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Jacobian determinants for nonlinear gradient of planar ∞-harmonic functions and applications

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Abstract. We introduce a distributional Jacobian determinant $\det DV_\beta(Dv)$ in dimension two for the nonlinear complex gradient $V_\beta(Dv) = |Dv|^\beta(v_{x_1}, -v_{x_2})$ for any $\beta > -1$, whenever $v \in W^{1,2}_{\mathrm{loc}}$ and $\beta|Dv|^{1+\beta} \in W^{1,2}_{\mathrm{loc}}$. This is new when $\beta \neq 0$. Given any planar ∞-harmonic function u, we show that such distributional Jacobian determinant $\det DV_\beta(Du)$ is a nonnegative Radon measure with some quantitative local lower and upper bounds. We also give the following two applications.

(i) Applying this result with $\beta = 0$, we develop an approach to build up a Liouville theorem, which improves that of Savin. Precisely, if u is an ∞-harmonic function in the whole $\mathbb{R}^2$ with

$$\liminf_{R\to\infty} \inf_{c\in\mathbb{R}} \frac{1}{R} \fint_{B(0,R)} |u(x) - c|\, dx < \infty,$$

then $u = b + a \cdot x$ for some $b \in \mathbb{R}$ and $a \in \mathbb{R}^2$.

(ii) Denoting by u_p the p-harmonic function having the same nonconstant boundary condition as u, we show that $\det DV_\beta(Du_p) \to \det DV_\beta(Du)$ as $p \to \infty$ in the weak-$\star$ sense in the space of Radon measure. Recall that $V_\beta(Du_p)$ is always quasiregular mappings, but $V_\beta(Du)$ is not in general.

1. Introduction

Let Ω be a domain (connected open subset) in $\mathbb{R}^n$. We say a function $u \in C^0(\Omega)$ is ∞-harmonic if it is a viscosity solution to the ∞-Laplace equation

$$\Delta_\infty u = D^2u\, Du \cdot Du = 0 \quad \text{in } \Omega.$$

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This equation was derived by Aronsson in the 1960s as the Euler–Lagrange equation for absolute minimizers with respect to the L^∞-functional

$$F(u, U) = \left\| \frac{1}{2}|Du|^2 \right\|_{L^\infty(U)} \quad \text{for domains } U \Subset \Omega.$$

See [1–3, 5]. For a probability interpretation (via Tug-of-War) of the ∞-Laplace equation, we refer the reader to [31]. Jensen [24] identified absolute minimizers with ∞-harmonic functions and, moreover, built up their existence and uniqueness in bounded domains with continuous boundary data. Since then, the regularity of ∞-harmonic functions has been a main issue in this field. Recall that ∞-harmonic functions are always locally Lipschitz and hence differentiable almost everywhere. In view of the ∞-harmonic function $w = x_1^{4/3} - x_2^{4/3}$ in $\mathbb{R}^n$ given by Aronsson [6], it was conjectured in the literature that ∞-harmonic functions have $C^{1,1/3}$ and also $W^{2,q}$ regularity with $q < 3/2$.

Towards this conjecture, Crandall–Evans [11] first obtained a linear approximation property for any ∞-harmonic function u: at each point x and for any sequence $\{r_j\}_{j\in\mathbb{N}}$ converging to 0, there are a subsequence $\{r_{j_k}\}_{k\in\mathbb{N}}$ and also a vector e depending on x and $\{r_{j_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sup_{z\in B(0,1)} \left| \frac{u(x + r_{j_k} z) - u(x)}{r_{j_k}} - e \cdot z \right| = 0$$

and $|e| = \operatorname{Lip} u(x)$, where and in the sequel the pointwise Lipschitz constant of u at x is defined as

$$\operatorname{Lip} u(x) = \limsup_{x\neq y\to x} \frac{|u(y) - u(x)|}{|x - y|}.$$

The vector e was then proved to be independent of the choice of subsequence, which implies that u is differentiable at any point x. See Savin [34] for dimension two based on a planar topological argument and Evans–Smart [20,21] for dimensions $n \geq 2$ via some PDE approach (flatness estimates). In dimension two, Savin [34] further proved the C^1 regularity of u, and Evans–Savin [19] obtained the $C^{1,\alpha}$-regularity of u for some $0 < \alpha \ll 1/3$. Recently, it was proved in [25] that $|Du|^\alpha \in W^{1,2}$ for any $\alpha > 0$, which is sharp as $\alpha \to 0$ as witted by $x_1^{4/3} - x_2^{4/3}$. Moreover, the distributional determinant of the Hessian, $-\det D^2 u$, was proved in [25] to be a Radon measure (in short, $-\det D^2 u \in \mathcal{M}(\Omega)$) enjoying the lower bound

$$-\det D^2 u \geq \big|D|Du|\big|^2\, dx$$

in measure sense, i.e.,

$$\int_\Omega -\det D^2 u \psi\, dx \geq \int_\Omega \big|D|Du|\big|^2 \psi\, dx \quad \text{for all } 0 \leq \psi \in C_c^0(\Omega), \tag{1.1}$$

and also the upper bound

$$\int_{\frac{1}{2}B} -\det D^2 u\, dx \leq C \fint_B |Du|^2\, dx \quad \text{for all balls } B \Subset \Omega.$$

Recall that, for any $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$, the distributional determinant $-\det D^2 v$ is given by

$$\int_\Omega \det D^2 v \psi\, dx = \frac{1}{2} \int_\Omega (D^2\psi\, Dv \cdot Dv)\, dx - \frac{1}{2} \int_\Omega |Dv|^2 \Delta\psi\, dx \quad \text{for all } \psi \in C_c^\infty(\Omega). \tag{1.2}$$

The main purpose of this paper is two-fold. First, via the distributional determinant of the Hessian, we develop a new approach to build up a gradient estimate and a Liouville theorem for planar ∞-harmonic functions. See Theorem 1.1 below. Recall that Aronsson [4] initiated the study of such Liouville theorems by proving that planar ∞-harmonic functions of $C^2(\mathbb{R}^2)$ must be affine functions. In the sequel, we denote by $\mathscr{P}$ the collection of affine functions $P(x) = b + a \cdot x$ for some $b \in \mathbb{R}$ and $a \in \mathbb{R}^2$. Evans [17] obtained an analogue result for ∞-harmonic functions of $C^4(\mathbb{R}^n)$ with $n \geq 3$. In all dimensions $n \geq 2$, Crandall–Evans–Gariepy [12] showed that any bounded ∞-harmonic function in $\mathbb{R}^n$ must be a constant, and also that any ∞-(sub)harmonic function in $\mathbb{R}^n$ bounded from above by some affine function P must be given by P. In the plane, from the C^1-regularity and a compactness argument, Savin [34] proved that any ∞-harmonic function u in $\mathbb{R}^2$ with the linear growth at ∞ (i.e., $|u(x)| \leq C(1 + |x|)$) must be an affine function. However, a high-dimensional analogue is quite open.

We obtain the following interior gradient estimate and Liouville theorem in the plane, the latter of which improves that of Savin [34] mentioned above.

Theorem 1.1. *The following statements hold.*

(i) *Let u be an ∞-harmonic function in a domain $\Omega \subset \mathbb{R}^2$. Then we have*

$$|Du(x)| \leq \frac{C}{r} \fint_{B(x,r)} |u| \, dy \quad \text{whenever } B(x,r) \subset \Omega,$$

and hence

$$\|Du\|_{L^\infty(B(x,r))} \leq \frac{C}{r^3} \|u\|_{L^1(B(x,2r))} \quad \text{whenever } B(x,2r) \subset \Omega.$$

(ii) *Let u be an ∞-harmonic function in $\mathbb{R}^2$ with*

$$\liminf_{R\to\infty} \inf_{c\in\mathbb{R}} \frac{1}{R} \fint_{B(0,R)} |u(x) - c| \, dx < \infty.$$

Then $u \in \mathscr{P}$, i.e., $u(x) = u(0) + a \cdot x$ in $\mathbb{R}^2$ for some vector $a \in \mathbb{R}^2$.

Our approach of the proof of the Liouville theorem is completely different from that of Savin [34]. The crucial point is that, given any ∞-harmonic function u in a planar domain Ω,

- in Lemma 2.2, we derive the identity

$$\begin{aligned}\int_\Omega (-\det D^2u) u^2 \psi \, dx = &-\int_\Omega |Du|^4 \psi \, dx - \int_\Omega |Du|^2 (Du \cdot D\psi) u \, dx \\ &+ \frac{1}{2} \int_\Omega u^2 [|Du|^2 \Delta\psi - D^2\psi \, Du \cdot Du] \, dx\end{aligned}$$

  for all $\psi \in C_c^\infty(\Omega)$, which implies Theorem 1.1 (i);
- in Proposition 2.5, we obtain an upper bound which improves the one in [25],

$$\int_{\frac{1}{4}B} -\det D^2u \, dx \leq C \inf_{P\in\mathscr{P}} \left\| \frac{u-P}{r} \right\|_{L^\infty(B)} \left[|DP| + \left\| \frac{u-P}{r} \right\|_{L^\infty(B)} \right]$$

  for all balls $B = B(x,r) \Subset \Omega$.

Both results have their own interests. These, together with (1.1) and some basic properties, allow us to obtain Theorem 1.1 (ii). In particular, we obtain an improved Caccioppoli-type estimate: whenever $B(x, 2r) \Subset \Omega$,

$$\int_{B(x,r)} \big|D|Du|\big|^2 \, dz \le C \inf_{P \in \mathcal{P}} \fint_{B(x,2r)} |Du - DP|^2 \, dz.$$

This is better than the one in [25] as we can subtract a constant vector field in the integrand on the right-hand side of the inequality above. However, on the left-hand side, we only have $D|Du|$ due to the degeneracy of the equation, for which heuristically there is only one direction of the Hessian of u. See Section 2 for details. We remark that, once a global Lipschitz bound of u is obtained, one can also appeal to the result in [34] mentioned above to conclude that u must be an affine function.

Next, inspired by the limiting behavior of planar p-harmonic functions (see the end of this section for details), we are interested in the Jacobian determinants of the nonlinear complex gradient

$$V_\beta(Du) = |Du|^\beta (u_{x_1}, -u_{x_2}) \quad \text{with } \beta > -1$$

for planar ∞-harmonic functions. In the special case $\beta = 0$, $\det DV_0(Du) = -\det D^2 u$ is already defined by (1.2) as a distribution. However, in the general case $\beta \ne 0$, since the Sobolev regularity of $|Du|^\beta Du$ is quite open, there is no appropriate definition for

$$\det DV_\beta(Dv) = -\det D[|Du|^\beta Du]$$

available in the literature. In this paper, we find out the following distributional definition, which has its own interest.

Definition 1.2. Let $\Omega \subset \mathbb{R}^2$ be a domain. For any $\beta > -1$ and $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfying $\beta |Dv|^{\beta+1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we define $\det DV_\beta(Dv)$ as a distribution by

$$\begin{aligned} \int_\Omega \det DV_\beta(Dv) \psi \, dx &:= -\frac{1}{2} \int_\Omega |Dv|^{2\beta} (D^2\psi \, Dv \cdot Dv) \, dx \\ &\quad + \frac{1}{2\beta+2} \int_\Omega |Dv|^{2\beta+2} \Delta\psi \, dx \\ &\quad - \frac{\beta}{\beta+1} \int_\Omega [D|Dv|^{\beta+1} \cdot Dv](Dv \cdot D\psi) |Dv|^{\beta-1} \, dx \end{aligned} \tag{1.3}$$

for all $\psi \in C_c^\infty(\Omega)$.

Compared to (1.2) in the case when $\beta = 0$, we need the additional assumption

$$|Du|^{1+\beta} \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

in the case when $\beta \ne 0$.

Before using Definition 1.2, we must first verify that it makes sense. To be precise, if $V_\beta(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ a priori, then we have a pointwise defined Jacobian determinant

$$\det DV_\beta(Dv) \in L^1_{\mathrm{loc}}(\Omega).$$

On the other hand, since $V_\beta(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ implies $|Dv|^{1+\beta} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, Definition 1.2 gives a distributional Jacobian determinant $\det DV_\beta(Dv)$. One has to show the coincidence between the pointwise definition and the distributional definition of $\det DV_\beta(Dv)$. In the case when $\beta = 0$, such coincidence is well known. Indeed, such coincidence holds for smooth functions $v \in C^3(\Omega)$ directly via the divergence of structure of $-\det D^2 v$, and for $v \in W^{2,2}_{\rm loc}(\Omega)$ via a standard approximation argument and linearity of Dv and $D^2 v$. However, in the general case when $\beta \neq 0$, due to several essential difficulties caused by the nonlinear structure of $V_\beta(Dv)$, essentially new ideas are required to get such coincidence. Eventually, we are able to prove such coincidence via

- a nonlinear second-order estimate to inhomogeneous $(\beta+2)$-Laplace equations given by Cianchi–Maz'ya [10],
- a fundamental structural identity and a divergence structure for
$$-\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv] \quad \text{with } v \in C^\infty(\Omega) \text{ and } \varepsilon > 0$$
(see Lemmas 3.7 and 3.2),
- a divergence structure of $-\det D^2 v$ with $v \in C^\infty$ and its connection with $\Delta_\infty v$ (see Lemmas 2.1 and 3.6).

See Section 3 for the proof. In Section 4, we present some useful properties of the distributional $\det DV_\beta(Dv)$.

For any planar ∞-harmonic function u and any $\beta > -1$, since $|Du|^{\beta+1} \in W^{1,2}_{\rm loc}$ as proved in [25], the distributional Jacobian determinant $\det DV_\beta(Dv)$ is defined by (1.3). Recalling that $D|Du|^{\beta+1} \cdot Du = 0$ almost everywhere (see [25]), for any $\psi \in C_c^\infty(\Omega)$, one has

$$\int_\Omega \det DV_\beta(Du)\psi\,dx = -\frac{1}{2}\int_\Omega |Du|^{2\beta}(D^2\psi\, Du\cdot Du)\,dx + \frac{1}{2\beta+2}\int_\Omega |Du|^{2\beta+2}\Delta\psi\,dx. \tag{1.4}$$

We then obtain the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a domain, and let $\beta > -1$. For any ∞-harmonic function u in Ω, we have* $\det DV_\beta(Du) \in \mathcal{M}(\Omega)$ *with the lower bound*

$$\int_\Omega \det DV_\beta(Du)\psi\,dx \geq \frac{1}{\beta+1}\int_\Omega \big|D|Du|^{\beta+1}\big|^2\psi\,dx \quad \textit{for all } 0 \leq \psi \in C_c^0(\Omega) \tag{1.5}$$

and the upper bound

$$\int_{\frac{1}{2}B} \det DV_\beta(Du)\,dx \leq C\Big(1+\frac{1}{\beta+1}\Big)\fint_B |Du|^{2+2\beta}\,dx \quad \textit{for all balls } B \Subset \Omega, \tag{1.6}$$

where $C > 0$ is a universal constant.

We prove Theorem 1.3 in Section 5. By using Lemmas 2.1, 3.6, 3.7, 4.1, and 4.2, we build up some analogue lower and upper bounds for exponential $e^{\frac{1}{2\varepsilon}|\xi|^2}$-harmonic functions u^ε in $U \Subset \Omega$, which are uniform in $\varepsilon \in (0,1)$. As a consequence, we conclude that

$$\det DV_\beta(Du) \in \mathcal{M}(U)$$

and $\det DV_\beta(Du^\varepsilon) \to \det DV_\beta(Du)$ in the weak-$\star$ sense in $\mathcal{M}(U)$.

Our original motivation of Definition 1.2 and Theorem 1.3 is to study planar p-harmonic functions u_p and their limiting behavior as $p \to \infty$. A function $v \in W^{1,p}_{\text{loc}}(\Omega)$ is called p-harmonic if it is a weak solution to

$$\Delta_p v = \operatorname{div}(|Dv|^{p-2} Dv) = 0 \quad \text{in } \Omega.$$

See [32] for a probability interpretation by using Tug-of-War with noise. We refer the reader to Iwaniec–Manfredi [23] and Aronsson [7] for the $C^{k,\alpha}$ and $W^{k+2,q}_{\text{loc}}$-regularity of u_p with optimal k, α, and q. In the literature, the interior regularity of p-harmonic functions in any dimension has been extensively studied. See [8, 13, 14, 16, 26, 29, 30, 33, 35–37].

Moreover, for each $\beta > -1$ and $1 < p < \infty$, the nonlinear complex gradient $V_\beta(Du_p)$ is well known to be a quasiregular mapping. Precisely, $V_\beta(Du_p) \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$|DV_\beta(Du_p)|^2 \le \Big[K(p,\beta) + \frac{1}{K(p,\beta)}\Big] \det DV_\beta(Du_p) \quad \text{a.e. in } \Omega, \tag{1.7}$$

where

$$K(p,\beta) = \max\Big\{\frac{p-1}{\beta+1}, \frac{\beta+1}{p-1}, \beta+1, \frac{1}{\beta+1}\Big\},$$

which leads to a pointwise defined Jacobian determinant $\det DV_\beta(Du_p) \in L^1_{\text{loc}}(\Omega)$. We refer the reader to Bojarski–Iwaniec [9], Manfredi [29], Iwaniec–Manfredi [23], and Aronsson [7]. See also Lemma A.3 and Remark A.4 for the sharpness of the constant in (1.7) by using some construction in [23].

Studying the Jacobian determinant and the corresponding Beltrami equation of

$$f = u_z = \frac{1}{2}(u_x - i u_y)$$

when u is a p-harmonic function leads to the optimal regularity of p-harmonic functions in the plane [23]. In a follow-up paper [15], the case for ∞-Laplacian was studied under an extra assumption that the solution is C^2. It was pointed out in [15, page 237] that if their calculation was formulated rigorously in the general setting, then the optimal regularity of ∞-Laplacian would be $C^{1,\frac{1}{3}}_{\text{loc}}$. Especially, one of the key problems is how to define (variants of) the Jacobian determinant of $f = u_z$ for ∞-harmonic functions.

In this paper, we obtain the following lower and upper bounds, where the constant is uniform in $p \ge 2$ and hence is quite different from the above quasiregular properties.

Theorem 1.4. *Let $\beta > -1$ and $1 < p < \infty$. For any p-harmonic functions u_p in a given planar domain Ω, one has the lower bound*

$$\det DV_\beta(Du_p) \ge \frac{\min\{1, p-1\}}{\beta+1} \big|D|Du_p|^{\beta+1}\big|^2 \quad a.e.$$

and upper bound

$$\int_{\frac{1}{2}B} \det DV_\beta(Du_p)\,dx \le C\Big[1 + \frac{1}{1+\beta} + \frac{1}{p-1}\frac{\beta^2}{\beta+1}\Big] \fint_B |Du_p|^{2+2\beta}\,dx$$

for all balls $B \Subset \Omega$.

Theorem 1.4 will be proved in Section 6 based on the approach in [25], Lemma 3.7, Lemma 2.1, Lemma 3.6, and Lemma 3.2. Note that the case $\beta = 0$ was already given in a similar but simpler way by Lindgren–Lindqvist [27] via the approach in [25].

If p-harmonic functions u_p in a bounded smooth domain Ω have the same boundary data $g \in C^{0,1}(\partial\Omega)$, then by using the variational approach and Jensen [24], there is a function $u_\infty \in C^{0,1}(\overline{\Omega})$, which is the unique ∞-harmonic function with boundary data g, so that $u_p \to u_\infty$ in $C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$ and weakly in $W^{1,q}_{\mathrm{loc}}(\Omega)$ for any $1 < q < \infty$.

Recently, based on the approach in [25] and Theorem 1.4 with $\beta = 0$, Lindgren–Lindqvist [27] deduced that $u_p \to u_\infty$ in $W^{1,q}_{\mathrm{loc}}(\Omega)$ for any $1 < q < \infty$. However, even though we already know that $u_\infty \in C^1(\Omega)$ and $C^{1,\alpha}(\Omega)$ for some $0 < \alpha < 1/3$ (see [19, 34]), it remains open whether $u_p \to u_\infty$ in either $C^1(\Omega)$ or $C^{1,\alpha}(\Omega)$. We also observe that, from p-harmonic to ∞-harmonic functions, the best possible regularity has a huge jump. For example, one always has $u_p \in W^{3,1}_{\mathrm{loc}}$ but does not necessarily have $u_\infty \in W^{2,3/2}_{\mathrm{loc}}$.

Because $Du_p \to Du_\infty$ in $L^q_{\mathrm{loc}}(\Omega)$ with $1 < q < \infty$, for $\beta > -1$, the limit of mappings $V_\beta(Du_p)$ as $p \to \infty$ is naturally expected to be the mapping $V_\beta(Du_\infty)$ in certain sense. However, since $K(p,\beta) \to \infty$ as $p \to \infty$, one cannot expect that $V_\beta(Du_\infty)$ is a quasiregular. Indeed, we do not necessarily have $V_\beta(Du_\infty) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ as witted by Aronsson's ∞-harmonic function $x_1^{4/3} - x_2^{4/3}$. Moreover, the $W^{1,1}_{\mathrm{loc}}$-regularity of $V_\beta(Du_\infty)$ is quite open and very difficult even in the case special when $\beta = 0$, and a pointwise Jacobian determinant $\det DV_\beta(Du_\infty)$ is unavailable.

Instead of the pointwise one, in the case when $\beta = 0$, we already have the distributional Jacobian determinant $\det DV_0(Du_\infty) = -\det D^2 u_\infty$ as in (1.2). Because $Du_p \to Du_\infty$ in $L^2_{\mathrm{loc}}(\Omega)$, one has

$$\det DV_0(Du_p) \to \det DV_0(Du_\infty)$$

in the sense of distributions. Since $\det DV_0(Du_\infty)$ was proved to be a Radon measure in [25], it is naturally expected that $\det DV_0(Du_p) \to \det DV_0(Du_\infty)$ in the weak-$\star$ sense in $\mathcal{M}(\Omega)$.

In the case when $0 \neq \beta > -1$, it is a basic question whether the limit

$$\lim_{p\to\infty} \det DV_\beta(Du_p)$$

exists in certain sense. If so, it is expected to be given by $\det DV_\beta(Du_\infty)$. However, unlike the case $\beta = 0$, a distributional definition for $\det DV_\beta(Du_\infty)$ is unavailable in the literature. This leads us to introduce the distributional $\det DV_\beta(Dv)$ as in Definition 1.2 and build up Theorem 1.5 below, which answer these questions.

Theorem 1.5. *Given any $\beta > -1$, as $p \to \infty$, one has that $V_\beta(Du_p) \to V_\beta(Du_\infty)$ in $L^q_{\mathrm{loc}}(\Omega)$ for any $q > 1$, and also that $\det DV_\beta(Du_p) \to \det DV_\beta(Du_\infty)$ in the weak-$\star$ sense in $\mathcal{M}(\Omega)$, that is,*

$$\int_\Omega \det DV_\beta(Du_\infty)\psi\,dx = \lim_{p\to\infty}\int_\Omega \det DV_\beta(Du_p)\psi\,dx \quad \textit{for all } \psi \in C^0_c(\Omega).$$

Theorem 1.5 follows from Theorems 1.3 and 1.4, and the convergence $Du_p \to Du_\infty$ in $L^q_{\mathrm{loc}}(\Omega)$ with $1 < q < \infty$ as given in [27]. See Section 6 for details.

2. Proof of Theorem 1.1

We begin with the following divergence structure of $-\det D^2 v$ for $v \in C^\infty$.

Lemma 2.1. *For any $v \in C^\infty(\Omega)$, one has*

$$-\det D^2 v = \frac{1}{2}[|D^2 v|^2 - (\Delta v)^2] = \frac{1}{2}\operatorname{div}(D^2 v\, Dv - \Delta v\, Dv) \tag{2.1}$$
$$= \frac{1}{2}[\Delta(|Dv|^2) - (v_{x_i} v_{x_j})_{x_i x_j}].$$

Consequently,

$$\begin{aligned}\int_\Omega -\det D^2 v \psi\, dx \\ &= -\frac{1}{2}\int_\Omega [D^2 v\, Dv - \Delta v\, Dv]\cdot D\psi\, dx \\ &= \frac{1}{2}\int_\Omega [|Dv|^2 \Delta\psi - D^2\psi\, Dv\cdot Dv]\, dx \quad \textit{for all } \psi \in C_c^\infty(\Omega).\end{aligned} \tag{2.2}$$

From this, we deduce the following formula for $(-\det D^2 u)u^2$ for ∞-harmonic functions u in $\Omega \subset \mathbb{R}^2$. Since u always enjoys locally Lipschitz regularity (see Jensen [24]), by Rademacher's theorem, u is differentiable almost everywhere, and hence, for almost all $x \in \Omega$, the derivative $Du(x)$ exists and $\operatorname{Lip} u(x) = |Du(x)|$. Moreover,

$$\|Du\|_{L^\infty(U)} = \sup_{x\in U} \operatorname{Lip} u(x) \quad \text{for any domain } U \subset \Omega.$$

As such, when there is no confusion, we slightly abuse the notation by writing $\operatorname{Lip} u$ instead of $|Du|$. We remark that, even though u is known to be everywhere differentiable (see Evans–Smart [20, 21]) and also C^1-regular (see Savin [34]), all the results in [25] and also all our results and proofs below do not rely on either the everywhere differentiability or the C^1-regularity of u.

Lemma 2.2. *If u is ∞-harmonic in Ω, then for any $\psi \in C_c^\infty(\Omega)$, one has*

$$\begin{aligned}\int_U (-\det D^2 u)u^2\psi\, dx + \int_U |Du|^4\psi\, dx \\ &= -\int_U |Du|^2 (Du\cdot D\psi)u\, dx \\ &\quad + \frac{1}{2}\int_U u^2[|Du|^2\Delta\psi - D^2\psi\, Du\cdot Du]\, dx.\end{aligned} \tag{2.3}$$

Proof. The proof is divided into the following two steps.

Step 1. Given any $v \in C^2(U)$ with $U \Subset \Omega$, we show that

$$\begin{aligned}\int_U (-\det D^2 v)v^2\psi\, dx = -\frac{3}{2}\int_U (D|Dv|^2\cdot Dv)v\psi\, dx - \int_U |Dv|^4\psi\, dx \\ - \int_U |Dv|^2 (Dv\cdot D\psi)v\, dx \\ + \frac{1}{2}\int_U v^2[|Dv|^2\Delta\psi - D^2\psi\, Dv\cdot Dv]\, dx.\end{aligned} \tag{2.4}$$

Indeed, by (2.1), one has

$$\int_U (-\det D^2 v) v^2 \psi \, dx = -\int_U D^2 v \, Dv \cdot \Big[v \, Dv \, \psi + \frac{1}{2} v^2 D\psi \Big] \, dx + \int_U \Delta v \, Dv \cdot \Big[v \, Dv \, \psi + \frac{1}{2} v^2 D\psi \Big] \, dx.$$

Note that

$$-\int_U D^2 v \, Dv \cdot \Big[v \, Dv \, \psi + \frac{1}{2} v^2 D\psi \Big] \, dx = -\frac{1}{2} \int_U (D|Dv|^2 \cdot Dv) v \psi \, dx - \frac{1}{4} \int_U (D|Dv|^2 \cdot D\psi) v^2 \, dx,$$

where, using integration by parts,

$$-\frac{1}{4} \int_U (D|Dv|^2 \cdot D\psi) v^2 \, dx = \int_U \Big[\frac{1}{4} |Dv|^2 \Delta\psi \, v^2 + \frac{1}{2} |Dv|^2 (Dv \cdot D\psi) v \Big] \, dx.$$

Moreover, integration by parts yields

$$\begin{aligned}
&\int_U \Delta v \, Dv \cdot \Big[v \, Dv \, \psi + \frac{1}{2} v^2 D\psi \Big] \, dx \\
&\quad = -\int_U Dv \cdot D(|Dv|^2 v \psi + \frac{1}{2} v^2 Dv \cdot D\psi) \, dx \\
&\quad = -\int_U (D|Dv|^2 \cdot Dv) v \psi \, dx - \int_U |Dv|^4 \psi \, dx - \frac{1}{4} \int_U v^2 D|Dv|^2 \cdot D\psi \, dx \\
&\qquad - \int_U \Big[2v |Dv|^2 Dv \cdot D\psi + \frac{1}{2} v^2 D^2\psi \, Dv \cdot Dv \Big] \, dx \\
&\quad = -\int_U (D|Dv|^2 \cdot Dv) v \psi \, dx - \int_U |Dv|^4 \psi \, dx + \frac{1}{4} \int_U v^2 |Dv|^2 \Delta\psi \, dx \\
&\qquad - \int_U \Big[\frac{3}{2} v |Dv|^2 Dv \cdot D\psi + \frac{1}{2} v^2 D^2\psi \, Dv \cdot Dv \Big] \, dx.
\end{aligned}$$

Combining these, we conclude (2.4).

Step 2. Given any smooth subdomain $U \Subset \Omega$, let $u^\varepsilon \in C^\infty(U) \cap C^0(\overline{U})$ be the solution to

$$\operatorname{div}(e^{\frac{1}{2\varepsilon}|Du^\varepsilon|^2} Du^\varepsilon) = \frac{1}{\varepsilon} e^{\frac{1}{2\varepsilon}|Du^\varepsilon|^2} (\Delta_\infty u^\varepsilon + \varepsilon \, \Delta u^\varepsilon) = 0 \quad \text{with } u^\varepsilon = u \text{ on } \partial U.$$

Given any $\psi \in C_c^\infty(U)$, we observe that

$$\int_U (-\det D^2 u) u^2 \psi \, dx = \lim_{\varepsilon \to 0} \int_U (-\det D^2 u^\varepsilon)(u^\varepsilon)^2 \psi \, dx.$$

Indeed,

$$\begin{aligned}
&\Big| \int_U (-\det D^2 u) u^2 \psi \, dx - \int_U (-\det D^2 u^\varepsilon)(u^\varepsilon)^2 \psi \, dx \Big| \\
&\quad \le \Big| \int_U (-\det D^2 u) u^2 \psi \, dx - \int_U (-\det D^2 u^\varepsilon) u^2 \psi \, dx \Big| \\
&\qquad + \Big| \int_U (-\det D^2 u^\varepsilon) [u^2 - (u^\varepsilon)^2] \psi \, dx \Big|.
\end{aligned}$$

By the definition of $-\det D^2u$ (see [25, Theorem 1.5] and also the proof of Theorem 1.3 below), the first term goes to 0 as $\varepsilon \to 0$. The second term is bounded by

$$\|\det D^2u^\varepsilon\|_{L^1(\operatorname{supp}\psi)}\|(u^\varepsilon)^2 - u^2\|_{L^\infty(\operatorname{supp}\psi)}.$$

By using the fact $-\det D^2u^\varepsilon \in L^1_{\mathrm{loc}}(U)$ uniformly in small $\varepsilon > 0$ (see [25, Theorem 1.5] and also the proof of Theorem 1.3 below), the second term also goes to 0 as $\varepsilon \to 0$.

Applying (2.4) proved in Step 1 to u^ε, one has

$$\begin{aligned}
\int_U (-\det D^2u^\varepsilon)(u^\varepsilon)^2\psi\,dx
&= -\frac{3}{2}\int_U (D|Du^\varepsilon|^2 \cdot Du^\varepsilon)u^\varepsilon\psi\,dx - \int_U |Du^\varepsilon|^4\psi\,dx\\
&\quad - \int_U |Du^\varepsilon|^2(Du^\varepsilon \cdot D\psi)u^\varepsilon\,dx\\
&\quad + \frac{1}{2}\int_U (u^\varepsilon)^2[|Du^\varepsilon|^2\Delta\psi - D^2\psi\, Du^\varepsilon \cdot Du^\varepsilon]\,dx.
\end{aligned}$$

Since $D|Du^\varepsilon|^2 \to D|Du|^2$ weakly in $L^2_{\mathrm{loc}}(U)$ and $u^\varepsilon \to u$ in $W^{1,q}_{\mathrm{loc}}(U)$ for any $1 < q < \infty$ (see [25]), sending $\varepsilon \to 0$ above and noting that $D|Du|^2 \cdot Du = 0$, one has the desired identity (2.3). □

Consequently, one has the following lemma.

Lemma 2.3. *If u is ∞-harmonic in Ω, then*

$$\begin{aligned}
\int_{B(x,r)} (-\det D^2u)u^2\,dz + \|Du\|^4_{L^4(B(x,r))}&\\
\le \frac{C}{r^4}\|u\|^4_{L^4(B(x,2r))} &\quad \textit{whenever } B(x,2r) \subset \Omega.
\end{aligned} \tag{2.5}$$

In particular, one has

$$\|Du\|_{L^4(B(x,r))} \le \frac{C}{r}\|u\|_{L^4(B(x,2r))} \quad \textit{whenever } B(x,2r) \subset \Omega. \tag{2.6}$$

Proof. Let $\phi \in C^\infty_c(B(x,2r))$ be a cut-off function satisfying

$$\phi = 1 \text{ in } B(x,r), \quad 0 \le \phi \le 1 \text{ in } B(x,2r), \quad |D\phi|^2 + |D^2\phi| \le \frac{C}{r^2} \text{ in } B(x,2r). \tag{2.7}$$

Taking $\psi = \phi^4$ in (2.3), one has

$$\begin{aligned}
\int_{B(x,2r)} (-\det D^2u)u^2\phi^4\,dz + \int_{B(x,2r)} |Du|^4\phi^4\,dz
&= -\int_{B(x,2r)} |Du|^2(Du \cdot D\phi^4)u\,dz\\
&\quad + \frac{1}{2}\int_{B(x,2r)} u^2[|Du|^2\Delta\phi^4 - D^2\phi^4 Du \cdot Du]\,dz.
\end{aligned}$$

By Young's inequality, the right-hand side of the above identity is bounded by

$$\frac{1}{2}\int_{B(x,2r)} |Du|^4\phi^4\,dz + C\int_{B(x,2r)} |u|^4[|D\phi|^4 + \phi^2|D^2\phi|^2]\,dz.$$

Thus,

$$\int_{B(x,2r)} (-\det D^2u)u^2\phi^4\,dz + \int_{B(x,2r)} |Du|^4\phi^4\,dz$$
$$\le C\int_{B(x,2r)} |u|^4[|D\phi|^4 + \phi^2|D^2\phi|^2]\,dz,$$

which together with (2.7) gives the desired (2.5). Finally, since $-\det D^2u \ge 0$ (see [25]), (2.6) follows from (2.5). □

From Lemma 2.3 and some basic properties of ∞-harmonic functions, we are able to get the following gradient estimate in Theorem 1.1 (i).

Proof of Theorem 1.1 (i). Since $u - c$ is also ∞-harmonic for any constant c, by Lemma 2.3,

$$\|Du\|_{L^4(B(x,r))} \le \frac{C}{r}\|u-c\|_{L^4(B(x,2r))} \quad \text{for all } c \in \mathbb{R}$$

whenever $B(x,2r) \subset \Omega$. Note that

$$\|u-c\|_{L^4(B(x,2r))} \le C\|u-c\|^{\frac14}_{L^1(B(x,2r))}\|u-c\|^{\frac34}_{L^\infty(B(x,2r))}.$$

Taking c as the average of u in $B(x,2r)$ and applying the Sobolev–Poincaré inequality, one has

$$\|u-c\|_{L^\infty(B(x,2r))} \le Cr^{\frac12}\|Du\|_{L^4(B(x,2r))}$$

and hence

$$\|Du\|_{L^4(B(x,r))} \le Cr^{-\frac58}\|u\|^{\frac14}_{L^1(B(x,2r))}\|Du\|^{\frac34}_{L^4(B(x,2r))}.$$

This and Young's inequality yield that, for any $\varepsilon \in (0,1)$, one has

$$r^{\frac52}\|Du\|_{L^4(B(x,r))} \le \varepsilon r^{\frac52}\|Du\|_{L^4(B(x,2r))} + C_\varepsilon\|u\|_{L^1(B(x,2r))}.$$

Via a standard iteration argument (see, for instance, [22, pp. 80–82]), we conclude that

$$r^{\frac52}\|Du\|_{L^4(B(0,r))} \le C\|u\|_{L^1(B(0,2r))}.$$

Applying Morrey's inequality, for any ball $B(x,2r) \subset \Omega$ and any $y \in \Omega$ with $|x-y| = r$, by the above, one has

$$|u(x)-u(y)| \le |x-y|^{\frac12}\|Du\|_{L^4(B(x,r))} \le \frac{C}{r^3}|x-y|\|u\|_{L^1(B(x,2r))}.$$

One obtains

$$u(y) - \frac{C}{r^3}|x-y|\|u\|_{L^1(B(x,2r))} \le u(x) \le u(y) + \frac{C}{r^3}|x-y|\|u\|_{L^1(B(x,2r))}$$

in $\partial(B(x,r)\setminus\{x\})$ and then, by the comparison property with cones, in $B(x,r)$. Since u is differentiable at x (see [21]), this yields that

$$|Du(x)| \le \frac{C}{r}\fint_{B(x,2r)} |u|\,dz.$$

Theorem 1.1 (i) is proved. □

Remark 2.4. Let u be an ∞-harmonic function in $\Omega \subset \mathbb{R}^2$.

(i) It was shown by Lindqvist and Manfredi [28] that

$$|Du(x)| \leq \frac{2}{r}\|u\|_{L^\infty(B(x,r))} \quad \text{in } B(x,r) \subset \Omega.$$

Theorem 1.1 (i) shows that $2\|u\|_{L^\infty(B(x,r))}$ can be relaxed to the average $C \fint_{B(x,r)}|u|\,dx$.

(ii) Via directly working on $-\det D^2u^\varepsilon$ for the $e^{\frac{1}{2\varepsilon}|\xi|^2}$-harmonic function u^ε and some tedious calculation, it was shown in [25, Theorem 1.5] that, for any $p > 2$,

$$\|Du\|_{L^p(B(x,r))} \leq \frac{C(p)}{r}\|u\|_{L^p(B(x,2r))} \quad \text{whenever } B(x,2r) \Subset \Omega.$$

In the case when $p = 4$, we derived a formula for $(-\det D^2u)u^2$ in Lemma 2.2, simplifying the argument and calculations in [25]. See Lemma 2.3 and its proof.

The following is crucial to prove Theorem 1.1 (ii). Recall that

$$\mathcal{P} = \{b + a \cdot x : b \in \mathbb{R},\ a \in \mathbb{R}^2\}.$$

Proposition 2.5. *If $u \in C^0(\Omega)$ is an ∞-harmonic function in a planar domain Ω, then*

$$\int_{\frac{1}{4}B} -\det D^2u\,dx \leq C \inf_{P\in\mathcal{P}}\left\|\frac{u-P}{r}\right\|_{L^\infty(B)}\left[|DP| + \left\|\frac{u-P}{r}\right\|_{L^\infty(B)}\right]$$

for all $B = B(x,r) \Subset \Omega$.

To prove this proposition, we need the following, which was proved by the comparison property with cones (see [12, Section 2]). For the reader's convenience, we briefly recall their proof below.

Lemma 2.6. *If u is ∞-harmonic in $B(0,2r)$, then*

$$\|Du\|_{L^\infty(B(0,r))} \leq \inf_{P\in\mathcal{P}}\left[|DP| + 2\left\|\frac{u-P}{r}\right\|_{L^\infty(B(0,2r))}\right]. \tag{2.8}$$

Proof. Given any $P(x) = b + a \cdot x \in \mathcal{P}$, write $\lambda = \|u - P\|_{L^\infty(B(0,2r))}$. Given any $x \in B(0,r)$, for any $y \in \mathbb{R}^2$ with $|x - y| = r$, we have $y \in B(0,2r)$ and

$$|u(y) - u(x) - a\cdot(y-x)| \leq |u(x) - b - a\cdot x| + |u(y) - b - a\cdot y| \leq 2\lambda,$$

which implies that

$$u(y) \leq u(x) + a\cdot(y-x) + 2\lambda \leq u(x) + \Big(|a| + 2\frac{\lambda}{r}\Big)|x-y|,$$
$$u(y) \geq u(x) + a\cdot(y-x) - 2\lambda \geq u(x) - \Big(|a| + 2\frac{\lambda}{r}\Big)|x-y|.$$

Applying the comparison property with cones, one has

$$|u(y) - u(x)| \leq \Big(|a| + 2\frac{\lambda}{r}\Big)|x-y| \quad \text{for all } y \in B(x,r),$$

which implies that $|Du(x)| \leq |a| + 2\lambda/r$, so (2.8) follows. □

The following property was observed in [21]. For the reader's convenience, we give the proof below.

Lemma 2.7. *If u is ∞-harmonic in $B(0,2r)$, then for any $P \in \mathcal{P}$, we have*

$$\fint_{B(0,r)} |Du - DP|^2\,dx \le 20\Big\|\frac{u-P}{r}\Big\|_{L^\infty(B(0,2r))}\Big[|DP| + \Big\|\frac{u-P}{r}\Big\|_{L^\infty(B(0,2r))}\Big].$$

Proof. By considering $u(rx)/r$ and $P(rx)/r \in \mathcal{P}$, we may assume that $r = 1$. Write $\lambda = \|u - P\|_{L^\infty(B(0,2))}$. If $\lambda \ge |DP|$, then by Lemma 2.6, one has

$$\begin{aligned}\fint_{B(0,1)} |Du - DP|^2\,dx &\le \fint_{B(0,1)} 2[|Du|^2 + |DP|^2]\,dx \\ &\le 2[(|DP| + 2\lambda)^2 + |DP|^2] \le 20\lambda^2.\end{aligned}$$

Below, assume that $0 < \lambda < |DP|$. Set

$$v = \frac{u}{|DP|}, \quad Q = \frac{P}{|DP|}, \quad \text{and} \quad \mu = \frac{\lambda}{|DP|} = \|v - Q\|_{L^\infty(B(0,2))} < 1.$$

Then $|DQ| = 1$. Up to a rotation, we may assume $DQ = e_2$. It then suffices to show

$$\fint_{B(0,1)} |Dv - e_2|^2\,dx \le 16\mu. \tag{2.9}$$

Indeed, this implies that

$$\fint_{B(0,1)} |Du - DP|^2\,dx \le 16\lambda|DP|,$$

as desired

To see (2.9), by (2.8) and $\mu < 1$, one has

$$\begin{aligned}|Dv(x) - e_2|^2 = |Dv(x)|^2 - 2v_{x_2} + 1 &\le (1+2\mu)^2 + 1 - 2v_{x_2} \\ &\le 2 + 4\mu + 4\mu^2 - 2v_{x_2} \\ &\le 2(1 + 4\mu - u_{x_2}).\end{aligned}$$

We therefore obtain

$$\begin{aligned}\int_{B(0,1)} |Dv(x) - e_2|^2\,dx &\le 2\int_{B(0,1)} (1 + 4\mu - v_{x_2})\,dx \\ &= 2\int_{|x_1|\le 1}\int_{-\sqrt{1-|x_1|^2}}^{\sqrt{1-|x_1|^2}} [1 + 4\mu - v_{x_2}]\,dx_2\,dx_1.\end{aligned}$$

Note that

$$\begin{aligned}&\int_{-\sqrt{1-|x_1|^2}}^{\sqrt{1-|x_1|^2}} [1 + 4\mu - v_{x_2}]\,dx_2 \\ &\quad = 2(1+4\mu)\sqrt{1-|x_1|^2} - \Big[v(x_1, \sqrt{1-|x_1|^2}) - v(x_1, -\sqrt{1-|x_1|^2})\Big] \\ &\quad = 8\mu\sqrt{1-|x_1|^2} - \Big\{\Big[v(x_1, \sqrt{1-|x_1|^2}) - Q(x_1, \sqrt{1-|x_1|^2})\Big] \\ &\qquad\qquad - \Big[v(x_1, -\sqrt{1-|x_1|^2}) - Q(x_1, -\sqrt{1-|x_1|^2})\Big]\Big\} \\ &\quad \le 8\mu.\end{aligned}$$

We therefore obtain (2.9). □

We are ready to prove Proposition 2.5.

Proof of Proposition 2.5. Let u be an ∞-harmonic function in a planar domain Ω. Fix any $x_0 \in \Omega$ and $r > 0$ so that $B(x_0, 4r) \subset \Omega$. Let $P \in \mathcal{P}$. Without loss of generality, we assume that $x_0 = 0$. By Lemma 2.7, we only need to prove

$$\int_{B(0,r)} -\det D^2 u\, dx \le C \inf_{P\in\mathcal{P}} \fint_{B(0,2r)} |Du - DP|^2\, dx. \tag{2.10}$$

Note that $D^2 v = D^2(v - P)$, and hence

$$-\det D^2 v = -\det D^2(v - P)$$

for any smooth function v. The distributional definition of $-\det D^2 v$ then must coincide with that of $-\det D^2(u - P)$, i.e.,

$$\int_\Omega (-\det D^2 u)\phi^2\, dx = \frac{1}{2}\int_\Omega [|D(u-P)|^2 \Delta\phi^2 - D^2\phi^2 D(u-P)\cdot D(u-P)]\, dx$$

for all $\phi \in C_c^\infty(\Omega)$. Thus,

$$\int_\Omega (-\det D^2 u)\phi^2\, dx \le C \int_\Omega |Du - DP|^2 (|D^2\phi||\phi| + |D\phi|^2)\, dx$$

for all $\phi \in C_c^\infty(\Omega)$. Since $-\det D^2 u \ge 0$ as proved in [25], by a choice of cut-off function ϕ as in (2.7), we have (2.10). □

To prove Theorem 1.1 (ii), we also need the following result by Crandall–Evans [11].

Lemma 2.8. *If u is ∞-harmonic in $\mathbb{R}^2$ and* $\operatorname{Lip} u(x_0) = \|Du\|_{L^\infty(\mathbb{R}^2)} < \infty$ *for some x_0, then $u \in \mathcal{P}$, i.e., $u(x) = b + a\cdot x$ in $\mathbb{R}^2$ for some $b \in \mathbb{R}$ and $a \in \mathbb{R}^2$.*

This allows us to get the following, via some argument much similar to that for the linear approximations in [11]. We give the details here for the reader's convenience.

Lemma 2.9. *Let u be an ∞-harmonic function in $\mathbb{R}^2$ with $0 < \|Du\|_{L^\infty(\mathbb{R}^2)} < \infty$. Then there exist a subsequence $\{m_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and a vector $a \in \mathbb{R}^2$ such that*

$$\lim_{j\to\infty} \sup_{B(0,4)} \left| \frac{u(m_j x)}{m_j} - a\cdot x \right| = 0. \tag{2.11}$$

Proof. Without loss of generality, we may assume that $u(0) = 0$. Write

$$u_m(x) = \frac{1}{m} u(mx) \quad \text{for } m \in \mathbb{N}.$$

Since $\|Du_m\|_{L^\infty(\mathbb{R}^2)} = \|Du\|_{L^\infty(\mathbb{R}^2)} < \infty$, we know that $\{u_m\}_{m\in\mathbb{N}}$ is equicontinuous and locally uniformly bounded in $\mathbb{R}^2$. Thus, there exists a subsequence $\{m_j\} \subset \mathbb{N}$ such that u_{m_j} converges locally uniformly to some continuous function $w \in C^0(\mathbb{R}^2)$ with $w(0) = 0$. Moreover, since $\|Du_m\|_{L^\infty(\mathbb{R}^2)} = \|Du\|_{L^\infty(\mathbb{R}^2)}$, one has

$$|w(x) - w(y)| = \lim_{k\to\infty} |u_{m_j}(x) - u_{m_j}(y)| \le \|Du\|_{L^\infty(\mathbb{R}^2)} |x - y| \quad \text{for all } x, y \in \mathbb{R}^2$$

and hence $\|Dw\|_{L^\infty(\mathbb{R}^2)} \le \|Du\|_{L^\infty(\mathbb{R}^2)} < \infty$. Due to the compactness of viscosity solutions, w is an ∞-harmonic function in $\mathbb{R}^2$. To see (2.11), it suffices to prove

$$\operatorname{Lip} w(0) = \|Dw\|_{L^\infty(\mathbb{R}^2)}.$$

This allows us to apply Lemma 2.8 to get that $w(x) = a \cdot x$ in $\mathbb{R}^2$ for some $a \in \mathbb{R}^2$. Hence u_{m_j} converges locally uniformly to $a \cdot x$, that is, (2.11) holds.

Finally, we show that $\operatorname{Lip} w(0) = \|Dw\|_{L^\infty(\mathbb{R}^2)}$. We always have

$$\operatorname{Lip} w(0) \le \|Dw\|_{L^\infty(\mathbb{R}^2)}.$$

To see the converse, recall that w has the linear approximation property at 0, that is, for any sequence $\{r_j\}_{j\in\mathbb{N}}$ converging to 0, there are a subsequence $\{r_{j_k}\}_{k\in\mathbb{N}}$ and also a vector e depending on $\{r_{j_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sup_{z\in B(0,1)} \left| \frac{w(r_{j_k} z)}{r_{j_k}} - e \cdot z \right| = 0$$

and $|e| = \operatorname{Lip} w(0)$. Therefore, for every $\lambda > 0$, there exists an $r_\lambda \in (0,1)$ such that

$$\sup_{B(0,2r_\lambda)} \frac{1}{r_\lambda} |w(x) - e \cdot x| \le \lambda.$$

On the other hand, since $u_{m_j} \to w$ uniformly in $B(0,2)$, there exists j_λ such that, for any $j \ge j_\lambda$,

$$|u_{m_j}(z) - w(z)| \le \lambda r_\lambda \quad \text{for all } z \in B(0,2).$$

Therefore,

$$\sup_{B(0,2r_\lambda)} \frac{1}{r_\lambda} |u_{m_j}(x) - e \cdot x| \le 2\lambda,$$

or equivalently,

$$\sup_{B(0,2m_j r_\lambda)} \frac{1}{2m_j r_\lambda} |u(x) - e \cdot x| \le \lambda.$$

By (2.8), one has $|Du(x)| \le |e| + 4\lambda$ for all $x \in B(0, m_j r_\lambda)$. By sending $j \to \infty$, we also have this inequality for all $x \in \mathbb{R}^2$. By the arbitrariness of $\lambda > 0$, we have $|e| \ge \|Du\|_{L^\infty(\mathbb{R}^n)}$ and hence $\operatorname{Lip} w(0) \ge \|Du\|_{L^\infty(\mathbb{R}^n)}$, as desired. □

Proof of Theorem 1.1 (ii). Let u be an ∞-harmonic function in $\mathbb{R}^2$ with

$$\liminf_{R\to\infty} \inf_{c\in\mathbb{R}} \frac{1}{R} \fint_{B(0,R)} |u(x) - c| \, dx < \infty.$$

For any $c \in \mathbb{R}$, since $u - c$ is an ∞-harmonic function in $\mathbb{R}^2$, by Theorem 1.1 (i), one has

$$\|Du\|_{L^\infty(B(0,R/2))} \le C \frac{1}{R} \fint_{B(0,R)} |u(x) - c| \, dx \quad \text{for all } R > 0.$$

Thus,

$$\|Du\|_{L^\infty(B(0,R/2))} \le C \inf_{c\in\mathbb{R}} \frac{1}{R} \fint_{B(0,R)} |u(x) - c| \, dx \quad \text{for all } R > 0.$$

This implies

$$\|Du\|_{L^\infty(\mathbb{R}^2)} = \liminf_{R\to\infty} \|Du\|_{L^\infty(B(0,R/2))} \le \liminf_{R\to\infty} \inf_{c\in\mathbb{R}} \frac{1}{R} \fint_{B(0,R)} |u(x)-c|\,dx < \infty.$$

We assume that $\|Du\|_{L^\infty(\mathbb{R}^2)} > 0$ because otherwise u is a constant. By Lemma 2.9, there exist a subsequence $\{m_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and a vector $a \in \mathbb{R}^2$ such that

$$\lim_{j\to\infty} \sup_{B(0,4)} \left|\frac{u(m_j x)}{m_j} - a\cdot x\right| = 0. \tag{2.12}$$

From (2.12), (1.1), and Proposition 2.5, we deduce

$$\begin{aligned}\int_{\mathbb{R}^2} \big|D|Du|\big|^2\,dx &\le C \int_{\mathbb{R}^2} -\det D^2u\,dx\\ &= C \lim_{j\to\infty} \int_{B(0,m_j)} -\det D^2u\,dx\\ &\le C \lim_{j\to\infty} \sup_{x\in B(0,4)} \left|\frac{u(m_j x)}{m_j} - a\cdot x\right| = 0.\end{aligned}$$

Thus, $|Du|$ is a constant almost everywhere, and hence $\|Du\|_{L^\infty(\mathbb{R}^2)} = |Du(x_0)|$ for some $x_0 \in \mathbb{R}^2$. By Lemma 2.8, we have $u \in \mathscr{P}$ and hence $u(x) = u(0) + a\cdot x$. □

3. A discussion of Definition 1.2

To see that Definition 1.2 makes sense, one must prove the following Proposition 3.1, which reads that, for any $\beta > -1$, if $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then it follows that $-\det D[|Dv|^\beta Dv]$ is defined almost everywhere, $-\det D[|Dv|^\beta Dv] \in L^1_{\mathrm{loc}}(\Omega)$, and $-\det D[|Dv|^\beta Dv]$ has a distributional representative as proved in Proposition 3.1 below.

Proposition 3.1. *For any $\beta > -1$, if $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then*

$$\begin{aligned}\int_\Omega -\det D[|Dv|^\beta Dv]\psi\,dx &= -\frac{1}{2}\int_\Omega |Dv|^{2\beta}(D^2\psi\, Dv\cdot Dv)\,dx\\ &\quad + \frac{1}{2\beta+2}\int_\Omega |Dv|^{2\beta+2}\Delta\psi\,dx\\ &\quad - \frac{\beta}{\beta+1}\int_\Omega [D|Dv|^{\beta+1}\cdot Dv](Dv\cdot D\psi)|Dv|^{\beta-1}\,dx\end{aligned} \tag{3.1}$$

for all $\psi \in C^\infty_c(\Omega)$.

When $\beta > -1$ and $\beta \ne 0$, the assumptions $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$ are minimal regularity conditions on v to guarantee that the right-hand side of (3.1) is finite. Under such minimal regularity, a pointwise definition $-\det D[|Dv|^\beta Dv]$ may not be available, but thanks to Proposition 3.1, we could use the right-hand side of (3.1) to define $-\det D[|Dv|^\beta Dv]$ in the distributional sense, as we did in Definition 1.2.

In the remaining part of Section 3, we will prove Proposition 3.1 by leveraging four auxiliary lemmas, namely Lemmas 3.2 through 3.5. The proofs of Lemmas 3.2 and 3.5 are presented in Sections 3.1 and 3.2, respectively. These proofs necessitate the introduction of several new concepts and approaches.

Recall that, in the case when $\beta = 0$, Proposition 3.1 follows from (2.2) and a standard approximation via smooth functions. Indeed, denote by $\{\eta_\varepsilon\}_{\varepsilon\in(0,1]}$ the standard smooth mollifier

$$\eta_\varepsilon(x) = \varepsilon^{-2}\eta(\varepsilon^{-1}x), \quad \text{where } \eta \in C^\infty(B(0,1)) \text{ satisfying} \qquad \eta \geq 0 \quad \text{and} \quad \int_{B(0,1)} \eta\, dx = 1. \tag{3.2}$$

Given any $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$, we know that (2.2) holds for $v * \eta_\varepsilon$. As $\varepsilon \to 0$, since

$$\begin{aligned} -\det D^2(v*\eta_\varepsilon) = -\det[(D^2v)*\eta_\varepsilon] &\to -\det D^2 v \quad \text{in } L^1_{\mathrm{loc}}(\Omega),\\ |D(v*\eta_\varepsilon)|^2 &\to |Dv|^2 \quad \text{in } L^1_{\mathrm{loc}}(\Omega),\\ D(v*\eta_\varepsilon)\otimes D(v*\eta_\varepsilon) &\to Dv\otimes Dv \quad \text{in } L^1_{\mathrm{loc}}(\Omega), \end{aligned}$$

we know that (2.2) holds for such v.

Below, we consider the case when $\beta \neq 0$. First, we note that, for any $v \in C^\infty(\Omega)$, in the case when $\beta \geq 0$, one has $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$, but when $-1 < \beta < 0$, we do not necessarily have $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Indeed, if $w(x) = x_1^2$ in $\mathbb{R}^2$, then a direct calculation leads to

$$|D[|Dw|^\beta Dw]|^2 = 4(\beta+1)^2|x_1|^{2\beta} \quad \text{when } x_1 \neq 0,$$

which does not belong to $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ when $\beta \leq -\frac{1}{2}$. For this reason, when $\beta < 0$, we consider

$$-\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}Dv] \quad \text{with } \varepsilon > 0.$$

We have the following result, whose proof is postponed to Section 3.1.

Lemma 3.2. *Let $v \in C^\infty(\Omega)$. Given any $\beta \geq 0$ and $\varepsilon \geq 0$, or given any $\beta \in (-1,0)$ and $\varepsilon > 0$, one has*

$$\begin{aligned} \int_\Omega -\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}Dv]\psi\, dx &= -\frac{1}{2}\int_\Omega (|Dv|^2+\varepsilon)^\beta (D^2\psi\, Dv\cdot Dv)\, dx\\ &\quad + \frac{1}{2\beta+2}\int_\Omega (|Dv|^2+\varepsilon)^{\beta+1}\Delta\psi\, dx\\ &\quad - \frac{\beta}{\beta+1}\int_\Omega (|Dv|^2+\varepsilon)^{\frac{\beta-1}{2}}[D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}}\cdot Dv]\\ &\qquad\qquad \times (Dv\cdot D\psi)\, dx \end{aligned} \tag{3.3}$$

for all $\psi \in C^\infty_c(\Omega)$.

We also have the following two divergence structural formulae.

Lemma 3.3. *Let $v \in C^\infty(\Omega)$. For any $\varepsilon > 0$ and $\beta > -1$, one has, for any $\psi \in C^\infty_c(\Omega)$,*

$$\begin{aligned} \int_\Omega -\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}Dv]\psi\, dx &= \int_\Omega \big\{[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}v_{x_2}]_{x_2}(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}v_{x_1}\psi_{x_1}\\ &\qquad - [(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}v_{x_2}]_{x_1}(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}v_{x_1}\psi_{x_2}\big\}\, dx. \end{aligned}$$

Proof. Write $F = (|Dv|^2 + \varepsilon)^{\frac{\beta}{2}} Dv$. By integration by parts, one has

$$\begin{aligned}\int_\Omega -\det(DF)\psi\,dx &= -\int_\Omega [(F_1)_{x_1}(F_2)_{x_2} - (F_1)_{x_2}(F_2)_{x_1}]\psi\,dx \\ &= \int_\Omega [F_1(F_2)_{x_2}\psi_{x_1}\,dx - F_1(F_2)_{x_1}\psi_{x_2}]\,dx,\end{aligned}$$

as desired. □

Lemma 3.4. *Let $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfy $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for some $\beta > -1$. One has*

$$\begin{aligned}&\int_\Omega -\det D[|Dv|^\beta Dv]\psi\,dx \\ &\quad= \int_\Omega [(|Dv|^\beta v_{x_2})_{x_2}|Dv|^\beta v_{x_1}\psi_{x_1} - (|Dv|^\beta v_{x_2})_{x_1}|Dv|^\beta v_{x_1}\psi_{x_2}]\,dx\end{aligned}$$

for all $\psi \in C^\infty_c(\Omega)$.

Proof. For $\varepsilon > 0$, let $F^\varepsilon = (F^\varepsilon_1, F^\varepsilon_2) = (|Dv|^\beta Dv) * \eta_\varepsilon \in C^\infty_{\mathrm{loc}}(\Omega)$, where η_ε is the standard smooth mollifier as in (3.2). By $|Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we know that $F^\varepsilon \to |Dv|^\beta Dv$ in $W^{1,2}_{\mathrm{loc}}(\Omega)$ as $\varepsilon \to 0$. By this, integration by parts and $(F^\varepsilon_2)_{x_1x_2} = (F^\varepsilon_2)_{x_2x_1}$, we have

$$\begin{aligned}&\int_\Omega -\det(D[|Dv|^\beta Dv])\psi\,dx \\ &\quad= \lim_{\varepsilon\to 0}\int_\Omega -\det(DF^\varepsilon)\psi\,dx \\ &\quad= -\lim_{\varepsilon\to 0}\int_\Omega [(F^\varepsilon_1)_{x_1}(F^\varepsilon_2)_{x_2} - (F^\varepsilon_1)_{x_2}(F^\varepsilon_2)_{x_1}]\psi\,dx \\ &\quad= \lim_{\varepsilon\to 0}\int_\Omega [F^\varepsilon_1(F^\varepsilon_2)_{x_2}\psi_{x_1}\,dx - F^\varepsilon_1(F^\varepsilon_2)_{x_1}\psi_{x_2}]\,dx \\ &\quad= \int_\Omega [(|Dv|^\beta v_{x_1})(|Dv|^\beta v_{x_2})_{x_2}\psi_{x_1}\,dx - (|Dv|^\beta v_{x_1})(|Dv|^\beta v_{x_2})_{x_1}\psi_{x_2}]\,dx\end{aligned}$$

for any $\psi \in C^\infty_c(\Omega)$. Hence we complete this proof. □

Moreover, we need the following approximation result. Given any $0 \neq \beta > -1$, let

$$v \in W^{1,1}_{\mathrm{loc}}(\Omega) \quad \text{satisfy} \quad |Dv|^\beta Dv \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

Write

$$g := \operatorname{div}(|Dv|^\beta Dv) \in L^2_{\mathrm{loc}}(\Omega).$$

Given any $B = B(x_0, r) \Subset 4B \Subset \Omega$, one has $g \in L^2(4B)$. For any $\varepsilon \in (0, r]$, set

$$g^\varepsilon(x) := g * \eta_\varepsilon(x) \quad \text{for all } x \in 3B,$$

where $\{\eta_\varepsilon\}_{\varepsilon\in(0,1]}$ is the standard smooth mollifier as in (3.2). Note that

$$g^\varepsilon \in C^\infty(3B), \quad g^\varepsilon \in L^2(3B) \quad \text{uniformly in } \varepsilon \in (0, r],$$

and $g^\varepsilon \to g$ in $L^2(3B)$ as $\varepsilon \to 0$. Since $|Dv|^\beta Dv \in W^{1,2}(4B)$, by the Sobolev embedding theorem, we have $|Dv|^\beta Dv \in L^q(4B)$, and hence $v \in W^{1,q}(4B)$, for any $1 < q < \infty$. Consider the Dirichlet problem for the inhomogeneous $(2+\beta)$-Laplace equation

$$\operatorname{div}\big((|Dw|^2+\varepsilon)^{\frac{\beta}{2}}Dw\big) = g^\varepsilon \text{ in } 2B, \quad w = v \text{ on } \partial(2B). \tag{3.4}$$

There exists a unique smooth solution $v^\varepsilon \in C^\infty(2B)\cap W^{1,2+\beta}(2B)\cap C^0(\overline{2B})$ to (3.4). The following convergence result plays a key role in the proof of Proposition 3.1. Its proof is postponed to Section 3.2.

Lemma 3.5. *We have*

$$Dv^\varepsilon \to Dv \qquad \text{in } L^{2+\beta}(2B) \text{ as } \varepsilon\to 0;$$

$$\begin{aligned} &[|Dv^\varepsilon|^2+\varepsilon]^{\frac{\beta}{2}}Dv^\varepsilon \in W^{1,2}(2B) && \text{uniformly in } \varepsilon\in(0,r],\\ &[|Dv^\varepsilon|^2+\varepsilon]^{\frac{\beta}{2}}Dv^\varepsilon \to |Dv|^\beta Dv && \text{in } L^q(B) \text{ for any } q\in(1,\infty) \text{ and}\\ &&& \text{weakly in } W^{1,2}(B) \text{ as } \varepsilon\to 0; \end{aligned}\tag{3.5}$$

$$\begin{aligned} &[|Dv^\varepsilon|^2+\varepsilon]^{\frac{\beta+1}{2}} \in W^{1,2}(B) && \text{uniformly in } \varepsilon\in(0,r],\\ &[|Dv^\varepsilon|^2+\varepsilon]^{\frac{\beta+1}{2}} \to |Dv|^{\beta+1} && \text{in } L^q(B) \text{ for any } q\in(1,\infty) \text{ and}\\ &&& \text{weakly in } W^{1,2}(B) \text{ as } \varepsilon\to 0. \end{aligned}\tag{3.6}$$

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\beta>-1$ but $\beta\neq 0$. Up to a partition of unit, we only need to show that (1.2) for all $\psi\in C_c^\infty(B)$ whenever $B=B(x_0,r)$ with $4B\subset\Omega$. Fix such a ball B. Let v^ε be as in Lemma 3.5. Since $[|Dv^\varepsilon|^2+\varepsilon]^{\frac{\beta}{2}}Dv^\varepsilon \to |Dv|^\beta Dv$ weakly in $W^{1,2}(B)$ as given in (3.5), by Lemmas 3.4 and 3.3, we have

$$\begin{aligned} &\int_B -\det D[|Dv|^\beta Dv]\psi\,dx\\ &\quad= \int_B [(|Dv|^\beta v_{x_2})_{x_2}|Dv|^\beta v_{x_1}\psi_{x_1} - (|Dv|^\beta v_{x_2})_{x_1}|Dv|^\beta v_{x_1}\psi_{x_2}]\,dx\\ &\quad= \lim_{\varepsilon\to 0}\int_B \Big[\big((|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}v^\varepsilon_{x_2}\big)_{x_2}(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}v^\varepsilon_{x_1}\psi_{x_1}\\ &\qquad\qquad - \big((|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}v^\varepsilon_{x_2}\big)_{x_1}(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}v^\varepsilon_{x_1}\psi_{x_2}\Big]\,dx\\ &\quad= \lim_{\varepsilon\to 0}\int_B -\det D[(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}Dv^\varepsilon]\psi\,dx \quad \text{for all } \psi\in C_c^\infty(B). \end{aligned}$$

Note that, by Lemma 3.2, (3.3) always holds v^ε and $\psi\in C_c^\infty(B)$. To get (3.1) for v, it then suffices to show that

$$\begin{aligned} &\int_B (|Dv^\varepsilon|^2+\varepsilon)^\beta (D^2\psi\, Dv^\varepsilon\cdot Dv^\varepsilon)\,dx\\ &\qquad\qquad \to \int_B |Dv|^{2\beta}(D^2\psi\, Dv\cdot Dv)\,dx, \end{aligned}\tag{3.7}$$

$$\int_B (|Dv^\varepsilon|^2+\varepsilon)^{\beta+1}\Delta\psi\,dx \to \int_B |Dv|^{2\beta+2}\Delta\psi\,dx, \tag{3.8}$$

$$\begin{aligned} &\int_B (|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta-1}{2}}[D(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}}\cdot Dv](Dv^\varepsilon\cdot D\psi)\,dx\\ &\qquad\qquad \to \int_B [D|Dv|^{\beta+1}\cdot Dv](Dv\cdot D\psi)|Dv|^{\beta-1}\,dx \end{aligned}\tag{3.9}$$

for any $\psi \in C_c^\infty(B)$. By (3.5), we have $(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \to |Dv|^\beta Dv$ in $L^2(B)$, which gives (3.8).

To get (3.7), we only need to show

$$(|Dv^\varepsilon|^2+\varepsilon)^{\beta} Dv^\varepsilon \otimes Dv^\varepsilon \to |Dv|^{2\beta} Dv \otimes Dv \quad \text{in } L^2(B). \tag{3.10}$$

Noting

$$|a\otimes a - b\otimes b| \le |a\otimes a - a\otimes b| + |a\otimes b - b\otimes b| \le |a-b|[|a|+|b|],$$

we have

$$\begin{aligned}
&\big|(|Dv^\varepsilon|^2+\varepsilon)^{\beta} Dv^\varepsilon \otimes Dv^\varepsilon - |Dv|^{2\beta} Dv \otimes Dv\big| \\
&\qquad \le \big|(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon - |Dv|^\beta Dv\big|[(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}} + |Dv|^{\beta+1}].
\end{aligned}$$

Since

$$\begin{aligned}
(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon &\to |Dv|^\beta Dv \quad && \text{in } L^2(B),\\
(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}} &\in L^2(B) && \text{uniformly in } \varepsilon>0,\\
|Dv|^{\beta+1} &\in L^2(B),
\end{aligned}$$

we obtain (3.10).

Finally, (3.9) follows from $D(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}} \to D|Dv|^{\beta+1}$ weakly in $L^2(B)$ as given in (3.6), and also

$$(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta-1}{2}} Dv^\varepsilon \otimes Dv^\varepsilon \to |Dv|^{\beta-1} Dv \otimes Dv \quad \text{in } L^2(B),$$

which is proved similarly to (3.10). □

3.1. Proof of Lemma 3.2. We first recall the following fundamental identity (3.11); see [14, 25].

Lemma 3.6. *For any $v \in C^\infty(\Omega)$, we have*

$$|D^2v\, Dv|^2 - \Delta v \Delta_\infty v = \frac{1}{2}[|D^2v|^2 - (\Delta v)^2]|Dv|^2 \quad \textit{in } \Omega. \tag{3.11}$$

Next we build up the following structural identity.

Lemma 3.7. *For any $v \in C^\infty(\Omega)$, $\beta\in\mathbb{R}$, and $\varepsilon>0$, we have*

$$\begin{aligned}
&-\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv] \\
&\qquad = \frac{1}{2}(|Dv|^2+\varepsilon)^{\beta}[|D^2v|^2 - (\Delta v)^2] \\
&\qquad\quad + \beta(|Dv|^2+\varepsilon)^{\beta-1}[|D^2v\, Dv|^2 - \Delta v\Delta_\infty v] \quad \textit{in } \Omega.
\end{aligned} \tag{3.12}$$

Moreover, if in addition $|Dv|>0$ in Ω, then (3.12) *holds with $\varepsilon=0$.*

Proof. For $1\le i,j\le 2$, one has

$$[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}} v_{x_i}]_{x_j} = (|Dv|^2+\varepsilon)^{\frac{\beta}{2}}\Big[v_{x_ix_j} + \beta(|Dv|^2+\varepsilon)^{-1}\Big(\frac{|Dv|^2}{2}\Big)_{x_j} v_{x_i}\Big].$$

Thus,

$$
\begin{aligned}
&\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv]\\
&\quad = (|Dv|^2+\varepsilon)^{\beta}\Big[v_{x_1x_1} + \beta(|Dv|^2+\varepsilon)^{-1}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1}\Big]\\
&\qquad\qquad \times \Big[v_{x_2x_2} + \beta(|Dv|^2+\varepsilon)^{-1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2}\Big]\\
&\qquad - (|Dv|^2+\varepsilon)^{\beta}\Big[v_{x_ix_j} + \beta(|Dv|^2+\varepsilon)^{-1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_1}\Big]\\
&\qquad\qquad \times \Big[v_{x_2x_1} + \beta(|Dv|^2+\varepsilon)^{-1}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_2}\Big]\\
&\quad = (|Dv|^2+\varepsilon)^{\beta}[v_{x_1x_1}v_{x_2x_2} - v_{x_1x_2}v_{x_2x_1}] + \beta(|Dv|^2+\varepsilon)^{\beta-1}\\
&\qquad\qquad \times \Big\{\Big[v_{x_1x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2} + v_{x_2x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1}\Big]\\
&\qquad\qquad\qquad - \Big[v_{x_1x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_2} + v_{x_2x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_1}\Big]\Big\}\\
&\qquad + \beta^2(|Dv|^2+\varepsilon)^{\beta-2}\Big[\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2}\\
&\qquad\qquad\qquad\qquad - \Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_2}\Big].
\end{aligned}
$$

Observe that the last term is 0, the first term equals $(|Dv|^2+\varepsilon)^{\beta}\det D^2v$. Regarding the second term, since

$$
\begin{aligned}
\Delta_\infty v &= \Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1} + \Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2},\\
|D^2v\, Dv|^2 &= v_{x_1x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1} + v_{x_2x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2}\\
&\qquad + v_{x_1x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_2} + v_{x_2x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_1},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\Big[v_{x_1x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2} + v_{x_2x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1}\Big]\\
&\qquad - \Big[v_{x_1x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_2} + v_{x_2x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_1}\Big]\\
&\quad = v_{x_1x_1}\Delta_\infty v + v_{x_2x_2}\Delta_\infty v - v_{x_1x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_1} - v_{x_2x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_2}\\
&\qquad - \Big[v_{x_1x_2}\Big(\frac{|Dv|^2}{2}\Big)_{x_1} v_{x_2} + v_{x_2x_1}\Big(\frac{|Dv|^2}{2}\Big)_{x_2} v_{x_1}\Big]\\
&\quad = \Delta v\Delta_\infty v - |D^2v\, Dv|^2.
\end{aligned}
$$

Thus, the second term equals $-\beta(|Dv|^2+\varepsilon)^{\beta-1}[|D^2v\, Dv|^2 - \Delta v\Delta_\infty v]$. We therefore obtain (3.12). Finally, we note that if $|Dv(x)| > 0$, the above argument holds with $\varepsilon = 0$. This completes the proof of Lemma 3.7. □

We are ready to prove Lemma 3.2.

Proof of Lemma 3.2. By (2.1) and integration by parts, for any $\psi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega (|Dv|^2+\varepsilon)^\beta [|D^2v|^2-(\Delta v)^2]\psi\,dx \\ &= \int_\Omega (|Dv|^2+\varepsilon)^\beta \operatorname{div}(D^2v\,Dv-\Delta v\,Dv)\psi\,dx \\ &= -2\beta\int_\Omega (|Dv|^2+\varepsilon)^{\beta-1}[|D^2v\,Dv|^2-\Delta_\infty v\Delta v]\psi\,dx \\ &\quad+\int_\Omega (|Dv|^2+\varepsilon)^\beta[\Delta v\,Dv\cdot D\psi-D^2v\,Dv\cdot D\psi]\,dx.\end{aligned}$$

From this and (3.12), it follows that

$$\begin{aligned}\int_\Omega -\det D[(|Dv|^2+\varepsilon)^{\frac{\beta}{2}}Dv]\psi\,dx \\ &= \frac12\int_\Omega (|Dv|^2+\varepsilon)^\beta[\Delta v\,Dv\cdot D\psi-D^2v\,Dv\cdot D\psi]\,dx.\end{aligned}$$

By integration by parts again, we have

$$\begin{aligned}\frac12\int_\Omega \Delta v(|Dv|^2+\varepsilon)^\beta(Dv\cdot D\psi)\,dx &= -\frac12\int_\Omega (|Dv|^2+\varepsilon)^\beta(D^2\psi\,Dv\cdot Dv)\,dx \\ &\quad-\frac12\int_\Omega (|Dv|^2+\varepsilon)^\beta(D^2v\,Dv\cdot D\psi)\,dx \\ &\quad-\frac12\int_\Omega [D(|Dv|^2+\varepsilon)^\beta\cdot Dv](Dv\cdot D\psi)\,dx.\end{aligned}$$

Noting

$$(|Dv|^2+\varepsilon)^\beta D^2v\,Dv = \frac{D(|Dv|^2+\varepsilon)^{\beta+1}}{2\beta+2},$$

one further gets

$$\begin{aligned}-\int_\Omega (|Dv|^2+\varepsilon)^\beta D^2v\,Dv\cdot D\psi\,dx &= -\frac{1}{2\beta+2}\int_\Omega [D(|Dv|^2+\varepsilon)^{\beta+1}\cdot D\psi]\,dx \\ &= \frac{1}{2\beta+2}\int_\Omega (|Dv|^2+\varepsilon)^{\beta+1}\Delta\psi\,dx.\end{aligned}$$

We also observe that

$$D(|Dv|^2+\varepsilon)^\beta = \frac{2\beta}{\beta+1}(|Dv|^2+\varepsilon)^{\frac{\beta-1}{2}}D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}}.$$

Thus,

$$\begin{aligned}-\frac12\int_\Omega [D(|Dv|^2+\varepsilon)^\beta\cdot Dv](Dv\cdot D\psi)\,dx \\ &= -\frac{\beta}{\beta+1}\int_\Omega (|Dv|^2+\varepsilon)^{\frac{\beta-1}{2}}[D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}}\cdot Dv](Dv\cdot D\psi)\,dx.\end{aligned}$$

Combining all the above, we obtain the desired identity (3.3). □

3.2. Proof of Lemma 3.5.

Proof of Lemma 3.5. Up to some scaling and translation, we assume that $x_0 = 0$ and $r = 1$. Write $B_m = B(0, m) = mB(0, 1)$ for $m \geq 1$. We divide the proof into four steps.

Step 1. Prove $v^\varepsilon \in L^2(B_2)$ and $Dv^\varepsilon \in L^{2+\beta}(B_2)$ uniformly in $\varepsilon \in (0, 1]$.

Since $v^\varepsilon - v \in W_0^{1,\beta+2}(B_2)$, by the Sobolev–Poincaré inequality, it suffices to prove that $Dv^\varepsilon \in L^{2+\beta}(B_2)$ uniformly in $\varepsilon \in (0, 1]$. Choosing the test function $v^\varepsilon - v \in W_0^{1,\beta+2}(B_2)$ to equation (3.4), we get

$$\int_{B_2} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \cdot (Dv^\varepsilon - Dv)\, dx = -\int_{B_2} g^\varepsilon (v^\varepsilon - v)\, dx, \tag{3.13}$$

or equivalently,

$$\int_{B_2} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} |Dv^\varepsilon|^2\, dx = \int_{B_2} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \cdot Dv\, dx - \int_{B_2} g^\varepsilon (v^\varepsilon - v)\, dx.$$

Young's inequality yields that

$$\begin{aligned}
\int_{B_2} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \cdot Dv\, dx &\leq \int_{B_2} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta+1}{2}} |Dv|\, dx \\
&\leq \frac{1}{2^{4+\beta}} \int_{B_2} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}+1}\, dx + C(\beta) \int_{B_2} |Dv|^{2+\beta}\, dx \\
&\leq \frac{1}{4} \int_{B_2} |Dv^\varepsilon|^{\beta+2}\, dx + \frac{1}{4} + C(\beta) \int_{B_2} |Dv|^{2+\beta}\, dx.
\end{aligned}$$

By Hölder's inequality, the Sobolev–Poincaré inequality, and Young's inequality, one has

$$\begin{aligned}
-\int_{B_2} g^\varepsilon (v^\varepsilon - v)\, dx &\leq \Big(\int_{B_2} (g^\varepsilon)^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{B_2} |v^\varepsilon - v|^2\, dx\Big)^{\frac{1}{2}} \\
&\leq C\Big(\int_{B_3} g^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{B_2} |Dv^\varepsilon - Dv|^{2+\beta}\, dx\Big)^{\frac{1}{2+\beta}} \\
&\leq C\Big(\int_{B_3} g^2\, dx\Big)^{\frac{2+\beta}{2(1+\beta)}} + \frac{1}{4} \int_{B_2} |Dv^\varepsilon|^{2+\beta}\, dx + C \int_{B_2} |Dv|^{2+\beta}\, dx.
\end{aligned}$$

Therefore, we obtain

$$\int_{B_2} |Dv^\varepsilon|^{\beta+2}\, dx \leq C(\beta) \int_{B_2} |Dv|^{\beta+2}\, dx + \Big(\int_{B_3} g^2\, dx\Big)^{\frac{2+\beta}{2(1+\beta)}} + C.$$

Thus, $Dv^\varepsilon \in L^{\beta+2}(B_2)$ uniformly in $\varepsilon \in (0, 1]$.

Step 2. Prove $v^\varepsilon \to v$ in $W^{1,2+\beta}(B_2)$ as $\varepsilon \to 0$.

Since $g = \operatorname{div}(|Dv|^\beta Dv)$ and $v^\varepsilon - v \in W_0^{1,p}(B_2)$, we have

$$-\int_{B_2} g(v^\varepsilon - v)\, dx = \int_{B_2} |Dv|^\beta Dv \cdot (Dv^\varepsilon - Dv)\, dx.$$

By this and (3.13), one has

$$\int_{B_2} (|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \cdot (Dv^\varepsilon - Dv)\,dx = \int_{B_2} |Dv|^\beta Dv\cdot (Dv^\varepsilon - Dv)\,dx + \int_{B_2} (g^\varepsilon - g)(v^\varepsilon - v)\,dx,$$

and hence

$$\int_{B_2} \big((|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon - (|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv\big)\cdot (Dv^\varepsilon - Dv)\,dx = \int_{B_2} \big(|Dv|^\beta Dv - (|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv\big)\cdot (Dv^\varepsilon - Dv)\,dx + \int_{B_2} (g^\varepsilon - g)(v^\varepsilon - v)\,dx.$$

Observe that, when $\beta > -1$, it holds that

$$(|\xi|^2+|\eta|^2+\varepsilon)^{\frac{\beta}{2}}|\xi-\eta|^2 \le C(\beta)\big((|\xi|^2+\varepsilon)^{\frac{\beta}{2}}\xi - (|\eta|^2+\varepsilon)^{\frac{\beta}{2}}\eta\big)\cdot(\xi-\eta) \quad \text{for all } \xi,\eta\in\mathbb{R}^2.$$

Thus,

$$\int_{B_2} (|Dv^\varepsilon|^2+|Dv|^2+\varepsilon)^{\frac{\beta}{2}}|Dv^\varepsilon - Dv|^2\,dx \le C(\beta)\int_{B_2} \big(|Dv|^\beta Dv - (|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv\big)\cdot (Dv^\varepsilon - Dv)\,dx + \int_{B_2} (g^\varepsilon - g)(v^\varepsilon - v)\,dx.$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{B_2} &(|Dv^\varepsilon|^2+|Dv|^2+\varepsilon)^{\frac{\beta}{2}}|Dv^\varepsilon - Dv|^2\,dx \\ &\le C(\beta)\Big(\int_{B_2}\big||Dv|^\beta Dv - (|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv\big|^{\frac{\beta+2}{\beta+1}}\,dx\Big)^{\frac{\beta+1}{\beta+2}} \times\Big(\int_{B_2}[|Dv^\varepsilon|^{\beta+2}+|Dv|^{\beta+2}]\Big)^{\frac{1}{\beta+2}} \\ &\quad + C(\beta)\Big(\int_{B_2}|g^\varepsilon-g|^2\,dx\Big)^{\frac12}\Big(\int_{B_2}[|v^\varepsilon|^2+|v|^2]\,dx\Big)^{\frac12}.\end{aligned} \tag{3.14}$$

Recalling that $v^\varepsilon\in W^{1,\beta+2}(B_2)$ uniformly in $\varepsilon\in(0,1]$ as given in the step 1, and noting that

$$(|Dv|^2+\varepsilon)^{\frac{\beta}{2}} Dv \to |Dv|^\beta Dv \quad \text{in } L^{\frac{\beta+2}{\beta+1}}(B_2) \text{ as } \varepsilon\to 0,$$

we deduce that the first term in the right-hand side of (3.14) tends to zero as $\varepsilon\to 0$. Since $v^\varepsilon\in W^{1,\beta+2}(B_2)$ uniformly in $\varepsilon\in(0,1]$ and recalling $g^\varepsilon\to g$ in $L^2(B_2)$, the second term of the right-hand side of (3.14) tends to 0 as $\varepsilon\to 0$. Thus,

$$\int_{B_2} (|Dv^\varepsilon|^2+|Dv|^2+\varepsilon)^{\frac{\beta}{2}}|Dv^\varepsilon - Dv|^2\,dx \to 0. \tag{3.15}$$

If $\beta>0$, since

$$|Dv^\varepsilon - Dv|^{2+\beta} \le C(\beta)(|Dv^\varepsilon|^2+|Dv|^2+\varepsilon)^{\frac{\beta}{2}}|Dv^\varepsilon - Dv|^2,$$

we have $Dv^\varepsilon \to Dv$ in $L^{2+\beta}(B_2)$ as $\varepsilon \to 0$. Thus, by $v^\varepsilon - v \in W_0^{1,2+\beta}(B_2)$ and using the Sobolev inequality, we always have $v^\varepsilon \to v$ in $W^{1,2+\beta}(B_2)$ as $\varepsilon \to 0$.

If $\beta \in (-1, 0)$, by Hölder's inequality, we get

$$\begin{aligned}\int_{B_2} &|Dv^\varepsilon - Dv|^{\beta+2}\,dx \\ &\le \Big(\int_{B_2} (|Dv^\varepsilon|^2 + |Dv|^2 + \varepsilon)^{\frac{\beta}{2}} |Dv^\varepsilon - Dv|^2\,dx\Big)^{\frac{\beta+2}{2}} \\ &\qquad \times \Big(\int_{B_2} (|Dv^\varepsilon|^2 + |Dv|^2 + \varepsilon)^{\frac{\beta+2}{2}}\,dx\Big)^{\frac{-\beta}{2}}.\end{aligned}$$

By Step 1, we have $v^\varepsilon \in W^{1,2+\beta}(B_2) \cap L^2(B_2)$ uniformly in $\varepsilon \in (0, 1)$. Then (3.15) yields that $Dv^\varepsilon \to Dv$ in $L^{2+\beta}(B_2)$ as $\varepsilon \to 0$. We further conclude $v^\varepsilon \to v$ in $W^{1,2+\beta}(B_2)$ as $\varepsilon \to 0$.

Step 3. Prove

$$\begin{aligned}&(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \in W^{1,2}(B_1) \quad \text{uniformly in } \varepsilon \in (0,1], \\ &(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \to |Dv|^\beta Dv \quad \text{weakly in } W^{1,2}(B_1) \text{ as } \varepsilon \to 0.\end{aligned}$$

By the local second-order estimates in [10, Theorem 2.1], we have

$$\int_{B_1} \big|D[(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon]\big|^2\,dx \le C_0 \int_{B_2} (g^\varepsilon)^2\,dx + C_0\Big(\int_{B_2} [|Dv^\varepsilon|^{\beta+1} + \varepsilon^{\frac{\beta+1}{2}}]\,dx\Big)^2.$$

Since $g^\varepsilon \in L^2(B_2)$ uniformly in $\varepsilon \in (0,1]$ and, by Step 1, $Dv^\varepsilon \in L^{2+\beta}(B_2)$ uniformly in $\varepsilon \in (0,1]$, one has $D[(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon] \in L^2(B_1)$ uniformly in $\varepsilon \in (0,1]$. By the Sobolev–Poincaré inequality, we also have

$$\begin{aligned}\int_{B_1} &\big|(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon\big|^2\,dx \\ &\le \int_{B_1} \Big|(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon - \fint_{B_1} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon\,dx\Big|^2\,dx \\ &\qquad + \Big|\int_{B_1} (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon\,dx\Big|^2 \\ &\le C_0 \int_{B_1} \big|D[(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon]\big|^2\,dx + C_0\Big|\int_{B_1} [|Dv^\varepsilon|^{\beta+1} + \varepsilon^{\frac{\beta+1}{2}}]\,dx\Big|^2.\end{aligned}$$

Thus, $(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \in L^2(B_1)$ uniformly in $\varepsilon \in (0,1]$.

By the weak compactness of the Sobolev space $W^{1,2}$, there exists $f \in W^{1,2}(B_1)$ such that, along a subsequence,

$$\begin{aligned}D[(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon] &\to Df \quad \text{weakly in } L^2(B_1), \\ (|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon &\to f \qquad \text{in } L^2(B_1).\end{aligned}$$

By Step 2, $Dv^\varepsilon \to Dv$ in $L^{2+\beta}(B_2)$ as $\varepsilon \to 0$, and hence $(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta}{2}} Dv^\varepsilon \to |Dv|^\beta Dv$ almost everywhere along a subsequence. We conclude that $f = |Dv|^\beta Dv$ in B_1, as desired.

Step 4. Prove

$$\begin{aligned}&(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta+1}{2}} \in W^{1,2}(B_1) \quad \text{uniformly in } \varepsilon \in (0,1], \\ &(|Dv^\varepsilon|^2 + \varepsilon)^{\frac{\beta+1}{2}} \to |Dv|^{\beta+1} \quad \text{weakly in } W^{1,2}(B_1) \text{ as } \varepsilon \to 0.\end{aligned}$$

Note that

$$|D(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}}|^2 = (\beta+1)^2(|Dv^\varepsilon|^2+\varepsilon)^{\beta-1}|D^2v^\varepsilon Dv^\varepsilon|^2,$$

$$\begin{aligned}\big|D[(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}Dv^\varepsilon]\big|^2 &= (|Dv^\varepsilon|^2+\varepsilon)^{\beta}|D^2v^\varepsilon|^2 + 2\beta(|Dv^\varepsilon|^2+\varepsilon)^{\beta-1}|D^2v^\varepsilon Dv^\varepsilon|^2\\ &\quad + \beta^2(|Dv^\varepsilon|^2+\varepsilon)^{\beta-2}|Dv^\varepsilon|^2|D^2v^\varepsilon Dv^\varepsilon|^2.\end{aligned}$$

If $\beta > 0$, then

$$\begin{aligned}|D(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}}|^2 &\le (\beta+1)^2(|Dv^\varepsilon|^2+\varepsilon)^{\beta}|D^2v^\varepsilon|^2\\ &\le C(\beta)\big|D[(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}Dv^\varepsilon]\big|^2.\end{aligned}$$

If $\beta \in (-1, 0)$,

$$\begin{aligned}&\big|D[(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta}{2}}Dv^\varepsilon]\big|^2\\ &\quad = (|Dv^\varepsilon|^2+\varepsilon)^{\beta-1}[|D^2v^\varepsilon|^2|Dv^\varepsilon|^2 + 2\beta|D^2v^\varepsilon Dv^\varepsilon|^2 + \beta^2|D^2v^\varepsilon Dv^\varepsilon|^2]\\ &\qquad + (|Dv^\varepsilon|^2+\varepsilon)^{\beta-2}[(|Dv^\varepsilon|^2+\varepsilon)\varepsilon|D^2v^\varepsilon|^2 - \beta^2\varepsilon|D^2v^\varepsilon Dv^\varepsilon|^2]\\ &\quad \ge (\beta+1)^2(|Dv^\varepsilon|^2+\varepsilon)^{\beta-1}|D^2v^\varepsilon Dv^\varepsilon|^2\\ &\quad = \big|D(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}}\big|^2.\end{aligned}$$

Thus, by Step 3,

$$\begin{aligned}&(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}} \in W^{1,2}(B_1) \quad \text{uniformly in } \varepsilon \in (0,1],\\ &(|Dv^\varepsilon|^2+\varepsilon)^{\frac{\beta+1}{2}} \to |Dv|^{\beta+1} \quad \text{weakly in } W^{1,2}(B_1) \text{ as } \varepsilon \to 0.\end{aligned}$$ □

4. Some properties of distributional Jacobian determinant

We build up the following stability result.

Lemma 4.1. *Let $\beta > -1$. If*

$$v_j \to v \text{ in } W^{1,2+\beta}_{\mathrm{loc}}(\Omega) \text{ as } j \to \infty \quad\text{and}\quad \beta|Dv_j|^{\beta+1} \in W^{1,2}_{\mathrm{loc}}(\Omega) \text{ uniformly in } j,$$

then

(i) *$\beta|Dv_j|^{\beta+1} \to \beta|Dv|^{\beta+1}$ in $L^q_{\mathrm{loc}}(\Omega)$ for any $q > 1$ and weakly in $W^{1,2}_{\mathrm{loc}}(\Omega)$;*

(ii) *$\beta|Dv_j|^{\beta}Dv_j \to \beta|Dv|^{\beta}Dv$ and $\beta|Dv_j|^{\beta-1}Dv_j \otimes Dv_j \to \beta|Dv|^{\beta-1}Dv \otimes Dv$ in $L^q_{\mathrm{loc}}(\Omega)$ for any $q > 1$.*

(iii) *$-\det D[|Dv_j|^{\beta}Dv_j] \to -\det D[|Dv|^{\beta}Dv]$ in the distributional sense, i.e.,*

$$\begin{aligned}\int_\Omega &-\det D[|Dv_j|^{\beta}v_j]\psi\,dx\\ &\to \int_\Omega -\det D[|Dv|^{\beta}Dv]\psi\,dx \quad \text{for all } \psi \in C^\infty_c(\Omega).\end{aligned} \tag{4.1}$$

Proof. The case $\beta = 0$ is easy. We only consider the case $\beta \neq 0$. Since

$$|Dv_j|^{\beta+1} \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

by the compact embedding theorem, there is a function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $|Dv_j|^{\beta+1} \to f$ in $L^q_{\mathrm{loc}}(\Omega)$ for any $1 < q < \infty$ and weakly in $W^{1,2}_{\mathrm{loc}}(\Omega)$ as $j \to \infty$ up to some subsequence. Since $Dv_j \to Dv$ in $L^1_{\mathrm{loc}}(\Omega)$, we know $f = |Dv|^{\beta+1}$. Thus, we have $|Dv|^{\beta+1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $|Dv_j|^{\beta+1}$ converges strongly to $|Dv|^{\beta+1}$ in $L^q_{\mathrm{loc}}(\Omega)$ for any $1 < q < \infty$ and weakly in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Therefore, (i) holds.

To see (ii), observe that if $Dv(x) = 0$, one has

$$\begin{aligned}\big||Dv_j|^\beta Dv_j - |Dv|^\beta Dv\big| &\le \big||Dv_j|^{\beta+1} - |Dv|^{\beta+1}\big|,\\ \big||Dv_j|^{\beta-1} Dv_j \otimes Dv_j - |Dv|^{\beta-1} Dv \otimes Dv\big| &\le \big||Dv_j|^{\beta+1} - |Dv|^{\beta+1}\big|.\end{aligned}$$

If $Dv(x) \neq 0$, one has

$$\begin{aligned}\big||Dv_j|^\beta Dv_j - |Dv|^\beta Dv\big| &\le \big||Dv_j|^{\beta+1} - |Dv|^{\beta+1}\big|\\ &\qquad + |Dv|^{\beta+1}|\overline{Dv_j} - \overline{Dv}|,\\ \big||Dv_j|^{\beta-1} Dv_j \otimes Dv_j - |Dv|^{\beta-1} Dv \otimes Dv\big| &\le \big||Dv_j|^{\beta+1} - |Dv|^{\beta+1}\big|\\ &\qquad + |Dv|^{\beta+1}|\overline{Dv_j} \otimes \overline{Dv_j} - \overline{Dv} \otimes \overline{Dv}|,\end{aligned}$$

where we set $\bar\xi = \xi/|\xi|$ when $\xi \neq 0$ and $\bar\xi = 0$ when $\xi = 0$. Since $Dv_j \to Dv$ almost everywhere along a subsequence, we know that $\overline{Dv_j} \otimes \overline{Dv_j} \to \overline{Dv} \otimes \overline{Dv}$ almost everywhere in $\Omega \setminus \{x \in \Omega : Dv(x) = 0\}$. By the Lebesgue dominated convergence, we conclude (ii) from above and (i).

Finally, note that (i) and (ii) imply

$$\begin{aligned}\int_\Omega |Dv_j|^{2\beta+2}\Delta\psi\,dx &\to \int_\Omega |Dv|^{2\beta+2}\Delta\psi\,dx,\\ \int_\Omega |Dv_j|^{2\beta}(D^2\psi\, Dv_j \cdot Dv_j)\,dx &\to \int_\Omega |Dv|^{2\beta}(D^2\psi\, Dv \cdot Dv)\,dx,\\ \int_\Omega [D|Dv_j|^{\beta+1} \cdot Dv_j](Dv_j \cdot D\psi)|Dv_j|^{\beta-1}\,dx&\\ &\to \int_\Omega [D|Dv|^{\beta+1} \cdot Du](Dv^\varepsilon \cdot D\psi)|Dv|^{\beta-1}\,dx\end{aligned}$$

for all $\psi \in C^\infty_c(\Omega)$. We conclude (4.1) from these and the definitions of $-\det D[|Dv_j|^\beta Dv_j]$ and $-\det D[|Dv|^\beta Dv]$. □

Lemma 4.2. *Let $v \in C^\infty(\Omega)$ satisfy $\beta|Dv|^{\beta+1} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for some $\beta > -1$. We have*

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega &-\det D[(|Du|^2+\varepsilon)^{\frac{\beta}{2}} Du]\psi\,dx\\ &\to \int_\Omega -\det D[|Du|^\beta Du]\psi\,dx \quad \textit{for all } \psi \in C^\infty_c(\Omega).\end{aligned}$$

Proof. The case $\beta = 0$ is easy. The case $\beta \neq 0$ would follow if we let $\varepsilon \to 0$ in (3.3) by Definition 1.2. To this end, it suffices to build up the following convergence.

Firstly, since $(|Dv|^2 + 1)^{\beta+1} \in L^q_{\mathrm{loc}}(\Omega)$ for $1 < q < \infty$, by the Lebesgue dominated convergence, one has

$$(|Dv|^2+\varepsilon)^\beta|Dv|^2 \to |Dv|^{2\beta+2} \quad \text{and} \quad (|Dv|^2+\varepsilon)^\beta Dv \otimes Dv \to |Dv|^{2\beta} Dv \otimes Dv$$

in $L^q_{\mathrm{loc}}(\Omega)$. Similarly,

$$(|Dv|^2+\varepsilon)^{\frac{\beta-1}{2}} Dv \otimes Dv \to |Dv|^{\frac{\beta-1}{2}} Dv \otimes Dv \quad \text{in } L^q_{\mathrm{loc}}(\Omega).$$

Moreover, we observe that

$$D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}} \to D|Dv|^{\beta+1} \quad \text{weakly in } L^2_{\mathrm{loc}}(\Omega). \tag{4.2}$$

Indeed, when $\beta>0$, since $v\in C^2(\Omega)$, one has

$$D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}} = C(\beta)(|Dv|^2+\varepsilon)^{\frac{\beta-1}{2}} D^2v\, Dv \in L^2_{\mathrm{loc}}(\Omega) \quad \text{uniformly in } \varepsilon\in(0,1),$$

and hence $D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}} \to D|Dv|^{\beta+1}$ weakly in $L^2_{\mathrm{loc}}(\Omega)$. When $-1<\beta<0$, by the assumption $D|Du|^{\beta+1}\in L^2_{\mathrm{loc}}(\Omega)$, we have

$$D(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}} = C(\beta)(|Dv|^2+\varepsilon)^{\frac{\beta-1}{2}}|Dv|^{1-\beta} D|Dv|^{\beta+1} \in L^2_{\mathrm{loc}}(\Omega)$$

uniformly in $\varepsilon\in(0,1)$. Thus, together with $(|Dv|^2+\varepsilon)^{\frac{\beta+1}{2}} \to |Dv|^{\beta+1}$ weakly in $L^2_{\mathrm{loc}}(\Omega)$, we conclude (4.2), as desired. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3. Let u be any ∞-harmonic function in planar domain Ω. Since

$$u\in C^{0,1}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad |Du|^{\beta+1}\in W^{1,2}_{\mathrm{loc}}(\Omega),$$

by Definition 1.2, the distributional Jacobi $-\det D[|Du|^\beta Du]$ is well defined. We proceed as below to show that $-\det D[|Du|^\beta Du]\in \mathcal{M}(\Omega)$ with the lower bound (1.5) and the upper bound (1.6).

Step 1. Given any smooth subdomain $U\Subset\Omega$, for any $\varepsilon\in(0,1)$, denote by

$$u^\varepsilon \in C^\infty(U)\cap C^0(\overline{U})$$

the unique solution to the equation

$$\operatorname{div}(e^{\frac{1}{2\varepsilon}|Du^\varepsilon|^2} Du^\varepsilon) = \frac{1}{\varepsilon} e^{\frac{1}{2\varepsilon}|Du^\varepsilon|^2}(\Delta_\infty u^\varepsilon + \varepsilon\,\Delta u^\varepsilon) = 0 \quad \text{with } u^\varepsilon = u \text{ on } \partial U.$$

It was shown in [18] that

$$\limsup_{\varepsilon\to 0}\|Du^\varepsilon\|_{L^\infty(V)} \le \|Du\|_{L^\infty(U)} \quad \text{for all } V\Subset U.$$

By [25], for any $\beta>-1$,

$$|Du^\varepsilon|^{\beta+1}\in W^{1,2}_{\mathrm{loc}}(U) \quad \text{uniformly in } \varepsilon>0$$

and

$$\lim_{\varepsilon\to 0}\|Du^\varepsilon - Du\|_{L^q(V)} \to 0 \quad \text{for all } 1<q<\infty \text{ and all } V\Subset U.$$

Thus, it follows that $|Du|^{\beta+1} \in W^{1,2}_{\mathrm{loc}}(U)$, and along a subsequence, $|Du^\varepsilon|^{\beta+1} \to |Du|^{\beta+1}$ strongly in $L^2_{\mathrm{loc}}(U)$ as $\varepsilon \to 0$.

In view of Definition 1.2, the distributional Jacobi $-\det D[|Du^\varepsilon|^\beta Du^\varepsilon]$ is well defined. By Lemma 4.2, one has

$$\lim_{\delta\to 0}\int_U -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}} Du^\varepsilon]\psi\,dx = \int_U -\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\psi\,dx \quad \text{for all } \psi \in C_c^\infty(U). \tag{5.1}$$

By Lemma 4.1, we know that

$$\lim_{\varepsilon\to 0}\int_U -\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\psi\,dx = \int_U -\det D[|Du|^\beta Du]\psi\,dx \quad \text{for all } \psi \in C_c^\infty(U). \tag{5.2}$$

Step 2. By Lemma 3.6 and $\Delta_\infty u^\varepsilon + \varepsilon\,\Delta u^\varepsilon = 0$ in U, we have

$$\begin{aligned}\frac{1}{2}[|D^2u^\varepsilon|^2 - (\Delta u^\varepsilon)^2]|Du^\varepsilon|^2 &= |D^2u^\varepsilon Du^\varepsilon|^2 - \Delta_\infty u^\varepsilon \Delta u^\varepsilon \\ &= |D^2u^\varepsilon Du^\varepsilon|^2 + \frac{1}{\varepsilon}(\Delta_\infty u^\varepsilon)^2 \quad \text{in } U.\end{aligned}$$

For $\delta > 0$, by this and Lemma 3.7, we obtain

$$\begin{aligned}&-\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}} Du^\varepsilon] \\ &\quad= \frac{1}{2}(|Du^\varepsilon|^2+\delta)^\beta[|D^2u^\varepsilon|^2 - (\Delta u^\varepsilon)^2] \\ &\qquad + \beta(|Du^\varepsilon|^2+\delta)^{\beta-1}[|D^2u^\varepsilon Du^\varepsilon|^2 - \Delta u^\varepsilon \Delta_\infty u^\varepsilon] \\ &\quad\ge (\beta+1)(|Du^\varepsilon|^2+\delta)^{\beta-1}\Big[|D^2u^\varepsilon Du^\varepsilon|^2 + \frac{1}{\varepsilon}(\Delta_\infty u^\varepsilon)^2\Big] \\ &\quad= \frac{1}{\beta+1}\big|D(|Du^\varepsilon|^2+\delta)^{\frac{\beta+1}{2}}\big|^2 \\ &\qquad + (\beta+1)\frac{1}{\varepsilon}(|Du^\varepsilon|^2+\delta)^{\beta-1}(\Delta_\infty u^\varepsilon)^2 \quad \text{in } U.\end{aligned} \tag{5.3}$$

For any $0 \le \psi \in C_c^\infty(U)$, by Lemma 4.2, one has

$$\begin{aligned}&\int_U -\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\psi\,dx \\ &\quad= \lim_{\delta\to 0}\int_U -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}} Du^\varepsilon]\psi\,dx \\ &\quad\ge \frac{1}{\beta+1}\liminf_{\delta\to 0}\int_U\Big[\big|D(|Du^\varepsilon|^2+\delta)^{\frac{\beta+1}{2}}\big|^2 \\ &\qquad\qquad + (\beta+1)\frac{1}{\varepsilon}(|Du^\varepsilon|^2+\delta)^{\beta-1}(\Delta_\infty u^\varepsilon)^2\Big]\psi\,dx \\ &\quad\ge \frac{1}{\beta+1}\int_U\Big[\big|D|Du^\varepsilon|^{\beta+1}\big|^2 + (\beta+1)\frac{1}{\varepsilon}|Du^\varepsilon|^{2\beta-2}(\Delta_\infty u^\varepsilon)^2\Big]\psi\,dx,\end{aligned} \tag{5.4}$$

where, in the last inequality, we used that $(|Du^\varepsilon|^2+\delta)^{\frac{\beta+1}{2}} \to |Du^\varepsilon|^{\beta+1}$ weakly in $W^{1,2}_{\mathrm{loc}}(U)$ as $\delta \to 0$, and also that $(|Du^\varepsilon|^2+\delta)^{\beta-1}(\Delta_\infty u^\varepsilon)^2 \to |Du^\varepsilon|^{2\beta-2}(\Delta_\infty u^\varepsilon)^2$ almost everywhere and it has a dominant function $|Du^\varepsilon|^{2\beta+2}|D^2u^\varepsilon|^2 \in L^1_{\mathrm{loc}}(U)$.

Letting $\varepsilon \to 0$, since $|Du^\varepsilon|^{\beta+1} \to |Du|^{\beta+1}$ weakly in $W^{1,2}_{\mathrm{loc}}(U)$, we further have

$$\begin{aligned}\int_U -\det D[|Du|^\beta Du]\psi\,dx &\ge \liminf_{\varepsilon\to 0} \frac{1}{\beta+1}\int_U \big|D|Du^\varepsilon|^{\beta+1}\big|^2\psi\,dx \\ &\ge \frac{1}{\beta+1}\int_U \big|D|Du|^{\beta+1}\big|^2\psi\,dx \quad \text{for all } 0\le\psi\in C_c^\infty(U),\end{aligned}$$

which gives the lower bound (1.5).

Step 3. For any $\phi \in C_c^\infty(U)$ and $\delta\in[0,1)$, we have

$$\begin{aligned}&\int_U -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}} Du^\varepsilon]\phi^2\,dx \\ &\qquad\le C\frac{1+\beta^2}{1+\beta}\int_U (|Du^\varepsilon|^2+\delta)^{\beta+1}[|\phi\, D^2\phi| + |D\phi|^2]\,dx.\end{aligned}\tag{5.5}$$

Indeed, by Lemma 3.2 for $\delta\in(0,1]$ and Definition 1.2 for $\delta=0$, and by Young's inequality, one has

$$\begin{aligned}&\int_U -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}} Du^\varepsilon]\phi^2\,dx \\ &\quad= -\frac12\int_U (|Du^\varepsilon|^2+\delta)^\beta (D^2\phi^2 Du^\varepsilon\cdot Du^\varepsilon)\,dx \\ &\qquad+\frac{1}{2\beta+2}\int_U (|Du^\varepsilon|^2+\delta)^{\beta+1}\Delta\phi^2\,dx \\ &\qquad-\frac{\beta}{2\beta+2}\int_U (|Du^\varepsilon|^2+\delta)^{\frac{\beta-1}{2}}[D(|Du^\varepsilon|^2+\delta)^{\frac{\beta+1}{2}}\cdot Du^\varepsilon](Du^\varepsilon\cdot D\phi^2)\,dx \\ &\quad\le C\Big(\frac12+\frac{1}{2+2\beta}+\frac{\beta^2}{2+2\beta}\Big)\int (|Du^\varepsilon|^2+\delta)^{\beta+1}[|\phi\, D^2\phi|+|D\phi|^2]\,dx \\ &\qquad+\frac{1}{2\beta+2}\int_U \big|D(|Du^\varepsilon|^2+\delta)^{\frac{\beta+1}{2}}\big|^2\phi^2\,dx.\end{aligned}$$

Applying (5.3) for $\delta\in(0,1]$ and (5.4) for $\delta=0$, one has

$$\frac{1}{2\beta+2}\int_U \big|D(|Du^\varepsilon|^2+\delta)^{\frac{\beta+1}{2}}\big|^2\phi^2\,dx \le \frac12\int_U -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}}Du^\varepsilon]\phi^2\,dx,$$

and therefore, we get (5.5).

Step 4. By (5.3) and (5.5), we know that

$$0\le -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}}Du^\varepsilon] \in L^1_{\mathrm{loc}}(U) \quad \text{uniformly in } \delta\in(0,1].\tag{5.6}$$

By (5.6), (5.1), and a density argument, we know that

$$\lim_{\delta\to 0}\int_U -\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}}Du^\varepsilon]\psi\,dx$$

always exists for all $\psi\in C_c^0(U)$, and is denoted by $\mu^\varepsilon(\psi)$. Moreover, μ^ε is a nonnegative Radon measure, i.e., $0\le\mu^\varepsilon\in\mathcal{M}(U)$, and $-\det D[(|Du^\varepsilon|^2+\delta)^{\frac{\beta}{2}}Du^\varepsilon]\,dx$ converges to μ^ε in the weak-$\star$ sense in $\mathcal{M}(U)$ as $\delta\to 0$. Note that (5.1) implies that μ^ε is induced by the distribution $-\det D[|Du^\varepsilon|^\beta Du^\varepsilon]$ uniquely. Therefore, we can view $-\det D[|Du^\varepsilon|^\beta Du^\varepsilon]$ as the measure μ^ε.

Moreover, by (5.5) with $\delta = 0$, given any subdomain $V \Subset U$, by a suitable choice of test function ϕ, we also have

$$\|-\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\|(V) \le C \frac{1}{[\mathrm{dist}(V, \partial U)]^2} \frac{1+\beta^2}{1+\beta} \int_W |Du^\varepsilon|^{2+2\beta}\, dx,$$

where $V \Subset W \Subset U$ with $\mathrm{dist}(W, \partial U) = \mathrm{dist}(V, \partial W) = \frac{1}{2}\mathrm{dist}(V, \partial U)$. As $|Du^\varepsilon| \in L^\infty(W)$ uniformly in $\varepsilon \in (0, \varepsilon_V)$ for some $\varepsilon_V > 0$, we have that $\|-\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\|(V)$ is bounded uniformly in $\varepsilon \in (0, \varepsilon_V)$. Since

$$0 \le -\det D[|Du^\varepsilon|^\beta Du^\varepsilon] \in \mathcal{M}(U),$$

by (5.2) and a density argument, we know that

$$\lim_{\varepsilon \to 0} \int_U -\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\psi\, dx$$

always exists for all $\psi \in C_c^0(U)$, and is denoted by $\mu(\psi)$. Moreover, $0 \le \mu \in \mathcal{M}(U)$, and $-\det D[|Du^\varepsilon|^\beta Du^\varepsilon]\, dx$ converges to μ in the weak-$\star$ sense in $\mathcal{M}(U)$ as $\varepsilon \to 0$. By (5.2), we know that μ is induced by the distribution $-\det D[|Du|^\beta Du]$ uniquely, and hence we can view $-\det D[|Du|^\beta Du]$ as μ.

By the arbitrariness of U, we know that $-\det D[|Du|^\beta Du] \in \mathcal{M}(\Omega)$. The upper bound (1.6) follows from (1.4) and a suitable choice of test functions $0 \le \psi \in C_c^\infty(\Omega)$. □

6. Proofs of Theorems 1.4 and 1.5

Given $p \in (1, \infty)$, let u_p be any nonconstant p-harmonic function in a planar domain Ω. For $\beta > -1$, one has $|Du_p|^\beta Du_p \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and hence $-\det D[|Du_p|^\beta Du_p] \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, we know that $E_{u_p} := \{x \in \Omega, Du_p(x) = 0\}$ is always discrete and hence is a null set, and $u \in C^\infty(\Omega \setminus E_{u_p})$. See [9, 29].

Lemma 6.1. *Let $\beta > -1$. Then*

$$-\det D[|Du_p|^\beta Du_p] = \frac{1}{\beta+1}\big|D|Du_p|^{\beta+1}\big|^2 + (\beta+1)(p-2)|Du_p|^{2\beta}\frac{(\Delta_\infty u_p)^2}{|Du_p|^4} \quad \text{in } \Omega \setminus E_{u_p}. \tag{6.1}$$

Consequently,

$$-\det D[|Du_p|^\beta Du_p] = \frac{1}{\beta+1}\big|D|Du_p|^{\beta+1}\big|^2 \quad \text{in } \Omega \setminus E_{u_p} \quad \text{if } p = 2;$$

$$-\det D[|Du_p|^\beta Du_p] \ge \frac{1}{\beta+1}\big|D|Du_p|^{\beta+1}\big|^2 \quad \text{in } \Omega \setminus E_{u_p} \quad \text{if } p > 2; \tag{6.2}$$

$$\frac{p-1}{\beta+1}\big|D|Du_p|^{\beta+1}\big|^2 \le -\det D[|Du_p|^\beta Du_p] \le \frac{1}{\beta+1}\big|D|Du_p|^{\beta+1}\big|^2 \quad \text{in } \Omega \setminus E_{u_p} \quad \text{if } 1 < p < 2. \tag{6.3}$$

Proof of Lemma 6.1. In $\Omega \setminus E_{u_p}$, applying Lemma 3.7, we have

$$\begin{aligned}-\det D[|Du_p|^\beta Du_p] = \frac{1}{2}|Du_p|^{2\beta}[|D^2u_p|^2 - (\Delta u_p)^2] \\ + \beta|Du_p|^{2(\beta-1)}[|D^2u_p Du_p|^2 - \Delta u_p \Delta_\infty u].\end{aligned}$$

Applying (3.11) to u_p, we have

$$\frac{1}{2}[|D^2u_p|^2 - (\Delta u_p)^2] = |Du_p|^{-2}[|D^2u_p Du_p|^2 - \Delta_\infty u_p \Delta u_p] \quad \text{in } \Omega \setminus E_{u_p},$$

and hence

$$-\det D[|Du_p|^\beta Du_p] = (\beta+1)|Du_p|^{2(\beta-1)}[|D^2u_p Du_p|^2 - \Delta u_p \Delta_\infty u_p] \quad \text{in } \Omega \setminus E_{u_p}.$$

Note

$$\Delta u_p = -(p-2)\frac{\Delta_\infty u_p}{|Du_p|^2} \quad \text{in } \Omega \setminus E_{u_p}.$$

For $\beta > -1$, one gets (6.1). When $2 < p < \infty$, (6.1) gives (6.2). When $1 < p < 2$ and $\beta > -1$, since

$$\begin{aligned}|D^2u_p Du_p|^2 - \Delta u \Delta_\infty u &= |D^2u_p Du_p|^2 + (p-2)|Du_p|^{-2}(\Delta_\infty u_p)^2 \\ &\geq (p-1)|D^2u_p Du_p|^2 \quad \text{in } \Omega \setminus E_{u_p},\end{aligned}$$

one has

$$-\det D[|Du_p|^\beta Du_p] \geq \frac{p-1}{\beta+1}\big|D|Du_p|^{\beta+1}\big|^2 \quad \text{in } \Omega \setminus E_{u_p},$$

as desired. □

Lemma 6.2. *For any $\phi \in C_c^\infty(\Omega)$, one has*

$$\begin{aligned}&\int_\Omega -\det D[|Du_p|^\beta Du_p]\phi^2\,dx \\ &\quad\leq C\Big[1 + \frac{1}{1+\beta} + \frac{1}{p-1}\frac{\beta^2}{\beta+1}\Big]\int_\Omega |Du_p|^{2\beta+2}[|\phi\, D^2\phi| + |D\phi|^2]\,dx.\end{aligned} \tag{6.4}$$

Proof. For all $\phi \in C_c^\infty(\Omega)$, write

$$\begin{aligned}&\int_\Omega -\det D[|Du_p|^\beta Du_p]\phi^2\,dx \\ &\quad= -\frac{1}{2}\int_\Omega |Du_p|^{2\beta}(D^2\phi^2 Du_p \cdot Du_p)\,dx + \frac{1}{2\beta+2}\int_\Omega |Du_p|^{2\beta+2}\Delta\phi^2\,dx \\ &\qquad - \frac{\beta}{\beta+1}\int_\Omega [D|Du_p|^{\beta+1} \cdot Du_p](Du_p \cdot D\phi^2)|Du_p|^{\beta-1}\,dx \\ &\quad=: I_1 + I_2 + I_3.\end{aligned}$$

Clearly,

$$I_1 + I_2 \leq C\Big(1 + \frac{1}{1+\beta}\Big)\int_\Omega |Du_p|^{2+2\beta}[|\phi\, D^2\phi| + |D\phi|^2]\,dx.$$

When $1 < p \le 2$, by Young's inequality and (6.3), one has

$$\begin{aligned} I_3 &\le \frac{p-1}{2+2\beta}\int_\Omega \big|D|Du_p|^{\beta+1}\big|^2\phi^2\,dx + \frac{4\beta^2}{(2+2\beta)(p-1)}\int_\Omega |Du_p|^{2+2\beta}|D\phi|^2\,dx \\ &\le \frac{1}{2}\int_\Omega -\det D[|Du_p|^\beta Du_p]\phi^2\,dx + C\frac{\beta^2}{(1+\beta)(p-1)}\int_\Omega |Du_p|^{2+2\beta}|D\phi|^2\,dx. \end{aligned}$$

When $2 < p \le 4$, by (6.1), similarly to the case $1 < p \le 2$, one also has

$$I_3 \le \frac{1}{2}\int_\Omega -\det D[|Du_p|^\beta Du_p]\phi^2\,dx + C\frac{\beta^2}{(1+\beta)(p-1)}\int_\Omega |Du_p|^{2+2\beta}|D\phi|^2\,dx.$$

When $p \ge 4$, by Young's inequality and (6.1), one has

$$\begin{aligned} I_3 &\le \beta\int_\Omega |\Delta_\infty u_p\,\phi\,D\phi||Du_p|^{2\beta-1}\,dx \\ &\le \frac{(p-2)(\beta+1)}{2}\int_\Omega |\Delta_\infty u_p|^2|Du_p|^{2\beta-4}\phi^2\,dx \\ &\quad + \frac{8|\beta|^2}{(p-2)(\beta+1)}\int_\Omega |D\phi|^2|Du_p|^{2\beta+2}\,dx \\ &\le \frac{1}{2}\int_\Omega -\det D[|Du_p|^\beta Du_p]\phi^2\,dx \\ &\quad + C\frac{\beta^2}{(1+\beta)(p-1)}\int_\Omega |Du_p|^{2+2\beta}|D\phi|^2\,dx. \end{aligned}$$

We therefore obtain (6.4). □

Proof of Theorem 1.4. Theorem 1.4 follows immediately from Lemmas 6.1 and 6.2. □

Proof of Theorem 1.5. Given any bounded smooth domain Ω and $g \in \mathrm{Lip}(\partial\Omega)$ denote by u_p for $1 < p \le \infty$ the unique p-harmonic functions in Ω with boundary g. Moreover, $u_p \to u_\infty$ in $C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$ and weakly in $W^{1,q}(\overline{\Omega})$ for any $1 < q < \infty$ as $p \to \infty$. This is well known; see for example [28]. For the reader's convenience, a proof is given below. Since u_∞ is the absolute minimizer with boundary g, we know that

$$\|Du\|_{L^\infty(\Omega)} = \|g\|_{\mathrm{Lip}(\partial\Omega)}.$$

Moreover, we may extend g to Ω with the same Lipschitz norm via the McShane extension. For $1 < p < \infty$, since u_p is the minimizer, we see that

$$\|Du_p\|_{L^p(\Omega)} \le \|Dg\|_{L^p(\Omega)} \le |\Omega|^{\frac{1}{p}}\|g\|_{\mathrm{Lip}(\partial\Omega)}.$$

Given any $1 < q < \infty$, for $p > q$, by the Hölder inequality, we know that

$$\|Du_p\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q}}\|g\|_{\mathrm{Lip}(\partial\Omega)},$$

and hence it is uniformly bounded. Then $u_p \in C^{0,1-n/q}(\overline{\Omega})$ for large $p > q$ uniformly. Thus, u_p converges to u in $C^{0,1-n/q}(\overline{\Omega})$ as $p \to \infty$ up to some subsequence. Since u_p is also a viscosity solution to $\Delta_\infty v + \frac{1}{p-2}\Delta v|Dv|^2 = 0$ in Ω with boundary g, by the compactness of viscosity solution, we see that u is a viscosity solution to $\Delta_\infty v = 0$. Observing that u and u_∞ satisfy the same boundary condition, by Jensen's uniqueness result, we know that $u = u_\infty$. Since $Du_p \in L^q(\Omega)$ for $p > q$ uniformly, we know that Du_p converges weakly to Du_∞ in $L^q(\Omega)$ as $p \to \infty$.

As $Du_p \in L^2(\Omega)$ uniformly in p, by Lemma 6.2, we know that $-\det D^2 u_p \in L^1_{\mathrm{loc}}(\Omega)$ uniformly in $p > 2$. By Lemma 6.1, $D|Du_p| \in L^2_{\mathrm{loc}}(\Omega)$ uniformly in $p > 2$. By the Sobolev embedding theorem, we know that $|Du_p|$ converges to some function h in $L^q_{\mathrm{loc}}(\Omega)$ and weakly in $W^{1,2}_{\mathrm{loc}}(\Omega)$ as $p \to \infty$. By building up a flatness estimates similarly to [25, Lemma 2.7] (here we omit the details; see [27]), one has $h = |Du_\infty|$. Since $Du_p \to Du_\infty$ weakly in $L^q(\Omega)$ with $q > 1$, we deduce that $Du_p \to Du_\infty$ in $L^q_{\mathrm{loc}}(\Omega)$ as $p \to \infty$.

Applying Lemma 4.1, we know that $|Du_p|^\beta Du_p \to |Du_\infty|^\beta Du_\infty$ in $L^q_{\mathrm{loc}}(\Omega)$ for any $1 < q < \infty$, and moreover,

$$
\begin{aligned}
(6.5) \qquad & \int_\Omega -\det D[|Du_\infty|^\beta Du_\infty]\psi \, dx \\
&= \lim_{p\to\infty} \int_\Omega -\det D[|Du_p|^\beta Du_p]\psi \, dx \quad \text{for all } \psi \in C_c^\infty(\Omega).
\end{aligned}
$$

Lemma 6.2 yields that $-\det D[|Du_p|^\beta Du_p] \in L^1(\Omega)$ uniformly in $p > 2$. By a density argument, we know that (6.5) holds for all $\psi \in C_c^0(\Omega)$, i.e.,

$$
-\det D[|Du_p|^\beta Du_p] \to -\det D[|Du_\infty|^\beta Du_\infty]
$$

in the weak-$\star$ sense in $\mathcal{M}(\Omega)$. □

Remark 6.3. One could also prove Theorem 1.3 via Lemma 6.1, Lemma 6.2, and (6.5).

A. Some sharpness in the plane

At the borderline case $\beta = -1$, we have the following result, which will be used later.

Lemma A.1. *Let $1 < p < \infty$. If u_p is a nonconstant p-harmonic function in a domain $\Omega \subset \mathbb{R}^2$, then*

$$
\big|D[|Du_p|^{-1} Du_p]\big|^2 = \big|D \log|Du_p|\big|^2 + (p-2)p\frac{(\Delta_\infty u_p)^2}{|Du_p|^4} \quad \text{a.e.}
$$

In particular,

$$
\begin{aligned}
\big|D[|Du_p|^{-1} Du_p]\big|^2 &= \big|D \log|Du_p|\big|^2 && \text{a.e.} \quad \text{if } p = 2; \\
\big|D[|Du_p|^{-1} Du_p]\big|^2 &\geq \big|D \log|Du_p|\big|^2 && \text{a.e.} \quad \text{if } p > 2; \\
\big|D \log|Du_p|\big|^2 \geq \big|D[|Du_p|^{-1} Du_p]\big|^2 &\geq (p-1)^2\big|D \log|Du_p|\big|^2 && \text{a.e.} \quad \text{if } 1 < p < 2.
\end{aligned}
$$

Proof. In $\Omega \setminus E_{u_p}$, one has

$$
\big|D \log|Du_p|\big|^2 = \frac{|D^2 u_p Du_p|^2}{|Du_p|^4} \leq \frac{|D^2 u_p|^2}{|Du_p|^2}
$$

and

$$\big|D[|Du_p|^{-1}Du_p]\big|^2 = \big||Du_p|^{-1}D^2u_p - |Du_p|^{-3}D^2u_pDu_p \otimes Du_p\big|^2 \\ = \frac{|D^2u_p|^2}{|Du_p|^2} - \frac{|D^2u_pDu_p|^2}{|Du_p|^4}.$$

Recall that

$$D^2u_pDu_p^2 - \Delta u_p\Delta_\infty u_p = \frac{1}{2}[|D^2u_p|^2 - (\Delta u_p)^2]|Du_p|^2 \quad \text{in } \Omega \setminus E_{u_p}.$$

Replacing Δu_p with $(p-2)\frac{\Delta_\infty u_p}{|Du_p|^2}$, one has

$$\frac{1}{2}\Big[|D^2u_p|^2 - (p-2)^2\frac{(\Delta_\infty u_p)^2}{|Du_p|^4}\Big] = \frac{|D^2u_pDu_p|^2}{|Du_p|^2} - (p-2)\frac{(\Delta_\infty u_p)^2}{|Du_p|^4},$$

or equivalently,

$$|D^2u_p|^2 = 2\frac{|D^2u_pDu_p|^2}{|Du_p|^2} + (p-2)p\frac{(\Delta_\infty u_p)^2}{|Du_p|^4}.$$

Thus,

$$\big|D[|Du_p|^{-1}Du_p]\big|^2 = \frac{|D^2u_pDu_p|^2}{|Du_p|^4} + (p-2)p\frac{(\Delta_\infty u_p)^2}{|Du_p|^6} \\ = \frac{1}{2}\frac{|D^2u_p|^2}{|Du_p|^2} + \frac{(p-2)p}{2}\frac{(\Delta_\infty u_p)^2}{|Du_p|^6}.$$

When $p = 2$, then

$$\big|D[|Du_p|^{-1}Du_p]\big|^2 = \frac{1}{2}\frac{|D^2u_p|^2}{|Du_p|^2} = \frac{|D^2u_pDu_p|^2}{|Du_p|^4} \quad \text{in } \Omega \setminus E_u;$$

When $p > 2$, we have

$$\big|D[|Du_p|^{-1}Du_p]\big|^2 \ge \frac{1}{2}\frac{|D^2u_p|^2}{|Du_p|^2} \ge \frac{|D^2u_pDu_p|^2}{|Du_p|^2},$$

while when $1 < p < 2$, we have

$$\frac{1}{2}\frac{|D^2u_p|^2}{|Du_p|^2} \ge \big|D[|Du_p|^{-1}Du_p]\big|^2 \ge \frac{(p-1)^2}{2}\frac{|D^2u_p|^2}{|Du_p|^2} \ge \frac{(p-1)^2}{2}\frac{|D^2u_pDu_p|^2}{|Du_p|^4}. \qquad \square$$

For $1 < p < \infty$, we recall the extremal p-harmonic function constructed by [23, Section 7]. Here we keep notation the same as therein. Let

$$H(\xi) = \Big(\frac{\xi}{|\xi|} + \varepsilon\frac{|\xi|^3}{\xi^3}\Big)|\xi|^{\frac{1}{d}} \quad \text{for all } \xi \in \mathbb{C} \tag{A.1}$$

with

$$\frac{1}{d} = \frac{1}{2}\Big(-p + \sqrt{16(p-1) + (p-2)^2}\Big) > 0 \quad \text{and} \quad \varepsilon = \frac{1-d}{1+3d}.$$

If $p = 2$, then $d = 1$ and $\varepsilon = 0$, and hence $H(\xi) = \xi$. If $p \neq 2$, then $d > 0$ and $\varepsilon \neq 0$, and H is a quasiconformal homeomorphism on the whole plane. According to [23, Theorem 2], $H(\xi)$ satisfies [23, (18) with $n = 1$], that is,

$$H_{\bar{\xi}} = \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big[\frac{\xi}{\bar{\xi}} H_{\xi} + \frac{\bar{\xi}}{\xi}\bar{H}_{\xi}\Big], \tag{A.2}$$

where $H_{\xi} = \frac{1}{2}(H_x - iH_y)$ and $H_{\bar{\xi}} = \frac{1}{2}(H_x + iH_y)$ for $\xi = x + iy$.

Let $f(z)$ denote the inverse of $H(\xi)$ in $\mathbb{C}$ so that $f(H(\xi)) = \xi$ and $H(f(z)) = z$ for all $z, \xi \in \mathbb{C}$. From (A.2), one deduces

$$f_{\bar{z}} = \Big(\frac{1}{p} - \frac{1}{2}\Big)\Big[\frac{f}{\bar{f}} \bar{f}_z + \frac{\bar{f}}{f} f_z\Big],$$

where $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$ for $z = x + iy$. This then defines a p-harmonic function w in the whole plane so that its complex derivative is $w_z = f$.

We have the following properties.

Lemma A.2. *One has* $\log|Dw| = \log|f| \notin W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ *and* $|Dw|^{-1}Dw \notin W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$.

Proof. By Lemma A.1, we only need to prove $\log|f| \notin W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$. We argue by contradiction. Assume that $\log|f| \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Note that $f(tz) = t^d f(z)$ for any $t \geq 0$. A direct calculation implies that

$$D|f|(z) = t^{1-d} D|f|(tz), \quad D\log|f|(z) = \frac{D|f|(z)}{|f(z)|}, \qquad z \in \mathbb{C} \setminus \{f^{-1}(0)\},\ t \geq 0, \tag{A.3}$$

where $f^{-1}(0) = \{z \in \mathbb{C} : f(z) = 0\}$. For each $R > 0$, we know that

$$f^{-1}(0) \cap \{z \in \mathbb{C} : |z| < R\}$$

is discrete, and from $f(tz) = t^d f(z)$, (A.3), we conclude that

$$\begin{aligned}
\int_{|z|<R} \big|D\log|f|(z)\big|^2 \,dz &= \int_{|z|<R} \frac{\big|D|f|(z)\big|^2}{|f(z)|^2}\,dz = \int_{|\xi|<tR} \frac{\big|D|f|(\frac{\xi}{t})\big|^2}{|f(\frac{\xi}{t})|^2}\,d\frac{\xi}{t} \\
&= \int_{|\xi|<tR} \frac{(\frac{1}{t})^{2d-2}\big|D|f|(\xi)\big|^2}{(\frac{1}{t})^{2d}|f(\xi)|^2}\frac{1}{t^2}\,d\xi \\
&= \int_{|\xi|<tR} \big|D\log|f|(\xi)\big|^2\,d\xi \quad \text{for all } t > 0.
\end{aligned}$$

Letting $t \to 0$, we conclude that $D\log|f|(z) = 0$ whenever $|z| < R$, and hence, by the arbitrariness of R, for all $z \in \mathbb{C}$. Thus, $|f|$ is a positive constant in the whole plane. This contradicts that $f(tz) = t^d f(z)$ for all $t > 0$ and $z \in \mathbb{C}$, where we recall that $d > 0$. □

Lemma A.3. *One has*

$$\sup_{\mathbb{C}\setminus\{0\}} \frac{|f_{\bar{z}}|}{|f_z|} = \frac{|p-2|}{p} = \frac{K(p)-1}{K(p)+1} \quad \textit{with } K(p) = \max\Big\{\frac{1}{p-1}, p-1\Big\}. \tag{A.4}$$

In general, for $\beta > -1$, writing $g = |f|^{\beta} f$, one has

$$\sup_{\mathbb{C}\setminus\{0\}} \frac{|g_{\bar z}|}{|g_z|} = \frac{K(p,\beta)-1}{K(p,\beta)+1} \tag{A.5}$$

$$\text{with } K(p,\beta) = \max\Big\{\frac{p-1}{\beta+1}, \frac{\beta+1}{p-1}, \beta+1, \frac{1}{\beta+1}\Big\}.$$

Proof. Since $H(\xi)$ is the inverse of $f(z)$, (A.4) is equivalent to

$$\sup_{\mathbb{C}\setminus\{0\}} \frac{|H_{\bar\xi}|}{|H_\xi|} = \frac{|p-2|}{p}.$$

We already have

$$|H_{\bar\xi}| = \frac{1}{2}\Big|\frac{\xi}{\bar\xi}H_\xi + \frac{\bar\xi}{\xi}\bar H_\xi\Big| \le \frac{|p-2|}{p}|H_\xi| \quad \text{in } \mathbb{C}\setminus\{0\}.$$

Taking the derivative ∂_ξ on both sides of (A.1), one has

$$H_\xi = \frac{1}{2}|\xi|^{\frac{1}{d}-1}\Big[\Big(\frac{1}{d}+1\Big) + \Big(\frac{1}{d}-3\Big)\varepsilon\frac{|\xi|^4}{\xi^4}\Big].$$

If $\xi \in \mathbb{R}$, then $H_\xi(\xi) \in \mathbb{R}$, and hence (A.1) gives $H_{\bar\xi} = \frac{p-2}{p}H_\xi$ as desired.

Next, for $\beta > -1$, write $g = |f|^{\beta} f$ and $G = g^{-1}$. By [9, Section 3], one has

$$g_{\bar z} = -\frac{1}{2}\Big(\frac{p-2-\beta}{p+\beta} + \frac{\beta}{\beta+2}\Big)\frac{\bar g}{g}g_z - \frac{1}{2}\Big(\frac{p-2-\beta}{p+\beta} - \frac{\beta}{\beta+2}\Big)\frac{g}{\bar g}\bar g_z$$

and hence

$$G_{\bar\xi} = \frac{1}{2}\Big(\frac{p-2-\beta}{p+\beta} + \frac{\beta}{\beta+2}\Big)\frac{\bar\xi}{\xi}G_\xi + \frac{1}{2}\Big(\frac{p-2-\beta}{p+\beta} - \frac{\beta}{\beta+2}\Big)\frac{\xi}{\bar\xi}\bar G_\xi. \tag{A.6}$$

Thus,

$$\sup_{\mathbb{C}\setminus\{0\}} \frac{|G_{\bar\xi}|}{|G_\xi|} \le \max\Big\{\frac{|p-2-\beta|}{p+\beta}, \frac{|\beta|}{\beta+2}\Big\} = \frac{K(p,\beta)-1}{K(p,\beta)+1}$$

with $K(p,\beta)$ as in (A.5). Moreover, note that

$$G(\xi) = H(|\xi|^{-\frac{\beta}{\beta+1}}\xi) = \Big(\frac{\xi}{|\xi|} + \varepsilon\frac{|\xi|^3}{\xi^3}\Big)|\xi|^{-\frac{1}{(\beta+1)d}} \quad \text{for all } \xi \in \mathbb{C},$$

and hence

$$G_\xi(\xi) = \frac{1}{2}|\xi|^{\frac{1}{(\beta+1)d}-1}\Big[\Big(\frac{1}{(\beta+1)d}+1\Big) + \Big(\frac{1}{(\beta+1)d}-3\Big)\varepsilon\frac{|\xi|^4}{\xi^4}\Big].$$

If

$$\frac{|p-2-\beta|}{p+\beta} \ge \frac{|\beta|}{\beta+2},$$

for $\xi \in \mathbb{R}$, we have $G_\xi(\xi) \in \mathbb{R}$, and therefore, (A.6) gives

$$G_{\bar\xi}(\xi) = \frac{p-2-\beta}{p+\beta}G_\xi(\xi),$$

as desired. If

$$\frac{|p-2-\beta|}{p+\beta} < \frac{|\beta|}{\beta+2},$$

for $\xi \in \mathbb{R} - i\mathbb{R}$, we have $\bar{\xi} = i\xi$, $G_\xi(\xi) \in \mathbb{R}$, and

$$\frac{\xi}{\bar{\xi}} G_\xi(\xi) = -\frac{\bar{\xi}}{\xi} \bar{G}_\xi(\xi) = -i G_\xi(\xi),$$

which together with (A.6) gives $G_{\bar{\xi}}(\xi) = \frac{\beta}{2+\beta} i G_\xi(\xi)$, as desired. □

Lemma A.3 gives the sharpness of constants in (1.7).

Remark A.4. By a standard calculation, (A.4) gives

$$\operatorname*{ess\,sup}_{\mathbb{C}} \frac{2[|f_z|^2 + |f_{\bar{z}}|^2]}{|f_z|^2 - |f_{\bar{z}}|^2} = \frac{(p-1)^2+1}{p-1}.$$

Since $|D^2 w|^2 = 2[|f_z|^2 + |f_{\bar{z}}|^2]$ and $-\det D^2 w = |f_z|^2 - |f_{\bar{z}}|^2$, we write this as

$$\operatorname*{ess\,sup}_{\mathbb{C}} \frac{|D^2 w|^2}{-\det D^2 w} = \frac{(p-1)^2+1}{p-1} = (p-1) + \frac{1}{p-1}.$$

Thus, the constant in (1.7) is sharp. Note that $(p-1) + \frac{1}{p-1}$ converges to ∞ as $p \to \infty$.

For $\beta > -1$, in a similar way, (A.5) gives

$$\operatorname*{ess\,sup} \frac{\left|D[|Dw|^\beta Dw]\right|^2}{-\det D[|Dw|^\beta Dw]} = \frac{K(p,\beta)^2+1}{K(p,\beta)} = K(p,\beta) + \frac{1}{K(p,\beta)},$$

and hence the constant in (1.7) is sharp. We also note that $K(p,\beta) + \frac{1}{K(p,\beta)}$ converges to ∞ as $p \to \infty$.

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