SURVEY ON THE SMOOTHNESS OF p-HARMONIC FUNCTIONS IN THE HEISENBERG GROUP

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ABSTRACT. We review the existing literature concerning regularity for the gradient of weak solutions of the p-Laplacian

$$\sum_{i=1}^{2n} X_i(|\nabla_0 u|^{p-2} X_i u),$$

in a domain Ω in the Heisenberg group $\mathbb{H}^n$, with $1 \le p < \infty$, and of its parabolic counterpart. We present some open problems and outline some of the difficulties they present.

CONTENTS

1. INTRODUCTION AND REVIEW OF THE LITERATURE

$$X_i\left(|\nabla_0 u|^{p-2} X_i u\right) = 0 \text{ in } \Omega, \tag{1.1}$$

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2. APPROXIMATION SCHEMES

2.1. **Hilbert-Haar approach.** In the setting of Euclidean spaces, the Hilbert-Haar theory gives the existence of Lipschitz continuous minimizers for convex functional if the boundary values satisfy the bounded slope condition. The smooth boundary values in strictly convex domain satisfy this condition. We refer to [12] for the detailed discussions in this setting.

In the setting of Heisenberg group, we have similar results as those in Euclidean spaces. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. We consider it as a domain in $\mathbb{R}^{2n+1}$. The following theorem was proved in [39].

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^{2n+1}$ be a bounded uniformly convex domain. Let ϕ be a C^2 function in $\overline{\Omega}$, and $u \in HW^{1,p}(\Omega)$ be the unique solution of*

$$\begin{cases} \sum_{i=1}^{2n} X_i(X_i f(\nabla_0 u)) = 0 & \text{in } \Omega; \\ u - \phi \in HW_0^{1,p}(\Omega). \end{cases} \tag{2.1}$$

Then u is Lipschitz continuous in Ω, that is,

$$\|\nabla_0 u\|_{L^\infty(\Omega)} \leq M,$$

where the constant M depends on n, Ω and $\max_{\overline{\Omega}}(|D\phi| + |D^2\phi|)$.

In the theorem, $D\phi$ is the Euclidean gradient of ϕ and $D^2\phi$ the Hessian matrix of ϕ. The proof of the above theorem is similar to that in the Euclidean setting. The essential point is that all linear functions

$$L(x) = \sum_{i=1}^{2n+1} a_i x_i, \quad x = (x_1, x_2, ..., x_{2n+1}), a_i \in \mathbb{R},$$

are solutions to equation (1.1) with its own boundary value. This is the case for the Heisenberg group. It was shown in [17] that this is also the case for the Carnot groups of step two, Goursat groups and all groups of step three with linearly independent third order commutators. So, the above theorem is also true for these groups.

A natural open problem arises: do we have similar theorem as the above one for the general Carnot groups? To answer this question, or to establish the Hilbert-Haar theory in the sub-Riemannian setting, one can not use the Euclidean approach and one probably needs to built up the "sub-Riemannian linear functions".

2.2. **Subriemannian approximation approach.** Subriemannian metrics can be seen as Gromov-Hausdorff limits of sequences of collapsing Riemannian metrics, where the non-horizontal directions are increasing penalized. This metric approximation in turns leads to an approximation of those subelliptic operators naturally arising from the concepts of energy, like the sub-Laplacian or the subelliptic $p-$Laplacian, by means of their Riemannian counterparts. This approximation is in fact a regularization scheme, and it has been used widely in the literature to prove a-priori estimates for weak solutions of subelliptic PDE.

In this section we will look at the specific case of the Heisenberg group. Consider the family of left-invariant Riemannian metrics g^ε in $\mathbb{H}^n$ parametrized by $\varepsilon > 0$, defined by setting the frame $\{X_1, ..., X_{2n}, \varepsilon X_{2n+1}\}$ to be $g^\varepsilon-$orthonormal. The g_ε gradient $\nabla_\varepsilon u$ of a function u has then length

$$|\nabla_\varepsilon u|_\varepsilon^2 = \sum_{i=1}^{2n} (X_i u)^2 + \varepsilon^2 (X_{2n+1} u)^2.$$

In order to study the balls associated to the associated control metric d_ε one can consider a regularized gauge function

$$N_\varepsilon^2(x) = \sum_{i=1}^{2n} x_i^2 + \min\left\{|x_{2n+1}|, \frac{x_{2n+1}^2}{\varepsilon^2}\right\}. \tag{2.2}$$

In [6, Lemma 2.13] it is proved that the pseudo distance function $d_{G,\varepsilon}(x,y) := N_\varepsilon(y^{-1}x)$ is equivalent to the distance function d_ε, i.e. there exists a constant $A > 0$ such that for all $x, y \in \mathbb{H}^n$, and for all $\varepsilon > 0$ one has

$$A^{-1} d_{G,\varepsilon}(x,y) \leq d_\varepsilon(x,y) \leq d_{G,\varepsilon}(x,y). \tag{2.3}$$

This estimate expresses in a quantitative fashion the fact that at short scale, with distance less than $\varepsilon > 0$, the metric d_ε is essentially the Euclidean metric, while at larger scales it behaves like a subriemannian metric.

From (2.3) it follows immediately that the doubling property

$$|B_\varepsilon(x, 2R)| \leq C|B_\varepsilon(x,R)|,$$

holds uniformly in $\varepsilon > 0$ with $C > 1$ independent of the choice of $\varepsilon > 0$. In [6] it is also shown that the Poincaré inequality holds uniformly with constants independent of $\varepsilon > 0$.

For $1 < p < \infty$, and $\Omega \subset \mathbb{H}^n$, the energy functionals associated to g_ε are

$$E_p(u,\Omega) = \int_\Omega |\nabla_\varepsilon u|^p dx$$

and their Euler-Lagrange equations give rise to regularized $p-$Laplacians

$$L_{p,\varepsilon} u = X_i^\varepsilon\left(|\nabla_\varepsilon u|^{p-2} X_i^\varepsilon u\right),$$

where $X_i^\varepsilon = X_i$ for $i = 1, ..., 2n$ and $X_{2n+1}^\varepsilon = \varepsilon X_{2n+1}$. The Euclidean and Riemannian elliptic and parabolic theory [18, 37, 13, 14] can then be invoked to provide interior $C^{1,\alpha}$ regularity for the solutions of

$$\partial_t u^\varepsilon(x,t) = L_{p,\varepsilon} u^\varepsilon(x,t) \text{ in } \Omega \times (0,T)$$

or its stationary counterpart

$$X_i^\varepsilon\left(|\nabla_\varepsilon u|^{p-2} X_i^\varepsilon u\right) = 0 \text{ in } \Omega, \tag{2.4}$$

or C^∞ smoothness for solutions corresponding to the regularized operators

$$X_i^\varepsilon\left([1 + |\nabla_\varepsilon u|^2]^{\frac{p-2}{2}} X_i^\varepsilon u\right),$$

with estimates that in principle may degenerate as $\varepsilon \to 0$. The approach to regularity in [6], [8], [7], [9] consists in proving regularity estimates that are stable as $\varepsilon \to 0$. Note that stable estimates for the interior C^α regularity of solutions u^ε follow from the stability of the doubling property and of the Poincaré estimates (see [6] where in addition one can find also stable estimates for the heat kernels).

In order to prove $C^{1,\alpha}$ estimates near a point $x \in \Omega$ for weak solutions u of (1.1), one considers for each $\varepsilon > 0$ a sequence of smooth solutions to the regularized Dirichlet problem

$$\begin{cases} X_i^\varepsilon\left([\delta + |\nabla_\varepsilon u_\varepsilon|^2]^{\frac{p-2}{2}} X_i^\varepsilon u_\varepsilon\right) = 0 \text{ in } B(x,r) \subset\subset \Omega \\ u_\varepsilon = u \text{ on } \partial B(x,r). \end{cases} \tag{2.5}$$

Uniform convergence $u^\varepsilon \to u$ in compacts subsets of $B(x,r)$ follows from the stable estimates on the C^α regularity of solutions u^ε, as mentioned above. The heart of the matter is then to prove higher regularity estimates for the u_ε which are uniform in both parameters ε and δ.

3. The linear growth case

The first higher regularity results for subelliptic quasilinear equations [2], dealt with equations of the form

$$\sum_{i=1}^{2n} X_i A_i(\nabla_0 u) = 0, \tag{3.1}$$

with A_i differentiable and satisfying the structure assumptions

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{2n} A_{i,\eta_j}(\eta)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

and $|A_i(\eta)| \leq C(1+|\eta|)$. In [2] the first named author proved smoothness of weak solutions, using fractional derivatives estimates inspired by Hörmander's work in [30]. This result in turn was then used in a regularization scheme by third named author in [39] to establish $C^{1,\alpha}$ estimates for $p-$harmonic functions. In this section we sketch two alternative approaches to the proof of the smoothness result for weak solutions of (3.1).

In both approaches one makes use of the Riemannian approximation scheme mentioned in the previous section. To approximate (3.1) we adopt a simplification of the process introduced in [9]:

For $\varepsilon > 0$ and $\xi = \sum_{i=1}^{2n+1} \xi_i X_i^\varepsilon \in \mathbb{R}^{2n+1}$, we set

$$A_i^\varepsilon(x,\xi) = \tilde{A}_i(x,\xi_H) + \lambda\xi_i, \tag{3.2}$$

for $i = 1, ..., 2N+1$. Here we have denoted $\xi_H = (\xi_1, ..., \xi_{2n})$, and $\tilde{A} = (A, 0) \in \mathbb{R}^{2n+1}$. We note that the equation

$$\sum_{i=1}^{2n+1} X_i^\varepsilon A_i^\varepsilon(\nabla_\varepsilon u) = 0 \tag{3.3}$$

is an elliptic equation with linear growth, and hence its weak solutions are smooth. Arguing as in the previous section, in order to prove smoothness for the weak solutions of (3.1) near a point $x \in \Omega$ we consider for each $\varepsilon > 0$ a sequence of smooth solutions to the regularized Dirichlet problem

$$\begin{cases} \sum_{i=1}^{2n+1} X_i^\varepsilon \mathcal{A}_i^\varepsilon(\nabla_\varepsilon u) = 0 \text{ in } B(x,r) \subset\subset \Omega \\ u_\varepsilon = u \text{ on } \partial B(x,r). \end{cases} \tag{3.4}$$

The heart of the matter rests on proving smoothness of the u_ε uniformly in ε.

Consider an arbitrary horizontal vector field X_k^ε for some $k \in \{1, ..., 2n\}$. Differentiating (3.3) in the direction X_k, and setting $w^\varepsilon = X_k^\varepsilon u^\varepsilon$ yields

$$\sum_{i,j=1}^{2n+1} X_i^\varepsilon \Big(A_{i,\eta_j}^\varepsilon(\nabla_0 u^\varepsilon) X_j^\varepsilon w^\varepsilon \Big) = \mathcal{B}, \tag{3.5}$$

where

$$\mathcal{B} = -\sum_{i=1}^{2n+1} [X_k^\varepsilon, X_i^\varepsilon] A_i^\varepsilon(\nabla_\varepsilon u) + \sum_{i,j=1}^{2n+1} X_i^\varepsilon \Big(A_{i,\eta_j}^\varepsilon(\nabla_0 u^\varepsilon)[X_j^\varepsilon, X_k^\varepsilon] u^\varepsilon \Big).$$

Note that (3.5) has the form

$$\sum_{i,j=1}^{2n+1} X_i^\varepsilon \left(a_{ij}^\varepsilon X_j^\varepsilon w^\varepsilon + a_i^e \right) = a^e \tag{3.6}$$

with $a_{ij}^\varepsilon = -A_{i,\eta_j}^\varepsilon(\nabla_0 u^\varepsilon)$, $a_i = a_{ij}^\varepsilon [X_j^\varepsilon, X_k^\varepsilon] u^\varepsilon$, and $a^\varepsilon = -\sum_{i,j=1}^{2n+1} a_{ij}[X_k^\varepsilon, X_i^\varepsilon] X_j^\varepsilon u^\varepsilon$. The coefficients of (3.6) satisfy the estimates

$$\begin{cases} \lambda|\xi|^2 & \le \sum_{i,j}^{2n+1} a_{ij}\eta_i\eta_j \le \Lambda\lambda|\xi|^2 \\ |a_i| & \le C|\nabla_\varepsilon Z u| \\ |a| & \le C|\nabla_\varepsilon Z u| \end{cases} \tag{3.7}$$

Differentiating along the center Z of the group one has

$$X_i^\varepsilon A_{i,\eta_j}^\varepsilon(\nabla_\varepsilon u) X_j^\varepsilon Z u^\varepsilon = 0. \tag{3.8}$$

In view of Caccioppoli inequality applied to (3.8), it would be sufficient to show that $Zu^\varepsilon \in L^2_{loc}$ uniformly in $\varepsilon > 0$, to deduce $|\nabla_\varepsilon Z u^\varepsilon| \in L^2_{\text{loc}}$ uniformly in $\varepsilon > 0$. From there, a standard argument involving Morrey classes would then yield the Hölder regularity of $\nabla_0 u^\varepsilon$ with estimates uniform in $\varepsilon > 0$ and in view of the Schauder theory (see [38]) conclude the smoothness of u^e, with estimates uniform as $\varepsilon \to 0$ (see [2] for details on the adaptation of Morrey's estimates and for further references).

There are two different approaches to prove that the solution u^ε is differentiable along Z and that Zu^ε is locally in L^2:

- **Option 1: Hörmander's method** In [30] Hörmander observed that in view of the Baker-Campbell-Hausdorff formula, one has that for every $\delta > 0$, there exists $C_\delta = C(n,\delta) > 0$ such that for every $w \in C_0^\infty(\mathbb{H}^n)$,

$$||Z^{\frac{1}{2}-\delta} w||_{L^2} \le C_\delta ||\nabla_0 w||_{L^2}.$$

Setting $\delta = 1/6$, and for any $\phi \in C_0^\infty(\mathbb{H}^n)$, one can write

$$||Z(u^\varepsilon\phi)||_{L^2} = ||Z^{\frac{1}{2}-\delta} Z^{\frac{5}{6}-\delta}(u^\varepsilon\phi)||_{L^2} le C_\delta ||\nabla_0 Z^{\frac{5}{6}-\delta}(u^\varepsilon\phi)||_{L^2} \tag{3.9}$$

- **Option 2: Zhong's method**

4. Lipschitz Regularity for Stationary PDE

In the section, we discuss the local Lipschitz continuity of weak solutions to equation (1.1) for the full range $1 < p < \infty$. Actually, as shown in [39], this result holds for weak solutions to equation (1.1) under suitable assumptions on f, see also [5] for more general equations.

Actually we will work on the approximation equation of (1.1), the δ-regularized p-Laplacian equation,

$$\sum_{i=1}^{2n} X_i\big((\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} X_i u\big) = 0, \tag{4.1}$$

where $\delta > 0$. The following theorem gives us the Lipschitz regularity for equation (4.1) with $\delta \ge 0$.

Theorem 4.1. *Let $1 < p < \infty$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta \geq 0$. Then $\nabla_0 u \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{2n})$. Moreover, for any ball $B_{2r} \subset \Omega$, we have that*

$$\sup_{B_r} |\nabla_0 u| \leq \frac{c}{r^{(2n+2)/p}} \Big(\int_{B_{2r}} (\delta + |\nabla_0 u|^2)^{\frac{p}{2}} \Big)^{1/p}, \tag{4.2}$$

where $c = c(n,p) > 0$.

In the above theorem, we state the Lipschitz regularity for equation (4.1) for all $\delta \geq 0$. Actually, we prove the theorem for the case $\delta > 0$ and let δ go to zero to obtain the result for the case $\delta = 0$. Notice that the constant c in estimate (4.2) does not depend on δ. This is also the case for all of the constants in the estimates in the remaining of this section. We first obtain the estimates for $\delta > 0$ and then that for $\delta = 0$ by letting δ go to zero.

4.1. **regularization procedure.** Let $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta > 0$. The existing approach to prove the Lipschitz continuity of u, even in the Riemannian setting, involves differentiating equation (4.1) to consider the the equations for $X_l u, l = 1, 2, ..., 2n$, and Zu. In order to achieve this rigorously, we have two different ways.

One is based on the Hilbert-Haar theory in section 2.1 as in [39] and another on the subriemannian approximation in section 2.2 as in [8].

We first consider the approach based on the Hilbert-Haar theory. Let $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta > 0$. By the Hilbert-Haar theory in section 2.1, we may assume that $\nabla_0 u \in L^\infty(\Omega;\mathbb{R}^{2n})$. With this assumption, equation (4.1) is uniformly elliptic, since we assume that $\delta > 0$. We may now apply the results of Capogna in [2]. From Theorem 1.1 and Theorem 3.1 of [2], we know that $\nabla_0 u$ and Zu are Hölder continuous in Ω and that

$$\nabla_0 u \in HW^{1,2}_{\rm loc}(\Omega;\mathbb{R}^{2n}), \quad Zu \in HW^{1,2}_{\rm loc}(\Omega;\mathbb{R}^{2n}).$$

Alternatively , one can apply the subriemannian approximation of equation (4.1). As in section 2.2, we consider for each $\varepsilon > 0$ to the subriemannian approximation equation of (4.1)

$$\begin{cases} \sum_{i=1}^{2n+1} X_i^\varepsilon \Big((\delta + |\nabla_\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}} X_i^\varepsilon u_\varepsilon \Big) = 0 & \text{in } \Omega; \\ u_\varepsilon = u & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $u \in HW^{1,p}(\Omega)$ is a weak solution of equation (4.1). Now since the solution u_ε belongs to $C^\infty(\Omega)$, we may formally differentiate equation (4.3) for u_ε to obtain the equations for $X_l u_\varepsilon$ and Zu_ε. Then we proceed as before to obtain the Caccioppoli inequality in Lemma 4.3 and also the reverse inequality in Lemma 4.7 for u_ε. Actually, the estimates are exactly the same if we replace ∇_0 by ∇_ε, and ∇_0^2 by ∇_ε^2, respectively. The rest of the proof is the same. Eventually, we let ε go to zero to obtain the estimates for u and we prove the Lipschitz continuity of u.

4.2. **Cacciopoli type inequalities satisfied by the derivatives.** With any of the previously recalled regularization techniques, we can assume that the solution is sufficiently regular, to allow differentiating the equation. With the above regularity, one can easily prove that Zu is a weak solution of the following equation

$$\sum_{i,j=1}^{2n} X_i \Big((\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} \big(\kappa_{ij} + (p-2) \frac{X_i u X_j u}{\delta + |\nabla_0 u|^2} \big) X_j(Zu) \Big) = 0, \tag{4.4}$$

where κ_{ij} is the Kronecker symbol, that is, $\kappa_{ij}=1$ if $i=j$ and $\kappa_{ij}=0$ if $i\neq j$. Here we use the notation κ_{ij}, instead of the standard Kronecker Delta symbol δ_{ij}. One can also prove that $X_l u, l=1,2,...,n$, is a weak solution of

$$\begin{aligned}&\sum_{i,j=1}^{2n} X_i\Big((\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}\big(\kappa_{ij}+(p-2)\frac{X_iuX_ju}{\delta+|\nabla_0 u|^2}\big)X_j(X_lu)\Big)\\ &+\sum_{i=1}^{2n} X_i\Big((\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}\big(\kappa_{i(n+l)}+(p-2)\frac{X_iuX_{n+l}u}{\delta+|\nabla_0 u|^2}\big)Zu\Big)\\ &+Z\big((\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}X_{n+l}u\big)=0,\end{aligned}\tag{4.5}$$

and that $X_{n+l}u, l=1,2,...,n$, is a weak solution of

$$\begin{aligned}&\sum_{i,j=1}^{2n} X_i\Big((\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}\big(\kappa_{ij}+(p-2)\frac{X_iuX_ju}{\delta+|\nabla_0 u|^2}\big)X_j(X_{n+l}u)\Big)\\ &-\sum_{i=1}^{2n} X_i\Big((\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}\big(\kappa_{il}+(p-2)\frac{X_iuX_lu}{\delta+|\nabla_0 u|^2}\big)Zu\Big)\\ &-Z\big((\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}X_lu\big)=0,\end{aligned}\tag{4.6}$$

Theorem 4.1 follows from the following surprising it is not clear why it is surprising if we do not start with lemma 4.3. Someone used to work in the elliptic setting probably find it standard Caccioppoli type inequality for $\nabla_0 u$, by the well-known Moser iteration. It is similar to that for weak solutions of p-Laplacian equation in the setting of Euclidean spaces.

Theorem 4.2. *Let $1<p<\infty$ and $u\in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta\geq 0$. Then for any $\beta\geq 2$ and every $\eta\in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^2(\delta+|\nabla_0 u|^2)^{\frac{p-2+\beta}{2}}|\nabla_0^2 u|^2\leq c\beta^{10}\big(||\nabla_0\eta||^2_{L^\infty}+||\eta Z\eta||_{L^\infty}\big)\int_{spt(\eta)}(\delta+|\nabla_0 u|^2)^{\frac{p+\beta}{2}},\tag{4.7}$$

where $c=c(n,p)>0$ and $spt(\eta)$ is the support of η.

To prove Theorem 4.2, we need the following Caccioppoli type inequality. Its proof is standard. We use $\varphi=\eta^2X_lu$ as a testing function to equation (4.5) or (4.5) to obtain the result.

Lemma 4.3. *Let $1<p<\infty$ and $u\in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta\geq 0$. Then for any $\beta\geq 0$ and every $\eta\in C_0^\infty(\Omega)$, we have that*

$$\begin{aligned}\int_\Omega \eta^2(\delta+|\nabla_0 u|^2)^{\frac{p-2+\beta}{2}}|\nabla_0^2 u|^2\leq& c\int_\Omega\big(|\nabla_0\eta|^2+|\eta||Z\eta|\big)(\delta+|\nabla_0 u|^2)^{\frac{p+\beta}{2}}\\ &+c(\beta+1)^4\int_\Omega\eta^2(\delta+|\nabla_0 u|^2)^{\frac{p-2+\beta}{2}}|Zu|^2,\end{aligned}\tag{4.8}$$

where $c=c(n,p)>0$.

To obtain estimate (4.7) from (4.8), we need to remove the last integral in (4.8).

In the special case $2\leq p\leq 4$ we will first study a Cacciopoli type inequality for the vector field Zu alone, and plug in the result in the previous equaiton. In the general case a much delicate procedure involving all the derivatives is needed

4.3. **The $2 \le p \le 4$ case.** The equation satisfied by Zu is a linear equation, with coefficients depending on $\nabla_0 u$. For this reason, a standard Caccioppoli type estimates is satisfied by the derivative Zu.

Lemma 4.4 (Lemma 3.4 [7])**.** *Let u be a solution of (1.1) in Q with $\delta > 0$. For every $\beta \ge 0$ and non-negative $\eta \in C_0^\infty(\Omega)$ vanishing on the parabolic boundary of Q, one has*

$$\int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |Zu|^\beta |\nabla_0 Zu|^2 \eta^{4+\beta} \le C \int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |Zu|^{\beta+2} |\nabla_0 \eta|^2 \eta^{2+\beta} \tag{4.9}$$

where $C = C(\lambda, \Lambda) > 0$.

The standard Moser technique, which ensures higher integrability and boundness of the solution, is composed by two steps: Caccioppoli inequality (which we have already established in Lemma 4.4), and Sobolev inequality. Here we replace the Sobolev inequality with a simple interpolation inequality inspired from [11]. This is where we use we make of the hypothesis $2 \le p \le 4$.

Lemma 4.5. *Assume that $2 \le p \le 4$ and let $u \in C^2(Q)$. There exists a constant $C > 0$ depending only on n, p such that for every $\beta \ge 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\begin{aligned} \int_\Omega |Zu|^{p+\beta} \eta^{p+\beta} \le C(p+\beta) \int_\Omega (\delta + |\nabla_0 u|^2)^{(p-2)/2} |Zu|^\beta |\nabla_0 Zu|^2 \eta^{4+\beta} \\ + C(p+\beta) ||\nabla_0 \eta||_{L^\infty} \int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \end{aligned} \tag{4.10}$$

Proof. We provide here the simple proof which directly follows from the definition of Z. Indeed $Zu = X_l X_{n+l} u - X_{n+l} X_l u$, so that we can write

$$|Zu|^{p+\beta} = |Zu|^{p-2+\beta} Zu (X_l X_{n+l} u - X_{n+l} X_l u).$$

Then integration by parts gives us

$$\begin{aligned} \int_\Omega |Zu|^{p+\beta} \eta^{p+\beta} &= \int_\Omega |Zu|^{p-2+\beta} Zu (X_l X_{n+l} u - X_{n+l} X_l u) \eta^{p+\beta} \\ &= -(p-1+\beta) \int_\Omega |Zu|^{p-2+\beta} (X_l Zu X_{n+l} u - X_{n+l} Zu X_l u) \eta^{p+\beta} \\ &\quad - (p+\beta) \int_\Omega |Zu|^{p-2+\beta} Zu (X_{n+l} u X_l \eta - X_l u X_{n+l} \eta) \eta^{p-1+\beta} \\ &\le 2(p+\beta) \int_\Omega |\nabla_0 u| |Zu|^{p-2+\beta} |\nabla_0 Zu| \eta^{p+\beta} \\ &\quad + 2(p+\beta) \int_\Omega |\nabla_0 u| |Zu|^{p-1+\beta} |\nabla_0 \eta| \eta^{p-1+\beta} = I_1 + I_2 \end{aligned} \tag{4.11}$$

Applyung the Young's inequality $abc \le \frac{1}{2} a^2 + \frac{4-p}{2(p+\beta)} b^{2(p+\beta)/(4-p)} + \frac{2p-4+\beta}{2(p+\beta)} c^{2(p+\beta)/(2p-4+\beta)}$ to I_1 and $ab \le \frac{1}{\beta+p} a^{\beta+p} + (\beta+p-1) b^{(\beta+p)/(\beta+p-1)}$ to I_2 we immediately obtain the thesis. □

We can now put together the Cacciopoli inequality **??** and the modified Poincaré-like inequality for Zu. we obtain an estimate very different from a typical step in the standard Moser iteration, which provides a gain of integrability for the solution. Precisely

Lemma 4.6. *Let u be a solution of (1.1) in Q, with $\delta > 0$ and $2 \le p \le 4$. Then for every $\beta \ge 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\int_\Omega |Zu|^{p+\beta}\eta^{p+\beta} \le C(p+\beta)^{p+\beta}||\nabla_0\eta||_{L^\infty}^{p+\beta}\int_{spt(\eta)}(\delta+|\nabla_0 u|^2)^{\frac{p+\beta}{2}} \tag{4.12}$$

We explicitly note that the inequality we have obtained can be considered as a gain of differentiability, the integral of a derivative of order 2 Zu is estimated which is a first order derivative $|\nabla_0 u|$.

Plugging this estimate in the Cacciopoli type inequality 4.3 we obtain Theorem 4.2 in the special case $2 \le p \le 4$

4.4. **The general case.** In order to handle the general case, the following estimate for Zu is essential. This is a reverse type inequality for Zu, associated with $\nabla_0 u$ and $\nabla_0^2 u$.

Lemma 4.7. *Let $1 < p < \infty$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta \ge 0$. Then for any $\beta \ge 2$ and every non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^{\beta+2}(\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}|Zu|^\beta|\nabla_0^2 u|^2 \le c\beta^2||\nabla_0\eta||_{L^\infty}^2\int_\Omega \eta^\beta(\delta+|\nabla_0 u|^2)^{\frac{p}{2}}|Zu|^{\beta-2}|\nabla_0^2 u|^2, \tag{4.13}$$

where $c = c(n,p) > 0$.

The proof of Lemma 4.7 involves a non-standard testing function $\varphi = \eta^{\beta+2}|Zu|^\beta X_l u$ to equation (4.5) or (4.6) for all $l = 1, 2, ..., 2n$ and sum up the estimates. One crucial point of the proof is that Zu is a weak solution to equation (4.4). Actually this is the reason that Zu enjoys this type of reverse inequality.

An immediate consequence of (4.13) by Hölder's inequality is the following inequality.

$$\int_\Omega \eta^{\beta+2}(\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}|Zu|^\beta|\nabla_0^2 u|^2 \le c^\beta\beta^\beta||\nabla_0\eta||_{L^\infty}^\beta\int_\Omega \eta^\beta(\delta+|\nabla_0 u|^2)^{\frac{p-2+\beta}{2}}|\nabla_0^2 u|^2, \tag{4.14}$$

where $c = c(n,p) > 0$.

Now we come back to the proof of (4.7). We only need to the last integral in (4.8). By Hölder's inequality, we have

$$\begin{aligned}&\int_\Omega \eta^2(\delta+|\nabla_0 u|^2)^{\frac{p-2+\beta}{2}}|Zu|^2\\ \le&\Big(\int_\Omega \eta^{\beta+2}(\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}|Zu|^{\beta+2}\Big)^{\frac{2}{\beta+2}}\Big(\int_{spt(\eta)}(\delta+|\nabla_0 u|^2)^{\frac{p+\beta}{2}}\Big)^{\frac{\beta}{\beta+2}}.\end{aligned}$$

Note that $|Zu| \le 2|\nabla_0 u|$. The first integral in the right hand side of the above inequality can be estimated by (4.14), which can be absorbed to the left hand side of (4.7). This finishes the proof of (4.7).

The following estimate for Zu follows from (4.14) and Theorem 4.2 is crucial for the proof of the Hölder continuity of the horizontal gradient of u in Section 4.

Corollary 4.8. *Let $1 < p < \infty$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta \ge 0$. Then for any $\beta \ge 2$ and every non-negative $\eta \in C_0^\infty(\Omega)$, we have that*

$$\int_\Omega \eta^{\beta+2}(\delta+|\nabla_0 u|^2)^{\frac{p-2}{2}}|Zu|^{\beta+2} \le cK^{\frac{\beta+2}{2}}\int_{spt(\eta)}(\delta+|\nabla_0 u|^2)^{\frac{p+\beta}{2}},$$

where $K = ||\nabla_0\eta||_{L^\infty}^2 + ||\eta Z\eta||_{L^\infty}$ and $c = c(n,p,\beta) > 0$.

5. $C^{1,\alpha}$ regularity for the stationary PDE

In this section, we discuss the Hölder continuity of horizontal gradient of solutions of equation (4.1) for $1 < p < \infty$. Actually, as shown in [39, 23], this result holds for weak solutions to equation (1.1) under suitable assumptions on f. We refer to [33] for more general equations, and also to [26] for equations with Hörmander vector fields of step two. The following theorem shows that the weak solutions of equation (4.1) is of class $C^{1,\alpha}$.

Theorem 5.1. *Let $1 < p < \infty$ and $u \in HW^{1,p}(\Omega)$ be a weak solution of equation (4.1) with $\delta \geq 0$. Then the horizontal gradient $\nabla_0 u$ is Hölder continuous. Moreover, there is a positive exponent $\alpha \leq 1$, depending only on n and p, such that for any ball $B_{r_0} \subset \Omega$ and any $0 < r \leq r_0/2$, we have that*

$$\max_{1\leq l\leq 2n} \operatorname{osc}_{B_r} X_l u \leq c\Big(\frac{r}{r_0}\Big)^{\alpha}\Big(\frac{1}{r_0^{2n+2}}\int_{B_{r_0}} (\delta + |\nabla_0 u|^2)^{\frac{p}{2}}\Big)^{1/p}, \tag{5.1}$$

where $c = c(n,p) > 0$.

For $p \neq 2$, it is well known that weak solutions of equations of type (1.1) in the Euclidean spaces are of the class $C^{1,\alpha}$, and the $C^{1,\alpha}$-regularity is optimal when $p > 2$. Ural'tseva [36] proved the $C^{1,\alpha}$-regularity in this setting for the range $p > 2$, see [35] for the systems and [29] for a new proof. For the full range $1 < p < \infty$, the $C^{1,\alpha}$-regularity is due to DiBenedetto [28], Lewis [31] and Tolksdorff [34], see also [32] for more general equations.

There are two existing approaches for the proof of Theorem 5.1. One is similar to that of DiBenedetto [28] and another to that of Tolksdorff [34] and Lieberman [32]. Both approaches are based on De Giorgi's method [27], where in this remarkable paper he proved the local boundedness and Hölder continuity for functions satisfying certain integral inequalities, nowadays known as De Giorgi's class of functions. In the remaining of this section, we discuss these two approaches.

First, the proof of Theorem 5.1 for the range $p > 2$ in [39] is in the same line as that in [28]. Let $u \in HW^{1,p}(\Omega)$ be a solution of equation (4.1) with $\delta > 0$. Fix a ball $B_r \subset \Omega$. We denote

$$\mu(r) = \max_{1\leq l\leq 2n} \sup_{B_r} |X_l u|.$$

and for $k \in \mathbb{R}$

$$A^+_{k,r} = \{x \in B_r : (u(x) - k)^+ > 0\}.$$

Theorem 5.1 follows, as in [28] with a minor modification, from a Caccioppoli inequality for $X_l u, l = 1, 2, \ldots, 2n$, see Lemma 4.3 of [39]. This Caccioppoli inequality shows that $X_l u$ belongs to a generalized version of De Giorgi's class. It states as follows. For any $q \geq 4$, there exists $c = c(n, p, L, q) > 0$ such that the following inequality holds for any $1 \leq l \leq 2n$, for any $k \in \mathbb{R}$ and for any $0 < r' < r \leq r_0/2$

$$\begin{aligned}&\int_{A^+_{k,r'}} (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |\nabla_0 X_l u|^2\, dx \\ \leq &\frac{c}{(r-r')^2}\int_{A^+_{k,r}} (\delta + |\nabla_0 u|^2)^{\frac{p-2}{2}} |(X_l u - k)^+|^2\, dx + cK|A^+_{k,r}|^{1-\frac{2}{q}}\end{aligned} \tag{5.2}$$

where $K = r_0^{-2}|B_{r_0}|^{\frac{2}{q}}(\delta + \mu(r_0)^2)^{\frac{p}{2}}$.

The proof of (5.2) is based on the estimate for Zu in Corollary 4.8. It also involves an iteration argument. We remark here that (5.2) holds for $p > 2$. In [28], there is a version of (5.2) for the case $1 < p < 2$ in the setting of Euclidean spaces. Unfortunately, we do not know how to prove the analog of that in the setting of Heisenberg group.

Second, the proof of Theorem 5.1 in [23] works for all $1 < p < \infty$, and is similar to that of Tolksdorff [34] and Lieberman [32] in the setting of Euclidean spaces. Following [34, 32], we consider the double truncation of the horizontal derivative $X_l u, l = 1, 2, ..., 2n$, of the weak solution u to equation (4.1) with $\delta > 0$

$$v = \min\big(\mu(r)/8, \max(\mu(r)/4 - X_l u, 0)\big).$$

The following Caccioppoli type inequality for v was proved in [23]. Let $\gamma > 1$ be a number. We have the Caccioppoli type inequality

$$\int_{B_r} \eta^{\beta+4} v^{\beta+2} |\nabla_0 v|^2 \, dx \le c(\beta+2)^2 \frac{|B_r|^{1-1/\gamma}}{r^2} \mu(r)^4 \Big(\int_{B_r} \eta^{\gamma\beta} v^{\gamma\beta} \, dx \Big)^{1/\gamma} \tag{5.3}$$

for all $\beta \ge 0$, where $c = c(n, p, \gamma) > 0$. Once we obtain (5.3), we may follow the same line as in [32] to prove Theorem 5.1.

The proof of (5.3) is also based on the integrability estimate for Zu in Corollary 4.8. We consider equations (4.5) and (4.6) for $X_l u$, and use the usual testing function

$$\varphi = \eta^{\beta+4} v^{\beta+3}$$

where $\beta \ge 0$. The proof of (5.3) in the case $p > 2$ is easy, while that in the case $1 < p < 2$ is more involved.

6. The degenerate parabolic case

6.1. **The $2 \le p \le 4$ case.** The L^∞ estimate of gradient in case $2 \le p \le 4$ the parabolic setting is similar to the stationary one. Again the problem was to face the lack of homogeneity of the equation. The main idea, due to Di Benedetto in the Euclidean setting, is to work on non homogeneous spheres, with the ratio between the spatian and temporal dimension compensates for the lack of homogeneity of the equation.

As in the stationary case, we start with an estimate of the derivative of the solution in the direction Z. Using this relation, a parabolic version of the Cacciopoli type estimate stated in Proposition 4.4 is establlished, which together with the Poincaré type inequality 4.5 leads to an estimate of $L^{p+\beta}-$norm of Zu

Lemma 6.1. *Let u be a solution of (1.1) in Q, with $\delta > 0$ and $2 \le p \le 4$. Then for every $\beta \ge 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\begin{aligned}\Big(\int_{t_1}^{t_2} \int_\Omega |Zu|^{p+\beta} \eta^{p+\beta} \Big)^{\frac{1}{p+\beta}} &\le C(p+\beta) ||\nabla_0 \eta||_{L^\infty} \Big(\int \int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \Big)^{\frac{1}{p+\beta}} \\ &+ C(p+\beta) ||\eta \partial_t \eta||_{L^\infty}^{\frac{1}{2}} |spt(\eta)|^{\frac{p-2}{2(p+\beta)}} \Big(\int \int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \Big)^{\frac{4-p}{2(p+\beta)}}\end{aligned} \tag{6.1}$$

Having established an estimate for the derivative with respect to Z, we can proceed to the estimate of the intrinsic gradient $(X_1 u, \cdots, X_m u)$. Differentiating the equation with respect to X_l we obtain the differential equation satisfied by $X_l u$, and from this, a Cacciopoli type inequality, analogous to the Lemma **??** established in the stationary setting. Precisely

Proposition 6.2. *Let u be a weak solution of (1.1) in Q, with $\delta > 0$ and $2 \le p \le 4$. Then for every $\beta \ge 0$ and non-negative $\eta \in C^1([0,T], C_0^\infty(\Omega))$ vanishing on the parabolic boundary of Q, we have*

$$\begin{aligned}
&\sup_{t_1<t<t_2} \int_\Omega (\delta + |\nabla_0 u|^2)^{\frac{\beta+2}{2}} \eta^2 + \int_{t_1}^{t_2} \int_\Omega (\delta + |\nabla_0 u|^2)^{(p-2+\beta)/2} |\nabla_0^2 u|^2 \eta^2 \\
\le& C(p+\beta)^7 \big(||\nabla_0 \eta||_{L^\infty}^2 + ||\eta Z \eta||_{L^\infty} \big) \int\int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \\
&+ C(p+\beta)^7 ||\eta \partial_t \eta||_{L^\infty} |spt(\eta)|^{\frac{p-2}{p+\beta}} \Big(\int\int_{spt(\eta)} (\delta + |\nabla_0 u|^2)^{\frac{p+\beta}{2}} \Big)^{\frac{\beta+2}{p+\beta}},
\end{aligned} \tag{6.2}$$

where $C = C(n, p, \lambda, \Lambda) > 0$.

This inequality reflects the lack of homogeneity we already observed in the equation. The gradient of the $p+\beta$ norm of Xu is estimated by a first term which contains a $p+\beta$ norm of the intrinsic gradient ∇u, and a second term which contains the same norm, to the power $(\beta+2)/(p+\beta)$. An ad hoc choice of an anysotropic ball, depending on the solution u will be used to overcome the problem. In particular if η is a cut off function, which is identically 1 on a ball of cilinder which is a ball of radius R times an interval of lenght R/μ, we will have

$$||\nabla_0 \eta||_{L^\infty}^2 \le \frac{1}{R^2}$$

$$||\eta \partial_t \eta||_{L^\infty} \le \frac{\mu^2}{R^2},$$

Choosing $\mu =$, we can make the second member homogeneous in the gradient,

7. Open problems

Some of the following extensions seem challenging, and we list them as open problems in increasing order of their perceived difficulty.

(1) Standard, but technically involved, modifications should allow to extend our work to the case of equations of the type

$$\partial_t u - X_i A_i(x, t, u, \nabla_0 u) = B(x, t, u, \nabla_0 u)$$

with structure conditions similar to those in [13, Section 1, Chapter VIII].

(2) We feel it should possible to weaken the bounds in the structure conditions for $\partial_{x_k} A_i$ and request instead only horizontal derivatives bounds, bounds on $X_k A_i$, although this would require some additional work in the proof of Lemma 4.5.

(3) This paper only deals with scalar equations, however in the Euclidean case the results continue to hold also for systems of equations with additional structure (see [13]). The extension in the subelliptic setting would involve first extending the results of [7], and all the regularity theory literature that is used there.

(4) Because our argument rests in a crucial way on Lemma 4.5, the Lipschitz regularity for the non-stationary case in the range $4 < p < \infty$ is currently beyond our reach. We conjecture that our main Caccioppoli inequality (6.2) still holds with exactly the same statement in this extended range.

(5) In the non-stationary case, the proof of the Hölder regularity of horizontal derivatives, in any range of $p \neq 2$ is still an open problem. Even in the range $2 \le p \le 4$ the methods of Zhong [39] and the Euclidean proofs in [13] break down and new ideas are needed.

(6) Just as in the Euclidean case, the regularity problem in the range $1 < p < 2$ is more challenging, and would require completely different arguments. In the stationary case this has been solved by Mukherjee and Zhong in [23] in the Heisenberg group, and similar arguments extend immediately to Carnot groups of step two. The problem is completely open in the non-stationary case and in settings more general than step two Carnot groups.

(7) Our work extends easily to any step two Carnot group. Beyond this setting, in the stationary case, there is promising work by Domokos and Manfredi [16] dealing with regularity in higher step groups, while the papers [8] and [26] show extensions beyond the group setting, but within the step two hypothesis. The problem is completely open in the non-stationary case.

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