  $r = 0$  is a novel feature here, as is the focus on causal solutions, rather than scattering theory.

particular, we show that there is an exact solution to this equation whose wavefront set is contained in the forward-in-time bicharacteristic flowout of WF  $f$  inside the characteristic set (i.e., the light cone), together with WF  $f$  itself. (Note that we have additionally allowed a nonnegative mass term  $m^2$ , and thus consider the more general Klein–Gordon equation.) By contrast we remark that the extensive treatment of solutions of the Tricomi equation in the gas dynamics literature (see e.g. [8]) tends to emphasize propagating data from either characteristic curves in the hyperbolic region (Tricomi problem) or from noncharacteristic surfaces such as would be locally given in our setup by  $r = \text{constant}$  (Frankl’s problem).<sup>2</sup>

We confess that dealing with single mode solutions, as we do here, essentially sidesteps the worst difficulties of causality violation: at high energy, the solutions we study have zero angular momentum (the angular momentum  $k$  is fixed, while the duals to  $t$  and  $r$  become infinite). As we will see below, the associated null-geodesic flow is thus well behaved and  $t$  is monotone, with the slight caveat that the null geodesics do have singularities at  $r = A$ . (The null bicharacteristics, corresponding to lifts of the geodesics to the cotangent bundle, remain nonsingular, in any event.) We intend to treat the full propagation of singularities for the wave equation on cosmic string backgrounds (i.e., not just for mode solutions) in a subsequent paper; this will entail a much more technical analysis of propagation of singularities through the string at  $r = 0$ .

## 2. FUNCTION SPACES AND MAPPING PROPERTIES

As we will be working on mode solutions, we could restrict our attention to functions defined on the space  $[0, \infty)_r \times \mathbb{R}_t$ , but this is potentially confusing owing to the artificial boundary at  $r = 0$  and the volume form  $r \, dr \, dt$ . Hence we will instead deal explicitly with  $k$ -equivariant functions on  $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}_x^2$ . To this end, we define adapted Sobolev spaces for our problem. Let  $\|\bullet\|$  denote  $L^2$  norm of a function on the spacetime  $\mathbb{R}^3$ .

*Definition 1.* Let  $\mathcal{D}$  denote the space of test functions  $\mathcal{C}_c^\infty(\mathbb{R}_t \times (\mathbb{R}_x^2 \setminus \{0\}))$ ; let  $\mathcal{D}'$  denote the dual space and let  $\mathcal{E}'$  denote the elements of  $\mathcal{D}'$  that are compactly supported in  $\mathbb{R}^3$ . Let  $\mathcal{D}(U)$  denote those test function supported in  $U$ . Let  $\mathcal{S}$  denote Schwartz functions supported in  $x \neq 0$  and  $\mathcal{S}'$  their dual. For  $\mathcal{X}$  any of the above spaces of distributions, we let  $\mathcal{X}_k$  denote the subspace of distributions that are annihilated by  $\partial_\varphi - ik$ .

We may now define Hilbert spaces adapted to our problem.

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<sup>2</sup>It is claimed in [2] that there exists a fundamental solution to the Tricomi equation that is *supported* in what the authors call Region III, which corresponds to our forward flowout, but the apparent contradiction with our results seems to be addressed by the erratum [3].

*Definition 2.* Fix  $k \in \mathbb{Z}$ . Let  $\mathcal{H}_k^1$  be the closure of the the  $S_\varphi^1$ -equivariant test functions  $\mathcal{D}_k$  with respect to the squared norm

$$\|\bullet\|_{\mathcal{H}_k^1}^2 = \|\bullet\|^2 + \|\partial_t \bullet\|^2 + \|\partial_r \bullet\|^2 + \|r^{-1}(\mathbf{A}\partial_t + ik)\bullet\|^2.$$

Let  $\mathcal{H}_k^{-1}$  denote the dual space with respect to the  $L^2$  inner product.

*Remark 3.* Away from  $r = 0$ ,  $\mathcal{H}_k^1$  is just equivariant functions in  $H^1$ ; at  $r = 0$ , though, membership in this space entails subtly different estimates than  $H^1$  regularity; for instance, smooth compactly supported functions of  $(t, r)$  times  $e^{ik\varphi}$  are not in  $\mathcal{H}_k^1$ .

We can (and will) identify  $\mathcal{H}_k^{-1}$  with a space of equivariant distributions.

**Lemma 4.** *For any  $K$  compact and  $S_\varphi^1$ -invariant, the inclusion*

$$\mathcal{H}_k^1 \cap \mathcal{E}'(K) \hookrightarrow L^2(K)$$

*is compact.*

*Proof.* We remark that elements of the space

$$e^{-ik\varphi}\mathcal{H}_k^1 = \{e^{-ik\varphi}u : u \in \mathcal{H}_k^1\}$$

are rotation-invariant in  $\varphi$ , hence annihilated by  $\partial_\varphi$  or even by  $r^{-1}\partial_\varphi$ . Thus if  $u_j$  are a sequence of elements in the unit ball in  $\mathcal{H}_k^1$ , supported in  $K$  then

$$v_j \equiv e^{-ik\varphi}u_j$$

enjoy the same support property and satisfy

$$\|v_j\|^2 + \|\partial_t v_j\|^2 + \|\partial_r v_j\|^2 + \|r^{-1}\partial_\varphi v_j\|^2 \leq 1,$$

where the last term on the LHS is of course zero. Recognizing that the LHS is now the usual  $H^1$  norm, we see that  $L^2$ -convergence of a subsequence follows from compact embedding of  $H^1 \cap \mathcal{E}'(K)$  in  $L^2$ .  $\square$

In discussing weak solutions to  $(\square_k + m^2)u = 0$  we must be careful about behavior near  $r = 0$ , since in fact  $\square_k$  does not a priori map even  $\mathcal{C}_c^\infty(\mathbb{R}^3)$  to distributions, owing to the singularity at  $r = 0$ . Hence in discussing distributional solutions to  $(\square_k + m^2)u = f$  we will mean weak solutions in the following sense.

*Definition 5.* For  $u \in L_k^2$ , and  $U \subset \mathbb{R}^3$  open and  $S_\varphi^1$ -invariant, we define  $(\square_k + m^2)u = f$  on  $U$  if

$$\langle u, (\square_k + m^2)\phi \rangle = \langle f, \phi \rangle$$

for all test functions

$$\phi \in \mathcal{D}_k(U).$$

(We will use the same definition of weak solution in dealing with the modified operators  $P, P^*$  defined below.)

With a notion of solutions and appropriate Sobolev spaces in hand, we can now state our main theorem. For  $q \in T^*\mathbb{R}^3$ , let  $\Phi^s(q)$  denote the Hamilton flow (with Hamiltonian given by the principal symbol of  $\square_k + m^2$ ) with parameter  $s$  starting at  $q$ ; note that the  $t$  variable may be increasing or decreasing along the flow according to the sign of its dual variable. For a set  $\Omega \subset T^*\mathbb{R}^3$  let

$$\Phi_+(\Omega) = \bigcup_{s \in \mathbb{R}} \{\Phi^s(q) : q \in \Omega, t(\Phi^s(q)) \geq t(q)\}$$

denote the *forward-in-time* flowout; projected to the base, this is a forward-in-time motion along radial geodesics for the cosmic string metric. Finally, let  $\Sigma$  denote the characteristic set of  $\square_k + m^2$ , intersected with that of  $\partial_\varphi - ik$ , i.e., the radial part of the light cone. (See Section 3 below for details on the Hamiltonian dynamics.)

**Theorem.** *Given  $m \in \mathbb{R}$ ,  $R_0 > 0$ , an  $S_\varphi^1$ -invariant compact set  $K \subset \mathbb{R}^3$ , and  $f \in \mathcal{H}_k^{-1}$  with  $\text{supp } f \subset \{r < R_0\}$ , there exists  $u \in L_k^2(K)$  such that*

$$(\square_k + m^2)u = f \quad \text{on } K^\circ$$

*and such that  $\text{WF } u \setminus \text{WF } f \subset \Phi_+(\text{WF } f \cap \Sigma)$ .*

*If, additionally,  $f \in L_k^2(\mathbb{R}^3)$ , then we further conclude that*

$$u \in L_k^2(K) \cap H_{loc}^1(K^\circ \setminus \{r = 0\}).$$

*The forward solution  $u$  is unique modulo an element of  $L_k^2 \cap \mathcal{C}^\infty(K^\circ \setminus \{r = 0\})$ .*

We begin with a unique continuation theorem that rules out solutions that are supported in the hyperbolic region.

**Lemma 6.** *Let  $u \in \mathcal{S}'_k(\mathbb{R}^3)$  and assume  $(\square_k + m^2)u = 0$  in  $\{r \in I\}$  where  $I \subset (0, \infty)$  is an open interval containing  $r = A$ . If  $u(t, r) = 0$  for  $r \in (0, A) \cap I$  then  $u \equiv 0$  on  $\{r \in I\}$ .*

Note that the same proof as that given here shows that a nontrivial solution to the Tricomi equation  $y\partial_x^2 + \partial_y^2$  near  $y = 0$  cannot identically vanish in the elliptic region  $y > 0$ .

*Proof.* Let  $\hat{u}(\lambda, r)$  denote the partial Fourier transform in  $t$ . Then

$$(2) \quad \left[ -\left(1 - \frac{A^2}{r^2}\right)\lambda^2 - \frac{1}{r^2}(r\partial_r)^2 + \frac{2Ak\lambda}{r^2} + \frac{k^2}{r^2} + m^2 \right] \hat{u} = 0, \quad r \in I.$$

Now replace  $\hat{u}$  by  $\hat{u}_\phi \equiv u * \phi$  for an arbitrary test function  $\phi(\lambda)$  to obtain a smooth (in  $\lambda$ ) solution to the above equation. The Picard–Lindelöf theorem applied to the ODE (2) means that for any fixed  $\lambda$ , if  $\hat{u}_\phi(\lambda, r) = 0$  for  $r \in I \cap (0, A)$ , then it is identically

zero. Thus,  $\hat{u}_\phi = 0$  identically for  $r \in I$ . Since  $\phi$  was arbitrary, the distribution  $u$  must vanish for  $r \in I$ .  $\square$

### 3. PROPAGATION OF SINGULARITIES

We now analyze the pair of operators  $(\square_k + m^2, \partial_\varphi - ik)$  from the perspective of microlocal analysis. Let  $(\lambda, \xi, \eta)$  denote canonical dual coordinates in  $T^*\mathbb{R}^3$  to the cylindrical coordinates  $(t, r, \varphi)$ . The principal symbol of  $\square_k$  (and likewise of  $\square_k + m^2$ ) is

$$\sigma_2(\square_k) = \frac{1}{r^2} A^2 \lambda^2 - \lambda^2 + \xi^2,$$

hence the Hamilton vector field of  $\sigma_2(\square_k)$  is

$$-2\lambda\left(1 - \frac{A^2}{r^2}\right)\partial_t + 2\xi\partial_r + \frac{2}{r^3}A^2\lambda^2\partial_\xi.$$

Meanwhile the operator  $\partial_\varphi - ik$ , on whose nullspace we work, is globally elliptic except at  $\eta = 0$ , hence we need only concern ourselves with this region of phase space. The system  $(\square_k + m^2, \partial_\varphi - ik)$  is then elliptic for  $r < A$ .

In the following, let  $\Sigma$  denote the *joint* characteristic set of  $(\square_k + m^2, \partial_\varphi - ik)$ , hence the subset of the complement of the zero-section given by  $\{\sigma_2(\square_k) = 0\} \cap \{\eta = 0\}$ . By standard elliptic regularity, for  $u \in \mathcal{S}'_k$ ,

$$\text{WF } u \subset \text{WF } ((\square_k + m^2)u) \cup \Sigma,$$

at least over  $r > 0$ . (We will develop an elliptic estimate below that is valid down to  $r = 0$ .)

*Remark 7.* Our system  $(\square_k + m^2, \partial_\varphi - ik)$  changes type abruptly across the hypersurface  $\{r = A\} \subset \mathbb{R}^3$ , hence the projection to the base of  $\Sigma$  has a boundary at  $r = A$ . By contrast, upstairs in the cotangent bundle

$$\Sigma = \{(r^2 - A^2)\lambda^2 = r^2\xi^2, \eta = 0\}$$

is nonetheless a smooth conic submanifold of  $T^*\mathbb{R}^3$ . It is only the projection to the base that is singular.

*Remark 8.* For later use, we note that on  $\Sigma$ , vanishing of  $\sigma_2(\square_k)$  gives

$$\lambda^2(1 - A^2/r^2) = \dot{r}^2/4,$$

with dot denoting derivative along the Hamilton flow. Hence along bicharacteristics with flow parameter  $s$ ,

$$r(s)^2 = A^2 + (2\lambda s + \text{const})^2,$$

i.e.  $r \rightarrow \infty$  (in a monotone fashion) as  $s \rightarrow \pm\infty$ . Meanwhile,  $\dot{t} = -2\lambda(1 - A^2/r^2)$  yields  $|\dot{t}| \geq |\lambda|$  for  $r$  sufficiently large, i.e.  $t$  is strictly monotone along the flow as  $s \rightarrow \pm\infty$ .

The integral curves are singular in the base (i.e.,  $t, r$  variables) when  $r = A$ , and  $t$  is stationary there, since  $dt/dr = -\lambda(r^2\xi)^{-1}(r^2 - A^2)$ . But the curves are smooth in the cosphere bundle: such points are not radial points since  $\dot{\xi} = 2r^{-3}A^2\lambda^2 \neq 0$ . (Note that  $\lambda \neq 0$  on  $\Sigma$ .)

Given a fixed compact  $K \subset \mathbb{R}^3$  and  $R_0 \in \mathbb{R}$ , choose  $R > \max\{A, R_0\}$  such that  $K \subset \{r < R\}$ . Note that over  $r > R$ , the characteristic set  $\Sigma$  separates into four components corresponding to choosing  $\lambda \gtrless 0$ ,  $\xi \gtrless 0$ . Let

$$\Sigma_- \equiv \Sigma \cap \{r > A\} \cap \{\text{sgn } \lambda = \text{sgn } \xi\}.$$

We now construct  $W \in \Psi^2(\mathbb{R}^3)$  enjoying the following properties:

- (1) For  $r > R + 1$ ,  $W$  is elliptic on  $\Sigma_-$ .
- (2) Where  $\sigma_2(W) \neq 0$ ,  

$$\text{sgn } \sigma_2(W) = -\text{sgn } \lambda.$$
- (3)  $\text{proj}_\bullet \text{supp } \kappa(W) \subset \{r > R\}$ , where  $\text{proj}_\bullet$  is projection to the left or right factor ( $\bullet = L$  or  $R$ ) and  $\kappa$  denotes Schwartz kernel.
- (4)  $[\partial_t, W] = [\partial_\varphi, W] = 0$ .

To produce such an operator, we begin by choosing  $R$  such that  $1 - A^2/R^2 > 9/10$ . Let  $\varrho, \psi, \chi$  be smooth functions such that

- $\varrho(s) = 1$  on  $(3/4, 5/4)$  and is supported on  $(2/3, 4/3)$ .
- $\psi(s) = 1$  on  $(-1/10, 1/10)$  and is supported on  $(-1/5, 1/5)$ .
- $\chi(r)$  is supported on  $(R, \infty)$  and equals 1 on  $(R + 1, \infty)$ .

Let  $w$  denote the homogeneous 2-symbol

$$w = -\text{sgn}(\lambda)\lambda^2\varrho(\xi/\lambda)\psi(|\eta/\lambda|)\chi(r).$$

Let  $W_0$  denote the Weyl quantization of  $w$ , and let

$$W = \tilde{\chi}W_0\tilde{\chi}$$

where  $\tilde{\chi}(r) = 1$  on  $\text{supp } \chi(r)$ , and  $\tilde{\chi}$  is supported in  $(R, \infty)$ . The cutoffs  $\tilde{\chi}$  enforce the support properties of the kernel of  $W$ . The ellipticity property follows from the fact that on  $\Sigma_- \cap \{r > R\}$ ,  $\xi^2 \in (9\lambda^2/10, \lambda^2)$  and  $\eta = 0$ , hence the  $\varrho$  and  $\psi$  cutoffs equal 1, and  $w = -\text{sgn}(\lambda)\lambda^2$  on this set.

We will consider solutions to  $Pu = f$  for the operator

$$P \equiv \square_k + m^2 - iW.$$

*Remark 9.*

- (1) The set  $\Sigma_-$  is *incoming* in forward time in the sense that under bicharacteristic flow,  $dr/dt < 0$  there.
- (2) We will prove a number of preliminary results that hold equally well for the operators  $P$  and  $P^*$ , hence we let  $P^{(*)}$  denote either of these operators.

- (3) By ellipticity of  $(P^{(*)}, \partial_\varphi - ik)$  on  $\{r > R + 1\} \cap \Sigma_-$ ,  
 $(\text{WF } u \setminus \text{WF } P^{(*)}u) \cap \{r > R + 1\} \cap \Sigma_- = \emptyset$   
 for  $u \in \mathcal{S}'_k$ .

**Lemma 10.** *The operators  $P, P^*$  enjoy the following mapping property:*

$$P^{(*)} : \mathcal{H}_k^1 \rightarrow \mathcal{H}_k^{-1}.$$

*Proof.* For test functions  $\phi, \psi \in \mathcal{D}$  with  $\text{supp } \phi \subset \{r < R\}$ ,

$$\langle P^{(*)}\phi, \psi \rangle = \langle r^{-1}(\mathbf{A}\partial_t + ik)\phi, r^{-1}(\mathbf{A}\partial_t + ik)\phi \rangle - \langle \partial_t \phi, \partial_t \psi \rangle + \langle \partial_r \phi, \partial_r \psi \rangle + m^2 \langle \phi, \psi \rangle.$$

Applying Cauchy–Schwarz to each term on the RHS, we may estimate it by a multiple of  $\|\phi\|_{\mathcal{H}_k^1} \|\psi\|_{\mathcal{H}_k^1}$ , hence the mapping property follows. For test functions with support in  $r > R/2$ , on the other hand, the estimate simply follows from boundedness of second order differential operators from  $H^1 \rightarrow H^{-1}$  since the norm on  $\mathcal{H}_k^1$  is equivalent to the  $H^1$  norm away from  $r = 0$ .

Thus, choosing a cutoff  $\chi(r)$  equal to 1 on  $[0, R/2)$  and supported in  $[0, 3R/4)$ , given any  $\phi \in \mathcal{D}$  we split

$$\phi = \chi\phi + (1 - \chi)\phi.$$

The operations of multiplication by  $\chi, 1 - \chi$  are bounded on  $\mathcal{H}_k^1$  since  $\chi = 1$  near the origin. Thus

$$P^{(*)}\phi = P^{(*)}(\chi\phi) + P^{(*)}((1 - \chi)\phi)$$

is bounded in  $\mathcal{H}_k^{-1}$  by a multiple of  $\|\phi\|_{\mathcal{H}_k^1}$  by applying the foregoing results to  $\chi\phi$  and  $(1 - \chi)\phi$ .  $\square$

The virtue of our construction of  $W$  is that owing to our choice of signs for  $W$ , regularity for solutions to the equation  $Pu = f$  propagates *forward* along null bicharacteristics of  $\text{Re } \sigma_2(P)$  in the hyperbolic region for  $\lambda < 0$  (since  $\sigma_2(W) \geq 0$ ) and *backward* for  $\lambda > 0$  (since  $\sigma_2(W) \leq 0$ ); we refer the reader to [11, Section 2.5] for a proof. Hence, since  $\dot{t} = -2\lambda(1 - \mathbf{A}^2/r^2)$  along the flow, and this has the same sign as  $-\lambda$  in the hyperbolic region, on every component of the characteristic set, regularity propagates *forward in time* for the operator  $P$ . Of course, when we consider  $P^*$ , the sign of the  $W$  term is reversed, and the reverse phenomenon therefore takes place: regularity propagates *backward in time* instead. We will use both of these propagation results in what follows: that for  $P^*$  to obtain solvability of the equation  $Pu = f$  and that for  $P$  to constrain the wavefront set of the resulting distribution  $u$ .

**Lemma 11.** *If  $u \in \mathcal{E}'_k(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$  and  $P^{(*)}u = 0$  then  $\text{supp } u \subset \{r \geq R\}$ .*

*Proof.* This follows from our uniqueness results (cf. [1]), since vanishing on an open subset of the elliptic set  $\{r \in (0, \mathbf{A})\}$  implies global vanishing on this set, by unique



continuation for elliptic operators, and in turn implies global vanishing on  $\{r \in (0, R)\}$  by Lemma 6.  $\square$

Of course for  $r \geq R$ , since  $P^{(*)} \neq \square_k + m^2$ , we cannot conclude vanishing anymore.

**Lemma 12.** *Given  $K \subset \mathbb{R}^3$  compact and  $S_\varphi^1$ -invariant, and  $R \gg 0$  chosen as above, there exists  $K' \supset K$ , also compact and invariant, such that for any radial null bicharacteristic  $\gamma(s) \subset \Sigma$  of  $\sigma_2(\square_k) = \text{Re } \sigma_2(P^{(*)})$  with  $\pi\gamma(0) \in K$ , there exists  $s_0$  such that*

- (1)  $\pi\gamma(s_0) \in K'$
- (2)  $t(\gamma(s_0)) < t(\gamma(0))$ ,
- (3)  $r(\gamma(s_0)) > R + 1$
- (4)  $\text{sgn } d(r \circ \gamma)/ds|_{s_0} = -\text{sgn } d(t \circ \gamma)/ds|_{s_0}$ .

That is to say, every null bicharacteristic starting in  $K$  hits the elliptic set of  $W$  in backward time, within the set  $K'$ , and does so at “incoming points” where  $dr/dt < 0$ . (See Figure 1.) The lemma follows directly from Remark 8, since for one choice of the sign, as the flow parameter  $s \rightarrow \pm\infty$  we have  $r \rightarrow +\infty$ ,  $t \rightarrow -\infty$ , with both functions monotone. Hence we may simply take  $K' = [0, R+2]_r \times [-T, T]_t \times S_\varphi^1$  with  $T$  sufficiently large that every bicharacteristic escapes to  $r > R+1$  in backward time no less than  $-T$ .

We additionally need an elliptic estimate valid down to  $r = 0$ : for  $u$  supported in  $\{\epsilon \leq r \leq A - \epsilon\}$ , standard elliptic estimates apply, but we will need uniformity down to  $r = 0$  as well. We begin with a coercivity estimate.

**Lemma 13.** *For all  $\phi \in \mathcal{D}_k$  supported in  $\{r \in (0, A/4)\}$ ,*

$$(3) \quad \|\phi\|_{\mathcal{H}_k^1}^2 \lesssim \|\phi\|_{L^2}^2 + \langle P^{(*)}\phi, \phi \rangle.$$

*Proof.* Pairing  $P^{(*)}\phi$  with  $\phi$  gives

$$\langle P^{(*)}\phi, \phi \rangle = \|r^{-1}(A\partial_t + ik)\phi\|^2 - \|\partial_t\phi\|^2 + \|\partial_r\phi\|^2 + m^2\|\phi\|^2.$$

Since for  $r < A/4$

$$\begin{aligned} \frac{1}{4}r^{-2}(A\lambda + k)^2 - \lambda^2 &\geq \left(2\lambda + \frac{2k}{A}\right)^2 - \lambda^2 \\ &= \lambda^2 + 2\lambda^2 + \frac{8\lambda k}{A} + \frac{4k^2}{A^2} \\ &= \lambda^2 + 2\left(\left(\lambda + \frac{2k}{A}\right)^2 - \frac{2k^2}{A^2}\right), \end{aligned}$$

Fourier transforming  $t \rightarrow \lambda$  and using Plancherel gives the elliptic estimate

$$\frac{1}{4}\|r^{-1}(A\partial_t + ik)\phi\|^2 - \|\partial_t\phi\|^2 \geq \|\partial_t\phi\|^2 - \frac{4k^2}{A^2}\|\phi\|^2.$$

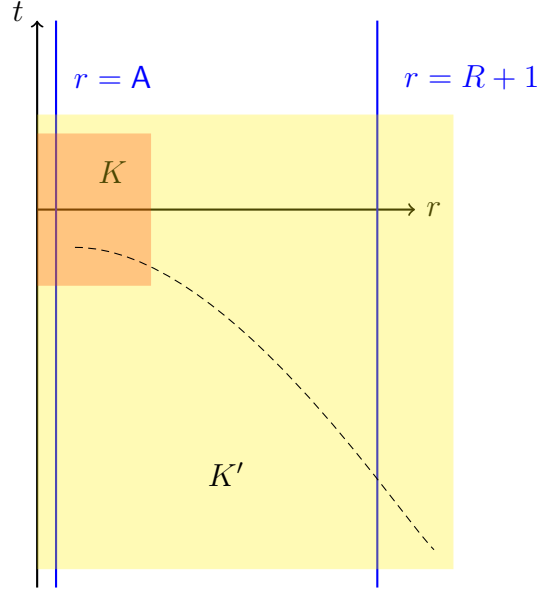


FIGURE 1. The sets  $K$  and  $K'$  with a sample (projected) bicharacteristic with one end in  $K$  arriving in backward time in the set  $K'$  inside  $r > R + 1$  (dashed curve).

Thus

$$\langle P^{(*)}\phi, \phi \rangle \geq \frac{3}{4} \|r^{-1}(\mathbf{A}\partial_t + ik)\phi\|^2 + \|\partial_t\phi\|^2 + \|\partial_r\phi\|^2 + \|\phi\|^2 - \left(1 + \frac{4k^2}{\mathbf{A}^2}\right) \|\phi\|^2$$

and we have obtained the desired estimate.  $\square$

**Lemma 14.** For  $u \in \mathcal{H}_k^1$ , supported in  $r < \mathbf{A}/4$ ,

$$(4) \quad \|\phi\|_{\mathcal{H}_k^1} \leq C\|\phi\|_{L^2} + C\|P^{(*)}\phi\|_{\mathcal{H}_k^{-1}}.$$

*Proof.* Apply Cauchy-Schwarz to the estimate (3) and use the density of test functions.  $\square$

*Remark 15.* The hypothesis that  $\phi \in \mathcal{H}_k^1$  may *not* be dispensed with in the preceding lemma. One might hope that a weaker a priori assumption, such as  $\phi \in L^2$ ,  $P^*\phi \in \mathcal{H}_k^{-1}$ , might *imply*  $\phi \in \mathcal{H}_k^1$ , but this is not so. For instance, as noted by Carrillo [4], take  $m = 0$  and fix  $\lambda$  such that  $\mathbf{A}\lambda + k \in (-1, 0)$  and consider

$$\phi(t, r, \varphi) = \chi(r)e^{i\lambda t}e^{ik\varphi}J_{\mathbf{A}\lambda+k}(\lambda r),$$

with  $\chi(r)$  a cutoff function equal to 1 on  $[0, \mathbf{A}/2)$  and supported in  $r < \mathbf{A}$ . This satisfies

$$P^*\phi = [\square_k, \chi]e^{i\lambda t}e^{ik\varphi}J_{\mathbf{A}\lambda+k}(\lambda r),$$

which is in  $\mathcal{C}^\infty(\mathbb{R}^3)$ , supported away from  $r = 0$ , hence certainly  $P^*\phi$  locally lies in  $\mathcal{H}_k^{-1}$ . Owing to our choice of  $A\lambda + k$ , we have moreover arranged that  $\phi \in L_{\text{loc}}^2$ . But  $\partial_r \phi$  is not in  $L_{\text{loc}}^2$  near  $r = 0$ . Thus, local finiteness of the RHS of the estimate (4) certainly cannot guarantee that  $u \in \mathcal{H}_{k,\text{loc}}^1$ . To see the global failure, we instead take a superposition of these examples. Take  $A > 0$  for simplicity of notation and construct

$$\phi(t, r, \varphi) = e^{ik\varphi} \int \zeta(\lambda) \chi(r) e^{i\lambda t} J_{A\lambda+k}(\lambda r) d\lambda,$$

where  $\zeta(\lambda)$  is a smooth function compactly supported in

$$-A^{-1}(3/4 + k) < \lambda < -A^{-1}(1/4 + k),$$

i.e., such that

$$\nu \equiv A\lambda + k \in (-3/4, -1/4) \text{ for } \lambda \in \text{supp } \zeta.$$

(We will further specify  $\zeta$  below.) Now again  $P^*\phi$  is compactly supported in  $r > 0$  but this time it is also Schwartz in  $t$ , hence  $P^*\phi \in L^2$ .

On the other hand, applying Taylor's theorem to the family of analytic functions of  $r$  given by

$$r^{-\nu} J_\nu(\lambda r), \quad \nu \in (-3/4, -1/4)$$

yields

$$J_\nu(\lambda r) = f_0(\nu) r^\nu + O(r)$$

with the remainder term estimated uniformly for  $\nu \in (-3/4, 1/4)$ ; here  $f_0(\nu)$  is a smooth (indeed, locally analytic) function of  $\nu = A\lambda + k$ , nonvanishing for  $\nu \notin \mathbb{Z}$  (and  $\lambda \neq 0$ ). Likewise

$$\partial_r J_\nu(\lambda r) = f_1(\nu) r^{\nu-1} + O(1)$$

for some other locally analytic function  $f_1(\nu)$  nonvanishing on  $\nu \in \mathbb{R} \setminus \mathbb{Z}$ ,  $\lambda \neq 0$ ; again we have uniform remainder bounds. Hence if we choose  $\zeta(\lambda)$  so that the product

$$\zeta(\lambda) f_1(A\lambda + k)$$

is a nonnegative cutoff function equal to 1 in  $-A^{-1}(2/3 + k) < \lambda < -A^{-1}(1/3 + k)$ , then applying Plancherel in  $\lambda$  shows that on the one hand,  $\phi \in L^2$ . On the other hand, it also yields

$$\|\partial_r \phi\|^2 \geq A^{-1} \int_0^{A/2} \int_{-2/3}^{-1/3} (r^{2(\nu-1)} + O(1)) d\nu r dr,$$

which diverges.

## 4. PROOF OF THE THEOREM

We now follow an approach similar to that of Payne [9] to show, using Duistermaat–Hörmander style microlocal energy estimates, that forward parametrices exist semiglobally. In particular, we now describe the crucial propagation estimate; from here our argument hews closely to [6, Theorem 6.3.1].

In what follows, the constant  $C$  will be allowed to change from line to line.

**Proposition 16.** *For  $\phi \in \mathcal{H}_k^1 \cap \mathcal{E}'(K')$ ,*

$$(5) \quad \|\phi\|_{\mathcal{H}_k^1} \leq C\|P^*\phi\|_{L^2} + C\|\phi\|_{L^2}.$$

*Proof.* Let  $\chi(r)$  be a cutoff function equal to 1 on  $r < A/8$  and supported in  $(-\infty, A/4)$ . Then applying Lemma 14 to  $\chi\phi$  gives

$$\|\chi\phi\|_{\mathcal{H}_k^1} \leq C\|\phi\|_{L^2} + C\|\chi P^*\phi\|_{\mathcal{H}_k^{-1}} + C\|[P^*, \chi]\phi\|_{\mathcal{H}_k^{-1}}.$$

Since  $[P^*, \chi]$  is an operator of order 1 with smooth coefficients, supported away from  $r = 0$  (hence  $\mathcal{H}_k^{-1}$  locally agrees with  $H^{-1}$ )

$$\|[P^*, \chi]\phi\|_{\mathcal{H}_k^{-1}} \lesssim \|\phi\|_{L^2}$$

and we conclude a fortiori (since  $L^2 \subset \mathcal{H}_k^{-1}$ ) that

$$\|\chi\phi\|_{\mathcal{H}_k^1} \leq C\|P^*\phi\|_{L^2} + C\|\phi\|_{L^2}.$$

It now suffices to additionally show that

$$(6) \quad \|(1 - \chi)\phi\|_{H^1} \leq C_1\|P^*\phi\|_{L^2} + C_2\|\phi\|_{L^2},$$

where we have switched to using the ordinary  $H^1$  Sobolev norm, since it agrees with the  $\mathcal{H}_k^1$  norm on the hyperbolic region. To show (6), let  $q_0 = (t_0, r_0, \lambda_0, \xi_0) \in \pi^{-1}(\text{supp}(1 - \chi)\phi) \subset S^*X$ . If  $q_0 \in \text{ell}(P^*)$ , then for  $A_{q_0} \in \Psi_c^0(X)$  microsupported sufficiently close to  $q_0$  (chosen, for use later, with nonnegative principal symbol)

$$\|A_{q_0}\phi\|_{H^1} \leq C\|\phi\|_{L^2} + C\|P^*\phi\|_{H^{-1}}$$

by standard elliptic estimates. (This estimate is stronger than needed, owing to the  $H^{-1}$  norm on the RHS, but the weaker estimate in the statement of the proposition will be as good as we can obtain on the hyperbolic set.) On the other hand, if  $q_0$  is in the characteristic set  $\Sigma$ , then either  $\lambda > 0$  or  $\lambda < 0$  along the whole null bicharacteristic  $\gamma$  of  $\text{Re } \sigma_2(P^*)$  through  $q_0$ , since  $\lambda = \lambda_0$  is conserved under the flow (owing to  $t$ -independence of the symbol), hence by our choice of the sign of  $\sigma_2(W)$  we have propagation of regularity *backwards in time* along null bicharacteristics of  $\text{Re } \sigma_2(P^*)$  by the results of [11, Section 2.5]. Since the flow eventually leaves  $K' \supset \text{supp } \phi$  as  $t \rightarrow +\infty$ ,

$$\|A_{q_0}\phi\|_{H^1} \leq C\|P^*\phi\|_{L^2} + C\|\phi\|_{L^2}$$

(cf. [11, Equation 2.18]). Piecing together these estimates for a finite cover of  $K' \cap \text{supp}(1 - \chi)$  by  $\text{ell } A_{q_1}, \dots, \text{ell } A_{q_N}$  and invoking elliptic regularity for  $\sum A_{q_j}$  yields the estimate (6).  $\square$

Now we claim that the second term on the RHS of (5) can be dropped if we restrict ourselves to a finite codimension subspace of  $\mathcal{H}_k^1 \cap \mathcal{E}'(K')$ : we let

$$N(P^*) = \{u \in \mathcal{H}_k^1 \cap \mathcal{E}'(K') : P^*u = 0\}.$$

The space  $N(P^*)$  is finite dimensional, since (5) implies that on this space  $\|\phi\|_{\mathcal{H}_k^1} \lesssim \|\phi\|_{L^2}$ , hence the unit ball of  $N(P^*)$  in the  $L^2$  topology is compact, by Lemma 4. Let  $N(P^*)^\perp$  denote the orthocomplement of this finite-dimensional space in  $L_k^2$ .

**Lemma 17.**

$$(7) \quad \|\phi\|_{\mathcal{H}_k^1} \leq C\|P^*\phi\|_{L^2}, \quad \phi \in \mathcal{H}_k^1 \cap \mathcal{E}'(K') \cap N(P^*)^\perp.$$

*Proof.* If (7) did not hold, there would exist  $\phi_j \in \mathcal{H}_k^1 \cap N(P^*)^\perp$  with support in  $K'$  such that

$$\|\phi_j\|_{\mathcal{H}_k^1} = 1, \quad \|P^*\phi_j\|_{L^2} \rightarrow 0.$$

Extracting a weakly convergent subsequence in  $\mathcal{H}_k^1$ , hence strongly convergent in  $L^2$  by Lemma 4, we get  $\phi_j \rightarrow \psi \in \mathcal{H}_k^1 \cap N(P^*)^\perp$  with convergence in the  $L^2$  sense. Recall that  $P^* : \mathcal{H}_k^1 \rightarrow \mathcal{H}_k^{-1}$  is continuous, so  $P^*\psi$  is the weak limit in  $\mathcal{H}_k^{-1}$  of  $P^*\phi_j$ . On the other hand  $P^*\phi_j \rightarrow 0$  in  $L^2$ , hence a fortiori in  $\mathcal{H}_k^{-1}$ , so in fact  $P^*\psi = 0$ . Consequently,  $\psi \in \mathcal{H}_k^1 \cap N(P^*)$ , i.e.  $\psi = 0$ . Thus  $\phi_j \rightarrow 0$  in  $L^2$ , and (5) for the sequence  $\phi_j$  reads

$$(8) \quad \|\phi_j\|_{\mathcal{H}_k^1} \leq C\|P^*\phi_j\|_{L^2} + C\|\phi_j\|_{L^2} \rightarrow 0,$$

contradicting the assumed normalization of the LHS.  $\square$

As a result of (7), if  $f \in N(P^*)^\perp$ , the map

$$T : P^*\phi \mapsto \langle \phi, f \rangle$$

is well-defined on the range of  $P^*$  on the test functions  $\mathcal{D}_k((K')^\circ)$  considered as a subset of  $L_k^2(K')$ ; (7) and the dual pairing of  $\mathcal{H}_k^1$  and  $\mathcal{H}_k^{-1}$  yields

$$|TP^*\phi| \lesssim \|P^*\phi\|_{L^2} \|f\|_{\mathcal{H}_k^{-1}}.$$

We now extend the map to the whole of  $L_k^2(K')$  by Hahn–Banach. The Riesz Lemma implies the existence of  $u \in L_k^2(K')$  with  $TP^*\phi = \langle P^*\phi, u \rangle$ , hence

$$\langle \phi, f \rangle = \langle \phi, Pu \rangle$$

for all test functions  $\phi$  supported on  $(K')^\circ \setminus \{r = 0\}$ . Hence  $u$  solves  $Pu = f$  on  $(K')^\circ$  (in the weak sense specified by Definition 5). Of course, we were restricted to  $f \in N(P^*)^\perp$  in making this construction. By Lemma 11, though, elements of  $N(P^*)$

are supported entirely in  $r \geq R$ , hence if we restrict to  $\text{supp } f \subset \{r < R_0\}$ , having chosen  $R > R_0$  ensures that  $f \in N(P^*)^\perp$  and the solvability result applies.

To see that  $u$  has the desired wavefront properties on  $K$ , we now bring to bear Lemma 12. Owing to our choice of the sign of  $W$ , recall that regularity propagates forward *in time* under bicharacteristic flow (away from  $\text{WF } f$ ). Ellipticity guarantees both  $\text{WF } u \subset \text{WF } f \cup \Sigma$  and also  $\text{WF } u \cap \{r > R + 1\} \cap \Sigma_- = \emptyset$ . Since every bicharacteristic passing through  $K$  reaches this latter set inside  $\pi^{-1}K'$  in backward time, no point may be in  $\text{WF } u \setminus \text{WF } f$  whose backward-in-time flow does not hit  $\text{WF } f$ . In other words,  $(\text{WF } u \setminus \text{WF } f) \cap \pi^{-1}K$  is contained in the *forward* flowout of  $\text{WF } f \cap \Sigma$ , as desired. Meanwhile,  $u$  does solve our original equation  $(\square_k + m^2)u = f$  on  $K$ , since  $W = 0$  on  $K$  (recall  $K \subset \{r < R\}$ ).

In case we have the increased regularity  $f \in L_k^2$ , we conclude that  $u \in H_{\text{loc}}^1$  by propagation of singularities in the hyperbolic region; on the elliptic set, we can even do better if desired ( $u \in H_{\text{loc}}^2$ ).

If  $u_1, u_2$  are both forward solutions to  $(\square_k + m^2)u_\bullet = f$ , then  $u_1 - u_2 \in \mathcal{C}^\infty$  microlocally on the elliptic set of  $P$ , by elliptic regularity. Elsewhere we obtain  $u_1 - u_2 \in \mathcal{C}^\infty$  by propagation of singularities, since  $\text{WF } u_\bullet \cap \{r > R + 1\} \cap \Sigma_- = \emptyset$  and  $(\square_k + m^2)(u_1 - u_2) = 0$ . The uniqueness assertion of the theorem follows.  $\square$

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