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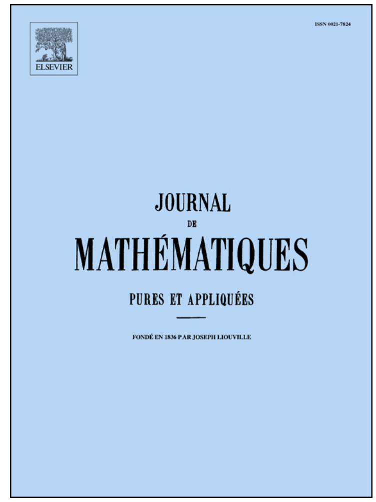

Gradient estimates for the insulated conductivity problem: the non-umbilical case

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GRADIENT ESTIMATES FOR THE INSULATED CONDUCTIVITY PROBLEM: THE NON-UMBILICAL CASE

HONGJIE DONG, YANYAN LI, AND ZHUOLUN YANG

ABSTRACT. We study the insulated conductivity problem with inclusions embedded in a bounded domain in $\mathbb{R}^n$, for $n \geq 3$. The gradient of solutions may blow up as ε, the distance between the inclusions, approaches to 0. We established in a recent paper optimal gradient estimates for a class of inclusions including balls. In this paper, we prove such gradient estimates for general strictly convex inclusions. Unlike the perfect conductivity problem, the estimates depend on the principal curvatures of the inclusions, and we show that these estimates are characterized by the first non-zero eigenvalue of a divergence form elliptic operator on $\mathbb{S}^{n-2}$.

1. INTRODUCTION AND MAIN RESULTS

In this paper, a continuation of [19], we establish gradient estimates for the insulated conductivity problem in the presence of multiple closely located inclusions in a bounded domain in $\mathbb{R}^n$, $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary containing two $C^{2,\gamma}$ $(0 < \gamma < 1)$ relatively strictly convex open sets D_1 and D_2 with $\operatorname{dist}(D_1 \cup D_2, \partial\Omega) > c > 0$. Denote $\widetilde{\Omega} := \Omega \setminus \overline{(D_1 \cup D_2)}$. The conductivity problem can be modeled by the following elliptic equation:

$$\begin{cases} \operatorname{div}\Big(a_k(x)\nabla u_k\Big) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

where a_k denotes the conductivity distribution, that is,

$$a_k = k\chi_{D_1 \cup D_2} + \chi_{\widetilde{\Omega}}.$$

Let

$$\varepsilon := \operatorname{dist}(D_1, D_2)$$

be small. When k is large or close to 0, the gradient of solutions may blow up, and it is significant to capture this singular behavior from an engineering point of view. The problem is motivated by the study of damage and fracture analysis of composite materials in the work of Babuška, Andersson, Smith, and Levin [6], where they studied the Lamé system and analyzed numerically that, when the ellipticity constants are bounded away from 0 and infinity, the gradient of solutions

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remains bounded independent of the distance between inclusions. Bonnetier and Vogelius [13] proved it in the context of conductivity problem when inclusions are two touching balls in $\mathbb{R}^2$. This result was extended by Li and Vogelius [34] to general second order elliptic equations of divergence form with piecewise Hölder coefficients and general shape of inclusions D_1 and D_2 in any dimension, and then by Li and Nirenberg [33] for general second order elliptic systems of divergence form, including the linear system of elasticity. Some higher order derivative estimates in dimension $n = 2$ were obtained in [18, 20, 24].

When k degenerates to ∞ (inclusions are perfect conductors) or 0 (insulators), it was shown in [14, 28, 38] that the gradient of solutions generally becomes unbounded as $\varepsilon \to 0$. For the perfect conductivity problem, it was known that

$$\begin{cases} \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 2, \\ \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C|\varepsilon \ln \varepsilon|^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 3, \\ \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n \ge 4, \end{cases}$$

see [4, 5, 7, 8, 40, 41]. These bounds were shown to be optimal and they are independent of the shape of inclusions, as long as the inclusions are relatively strictly convex. Moreover, many works have been done in characterizing the singular behavior of ∇u, which are significant in practical applications. For further works on the perfect conductivity problem and closely related works, see e.g. [1–3, 9–12, 16, 18, 20, 23, 26, 27, 29–32, 37] and the references therein.

For the insulated conductivity problem, it was proved in [8] that

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)} \quad \text{when } n \ge 2. \tag{1.1}$$

The upper bound is optimal for $n = 2$. Yun [42] studied the following free space insulated conductivity problem in $\mathbb{R}^3$: Let H be a harmonic function in $\mathbb{R}^3$, $D_1 = B_1\left(0, 0, 1 + \frac{\varepsilon}{2}\right)$, and $D_2 = B_1\left(0, 0, -1 - \frac{\varepsilon}{2}\right)$,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \ i = 1, 2, \\ u(x) - H(x) = O(|x|^{-2}) & \text{as } |x| \to \infty. \end{cases}$$

He proved that for some positive constant C independent of ε,

$$\max_{|x_3| \le \varepsilon/2} |\nabla u(0, 0, x_3)| \le C\varepsilon^{\frac{\sqrt{2}-2}{2}}.$$

He also showed that this upper bound of $|\nabla u|$ on the ε-segment connecting D_1 and D_2 is optimal for $H(x) \equiv x_1$. However, this result does not provide an upper bound of $|\nabla u|$ in the complement of the ε-segment. The upper bound (1.1) was improved by Li and Yang [36] to

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1/2+\beta}\|\varphi\|_{C^2(\partial\Omega)} \quad \text{when } n \ge 3,$$

for some $\beta > 0$. See [35] for flatter insulator case. When insulators are unit balls, a more explicit constant $\beta(n)$ was given by Weinkove in [39] for $n \ge 4$ by a different method. The constant $\beta(n)$ obtained in [39] presumably improves that in [36]. In particular, it was proved in [39] that $\beta(n)$ approaches $1/2$ from below as $n \to \infty$. Despite the significant progress on the conductivity problem that has been made in the past three decades or so, the optimal blow-up rate for the insulated conductivity problem in dimensions $n \geq 3$ remains unknown, and it is described as an outstanding open problem in [25].

In [19], we established optimal gradient estimates for a certain class of inclusions including two balls of any size in dimensions $n \geq 3$. In this paper, we study the insulated conductivity problem with C^γ coefficients in dimensions $n \geq 3$, with any $C^{2,\gamma}$ relatively strictly convex inclusions:

$$\begin{cases} -\partial_i(A^{ij}(x)\partial_j u(x)) = 0 & \text{in } \widetilde{\Omega}, \\ A^{ij}(x)\partial_j u(x)\nu_i = 0 & \text{on } \partial D_i,\ i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $0 < \gamma < 1$ and $(A^{ij}(x))$ satisfies, for some constants $\sigma > 0$,

$$(A^{ij}(x)) \in C^\gamma \text{ is symmetric}, \quad \sigma I \leq A(x) \leq \frac{1}{\sigma} I, \tag{1.3}$$

$\varphi \in C^2(\partial\Omega)$ is given, and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the inner normal vector on $\partial D_1 \cup \partial D_2$. We use the notation $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$. After choosing a coordinate system properly, we can assume that near the origin, the part of ∂D_1 and ∂D_2, denoted by Γ_+ and Γ_-, are respectively the graphs of two $C^{2,\gamma}$ functions in terms of x'. That is, for some $R_0 > 0$,

$$\Gamma_+ = \left\{ x_n = \frac{\varepsilon}{2} + f(x'),\ |x'| < R_0 \right\} \text{ and } \Gamma_- = \left\{ x_n = -\frac{\varepsilon}{2} + g(x'),\ |x'| < R_0 \right\}, \tag{1.4}$$

where f and g are $C^{2,\gamma}(0 < \gamma < 1)$ functions satisfying

$$f(x') > g(x') \quad \text{for } 0 < |x'| < R_0, \tag{1.5}$$

$$f(0') = g(0') = 0, \quad \nabla_{x'} f(0') = \nabla_{x'} g(0') = 0, \quad D^2(f - g)(0') > 0. \tag{1.6}$$

For $0 < r \leq R_0$, we denote

$$\Omega_r := \left\{ (x', x_n) \in \widetilde{\Omega} \;\Big|\; -\frac{\varepsilon}{2} + g(x') < x_n < \frac{\varepsilon}{2} + f(x'),\ |x'| < r \right\}. \tag{1.7}$$

We will focus on the following problem near the origin:

$$\begin{cases} -\partial_i(A^{ij}(x)\partial_j u(x)) = 0 & \text{in } \Omega_{R_0}, \\ A^{ij}(x)\partial_j u(x)\nu_i = 0 & \text{on } \Gamma_+ \cup \Gamma_-. \end{cases} \tag{1.8}$$

It was proved in [8] that for $u \in H^1(\Omega_{R_0})$ satisfying (1.8),

$$|\nabla u(x)| \leq C\|u\|_{L^\infty(\Omega_{R_0})}(\varepsilon + |x'|^2)^{-1/2}, \quad \forall x \in \Omega_{R_0/2}, \tag{1.9}$$

where C is a positive constant depending only on $n, R_0, \gamma, \sigma, \|A\|_{C^\gamma}, \|f\|_{C^{2,\gamma}}$, and $\|g\|_{C^{2,\gamma}}$, and is in particular independent of ε.

In this paper, we show that the optimal exponent of the gradient estimates of the insulated conductivity problem (1.8) is closely related to the following eigenvalue problem on $\mathbb{S}^{n-2}$. Consider

$$-\operatorname{div}_{\mathbb{S}^{n-2}}\left(a(\xi)\nabla_{\mathbb{S}^{n-2}} u(\xi) \right) = \lambda a(\xi) u(\xi), \quad \xi \in \mathbb{S}^{n-2}, \tag{1.10}$$

where $a(\xi)$ is a positive function on $\mathbb{S}^{n-2}$ with $\ln a \in L^\infty(\mathbb{S}^{n-2})$. Denote

$$\langle u, v \rangle_{\mathbb{S}^{n-2}} = \fint_{\mathbb{S}^{n-2}} a(\xi) uv \, d\sigma. \tag{1.11}$$

From the classical theory, all eigenvalues are real, and the corresponding eigenfunctions can be normalized to form an orthonormal basis of $L^2(\mathbb{S}^{n-2})$ under the inner-product defined above. The first nonzero eigenvalue λ_1 of this problem is given by the Rayleigh quotient:

$$\lambda_1 = \inf_{u\not\equiv 0, \langle u,1\rangle_{\mathbb{S}^{n-2}}=0} \frac{\fint_{\mathbb{S}^{n-2}} a(\xi)|\nabla_{\mathbb{S}^{n-2}} u|^2\, d\sigma}{\fint_{\mathbb{S}^{n-2}} a(\xi)|u|^2\, d\sigma}. \tag{1.12}$$

Let $\alpha(\lambda_1)$ be the positive root of the quadratic polynomial $\alpha^2 + (n-1)\alpha - \lambda_1$, that is,

$$\alpha(\lambda_1) = \frac{-(n-1) + \sqrt{(n-1)^2 + 4\lambda_1}}{2}. \tag{1.13}$$

First, we consider the case when $A^{ij}(0) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. When two inclusions touch, namely, $\varepsilon = 0$, we prove the following gradient estimates.

Theorem 1.1. *For $n \geq 3$, $R_0 > 0$, and $\varepsilon = 0$, let $f, g \in C^{2,\gamma}(0 < \gamma < 1)$ satisfy (1.5) and (1.6), $(A^{ij}(x))$ satisfy (1.3) with $\sigma > 0$ in Ω_{R_0}, and $A^{ij}(0) = \delta_{ij}$. For any solution $u \in H^1(\Omega_{R_0})$ of (1.8), we have*

$$|\nabla u(x)| \leq C\|u\|_{L^\infty(\Omega_{R_0})}|x'|^{\alpha(\lambda_1)-1} \quad \forall x \in \Omega_{R_0/2} \setminus \{0\}, \tag{1.14}$$

where λ_1 and $\alpha(\lambda_1)$ are given by (1.12) and (1.13) with $a(\xi) = \xi^t \left(D^2(f-g)(0')\right)\xi$, and C is a positive constant depending only on n, R_0, γ, σ, a positive lower bound of the eigenvalues of $D^2(f-g)(0')$, and upper bounds of $\|A\|_{C^\gamma}$, $\|f\|_{C^{2,\gamma}}$, and $\|g\|_{C^{2,\gamma}}$.

Remark 1.2. When $a(\xi) > 0$ a.e. satisfies $\ln a \in L^\infty(\mathbb{S}^{n-2})$ and $\int_{\mathbb{S}^{n-2}} a x_i = 0$ for all $i = 1, \ldots, n-1$, it will be shown that $\lambda_1 \leq n-2$ (see Lemma 5.1), and hence $\alpha(\lambda_1) \in (0, 1)$.

When $\varepsilon > 0$, the following gradient estimate is proved.

Theorem 1.3. *For $n \geq 3$, $R_0 > 0$, and $\varepsilon \in (0, 1/4)$, let $(A^{ij}(x))$, f, g, λ_1, and $\alpha(\lambda_1)$ be the same as in Theorem 1.1. For any solution $u \in H^1(\Omega_{R_0})$ of (1.8), we have, for any $0 \leq \alpha < \alpha(\lambda_1)$,*

$$|\nabla u(x)| \leq C\|u\|_{L^\infty(\Omega_{R_0})}(\varepsilon + |x'|^2)^{\frac{\alpha-1}{2}} \quad \forall x \in \Omega_{R_0/2},$$

where C is a positive constant depending only on n, R_0, γ, σ, a positive lower bound of $\alpha(\lambda_1) - \alpha$, a positive lower bound of the eigenvalues of $D^2(f-g)(0')$, and upper bounds of $\|A\|_{C^\gamma}$, $\|f\|_{C^{2,\gamma}}$, and $\|g\|_{C^{2,\gamma}}$.

We show that the estimate (1.14) is optimal in the following sense. Note that in the next three theorems, ∂D_1 and ∂D_2 near the origin are represented by the graphs of f and g respectively.

Theorem 1.4. *For $n = 3$, $A^{ij}(x) \equiv \delta_{ij}$, $\varepsilon = 0$, and for any positive definite matrix M, there exist smooth strictly convex inclusions D_1, D_2 inside $\Omega = B_5$ with $D^2(f-g)(0') = M$, and a boundary data $\varphi \in C^\infty(\partial\Omega)$ with $\|\varphi\|_{L^\infty(\partial\Omega)} = 1$, such that the solution $u \in H^1(\widetilde{\Omega})$ of (1.2) satisfies*

$$\limsup_{x\in\widetilde{\Omega}, |x|\to 0} |x'|^{1-\alpha(\lambda_1)}|\nabla u(x)| > \frac{1}{C},$$

where λ_1 *and* $\alpha(\lambda_1)$ *are given by* (1.12) *and* (1.13) *with* $a(\xi) = \xi^t M\xi$*, and* C *is some positive constant depending only on the positive lower bounds of the eigenvalues of* M*, and upper bounds of* $\|\partial D_1\|_{C^4}$ *and* $\|\partial D_2\|_{C^4}$*.*

Theorem 1.5. *For* $n \geq 4$, $A^{ij}(x) \equiv \delta_{ij}$*, and* $\varepsilon = 0$*, there exists an* $\varepsilon_0 = \varepsilon_0(n) \in (0, 1/2)$ *such that for any positive definite matrix* M *satisfying*

$$(1-\varepsilon_0)\frac{I}{\|I\|} \leq \frac{M}{\|M\|} \leq (1+\varepsilon_0)\frac{I}{\|I\|},$$

there exist smooth strictly convex inclusions D_1, D_2 *inside* $\Omega = B_5$ *with* $D^2(f-g)(0') = M$*, and a boundary data* $\varphi \in C^\infty(\partial\Omega)$ *with* $\|\varphi\|_{L^\infty(\partial\Omega)} = 1$*, such that the solution* $u \in H^1(\widetilde{\Omega})$ *of* (1.2) *satisfies*

$$\limsup_{x\in\widetilde{\Omega}, |x|\to 0} |x'|^{1-\alpha(\lambda_1)}|\nabla u(x)| > \frac{1}{C},$$

where λ_1 *and* $\alpha(\lambda_1)$ *are given by* (1.12) *and* (1.13) *with* $a(\xi) = \xi^t M\xi$*, and* C *is a positive constant depending only on* n*,* $\|M\|$*, and upper bounds of* $\|\partial D_1\|_{C^4}$ *and* $\|\partial D_2\|_{C^4}$*.*

In the above, $\|M\|$ and $\|I\|$ denote the standard norm of the matrices. Theorems 1.4 and 1.5 are consequences of the following more general theorem.

Let D_1, D_2 be two strictly convex smooth domains in $B_4 \subset \mathbb{R}^n$, which are symmetric in x_j for each $1 \leq j \leq n-1$, and $\overline{D}_1 \cap \overline{D}_2 = \{0\}$. Let $\Omega = B_5$ and $\widetilde{\Omega} = \Omega \setminus \{\overline{D_1 \cup D_2}\}$.

Theorem 1.6. *For* $n \geq 3$*, let* D_1*,* D_2*, and* Ω *be as above,* $A^{ij}(x) \equiv \delta_{ij}$*,* λ_1 *and* $\alpha(\lambda_1)$ *be given by* (1.12) *and* (1.13) *with* $a(\xi) = \xi^t\left(D^2(f-g)(0')\right)\xi$*. Assume that the eigenspace corresponding to* λ_1 *contains a function which is odd in* x_j *for some* $1 \leq j \leq n-1$*. Let* $\varphi = x_j$ *and* $u \in H^1(\widetilde{\Omega})$ *be the solution of* (1.2)*. Then*

$$\limsup_{x\in\widetilde{\Omega}, |x|\to 0} |x'|^{1-\alpha(\lambda_1)}|\nabla u(x)| > \frac{1}{C},$$

where C *is some positive constant depending only on* n*, a positive lower bound of the eigenvalues of* $D^2(f-g)(0')$*, and upper bounds of* $\|\partial D_1\|_{C^4}$ *and* $\|\partial D_2\|_{C^4}$*.*

We will show in Section 5 (see Theorem 5.2 and Corollary 5.7) that the conditions in Theorems 1.4 and 1.5 imply the condition in Theorem 1.6.

The rest of this paper is organized as follows. In Section 2, we establish some estimates for the associated degenerate elliptic operator

$$L_\varepsilon := \operatorname{div}\left[\left(\varepsilon + a\Big(\frac{x'}{|x'|}\Big)|x'|^2\right)\nabla\right],$$

which play an important role in proving Theorems 1.1 and 1.3. Theorems 1.1 and 1.3 are proved in Sections 3 and 4, respectively. Some properties of λ_1, the first nonzero eigenvalue of (1.10), and its corresponding eigenspace, are established in Section 5. Theorem 1.6 is proved in Section 6, and therefore, Theorems 1.4 and 1.5 follow. Finally in Section 7, we discuss the case when $A_{ij}(0) \neq \delta_{ij}$, and give a reduction to the case when $A_{ij}(0) = \delta_{ij}$.

2. Some estimates on the associated degenerate elliptic operator

In this section, we establish some estimates that are useful in proving Theorems 1.1 and 1.3. Throughout the section, we work in the domain $B_{R_0} \subset \mathbb{R}^{n-1}$ for some $R_0 > 0$ and $n \geq 3$. Let a be a function on $\mathbb{S}^{n-2}$ satisfying

$$a > 0 \text{ a.e.} \quad \text{and} \quad \ln a \in L^\infty(\mathbb{S}^{n-2}), \tag{2.1}$$

and let λ_1 and $\alpha(\lambda_1)$ be given by (1.12) and (1.13).

Here are some notation we will be using throughout this paper: For $\sigma, s \in \mathbb{R}$, we introduce the following norm

$$\|F\|_{\varepsilon,\sigma,s,B_{R_0}} := \sup_{x' \in B_{R_0}} \frac{|F(x')|}{|x'|^\sigma (\varepsilon + |x'|^2)^{1-s}}. \tag{2.2}$$

For any bounded set $\Omega \subset \mathbb{R}^{n-1}$, we denote $H^1(\Omega, |x'|^2 dx')$ to be the following weighted H^1 norm:

$$\|f\|_{H^1(\Omega,|x'|^2dx')} = \Big(\int_\Omega |f|^2|x'|^2\,dx'\Big)^{\frac12} + \Big(\int_\Omega |\nabla f|^2|x'|^2\,dx'\Big)^{\frac12}.$$

For any $0 < \rho < R_0$, we denote

$$(u)^a_{\partial B_\rho} := \Big(\int_{\partial B_\rho} a\Big(\frac{x'}{|x'|}\Big)\,d\sigma\Big)^{-1} \int_{\partial B_\rho} a\Big(\frac{x'}{|x'|}\Big) u(x')\,d\sigma,$$

$$(u)^a_{B_\rho} := \Big(\int_{B_\rho} a\Big(\frac{x'}{|x'|}\Big)\,dx'\Big)^{-1} \int_{B_\rho} a\Big(\frac{x'}{|x'|}\Big) u(x')\,dx'.$$

Proposition 2.1. *For $n \geq 3$, let a satisfy* (2.1)*, λ_1 and $\alpha(\lambda_1)$ be given by* (1.12) *and* (1.13)*. For $\sigma > 1$, $\sigma - 1 \neq \alpha(\lambda_1)$, let $\bar u \in H^1(B_{R_0}, |x'|^2 dx')$ be a solution of*

$$\operatorname{div}\Big[a\Big(\frac{x'}{|x'|}\Big)|x'|^2\nabla \bar u\Big] = \operatorname{div} F + G \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1},$$

where $F, G \in L^\infty(B_{R_0})$ satisfy

$$\|F\|_{\varepsilon,\sigma,1,B_{R_0}} < \infty, \quad \|G\|_{\varepsilon,\sigma-1,1,B_{R_0}} < \infty. \tag{2.3}$$

Then $\bar u \in C^\beta(B_{R_0})$ for some $\beta \in (0,1)$. Moreover, for any $|x'| \leq R_0/2$, we have

$$\begin{aligned}&|\bar u(x') - \bar u(0)| \\ &\leq C(\|F\|_{\varepsilon,\sigma,1,B_{R_0}} + \|G\|_{\varepsilon,\sigma-1,1,B_{R_0}} + \|\bar u - \bar u(0)\|_{L^2(\partial B_{R_0})})|x'|^{\tilde\alpha}\end{aligned} \tag{2.4}$$

where $\tilde\alpha := \min\{\alpha(\lambda_1), \sigma - 1\}$, and C is some positive constant depending only on n, σ, R_0, an upper bound of $\|\ln a\|_{L^\infty}$, and is independent of ε.

For the proof, we use an iteration argument based on the following two lemmas.

Lemma 2.2. *For $n \geq 3$, let a satisfy* (2.1)*, λ_1 and $\alpha(\lambda_1)$ be given by* (1.12) *and* (1.13)*, and $v_1 \in H^1(B_{R_0}, |x'|^2 dx')$ satisfy*

$$\operatorname{div}\Big[a\Big(\frac{x'}{|x'|}\Big)|x'|^2\nabla v_1\Big] = 0 \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}. \tag{2.5}$$

Then $v_1 \in C^\beta(B_{R_0})$, for some $\beta > 0$ depending only on n and $\|\ln a\|_{L^\infty}$. Moreover, for any $0 < \rho < R \leq R_0$, we have

$$v_1(0) = (v_1)^a_{\partial B_\rho},$$

$$\left(\fint_{\partial B_\rho} a\left(\frac{x'}{|x'|}\right)|v_1(x')-v_1(0)|^2\,d\sigma\right)^{\frac{1}{2}}$$
$$\leq\left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)}\left(\fint_{\partial B_R} a\left(\frac{x'}{|x'|}\right)|v_1(x')-v_1(0)|^2\,d\sigma\right)^{\frac{1}{2}}, \tag{2.6}$$

and for any $x'\in B_{R_0/2}$,

$$|v_1(x')-v_1(0)|\leq CR_0^{-\alpha(\lambda_1)-\frac{n-1}{2}}\|v_1-v_1(0)\|_{L^2(\partial B_{R_0})}|x'|^{\alpha(\lambda_1)}, \tag{2.7}$$

where C *is some positive constant depending only on* n *and* $\|\ln a\|_{L^\infty}$.

Proof. By [21, Theorem 2.3.12], $v_1\in C^\beta(B_{R_0})$ for some $\beta>0$. It should be noted that when $n=3$, the weight $|x'|^2$ does not satisfy the A_2 condition (in $\mathbb{R}^{n-1}$) required in [21, Theorem 2.3.12]. Nevertheless, it satisfies the conditions in [21, Section 3, pp. 106]. Therefore, the Hölder estimate still holds. Without loss of generality, it suffices to prove (2.6) and (2.7) for $a\in C^\infty(\mathbb{S}^{n-2})$ and $R=R_0=1$. In the polar coordinates, we write $x'=(r,\xi)$ with $0<r<1,\xi\in\mathbb{S}^{n-2}$. Let $\varphi(r)\in C_0^\infty((0,1))$ and $\psi(\xi)\in C^\infty(\mathbb{S}^{n-2})$. Multiplying (2.5) by $\varphi\psi$ and integrating by parts gives

$$\int_{B_1} a(\xi)r^2\nabla v_1\cdot\nabla(\varphi\psi)=\int_0^1\int_{\mathbb{S}^{n-2}} ar^n\partial_r v_1\varphi'\psi+ar^{n-2}\nabla_{\mathbb{S}^{n-2}}v_1\cdot\nabla_{\mathbb{S}^{n-2}}\psi\varphi\,d\xi dr$$
$$=-\int_{B_1}[ar^2\partial_{rr}v_1+nar\partial_r v+\mathrm{div}_{\mathbb{S}^{n-2}}(a\nabla_{\mathbb{S}^{n-2}}v_1)]\varphi\psi.$$

Therefore, we can write (2.5) in polar coordinates as

$$\partial_{rr}v_1+\frac{n}{r}\partial_r v_1+\frac{1}{a(\xi)r^2}\mathrm{div}_{\mathbb{S}^{n-2}}\Big(a(\xi)\nabla_{\mathbb{S}^{n-2}}v_1\Big)=0\quad\text{in } B_1\setminus\{0\}, \tag{2.8}$$

Let $\lambda_0=0$, $\{\lambda_k\}_{k=1}^\infty$ be the set of all positive eigenvalues of (1.10) satisfying $\lambda_k<\lambda_{k+1}$ for all $k\in\mathbb{N}\cup\{0\}$. Let Y_0 be the positive constant satisfying $\langle Y_0,Y_0\rangle_{\mathbb{S}^{n-2}}=1$, $Y_{k,i}$ be an eigenfunction corresponding to λ_k, that is,

$$\mathrm{div}_{\mathbb{S}^{n-2}}\Big(a(\xi)\nabla_{\mathbb{S}^{n-2}}Y_{k,i}\Big)=-\lambda_k a(\xi)Y_{k,i},$$

and $\{Y_{k,i}\}_{k,i}\cup\{Y_0\}$ forms an orthonormal basis of $L^2(\mathbb{S}^{n-2})$ with respect to the inner product (1.11).

For $0<r<1$, take the decomposition

$$v_1(r,\xi)=V_0(r)Y_0+\sum_{k=1}^\infty\sum_{i=1}^{N(k)}V_{k,i}(r)Y_{k,i}(\xi)\quad\text{in } L^2(\mathbb{S}^{n-2}), \tag{2.9}$$

where $V_0(r),V_{k,i}(r)\in C^2(0,1)$ are given by

$$V_0(r)=\fint_{\mathbb{S}^{n-2}}a(\xi)v_1(r,\xi)Y_0\,d\xi,\quad V_{k,i}(r)=\fint_{\mathbb{S}^{n-2}}a(\xi)v_1(r,\xi)Y_{k,i}(\xi)\,d\xi.$$

Multiplying (2.8) by $a(\xi)Y_0$ and $a(\xi)Y_{k,i}(\xi)$ respectively and integrate over $\mathbb{S}^{n-2}$, we see that $V_0(r)$ and $V_{k,i}(r)$ satisfy

$$V_0''+\frac{n}{r}V_0'=0\quad\text{and}\quad V_{k,i}''+\frac{n}{r}V_{k,i}'-\frac{\lambda_k}{r^2}V_{k,i}=0\quad\text{in } (0,1).$$

Therefore $V_0 = c_1 + c_2 r^{1-n}$ and $V_{k,i} = c_3 r^{\alpha(\lambda_k)_+} + c_4 r^{\alpha(\lambda_k)_-}$ for some constants c_1, c_2, c_3, and c_4, where

$$\alpha(\lambda_k)_\pm := \frac{-(n-1) \pm \sqrt{(n-1)^2 + 4\lambda_k}}{2}.$$

Since $v_1 \in H^1(B_1, r^2 dx')$, we have for any $0 < \delta < 1$,

$$\begin{aligned}\infty > \int_{B_1 \setminus B_\delta} a(\xi) v_1^2 r^2 \, dx' &\geq \frac{1}{C} \int_{B_1 \setminus B_\delta} a(\xi) V_0(r)^2 r^2 \, dx' \\ &\geq \frac{1}{C} \int_\delta^1 |c_1 + c_2 r^{1-n}|^2 r^{n+1} \, dr,\end{aligned}$$

which implies $c_2 \equiv 0$. Hence $V_0(r) \equiv V_0(1)$. Similarly, we have $c_4 \equiv 0$, and hence

$$V_{k,i}(r) = r^{\alpha(\lambda_k)_+} V_{k,i}(1).$$

By (2.9), for any $0 < \rho < 1$,

$$\begin{aligned}\fint_{\partial B_\rho} a\left(\frac{x'}{|x'|}\right) |v_1(x') - V_0 Y_0|^2 \, d\sigma &= \sum_{k=1}^\infty \sum_{i=1}^{N(k)} |V_{k,i}(\rho)|^2 \\ &\leq \rho^{2\alpha(\lambda_1)} \sum_{k=1}^\infty \sum_{i=1}^{N(k)} |V_{k,i}(1)|^2 \\ &= \rho^{2\alpha(\lambda_1)} \fint_{\partial B_1} a\left(\frac{x'}{|x'|}\right) |v_1(x') - V_0 Y_0|^2 \, d\sigma.\end{aligned}$$

Therefore, $v_1(0) = V_0(0) Y_0 = V_0(\rho) Y_0 = (v_1)^a_{\partial B_\rho}$ for any $\rho \in (0,1)$, and

$$\left(\fint_{\partial B_\rho} |v_1(x') - v_1(0)|^2 \, d\sigma\right)^{1/2} \leq C\rho^{\alpha(\lambda_1)} \left(\fint_{\partial B_1} |v_1(x') - v_1(0)|^2 \, d\sigma\right)^{1/2},$$

which implies (2.7) by the interior elliptic estimate applied to $B_\rho \setminus \overline{B}_{\rho/2}$. □

Lemma 2.3. *For $n \geq 3$ and $\sigma > 1$. Let a satisfy (2.1), λ_1 and $\alpha(\lambda_1)$ be given by (1.12) and (1.13), and $v_2 \in H_0^1(B_{R_0}, |x'|^2 dx')$ satisfy*

$$\operatorname{div}\left[a\Big(\frac{x'}{|x'|}\Big)|x'|^2 \nabla v_2\right] = \operatorname{div} F + G \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}, \tag{2.10}$$

where $F, G \in L^\infty(B_{R_0})$ satisfy (2.3). Then we have

$$\|v_2\|_{L_\infty(B_{R_0})} \leq C(\|F\|_{\varepsilon,\sigma,1,B_{R_0}} + \|G\|_{\varepsilon,\sigma-1,1,B_{R_0}}),$$

where $C > 0$ depends only on n, σ, and an upper bound of $\|\ln a\|_{L^\infty}$ and is in particular independent of ε.

Proof. Without loss of generality, we assume $R_0 = 1$ and

$$\|F\|_{\varepsilon,\sigma,1,B_1} + \|G\|_{\varepsilon,\sigma-1,1,B_1} = 1.$$

For $p \geq 2$, we multiply the equation (2.10) with $-|v_2|^{p-2}v_2$ and integrate by parts to obtain

$$(p-1)\int_{B_1} a\Big(\frac{x'}{|x'|}\Big)|x'|^2|\nabla v_2|^2|v_2|^{p-2}\,dx' \\ = (p-1)\int_{B_1} F\cdot\nabla v_2|v_2|^{p-2}\,dx' - \int_{B_1} Gv_2|v_2|^{p-2}\,dx'.$$

By the definition in (2.2),

$$|F(x')| \leq |x'|^{\sigma}\|F\|_{\varepsilon,\sigma,1,B_1} \quad \text{and} \quad |G(x')| \leq |x'|^{\sigma-1}\|G\|_{\varepsilon,\sigma-1,1,B_1} \quad \text{for } x' \in B_1.$$

Therefore, by Young's inequality, Hölder's inequality, and using $\sigma > 1$,

$$(p-1)\left|\int_{B_1} F\cdot\nabla v_2|v_2|^{p-2}\,dx'\right| \\ \leq \frac{(p-1)\delta}{2}\int_{B_1}|x'|^2|\nabla v_2|^2|v_2|^{p-2}\,dx' + C(p-1)\int_{B_1}|x'|^{2\sigma-2}|v_2|^{p-2}\,dx' \\ \leq \frac{(p-1)\delta}{2}\int_{B_1}|x'|^2|\nabla v_2|^2|v_2|^{p-2}\,dx' + \\ + C(p-1)\left(\int_{B_1}|x'|^{(\sigma-1)(n+1+2\mu)-(n-1+2\mu)}\,dx'\right)^{\frac{2}{n+1+2\mu}} \times \\ \times\left(\int_{B_1}|x'|^2|v_2|^{(p-2)\frac{n+1+2\mu}{n-1+2\mu}}\,dx'\right)^{\frac{n-1+2\mu}{n+1+2\mu}},$$

and

$$\left|\int_{B_1} Gv_2|v_2|^{p-2}\,dx'\right| \leq C\left(\int_{B_1}|x'|^{(\sigma-1)(n+1+2\mu)/2-(n-1+2\mu)}\,dx'\right)^{\frac{2}{n+1+2\mu}} \times \\ \times\left(\int_{B_1}|x'|^2|v_2|^{(p-1)\frac{n+1+2\mu}{n-1+2\mu}}\,dx'\right)^{\frac{n-1+2\mu}{n+1+2\mu}},$$

where $\mu > 0$ is chosen sufficiently small so that

$$\int_{B_1}|x'|^{(\sigma-1)(n+1+2\mu)/2-(n-1+2\mu)}\,dx' < \infty.$$

Hence,

$$\frac{4(p-1)}{p^2}\int_{B_1}|x'|^2\left|\nabla|v_2|^{\frac{p}{2}}\right|^2\,dx' = (p-1)\int_{B_1}|x'|^2|\nabla v_2|^2|v_2|^{p-2}\,dx' \\ \leq C(p-1)\|v_2^{p-2}\|_{L^{\frac{n+1+2\mu}{n-1+2\mu}}(B_1,|x'|^2dx')} + C\|v_2^{p-1}\|_{L^{\frac{n+1+2\mu}{n-1+2\mu}}(B_1,|x'|^2dx')}. \tag{2.11}$$

We use the following version of the Caffarelli-Kohn-Nirenberg inequality (see [15]):

$$\|u\|_{L^{\frac{2(n+1)}{n-1}}(B_1,|x'|^2dx')} \leq C\|\nabla u\|_{L^2(B_1,|x'|^2dx')} \quad \forall u \in H_0^1(B_1,|x'|^2dx'). \tag{2.12}$$

Taking $p = 2$ in (2.11), we have, by (2.12) with $u = |v_2|$ and Hölder's inequality,

$$\|v_2\|_{L^{\frac{2(n+1+2\mu)}{n-1+2\mu}}(B_1,|x'|^2dx')} \leq C. \tag{2.13}$$

For $p \geq 2$, from (2.11), by (2.12) with $u = |v_2|^{\frac{p}{2}}$ and Hölder's inequality,

$$\begin{aligned}
&\|v_2\|^p_{L^{\frac{(n+1)p}{n-1}}(B_1,|x'|^2dx')} \leq C\|\nabla|v_2|^{\frac{p}{2}}\|^2_{L^2(B_1,|x'|^2dx')} \\
&\leq Cp^2\|v_2\|^{p-2}_{L^{\frac{n+1+2\mu}{n-1+2\mu}(p-2)}(B_1,|x'|^2dx')} + Cp\|v_2\|^{p-1}_{L^{\frac{n+1+2\mu}{n-1+2\mu}(p-1)}(B_1,|x'|^2dx')} \\
&\leq \max_{i=1,2} Cp^i\|v_2\|^{p-i}_{L^{\frac{n+1+2\mu}{n-1+2\mu}p}(B_1,|x'|^2dx')}.
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}
\|v_2\|_{L^{\frac{(n+1)p}{n-1}}(B_1,|x'|^2dx')} &\leq \max_{i=1,2}(Cp^i)^{1/p}\left(\frac{p-i}{p}\|v_2\|_{L^{\frac{n+1+2\mu}{n-1+2\mu}p}(B_1,|x'|^2dx')} + \frac{i}{p}\right) \\
&\leq (Cp^2)^{1/p}\left(\|v_2\|_{L^{\frac{n+1+2\mu}{n-1+2\mu}p}(B_1,|x'|^2dx')} + \frac{2}{p}\right).
\end{aligned}$$

For $k \geq 0$, let

$$p_k = 2\left(\frac{n+1}{n-1}\cdot\frac{n-1+2\mu}{n+1+2\mu}\right)^k \frac{n+1+2\mu}{n-1+2\mu}.$$

Iterating the relations above, we have, by (2.13),

$$\begin{aligned}
\|v_2\|_{L^{p_k}(B_1,|x'|^2dx')} &\leq \prod_{i=0}^{k-1}\left(Cp_i^2\right)^{2/p_i}\|v_2\|_{L^{p_0}(B_1,|x'|^2dx')} \\
&\quad + \sum_{i=0}^{k-1}\prod_{j=i}^{k-1}\left(Cp_j^2\right)^{2/p_j}\frac{4}{p_i} \\
&\leq C\|v_2\|_{L^{\frac{2(n-1+2\mu)}{n-3+2\mu}}(B_1,|x'|^2dx')} + C\sum_{i=0}^{k-1}\frac{1}{p_i} \leq C, \qquad (2.14)
\end{aligned}$$

where C is a positive constant depending on n and σ, and is in particular independent of k. The lemma is concluded by taking $k \to \infty$ in (2.14). □

Now we are in a position to prove Proposition 2.1.

Proof of Proposition 2.1. We first show the (Hölder) continuity of $\bar{u}$. By Lemma 2.2, v_1 is locally Hölder continuous. In particular, it satisfies the estimate in [21, Lemma 2.3.11]. Now for F and G such that

$$Fr^{-2} \in L^p(B_{R_0}, r^2\,dx') \quad \text{and} \quad Gr^{-2} \in L^{p/2}(B_{R_0}, r^2\,dx')$$

with some $p > n+2$, which are satisfied by the condition of the proposition, we can apply the Moser iteration to get an L^∞ estimate of v_2 as in [21, Lemma 2.3.14]. By combining these two estimates and using the standard iteration argument, we get the local Hölder continuity of $\bar{u}$ with a small exponent. The proof of (2.4) below essentially follows this scheme, using the more precise L^2 oscillation estimate obtained in Lemma 2.2.

Without loss of generality, we assume that $\bar{u}(0) = 0$ and

$$\|F\|_{\varepsilon,\sigma,1,B_{R_0}} + \|G\|_{\varepsilon,\sigma-1,1,B_{R_0}} + \|\bar{u}\|_{L^2(\partial B_{R_0})} = 1.$$

Consider

$$\omega(\rho) := \left(\fint_{\partial B_\rho} a\Big(\frac{x'}{|x'|}\Big)|\bar{u}(x')|^2\,d\sigma\right)^{1/2}.$$

For $0 < \rho \le R/2 \le R_0/2$, we write $\bar{u} = v_1 + v_2$ in B_R, where v_2 satisfies

$$\operatorname{div}\Big[a\Big(\frac{x'}{|x'|}\Big)|x'|^2\nabla v_2\Big] = \operatorname{div} F + G \quad \text{in } B_R$$

and $v_2 = 0$ on ∂B_R. Thus v_1 satisfies

$$\operatorname{div}\Big[a\Big(\frac{x'}{|x'|}\Big)|x'|^2\nabla v_1\Big] = 0 \quad \text{in } B_R$$

and $v_1 = \bar{u}$ on ∂B_R. By Lemma 2.2,

$$\begin{aligned}&\left(\fint_{\partial B_\rho} a\left(\frac{x'}{|x'|}\right)|v_1(x') - v_1(0)|^2\, d\sigma\right)^{\frac12}\\ &\le \left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)}\left(\fint_{\partial B_R} a\left(\frac{x'}{|x'|}\right)|v_1(x') - v_1(0)|^2\, d\sigma\right)^{\frac12}.\end{aligned} \tag{2.15}$$

Since $\tilde{v}_2(x') := v_2(Rx')$ satisfies

$$\operatorname{div}\Big[a\Big(\frac{x'}{|x'|}\Big)|x'|^2\nabla \tilde{v}_2\Big] = \operatorname{div} \tilde{F} + \tilde{G} \quad \text{in } B_1,$$

where $\tilde{F}(x') := R^{-1}F(Rx')$ and $\tilde{G}(x') := G(Rx')$ satisfy

$$\begin{aligned}\|\tilde{F}\|_{R^{-2}\varepsilon,\sigma,1,B_1} &= R^{\sigma-1}\|F\|_{\varepsilon,\sigma,1,B_R},\\ \|\tilde{G}\|_{R^{-2}\varepsilon,\sigma-1,1,B_1} &= R^{\sigma-1}\|G\|_{\varepsilon,\sigma-1,1,B_R},\end{aligned}$$

we apply Lemma 2.3 with $R_0 = 1$ to $\tilde{v}_2$ to obtain

$$\|v_2\|_{L_\infty(B_R)} \le CR^{\sigma-1}. \tag{2.16}$$

Since $\bar{u}(0) = v_1(0) + v_2(0) = 0$, we have $|v_1(0)| = |v_2(0)|$. Combining (2.15) and (2.16) yields, using $\bar{u} = v_1 + v_2$, and $\bar{u} = v_1$ on ∂B_R,

$$\begin{aligned}&\omega(\rho)\\ &\le \left(\fint_{\partial B_\rho} a\left(\frac{x'}{|x'|}\right)|v_1(x') - v_1(0)|^2\, d\sigma\right)^{\frac12} + \left(\fint_{\partial B_\rho} a\left(\frac{x'}{|x'|}\right)|v_2(x') - v_2(0)|^2\, d\sigma\right)^{\frac12}\\ &\le \left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)}\left(\fint_{\partial B_R} a\left(\frac{x'}{|x'|}\right)|v_1(x')|^2\, d\sigma\right)^{\frac12} + C|v_1(0)| + C\|v_2\|_{L_\infty(B_R)}\\ &\le \left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)}\omega(R) + CR^{\sigma-1}.\end{aligned} \tag{2.17}$$

For a positive integer k, we take $\rho = 2^{-i-1}R_0$ and $R = 2^{-i}R_0$ in (2.17) and iterate from $i = 0$ to $k-1$. We have, using $\sigma - 1 \ne \alpha(\lambda_1)$,

$$\begin{aligned}\omega(2^{-k}R_0) &\le 2^{-k\alpha(\lambda_1)}\omega(R_0) + C\sum_{i=1}^{k} 2^{-(k-i)\alpha(\lambda_1)}(2^{1-i}R_0)^{\sigma-1}\\ &\le 2^{-k\alpha(\lambda_1)}\omega(R_0) + C2^{-k\alpha(\lambda_1)}R_0^{\sigma-1}\frac{1 - 2^{k(\alpha(\lambda_1)-\sigma+1)}}{1 - 2^{\alpha(\lambda_1)-\sigma+1}}.\end{aligned}$$

It follows that

$$\omega(2^{-k}R_0) \le 2^{-k\tilde{\alpha}}\left(\omega(R_0) + CR_0^{\sigma-1}\right),$$

where $\tilde{\alpha} = \min\{\sigma - 1, \alpha(\lambda_1)\}$. For any $\rho \in (0, R_0/2)$, let k be the positive integer such that $2^{-k-1}R_0 < \rho \leq 2^{-k}R_0$. Then

$$\omega(\rho) \leq C\rho^{\tilde{\alpha}}, \quad \forall \rho \in (0, R_0/2),$$

and hence

$$\Big(\fint_{\partial B_\rho} |\bar{u}(x')|^2 \, d\sigma\Big)^{1/2} \leq C\rho^{\tilde{\alpha}}, \quad \forall \rho \in (0, R_0/2).$$

The proposition then follows from the standard interior elliptic estimate applied to $B_\rho \setminus \overline{B}_{\rho/2}$. □

In the remaining part of this section, we consider the case when $\varepsilon > 0$.

Proposition 2.4. *For $n \geq 3$, $s > -(n-1)/2$, $0 < t \leq 1$, and $\varepsilon > 0$. Let a satisfy (2.1) and be Hölder continuous, λ_1 and $\alpha(\lambda_1)$ be given by (1.12) and (1.13), and $\bar{v} \in H^1(B_{R_0})$ be a solution of*

$$\operatorname{div}\Big[\Big(\varepsilon + a\Big(\frac{x'}{|x'|}\Big)|x'|^2\Big)\nabla \bar{v}\Big] = \operatorname{div} F \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}$$

satisfying

$$\|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}} < \infty,$$

where $F \in L^\infty(B_{R_0})$ satisfy

$$\|F\|_{\varepsilon,s,0,B_{R_0}} < \infty.$$

Then for any $0 < \rho < \frac{1}{4}R \leq \frac{1}{4}R_0$, we have

$$\begin{aligned}
&\Big(\fint_{B_\rho \setminus B_{\rho/2}} |\bar{v}(x') - (\bar{v})^a_{B_\rho \setminus B_{\rho/2}}|^2 \, d\sigma\Big)^{1/2} \\
&\leq C\left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)} \Big(\fint_{B_R \setminus B_{R/2}} |\bar{v}(x') - (\bar{v})^a_{B_R \setminus B_{R/2}}|^2 \, d\sigma\Big)^{1/2} \\
&\quad + C\left(\frac{R}{\rho}\right)^{\frac{n}{2}} \left(R^{1+s}\left(\frac{\sqrt{\varepsilon}}{R} + 1\right)\|F\|_{\varepsilon,s,0,B_{R_0}} + \left(\frac{\varepsilon}{R^2}\right)^{\tilde{\alpha}(\lambda_1)} R^{1-t}\|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}}\right),
\end{aligned} \tag{2.18}$$

where

$$\tilde{\alpha}(\lambda_1) = \begin{cases} \Big(\alpha(\lambda_1) + \dfrac{n-1}{2}\Big)\Big/2, & \text{when } n < 5 - 2\alpha(\lambda_1); \\ \text{any } \alpha < 1, & \text{when } n = 5 - 2\alpha(\lambda_1); \\ 1, & \text{when } n > 5 - 2\alpha(\lambda_1), \end{cases} \tag{2.19}$$

and C is some positive constant depending only on n, s, t, R_0, and an upper bound of $\|\ln a\|_{L^\infty}$, and is independent of ε.

Proposition 2.4 will follow from Lemma 2.2 and the following lemma.

Lemma 2.5. *For $n \geq 3$, $B_1 \subset \mathbb{R}^{n-1}$, and $\beta < 1$, let $v \in H^1_0(B_1, |x'|^{2+\beta}dx')$. There exists a positive constant C depending only on n and β, such that*

$$\sup_{0<r<1} r^n \fint_{\partial B_r} |v|^2 \, d\sigma \leq C \int_{B_1} |x'|^{2+\beta} |\nabla v|^2 \, dx'.$$

Proof. Without loss of generality, we may assume $v \in C_0^1(B_1)$. Then we have

$$r^2 \int_{\partial B_r} |v|^2 \, d\sigma \le C \int_{\mathbb{S}^{n-2}} r^n |v(r,\xi)|^2 \, d\xi = C \int_{\mathbb{S}^{n-2}} r^n \left(\int_r^1 \partial_s v(s,\xi) \, ds \right)^2 d\xi$$
$$\le C \int_{\mathbb{S}^{n-2}} r^n \left(\int_r^1 |\partial_s v(s,\xi)|^2 s^\beta \, ds \right) \left(\int_r^1 s^{-\beta} \, ds \right) d\xi$$
$$\le C \int_0^1 \int_{\mathbb{S}^{n-2}} s^{n+\beta} |\partial_s v(s,\xi)|^2 \, ds d\xi \le C \int_{B_1} |x'|^{2+\beta} |\nabla v|^2 \, dx',$$

where in the last two lines, we used Hölder's inequality and the Fubini theorem. □

Now we are in a position to prove Proposition 2.4.

Proof of Proposition 2.4. In this proof, we denote $\alpha = \alpha(\lambda_1)$ and $\tilde{\alpha} = \tilde{\alpha}(\lambda_1)$ for simplicity. Without loss of generality, we may assume $\bar{v}(0) = 0$. Then by the mean value formula,

$$|\bar{v}(x')| \le |x'|^{1-t} \|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}} \quad \text{for } x' \in B_{R_0}. \tag{2.20}$$

For any $0 < R < R_0$, we write $\bar{v} = v_1 + v_2$ so that $v_1 \in H^1(B_R, |x'|^2 dx')$ satisfies

$$\operatorname{div} \left[a\Big(\frac{x'}{|x'|}\Big) |x'|^2 \nabla v_1 \right] = 0 \quad \text{in } B_R,$$

with $v_1 = \bar{v}$ on ∂B_R. By Lemma 2.2 and (2.20), we have, for $0 < \rho < R$,

$$\left(\fint_{\partial B_\rho} |v_1(x') - v_1(0)|^2 \, d\sigma \right)^{\frac{1}{2}} \le C \left(\frac{\rho}{R} \right)^\alpha R^{1-t} \|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}}.$$

Since a is Hölder continuous, by the interior gradient estimate applied in $B_\rho \setminus \overline{B}_{\rho/2}$ for $0 < \rho < R$,

$$|\nabla v_1(x')| \le C \|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}} |x'|^{\alpha-1} R^{1-t-\alpha} \quad \text{for } x' \in B_{R/2} \setminus \{0\}. \tag{2.21}$$

From the maximum principle, Lemma 2.2, and (2.20), we know that

$$\|v_1\|_{L^\infty(B_R)} = \sup_{x' \in \partial B_R \cup \{0\}} |v_1(x')| \le C R^{1-t} \|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}}.$$

Therefore, by the boundary gradient estimate,

$$|\nabla v_1(x')| \le C \|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}} R^{-t} \quad \text{for } x' \in B_R \setminus B_{R/2}. \tag{2.22}$$

In particular, $v_1 \in H^1(B_R)$. Therefore, $v_2 \in H_0^1(B_R)$ satisfies

$$\operatorname{div} \left[\Big(\varepsilon + a\Big(\frac{x'}{|x'|}\Big) |x'|^2 \Big) \nabla v_2 \right] = \operatorname{div} F - \varepsilon \Delta v_1 \quad \text{in } B_R.$$

Let $\tilde{v}_1(y') = v_1(Ry')$, $\tilde{v}_2(y') = v_2(Ry')$, $\tilde{F}(y') = R^{-1} F(Ry')$, and $\tilde{\varepsilon} = \varepsilon R^{-2}$. Then by (2.21) and (2.22),

$$\|\tilde{F}\|_{\tilde{\varepsilon},s,0,B_1} = R^{1+s} \|F\|_{\varepsilon,s,0,B_R}, \quad \|\nabla \tilde{v}_1\|_{\tilde{\varepsilon},\alpha-1,1,B_1} \le C R^{1-t} \|\nabla \bar{v}\|_{\varepsilon,-t,1,B_{R_0}}, \tag{2.23}$$

and $\tilde{v}_2$ satifies

$$\operatorname{div} \left[\Big(\tilde{\varepsilon} + a\Big(\frac{x'}{|x'|}\Big) |x'|^2 \Big) \nabla \tilde{v}_2 \right] = \operatorname{div} \tilde{F} - \tilde{\varepsilon} \Delta \tilde{v}_1 \quad \text{in } B_1. \tag{2.24}$$

Denote $x' = (r, \xi)$ in the polar coordinates, where $0 \le r \le 1$ and $\xi \in \mathbb{S}^{n-2}$. We multiply (2.24) by $\tilde{v}_2$ and integrate by parts to get

$$\int_{B_1} (\tilde{\varepsilon} + a(\xi) r^2) |\nabla \tilde{v}_2|^2 = \int_{B_1} \tilde{F} \cdot \nabla v_2 + \tilde{\varepsilon} \int_{B_1} \nabla \tilde{v}_1 \cdot \nabla \tilde{v}_2.$$

By Young's inequality and the definitions of $\|\tilde{F}\| = \|\tilde{F}\|_{\varepsilon,s,0,B_1}$ and $\|\nabla \tilde{v}_1\| = \|\nabla \tilde{v}_1\|_{\tilde{\varepsilon},\alpha-1,1,B_1}$, since $2s + n - 2 > -1$,

$$\begin{aligned}\int_{B_1} (\tilde{\varepsilon} + r^2) |\nabla \tilde{v}_2|^2 &\le C\|\tilde{F}\|^2 \int_{B_1} r^{2s} (\tilde{\varepsilon} + r^2) + C \|\nabla \tilde{v}_1\|^2 \int_{B_1} \frac{\tilde{\varepsilon}^2 r^{2\alpha-2}}{\tilde{\varepsilon} + r^2} \\ &\le C\|\tilde{F}\|^2 (\tilde{\varepsilon} + 1) + C\|\nabla \tilde{v}_1\|^2 \begin{cases} \tilde{\varepsilon}^{\alpha + \frac{n-1}{2}}, & \text{when } n < 5 - 2\alpha; \\ \tilde{\varepsilon}^2 (|\ln \tilde{\varepsilon}| + 1), & \text{when } n = 5 - 2\alpha; \\ \tilde{\varepsilon}^2, & \text{when } n > 5 - 2\alpha, \end{cases}\end{aligned}$$

By the inequality above and Lemma 2.5 with $\beta = 0$,

$$\sup_{0<r<1} r^n \fint_{\partial B_r} |\tilde{v}_2|^2 \, d\sigma \le C \left((\tilde{\varepsilon} + 1) \|\tilde{F}\|^2_{\tilde{\varepsilon},s,0,B_1} + \tilde{\varepsilon}^{2\tilde{\alpha}} \|\nabla \tilde{v}_1\|^2_{\tilde{\varepsilon},\alpha-1,1,B_1} \right),$$

which together with (2.23) implies, for any $0 < \rho < R$,

$$\begin{aligned}&\fint_{\partial B_\rho} |v_2(x')|^2 \, d\sigma \\ &\le C \left(\frac{R}{\rho} \right)^n \left(R^{2+2s} \left(\frac{\varepsilon}{R^2} + 1 \right) \|F\|^2_{\varepsilon,s,0,B_R} + \left(\frac{\varepsilon}{R^2} \right)^{2\tilde{\alpha}} R^{2-2t} \|\nabla \bar{v}\|^2_{\varepsilon,-t,1,B_{R_0}} \right).\end{aligned} \tag{2.25}$$

Denote

$$\omega(\rho) := \left(\fint_{B_\rho \setminus B_{\rho/2}} a\Big(\frac{x'}{|x'|}\Big) |\bar{v}(x') - (\bar{v})^a_{B_\rho \setminus B_{\rho/2}}|^2 \, d\sigma \right)^{1/2}. \tag{2.26}$$

By Lemma 2.2 and (2.25), for any $0 < \rho < R/4$,

$$\begin{aligned}&\fint_{\partial B_\rho} a\Big(\frac{x'}{|x'|}\Big) |\bar{v}(x') - v_1(0)|^2 \, d\sigma \\ &\le C \fint_{\partial B_\rho} a\Big(\frac{x'}{|x'|}\Big) |v_1(x') - v_1(0)|^2 \, d\sigma + C \fint_{\partial B_\rho} a\Big(\frac{x'}{|x'|}\Big) |v_2(x')|^2 \, d\sigma \\ &\le C \left(\frac{\rho}{R} \right)^{2\alpha} \fint_{\partial B_R} a\Big(\frac{x'}{|x'|}\Big) |\bar{v}(x') - (\bar{v})^a_{\partial B_R}|^2 \, d\sigma \\ &\quad + C \left(\frac{R}{\rho} \right)^n \left(R^{2+2s} \left(\frac{\varepsilon}{R^2} + 1 \right) \|F\|^2_{\varepsilon,s,0,B_R} + \left(\frac{\varepsilon}{R^2} \right)^{2\tilde{\alpha}} R^{2-2t} \|\nabla \bar{v}\|^2_{\varepsilon,-t,1,B_{R_0}} \right).\end{aligned}$$

Multiplying both sides by ρ^{n-2}, integrating over $(\rho/2, \rho)$ and dividing both sides by ρ^{n-1}, we have for any $0 < \rho < \frac{r}{4} \leq \frac{R}{4}$,

$$\begin{aligned}\omega(\rho)^2 &\leq \fint_{B_\rho \setminus B_{\rho/2}} a\Big(\frac{x'}{|x'|}\Big)|\bar{v}(x') - v_1(0)|^2 \, d\sigma \\ &\leq C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{\partial B_r} a\Big(\frac{x'}{|x'|}\Big)|\bar{v}(x') - (\bar{v})^a_{\partial B_r}|^2 \, d\sigma \\ &\quad + C\left(\frac{r}{\rho}\right)^n \left(r^{2+2s}\left(\frac{\varepsilon}{r^2}+1\right)\|F\|^2_{\varepsilon,s,0,B_r} + \left(\frac{\varepsilon}{r^2}\right)^{2\tilde{\alpha}} r^{2-2t}\|\nabla\bar{v}\|^2_{\varepsilon,-t,1,B_{R_0}}\right) \\ &\leq C\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{\partial B_r} a\Big(\frac{x'}{|x'|}\Big)|\bar{v}(x') - (\bar{v})^a_{B_R \setminus B_{R/2}}|^2 \, d\sigma \\ &\quad + C\left(\frac{r}{\rho}\right)^n \left(r^{2+2s}\left(\frac{\varepsilon}{r^2}+1\right)\|F\|^2_{\varepsilon,s,0,B_R} + \left(\frac{\varepsilon}{r^2}\right)^{2\tilde{\alpha}} r^{2-2t}\|\nabla\bar{v}\|^2_{\varepsilon,-t,1,B_{R_0}}\right).\end{aligned}$$

Multiplying both sides by r^{n-2}, integrating r over $(R/2, R)$, and dividing both sides by R^{n-1} give (2.18). □

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. In this case, Γ_+ and Γ_- touch at the origin. After a suitable rotation in $\mathbb{R}^{n-1}$, we may assume without loss of generality that $D^2(f-g)(0')$ is a diagonal matrix whose entries are denoted by $a_1, a_2, \ldots, a_{n-1} > 0$. Therefore,

$$f(x') - g(x') = \sum_{i=1}^{n-1} a_i x_i^2 + e(x'), \tag{3.1}$$

where $e(x')$ satisfies $|e(x')| \leq C|x'|^{2+\gamma}$.

Let $u \in H^1(\Omega_{R_0})$ be a solution of (1.8). For $x \in \Omega_{R_0}$, we consider, as in [19],

$$\bar{u}(x') := \fint_{g(x')}^{f(x')} u(x', x_n) \, dx_n. \tag{3.2}$$

It follows from a direct computation that $\bar{u} \in H^1(B_{R_0}, |x'|^2 dx')$. For any $0 < R \leq R_0/2$, we make a change of variables by setting

$$\begin{cases} y' = x', \\ y_n = 2R^2\left(\dfrac{x_n - g(x')}{f(x') - g(x')} - \dfrac{1}{2}\right), \end{cases} \quad \forall (x', x_n) \in \Omega_{2R} \setminus \Omega_R.$$

This change of variables maps the domain $\Omega_{2R} \setminus \Omega_R$ to $Q_{2R,R^2} \setminus Q_{R,R^2}$, where

$$Q_{s,t} := \{y = (y', y_n) \in \mathbb{R}^n \mid |y'| < s, |y_n| < t\} \tag{3.3}$$

for $s, t > 0$. Let $v(y) = u(x)$, so that $v(y)$ satisfies

$$\begin{cases} -\partial_i(B^{ij}(y)\partial_j v(y)) = 0 & \text{in } Q_{2R,R^2} \setminus Q_{R,R^2}, \\ B^{nj}(y)\partial_j v(y) = 0 & \text{on } \{z_n = -R^2\} \cup \{z_n = R^2\}, \end{cases}$$

where

$$(B^{ij}(y)) = \frac{2R^2(\partial_x y)(A^{ij}(x(y)))(\partial_x y)^t}{\det(\partial_x y)}$$
$$= \frac{2R^2(\partial_x y)(\partial_x y)^t}{\det(\partial_x y)} + \frac{2R^2(\partial_x y)(A^{ij}(x(y)) - \delta_{ij})(\partial_x y)^t}{\det(\partial_x y)} =: (C^{ij}(y)) + (D^{ij}(y))$$

and

$$\det(\partial_x y) = 2R^2(f(y') - g(y'))^{-1}.$$

Note that the top left $(n-1)\times(n-1)$ part of $(C^{ij}(y))$ is $(f(y')-g(y'))I_{(n-1)\times(n-1)}$. Let

$$\bar{v}(y') = \fint_{-R^2}^{R^2} v(y', y_n)\, dy_n (= \bar{u}(y')).$$

Then $\bar{v}$ satisfies in $B_{2R} \setminus B_R \subset \mathbb{R}^{n-1}$ that

$$\operatorname{div}\Big[\Big(f(y') - g(y')\Big)\nabla\bar{v}\Big] = -\sum_{i=1}^{n-1} \partial_i \overline{C^{in}\partial_n v} - \sum_{i=1}^{n-1}\sum_{j=1}^{n} \partial_i \overline{D^{ij}\partial_j v},$$

where $\overline{h}$ denotes the average of h with respect to y_n in the interval $(-R^2, R^2)$. Reversing the change of variables, one can see that $\bar{u}$ satisfies in $B_{R_0} \setminus \{0\} \subset \mathbb{R}^{n-1}$ that

$$\operatorname{div}\Big[\Big(f(x') - g(x')\Big)\nabla\bar{u}\Big] = -\sum_{i=1}^{n-1} \partial_i \overline{b^i\partial_n u} - \sum_{i=1}^{n-1}\sum_{j=1}^{n} \partial_i \overline{c^{ij}\partial_j u},$$

where for $1 \le i \le n-1$,

$$b^i(x) = (f(x') - g(x'))\partial_i g(x') + (x_n - g(x'))\partial_i(f(x') - g(x')),$$
$$c^{ij}(x) = \big(A^{ij}(x) - A^{ij}(0)\big)\,(f(x') - g(x')) \quad \text{for } 1 \le j \le n-1,$$
$$c^{in}(x) = \sum_{k=1}^{n-1} \big(A^{ik}(x) - A^{ik}(0)\big)\, b^k + \big(A^{in}(x) - A^{in}(0)\big)\,(f(x') - g(x'))\,,$$

and $\overline{h}$ denotes the average of h with respect to x_n in the interval $(g(x'), f(x'))$ as in (3.2). By the weak formulation and $\bar{u} \in H^1(B_{R_0}, |x'|^2 dx')$, one can see that $\bar{u}$ satisfies the above equation in B_{R_0}. Therefore, $\bar{u}$ satisfies

$$\operatorname{div}\Big[\Big(\sum_{i=1}^{n-1} a_i|x_i|^2\Big)\nabla\bar{u}\Big] = \operatorname{div} F \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}, \tag{3.4}$$

where $F_i = -\overline{b^i\partial_n u} - e\partial_i\bar{u} - \sum_{j=1}^{n} \overline{c^{ij}\partial_j u}$ and e is given in (3.1). From the assumptions (1.3), (1.5), and (1.6), we have

$$|b^i(x)| \le C|x'|^3, \quad |c^{ij}(x)| \le C|x'|^{2+\gamma} \quad \text{for } i = 1, \dots, n-1, j = 1, \dots, n.$$

Hence

$$|F(x')| \le C|x'|^{2+\gamma}\overline{|\nabla u(x')|} \quad \text{for } x' \in B_{R_0}.$$

Proof of Theorem 1.1. Without loss of generality, we assume that $\|u\|_{L^\infty(\Omega_{R_0})} = 1$. Let $\bar{u}$ be defined as in (3.2). By (1.9) with $\varepsilon = 0$,

$$\|\nabla\bar{u}\|_{0,-s_0,1,B_{R_0}} < \infty,$$

where $s_0 = 1$. Then $\bar{u}$ satisfies the equation (3.4) with F satisfying

$$\|F\|_{0,2+\gamma-s_0,1,B_{R_0}} < \infty.$$

By (1.9) with $\varepsilon = 0$,

$$|u(x', x_n) - \bar{u}(x')| \le (f(x') - g(x')) \max_{x_n \in (g(x'), f(x'))} |\partial_n u(x', x_n)| \le C|x'| \quad \text{in } \Omega_{R_0}. \tag{3.5}$$

By Proposition 2.1 and (1.9), both u and $\bar{u}$ are Hölder continuous. Indeed, for any $x, y \in \Omega_{R_0}$ such that $|x'| \le |y'|$, we denote $r = |x - y|$. When $r \le |x'|^2$, by (1.9) and the mean value formula, we have

$$|u(x) - u(y)| \le Cr|x'|^{-1} \le Cr^{1/2}.$$

When $r > |x'|^2$, by (3.5) and using the C^β regularity of $\bar{u}$, we also have

$$\begin{aligned}|u(x) - u(y)| &\le |u(x) - \bar{u}(x')| + |u(y) - \bar{u}(y')| + |\bar{u}(x') - \bar{u}(y')| \\ &\le C|x'| + Cr^\beta \le C(r^{1/2} + r^\beta).\end{aligned}$$

Combining the above two estimates, we see the Hölder continuity of u. Thus, we may further assume, without loss of generality, that $u(0) = \bar{u}(0) = 0$. By decreasing γ if necessary, we may assume that $1 + \gamma - s_0 = \gamma < \alpha(\lambda_1)$. By Proposition 2.1 and (3.5), we have, for any $0 < R < R_0/4$,

$$\fint_{\Omega_{4R} \setminus \Omega_{R/2}} |u|^2 \, dx \le C \fint_{\Omega_{4R} \setminus \Omega_{R/2}} |u - \bar{u}|^2 \, dx + C \fint_{\Omega_{4R} \setminus \Omega_{R/2}} |\bar{u}|^2 \, dx \le CR^{2\tilde{\alpha}},$$

where $\tilde{\alpha} = \min\{\alpha(\lambda_1), 1 + \gamma - s_0\}$. We make a change of variables by setting

$$\begin{cases} z' = x', \\ z_n = 2R^2 \left(\dfrac{x_n - g(x')}{f(x') - g(x')} - \dfrac{1}{2} \right), \end{cases} \quad \forall (x', x_n) \in \Omega_{4R} \setminus \Omega_{R/2}.$$

This change of variables maps the domain $\Omega_{4R} \setminus \Omega_{R/2}$ to $Q_{4R,R^2} \setminus Q_{R/2,R^2}$, where $Q_{s,t}$ is defined as (3.3). Let $w(z) = u(x)$, so that $w(z)$ satisfies

$$\begin{cases} -\partial_i (b^{ij}(z) \partial_j w(z)) = 0 & \text{in } Q_{4R,R^2} \setminus Q_{R/2,R^2}, \\ b^{nj}(z) \partial_j w(z) = 0 & \text{on } \{z_n = -R^2\} \cup \{z_n = R^2\}, \end{cases}$$

where

$$(b^{ij}(z)) = \frac{(\partial_x z)(A^{ij}(x(z)))(\partial_x z)^t}{\det(\partial_x z)}.$$

It is straightforward to verify that

$$\frac{I}{C} \le b(z) \le CI \quad \text{and} \quad \|b\|_{C^\gamma(Q_{4R,R^2} \setminus Q_{R/2,R^2})} \le CR^{-\gamma}.$$

Let $\tilde{b}^{ij}(z) = b^{ij}(Rz)$ and $\tilde{w}(z) = w(Rz)$. Then $\tilde{w}$ satisfies

$$\begin{cases} -\partial_i (\tilde{b}^{ij}(z) \partial_j \tilde{w}(z)) = 0 & \text{in } Q_{4,R} \setminus Q_{1/2,R}, \\ \tilde{b}^{nj}(z) \partial_j \tilde{w}(z) = 0 & \text{on } \{z_n = -R\} \cup \{z_n = R\}, \end{cases}$$

with

$$\frac{I}{C} \le \tilde{b} \le CI \quad \text{and} \quad \|\tilde{b}\|_{C^\gamma(Q_{4,R} \setminus Q_{1/2,R})} \le C.$$

Now we define

$$S_l := \left\{ z \in \mathbb{R}^n \;\middle|\; 1/2 < |z'| < 4, \; (2l-1)R < z_n < (2l+1)R \right\}$$

for any integer l, and

$$S_{s,t}^m := \left\{ z \in \mathbb{R}^n \;\middle|\; s < |z'| < t, \; |z_n| < m \right\}.$$

Note that $Q_{4,R} \setminus Q_{1/2,R} = S_0$. We take the even extension of $\tilde{w}$ with respect to $y_n = R$ and then take the periodic extension (so that the period is equal to $4R$). More precisely, we define, for any $l \in \mathbb{Z}$, a new function $\hat{w}$ by setting

$$\hat{w}(z) := \tilde{w}\left(z', (-1)^l (y_n - 2lR)\right), \quad \forall z \in S_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \cdots, n-1$,

$$\hat{b}^{nk}(z) = \hat{b}^{kn}(z) := (-1)^l \tilde{b}^{nk}\left(z', (-1)^l (z_n - 2l\rho)\right), \quad \forall z \in S_l,$$

and for other indices,

$$\hat{b}^{ij}(z) := \tilde{b}^{ij}\left(z', (-1)^l (z_n - 2l\rho)\right), \quad \forall y \in S_l.$$

Then $\hat{w}$ and $\hat{c}^{ij}$ are defined in the infinite ring $Q_{4,\infty} \setminus Q_{1/2,\infty}$. In particular, $\hat{w}$ satisfies the equation

$$\partial_i(\hat{b}^{ij}\partial_j \hat{w}) = 0 \quad \text{in } S^2_{1/2,4}.$$

By [33, Proposition 4.1] and [36, Lemma 2.1], we have

$$\|\nabla \hat{w}\|_{L^\infty(S^1_{1,2})} \le C\|\hat{w}\|_{L^2(S^2_{1/2,4})} \le CR^{\tilde{\alpha}},$$

which, after reversing the changes of variables, implies,

$$\|\nabla u\|_{L^\infty(\Omega_{2R}\setminus\Omega_R)} \le CR^{\tilde{\alpha}-1}.$$

Therefore, we have improved the upper bound $|\nabla u(x)| \le C|x'|^{-s_0}$ to $|\nabla u(x)| \le C|x'|^{\tilde{\alpha}-1}$, where $\tilde{\alpha} - 1 = \min\{\alpha(\lambda_1) - 1, -s_0 + \gamma\}$. If $-s_0 + \gamma < \alpha(\lambda_1) - 1$, we take $s_1 = s_0 - \gamma$ and repeat the argument above. We may decrease γ if necessary so that $\alpha(\lambda_1) - 1 \ne -s_0 + k\gamma$ for any $k = 1, 2, \ldots$. After repeating the argument finitely many times, we obtain the estimate (1.14). □

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. Without loss of generality, we assume $\|u\|_{L^\infty(\Omega_{R_0})} = 1$. We perform a change of variables by setting

$$\begin{cases} y' = x', \\ y_n = 2\varepsilon\left(\dfrac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \dfrac{1}{2}\right), \end{cases} \quad \forall (x', x_n) \in \Omega_{R_0}. \tag{4.1}$$

This change of variables maps the domain Ω_{R_0} to $Q_{R_0,\varepsilon}$, where $Q_{s,t}$ is defined as in (3.3). Moreover,

$$\det(\partial_x y) = 2\varepsilon(\varepsilon + f(x') - g(x'))^{-1}. \tag{4.2}$$

After a suitable rotation in $\mathbb{R}^{n-1}$, we may assume without loss of generality that $D^2(f-g)(0')$ is a diagonal matrix whose entries are denoted by $a_1, a_2, \ldots, a_{n-1} > 0$ and (3.1) holds. Let $u \in H^1(B_{R_0})$ be a solution of (1.8), and let $v(y) = u(x)$. Then v satisfies

$$\begin{cases} -\partial_i(b^{ij}(y)\partial_j v(y)) = 0 & \text{in } Q_{R_0,\varepsilon}, \\ b^{nj}(y)\partial_j v(y) = 0 & \text{on } \{y_n = -\varepsilon\} \cup \{y_n = \varepsilon\} \end{cases} \tag{4.3}$$

with $\|v\|_{L^\infty(Q_{R_0,\varepsilon})} = 1$, where the matrix $(b^{ij}(y))$ is given by

$$
\begin{aligned}
&(b^{ij}(y)) \\
&= \frac{2\varepsilon(\partial_x y)(A^{ij}(x(y)))(\partial_x y)^t}{\det(\partial_x y)} = \frac{2\varepsilon(\partial_x y)(\partial_x y)^t}{\det(\partial_x y)} + \frac{2\varepsilon(\partial_x y)(A^{ij}(x(y)) - \delta_{ij})(\partial_x y)^t}{\det(\partial_x y)} \\
&= \begin{pmatrix}
\varepsilon + \sum_{j=1}^{n-1} a_j y_j^2 & 0 & \cdots & 0 & b^{1n} \\
0 & \varepsilon + \sum_{j=1}^{n-1} a_j y_j^2 & \cdots & 0 & b^{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \varepsilon + \sum_{j=1}^{n-1} a_j y_j^2 & b^{n-1,n} \\
b^{n1} & b^{n2} & \cdots & b^{n,n-1} & \frac{\sum_{j=1}^{n-1} |b^{jn}|^2 + 4\varepsilon^2}{\varepsilon + f(y') - g(y')}
\end{pmatrix} \\
&\quad + \begin{pmatrix}
e^1 & 0 & \cdots & 0 & 0 \\
0 & e^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & e^{n-1} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix} + \begin{pmatrix}
c^{11} & c^{12} & \cdots & c^{1n} \\
c^{21} & c^{22} & \cdots & c^{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c^{n1} & c^{n2} & \cdots & c^{nn}
\end{pmatrix},
\end{aligned}
$$

and for $i = 1, \ldots, n-1$,

$$
\begin{aligned}
b^{ni} = b^{in} &= -2\varepsilon \partial_i g(y') - (y_n + \varepsilon)\partial_i (f(y') - g(y')), \\
e^i &= f(y') - g(y') - \sum_{j=1}^{n-1} a_j y_j^2,
\end{aligned}
$$

the matrix $\{c^{ij}\}$ is given by

$$
\frac{2\varepsilon(\partial_x y)(A^{ij}(y) - \delta_{ij})(\partial_x y)^t}{\det(\partial_x y)}.
$$

By (1.3), (1.5), and (1.6), we know for $i = 1, \ldots, n-1$,

$$
|b^{ni}(y)| = |b^{in}(y)| \le C\varepsilon |y'| \quad \text{and} \quad |e^i(y')| \le C|y'|^{2+\gamma}, \tag{4.4}
$$

and for $i = 1, \ldots, n-1$, $j = 1, \ldots, n-1$,

$$
\begin{aligned}
|c^{ij}(y)| &\le C(\varepsilon + |y'|^2)\left(|y'|^\gamma + (\varepsilon + |y'|^2)^\gamma\right), \\
|c^{in}(y)| &\le C\varepsilon\left(|y'|^\gamma + (\varepsilon + |y'|^2)^\gamma\right).
\end{aligned} \tag{4.5}
$$

Note that $e^1(y), \ldots, e^{n-1}(y)$ depend only on y' and are independent of y_n. We define

$$
\bar{v}(y') := \fint_{-\varepsilon}^{\varepsilon} v(y', y_n)\, dy_n. \tag{4.6}
$$

It is straightforward to verify that $\bar{v}$ satisfies in $B_{R_0} \subset \mathbb{R}^{n-1}$,

$$
\operatorname{div}\left[\Big(\varepsilon + \sum_{i=1}^{n-1} a_i y_i^2\Big)\nabla \bar{v}\right] = -\sum_{i=1}^{n-1} \partial_i \overline{b^{in}\partial_n v} - \sum_{i=1}^{n-1} \partial_i (e^i \partial_i \bar{v}) - \sum_{i=1}^{n-1}\sum_{j=1}^{n} \partial_i \overline{c^{ij}\partial_j v}, \tag{4.7}
$$

with $\|\bar{v}\|_{L^\infty(B_{R_0})} \le 1$, where $\overline{b^{in}\partial_n v}$ and $\overline{c^{ij}\partial_j v}$ are the average of $b^{in}\partial_n v$ and $c^{ij}\partial_j v$ with respect to y_n in $(-\varepsilon, \varepsilon)$ as in (4.6).

Proof of Theorem 1.3. We make the change of variables (4.1), and let $v(y) = u(x)$. Then v satisfies (4.3). Let $\bar{v}$ be defined as in (4.6). By (1.9),

$$\|\nabla \bar{v}\|_{\varepsilon,-2s_0,1,B_{R_0}} < \infty,$$

where $s_0 = \frac{1}{2}$. Then $\bar{v}$ satisfies the equation (4.7), that is

$$\operatorname{div}\left[(\varepsilon + a(\xi)r^2)\nabla \bar{v}\right] = \operatorname{div} F \quad \text{in } B_{R_0} \subset \mathbb{R}^{n-1}$$

with

$$F_i = -\overline{b^{in}\partial_n v} - e^i \partial_i \bar{v} - \overline{c^{ij}\partial_j v}, \quad i = 1, \ldots, n-1.$$

By (1.9) and (4.2),

$$|\partial_n v| \le C(\varepsilon + |y'|^2)^{1-s_0}/\varepsilon \quad \text{and} \quad |\nabla_{y'} v| \le C(\varepsilon + |y'|^2)^{-s_0} \quad \text{in } Q_{R_0,\varepsilon}. \tag{4.8}$$

Therefore, by (4.4) and (4.5),

$$\|F\|_{\varepsilon,\gamma-2s_0,0,B_{R_0}} < \infty.$$

Denote the left-hand side of (2.18) by $\omega(\rho)$. By Proposition 2.4 with $s = \gamma - 2s_0$ and $t = 2s_0$, for $0 < \rho < R/4 \le R_0/4$,

$$\omega(\rho) \le C\left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)} \omega(R) + C\left(\frac{R}{\rho}\right)^{\frac{n}{2}} R^{1-2s_0}\left(R^{\gamma}\left(\frac{\sqrt{\varepsilon}}{R} + 1\right) + \left(\frac{\varepsilon}{R^2}\right)^{\tilde{\alpha}(\lambda_1)}\right),$$

where $\tilde{\alpha}(\lambda_1) > \alpha(\lambda_1)$ is given by (2.19) and ω is defined in (2.26). Fix a $\bar{\mu} > 0$ satisfying $\bar{\mu}\tilde{\alpha}(\lambda_1) < \gamma$. For any $0 < \mu < \bar{\mu}$, $0 < \rho < R/4$, and $\varepsilon^{\frac{1}{2+\mu}} < R/4 \le R_0/4$, we have

$$\omega(\rho) \le C\left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)} \omega(R) + C\left(\frac{R}{\rho}\right)^{\frac{n}{2}} R^{1-2s_0+\mu\tilde{\alpha}(\lambda_1)}.$$

If $1 - 2s_0 + \mu\tilde{\alpha}(\lambda_1) < \alpha(\lambda_1)$, by [22, Lemma 5.13],

$$\omega(\rho) \le C\rho^{1-2s_0+\mu\tilde{\alpha}(\lambda_1)} \quad \forall\, \varepsilon^{\frac{1}{2+\mu}} \le \rho < R_0. \tag{4.9}$$

By (4.8),

$$|v(y', y_n) - \bar{v}(y')| \le 2\varepsilon \max_{y_n \in (-\varepsilon,\varepsilon)} |\partial_n v(y', y_n)| \le C(\varepsilon + |y'|^2)^{1-s_0} \quad \text{in } Q_{R_0,\varepsilon}. \tag{4.10}$$

Therefore, by (4.9) and (4.10),

$$\begin{aligned}
&\left(\fint_{Q_{\rho,\varepsilon}\setminus Q_{\rho/2,\varepsilon}} \left|v(y) - (v)_{Q_{\rho,\varepsilon}\setminus Q_{\rho/2,\varepsilon}}\right|^2 dy\right)^{\frac{1}{2}} \\
&\le \left(\fint_{Q_{\rho,\varepsilon}\setminus Q_{\rho/2,\varepsilon}} |v - \bar{v}|^2\, dy\right)^{\frac{1}{2}} + \omega(\rho) \\
&\le C\rho^{1-2s_0+\mu\tilde{\alpha}(\lambda_1)} \quad \forall\, \varepsilon^{\frac{1}{2+\mu}} \le \rho < R_0,
\end{aligned}$$

where

$$(v)_{Q_{\rho,\varepsilon}\setminus Q_{\rho/2,\varepsilon}} := \fint_{Q_{\rho,\varepsilon}\setminus Q_{\rho/2,\varepsilon}} v(y)\, dy.$$

This implies

$$\left(\fint_{\Omega_{4\rho}\setminus\Omega_{\rho/2}} \left|u(x) - (u)_{\Omega_{4\rho}\setminus\Omega_{\rho/2}}\right|^2 dx\right)^{\frac{1}{2}} \le C\rho^{1-2s_0+\mu\tilde{\alpha}(\lambda_1)} \quad \forall\, \varepsilon^{\frac{1}{2+\mu}} \le \rho < R_0/4. \tag{4.11}$$

We will show that

$$|\nabla u(x)| \le C(\varepsilon + |x'|^2)^{-\frac{1}{2}+\frac{1-2s_0+\mu\tilde{\alpha}(\lambda_1)}{2+\mu}} \quad \text{for } x \in \Omega_{R_0/4}. \tag{4.12}$$

For any $\varepsilon^{\frac{1}{2+\mu}} \le \rho < \frac{R_0}{4}$, we make a change of variables by setting

$$\begin{cases} z' = x', \\ z_n = 2\rho^2 \left(\dfrac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \dfrac{1}{2} \right), \end{cases} \quad \forall (x', x_n) \in \Omega_{4\rho} \setminus \Omega_{\rho/2}. \tag{4.13}$$

This change of variables maps the domain $\Omega_{4\rho} \setminus \Omega_{\rho/2}$ to $Q_{4\rho,\rho^2} \setminus Q_{\rho/2,\rho^2}$. Let $w(z) = u(x) - (u)_{\Omega_{4\rho}\setminus\Omega_{\rho/2}}$, so that $w(z)$ satisfies

$$\begin{cases} -\partial_i(d^{ij}(z)\partial_j w(z)) = 0 & \text{in } Q_{4\rho,\rho^2} \setminus Q_{\rho/2,\rho^2}, \\ d^{nj}(z)\partial_j w(z) = 0 & \text{on } \{z_n = -\rho^2\} \cup \{z_n = \rho^2\}, \end{cases}$$

where

$$(d^{ij}(z)) = \frac{(\partial_x z)(A^{ij}(x(z)))(\partial_x z)^t}{\det(\partial_x z)}.$$

Let $\tilde{d}^{ij}(z) = d^{ij}(\rho z)$ and $\tilde{w}(z) = w(\rho z)$. Then $\tilde{w}$ satisfies

$$\begin{cases} -\partial_i(\tilde{d}^{ij}(z)\partial_j \tilde{w}(z)) = 0 & \text{in } Q_{4,\rho} \setminus Q_{1/2,\rho}, \\ \tilde{d}^{nj}(z)\partial_j \tilde{w}(z) = 0 & \text{on } \{z_n = -\rho\} \cup \{z_n = \rho\}. \end{cases}$$

It is straightforward to verify that

$$\frac{I}{C} \le \tilde{d} \le CI \quad \text{and} \quad \|\tilde{d}\|_{C^\gamma(Q_{4,\rho}\setminus Q_{1/2,\rho})} \le C.$$

Using the similar "flipping argument" as in the proof of Theorem 1.1, we have, by (4.11),

$$|\nabla u(x)| \le C|x'|^{-2s_0+\mu\tilde{\alpha}(\lambda_1)} \quad \text{for } \varepsilon^{\frac{1}{2+\mu}} \le |x'| < R_0/4 \tag{4.14}$$

and

$$\operatorname{osc}_{\Omega_{2\rho}\setminus\Omega_\rho} u \le C\rho^{1-2s_0+\mu\tilde{\alpha}(\lambda_1)} \quad \text{for } \varepsilon^{\frac{1}{2+\mu}} \le \rho < R_0/4. \tag{4.15}$$

By the maximum principle and (4.15), we have

$$\operatorname{osc}_{\Omega_{2\varepsilon^{\frac{1}{2+\mu}}}} u \le C\varepsilon^{\frac{1-2s_0+\mu\tilde{\alpha}(\lambda_1)}{2+\mu}}. \tag{4.16}$$

For $\varepsilon^{\frac{1}{2}} \le \rho < \frac{1}{2}\varepsilon^{\frac{1}{2+\mu}}$, we consider u in $\Omega_{4\rho}\setminus\Omega_{\rho/2}$. By the change of variables (4.13), the same "flipping argument" as above, and (4.16), we have

$$|\nabla u(x)| \le C|x'|^{-1}\varepsilon^{\frac{1-2s_0+\mu\tilde{\alpha}(\lambda_1)}{2+\mu}} \quad \text{for } \varepsilon^{\frac{1}{2}} \le |x'| < \varepsilon^{\frac{1}{2+\mu}}. \tag{4.17}$$

Finally, we consider $u \in \Omega_{2\sqrt{\varepsilon}}$, and make changes of variables (4.1). By the same "flipping argument" and (4.16), we have

$$|\nabla u(x)| \le C\varepsilon^{-\frac{1}{2}+\frac{1-2s_0+\mu\tilde{\alpha}(\lambda_1)}{2+\mu}} \quad \text{for } |x'| < \varepsilon^{\frac{1}{2}}. \tag{4.18}$$

Therefore, (4.12) is concluded from (4.14), (4.17), and (4.18).

We have improved the upper bound of $|\nabla u(x)| \le C(\varepsilon + |x'|^2)^{-s_0}$ to $|\nabla u(x)| \le C(\varepsilon + |x'|^2)^{-s_1}$, where $s_1 = \frac{1}{2} - \frac{1-2s_0+\mu\tilde{\alpha}(\lambda_1)}{2+\mu}$. We can repeat the argument with

$$\|\nabla \bar{v}\|_{\varepsilon,-2s_1,1,B_{R_0}} + \|F\|_{\varepsilon,\gamma-2s_1,0,B_{R_0}} < \infty.$$

Let

$$s_{i+1} = \frac{1}{2} - \frac{1 - 2s_i + \mu\tilde{\alpha}(\lambda_1)}{2+\mu},$$

which is equivalent to

$$s_{i+1} - \frac{1}{2} = \frac{2}{2+\mu}\left(s_i - \frac{1}{2}\right) - \frac{\mu\tilde{\alpha}(\lambda_1)}{2+\mu}.$$

Since $s_0 = \frac{1}{2}$, iterating the equation above gives

$$s_k = \frac{1}{2} - \frac{\mu\tilde{\alpha}(\lambda_1)}{2+\mu}\sum_{i=0}^{k-1}\left(\frac{2}{2+\mu}\right)^i \quad \forall k \in \mathbb{N}.$$

After repeating this argument k times, we have

$$\omega(\rho) \le C\left(\frac{\rho}{R}\right)^{\alpha(\lambda_1)}\omega(R) + C\left(\frac{R}{\rho}\right)^{\frac{n}{2}} R^{1-2s_k+\mu\tilde{\alpha}(\lambda_1)} \quad \forall\, \varepsilon^{\frac{1}{2+\mu}} \le \rho < \frac{R}{4} \le \frac{R_0}{4}, \tag{4.19}$$

provided that

$$1 - 2s_{k-1} + \mu\tilde{\alpha}(\lambda_1) = \mu\tilde{\alpha}(\lambda_1)\sum_{i=0}^{k-1}\left(\frac{2}{2+\mu}\right)^i < \alpha(\lambda_1).$$

Since

$$\mu\tilde{\alpha}(\lambda_1)\sum_{i=0}^{\infty}\left(\frac{2}{2+\mu}\right)^i = (2+\mu)\tilde{\alpha}(\lambda_1) > \alpha(\lambda_1),$$

there exists a $k \in \mathbb{N}$ such that

$$\mu\tilde{\alpha}(\lambda_1)\sum_{i=0}^{k-1}\left(\frac{2}{2+\mu}\right)^i < \alpha(\lambda_1) \le \mu\tilde{\alpha}(\lambda_1)\sum_{i=0}^{k}\left(\frac{2}{2+\mu}\right)^i = 1 - 2s_k + \mu\tilde{\alpha}(\lambda_1).$$

For such k, (4.19) implies that for any $\alpha < \alpha(\lambda_1)$,

$$\omega(\rho) \le C(\alpha)\rho^{\alpha} \quad \forall\, \varepsilon^{\frac{1}{2+\mu}} \le \rho < \frac{R_0}{4}.$$

By the same argument of proving (4.12), we can conclude that

$$|\nabla u(x)| \le C(\varepsilon + |x'|^2)^{-\frac{1}{2}+\frac{\alpha}{2+\mu}} \quad \text{for } x \in \Omega_{R_0/4}.$$

By taking μ sufficiently small, this concludes the proof. □

5. Properties of λ_1 and its corresponding eigenspace

In this section, we consider the eigenvalue problem (1.10) with $a(\xi) = \xi^t M\xi$ for some positive definite $(n-1)\times(n-1)$ matrix M. We study the properties of λ_1, the first nonzero eigenvalue of (1.10), and the properties of its corresponding eigenspace. After a suitable rotation in $\mathbb{R}^{n-1}$, we may assume without loss of generality that

$$(x')^t M x' = \sum_{j=1}^{n-1} a_j x_j^2, \quad a_1 \ge \ldots \ge a_{n-1} > 0. \tag{5.1}$$

Recall that

$$\mathbb{S}^{n-2} = \left\{ x' = (x_1, \dots x_{n-1}) \in \mathbb{R}^{n-1} \ \Big| \ \sum_{j=1}^{n-1} x_j^2 = 1 \right\}.$$

First we prove an estimate on λ_1 under a more general assumption on $a(\xi)$.

Lemma 5.1. *For $n \geq 3$, let λ_1 be the first nonzero eigenvalue of the eigenvalue problem* (1.10) *with $a(\xi) > 0$ a.e. satisfying $\ln a \in L^\infty(\mathbb{S}^{n-2})$ and $\int_{\mathbb{S}^{n-2}} a x_i = 0$ for all $i = 1, \dots, n-1$. Then $\lambda_1 \leq n-2$, and the equality holds if and only if a is constant.*

Proof. Since

$$-\Delta_{\mathbb{S}^{n-2}} x_i = (n-2) x_i \quad \text{on } \mathbb{S}^{n-2} \tag{5.2}$$

for $i = 1, \dots, n-1$, multiplying the above equation by $a x_i$, and integrating over $\mathbb{S}^{n-2}$, we have, by the identity $x_i \Delta_{\mathbb{S}^{n-2}} x_i = -|\nabla_{\mathbb{S}^{n-2}} x_i|^2 + \frac{1}{2} \Delta_{\mathbb{S}^{n-2}} (x_i^2)$,

$$\begin{aligned} (n-2) \int_{\mathbb{S}^{n-2}} a x_i^2 &= - \int_{\mathbb{S}^{n-2}} a x_i \Delta_{\mathbb{S}^{n-2}} x_i \\ &= \int_{\mathbb{S}^{n-2}} a |\nabla_{\mathbb{S}^{n-2}} x_i|^2 - \frac{1}{2} \int_{\mathbb{S}^{n-2}} a \Delta_{\mathbb{S}^{n-2}} (x_i^2). \end{aligned}$$

Summing over $i = 1, \dots, n-1$, since $\sum_{i=1}^{n-1} x_i = 1$ on $\mathbb{S}^{n-2}$, we have

$$(n-2) \sum_{i=1}^{n-1} \int_{\mathbb{S}^{n-2}} a x_i^2 = \sum_{i=1}^{n-1} \int_{\mathbb{S}^{n-2}} a |\nabla_{\mathbb{S}^{n-2}} x_i|^2.$$

Thus for at least one i,

$$\int_{\mathbb{S}^{n-2}} a |\nabla_{\mathbb{S}^{n-2}} x_i|^2 \leq (n-2) \int_{\mathbb{S}^{n-2}} a x_i^2,$$

which implies $\lambda_1 \leq n-2$. If $\lambda_1 = n-2$, then by the Rayleigh quotient formula,

$$\int_{\mathbb{S}^{n-2}} a |\nabla_{\mathbb{S}^{n-2}} x_i|^2 = (n-2) \int_{\mathbb{S}^{n-2}} a x_i^2$$

for all $i = 1, \dots, n-1$. This implies

$$-\operatorname{div}_{\mathbb{S}^{n-2}} \Big(a \nabla_{\mathbb{S}^{n-2}} x_i \Big) = (n-2) a x_i \quad \text{for } i = 1, \dots, n-1. \tag{5.3}$$

By an orthogonal transformation, we have

$$-\operatorname{div}_{\mathbb{S}^{n-2}} \Big(a \nabla_{\mathbb{S}^{n-2}} (e \cdot \xi) \Big) = (n-2) a (e \cdot \xi) \quad \text{on } \mathbb{S}^{n-2}$$

for any unit vector $e \in \mathbb{R}^{n-1}$. Let $\eta \in C^\infty(\mathbb{S}^{n-2})$. Multiplying the above equation by η and integrating over $\mathbb{S}^{n-2}$, we have

$$\begin{aligned} (n-2) \int_{\mathbb{S}^{n-2}} a (e \cdot \xi) \eta &= \int_{\mathbb{S}^{n-2}} a \nabla_{\mathbb{S}^{n-2}} (e \cdot \xi) \cdot \nabla_{\mathbb{S}^{n-2}} \eta \\ &= \int_{\mathbb{S}^{n-2}} a (e - (e \cdot \xi) \xi) \cdot \nabla_{\mathbb{S}^{n-2}} \eta = \int_{\mathbb{S}^{n-2}} a e \cdot \nabla_{\mathbb{S}^{n-2}} \eta. \end{aligned}$$

This implies

$$\left| \int_{\mathbb{S}^{n-2}} a e \cdot \nabla_{\mathbb{S}^{n-2}} \eta \right| \leq C \|\eta\|_{L^1(\mathbb{S}^{n-2})} \quad \forall \eta \in C^\infty(\mathbb{S}^{n-2}).$$

Therefore, $ae \in W^{1,\infty}(\mathbb{S}^{n-2})$ and hence $a \in W^{1,\infty}(\mathbb{S}^{n-2})$. Multiplying (5.2) by a and subtracting (5.3), we have

$$\nabla_{\mathbb{S}^{n-2}} a \cdot \nabla_{\mathbb{S}^{n-2}} x_i = 0 \text{ a.e.} \quad \text{for } i = 1, \dots, n-1.$$

Since the span of $\{\nabla_{\mathbb{S}^{n-2}} x_1, \dots, \nabla_{\mathbb{S}^{n-2}} x_{n-1}\}$ is the tangent space of $\mathbb{S}^{n-2}$ at x, we have $\nabla_{\mathbb{S}^{n-2}} a = 0$ a.e. Therefore, a is constant. □

In the sequel, we will first discuss the case when $n = 3$ and then the case when $n \geq 4$.

5.1. **The case when** $n = 3$. We write $x_1 = \cos\theta$ and $x_2 = \sin\theta$, so that (5.1) takes the form

$$(x')^t M x' = \sum_{j=1}^{2} a_j x_j^2 = \frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2}\cos(2\theta), \quad a_1 \geq a_2 > 0. \tag{5.4}$$

Theorem 5.2. *For $n = 3$, let λ_1 be the first nonzero eigenvalue of the eigenvalue problem* (1.10) *with $a(\xi) = \xi^t M \xi$, where M satisfies* (5.4)*. Then λ_1 is strictly decreasing with respect to $\frac{a_1}{a_2} \in [1, \infty)$, and satisfies*

$$\frac{C_1 a_2^{1/2}}{(a_1 + a_2)^{1/2}} \leq \lambda_1 \leq \frac{C_2 a_2^{1/2}}{(a_1 + a_2)^{1/2}} \quad \text{and} \quad \lambda_1 \leq \frac{a_1 + 3a_2}{3a_1 + a_2}$$

for some positive constants C_1, C_2 independent of M. Moreover, when $a_1 > a_2$, the eigenspace corresponding to λ_1 is one dimensional, the corresponding eigenfunctions have exactly two zeros at $\theta = \pi/2$ and $3\pi/2$, and they are odd with respect to $\theta = \pi/2$ and $3\pi/2$.

We prove Theorem 5.2 through the following two lemmas. Denote $\beta = \frac{a_1 - a_2}{a_1 + a_2}$. Then $\beta \in [0, 1)$ and the eigenvalue problem (1.10) becomes

$$[(1 + \beta\cos(2\theta))u'(\theta)]' = -\lambda(1 + \beta\cos(2\theta))u(\theta) \quad \text{on } (0, 2\pi), \tag{5.5}$$

with periodic boundary condition. When $\beta = 0$, it is easy to see that $\lambda_1 = 1$.

Lemma 5.3. *For $\beta \in (0, 1)$, consider the eigenvalue problem* (5.5)*. If the first nonzero eigenvalue $\lambda_1(\beta)$ is simple, then the eigenfunctions corresponding to $\lambda_1(\beta)$ must have zeros at $\theta = 0, \pi$ or $\theta = \pi/2, 3\pi/2$.*

Proof. By [17, Theorem 3.1 in Chapter 8], the problem (5.5) has eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$. Namely, $\lambda_{2i} < \lambda_{2i+1} \leq \lambda_{2i+2}$ for $i = 0, 1, 2, \dots$. Moreover, an eigenfunction corresponding to λ_{2i+1} or λ_{2i+2} must have exactly $2i+2$ zeros on $[0, 2\pi)$. To conclude the lemma, we only need to construct two solutions u_1 and u_2 of (5.5), whose zeros are at $\theta = 0, \pi$ and $\theta = \pi/2, 3\pi/2$, respectively.

First we consider the Dirichlet problem on $(0, \pi)$:

$$\begin{cases} \big[(1 + \beta\cos(2\theta))u'(\theta)\big]' = -\mu(1 + \beta\cos(2\theta))u(\theta) & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases}$$

From the standard Sturm-Liouville theory, the first eigenvalue $\mu_1 > 0$ is simple and there exists an eigenfunction $u_1 > 0$ in $(0, \pi)$. Taking the odd extension of u_1, since $\cos(2\theta)$ is even, we know that u_1 satisfies (5.5) on $\mathbb{S}_1$ with $\lambda = \mu_1$, and u_1 only has zeros at $\theta = 0, \pi$ on $[0, 2\pi)$.

Then we consider the following Dirichlet problem on $(\pi/2, 3\pi/2)$:

$$\begin{cases} \big[(1+\beta\cos(2\theta))u'(\theta)\big]' = -\mu(1+\beta\cos(2\theta))u(\theta) & \text{in } (\pi/2, 3\pi/2), \\ u(\pi/2) = u(3\pi/2) = 0. \end{cases}$$

Let $v(\theta) = u(\theta + \pi/2)$. Then v satisfies

$$\begin{cases} \big[(1-\beta\cos(2\theta))v'(\theta)\big]' = -\mu(1-\beta\cos(2\theta))v(\theta) & \text{in } (0, \pi), \\ v(0) = v(\pi) = 0. \end{cases}$$

By the same argument as above, we know that there exist u_2 and the first eigenvalue $\mu_2 > 0$, such that u_2 satisfies (5.5) with $\lambda = \mu_2$, and u_2 only has zeros at $\theta = \pi/2, 3\pi/2$ on $[0, 2\pi)$. Therefore, $\{\mu_1, \mu_2\} = \{\lambda_1, \lambda_2\}$ and u_1, u_2 are eigenfunctions corresponding to μ_1, μ_2 respectively. $\square$

As a consequence of this lemma, the problem (5.5) can be reduced to the following Dirichlet problem in case of studying the first nonzero eigenvalue:

$$\begin{cases} \big[(1+\tilde\beta\cos(2\theta))u'(\theta)\big]' = -\mu(1+\tilde\beta\cos(2\theta))u(\theta) & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{5.6}$$

with $\tilde\beta \in (-1, 1]$. Denote $\mu_1(\tilde\beta)$ to be the first eigenvalue of (5.6), which is given by the follow Rayleigh quotient

$$\mu_1(\tilde\beta) = \inf_{u>0 \in H_0^1((0,\pi))} \frac{\int_0^\pi (1+\tilde\beta\cos(2\theta))u'(\theta)^2\, d\theta}{\int_0^\pi (1+\tilde\beta\cos(2\theta))u(\theta)^2\, d\theta}. \tag{5.7}$$

Lemma 5.4. *Consider the eigenvalue problem* (5.6) *and let $\mu_1(\tilde\beta)$ be as above. The function $\mu_1(\tilde\beta)$ is strictly increasing with respect to $\tilde\beta \in (-1, 1]$, $\mu_1(1) = 3$, and $\lim_{\tilde\beta\to -1} \mu_1(\tilde\beta) = 0$. Moreover, we have*

$$C_1(1+\tilde\beta)^{1/2} \le \mu_1(\tilde\beta) \le C_2(1+\tilde\beta)^{1/2} \quad \text{and} \quad \mu_1(\tilde\beta) \le \frac{2+\tilde\beta}{2-\tilde\beta} \tag{5.8}$$

for some constants $C_1, C_2 > 0$ independent of $\tilde\beta$.

Proof. First, suppose that $u_{\tilde\beta}$ is an eigenfunction corresponding to $\mu_1(\tilde\beta)$, which is positive on $(0, \pi)$. Since $\cos(2\theta) = \cos(2(\pi-\theta))$, it is easily seen that $u_\alpha(\pi - \cdot)$ is also an eigenfunction. Therefore, $u_{\tilde\beta}(\pi - \cdot)$ is a multiple of $u_{\tilde\beta}$. Because $\max u_{\tilde\beta}(\pi - \cdot) = \max u_{\tilde\beta}$ and both are nonnegative, we get $u_{\tilde\beta}(\pi - \cdot) = u_{\tilde\beta}$. This implies that $u_{\tilde\beta}$ can be written as an expansion of $\sin(k\theta), k = 1, 3, 5, \dots$ on $[0, \pi]$.

We define

$$A_{\tilde\beta} := \int_0^\pi |u'_{\tilde\beta}|^2 - \mu_1(\tilde\beta)\int_0^\pi |u_{\tilde\beta}|^2,$$
$$B_{\tilde\beta} := \int_0^\pi \cos(2\theta)|u'_{\tilde\beta}|^2 - \mu_1(\tilde\beta)\int_0^\pi \cos(2\theta)|u_{\tilde\beta}|^2.$$

Because $u_{\tilde\beta}$ is a solution of (5.6) with $\mu = \mu_1(\tilde\beta)$, we have $A_{\tilde\beta} = -\tilde\beta B_{\tilde\beta}$. For $\tilde\beta \in (-1, 1]$, by taking $u = \sin(\theta)$ in the Rayleigh quotient (5.7), we see that

$$\mu_1(\tilde\beta) \le (2+\tilde\beta)/(2-\tilde\beta).$$

This concludes the second inequality in (5.8). When $\tilde\beta = 1$, $u = \sin(\theta)$ is a solution to (5.6) with $\mu = 3$. Since $\sin(\theta)$ is strictly positive on $(0, \pi)$, we infer that $\mu_1(1) = 3$. Thus $\mu_1(\tilde\beta) < \mu_1(1) = 3$, and

$$\int_0^\pi (1 + \cos(2\theta))|u'_{\tilde\beta}|^2 \ge 3\int_0^\pi (1 + \cos(2\theta))|u_{\tilde\beta}|^2 > \mu_1(\tilde\beta)\int_0^\pi (1 + \cos(2\theta))|u_{\tilde\beta}|^2.$$

Namely, $A_{\tilde\beta} + B_{\tilde\beta} > 0$. Therefore, $(1 - \tilde\beta)B_{\tilde\beta} > 0$, which implies $B_{\tilde\beta} > 0$ for any $\tilde\beta \in (-1, 1)$. For any $-1 < \tilde\beta_1 < \tilde\beta < 1$, we have $A_{\tilde\beta} = -\tilde\beta B_{\tilde\beta} < -\tilde\beta_1 B_{\tilde\beta}$. Namely,

$$\int_0^\pi (1 + \tilde\beta_1 \cos(2\theta))|u'_{\tilde\beta}|^2 < \mu_1(\tilde\beta)\int_0^\pi (1 + \tilde\beta_1 \cos(2\theta))|u_{\tilde\beta}|^2.$$

Therefore, $\mu_1(\tilde\beta_1) < \mu_1(\tilde\beta)$. Next, we show the first inequality in (5.8). We may certainly assume that $\tilde\beta \in (-1, -1/2)$. Let $\varepsilon = 1 + \tilde\beta \in (0, 1/2)$. Define $u(\theta) = \varepsilon^{-1/2}\theta$ when $\theta \in [0, \varepsilon^{1/2}]$, $u(\theta) = 1$ when $\theta \in (\varepsilon^{1/2}, \pi - \varepsilon^{1/2})$, and $u(\theta) = \varepsilon^{-1/2}(\pi - \theta)$ when $\theta \in [\pi - \varepsilon^{1/2}, \pi]$. Then

$$\begin{aligned}
&\int_0^\pi (1 + \tilde\beta \cos(2\theta))|u'|^2\, d\theta = 2\int_0^{\sqrt{\varepsilon}} (1 + \tilde\beta\cos(2\theta))\varepsilon^{-1}\, d\theta \\
&= 2\int_0^{\sqrt{\varepsilon}} (\varepsilon - 2\tilde\beta \sin^2\theta)\varepsilon^{-1}\, d\theta \le C\int_0^{\sqrt{\varepsilon}} (1 - \tilde\beta\theta^2\varepsilon^{-1})\, d\theta \le C\sqrt{\varepsilon}.
\end{aligned}$$

This together with the obvious inequality

$$\int_0^\pi (1 + \tilde\beta\cos(2\theta))|u|^2\, d\theta \ge C$$

and (5.7) imply the upper bound of first inequality in (5.8). To see the lower bound, without loss of generality, we assume that

$$u_{\tilde\beta}(\theta_0) = \max_{\theta\in[0,\pi]} u_{\tilde\beta}(\theta) = 1.$$

By symmetry, we may also assume that $\theta_0 \le \pi/2$. Then by Hölder's inequality,

$$\begin{aligned}
1 = u^2_{\tilde\beta}(\theta_0) &\le \Big(\int_0^{\theta_0} |u'_{\tilde\beta}|\, d\theta\Big)^2 \\
&\le \Big(\int_0^{\theta_0} (1 + \tilde\beta\cos(2\theta))|u'_{\tilde\beta}|^2\, d\theta\Big)\Big(\int_0^{\theta_0} (1 + \tilde\beta\cos(2\theta))^{-1}\, d\theta\Big). \qquad (5.9)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^{\theta_0} (1 + \tilde\beta\cos(2\theta))^{-1}\, d\theta = \int_0^{\theta_0} (\varepsilon - 2\tilde\beta\sin^2\theta)^{-1}\, d\theta \\
&\le C\int_0^{\sqrt{\varepsilon}} \varepsilon^{-1}\, d\theta + C\int_{\sqrt{\varepsilon}}^{\theta_0} \theta^{-2}\, d\theta \le C\varepsilon^{-1/2}.
\end{aligned}$$

Thus from (5.9), we get

$$\int_0^{\theta_0} (1 + \tilde\beta\cos(2\theta))|u'_{\tilde\beta}|^2\, d\theta \ge C\varepsilon^{1/2},$$

which together with the obvious inequality

$$\int_0^\pi (1 + \tilde\beta\cos(2\theta))|u|^2\, d\theta \le C$$

and (5.7) imply the lower bound of first inequality in (5.8). Finally, from (5.8), we conclude that $\lim_{\tilde{\beta}\to -1} \mu_1(\tilde{\beta}) = 0$. The lemma is proved. □

Proof of Theorem 5.2. Let $0 < \lambda_1(\beta) \leq \lambda_2(\beta)$ denote the first and the second nonzero eigenvalues of the problem (5.5), respectively. By Lemma 5.4, $\mu_1(\tilde{\beta})$ is strictly increasing in $(-1, 1]$, so we know that $\lambda_1(\beta) = \mu_1(-\beta)$, $\lambda_2(\beta) = \mu_1(\beta)$. Therefore, λ_1 being strictly decreasing with respect to $\frac{a_1}{a_2} \in [1, \infty)$ follows from the monotonicity of $\mu_1(\tilde{\beta})$ for $\tilde{\beta} \in (-1, 0)$. The inequalities in Theorem 5.2 follow from (5.8) with $\tilde{\beta} = -\beta = -\frac{a_1 - a_2}{a_1 + a_2}$. Finally, when $\beta > 0$, we can see from the proof of Lemma 5.3 that the eigenspace corresponding to λ_1 is one dimensional, the corresponding eigenfunctions have exactly two zeros at $\theta = \pi/2$ and $3\pi/2$, and they are odd with respect to $\theta = \pi/2$ and $3\pi/2$. □

5.2. **Higher dimensional case.** In this subsection, we consider the case when $n \geq 4$. We will show that there exists a small constant ε_0, depending only on n, such that if

$$(1 - \varepsilon_0)\frac{I}{\|I\|} \leq \frac{M}{\|M\|} \leq (1 + \varepsilon_0)\frac{I}{\|I\|},$$

the eigenspace corresponding to the first nonzero eigenvalue λ_1 of (1.10) satisfies the property O, which is defined as follows.

Definition 5.5. We say that a function space on $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$ satisfies the property O if it is the span of functions which are odd in one of the x_i variables and even with respect to other variables.

Indeed, we consider the following operator on $\mathbb{S}^{n-2}$:

$$L_\mu = -\mathrm{div}_{\mathbb{S}^{n-2}}\big((1 + \mu b(x))\nabla_{\mathbb{S}^{n-2}}\big) \quad \text{for } \mu \in \mathbb{R},$$

where $b(x) \in L^\infty(\mathbb{S}^{n-2})$. Let $\lambda_{1,\mu}$ and $V_{1,\mu}$ be the corresponding first nonzero eigenvalue and the eigenspace of the eigenvalue problem

$$L_\mu u = \lambda(1 + \mu b(x))u.$$

Proposition 5.6. *Consider the above eigenvalue problem, and assume that $b(x)$ is even with respect all variables, and V_{1,μ_0} satisfies the property O for some $\mu_0 \in \mathbb{R}$. Then there exists a constant ε_0, depending only on n, an upper bound of $\|b\|_{L^\infty}$, and μ_0, such that $V_{1,\mu}$ also satisfies the property O for any $\mu \in (\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0)$.*

Proof. Suppose that an orthogonal basis of V_{1,μ_0} is given by $\{f_1, \ldots, f_m\}$ with $m \in \{1, \ldots, n-1\}$, where for $j = 1, \ldots, m$, f_j is odd in x_j and even in other variables. Let $\varepsilon_0 > 0$ be a small constant to be specified later. The perturbation argument below gives all the eigenfunctions of L_μ close to λ_{1,μ_0} when μ is in a small neighborhood of μ_0.

For any $\mu \in (\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0)$, we consider the expansions

$$\tilde{\lambda}_1 = \lambda_{1,\mu_0} + \sum_{k=1}^{\infty} \varepsilon^k c_k, \quad \tilde{f}_1 = f_1 + \sum_{k=1}^{\infty} \varepsilon^k v_k, \tag{5.10}$$

where $\varepsilon = \mu - \mu_0 \in (-\varepsilon_0, \varepsilon_0)$. Then

$$L_\mu \tilde{f}_1 = \tilde{\lambda}_1(1 + \mu b(x))\tilde{f}_1$$

is equivalent to

$$\begin{aligned}&\Big[\Big(\lambda_{1,\mu_0}+\sum_{k=1}^{\infty}\varepsilon^k c_k\Big)\Big(1+(\mu_0+\varepsilon)b(x)\Big)-L_{\mu_0}\Big]\Big(f_1+\sum_{k=1}^{\infty}\varepsilon^k v_k\Big)\\&=-\varepsilon\mathrm{div}_{\mathbb{S}^{n-2}}\Big(b(x)\nabla_{\mathbb{S}^{n-2}}\Big(f_1+\sum_{k=1}^{\infty}\varepsilon^k v_k\Big)\Big).\end{aligned}\tag{5.11}$$

To solve for $c_k, v_k, k=1,\ldots$, we compare the coefficients of ε^k on both sides of (5.11). The zeroth order term on the left-hand side is equal to zero because f_1 is an eigenfunction of L_{μ_0} with the eigenvalue λ_{1,μ_0}.

Considering the first order terms, we get

$$\begin{aligned}&\big(\lambda_{1,\mu_0}(1+\mu_0 b(x))-L_{\mu_0}\big)v_1\\&=-(\lambda_{1,\mu_0}b(x)+c_1(1+\mu_0 b(x)))f_1-\mathrm{div}_{\mathbb{S}^{n-2}}\big(b(x)\nabla_{\mathbb{S}^{n-2}}f_1\big).\end{aligned}\tag{5.12}$$

Let X_0 and X_2 be the orthogonal (complement) spaces of $V_{1,\mu}$ in $L^2(\mathbb{S}^{n-2})$ and $H^2(\mathbb{S}^{n-2})$, respectively. Then because $\lambda_{1,\mu_0}(1+\mu_0 b(x))-L_{\mu_0}$ is self-adjoint, it is easily seen that

$$(\lambda_{1,\mu_0}(1+\mu_0 b(x))-L_{\mu_0})X_2\subset X_0.$$

Moreover, the mapping $\lambda_{1,\mu_0}(1+\mu_0 b(x))-L_{\mu_0}: X_2\to X_0$ is injective. By the Fredholm theorem, we also know that the mapping is surjective. Therefore, $\lambda_{1,\mu_0}(1+\mu_0 b(x))-L_{\mu_0}$ has a bounded inverse $\mathcal{R}$ from X_0 to X_2, and (5.12) has a unique solution $v_1\in X_2$ if and only if its right-hand side is orthogonal to $f_1,\ldots,f_m$. Since the right-hand side is odd in x_1 and even in the other variables, we know that it is orthogonal to $f_2,\ldots,f_m$. To make it to be also orthogonal to f_1, we find a unique c_1 given by

$$c_1=\frac{\int_{\mathbb{S}^{n-2}}\Big(b(x)|\nabla_{\mathbb{S}^{n-2}}f_1|^2-\lambda_{1,\mu_0}b(x)f_1^2\Big)}{\int_{\mathbb{S}^{n-2}}(1+\mu_0 b(x))f_1^2}.$$

Then

$$v_1=-\mathcal{R}\Big((\lambda_{1,\mu_0}b(x)+c_1(1+\mu_0 b(x)))f_1+\mathrm{div}_{\mathbb{S}^{n-2}}\big(b(x)\nabla_{\mathbb{S}^{n-2}}f_1\big)\Big)\in X_2,$$

which is odd in x_1 and even in other variables because $b(x)$ is even with respect to all variables.

Now considering the second order terms, we get

$$\begin{aligned}(\lambda_{1,\mu_0}(1+\mu_0 b(x))-L_{\mu_0})v_2=&-\big(c_2(1+\mu_0 b(x))+c_1 b(x)\big)f_1\\&-(\lambda_{1,\mu_0}b(x)+c_1(1+\mu_0 b(x)))v_1-\mathrm{div}_{\mathbb{S}_{n-2}}\big(b(x)\nabla_{\mathbb{S}_{n-2}}v_1\big).\end{aligned}$$

As before, the right-hand side above is orthogonal to $f_2,\ldots,f_m$. To make it to be also orthogonal to f_1, we find a unique c_2 given by

$$c_2=\frac{\int_{\mathbb{S}_{n-2}}\Big(b\nabla_{\mathbb{S}_{n-2}}v_1\cdot\nabla_{\mathbb{S}_{n-2}}f_1-(\lambda_{1,\mu_0}b+c_1(1+\mu_0 b))v_1 f_1-c_1 b f_1^2\Big)}{\int_{\mathbb{S}_{n-2}}(1+\mu_0 b)f_1^2}.$$

Then

$$\begin{aligned}v_2=-\mathcal{R}\Big(&\big(c_2(1+\mu_0 b(x))+c_1 b(x)\big)f_1\\&+(\lambda_{1,\mu_0}b(x)+c_1(1+\mu_0 b(x)))v_1+\mathrm{div}_{\mathbb{S}_{n-2}}\big(b(x)\nabla_{\mathbb{S}_{n-2}}v_1\big)\Big)\in X_2,\end{aligned}$$

which is odd in x_1 and even in other variables.

We can repeat this procedure and solve all the c_k and v_k's inductively. Moreover, all the v_k's are odd in x_1 and even in other variables. Note that $|c_k|$ and the H^2 norm of v_k can be bounded by C^k (C to the power of k), where C is some positive constant depending only on f_1, $b(x)$, and the norm of $\mathcal{R}$. Therefore, by taking ε_0 sufficiently small (with the same dependence), both series in (5.10) are convergent in $\mathbb{R}$ and $H^2(\mathbb{S}^{n-2})$ respectively, and $\tilde{f}_1$ is odd in x_1 and even in other variables. Similarly, we can find the eigenpairs $(\tilde{\lambda}_j, \tilde{f}_j)$ for $j = 2, \ldots, m$. From the min-max formula of the eigenvalues, we know that every eigenvalue is Lipschitz in μ. In particular, for ε_0 sufficiently small, we know that $\lambda_{1,\mu} = \min\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m\}$ and $V_{1,\mu}$ is spanned by $\tilde{f}_j$'s for those j such that $\tilde{\lambda}_j = \lambda_{1,\mu}$. Therefore, $V_{1,\mu}$ satisfies the property O. □

Applying Proposition 5.6 with $n \geq 4$, $\|b\|_{L^\infty} \leq 8$, $\mu_0 = 0$, we obtain an ε_0 such that $V_{1,\mu}$ also satisfies the property O for any $\mu \in (-\varepsilon_0, \varepsilon_0)$. Setting

$$b(x) = \sum_{i=1}^{n-2} \frac{a_i - a_{n-1}}{\varepsilon_0 a_{n-1}} 2x_i^2 \quad \text{and} \quad \mu = \frac{\varepsilon_0}{2},$$

we have

$$1 + \mu b(x) = \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} a_i x_i^2.$$

Therefore, the following corollary follows.

Corollary 5.7. *For $n \geq 4$, there exists a small constant ε_0 depending only on n, such that if*

$$(1 - \varepsilon_0)\frac{I}{\|I\|} \leq \frac{M}{\|M\|} \leq (1 + \varepsilon_0)\frac{I}{\|I\|},$$

the eigenspace corresponding to the first nonzero eigenvalue λ_1 of (1.10) *with $a(\xi) = \xi^t M \xi$ satisfies the property O.*

The following lemma will not be used in this paper.

Lemma 5.8. *For $n \geq 4$, let λ_1 be the first nonzero eigenvalue to the eigenvalue problem* (1.10) *with $a(\xi) = \xi^t M \xi$, where M satisfies* (5.1)*. Then for any $a_2 \geq \ldots \geq a_{n-1} > 0$, we have $\lambda_1 \to 0$ as $a_1 \to +\infty$.*

Proof. We consider in the spherical coordinate: for $x' \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, we can write

$$x_1 = \cos\theta_1, \quad x_2 = \sin\theta_1 \cos\theta_2, \quad x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3, \ldots,$$

$$x_{n-2} = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \cos\theta_{n-2}, \quad x_{n-1} = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \sin\theta_{n-2},$$

where $\theta_1, \theta_2, \ldots, \theta_{n-3} \in [0, \pi]$ and $\theta_{n-2} \in [0, 2\pi)$. Then the proof is similar to that of the first upper bound in (5.8) by considering $u(\theta_1) = \max\{-1, \min\{1, \varepsilon^{-1}(\theta_1 - \pi/2)\}\}$. □

6. Proof of Theorem 1.6

Proof of Theorem 1.6. In this proof, we denote $\alpha = \alpha(\lambda_1)$ for simplicity. After a suitable rotation in $\mathbb{R}^{n-1}$, we may assume without loss of generality that

$$(f - g)(x') = \sum_{j=1}^{n-1} a_j x_j^2 + e(x'),$$

where $|e(x')| \le C|x'|^4$.

Step 1. Since $\widetilde{\Omega}$ is symmetric in x_i and φ is odd in x_j, by the uniqueness of solutions, we know that u is odd in x_j. In Ω_1, where Ω_r is defined as in (1.7), let $\bar{u}$ be defined as (3.2). By a similar argument as in Section 3 and Theorem 1.1, we know that $\bar{u}$ satisfies

$$\operatorname{div}\Big[\Big(\sum_{i=1}^{n-1} a_i|x_i|^2\Big)\nabla\bar{u}\Big] = \operatorname{div} F \quad \text{in } B_1 \subset \mathbb{R}^{n-1},$$

where F satisfies

$$|F(x')| \le C|x'|^{2+\alpha} \quad \text{for } x' \in B_1, \tag{6.1}$$

C is a positive constant depending only on n and upper bounds of $\|\partial D_1\|_{C^4}$ and $\|\partial D_2\|_{C^4}$. Again, we denote $Y_{k,i}$ to be a normalized eigenfunction corresponding to $(k+1)$-th eigenvalue λ_k of the problem (1.10), so that $\{Y_{k,i}\}_{k,i}$ forms an orthonormal basis of $L^2(\mathbb{S}^1)$ under the inner product (1.11). By the assumption, we denote $Y_{1,j}$ to be the eigenfunction that is odd in x_j. It is easily seen that $Y_{1,j}$ is an eigenfunction corresponding to λ_1 in the half sphere $\mathbb{S}^{n-2}\cap\{x_j>0\}$ with zero Dirichlet boundary condition. Since λ_1 is the first nonzero eigenvalue of the eigenvalue problem in the sphere, it must be the first eigenvalue of the eigenvalue problem in the half sphere. Therefore, it is simple and $Y_{1,j}$ does not change its sign in the half sphere. Without loss of generality, we assume $Y_{1,j}$ is positive in $\{x_j>0\}$ and negative in $\{x_j<0\}$. Since $\bar{u}$ is odd with respect to $x_j=0$, and in particular $\bar{u}(0)=0$, we have the following decomposition

$$\bar{u}(x') = \sum_{k=1}^{\infty}\sum_{i=1}^{N(k)} U_{k,i}(r)Y_{k,i}(\xi), \quad x' \in B_1\setminus\{0\}, \tag{6.2}$$

where $U_{k,i}(r) = \fint_{\mathbb{S}^1} a(\xi)\bar{u}(r,\xi)Y_{k,i}(\xi)\,d\xi$ and $U_{k,i}\in C([0,1))\cap C^\infty((0,1))$. Then $U_{1,j}$ satisfies $U_{1,j}(0)=0$ and

$$LU_{1,j} := U''_{1,j}(r) + \frac{n}{r}U'_{1,j}(r) - \frac{\lambda_1}{r^2}U_{1,j}(r) = H(r), \quad 0<r<1,$$

where

$$\begin{aligned} H(r) &= \int_{\mathbb{S}^{n-2}} \frac{(\operatorname{div} F)Y_{1,1}(\xi)}{a(\xi)r^2}\,d\xi = \int_{\mathbb{S}^{n-2}} \frac{\partial_r F_r + \frac{1}{r}\nabla_\xi F_\xi}{a(\xi)r^2}Y_{1,1}(\xi)\,d\xi \\ &= \partial_r\Big(\int_{\mathbb{S}^{n-2}} \frac{F_r}{a(\xi)r^2}Y_{1,1}(\xi)\,d\xi\Big) + \int_{\mathbb{S}^{n-2}} \frac{2F_rY_{1,1}}{a(\xi)r^3} - \frac{F_\xi}{r^3}\nabla_\xi\Big(\frac{Y_{1,1}(\xi)}{a(\xi)}\Big)\,d\xi \\ &=: A'(r) + B(r), \quad 0<r<1, \end{aligned}$$

and $A(r), B(r)\in C^1([0,1))$ satisfy, in view of (6.1), that

$$|A(r)| \le C(n)r^\alpha, \quad |B(r)| \le C(n)r^{\alpha-1}, \quad 0<r<1. \tag{6.3}$$

Step 2. We will prove, for some constant C_1, that

$$U_{1,j}(r) = C_1 r^\alpha + v(r), \quad 0<r<1, \tag{6.4}$$

where $|v(r)|\le Cr^{1+\alpha}$. We use the method of reduction of order to find a bounded solution v satisfying $Lv=H$ in $(0,1)$, and then show that $|v(r)|\le Cr^{1+\alpha}$. Note that $h=r^\alpha$ is a solution of $Lh=0$. Let $v=hw$ and

$$w(r) := \int_0^r \frac{1}{s^{n+2\alpha}}\int_0^s \tau^{n+\alpha}H(\tau)\,d\tau ds, \quad 0<r<1.$$

By a direct computation,

$$Lv = L(hw) = hw'' + \left(2h' + \frac{n}{r}h\right) w' = H.$$

By (6.3), we can estimate $|w(r)| \le Cr$. Therefore, $|v(r)| \le Cr^{1+\alpha}$. Since $U_{1,j} - v$ is bounded and satisfies $L(U_{1,j} - v) = 0$ in $(0,1)$, we know that $U_{1,j} = C_1 h + v$ and (6.4) follows.

Step 3. Completion of the proof.

Since D_1 and D_2 are strictly convex and symmetric in $x_1, \dots, x_{n-1}$, it is easy to see that $\partial_\nu x_j \ge 0$ in $\{x_j \ge 0\}$ and $\partial_\nu x_j \le 0$ in $\{x_j \le 0\}$. Therefore, under the assumptions of Theorem 1.6, x_j is a subsolution of (1.2) in $\{x_j \ge 0\}$, and is a supersolution of (1.2) in $\{x_j \le 0\}$. Hence, $u \ge x_j$ in $\{x_j \ge 0\}$ and $u \le x_j$ in $\{x_j \le 0\}$. Then, $|\bar{u}(x')| \ge |x_j|$ in $B_1 \subset \mathbb{R}^{n-1}$. Since $Y_{1,j}$ has the same sign as x_j, we have

$$U_{1,j} = \fint_{\mathbb{S}^{n-2}} a(\xi)\bar{u}(r,\xi) Y_{1,j}(\xi)\, d\xi \ge Cr$$

for some positive constant C. This implies $C_1 > 0$. By (6.2) and (6.4), we have

$$\left(\int_{\mathbb{S}^{n-2}} a(\xi) |\bar{u}(r,\xi)|^2 \, d\xi \right)^{1/2} \ge |U_{1,j}(r)| \ge \frac{C_1}{2} r^\alpha \quad \text{for } 0 < r < r_0,$$

where r_0 is some small positive constant. Then, for any $r \in (0, r_0)$, there exists a $\xi_0(r) \in \mathbb{S}^{n-2}$ such that

$$|\bar{u}(r, \xi_0(r))| \ge \frac{1}{C_2} r^\alpha$$

for some positive constant C_2. Since $\bar{u}$ is the average of u in the x_n direction, by (1.9) with $\varepsilon = 0$, we have

$$|u(r, \xi_0(r), 0) - \bar{u}(r, \xi_0(r))| \le Cr^2 \sup_{x_n \in (g(x'), f(x'))} |\partial_{x_n} u(r, \xi_0(r), x_n)| \le Cr.$$

Therefore, there exists a small constant r_1 such that for any $r \in (0, r_1)$,

$$|u(r, \xi_0(r), 0)| \ge \frac{1}{2C_2} r^\alpha.$$

We denote $x_0 = (r, \xi_0(r), 0)$. For a sufficiently large constant C_3, independent of x_0, we have, by Theorem 1.1,

$$\left| u\left(\frac{x_0}{C_3}\right)\right| \le C \left(\frac{|x_0'|}{C_3}\right)^\alpha \le \frac{1}{4C_2} |x_0'|^\alpha.$$

Therefore, there exists an x on the line segment between x_0 and x_0/C_3, such that

$$|\nabla u(x)| \ge \frac{1}{C} |x'|^{\alpha - 1}$$

for some positive constant C depending only on n, a positive lower bound of the eigenvalues of $D^2(f-g)(0')$, and upper bounds of $\|\partial D_1\|_{C^4}$ and $\|\partial D_2\|_{C^4}$. This concludes the proof. □

7. The variable coefficients case

In this section, we study the insulated problem with variable coefficients in dimension $n \geq 3$:

$$\begin{cases} -\partial_i(A^{ij}(x)\partial_j u(x)) = 0 & \text{in } \Omega_{R_0}, \\ A^{ij}(x)\partial_j u(x)\nu_i = 0 & \text{on } \Gamma_+ \cup \Gamma_-, \\ \|u\|_{L^\infty(\Omega_{R_0})} \leq 1, \end{cases} \tag{7.1}$$

where Γ_+ and Γ_- are given in (1.4), $(A^{ij}(x)) \in C^\gamma(\Omega_{R_0}), \gamma > 0$ is symmetric and uniformly elliptic with the Lipschitz constant σ, i.e.

$$A^{ij}(x) = A^{ji}(x), \quad \sigma I \leq A(x) \leq \frac{1}{\sigma} I.$$

We want to find a point x_0 and a linear transformation l, so that after the linear transformation, the coefficients A^{ij} becomes δ_{ij} at the point $l(x_0)$, and $l(x_0)$ is the middle point of the closet points of $\widetilde{\Gamma}_+ := l(\Gamma_+)$ and $\widetilde{\Gamma}_- := l(\Gamma_-)$. Then we can apply Theorem 1.1 or 1.3 to get the gradient estimates.

When $\varepsilon = 0$, x_0 is the origin, and $l = C^{-1}(0)$, where $C(x) = \sqrt{A(x)}$. When $\varepsilon \neq 0$, by the change of variables

$$\begin{cases} y' = x', \\ y_n = x_n - g(x'), \end{cases} \quad \forall (x', x_n) \in \Omega_{R_0},$$

we may assume that $g \equiv 0$. Then any linear transformation l (with no translation) maps the lower boundary Γ_- to a hyperplane $\widetilde{\Gamma}_-$. It also maps the tangent plane $x_n = \varepsilon/2$ of the upper boundary Γ_+ to the tangent plane of $\widetilde{\Gamma}_+$, which is paralleled to $\widetilde{\Gamma}_-$ as the mapping is linear. Then $l(e_n\varepsilon/2)$ is the closest point on $\widetilde{\Gamma}_+$ to $\widetilde{\Gamma}_-$. Let $C(x) = \sqrt{A(x)}$ and C_n be the last column of $C(x)$. We have the following Lemma.

Lemma 7.1. *Under the settings above, let $R = \sqrt{n-1}\varepsilon/(2\sigma^2)$, there exists $x_0 \in \overline{B}_R \cap \{x_n = 0\}$ such that with the mapping $l = C^{-1}(x_0)$, $l(x_0)$ is the middle point of the closet points of $\widetilde{\Gamma}_+$ and $\widetilde{\Gamma}_-$.*

Proof. It is easily seen that the normal direction of $\widetilde{\Gamma}_-$ is given by $C_n(x_0)$. By linearity, the distance from $l(x_0)$ to $l(\{x_n = \varepsilon/2\})$ is equal to the distance from $l(x_0)$ to $\widetilde{\Gamma}_-$. Thus, it suffices to have $l(x_0 - e_n\varepsilon/2) \| C_n(x_0)$. This is equivalent to $x_0 - e_n\varepsilon/2 \| C(x_0)C_n(x_0)$, where $C(x_0)C_n(x_0) =: A_n(x_0)$ is the last column of $A(x_0)$. Thus, we only need to have

$$(x_0)' = -\frac{\varepsilon}{2}\frac{(A_n(x_0))'}{A^{nn}(x_0)},$$

where $(x_0)'$ and $(A_n(x_0))'$ are the first $n-1$ components of x_0 and $A_n(x_0)$, respectively. Now we define a mapping T on R^{n-1} by

$$Ty = -\frac{\varepsilon}{2}\frac{(A_n(y,0))'}{A^{nn}(y,0)}.$$

Clearly, T is continuous. Since $A^{nn} \geq \sigma$ and $|A^{ni}| \leq 1/\sigma$, for $i = 1, 2, \ldots, n-1$, we have $|Ty| \leq R$ for any $y \in \overline{B}_R$. By the Brouwer fixed point theorem, T has a fixed point $(x_0)' \in \overline{B}_R$. □

After applying this linear transform and picking an appropriate coordinate system, we reduce the problem (7.1) to the case when $A^{ij}(0) = \delta_{ij}$. Therefore, Theorems 1.1 and 1.3 apply.

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