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The geometry of near ground states in Gaussian polymer models

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Abstract

The energy and geometry of maximizing paths in integrable last passage percolation models are governed by the characteristic KPZ scaling exponents of one-third and two-thirds. When represented in scaled coordinates that respect these exponents, this random field of paths may be viewed as a complex energy landscape. We investigate the structure of valleys and connecting pathways in this landscape. The routed weight profile $\mathbb{R} \to \mathbb{R}$ associates to $x \in \mathbb{R}$ the maximum scaled energy obtainable by a path whose scaled journey from $(0,0)$ to $(0,1)$ passes through the point $(x, 1/2)$. Developing tools of Brownian Gibbs analysis from [Ham22] and [CHH19], we prove an assertion of strong similarity of this profile for Brownian last passage percolation to Brownian motion of rate two on the unit-order scale. A sharp estimate on the rarity that two macroscopically different routes in the energy landscape offer energies close to the global maximum results. We prove robust assertions concerning modulus of continuity for the energy and geometry of scaled maximizing paths, that develop the results and approach of [HS20], delivering estimates valid on all scales above the microscopic. The geometry of excursions of near ground states about the maximizing path is investigated: indeed, we estimate the energetic shortfall of scaled paths forced to closely mimic the geometry of the maximizing route while remaining disjoint from it. We also provide bounds on the approximate gradient of the maximizing path, viewed as a function, ruling out sharp steep movement down to the microscopic scale. Our results find application in a companion study [GH20a] of the stability, and fragility, of last passage percolation under a dynamical perturbation.

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Contents

1 Introduction

1.1 KPZ universality, last passage percolation models, and scaled coordinates

The $1+1$ dimensional Kardar-Parisi-Zhang [KPZ] universality class includes many microscopic models in which a random interface is suspended over a one-dimensional domain, whose growth in a direction normal to the surface competes with a smoothening

surface tension in the presence of a local force that randomly roughens the surface. Many planar last passage percolation [LPP] models exhibit these characteristics. In a planar LPP model, directed paths, moving in directions in the first quadrant, are assigned energy via a random environment, which is independent in disjoint regions. This energy is assigned by integrating the environment's value along the path. For a given pair of planar points, the directed path between them attaining the maximum energy is called a geodesic.

For LPP models lying in the KPZ class, a geodesic moving in a non-axial direction, crossing a large distance n has an energy that is typically linear in n with a standard deviation of order $n^{1/3}$. The associated random interface mentioned at the outset is the function obtained when the lower geodesic endpoint is held fixed, and the geodesic energy is a function of the other endpoint varying horizontally. In this particular case, where the first endpoint is fixed, the energy profile is termed 'narrow wedge'. Non-trivial correlations in the interface occur between points with separation of order $n^{2/3}$. The same exponent governs the related notion of transversal fluctuation of the geodesic from the straight line joining its endpoints. Despite the predicted universality, these assertions have been rigorously demonstrated for only a few LPP models with certain exactly solvable features: the seminal work of Baik, Deift and Johansson [BDJ99] established the one-third exponent, and the GUE Tracy-Widom distributional limit, for the case of Poissonian last passage percolation, while Johansson [Joh00] derived the two-thirds power law for maximal transversal fluctuation for this model.

In view of these facts, it is natural to represent the field of geodesics in a scaled system of coordinates. Under this scaling, a northeasterly displacement of order n becomes a vertical displacement of one unit, while a horizontal displacement of order $n^{2/3}$ becomes a unit horizontal displacement. The system of energies also transfers to scaled coordinates, with the scaled geodesic energy being specified by centring about the mean value and normalizing by the typical scale of $n^{1/3}$.

In this way, the LPP geodesic that begins at $(0,0)$ and ends at (n,n) has a scaled counterpart, which we will refer to as a polymer and label $\rho\big[(0,0)\to(0,1)\big]$, that travels between $(0,0)$ and $(0,1)$. This polymer has a scaled energy, or weight, that we denote by $\mathsf{Wgt}\big[(0,0)\to(0,1)\big]$, which in the high n limit is distributed according to the GUE Tracy-Widom distribution. In the scaled LPP description more generally, a polymer $\rho\big[(x,s)\to(y,t)\big]$ is associated to each pair of planar points (x,s) and (y,t) with $s<t$. The polymer's weight is denoted by $\mathsf{Wgt}\big[(x,s)\to(y,t)\big]$.

1.2 The energy landscape of scaled LPP and the structure of its valleys

Many statistical mechanical systems may be described by a probability measure whose density $e^{-H(x)}$ with respect to a background measure μ, supported on a space X, is specified by a Hamiltonian $H:X\to\mathbb{R}$ which may be viewed as an energy landscape over X. Such a system may be viewed as a particle that dwells randomly in X; the system is held at equilibrium by a Markovian dynamics in which the present state evolves locally according to a Metropolis rule governed by the relative values of the function $e^{-H(x)}$. The present state is thus a snapshot of a particle wandering in the energy landscape, which is typically attracted into the landscape's local valleys; its long-term behaviour is governed by the structure of valleys—their number; depths; and the heights and geometry of mountain passes that connect them. For Gaussian models of disorder, including Gaussian polymers, [Cha14] proved via an interpolation method that certain strong concentration properties exhibited by such systems are equivalent to an abundance of well-separated valleys—which abundance entails chaotic behaviour of observables when the system is slightly perturbed. The landscape geometry of several models has since then been studied. Landscapes of general smooth Gaussian functions

on the sphere in high dimensions as well as those related to spin-glass models have been studied in [AAČ13, ABA13]. More recently, refined results for the number of valleys for spin-glasses have been obtained in [DEZ15, CHL18, Eld20].

In this article, we investigate the structure of the energy landscape of *Brownian* last passage percolation, a semi-discrete polymer wandering through Gaussian noise, in its scaled coordinate description. This will find application in a companion study [GH20a] of the transition from stability to chaos of this model subject to a dynamical perturbation. Brownian LPP model will be recalled shortly; its noise environment is comprised of Brownian randomness. The model carries a parameter $n \in \mathbb{N}$ which in rising approximates a limiting scaled description; in our present heuristic purpose, we omit mention of it; indeed we already did so, in indicating the meaning of the field of weights $\mathsf{Wgt}\big[(x,s) \to (y,t)\big]$.

Let $x \in \mathbb{R}$ and $a \in (0,1)$. Set $Z(x,a)$ equal to the supremum of weights of scaled paths on the route from $(0,0)$ to $(0,1)$ that pass through (x,a). That is,

$$Z(x,a) = \mathsf{Wgt}\big[(0,0) \to (x,a)\big] + \mathsf{Wgt}\big[(x,a) \to (0,1)\big]. \tag{1.1}$$

We refer to the random process $\mathbb{R} \to \mathbb{R} : x \to Z(x,a)$ as the *routed weight profile* at height a, because this process records weights of paths that are routed through a given location at this height. This profile is a cross-section of the LPP energy landscape that is pertinent for understanding the geometry of the polymer $\rho\big[(0,0) \to (0,1)\big]$ and how effectively scaled paths that share the polymer's endpoints but that take alternative routes compete in weight with the polymer. For example, the horizontal location at which $\rho\big[(0,0) \to (0,1)\big]$ traverses height a is the maximizer M of the random function $\mathbb{R} \to \mathbb{R} : x \to Z(x,a)$ (an almost surely unique location, as we will indicate in Lemma 2.1(2)). For $x \in \mathbb{R}$, the quantity $Z(M,a) - Z(x,a) \geq 0$ is the shortfall in weight relative to the polymer's of a scaled LPP path from $(0,0)$ to $(0,1)$ that is constrained to pass via the point (x,a).

1.3 Principal conclusions and themes in overview

In this article, we will prove several conclusions concerning the energy landscape of scaled Brownian LPP. As we informally summarise them now, we continue to omit mention of the scaling parameter $n \in \mathbb{N}$: roughly, our assertions should be understood uniformly in high choices of this parameter.

1.3.1 Brownianity of the routed weight profile

On scales larger the unit scale, the profile $x \to Z(x,a)$ is curved, following the parabola $x \to -2^{-1/2}a^{-1}(1-a)^{-1}x^2$. On the unit scale, however, it resembles Brownian motion. Building on a Brownian comparison result for narrow wedge weight profiles from [CHH19], our first result, Theorem 1.2, offers a strong attestation of this resemblance. For $a \in (0,1)$, the profile $x \to Z(x,a)$ enjoys a strong similarity with Brownian motion B *of rate two*. Indeed, if A denotes a collection of continuous functions on $[-1,1]$ that vanish at -1 for which the probability that $[-1,1] \to \mathbb{R} : x \to B(x) - B(-1)$ belongs to A is denoted by η, then the probability that the profile $[-1,1] \to \mathbb{R} : x \to Z(x,a) - Z(-1,a)$ belongs to A is at most an expression of the form $\eta \cdot \exp\big\{\Theta(1)(\log \eta^{-1})^{5/6}\big\}$. The latter, correction, term grows much more slowly than any inverse power of η in the limit of $\eta \searrow 0$. The constant implied by use of the notation $\Theta(1)$ may be chosen uniformly as a varies over any given compact set in $(0,1)$. In general, the notation $\Theta(1)$ will be used to refer to positive constants that are bounded away from zero and infinity in a manner that is independent of variable parameters in the context in question.

1.3.2 The rarity of twin peaks

If the maximizer M of $x \to Z(x, 1/2)$ lies to the right of the origin, so that say the positive probability event $M \in [1, 3]$ is satisfied, the random variable

$$Z(M, 1/2) \; - \; \sup\big\{Z(x, 1/2) : x \in [-3, -1]\big\} \geq 0$$

equals the shortfall in weight relative to the polymer's along those scaled paths from $(0, 0)$ to $(0, 1)$ that instead pass on the left, via locations in $[-3, -1]$, at the mid-life time one-half. If this random variable is less than a given small quantity $\sigma > 0$, the profile $x \to Z(x, 1/2)$ resembles a pair of peaks, with the left hill's height rivalling the right hill's to within a distance of σ. When this *twin peaks'* event occurs, a local valley in the LPP energy landscape lies at a significant remove from the global valley while succeeding to rival the latter's depth. An upper bound on this event's probability thus sheds light on the landscape's geometry. Given the strong resemblance of the profile to Brownian motion, the probability of the twin peaks' event is inherited from the counterpart Brownian probability. Our second principal conclusion, Theorem 1.3, asserts that twin peaks with discrepancy σ arise in the routed weight profile with probability at most $\sigma \cdot \exp\big\{\Theta(1)(\log \sigma^{-1})^{5/6}\big\}$. Figure 1 offers a guide to twin peaks in the energy landscape via the equivalent notion of *near touch* for a natural decomposition of the routed weight profile.

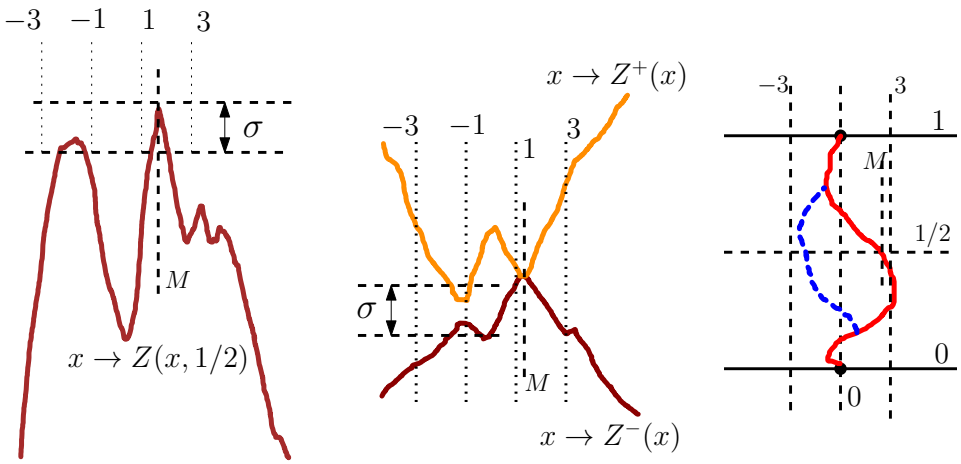

Figure 1: *Left:* The routed weight profile $x \to Z(x, 1/2)$ realizes the twin peaks' event. *Middle:* Let $Z^-(x) = \mathsf{Wgt}_n\big[(0, 0) \to (x, 1/2)\big]$; and let $x \to Z^+(x)$ denote the vertical translate of $x \to -\mathsf{Wgt}_n\big[(x, 1/2) \to (0, 1)\big]$ such that the graphs of Z^- and Z^+ touch, but do not cross. The horizontal coordinate of the point of touch is M, the maximizer of $Z(\cdot, 1/2)$; the occurrence of twin peaks is now represented by a near touch on the part of the two graphs in the strip $[-3, -1] \times \mathbb{R}$. *Right:* The bold polymer $\rho\big[(0, 0) \to (0, 1)\big]$ has horizontal coordinate $M \in [1, 3]$ at height one-half. The rival path, following the bold-dashed-bold route, attains a weight within σ of the polymer's while swinging left, into $[-3, -1]$, at the mid-life time one-half.

1.3.3 Robust assertions of modulus of continuity for geometry and weight of polymers

Polymers such as $\rho = \rho\big[(0, 0) \to (0, 1)\big]$ may be viewed as functions of the vertical coordinate; in this way, we interpret $\rho : [0, 1] \to \mathbb{R}$ as a random real-valued function. A

modulus of continuity for this function is known by [HS20, DOV22] to take the form of a large constant multiple of $z \to z^{2/3}\big(\log z^{-1}\big)^{1/3}$. The weight of the polymer restricted to $[0,t]$ may also be viewed as a function of $t \in [0,1]$; these same references prove a modulus of continuity for this weight profile of order $z \to z^{1/3}\big(\log z^{-1}\big)^{2/3}$. We provide robust forms of such assertions in Theorems 1.4 and 1.6. These results are valid in Brownian LPP uniformly in high values of its parameter $n \in \mathbb{N}$ and in variation of polymer endpoints over compact regions. Just as significantly, they control variation in polymer weight and geometry not merely in response to small n-independent changes in the vertical coordinate as do [HS20] and [DOV22], but on any vertical scale down to the microscopic separation n^{-1}. A related local fluctuation result proved in [BSS17a] was also a crucial ingredient in [BSS17b].

1.3.4 Slender excursions around the polymer are typically uncompetitive

The weight $Z(M,1/2)$ is realized by the maximum weight path on the route from $(0,0)$ to $(0,1)$, namely the polymer $\rho = \rho\big[(0,0) \to (0,1)\big]$. For $z \in \mathbb{R}$ small, the weight $Z(M+z,1/2)$ is realized by a scaled path ϕ that begins from $(0,0)$ by following the course of ρ; that departs this course at some height $h_1 \in (0,1/2)$ to visit $M+z$ at height one-half before rejoining ρ at some height $h_2 \in (1/2,1)$; and that then follows the course of ρ until $(0,1)$. As such, $[h_1,h_2]$ is an interval during which ϕ makes an excursion away from ρ. The KPZ exponent of two-thirds for polymer geometry indicates that $(h_2-h_1)^{2/3}$ has typical order z. That is, we expect ϕ to make an excursion, on an interval that contains one-half, for a duration h_2-h_1 of order $z^{3/2}$; to maintain a horizontal distance from ρ of order z during much of the excursion's duration; and, in view of the Gaussian-order increment of the routed weight profile, to accrue a shortfall in weight relative to ρ of order $z^{1/2}$.

We present a conclusion, Theorem 1.10, that validates this heuristic view. It considers the maximum weight accrued by a path ϕ that makes an excursion of duration $h \in (0,1)$ from the polymer ρ in such a way that the horizontal discrepancy between ρ and ϕ is at most $h^{2/3}\theta$ at most moments during the excursion's lifetime. The parameter $\theta > 0$ will be chosen to be small, so that a *slender* excursion is being demanded, one that deviates horizontally from the polymer by a factor of θ less than is expected. We will prove that the weight of any such path ϕ is highly likely to fall short of the polymer weight by an order of at least $h^{1/3}\theta^{-1}$. When θ is of unit order, this shortfall is predicted by the KPZ exponent of one-third for polymer weight; when $\theta \ll 1$, the factor of θ^{-1} represents a weight penalty for the forcibly confined geometry endured by the excursion. While results of a similar flavor have appeared before in [BGH18, BGHH20, BHS18, GH20b, BB21, DJP18], all of them investigated energetic shortfalls of paths restricted to lie in deterministic thin cylinders. In contrast, Theorem 1.10 offers a more robust setting treating paths restricted to be in a random region determined by the polymer.

1.3.5 The polymer advances in a regular fashion microscopically

Our final main result, Theorem 1.11, concerns the microscopic structure of the trajectory of the polymer in Brownian LPP, and is more vividly expressed in unscaled coordinates. Consider then the Brownian LPP geodesic Γ_n that runs from $(0,0)$ to (n,n). The geodesic's progression is globally diagonal; we prove that, even on the shortest of scales, this progression is manifest. A *cliff* in Γ is a subpath of Γ in which Γ advances horizontally by one unit while advancing vertically by A units. We prove that, when the positive parameter A is fixed at a high value, and except on an event of probability that decays at an exponential rate in n, the proportion of Γ that is comprised of cliffs is bounded away from one.

1.4 Probabilistic and geometric inquiry into KPZ universality

The study of KPZ universality has advanced through physical insights, numerical analysis, and several techniques of integrable or algebraic origin. Rather than hazard a summary of literature to support this one sentence history, we refer to the reader to [Cor12] for a KPZ survey from 2012; in fact, integrable and analytic approaches to KPZ have attracted huge interest around and since that time. A recent wave of KPZ research has brought probabilistic and geometric tools to the fore, making use of integrable aspects of the models of study as occasional inputs in arguments, albeit essential ones. Three examples are the solution [BSS16] of the slow bond problem, in which the integrable model of exponential LPP is perturbed by altering the random environment along a one-dimensional subspace, and the resulting geometry and energy of geodesics is studied; the construction [CH14] of the Airy line ensemble, a KPZ universal object encoding polymer weights in the narrow wedge case as a continuous system of mutually avoiding random curves; and the construction [DOV22] of the Airy sheet (or the directed landscape), a rich scaling limit for the weights of KPZ polymers in which these weights are obtained as LPP values in an environment specified by the Airy line ensemble after the subtraction of a parabola.

The present article pursues the study of problems in KPZ in a probabilistic and geometric vein. In particular, our results Theorems 1.2 and 1.3 on Brownianity and twin peaks' rarity for the routed weight profile lie in the domain of *Brownian Gibbs* analysis of LPP. The parabolic Airy line ensemble is in essence a mutually avoiding system of Brownian motions, subject to suitable boundary conditions. As we will indicate more clearly early in Section 2, this ensemble of random curves thus satisfies the Brownian Gibbs property, a simple and attractive resampling property involving Brownian motion and avoidance. The Brownian Gibbs technique led to the construction of the Airy line ensemble in [CH14]. The technique has been pursued in [Ham22] and [CHH19] to yield strong inferences regarding the similarity to Brownian motion of the Airy_2 process, which is, after a parabolic shift, the scaling limit of the narrow wedge polymer weight profile in integrable LPP models.

Our results on Brownianity and twin peaks' rarity develop this strand of research, begun in [CH14], and pursued in [Ham22] and [CHH19], so that results such as Theorems 1.2 and 1.3 pertinent to the LPP energy landscape now become available.

By means of a more algebraic approach that analyses representations involving Fredholm determinants, strong Brownian comparison estimates have also been obtained in [MQR17]. This work constructs a universal Markov process called the KPZ fixed point that describes the evolution of geodesic energy profiles starting from rather arbitrary initial data; an earlier result in [QR19] identified domains of attraction for the one point fluctuations of the KPZ equation starting from general initial data. The assertion that the Airy_2 process closely resembles Brownian motion on the unit-order scale, and counterpart results for scaled geodesic energy profiles in LPP models, have been instrumental in several recent inquiries into geometric and fractal properties of the KPZ fixed point. In [BG18, BGZ19, FO19], exponents governing temporal correlations induced by various initial data have been determined, thereby settling conjectures by Ferrari and Spohn from [FS$^+$16]. (In [CGH19], an analogous result for the KPZ equation is derived.) Profile Brownianity also drives the fractal geometry and Hausdorff dimension results for exceptional sets found in the space-time Airy sheet that are the subject of [**?**, BGH19].

1.5 Stability and chaos in dynamical Brownian LPP

Control on polymer geometry and weight; on the excursion geometry of LPP paths that in weight are competitive with the maximum; on the microscopic structure of geodesics—these geometric inferences are, we believe, robust tools that will serve to advance the analysis of LPP; its scaling limit; and its reaction to perturbation. Indeed, this article's results find application in a companion study [GH20a] of the stability, and fragility, of Brownian LPP under dynamical perturbation. Very shortly, we will define Brownian LPP; for now, we note merely that its noise environment is specified by a countable system of independent Brownian motions. A dynamics may be introduced that leaves Brownian LPP invariant, in which each of these constituent Brownian motions is updated according to Ornstein-Uhlenbeck dynamics. In [GH20a] is identified the time-scale that heralds the transition from stability to chaos for dynamical Brownian LPP—the polymer from $(0,0)$ to $(0,1)$ is largely unperturbed in the stable zone and is profoundly altered in the chaotic phase. This time-scale takes the form $n^{-1/3+o(1)}$ when a geodesic of extension $n \in \mathbb{N}$ is considered; this corresponds to updating $n^{2/3+o(1)}$ bits along the geodesic in a discrete LPP model. Every one of the results that we have indicated in the preceding overview has a role to play in proving this transition in [GH20a]. The robust probabilistic and geometric results and technique that we present undergird the companion dynamical LPP analysis and, we believe, will find further application in the study of scaled KPZ structure.

In the next two sections, we define Brownian LPP and introduce some of its basic objects; and we specify the transformation that specifies the scaled coordinates in which we couch our principal results and proofs. In the remaining introductory sections, we then present the statements of our main results, in the same order in which we have just summarized them.

1.6 Brownian last passage percolation

On a probability space equipped with a law labelled $\mathbb{P}$, let $B : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ denote a collection of independent two-sided standard Brownian motions $B(\cdot, k) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$. The indexing of the domain in the form $\mathbb{R} \times \mathbb{Z}$ is unusual, with the other choice $\mathbb{Z} \times \mathbb{R}$ being more conventional. The choice of $\mathbb{R} \times \mathbb{Z}$ is made because it permits us to visualize this index set for the ensemble B's curves as a subset of $\mathbb{R}^2$ with the usual Cartesian coordinate order being respected by the notation.

Let $i, j \in \mathbb{Z}$ with $i \leq j$. $[\![i, j]\!]$ will denote the integer interval $\{i, \cdots, j\}$. For $x, y \in \mathbb{R}$ with $x \leq y$, consider the collection of non-decreasing lists $\big\{z_k : k \in [\![i+1, j]\!]\big\}$ of values $z_k \in [x, y]$. Adopting the convention that $z_i = x$ and $z_{j+1} = y$, we associate to any such list the energy $\sum_{k=i}^{j} \big(B(z_{k+1}, k) - B(z_k, k)\big)$. We then define the maximum energy to be

$$M\big[(x,i) \to (y,j)\big] \;=\; \sup \left\{ \sum_{k=i}^{j} \Big(B(z_{k+1}, k) - B(z_k, k) \Big) \right\},$$

where the supremum is taken over all such lists. The random process $M\big[(0,1) \to (\cdot, n)\big] : [0, \infty) \to \mathbb{R}$ was introduced by [GW91] and further studied in [OY02].

1.6.1 Staircases

Set $\mathbb{N} = \{0, 1, \cdots\}$. For $i, j \in \mathbb{N}$ with $i \leq j$, and $x, y \in \mathbb{R}$ with $x \leq y$, an energy has been ascribed to any non-decreasing list $\big\{z_k : k \in [\![i+1, j]\!]\big\}$ of values $z_k \in [x, y]$. In order to emphasize the geometric aspects of this definition, we associate to each list a subset of $[x, y] \times [i, j] \subset \mathbb{R}^2$, that we call a staircase, which will be the range of a piecewise affine path.

To define the staircase above we again adopt the convention that $z_i = x$ and $z_{j+1} = y$. The staircase will be specified as the union of certain horizontal planar line segments, and certain vertical ones. The horizontal segments take the form $[z_k, z_{k+1}] \times \{k\}$ for $k \in [\![i, j]\!]$. A vertical planar line segment of unit length connects the right and left endpoints of each consecutive pair of horizontal segments. It is this collection of vertical line segments that form the vertical segments of the staircase.

The resulting staircase may be depicted as the range of an alternately rightward and upward moving path from starting point (x, i) to ending point (y, j). The set of staircases with these starting and ending points will be denoted by $SC\big[(x, i) \to (y, j)\big]$. Since such staircases are in bijection with the collection of non-decreasing lists, any staircase $\phi \in SC\big[(x, i) \to (y, j)\big]$ is assigned an energy $E(\phi) = \sum_{k=i}^{j} \big(B(z_{k+1}, k) - B(z_k, k)\big)$ via the associated z-list.

1.6.2 Energy maximizing staircases are called geodesics.

A staircase $\phi \in SC\big[(x, i) \to (y, j)\big]$ whose energy attains the maximum value $M\big[(x, i) \to (y, j)\big]$ is called a geodesic from (x, i) to (y, j). That this geodesic exists for all choices of $x, y \in \mathbb{R}$ with $x \le y$, is a simple consequence of the continuity of the constituent Brownian paths $B(k, \cdot)$. Further, for any given such choice of the pair (x, y), by [Ham19b, Lemma $A.1$], there is an almost surely unique geodesic from (x, i) to (y, j). We denote it by $\Gamma\big[(x, i) \to (y, j)\big]$.

1.7 Scaled coordinates for Brownian LPP

Members of the KPZ universality class enjoy scalings represented by the characteristic exponents of one-third and two-thirds. The one-third exponent governs the energetic fluctuation of the geodesic between $(0, 0)$ and (n, n), that is, if we write

$$M\big[(0, 0) \to (n, n)\big] = 2n + n^{1/3}\mathsf{Wgt}_n\big[(0, 0) \to (0, 1)\big], \tag{1.2}$$

then the term $\mathsf{Wgt}_n\big[(0, 0) \to (0, 1)\big]$ is a random, tight in n unit-order quantity. (See the sixth paragraph of [CHH19, Subsection 3.2.1] for a discussion of this tightness.) This is the scaled geodesic energy, which we will call *weight*. The exponent two-thirds appears in the fact that when geodesic energy $[0, \infty) \to \mathbb{R} : x \to M\big[(0, 0) \to (x, n)\big]$ is varied from $x = n$, it is changes of order $n^{2/3}$ in x that result in non-trivial correlation.

Given the above, it is natural to work in scaled coordinates under which the journey between $(0, 0)$ and (n, n) corresponds to the unit vertical journey between $(0, 0)$ and $(0, 1)$, while horizontal perturbation of the endpoint (n, n) by magnitude $n^{2/3}$ corresponds to unit-order scaled horizontal perturbation. This will lead to the notion of scaled energy, or weight, associated to the image of any path in scaled coordinates. This is done next, namely, we specify the scaling map $R_n : \mathbb{R}^2 \to \mathbb{R}^2$ whose range specifies scaled coordinates; introduce notation for scaled paths; and specify the form of scaled energy.

1.7.1 The scaling map.

For $n \in \mathbb{N}$, consider the n-indexed *scaling* map $R_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$R_n\big(v_1, v_2\big) = \Big(2^{-1}n^{-2/3}(v_1 - v_2)\,,\, v_2/n\Big). \tag{1.3}$$

The scaling map naturally acts on subsets C of $\mathbb{R}^2$ with $R_n(C) = \big\{R_n(x) : x \in C\big\}$.

1.7.2 Scaling transforms staircases to zigzags.

The image of any staircase under R_n will be called an n-zigzag. The starting and ending points of an n-zigzag Z are defined to be the image under R_n of the corresponding points for the staircase S such that $Z = R_n(S)$.

Note that the set of horizontal lines is invariant under R_n, while vertical lines are mapped to lines of gradient $-2n^{-1/3}$. Thus, an n-zigzag is the range of a piecewise affine path from the starting point to the ending point which alternately moves rightwards along horizontal line segments and northwesterly along sloping line segments with gradient $-2n^{-1/3}$.

Note for example that, for given real choices of x and y, a journey which in the original coordinates occurs between $(2n^{2/3}x, 0)$ and $(n + 2n^{2/3}y, n)$ takes place in scaled coordinates between $(x, 0)$ and $(y, 1)$. We may view the first coordinate as space and the second as time, though the latter interpretation should not be confused with dynamic time t; with this view in mind, the journey at hand is between x and y over the unit lifetime $[0, 1]$.

1.7.3 Compatible triples

Let $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$, where we write $R^2_{\leq} = \big\{(s_1, s_2) \in \mathbb{R}^2 : s_1 \leq s_2\big\}$. Taking $x, y \in \mathbb{R}$, does there exist an n-zigzag from (x, s_1) and (y, s_2)? Two conditions must be satisfied for an affirmative answer.

First: as far as the data (n, s_1, s_2) is concerned, such an n-zigzag may exist only if

$$s_1 \text{ and } s_2 \text{ are integer multiplies of } n^{-1}\,. \tag{1.4}$$

We say that data $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$ is a *compatible triple* if the above holds. We will consistently impose this condition, whenever we seek to study n-zigzags whose lifetime is $[s_1, s_2]$. The use of compatible triples should be thought of as a fairly minor detail. As the index n increases, the n^{-1}-mesh becomes finer, so that the space of n- zigzags better approximates a field of functions, defined on arbitrary finite intervals of the vertical coordinate, and taking values in the horizontal coordinate.

Associated to a compatible triple is the notation $s_{1,2}$, which will denote the difference $s_2 - s_1$. The law of the underlying Brownian ensemble $B : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ is invariant under integer shifts in the latter, curve indexing, coordinate. This translates to an invariance in law of scaled objects under vertical shifts by multiples of n^{-1}, thus making the parameter $s_{1,2}$ of far greater relevance than the individual values s_1 or s_2.

Returning to the above posed question, the second needed condition is that the horizontal coordinate of the unscaled counterpart of the latter endpoint must be at least the former which translates to the condition

$$y - x \geq -2^{-1}n^{1/3}s_{1,2}\,. \tag{1.5}$$

1.7.4 Zigzag subpaths

Let ϕ denote an n-zigzag between elements (x, s_1) and (y, s_2) in $\mathbb{R} \times n^{-1}\mathbb{Z}$. Let (u, s_3) and (v, s_4) be elements in $\phi \cap \big([s_1, s_2] \cap n^{-1}\mathbb{Z}\big)$. Suppose that $s_3 \leq s_4$ (and that $u \leq v$ if equality here holds), so that (u, s_3) is encountered before (v, s_4) in the journey along ϕ. The removal of (u, s_3) and (v, s_4) from ϕ results in three connected components. The closure of one of these contains these two points and this closure will be denoted by $\phi_{(u,s_3)\to(v,s_4)}$. This is the zigzag subpath, or sub-zigzag, of ϕ between (u, s_3) and (v, s_4).

1.7.5 Staircase energy scales to zigzag weight.

Let $n \in \mathbb{N}$ and $i, j \in \mathbb{N}$ satisfy $i < j$. Any n-zigzag Z from $(x, i/n)$ to $(y, j/n)$ is ascribed a scaled energy, which we will refer to as its weight, $\mathsf{Wgt}(Z) = \mathsf{Wgt}_n(Z)$, given by

$$\mathsf{Wgt}(Z) = 2^{-1/2} n^{-1/3} \Big(E(S) - 2(j-i) - 2n^{2/3}(y-x) \Big) \tag{1.6}$$

where Z is the image under R_n of the staircase S.

1.7.6 Maximum weight.

Let $n \in \mathbb{N}$. The quantity $\mathsf{Wgt}_n\big[(0,0) \to (0,1)\big]$ specified in (1.2) is simply the maximum weight ascribed to any n-zigzag from $(0,0)$ to $(0,1)$.

Let $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$ be a compatible triple. Suppose that $x, y \in \mathbb{R}$ satisfy $y \geq x - 2^{-1} n^{1/3} s_{1,2}$. We will now define $\mathsf{Wgt}_n\big[(x, s_1) \to (y, s_2)\big]$ in a way such that this quantity equals the maximum weight of any n-zigzag from (x, s_1) to (y, s_2). We must set

$$\begin{aligned} &\mathsf{Wgt}_n\big[(x, s_1) \to (y, s_2)\big] \\ &= 2^{-1/2} n^{-1/3} \Big(M\big[(ns_1 + 2n^{2/3}x, ns_1) \to (ns_2 + 2n^{2/3}y, ns_2)\big] - 2ns_{1,2} - 2n^{2/3}(y-x) \Big). \end{aligned} \tag{1.7}$$

The quantity $\mathsf{Wgt}_n\big[(x, s_1) \to (y, s_2)\big]$ may be expected to be, for given real choices of x and y that differ by order $s_{1,2}^{2/3}$, a unit-order random quantity; this collection of random variables is tight in the scaling parameter $n \in \mathbb{N}$ and in such choices of $s_1, s_2 \in n^{-1}\mathbb{Z}$ and $x, y \in \mathbb{R}$.

1.7.7 Highest weight zigzags are called polymers.

An n-zigzag that attains the maximum weight given its endpoints will be called an n-polymer, or, usually, simply a polymer. Thus, under the scaling map, geodesics map to polymers. As we recalled in Subsection 1.6.2, the geodesic with any given pair of endpoints is almost surely unique. For $x, y \in \mathbb{R}$ and $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$ a compatible triple, the almost surely unique n-polymer from (x, s_1) to (y, s_2) will be denoted by $\rho_n\big[(x, s_1) \to (y, s_2)\big]$; see Figure 2. The shorthand $\rho_n = \rho_n\big[(0,0) \to (0,1)\big]$ will be used.

Though not standard, since the term 'polymer' is often used to refer to typical realizations of the path measure in LPP models at positive temperature, the above usage of the term 'polymer' for 'scaled geodesic' is quite apt for our study, owing to the central role played by these objects.

1.7.8 Zigzags as near functions of the vertical coordinate

Suppose again that ϕ is an n-zigzag between points (x, s_1) and (y, s_2) in $\mathbb{R} \times n^{-1}\mathbb{Z}$. For $s \in [s_1, s_2] \cap n^{-1}\mathbb{Z}$, we will write $\phi(s)$ for the supremum of values $x \in \mathbb{R}$ for which $(x, s) \in \phi$. This abuse of notation permits $\phi(s)$ to denote the horizontal coordinate of the point of departure from vertical coordinate s in the journey along ϕ from (x, s_1) to (y, s_2). This convention is adopted partly because it captures the notion that the typical zigzags ϕ we will consider—polymers or concatenations thereof—are closely approximable by a real-valued function of the vertical coordinate $s \in [s_1, s_2]$, at least when n is high—indeed, the maximum length of the horizontal line segments in an n-polymer is readily seen to decay to zero in n with high probability. (Our few cliffs' Theorem 1.11 quantifies this assertion.)

Note that this notational device will be used alongside the shorthand ρ_n from the preceding subsection, so that, for $a \in n^{-1}\mathbb{Z} \cap [0,1]$, $\rho_n(a)$ is the maximum x-coordinate of a point at y-coordinate a on the polymer $\rho_n\big[(0,0) \to (0,1)\big]$.

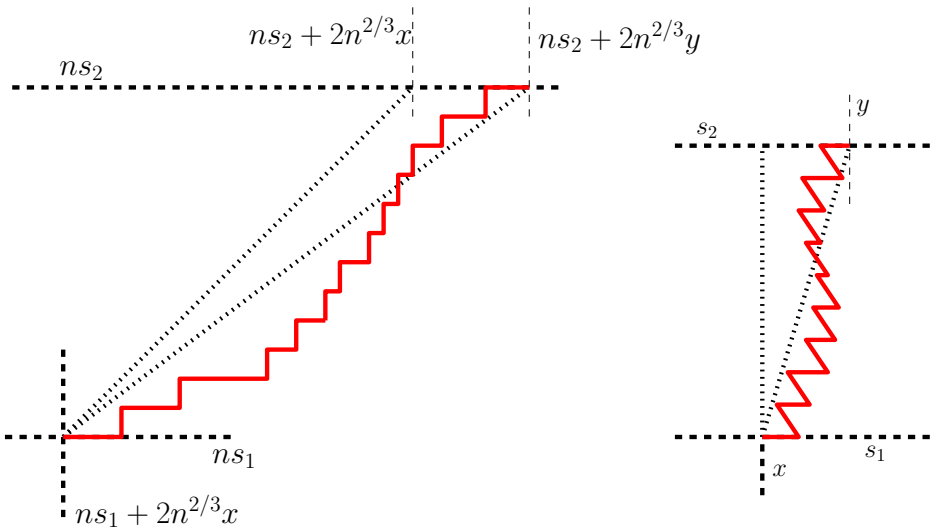

Figure 2: Let (n, s_1, s_2) be a compatible triple and let $x, y \in \mathbb{R}$. The endpoints of the geodesic in the left sketch are such that, when the scaling map R_n is applied to produce the right sketch, the result is the n-polymer $\rho_n\big[(x, s_1) \to (y, s_2)\big]$ from (x, s_1) to (y, s_2).

1.8 Brownianity and twin peaks' rarity for the routed weight profile

To specify the routed weight profile for scaled Brownian LPP, let $n \in \mathbb{N}$ and $a \in n^{-1}\mathbb{Z} \cap (0,1)$. For $x \in \mathbb{R}$, let $\Psi_n(x,a)$ denote the set of n-zigzags ϕ that begin at $(0,0)$; end at $(0,1)$; and for which $x = \sup\big\{z \in \mathbb{R} : (z,a) \in \phi\big\}$. In other words, $\Psi_n(x,a)$ comprises those n-zigzags on the route from $(0,0)$ to $(0,1)$ whose point of departure from level a occurs at (x,a). We set $Z_n(x,a)$ equal to the supremum of the weights of elements of $\Psi_n(x,a)$. In this way, the *routed weight profile* $Z_n(\cdot, a) : \mathbb{R} \to \mathbb{R}$ records the maximum weight of zigzags that are constrained to exit level a at a given horizontal location. We did not allude to this exit constraint in the heuristic discussion of Subsection 1.3.1: in the microscopic model, where $n \in \mathbb{N}$ is finite, this definition renders the maximizer location $M \in \mathbb{R}$ at which $Z_n(M,a) = \sup_{x \in \mathbb{R}} Z_n(x,a)$ unique, while maintaining that $Z_n(M,a) = \mathsf{Wgt}_n\big[(0,0) \to (0,1)\big]$; in Lemma 2.1, we will moreover see that, in the counterpart expression to (1.1), the two right-hand terms are independent, even when n is finite, when the present definition is adopted.

Next we make precise the notion of comparison that we will make to Brownian motion.

Definition 1.1. *Let $K \in \mathbb{R}$ and $d > 0$. Let I denote the interval $[K-d, K+d]$. We denote by $\mathcal{C}_{0,*}(I, \mathbb{R})$ the space of continuous functions $f : I \to \mathbb{R}$ such that $f(K-d) = 0$. For $\nu > 0$, let $\mathcal{B}_{0,*}^{\nu;I}$ denote the law on this function space given by Brownian motion $B : I \to \mathbb{R}$, $B(K-d) = 0$, of diffusion rate ν.*

Let g and G be positive real parameters; and let $m \in \mathbb{N}$. A continuous random function $X : I \to \mathbb{R}$ defined under a law $\mathbb{P}$ is said to be $\big(g, G, d, m, \nu\big)$-Brownian if the following holds. Let A denote a Borel measurable subset of $\mathcal{C}_{0,*}(I, \mathbb{R})$. Set $\eta = \mathcal{B}_{0,*}^{\nu;I}(A)$. Then the condition that $e^{-gm^{1/12}} \le \eta \le g \wedge e^{-Gd^6}$ implies that

$$\mathbb{P}\Big(I \to \mathbb{R} : x \to X(x) - X(K-d) \text{ belongs to } A\Big) \le \eta \cdot G \exp\Big\{Gd\big(\log \eta^{-1}\big)^{5/6}\Big\}. \quad (1.8)$$

In heuristic overview, we compared the routed weight profile to Brownian motion of rate two on $[-1,1]$. Our rigorous result crucially relying on the main result in [CHH19]

makes the comparison on any compact interval: after the addition of the linear term $2^{1/2}\big(a(1-a)\big)^{-1}Rx$, the profile $x \to Z_n(x,a)$ is very similar to Brownian motion of rate two, in the locale of any given $R \in \mathbb{R}$.

Theorem 1.2. *There exist positive constants c, G and g such that the following holds. Let $n \in \mathbb{N}$ and $a \in n^{-1}\mathbb{Z} \cap (0,1)$. Suppose that $|R| \le 2^{-1}cn^{1/9}\big(a \wedge (1-a-n^{-1})\big)^{7/9}$. Let $\ell \in \mathbb{R}$ satisfy $-2^{-1}n^{1/3}a \le R-\ell$ and $R+\ell \le 2^{-1}n^{1/3}(1-a-n^{-1})$. The process $Z_n(\cdot,a) : [R-\ell, R+\ell] \to \mathbb{R}$ may be expressed in the form*

$$Z_n(x,a) = X(x) - \Big(2^{1/2}\big(a(1-a)\big)^{-1}R + \varepsilon\Big)x\,,$$

where $X : [R-\ell, R+\ell] \to \mathbb{R}$ is $\big(g, G'\ell^6, \ell, \min\{a,1-a\}n, 2\big)$-Brownian. Here, the constant G' is up to an absolute positive factor equal to $G^{17/6}g^{-5/6}\big(a \wedge (1-a-n^{-1})\big)^{-34/3}$; and $\varepsilon = \varepsilon(a,R,n)$, given by $\varepsilon = 2^{1/2}\big((1-a-n^{-1})(1-a)\big)^{-1}Rn^{-1}$, is an error term without dependence on x.

Given the above quantitative comparison to Brownian motion, the next result presents our conclusion regarding the rarity of *twin peaks*. The probability that there exists $x \in \mathbb{R}$ such that $Z_n(x,a)$ rivals the maximum value of $Z_n(\cdot,a)$, with $Z_n(x,a)$ being less than this maximum by a small multiple σ of the square-root distance $\big(x-\rho_n(a)\big)^{1/2}$ is bounded above by the product of σ and a lower-order correction $\exp\big\{\Theta(1)\big(\log\sigma^{-1}\big)^{5/6}\big\}$; a further factor of $e^{-\Theta(1)R^2\ell}$ penalizes the maximizer for being of a large order $R > 0$.

Theorem 1.3. *For K any compact interval of $(0,1)$, there exist positive constants $H = H(K)$ and $h = h(K)$ and an integer $n_0 = n_0(K)$ such that the following holds. Let $n \in \mathbb{N}$, $R \in \mathbb{R}$, $\ell \ge 1$, $\ell' > 0$, $a \in n^{-1}\mathbb{Z} \cap K$, $\sigma > 0$ and $\varepsilon > 0$. Suppose that $n \ge n_0$, $|R| \le hn^{1/9}$, $\ell \in (3\varepsilon, hn^{1/1370})$ and $\ell' \in (3\varepsilon, \ell]$. Denoting $\sigma \wedge 1$ by σ_*, we have that*

$$\begin{aligned}&\mathbb{P}\Big(M \in [R-\ell/3, R+\ell/3]\,, \sup_{x\in\mathbb{R}:|x-M|\in[\varepsilon,\ell'/3]}\big(Z_n(x,a)+\sigma(x-M)^{1/2}\big) \ge Z_n(M,a)\Big)\\ \le\ & \log\big(\ell'\varepsilon^{-1}\big)\max\Big\{\sigma_*\cdot\exp\big\{-hR^2\ell + H\ell^{19}\big(1+R^2+\log\sigma_*^{-1}\big)^{5/6}\big\}, \exp\big\{-hn^{1/12}\big\}\Big\}\,,\end{aligned}$$

where M denotes $\rho_n(a)$, the almost surely unique maximizer of $x \to Z_n(x,a)$.

The right-hand factor of $\log\big(\ell'\varepsilon^{-1}\big)$ reflects a union bound indexed by dyadic scales intersecting the interval $[\varepsilon, \ell'/3]$. There is no non-smallness condition on the scale ε. The probability upper bound $\exp\big\{-hn^{1/12}\big\}$ becomes operative for extremely small values of σ.

1.9 Robust modulus of continuity for the geometry and weight of polymers

The next result offers a quantified prelimiting expression for the $z \to z^{2/3}\big(\log z^{-1}\big)^{1/3}$ modulus of continuity for polymer geometry in a fashion that is uniform as the polymer's endpoints vary over a compact region and that holds on all scales above the microscopic separation n^{-1}.

Theorem 1.4.

1. *There exist positive H, h and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ satisfies $n \ge n_0$, $k \in \mathbb{N}$ satisfies $2^k \le hn$ and $r \in \mathbb{R}$ satisfies $r_0 \le r \le n^{1/10}$, it is with probability at least $1 - H\exp\big\{-hr^3k\big\}$ that the following event occurs. Let $x, y \in \mathbb{R}$ be of absolute value at most r. Let $h_1, h_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ satisfy $h_{1,2} \in (2^{-k-1}, 2^{-k}]$ and let $u, v \in \mathbb{R}$ be such that (u,h_1) and (v,h_2) belong to $\rho_n\big[(x,0) \to (y,1)\big]$. Then*

$$\big|v-u\big| \le Hh_{1,2}^{2/3}\big(\log(1+h_{1,2}^{-1})\big)^{1/3}r\,.$$

2. *There exist positive G, H, h and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ satisfies $n \geq n_0$, and $r \in \mathbb{R}$ satisfies $r_0 \leq r \leq n^{1/10}$, it is with probability at least $1 - Hn^{-hr^3}$ that the following event occurs. As above, let $x, y \in \mathbb{R}$ be of absolute value at most r, and let $u, v \in \mathbb{R}$ be such that (u, h_1) and (v, h_2) belong to $\rho_n\big[(x,0) \to (y,1)\big]$. Consider any $h_1, h_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ that satisfy $h_{1,2} < Hn^{-1}$. Then*

$$\big|v - u\big| \leq Gn^{-2/3}(\log n)^{1/3} r\,.$$

As a special case, we gain control on the maximum fluctuation of such polymers.

Corollary 1.5. *There exist positive H, h and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ satisfies $n \geq n_0$, and $r \in \mathbb{R}$ satisfies $r_0 \leq r \leq n^{1/10}$, it is with probability at least $1 - H\exp\big\{-hr^3\big\}$ that the following holds. Let $x, y \in \mathbb{R}$ be of absolute value at most r. If $(u, h') \in \mathbb{R} \times \big(n^{-1}\mathbb{Z} \cap [0,1]\big)$ lies in $\rho_n\big[(x,0) \to (y,1)\big]$, then $|u| \leq Hr$.*

A control, similar to that offered by Theorem 1.4, on the $z \to z^{1/3}\big(\log z^{-1}\big)^{2/3}$ modulus of continuity for polymer weight is available.

Theorem 1.6.

1. *There exist positive H, h and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ satisfies $n \geq n_0$; $k \in \mathbb{N}$ satisfies $2^k \leq hn$; and $r \in \mathbb{R}$ satisfies $r \geq r_0$, it is with probability at least $1 - H\exp\big\{-hr^3k\big\}$ that the following occurs. Let $h_1, h_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ satisfy $h_{1,2} \in (2^{-k-1}, 2^{-k}]$ and $r \leq (nh_{1,2})^{1/64}$; let $x, y \in \mathbb{R}$ be of absolute value at most r; and let $u, v \in \mathbb{R}$ be such that (u, h_1) and (v, h_2) belong to $\rho_n\big[(x,0) \to (y,1)\big]$. Then*

$$\big|\mathsf{Wgt}_n\big[(u,h_1) \to (v,h_2)\big]\big| \leq H^2 r^2 \cdot h_{1,2}^{1/3}\big(\log h_{1,2}^{-1}\big)^{2/3}\,.$$

2. *There exist positive G, H and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ satisfies $n \geq n_0$, and $r \in \mathbb{R}$ satisfies $r \geq r_0$, it is with probability at least $1 - Hn^{-hr^3}$ that, if $h_{1,2} < Hn^{-1}$; $x, y \in \mathbb{R}$ have absolute value at most r; and $u, v \in \mathbb{R}$ are such that (u, h_1) and (v, h_2) belong to $\rho_n\big[(x,0) \to (y,1)\big]$; then $\big|\mathsf{Wgt}_n\big[(u,h_1) \to (v,h_2)\big]\big| \leq Gr^2 \cdot n^{-1/3}(\log n)^{2/3}$.*

We present two further results, which emerge in the course of the proof of Theorem 1.4. The former offers uniform control on the maximum fluctuation of polymers, in which we permit to vary the polymer endpoints and the moment during the lifetime of the polymer at which fluctuation is measured. The latter proves the rarity of short polymers of extreme weight that begin and end in a unit-order region.

In order to state the first result, we specify a measure of the fluctuation of the polymer $\rho_n\big[(x,s_1) \to (y,s_2)\big]$ at the intermediate moment $h \in [s_1, s_2] \cap n^{-1}\mathbb{Z}$, measuring the horizontal distance between the polymer at this height h relative to the height-h location $\frac{s_2 - h}{s_{1,2}}x + \frac{h - s_1}{s_{1,2}}y$ of the line that interpolates (x, s_1) and (y, s_2). We set

$$\mathsf{Fluc}_n\big[(x,s_1) \to (y,s_2); h\big] = \sup\Big\{\big|u - \tfrac{s_2-h}{s_{1,2}}x - \tfrac{h-s_1}{s_{1,2}}y\big| : u \in \mathbb{R}\,,\ (u,h) \in \rho_n\big[(x,s_1) \to (y,s_2)\big]\Big\}. \tag{1.9}$$

The typical order of this quantity is $\lambda^{2/3}$, where λ equals $(h - s_1) \wedge (s_2 - h)$, with $\wedge$ denoting minimum.

Theorem 1.7. *Let $K > 0$, $r \geq r_0$, $a \in (0, 1/4]$, $n \in \mathbb{N}$ and $s_1, s_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ satisfy $s_1 \leq s_2$; $ns_{1,2}a$ and $ns_{1,2}(1-a)$ are at least $\Theta(1)$; $Ka^{1/3} \leq \Theta(1)$; and $|K| \leq (ns_{1,2})^{2/3}$. Then*

$$\mathbb{P}\Big(\sup \mathsf{Fluc}_n\big[(x,h_1) \to (y,h_2); h\big] \geq r(as_{1,2})^{2/3}\big(\log a^{-1}\big)^{1/3}\Big) \leq \Theta(1)K^2 a^{-10/3} a^{\Theta(1)r^3}\,, \tag{1.10}$$

where the supremum is taken over $x, y \in [-K, K] \cdot s_{1,2}^{2/3}$, $h_1 \in n^{-1}\mathbb{Z} \cap [s_1, s_1 + s_{1,2}/3]$, $h_2 \in n^{-1}\mathbb{Z} \cap [s_2 - s_{1,2}/3, s_2]$ and $h \in n^{-1}\mathbb{Z}$ such that $\frac{h - h_1}{h_{1,2}} \in [a, 2a] \cup [1-2a, 1-a]$.

To express our result on weights, let $\mathsf{Wgt}_n^{\cup}\big[(x,h_1)\to(y,h_2)\big]$ denote the parabolically adjusted weight $\mathsf{Wgt}_n\big[(x,h_1)\to(y,h_2)\big]+2^{-1/2}(y-x)^2h_{1,2}^{-1}$. We denote by $\mathsf{Low}_n(\zeta,\ell,L,M)$ the event that

$$s_{1,2}^{-1/3}\mathsf{Wgt}_n^{\cup}\big[(x,s_1)\to(y,s_2)\big]$$

is less than $-\zeta$ for some pair $(x,s_1),(y,s_2)\in\mathbb{R}\times n^{-1}\mathbb{Z}\cap[0,1]$ with $|x|\vee|y|\le M$, $|x-y|\le 2^{-2\ell/3}L$ and $s_{1,2}\in(2^{-\ell-1},2^{-\ell}]$. Let $\mathsf{High}_n(\zeta,\ell,L,M)$ denote the event that the displayed quantity exceeds ζ for some such pair.

Proposition 1.8. *When $n\ge\Theta(1)2^{\ell}$, $L\le\Theta(1)(n2^{-\ell})^{1/46}$, $\Theta(1)\le\zeta\le\Theta(1)(n2^{-\ell})^{1/30}$ and $M>0$,*

$$\mathbb{P}\Big(\mathsf{Low}_n\big(\zeta,\ell,L,M\big)\cup\mathsf{High}_n\big(\zeta,\ell,L,M\big)\Big)\;\le\;\Theta(1)2^{5\ell/3}ML\exp\big\{-\Theta(1)\zeta^{3/2}\big\}.$$

1.10 Slim pickings for slender excursions

Here we present results asserting that zigzags constrained to stay close to a deterministic path or the polymer are typically uncompetitive in weight.

Consider a given zigzag ϕ from $(0,0)$ to $(0,1)$. Let $(x,s_1),(y,s_2)\in\mathbb{R}\times n^{-1}\mathbb{Z}\cap[0,1]$, $s_1<s_2$, be two points, neither of which necessarily lies in ϕ. A zigzag ψ from (x,s_1) to (y,s_2) that is disjoint from ρ_n will be called an excursion, even though this name might more properly be reserved for the case where ψ's endpoints lie in ϕ. For an excursion ψ, consider the set of $s\in[s_1,s_2]\cap n^{-1}\mathbb{Z}$ for which, to use the language of Subsection 1.7.8, $|\psi(s)-\phi(s)|$ is at most $s_{1,2}^{2/3}\theta$, where $\theta>0$ is given. If this set has cardinality at least $(1-\chi)\big|[s_1,s_2]\cap n^{-1}\mathbb{Z}\big|$ and contains the values s_1 and s_2, then the excursion ψ is called $(\phi,\theta,1-\chi)$-close. The parameter $\theta>0$ measures constraint in movement beyond the factor $s_{1,2}^{2/3}$ that is dictated by KPZ scaling, and our notion of closeness indicates that this constraint is satisfied at a high percentage of levels in $[s_1,s_2]$. See Figure 3.

The supremum of the weights of $(\phi,\theta,1-\chi)$-close excursions will be denoted by

$$\mathsf{Wgt}_n\big[(x,s_1)\to(y,s_2)\,;\,(\phi,\theta,1-\chi)\text{-close}\big].$$

For $\ell\in\mathbb{N}$ and $d_0>0$, let $\mathsf{LowSlenderExcursion}\big(\ell,\theta,1-\chi;\phi\big)$ denote the event that

$$\sup\,s_{1,2}^{-1/3}\mathsf{Wgt}_n\big[(x,s_1)\to(y,s_2)\,;\,(\phi,\theta,1-\chi)\text{-close}\big]\le-d_0\theta^{-1},$$

where the supremum is taken by varying the points $(x,s_1),(y,s_2)\in\mathbb{R}\times[0,1]\cap n^{-1}\mathbb{Z}$ over choices such that $2^{-1-\ell}\le s_{1,2}\le 2^{-\ell}$.

Our first assertion of slender slim pickings concerns excursions about the polymer ρ_n.

Theorem 1.9. *There exist constants $d_0,C>0$ such that we may find $\chi_0\in(0,1)$, $d_2>0$ and $n_0\in\mathbb{N}$ for which $\chi\in(0,\chi_0)$ and $n\ge n_0$ imply that when $\theta^{-1/4}>C\log n$, and $\ell\in\mathbb{N}$ satisfies $2^{\ell}\le n\theta^{40}$,*

$$\mathbb{P}\Big(\neg\,\mathsf{LowSlenderExcursion}(\ell,\theta,1-\chi;\rho_n)\Big)\le\exp\Big\{-d_2\theta^{-1/2}\Big\}.$$

Theorem 1.9 will follow from our second assertion: any zigzag that is constrained to stay close to a deterministic zigzag ϕ is typically uncompetitive in weight. To this end, for ϕ any n-zigzag from $(0,0)$ to $(0,1)$, let $\mathsf{Wgt}_n^{*}\big[(x,s_1)\to(y,s_2)\,;\,(\phi,\theta,1-\chi)\text{-close}\big]$ denote the supremum of the weights of $(\phi,\theta,1-\chi)$-close zigzags ψ. Note that ψ varies over a class of zigzags; it is not assumed to be an excursion about ϕ, and may intersect ϕ. The superscript asterisk in the new notation indicates this distinction.

Let $\mathsf{LowSlenderWeight}^{*}(\ell,\theta,1-\chi;\phi)$ denote the event specified above, with Wgt_n replaced by Wgt_n^{*}.

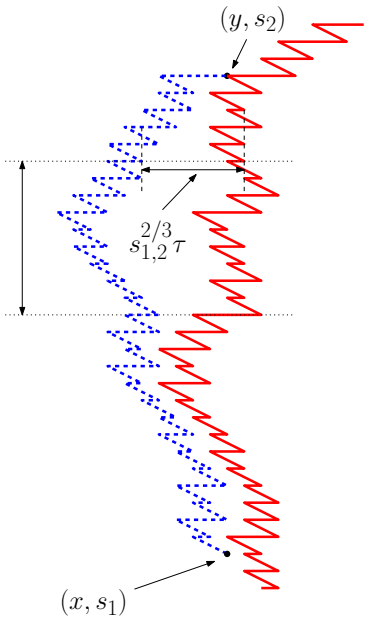

Figure 3: The dashed zigzag ψ from (x, s_1) to (y, s_2) is an excursion around the bold zigzag ϕ. The vertical double-arrowed line indicates the set of coordinates—an interval in this instance—at which departures from vertical levels differ by more than the quantity appearing in the definition of a slender excursion. Indeed, if such levels exclude s_1 and s_2 and constitute a fraction less than χ of all levels in $[s_1, s_2]$, the excursion is $(\phi, \theta, 1-\chi)$-close.

The upcoming assertion, that $\mathsf{LowSlenderWeight}^*$ is typical, will suppose a certain regularity on ϕ.

We say that an n-zigzag ϕ from $(0,0)$ to $(0,1)$ is $\mathfrak{R}$-*regular* if, whenever $h_1, h_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ and $u, v \in \mathbb{R}$ satisfy $(u, h_1), (v, h_2) \in \phi$, we have that

$$\big|v - u\big| \le h_{1,2}^{2/3}\mathfrak{R}\,. \tag{1.11}$$

Theorem 1.10. *There exist constants $d_0, C > 0$ such that we may find $\chi_0 \in (0,1)$, $d_2 > 0$ and $n_0 \in \mathbb{N}$ for which $\chi \in (0, \chi_0)$ and $n \ge n_0$ imply that, when $\theta^{-1/4} > C \log n$ and $\ell \in \mathbb{N}$ satisfies $2^\ell \le n\theta^{40}$,*

$$\mathbb{P}\Big(\neg\, \mathsf{LowSlenderWeight}^*(\ell, \theta, 1-\chi; \phi)\Big) \le \exp\big\{-d_2\theta^{-3/2}\big\}$$

for any given *zigzag ϕ from $(0,0)$ to $(0,1)$ which is $\theta^{-1/4}$-regular.*

Theorem 1.9 will follow from Theorem 1.10 and an application of the FKG inequality to the effect that conditioning on ρ_n is negative on the environment exterior to ρ_n. This is why we consider excursions, namely zigzags that are disjoint from ρ_n. Theorem 1.10 is then applicable because $\mathsf{LowSlenderExcursion}(\ell, \theta, 1-\chi; \rho_n)$ is a decreasing event on this exterior environment. Theorem 1.4 will show that ρ_n typically is suitably regular.

1.11 There are few cliffs along the geodesic

The geodesic Γ_n from $(0,0)$ to (n,n) progresses in a roughly diagonal fashion, even at the microscopic scale. We now state this *few cliffs'* assertion, using unscaled coordinates because these are suited to discussing the microscopic scale.

For $n \in \mathbb{N}$, let $\Gamma_n \subset [0,n]^2$ denote the almost surely unique staircase of maximum energy with starting and ending points $(0,0)$ and (n,n) in static Brownian LPP. For $A \in \mathbb{N}$ a positive integer, we divide the vertical coordinate interval $[0,n]$ into consecutive subintervals of lengths A, as well as a remaining subinterval of shorter length if need be. We set m to be the greatest integer *strictly* less than n/A, so that there are m subintervals of length A. There is also one remaining interval, whose length is at least one and at most A.

We record a sequence $\{X_i : i \in [\![0, m+1]\!]\}$ of horizontal coordinates of departure of Γ_n from the horizontal borders of the consecutive strips indexed by the vertical subintervals that we have just defined. Formally, when $i \in [\![0,m]\!]$, X_i is equal to the supremum of those $x \in [0,n]$ for which $(x, iA) \in \Gamma_n$; while for $i = m+1$, $X_i = \sup\{x \in [0,n] : (x,n) \in \Gamma_n\}$, so that $X_i = n$.

Now set, for each such index i, $Z_i = \lfloor X_i \rfloor$. The sequence $\{Z_i : i \in [\![0, m+1]\!]\}$ is a non-decreasing list of integers lying in $[\![0,n]\!]$ that offers a unit-scale coarse-grained description of the horizontal progress of the staircase Γ_n as consecutive vertical milestones at separation A are passed. This description is equally captured by the difference function

$$\Psi : [\![0,m]\!] \to [\![0,n]\!]\,,\ \Psi(i) = Z_{i+1} - Z_i\,. \tag{1.12}$$

Note that $\sum_{i=0}^{m} \Psi(i) = Z_{m+1} - Z_0$ is at most n. We now specify a set $\mathcal{I}$ of indices marking slow horizontal advance—cliffs in the graph of Γ_n; we set $\mathcal{I}(\Gamma_n)$ equal to the set of $i \in [\![0,m]\!]$ for which $\Psi(i)$ is at most two.

Theorem 1.11. *There exist $A > 0$, $\alpha_0 \in (1/2, 1)$, $h > 0$ and $n_0 \in \mathbb{N}$, such that, when $n \geq n_0$ and $\alpha \geq \alpha_0$ satisfies $\alpha m \in \mathbb{N}$ where $m = \lfloor n/A \rfloor$,*

$$\mathbb{P}\Big(\big|\mathcal{I}(\Gamma_n)\big| < \alpha m\Big) \geq 1 - e^{-hn}\,.$$

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2 Brownian regularity and twin peaks

Here we prove the Brownianity and twin peaks' rarity assertions Theorems 1.2 and 1.3 concerning the routed weight profile $x \to Z_n(x,a)$. We begin by noting some similiarities between the specification (1.1) of this profile in heuristic discussion and the actual definition at the start of Section 1.8. Recall the notation $\rho_n(a)$ that has been specified in Subsections 1.7.7 and 1.7.8.

Lemma 2.1. *Set $a_+ = a + n^{-1}$ and $x^- = x - 2^{-1}n^{-2/3}$.*

1. *The routed weight profile is given by a sum of independent weight profiles,*

$$Z_n(x,a) \,=\, \mathsf{Wgt}_n\big[(0,0) \to (x,a)\big] \,+\, \mathsf{Wgt}_n\big[(x^-, a_+) \to (0,1)\big]\,. \tag{2.1}$$

2. *Almost surely, the maximizer of $Z_n(\cdot, a)$, namely the value of $x \in \mathbb{R}$ for which $Z_n(x,a)$ equals the supremum of $Z_n(z,a)$ over $z \in \mathbb{R}$, is unique and equals $\rho_n(a)$.*

Proof: (1). Let ψ denote an n-zigzag that begins at $(0,0)$, ends at $(0,1)$, and for which $x = \sup\{z \in \mathbb{R} : (z,a) \in \psi\}$. Let ψ^- denote the initial zigzag of ψ that ends at (x,a). Note that ψ reaches $\mathbb{R} \times \{a + n^{-1}\}$ at (x^-, a_+). Let ψ^+ denote the final sub-zigzag of ψ

that begins at (x^-, a_+). Thus, $\mathsf{Wgt}_n(\psi) = \mathsf{Wgt}_n(\psi^-) + \mathsf{Wgt}_n(\psi^+)$. By definition, $Z_n(x,a)$ equals the supremum of $\mathsf{Wgt}_n(\psi)$ over such ψ. We see that $Z_n(x,a)$ is at most the right-hand side of (2.1). But equality may be obtained by varying (ψ^-, ψ^+) subject to the endpoint constraints that specify this pair. Moreover, the two right-hand terms in (2.1) are independent because they are respectively measurable with respect to randomness indexed by the disjoint regions $\mathbb{R} \times [0,a]$ and $\mathbb{R} \times [a + n^{-1}, 1]$.

(2). The polymer ρ_n is almost surely unique by [Ham19b, Lemma $4.6(1)$]. Since $\rho_n(a)$ is by definition the location of departure of the polymer ρ_n from $\mathbb{R} \times \{a\}$, we see that it is the maximizer of $x \to Z_n(x,a)$. □

The notation a_+ and x^- is adopted henceforth. It reflects the two denoted quantities being merely microscopically perturbed copies of a and x.

The proof of Theorem 1.3 will harness Theorem 1.2. The derivation of the latter result is comprised of four steps; two further steps will yield the former.

1. The two right-hand weights in the formula (2.1) for the routed weight profile may be viewed as functions of the variable $x \in \mathbb{R}$. In a simple a-dependent change of coordinates, we will present *normalized* counterparts to these profiles. These normalized profiles are Brownian of rate one on the unit scale in a sense that is uniform in a and n; they are globally governed by the curvature of a shared parabola, $x \to -2^{-1/2}x^2$.
2. We will recall from [Ham20] that any normalized profile may be embedded via the Robinson-Schensted-Knuth correspondence as the uppermost curve of a *regular* ensemble. The latter object is a random ordered system of continuous curves which enjoy an attractive probabilistic resampling, the Brownian Gibbs property, alongside certain boundary conditions.
3. We will recall from [CHH19] that the curves in a regular ensemble enjoy strong similarity to Brownian motion of rate one, in the sense of Definition 1.1.
4. The profile $x \to Z_n(x,a)$ is thus seen via (2.1) to be the sum of two independent processes that bear a demanding comparison to standard Brownian motion; it would thus seem—and we will prove—that this profile withstands such a comparison to Brownian motion of rate two. In this way, we will obtain Theorem 1.2.
5. Twin peaks are rare for Brownian motion (of rate two).
6. Thus, and as Theorem 1.3 asserts, they are also rare for the profile $x \to Z_n(x,a)$.

The six ensuing subsections give rigorous renderings of these respective steps.

2.1 Forward and backward weight profiles

By the formula (2.1), the routed weight profile is exhibited as a sum of two independent random processes. The first may be labelled 'forward', because the origin $(0,0)$ is fixed, and the spatial variable x is attached to the more advanced height $a \in (0,1)$. The latter process may be called 'backward', because the fixed point $(0,1)$ is more advanced that the height $a^+ = a + n^{-1}$ of the endpoint (x^-, a^+) that varies with x.

It is useful to discuss further the forward and backward processes, and we consider a given compatible triple $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$ in order to do so. Consider the forward weight profile, given by

$$\mathsf{Wgt}_n\big[(0,s_1) \to (\cdot, s_2)\big] : \big[-2^{-1}n^{1/3}s_{1,2}, \infty\big) \to \mathbb{R}\,;$$

and the backward profile,

$$\mathsf{Wgt}_n\big[(\cdot,s_1) \to (0, s_2)\big] : \big(-\infty, 2^{-1}n^{1/3}s_{1,2}\big] \to \mathbb{R}\,.$$

It is valuable to vividly picture these two profiles. Each locally resembles Brownian motion but globally follows the contour of the parabola $-2^{-1/2}(y-x)^2 s_{1,2}^{-1}$ as a function of y or x in the forward or backward case. Each profile adopts values of order $s_{1,2}^{1/3}$ when x and y differ by an order of $s_{1,2}^{2/3}$. More negative values, dictated by parabolic curvature, are witnessed outside this region. This description holds sway in a region that expands from the origin as the parameter n rises.

Clearly, then, our profiles have fundamental differences according to the value of $s_{1,2}$: sharply peaked ensemble curves when $s_{1,2}$ is small, and much flatter curves when $s_{1,2}$ is large. A simple further parabolic transformation will serve to put the profiles on a much more equal footing. Since the profiles are already scaled objects, we will use the term 'normalized' to allude to the newly transformed counterparts.

That is, we define the *normalized* forward profile

$$\mathsf{Nr}\mathcal{L}^{\uparrow;s_2}_{n;(x,s_1)} : \big[-2^{-1}(ns_{1,2})^{1/3}, \infty\big) \to \mathbb{R}\,,$$

setting, for $z \geq -2^{-1}n^{1/3}s_{1,2}^{-2/3}$,

$$\mathsf{Nr}\mathcal{L}^{\uparrow;s_2}_{n;(x,s_1)}(z) = s_{1,2}^{-1/3}\,\mathsf{Wgt}_n\big[(x,s_1) \to (x + s_{1,2}^{2/3}z, s_2)\big]\,. \tag{2.2}$$

The normalized backward profile $\mathsf{Nr}\mathcal{L}^{\downarrow;(y,s_2)}_{n;s_1} : \big(-\infty, 2^{-1}(ns_{1,2})^{1/3}\big] \to \mathbb{R}$ is specified by setting

$$\mathsf{Nr}\mathcal{L}^{\downarrow;(y,s_2)}_{n;s_1}(z) = s_{1,2}^{-1/3}\,\mathsf{Wgt}_n\big[(y + s_{1,2}^{2/3}z, s_1) \to (y, s_2)\big] \tag{2.3}$$

for $z \leq 2^{-1}n^{1/3}s_{1,2}^{-2/3}$.

The new curves locally resemble Brownian motion as before, but they have been centred and squeezed so that now the parabola that dictates their overall shape is $-2^{-1/2}z^2$. This picture is accurate in a region that expands as the parameter $ns_{1,2}$ rises.

2.2 Brownian Gibbs line ensembles

The Robinson-Schensted-Knuth correspondence permits any given forward and backward weight profile to be embedded as the uppermost curve in an ordered system—or line ensemble—of random continuous curves that enjoy an attractive and valuable probabilistic resampling called the Brownian Gibbs property. The notion of a Brownian Gibbs line ensemble was introduced in [CH14] to capture a system of ordered curves that arise by conditioning Brownian motions or bridges on mutual avoidance. The precise definition is not logically needed in this article, but we offer an informal summary next, and then indicate how our normalized profiles satisfy this definition.

2.2.1 An overview

Let $n \in \mathbb{N}$ and let I be a closed interval in the real line. A $[\![1,n]\!]$-indexed line ensemble defined on I is a random collection of continuous curves $\mathcal{L} : [\![1,n]\!] \times I \to \mathbb{R}$ specified under a probability measure $\mathbb{P}$. The i^{th} curve is thus $\mathcal{L}(i,\cdot) : I \to \mathbb{R}$. (The adjective 'line' has been applied to these systems perhaps because of their origin in such models as Poissonian LPP, where the counterpart object has piecewise constant curves. We will omit it henceforth.) An ensemble is called *ordered* if $\mathcal{L}(i,x) > \mathcal{L}(i+1,x)$ whenever $i \in [\![1,n-1]\!]$ and x lies in the interior of I. The curves may thus assume a common value at any finite endpoint of I. We will consider ordered ensembles that satisfy a condition called the Brownian Gibbs property. Colloquially, we may say that an ordered ensemble is called Brownian Gibbs if it arises from a system of Brownian bridges or Brownian motions defined on I by conditioning on the mutual avoidance of the curves at all times in I.

2.2.2 Defining (c, C)-regular ensembles

We are interested in ensembles that are not merely Brownian Gibbs but that hew to the shape of a parabola and have one-point distributions for the uppermost curve that enjoy tightness properties. We will employ the next definition, which specifies a $(\bar{\phi}, c, C)$-regular ensemble from [Ham22, Definition 2.4], in the special case where the vector $\bar{\phi}$ equals $(1/3, 1/9, 1/3)$.

Definition 2.2. *Consider a Brownian Gibbs ensemble of the form*

$$\mathcal{L} : [\![1, N]\!] \times \big[-z_{\mathcal{L}}, \infty\big) \to \mathbb{R}\,,$$

and which is defined on a probability space under the law $\mathbb{P}$. The number $N = N(\mathcal{L})$ of ensemble curves and the absolute value $z_{\mathcal{L}}$ of the finite endpoint may take any values in $\mathbb{N}$ and $[0, \infty)$.

Let $Q : \mathbb{R} \to \mathbb{R}$ denote the parabola $Q(x) = 2^{-1/2}x^2$.

Let C and c be two positive constants. The ensemble $\mathcal{L}$ is said to be (c, C)-regular if the following conditions are satisfied.

1. **Endpoint escape.** $z_{\mathcal{L}} \geq cN^{1/3}$.
2. **One-point lower tail.** *If $z \in [-z_{\mathcal{L}}, \infty)$ satisfies $|z| \leq cN^{1/9}$, then*
$$\mathbb{P}\Big(\mathcal{L}\big(1, z\big) + Q(z) \leq -s\Big) \leq C \exp\big\{ - cs^{3/2}\big\}$$
for all $s \in \big[1, N^{1/3}\big]$.
3. **One-point upper tail.** *If $z \in [-z_{\mathcal{L}}, \infty)$ satisfies $|z| \leq cN^{1/9}$, then*
$$\mathbb{P}\Big(\mathcal{L}\big(1, z\big) + Q(z) \geq s\Big) \leq C \exp\big\{ - cs^{3/2}\big\}$$
for all $s \in [1, \infty)$.

We will call these conditions $\mathrm{Reg}(1)$, $\mathrm{Reg}(2)$ and $\mathrm{Reg}(3)$.

A Brownian Gibbs ensemble of the form

$$\mathcal{L} : [\![1, N]\!] \times \big(-\infty, z_{\mathcal{L}}\big] \to \mathbb{R}$$

is also said to be (c, C)-regular if the reflected ensemble $\mathcal{L}(\cdot, -\cdot)$ is. This is equivalent to the above conditions when instances of $[-z_{\mathcal{L}}, \infty)$ are replaced by $(-\infty, z_{\mathcal{L}}]$.

2.2.3 The normalized forward and backward profiles may be embedded in regular ensembles

We say that a random function of the form $\mathcal{L} : [-z_{\mathcal{L}}, \infty) \to \mathbb{R}$ or $\mathcal{L} : (-\infty, z_{\mathcal{L}}] \to \mathbb{R}$ is (c, C, m)-regular if there exists an m-curve (c, C)-regular ensemble of which it is the lowest indexed curve.

Our reason for invoking the theory of regular Brownian Gibbs ensembles is that the normalized Brownian LPP profiles are regular.

Proposition 2.3. *There exist values for the positive parameters C and c such that, for $n \in \mathbb{N}$ and $a \in n^{-1}\mathbb{Z} \cap (0, 1)$, the following hold.*

1. *The process $\mathsf{Nr}\mathcal{L}^{\uparrow;a}_{n;(0,0)} : \big[-2^{-1}(na)^{1/3}, \infty\big) \to \mathbb{R}$ is $\big(c, C, na + 1\big)$-regular.*
2. *The process $\mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a_+} : \big(-\infty, 2^{-1}\big(n(1 - a_+)\big)^{1/3}\big] \to \mathbb{R}$ is $\big(c, C, n(1 - a)\big)$-regular.*

Proof. Values of C and c that validate these two statements are offered by [Ham20, Proposition 4.2]. □

The reader may consult the fifth paragraph of Section 5.8 of [Ham20] for a point of departure to the proof of [Ham20, Proposition 4.2].

2.3 The Brownianity of the narrow wedge weight profile

We now state the principal result of [CHH19], asserting the Brownianity of scaled Brownian LPP polymer weight profiles in the narrow wedge case (recall the notion of approximate Brownianity from Definition 1.1), when one endpoint is fixed, and the other varies horizontally. This conclusion is expressed in terms of regular ensembles in [CHH19]. Our concern is merely with the uppermost curve and we record the result only in this case.

Theorem 2.4. *Suppose that $\mathcal{L}_m$ is an m-curve (c,C)-regular ensemble for some $m \in \mathbb{N}$ and $C, c \in (0,\infty)$. Let $d \geq 1$ and $K \in \mathbb{R}$ satisfy $[K-d, K+d] \subset c/2 \cdot [-m^{1/9}, m^{1/9}]$. There exist values of the positive parameters g and G, chosen without dependence on m, K or d, such that the random function $[K-d, K+d] \to \mathbb{R} : x \to \mathcal{L}_m(1,x) + 2^{1/2}Kx$ is $\big(g, G, d, m, 1\big)$-Brownian.*

Proof. The result follows from [CHH19, Theorem 3.11] by considering the curve with the lowest index $k = 1$ in the ensemble $\mathcal{L}_m$. We now indicate conditions on the claimed constants G and g that render valid the application of this theorem—a mundane check phrased in terms of parameters D_1 and C_1 from the quoted result. We choose G to be at least the value of this parameter as specified in [CHH19, Theorem 3.11] while satisfying $G \geq 24^6 D_1^{-3}$ and $G \geq 4932 D_1^{5/2}$. We choose $g > 0$ to satisfy $g \leq e^{-1} \wedge (17)^{-1} C_1^{-1} D_1^{-1}$. We further demand that $g \leq \big(c/2 \wedge 2^{1/2}\big) D_1^{-1}$; and ensure that this parameter is small enough that $\exp(-g m_0^{1/2}) \geq e^{-1}$ for $m_0 = (c/3)^{-18} \vee 6^{36}$, in order that the condition $e^{-g m^{1/12}} \leq \eta$ be impossible to satisfy unless $m \geq m_0$. We impose the lower bound of e^{-1} since the upper bound that η is assumed to satisfy in Definition 1.1 implies that $\eta \leq e^{-1}$ in view of $G, d \geq 1$. □

2.4 Brownianity for the routed weight profile

We now prove Theorem 1.2. We begin by rewriting the basic formula Lemma 2.1(1) in normalized form:

$$Z_n(x,a) = a^{1/3} \mathsf{Nr}\mathcal{L}^{\uparrow;a}_{n;(0,0)}\big(a^{-2/3}x\big) + (1-a_+)^{1/3} \mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a_+}\big((1-a_+)^{-2/3}x^-\big)\,. \tag{2.4}$$

Theorem 1.2 is concerned with the Brownian character of the profile $x \to Z_n(x,a)$ in a neighbourhood of a given $R \in \mathbb{R}$. This profile is curved parabolically in a global sense, so that a suitable drift must be identified for the Brownian motion with which we seek to make comparison. We find the drift by determining the counterpart drifts for the two right-hand terms in (2.4). We are thus led to make a further simple change of coordinates.

Set $\ell : \mathbb{R}^2 \to \mathbb{R}$ to equal $\ell(x,y) = -2^{-1/2}x^2 - 2^{1/2}x(y-x)$; thus, $y \to \ell(x,y)$ is the line tangent at $x \in \mathbb{R}$ to the parabola $z \to -2^{-1/2}z^2$.

Let $s \in n^{-1}\mathbb{Z} \cap (0,1)$. For $x \in \mathbb{R}$, define the *shifted* forward profile

$$\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\uparrow;s}_{n;x;(0,0)} : \big[-2^{-1}(ns)^{1/3} - x, \infty\big) \to \mathbb{R}$$

by setting

$$\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\uparrow;s}_{n;x;(0,0)}(z) = \mathsf{Nr}\mathcal{L}^{\uparrow;s}_{n;(0,0)}(x+z) - \ell\big(x, x+z\big)\,.$$

For $y \in \mathbb{R}$, define the *shifted* backward profile $\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\downarrow;(0,1)}_{n;y;a} : \big(-\infty, 2^{-1}(ns_{1,2})^{1/3} - y\big] \to \mathbb{R}$ by setting

$$\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\downarrow;(0,1)}_{n;y;a}(z) = \mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a}(y+z) - \ell\big(y, y+z\big)\,.$$

Further set

$$Z^{\uparrow}_n(x,a) = a^{1/3} \mathsf{Nr}\mathcal{L}^{\mathrm{shift}\uparrow;a}_{n;a^{-2/3}R;(0,0)}\big(a^{-2/3}(x-R)\big) \tag{2.5}$$

and

$$Z_n^{\downarrow}(x,a) = (1-a_+)^{1/3}\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\downarrow;(0,1)}_{n;(1-a_+)^{-2/3}R;a_+}\Big(\big(1-a_+\big)^{-2/3}(x^- - R)\Big)\,.$$

Since $\ell(x,y) = 2^{-1/2}x^2 - 2^{1/2}xy$, we find then from (2.4) that

$$Z_n(x,a) = Z_n^{\uparrow}(x,a) + Z_n^{\downarrow}(x,a) + \Theta(x) + \mathcal{E}(x)\,, \tag{2.6}$$

where $\Theta(x) = 2^{-1/2}\big(a(1-a)\big)^{-1}R(R-2x)$ and

$$\mathcal{E}(x) = 2^{-1/2}\big((1-a_+)(1-a)\big)^{-1}R(R-2x)n^{-1} + 2^{-1/2}(1-a_+)^{-1}Rn^{-2/3}\,.$$

In the next result, we see how shifted coordinates, which have made possible the formula (2.6), put us in excellent shape to derive Theorem 1.2. Indeed, as we will argue shortly, largely on the basis of the upcoming lemma and Theorem 2.4, the first two right-hand terms of this formula are independent processes that are very similar to Brownian motion of rate one; while the third term Θ records the drift inherited from parabolic curvature that is manifest in the locale of the location $R \in \mathbb{R}$.

Lemma 2.5. *There exist values for the positive parameters C, c, G and g such that, for $n \in \mathbb{N}$ and $a \in n^{-1}\mathbb{Z} \cap (0,1)$, the following hold.*

1. *Suppose that $|x| \le c/2 \cdot (na+1)^{1/9}$. Then the process $\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\uparrow;a}_{n;x;(0,0)} : \big[-2^{-1}(na)^{1/3} - x, \infty\big) \to \mathbb{R}$ is $\big(c/2, C, na+1\big)$-regular.*
2. *Suppose that $|y| \le c/2 \cdot \big(n(1-a)\big)^{1/9}$. Then the process*

$$\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\downarrow;(0,1)}_{n;y;a} : \big(-\infty, 2^{-1}\big(n(1-a)\big)^{1/3} - y\big] \to \mathbb{R}$$

   is $\big(c/2, C, n(1-a)\big)$-regular.
3. *The processes $Z_n^{\uparrow}(\cdot,a), Z_n^{\downarrow}(\cdot,a) : [R, R+\ell] \to \mathbb{R}$ are independent.*
4. *Suppose that $|R| \le 2^{-1}cn^{1/9}a^{7/9}$. Let $\ell \in \mathbb{R}$ satisfy $\ell - R \le 2^{-1}n^{1/3}a$. Then the process $Z_n^{\uparrow}(\cdot,a) : [R-\ell, R+\ell] \to \mathbb{R}$ is $\big(g, Ga^{-4}, \ell, an+1, 1\big)$-Brownian.*
5. *Suppose that $|R| \le 2^{-1}cn^{1/9}(1-a_+)^{7/9}$. Let $\ell \in \mathbb{R}$ satisfy $R + \ell \le 2^{-1}n^{1/3}(1-a_+)$. Then the process $Z_n^{\downarrow}(\cdot,a) : [R-\ell, R+\ell] \to \mathbb{R}$ is $\big(g, G(1-a_+)^{-4}, \ell, (1-a)n, 1\big)$-Brownian.*

Proof. (1,2). These are due to [Ham19a, Lemma 3.4] or [Ham22, Lemma 2.26].

(3). The regions $\mathbb{R} \times [0,a]$ and $\mathbb{R} \times [a+n^{-1}, 1]$ that respectively specify $Z_n^{\uparrow}(\cdot,a)$ and $Z_n^{\downarrow}(\cdot,a)$ are disjoint, so that these processes are independent.

(4). By the third part of the lemma, and Theorem 2.4, there exist positive values G and g such that $\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\uparrow;a}_{n;a^{-2/3}R;(0,0)} : [-a^{-2/3}\ell, a^{-2/3}\ell] \to \mathbb{R}$ is $\big(g, G, a^{-2/3}\ell, an+1, 1\big)$-Brownian, where we have used $[-a^{-2/3}\ell, a^{-2/3}\ell] \subset \big[-2^{-1}(na)^{1/3} - a^{-2/3}R, \infty\big)$ and $a^{-2/3}|R| \le 2^{-1}c(na+1)^{1/9}$. The former condition is implied by $a^{-2/3}\ell \le 2^{-1}(na)^{1/3} + a^{-2/3}R$ and thus by our hypothesis that $\ell - R \le 2^{-1}n^{1/3}a$. The latter is implied by $|R| \le 2^{-1}cn^{1/9}a^{7/9}$.

In light of this, and (2.5), we may apply the next presented Lemma 2.6 with $\kappa = a$ to verify Lemma 2.5(6) holds.

(5). Invoking similarly the fourth part of the lemma,

$$\mathsf{Nr}\mathcal{L}^{\mathrm{shift}\downarrow;(0,1)}_{n;(1-a_+)^{-2/3}R;a_+} : \big[-(1-a_+)^{-2/3}\ell, (1-a_+)^{-2/3}\ell\big] \to \mathbb{R}$$

is seen to be $\big(g, G, (1-a_+)^{-2/3}\ell, (1-a_+)n, 1\big)$-Brownian. This time, we need

$$\big[-(1-a_+)^{-2/3}\ell, (1-a_+)^{-2/3}\ell\big] \subset \big(-\infty, 2^{-1}\big(n(1-a_+)\big)^{1/3} - (1-a_+)^{-2/3}R\big]$$

and $(1-a_+)^{-2/3}|R| \le 2^{-1}c(n(1-a_+)+1)^{1/9}$. The former condition is implied by $R+\ell \le 2^{-1}n^{1/3}(1-a_+)$; the latter by $|R| \le 2^{-1}cn^{1/9}(1-a_+)^{7/9}$. □

The a-dependent spatial-temporal scaling in (2.5) respects Brownian motion; it is unsurprising then that this scaling in essence leaves invariant a property of similarity to Brownian motion. The next result is a rigorous interpretation of this notion.

Lemma 2.6. *Let $G, g, \ell > 0$, $\kappa \in (0,1)$, $m \in \mathbb{N}$ and $R \in \mathbb{R}$. Suppose that the random function $\mathcal{L} : \big[-\kappa^{-2/3}\ell, \kappa^{-2/3}\ell\big] \to \mathbb{R}$ is $\big(g, G, \kappa^{-2/3}\ell, m, 1\big)$-Brownian. Set $\mathcal{L}^\kappa(x) = \kappa^{1/3}\mathcal{L}\big(\kappa^{-2/3}(x-R)\big)$. Then $\mathcal{L}^\kappa : [R-\ell, R+\ell] \to \mathbb{R}$ is $\big(g, G\kappa^{-4}, \ell, m, 1\big)$-Brownian.*

Proof. Recalling the notation of Definition 1.1, let $B \subseteq \mathcal{C}_{0,*}\big([R-\ell, R+\ell], \mathbb{R}\big)$. Set

$$B^* = \Big\{ f \in \mathcal{C}_{0,*}\big([-\kappa^{-2/3}\ell, \kappa^{-2/3}\ell], \mathbb{R}\big) : x \to \kappa^{1/3} f\big(\kappa^{-2/3}(x-R)\big) \in B \Big\}.$$

Then Brownian scaling implies that, when B is Borel measurable, $\mathcal{B}_{0,*}^{[0,\kappa^{-2/3}\ell]}(B^*) = \mathcal{B}_{0,*}^{[R,R+\ell]}(B)$.

Taking $I = \big[-\kappa^{-2/3}\ell, \kappa^{-2/3}\ell\big]$, the process $\mathcal{L} : \big[-\kappa^{-2/3}\ell, \kappa^{-2/3}\ell\big] \to \mathbb{R}$ meets the condition on $X : I \to \mathbb{R}$ in Definition 1.1 for the parameter quintet $\big(g, G, \kappa^{-2/3}\ell, m, 1\big)$. It is our task to verify that $\mathcal{L}^\kappa : [R-\ell, R+\ell] \to \mathbb{R}$ does so for the quintet $\big(g, G\kappa^{-4}, \ell, m, 1\big)$. The third parameter, which is one-half the length of the domain interval, has decreased, by a factor of $\kappa^{2/3}$. By the preceding paragraph, the value of the Brownian probability η is shared in the definition as it applies to the processes $\mathcal{L}$ and $\mathcal{L}^\kappa$. If we denote the second element of the latter quintet by G', we may note that we must demand of it that $G' \ge G\big(\kappa^{-2/3}\big)^6 = G\kappa^{-4}$—so that the hypothesis $\eta \le e^{-G'\ell^6}$ is implied by $\eta \le \exp\big\{-G\big(\kappa^{-2/3}\ell\big)^6\big\}$; and that $G' \ge G\kappa^{-2/3}$—so that $G'\ell \ge G\kappa^{-2/3}\ell$ may be applied to obtain the right-hand side in the display in Definition 1.1. The choice $G' = G\kappa^{-4}$ meets these two requirements. Since the three further parameters, g, G and m, transmit unaltered, we obtain Lemma 2.6. □

Each profile $x \to Z_n^{\uparrow}(x,a)$ and $x \to Z_n^{\downarrow}(x,a)$ will shortly be shown to be very similar to standard Brownian motion by an argument that harnesses Lemmas 2.5 and 2.6 to the fundamental estimate Theorem 2.4. The profile $x \to Z_n(x,a)$, after linear adjustment, will then be seen via (2.6) to resemble Brownian motion of rate two (as Theorem 1.2 asserts), provided that we argue that Brownianity in the sense of Definition 1.1 is stable under addition of processes. After we establish this in Lemma 2.8, we will be ready to give a short proof of Theorem 1.2. First, however, we present a result that permits us to dispense with Definition 1.1's inconsequential but practically irksome Brownian probability hypothesis $\eta \le g \wedge e^{-Gd^d}$.

Lemma 2.7. *Under the circumstances of Definition 1.1, suppose on the parameter η, instead of the condition $e^{-gm^{1/12}} \le \eta \le g \wedge e^{-Gd^6}$, that merely the lower bound $\eta \ge e^{-gm^{1/12}}$ holds. Suppose also that $g \in (0,1)$ and that $G, d \ge 1$. Then*

$$\begin{aligned} &\mathbb{P}\Big(I \to \mathbb{R} : x \to X(x) - X(K-d) \text{ belongs to } A\Big) \\ \le\ & \eta \cdot \exp\Big\{5G^{11/6}d^6 g^{-5/6}\Big\} \exp\Big\{Gd\big(\log \eta^{-1}\big)^{5/6}\Big\}. \end{aligned}$$

Proof. Note that by Theorem 2.4 the condition $e^{-gm^{1/12}} \le \eta$ implies that

$$\mathbb{P}\Big(I \to \mathbb{R} : x \to X(x) - X(K-d) \text{ belongs to } A\Big) \le \eta \cdot HG \exp\Big\{Gd\big(\log \eta^{-1}\big)^{5/6}\Big\}, \quad (2.7)$$

where $H = \big(\max\big\{g^{-1}, e^{Gd^6}\big\} + 1\big) \exp\Big\{Gd\big(\log \max\big\{g^{-1}, e^{Gd^6}\big\}\big)^{5/6}\Big\}$. The above follows from Theorem 2.4 since $H \ge 1$ and moreover for $\eta \ge g \wedge e^{-Gd^6}$, we have $\eta H \ge 1$ while $G \exp\Big\{Gd\big(\log \eta^{-1}\big)^{5/6}\Big\}$ is always at least 1 since $G \ge 1$ by hypothesis.

Since $g \in (0,1)$, note that

$$H \le 2g^{-1}e^{Gd^6}\exp\Big\{Gd\big(\log(g^{-1}e^{Gd^6})\big)^{5/6}\Big\} \le 2g^{-1}e^{Gd^6}\exp\Big\{G^{11/6}d^6g^{-5/6}\Big\},$$

where we have applied $ae^x \le e^{ax}$ for $a, x \ge 1$ with $a = g^{-1}$ and $x = Gd^6$. Thus,

$$\begin{aligned} GH &\le 2g^{-1}Ge^{Gd^6}\exp\Big\{G^{11/6}d^6g^{-5/6}\Big\} \\ &\le g^{-1}e^{3Gd^6}\exp\Big\{G^{11/6}d^6g^{-5/6}\Big\} \le g^{-1}\exp\Big\{4G^{11/6}d^6g^{-5/6}\Big\}, \end{aligned}$$

where we used $G \ge 1 \ge g$ and $d \ge 1$. Noting that $g^{-1} \le \exp\{g^{-5/6}\}$ alongside $G, d \ge 1$ completes the proof. □

Recall Definition 1.1.

Lemma 2.8 (Additive stability for Brownian regularity)**.** *Let $\nu_i \in (0,\infty)$; $n_i \in \mathbb{N}$ for $i \in \{1,2\}$; $g, G, \ell > 0$; and $R \in \mathbb{R}$. Let the random functions $X, Y : [R-\ell, R+\ell] \to \mathbb{R}$ be independent under the law $\mathbb{P}$. Suppose that X is $\big(g, G, \ell, m_1, \nu_1\big)$-Brownian, and that Y is $\big(g, G, \ell, m_2, \nu_2\big)$-Brownian. Then $X + Y$ is $\big(g, G', \ell, m_1 \wedge m_2, \nu_1 + \nu_2\big)$-Brownian, where G' is a multiple of $G^{17/6}g^{-5/6}\ell^6$ by an absolute positive factor.*

Proof. Write $\mathcal{C} = \mathcal{C}_{0,*}\big(I, \mathbb{R}\big)$ where $I = [R-\ell, R+\ell]$. Our argument will rely on analysing the different values of the Radon-Nikodym derivative of $\hat{X} := X(\cdot) - X(R-\ell)$ with respect to Brownian motion. So we first apply the Lebesgue decomposition theorem to obtain a Borel set $\mathcal{X}_1 \subset \mathcal{C}$ such that $\hat{X}$ is absolutely continuous with respect to $\mathcal{B}_{0,*}^{\nu_1+\nu_2;I}$ on $\mathcal{X}_1$ and $\mathcal{B}_{0,*}^{\nu_1+\nu_2;I}(\mathcal{X}_1^c) = 0$. Similarly for $\hat{Y}$, we define $\mathcal{X}_2$.

We first claim the following bounds:

$$\begin{aligned} \mathbb{P}(\hat{X} \in \mathcal{X}_1^c) &\le G\exp\Big\{5G\ell g^{-5/6}m_1^{5/72}\Big\}e^{-gm_1^{1/12}} \\ \mathbb{P}(\hat{Y} \in \mathcal{X}_2^c) &\le G\exp\Big\{5G\ell g^{-5/6}m_2^{5/72}\Big\}e^{-gm_2^{1/12}}. \end{aligned} \tag{2.8}$$

We only discuss the first bound since the argument for the second is similar. Note that since $\mathcal{B}_{0,*}^{\nu_1+\nu_2;I}(\mathcal{X}_1^c) = 0$, one cannot directly appeal to the fact that X is $\big(g, G, \ell, m_1, \nu_1\big)$-Brownian. However, a simple enlargement argument gives a Borel set S such that $\mathcal{X}_1^c \subset S$ and $\mathcal{B}_{0,*}^{\nu_1+\nu_2;I}(S) = e^{-gm_1^{1/12}}$. Thus we get

$$\mathbb{P}(\hat{X} \in \mathcal{X}_1^c) \le \mathbb{P}(\hat{X} \in S) \le G\exp\Big\{G\ell g^{5/6}m_1^{5/72}\Big\}e^{-gm_1^{1/12}}.$$

Next, let $A \subset \mathcal{C}$ be Borel measurable. Set $\sigma = \mathcal{B}_{0,*}^{\nu_1+\nu_2;I}(A)$, and suppose that σ is at least the quantity $\exp\big\{-g(m_1 \wedge m_2)^{1/12}\big\}$. For $f \in \mathcal{C}$, set $A_f = \big\{g \in \mathcal{C} \cap \mathcal{X}_2 : f + g \in A\big\}$.

Now let $F : \mathcal{C} \to [0,\infty)$ denote the Radon-Nikodym derivative of the law of $\hat{X}$ with respect to the law $\mathcal{B}_{0,*}^{\nu_1;I}$ on $\mathcal{X}_1$; and let $G : \mathcal{C} \to [0,\infty)$ denote the counterpart with the replacements $\hat{X} \to \hat{Y}$, $\nu_1 \to \nu_2$ and $\mathcal{X}_1 \to \mathcal{X}_2$ made.

For $k \in \mathbb{N}$, write $D_k = \big\{f \in \mathcal{C} : 2^k \le G(f) < 2^{k+1}\big\}$ for $k \ge 1$ and let $D_0 = \big\{f \in \mathcal{C} : G(f) < 2\big\}$. Set $\eta_k = \mathcal{B}_{0,*}^{\nu_2;I}\big(\cup_{j=k}^{\infty} D_j\big)$.

Note that $\mathbb{P}\big(\hat{Y} \in \cup_{j=k}^{\infty} D_j\big) \ge 2^k\eta_k$. However, by Lemma 2.7, the condition

$$e^{-gm_2^{1/12}} \le \eta_k\,, \tag{2.9}$$

implies that

$$\mathbb{P}\big(\hat{Y} \in \cup_{j=k}^{\infty} D_j\big) \;\le\; \exp\Big\{5G^{11/6}\ell^6 g^{-5/6}\Big\}\exp\Big\{G\ell\big(\log\eta_k^{-1}\big)^{5/6}\Big\}\eta_k\,. \tag{2.10}$$

Thus, when (2.9) holds, $2^k \le \hat{G}\exp\{G\ell(\log\eta_k^{-1})^{5/6}\}$ with $\hat{G} = \exp\{5G^{11/6}\ell^6 g^{-5/6}\}$, whence

$$\eta_k \le \exp\{-(G\ell)^{-6/5}\big(k\log 2 - \log\hat{G}\big)^{6/5}\}\,.$$

Hence, $\eta_k \le \exp\{-(G\ell)^{-6/5}\big(k\log 2 - \log\hat{G}\big)^{6/5}\} \vee e^{-gm_2^{1/12}}$, whether or not (2.9) holds.

Set $J \in \mathbb{N}$ to be minimal such that $\exp\{-(G\ell)^{-6/5}\big(J\log 2 - \log\hat{G}\big)^{6/5}\} \le \sigma$. Note that J is at most $2G\ell(\log\sigma^{-1})^{5/6}$ provided that $G\ell(\log\sigma^{-1})^{5/6} \ge \log\hat{G}$.

If $\eta_J \ge e^{-gm_2^{1/12}}$, then $\eta_J \le \sigma$, so that

$$\mathbb{P}\big(\hat{Y} \in \cup_{j=J}^{\infty} D_j\big) \le \exp\{5G^{11/6}\ell^6 g^{-5/6}\}\exp\{G\ell(\log\sigma^{-1})^{5/6}\}\sigma\,, \tag{2.11}$$

where we used that $\sigma \in (0,\sigma_0)$ for $\sigma_0 > 0$ small enough that $(0,\sigma_0) \to \mathbb{R} : x \to \exp\{G\ell(\log x^{-1})^{5/6}\}x$ is increasing. If $\eta_J < e^{-gm_2^{1/12}}$, then, by an enlargement argument,

$$\mathbb{P}\big(\hat{Y} \in \cup_{j=J}^{\infty} D_j\big) \le \exp\{5G^{11/6}\ell^6 g^{-5/6}\}\exp\Big\{G\ell g^{5/6} m_2^{5/72}\Big\}e^{-gm_2^{1/12}}\,.$$

Since $\sigma \ge e^{-gm_2^{1/12}}$, we find that (2.11) holds, whether or not $\eta_J \ge e^{-gm_2^{1/12}}$.

Writing $B : I \to \mathbb{R}$, $B(R-\ell) = 0$, to denote a Brownian motion of diffusion rate one under the law $\mathbb{P}$, independent of X, note that

$$\begin{aligned}\mathbb{P}\big(\hat{X}+\hat{Y} \in A\big) &= \mathbb{P}\big(\hat{Y} \in A_{\hat{X}}\big) \\ &\le \mathbb{P}(\hat{X} \in \mathcal{X}_1^c) + \mathbb{P}(\hat{Y} \in \mathcal{X}_2^c) + \sum_{k=0}^{J-1}\mathbb{P}\big(\hat{Y} \in A_{\hat{X}} \cap D_k, \hat{Y} \in \mathcal{X}_2, \hat{X} \in \mathcal{X}_1\big) \\ &\quad + \mathbb{P}\big(\hat{Y} \in \cup_{j=J}^{\infty} D_j, \hat{Y} \in \mathcal{X}_2\big)\,.\end{aligned} \tag{2.12}$$

The first two terms will be bounded using (2.8). For the next two terms, observe that

$$\begin{aligned}&\sum_{k=0}^{J-1}\mathbb{P}\big(\hat{Y} \in A_{\hat{X}} \cap D_k, \hat{X} \in \mathcal{X}_1\big) + \mathbb{P}\big(\hat{Y} \in \cup_{j=J}^{\infty} D_j\big) \\ &\le \sum_{k=0}^{J-1} 2^{k+1}\mathbb{P}\big(B \in A_{\hat{X}} \cap D_k, \hat{X} \in \mathcal{X}_1\big) \;+\; \exp\{4G^{11/6}\ell^6 g^{-5/6}\}\exp\{G\ell(\log\sigma^{-1})^{5/6}\}\sigma \\ &\le 2^J\mathbb{P}(B \in A_{\hat{X}}, \hat{X} \in \mathcal{X}_1) \;+\; \exp\{5G^{11/6}\ell^6 g^{-5/6}\}\exp\{G\ell(\log\sigma^{-1})^{5/6}\}\sigma \\ &\le 2^{2G\ell(\log\sigma^{-1})^{5/6}}\mathbb{P}(B \in A_{\hat{X}}, \hat{X} \in \mathcal{X}_1) \;+\; \exp\{5G^{11/6}\ell^6 g^{-5/6}\}\exp\{G\ell(\log\sigma^{-1})^{5/6}\}\sigma\,.\end{aligned} \tag{2.13}$$

In a specification similar to that of D_k, set $E_k = \{f \in \mathcal{C} : 2^k \le F(f) < 2^{k+1}\}$ for $k \ge 1$, and $E_0 = \{f \in \mathcal{C} : F(f) \le 2\}$. By a verbatim argument that invokes $\sigma \ge e^{-gm_1^{1/12}}$, we see that $\mathbb{P}\big(\hat{X} \in \cup_{j=J}^{\infty} E_j\big)$ is at most the right-hand side of (2.11).

Write $B' : I \to \mathbb{R}$, $B'(R-\ell) = 0$, for a further Brownian motion of diffusion rate one,

defined under the law $\mathbb{P}$ and chosen independently of B. Note then that

$$\begin{aligned}
&\mathbb{P}\big(B \in A_{\hat{X}}, \hat{X} \in \mathcal{X}_1\big) \\
= &\sum_{k=0}^{J-1} \mathbb{P}\big(B \in A_{\hat{X}}, \hat{X} \in E_k\big) \ + \ \mathbb{P}\big(\hat{X} \in \cup_{j=J}^{\infty} E_j\big) \\
\leq &\sum_{k=0}^{J-1} 2^{k+1} \mathbb{P}\big(B \in A_{B'}, B' \in E_k\big) \ + \ \exp\big\{5G^{11/6}\ell^6 g^{-5/6}\big\} \exp\big\{G\ell\big(\log\sigma^{-1}\big)^{5/6}\big\}\sigma \\
\leq &\ 2^J \mathbb{P}\big(B \in A_{B'}\big) \ + \ \exp\big\{5G^{11/6}\ell^6 g^{-5/6}\big\} \exp\big\{G\ell\big(\log\sigma^{-1}\big)^{5/6}\big\}\sigma \\
= &\ 2^{2G\ell(\log\sigma^{-1})^{5/6}} \mathbb{P}\big(B + B' \in A\big) \ + \ \exp\big\{5G^{11/6}\ell^6 g^{-5/6}\big\} \exp\big\{G\ell\big(\log a^{-1}\big)^{5/6}\big\}\sigma \\
= &\ \Big(\exp\big\{2\log 2 \cdot G\ell\big(\log\sigma^{-1}\big)^{5/6}\big\} \ + \ \exp\big\{5G^{11/6}\ell^6 g^{-5/6}\big\} \exp\big\{G\ell\big(\log\sigma^{-1}\big)^{5/6}\big\}\Big)\,\sigma \\
\leq &\ \exp\big\{2 \cdot 5G^{11/6}\ell^6 g^{-5/6} \cdot 2\log 2 \cdot G\ell\big(\log\sigma^{-1}\big)^{5/6}\big\}\,\sigma \\
= &\ \exp\big\{20\log 2 \cdot G^{17/6} g^{-5/6}\ell^7\big(\log\sigma^{-1}\big)^{5/6}\big\} \cdot \sigma\,,
\end{aligned}$$

where to get the inequality in the second line, for the term involving the sum, we simply use the definition of E_k, and independence of B and X as well as of B and B', to pass from $\mathbb{P}\big(B \in A_{\hat{X}}, \hat{X} \in E_k\big)$ to $2^{k+1}\mathbb{P}\big(B \in A'_B, B' \in E_k\big)$. The second term is bounded using the already stated bound on $\mathbb{P}\big(\hat{X} \in \cup_{j=J}^{\infty} E_j\big)$. The final inequality uses $G, \ell \geq 1$, $g \leq 1$, $\sigma \leq e^{-1}$ and $2e^x \leq e^{2x}$ for $x \geq 1$.

The preceding display and (2.13) yield

$$\begin{aligned}
&\sum_{k=0}^{J-1} \mathbb{P}\big(\hat{Y} \in A_{\hat{X}} \cap D_k, \hat{X} \in \mathcal{X}_1\big) + \mathbb{P}\big(\hat{Y} \in \cup_{j=J}^{\infty} D_j\big) \\
&\leq 2^{2G\ell(\log\sigma^{-1})^{5/6}} \exp\big\{20\log 2 \cdot G^{17/6} g^{-5/6}\ell^7\big(\log\sigma^{-1}\big)^{5/6}\big\} \cdot \sigma \\
&\qquad\qquad + \exp\big\{5G^{11/6}\ell^6 g^{-5/6}\big\} \exp\big\{G\ell\big(\log\sigma^{-1}\big)^{5/6}\big\}\sigma\,.
\end{aligned}$$

This, along with (2.12) and (2.8), implies that

$$\mathbb{P}\big(\hat{X} + \hat{Y} \in A\big) \leq \exp\big\{\Theta(1) G^{17/6} g^{-5/6}\ell^7\big(\log\sigma^{-1}\big)^{5/6}\big\}\sigma$$

and hence completes the proof of Lemma 2.8. □

Proof of Theorem 1.2. If we set $X(x) = Z_n^{\uparrow}(x,a) + Z_n^{\downarrow}(x,a)$, then (2.6) implies that

$$Z_n(x,a) = X(x) - \Big(2^{1/2}\big(a(1-a)\big)^{-1} R + \varepsilon\Big)x + c(a,R,n)$$

where $\varepsilon = 2^{1/2}\big((1-a-n^{-1})(1-a)\big)^{-1} R n^{-1}$ and $c(a,R,n)$ is a constant. The process X is found to be $\big(g, G', \max\{a^{-2/3}, (1-a)^{-2/3}\}\ell, \min\{a, 1-a\}n, 2\big)$-Brownian for the value of G' given in Theorem 1.2 by means of Lemma 2.5(3,4,5) and Lemma 2.8. It remains only to remove the constant $c(a,R,n)$ from the representation of $Z_n(x,a)$ in order to complete the proof of Theorem 1.2. The stated Brownian property of X is unaltered by the addition of a constant to this process; thus, we may absorb the constant by adding it to X. □

2.5 The rarity of twin peaks in the Brownian case

Here, we state and prove our Brownian twin peaks' rarity result, Proposition 2.9. Although the result is not new, we could not locate a reference; the following quantitative form is suitable for our applications.

The argument rests in part on Proposition 2.11, a near-return probability estimate for Brownian meander, which is stated and proved in the later part of the section.

Let K and r be positive. Let $B : [-r, r] \to \mathbb{R}$, $B(0) = 0$, denote standard Brownian motion under a law labelled $\mathbb{P}$. Set $W : [-r, r] \to \mathbb{R}$, $W(x) = B(x) + Kx$. Let $M \in [-r, r]$ denote the almost surely unique point at which the process W attains its maximum. Let Mid denote the *middle-third event* that $M \in [-r/3, r/3]$. For parameters $\varepsilon \in (0, r/6)$ and $\sigma \in (0, 1)$, let $\mathsf{NT} = \mathsf{NT}(B)$ denote the *near-touch event* that there exists a value of $z \in [-r, r]$ such that $|z - M| \in [\varepsilon, 2\varepsilon]$ for which $W(z) \geq W(M) - \sigma\varepsilon^{1/2}$. The near-touch event will be considered only when the middle-third event occurs, so that the condition that $z \in [-r, r]$ will be implied by the demand that $|z - M| \leq 2\varepsilon$.

Proposition 2.9. *For a constant $D > 0$ that is independent of $K \geq 0$, $r > 0$, $\varepsilon \in (0, r/6)$ and $\sigma \in (0, 1)$, we have that*

$$\mathbb{P}\big(\mathsf{NT} \cap \mathsf{Mid}\big) \leq DK^{-1}r^{-1/2} \exp\big\{ - K^2 r/18\big\} \min\big\{\sigma, 1\big\},$$

as well as $\mathbb{P}\big(\mathsf{NT} \cap \mathsf{Mid}\big) \leq D\sigma$.

Proof. The case of general $r > 0$ may be reduced to that where $r = 1$ by considering $r^{-1/2}W(rx) : [-1, 1] \to \mathbb{R}$. Brownian scaling entails that the $r = 1$ result implies the general result when the replacement of K by $Kr^{1/2}$ is made. Thus, it suffices to prove the lemma with r set equal to one, a choice which we now make.

We will first argue that

$$\mathbb{P}(\mathsf{Mid}) \leq 3 \cdot 2^{3/2}\pi^{-1/2}K^{-1} \exp\big\{ - \tfrac{K^2}{18}\big\}. \tag{2.14}$$

Note that Mid entails that $W(1) < W(M)$ with $M \in [-1/3, 1/3]$, so that $B(1) < B(M) - 2K/3$, which forces one of $B(1)$ and $B(M)$ to exceed $K/3$ in absolute value. Thus,

$$\mathsf{Mid} \subseteq \Big\{ \sup\big\{|B(x)| : x \in [-1, 1]\big\} \geq K/3\Big\}.$$

By symmetry and the reflection principle, the probability of this right-hand event is at most

$$2{\cdot}2\mathbb{P}\big(B(1) \geq K/3\big) \leq 4(2\pi)^{-1/2}(K/3)^{-1} \exp\big\{-2^{-1}(K/3)^2\big\} = 3{\cdot}2^{3/2}\pi^{-1/2}K^{-1} \exp\big\{-\tfrac{K^2}{18}\big\}, \tag{2.15}$$

the displayed inequality by a standard upper bound on the tail of a Gaussian random variable. Thus, we obtain (2.14).

The next result will be important as we turn to analysing the conditional probability of NT given Mid. For $r > 0$ and $y \leq 0$, we write $\mathcal{B}_{0,y}^{[0,r]}$ for the law of Brownian bridge B of diffusion rate one on $[0, r]$ with $B(0) = 0$ and $B(r) = y$; thus, $\mathcal{B}_{0,y}^{[0,r]}\big(\cdot \,\big|B < 0\big)$ is the law resulting from conditioning B on $B(x) < 0$ for $x \in (0, r]$.

Lemma 2.10. *Under the conditional law $\mathbb{P}\big(\cdot \,\big|\mathsf{Mid}\big)$, consider the processes $X^- : [0, M + 1] \to \mathbb{R}$ and $X^+ : [0, 1 - M] \to \mathbb{R}$, given by $X^-(x) = B(M - x) - B(M)$ and $X^+(x) = B(M + x) - B(M)$.*

1. *Conditionally on the value of $M \in [-1/3, 1/3]$ and on $X^+(1 - M)$ being any given value $y \leq 0$, the conditional distribution of X^+ is given by the law $\mathcal{B}_{0,y}^{[0,1-M]}\big(\cdot\big|B < 0\big)$.*
2. *Conditionally on the value of $M \in [-1/3, 1/3]$ and on $X^-(M + 1)$ being any given value $y \leq 0$, the conditional distribution of X^- is given by the law $\mathcal{B}_{0,y}^{[0,M+1]}\big(\cdot\big|B < 0\big)$.*

Proof: (1). Under the stated conditioning, the process $X^+ : [0, 1 - M] \to \mathbb{R}$ is given by $[0, 1-M] \to \mathbb{R} : x \to X(x)$, where $X(x) = B(x)+Kx$, with B being standard Brownian motion, conditioned on $X(1 - M) = y$ and on $X(x) < 0$ for $x \in (0, 1 - M)$. Conditioning

X as given by the preceding formula on $X(1-M)=y$ results in the Brownian bridge law $\mathcal{B}_{0,y}^{[0,1-M]}$, as we may readily verify by decomposing $B:[0,1-M]\to\mathbb{R}$ as the sum of a Brownian bridge and a linear term with an independent Gaussian coefficient. The further conditioning $X(x)<0$ results in the conditional distribution stated in the lemma. **(2).** This almost verbatim argument is omitted. □

We wish to apply the shortly upcoming Proposition 2.11 alongside Lemma 2.10 to find that there exists $D>0$ such that, for any $y<0$,

$$\mathcal{B}_{0,y}^{[0,1-M]}\big(\mathsf{NT}\big|B<0\big)+\mathcal{B}_{0,y}^{[0,M+1]}\big(\mathsf{NT}\big|B<0\big)\le D\sigma.$$

Above, we are abusing notation a little in denoting by NT the event that there exists a value of $z\in[-\varepsilon,2\varepsilon]$ such that for which $B(z)\ge-\sigma\varepsilon^{1/2}$.

We set the proposition's parameters: $r=\varepsilon$ and $s=1\pm M$. The proposition's hypothesis $s\ge 3r$ is valid because because $M\le 1/3$ when Mid occurs, so that ε is merely supposed to be at most a given positive constant. Equipped with the just stated outcome, we find that

$$\begin{aligned}&\mathbb{P}\big(\mathsf{Mid}\cap\mathsf{NT}\big)\;=\;\mathbb{P}\big(\mathsf{Mid}\big)\mathbb{P}\big(\mathsf{NT}\big|\mathsf{Mid}\big)\\ \le\;&3\cdot 2^{3/2}\pi^{-1/2}K^{-1}\exp\big\{-\tfrac{K^2}{18}\big\}\mathbb{E}\Big(\mathcal{B}_{0,y}^{[0,1-M]}\big(\mathsf{NT}\big|B<0\big)+\mathcal{B}_{0,y}^{[0,M+1]}\big(\mathsf{NT}\big|B<0\big)\Big)\\ \le\;&3\cdot 2^{3/2}\pi^{-1/2}K^{-1}\exp\big\{-\tfrac{K^2}{18}\big\}D\sigma\,.\end{aligned}$$

It is in the second inequality that the conclusion of Proposition 2.11 is used. The first inequality is due to (2.14), and the mean $\mathbb{E}$ on this inequality's right-hand side is taken over y and M. Note that the displayed assertion in Proposition 2.9 may be viewed as a pair of bounds due to the right-hand factor of $\min\{\sigma,1\}$. The proof of the bound including the factor of σ has just been completed, while the bound without this factor is implied by (2.14). The latter assertion of Proposition 2.9 follows from $\mathbb{P}\big(\mathsf{NT}\big|\mathsf{Mid}\big)\le D\sigma$. □

Proposition 2.11. *Let s, r be positive, with $s\ge 3r$; let $y\le 0$; and let $\varepsilon>0$. Let $X:[0,s]\to\mathbb{R}$ be a random process specified under the law $\mathbb{P}$ whose law is $\mathcal{B}_{0,y}^{[0,s]}\big(\cdot\big|B<0\big)$. Let $E=E(X,r,\varepsilon)$ denote the event that $\sup_{x\in[r,2r]}X(x)$ is at least $-r^{1/2}\varepsilon$. There exists a constant $D>0$ such that, for any such s, r, y and ε, $\mathbb{P}(E)\le D\varepsilon$.*

Some preliminaries will be of aid in proving this proposition.

Let $f:[u_0,v_0]\to\mathbb{R}$ denote a continuous function defined on a compact real interval. For any closed subinterval $[u,v]\subseteq[u_0,v_0]$, let $f^{[u,v]}:[u,v]\to\mathbb{R}$ denote the *bridge*—that is, the continuous function with vanishing endpoint values—that is an affine translate of f's restriction to $[u,v]$. Namely,

$$f^{[u,v]}(x)\;=\;f(x)-\tfrac{v-x}{v-u}f(u)-\tfrac{x-u}{v-u}f(v)\;\;\text{for}\;\;x\in[u,v]\,.$$

Let $x,y\in[u_0,v_0]$ satisfy $x<y$. We view the interval $[x,y]$ as a union $L\cup R$ of a left and a right subinterval, setting $L=[x,m]$ and $R=[m,y]$, where $m=(x+y)/2$.

Note that the function f is characterized by the list of data:

- the restriction of f to $[u_0,x]\cup[y,v_0]$;
- the left bridge f^L and the right bridge f^R; and
- the relative midpoint value $f\big((x+y)/2\big)-\big(f(x)+f(y)\big)/2$.

Indeed, given the listed data, f may be reconstructed by recording its values on $[u_0,x]\cup[y,v_0]$; by recording the value $f\big((x+y)/2\big)$ via the first and third items; and by recovering its remaining values by adding to the affine interpolations of the endpoint values on L and R the respective bridges f^L and f^R.

Lemma 2.12. *Let $B : [u_0, v_0] \to \mathbb{R}$ have the law $\mathcal{B}_{0,*}^{[u_0,v_0]}$. The elements in the three-item list for B are independent. The bridges in the second item have respective laws—$\mathcal{B}_{0,0}^{L}$ and $\mathcal{B}_{0,0}^{R}$—of Brownian bridge on L and R with vanishing endpoint values. The law of the real random variable in the third item is Gaussian with mean zero and variance $(y-x)/4$.*

Proof. Given the first element in the three item list for B, the third item takes the indicated form by explicit computation. By Lévy's construction of Brownian motion [MP10], the two second item bridges are then independent standard Brownian bridges. □

Proof of Proposition 2.11. For $t > 0$, set $X_t : [0, s/t] \to \mathbb{R}$, $X_t(x) = t^{-1/2}X(tx)$. Note that $X_t(s/t) = t^{-1/2}y$. By Brownian scaling, we note that X_r has the law of X indexed by parameters $(r/t, s/t, t^{-1/2}y, \varepsilon)$ in place of (r, s, y, ε). Moreover, the spatial-temporal scaling $(x, y) \to \big(tx, t^{1/2}y\big)$ sends the event $E(X, r, \varepsilon)$ to the event $E(X_t, r/t, \varepsilon)$. It is thus enough to prove the proposition for a given value of $r > 0$. We will do so with $r = 2$.

Set $L = [1, 3]$, $R = [3, 5]$, and write $N = X(3) - \big(X(1) + X(5)\big)/2$. Consider the three-item list that represents X in the case that $[x, y] = [1, 5]$. The two bridges in the second item are X^L and X^R, and the relative value in the third item equals N.

We will prove the proposition by analysing a random experiment in which the process X is sampled and then altered to produce a coupled process X^r. This resampled process X^r will share the law of X. The discussion of the experiment will include four claims each of whose proofs is given straight after the claim in question is stated.

In the experiment, X is first sampled and represented in the format of the three-item list. The third element is discarded and resampled to equal N^r, a random variable that is selected independently according to the standard Gaussian law. Let $X^r : [0, s] \to \mathbb{R}$ denote the process arising from the resampled three-item list.

Let S denote the event that $X^r(x) < 0$ for all $x \in (0, s)$. If S occurs, we set $Z = X^r$. In the other case, we sample an independent copy of the process X and repeat the procedure. This process continues until $Z : [0, \infty) \to \mathbb{R}$ is specified.

Claim 1. The process $Z : [0, s] \to \mathbb{R}$ has the law of X.

Proof. It is enough to argue that the conditional distribution of X^r given the occurrence of S equals the law of X. We will establish the *stronger assertion* that, given the first and second items in the three-item list that specifies X, the conditional distribution of the third item that specifies X, and of the third item that specifies X^r given S, coincide.

Thus suppose given the first and second items that specify X. Let Y denote the random height, measurable with respect to these items, such that setting the third item value equal to Y ensures that $X : [0, s] \to \mathbb{R}$ assumes the value zero in $[1, 5]$, but is never positive in this interval. Since the value $(y - x)/4$ in Lemma 2.12 equals one in the present case, this lemma implies that, given the first and second items, the conditional distribution of the third item in the specification of X is the law of a standard Gaussian random variable conditioned to be less than Y. But the characterization of the event S given the first and second items is simply that the standard Gaussian random variable N^r be less than Y. This confirms the stronger assertion and completes the proof of Claim 1. □

In view of Claim 1, it suffices for the proof of Proposition 2.11 to argue that, for some $D > 0$, the bound $\mathbb{P}\big(E(Z, 2, \varepsilon)\big) \le D\varepsilon$ holds for all $\varepsilon > 0$.

Let $K \in \mathbb{N}^+$ denote the step at which the procedure terminates. Thus, $S = \{K = 1\}$. Let $\mathcal{F}$ denote the σ-algebra generated by the first and second items in the three-item list for X. Let $H \subset \mathbb{R}$ denote the $\mathcal{F}$-measurable random set of $h \in \mathbb{R}$ such that the specification of the third item value to h alongside the given first and second items causes the event $S \cap E\big(X^r, 2, \varepsilon\big)$ to occur.

Claim 2. The random set H is an $\mathcal{F}$-measurable interval whose length is almost surely at most $2^{3/2}\varepsilon$.

Proof. Given $\mathcal{F}$, the event $S \cap E\big(X^r, 2, \varepsilon\big)$ is characterized by the condition that $\sup\big\{X^r(z) : z \in [2,4]\big\} \in (-2^{1/2}\varepsilon, 0)$. When $N^r = Y$, this supremum equals zero; let $Z \in [2,4]$ satisfy $X^r(Z) = 0$ when $N^r = Y$. Then $X^r(Z) = -\frac{2-|Z-3|}{2}\lambda \leq -2^{-1}\lambda$ when $N^r = Y - \lambda$, so that the supremum is at most $-2^{1/2}\varepsilon$ when $\lambda \geq 2^{3/2}\varepsilon$. Thus, $H \subset [Y - 2^{3/2}\varepsilon, Y]$. The monotonicity of X^r in N^r implies that H is an interval, so that Claim 2 is validated. □

Denoting by μ the standard Gaussian law, note that

$$\mathbb{P}\big(E(Z,2,\varepsilon) \,\big|\, K = 1\big) = \frac{\mathbb{P}(N_r \in H)}{\mathbb{P}(S)} = \frac{\mu(H)}{\mathbb{P}(S)}\,. \tag{2.16}$$

Claim 3. $\mathbb{P}(S) \geq 4^{-1}\mu(1,\infty)e^{-2}$.

Proof. Consider four independent events: $\big\{X(1) \leq -1\big\} \cap \big\{X(5) < X(1)\big\}$; the supremum of the bridge $X^{[1,3]}$ is at most one; likewise for the bridge $X^{[3,5]}$; and $N^r < 0$. Lower bounds on the probabilities of these four events are: $2^{-1}\mu(1,\infty)$; e^{-1}; e^{-1}; and $1/2$. Indeed, $X(1)$ is stochastically dominated by a standard Gaussian random variable, as is $X(5) - X(1)$ conditionally on the value of $X(1)$; $\mathcal{B}_{0,0}^{[1,3]}\big(\sup_{x\in[1,3]} B(x) \geq r\big) = e^{-r^2}$ for $r \geq 0$ by Brownian scaling and equation (3.40) in [KS88, Chapter 4]; while the third bound follows similarly to the second, and the fourth is trivial. The four events ensure that S occurs, whence Claim 3. □

Claim 4. For $x \in \mathbb{R}$ and $a > 0$, $\mu\big[x, x+a\big] \leq \big(2\pi\big)^{-1/2}a$.

Proof. The standard Gaussian density is at most $(2\pi)^{-1/2}$. □

Applying Claims 2, 3 and 4 to (2.16), we learn that

$$\mathbb{P}\big(E(Z,2,\varepsilon) \,\big|\, K = 1\big) \leq D\varepsilon$$

where $D = 2\pi^{-1/2} \cdot 4\mu(1,\infty)^{-1}e^2$. Note then that

$$\mathbb{P}\big(E(X,2,\varepsilon)\big) = \mathbb{P}\big(E(Z,2,\varepsilon)\big) = \mathbb{P}\big(E(Z,2,\varepsilon) \,\big|\, K = 1\big)\,,$$

so that $\mathbb{P}\big(E(X,2,\varepsilon)\big) \leq D\varepsilon$ is seen to hold for the same choice of $D > 0$. Thus we obtain Proposition 2.11. □

2.6 The rarity of twin peaks for the routed weight profile

Theorem 1.3 would now seem to be readily at hand on the basis of Theorem 1.2 and Proposition 2.9. There is a gap to be bridged, however. To explain this, let $X = Y^\perp$ mean that 'the random function X is very similar to the random function Y in the sense of Definition 1.1'. By Theorem 1.2, $Z_n(\cdot, a) : [R-\ell, R+\ell] \to \mathbb{R}$ has (up to an additive shift) the form $B^\perp + \ell$, where B is Brownian motion of rate two and ℓ is the linear function $\ell(x) = Kx$, with $K = 2^{1/2}\big(a(1-a)\big)^{-1}R$. Proposition 2.9 delivers pertinent information about the process $B + \ell$. The extra little element we thus need is to understand that a random function that has the form $B^\perp + \ell$ also has the form $(B+\ell)^\perp$. This element is furnished in the proof of the next result, which translates Proposition 2.9 into a form which in unison with Theorem 1.2 will then readily deliver Theorem 1.3.

Corollary 2.13. *Let $R \in \mathbb{R}$, $\ell \geq 1$, $a \in n^{-1}\mathbb{Z} \cap (0,1)$, $\sigma > 0$ and $\varepsilon > 0$. Let M denote the maximizer of $Z_n(\cdot, a) : [R-\ell, R+\ell] \to \mathbb{R}$. Suppose that $|R| \leq 2^{-1}cn^{1/9}\big(a \wedge (1-a-n^{-1})\big)^{7/9}$, where c appears in Theorem 1.2. Let $\ell > 0$ satisfy $-2^{-1}n^{1/3}a \leq R - \ell$ and $R + \ell \leq 2^{-1}n^{1/3}(1-a-n^{-1})$. Abbreviating $Z = Z_n(\cdot, a)$, let $\mathsf{Mid}(Z)$ denote the event that M lies in $[R-\ell/3, R+\ell/3]$. Let $\mathsf{NT}(Z)$ denote the event that there exists $x \in [R-\ell, R+\ell]$ such that $|x - M| \in [\varepsilon, 2\varepsilon]$ and $Z(x) \geq Z(M) - \sigma\varepsilon^{1/2}$. When a lies in a compact interval in $(0,1)$, then there exist constants $H, h > 0$ and $n_0 \in \mathbb{N}$ determined by this compact interval such that, if we further suppose that $\ell \leq hn^{1/1370}$ and $n \geq n_0$, then*

$$\mathbb{P}\Big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\Big) \;\leq\; \max\Big\{\sigma_* \cdot \exp\big\{-hR^2\ell + H\ell^{19}\big(1 + R^2 + \log\sigma_*^{-1}\big)^{5/6}\big\}, \exp\big\{-hn^{1/12}\big\}\Big\}\,,$$

where we denote $\sigma_* = \min\{\sigma, 1\}$.

Proof. Set $X^* : [R-\ell, R+\ell] \to \mathbb{R}$, $X^*(R) = 0$, to be Brownian motion of diffusion rate two. Thus X^* is the pure counterpart to the process $X : [R-\ell, R+\ell] \to \mathbb{R}$ from Theorem 1.2 which is $\big(g, G'\ell^6, \ell, \min\{a, 1-a\}n, 2\big)$-Brownian for $G' = \Theta(1)G^{17/6}g^{-5/6}\big(a \wedge (1-a-n^{-1})\big)^{-34/3}$.

Now define $\mathsf{MidNT}^*(X^*)$ to be the event that the process $[R-\ell, R+\ell] \to \mathbb{R} : x \to X^*(x) + Kx$ realizes the event $\mathsf{Mid} \cap \mathsf{NT}$. That is, MidNT^* is the set of those continuous functions $f : [R-\ell, R+\ell] \to \mathbb{R}$ that vanish at R and for which

$$\text{the map } [R-\ell, R+\ell] \to \mathbb{R} : x \to f(x) + K(x-R) \text{ belongs to } \mathsf{Mid} \cap \mathsf{NT}\,.$$

Let $Z^* : [R-\ell, R+\ell] \to \mathbb{R}$ be given by $Z^*(x) = X^*(x) + K(x-R)$, where $K = -2^{1/2}\big(a(1-a)\big)^{-1}R + \varepsilon$, with ε specified by Theorem 1.2. Given the form of ε, we see that, provided that $n \geq n_0$, we have $d_1|R| \leq |K| \leq D_1|R|$, where the positive constants d_1 and D_1 and the natural number n_0 are determined by the compact interval in $(0,1)$ in which $a \in n^{-1}\mathbb{Z}$ is supposed to lie.

Note that, by definition,

$$\mathbb{P}\Big(\mathsf{Mid}(Z^*) \cap \mathsf{NT}(Z^*)\Big) = \mathbb{P}\Big(\mathsf{MidNT}^*(X^*)\Big)\,.$$

Let $p \in (0,1)$ denote this probability. Recalling from above that the process $X : [R-\ell, R+\ell] \to \mathbb{R}$ is $\big(g, G'\ell^6, \ell, \min\{a, 1-a\}n, 2\big)$-Brownian for $G' = \Theta(1)G^{17/6}g^{-5/6}(a \wedge (1-a-n^{-1}))^{-34/3}$. By Lemma 2.7, we thus have that, when $\exp\big\{-g\big(\min\{a, 1-a\}n\big)^{1/12}\big\} \leq p$,

$$\mathbb{P}\Big(\mathsf{MidNT}^*(X)\Big) \;\leq\; q \cdot \exp\big\{5(G')^{11/6}\ell^{6+77/6}g^{-5/6}\big\} \exp\Big\{G'\ell\big(\log q^{-1}\big)^{5/6}\Big\}\,.$$

for any value $q \in [p, 1]$. This left-hand side equals $\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big)$ by definition, while the value of p satisfies

$$p \leq DK^{-1}\ell^{-1/2} \exp\big\{-K^2\ell/18\big\} \min\{\sigma, 1\}$$

as well as $p \leq D\sigma$ by Proposition 2.9. Choose $q = DK^{-1}\ell^{-1/2} \exp\big\{-K^2\ell/18\big\} \min\{\sigma, 1\}$. Recalling the notation $\sigma_* = \sigma \wedge 1$, we find that

$$\begin{aligned}\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big) \;\leq\; & DK^{-1}\ell^{-1/2} \exp\big\{-K^2\ell/18\big\}\sigma_* \cdot g^{-1} \exp\big\{5(G')^{11/6}\ell^{113/6}g^{-5/6}\big\} \\ & \times \exp\Big\{G'\ell\big(\log\big(D^{-1}K\ell^{1/2} \exp\big\{K^2\ell/18\big\}\sigma_*^{-1}\big)\big)^{5/6}\Big\}\,.\end{aligned}$$

Using $d_1|R| \leq |K| \leq D_1|R|$, and absorbing the $D^{-1}K\ell^{1/2}$ factor in the third exponential term into $\exp\big\{K^2\ell/18\big\}$, we obtain, for positive constants H and h determined by the compact interval in $(0,1)$ in which a is supposed to lie,

$$\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big) \leq H \exp\big\{-hR^2\ell\big\} \exp\big\{H\ell^{19}g^{-5/6}\big\} \exp\big\{H\ell^7\big(R^2\ell + \log\sigma_*^{-1}\big)^{5/6}\big\}\,\sigma_*\,.$$

(The dependence on G and g has been absorbed by H.) Further note that, the factor $K^{-1}\ell^{-1/2}$, which is problematic for small $K\ell^{1/2} \geq 0$, has been omitted by making H large enough; indeed, $\exp\big\{-hR^2\ell\big\}$ approaches one as $K\ell^{1/2} \searrow 0$, while the other right-hand exponential terms are at least one, and the left-hand side, being a probability, is at most one.

Noting that $\ell \geq 1$, and suitably increasing H, this upper bound is at most

$$\exp\Big\{-hR^2\ell + H\ell^{19}\big(1 + R^2 + \log\sigma_*^{-1}\big)^{5/6}\Big\}\,\sigma_*\,.$$

This completes the proof of Corollary 2.13 in the case that $\exp\big\{-g\big(\min\{a,1-a\}n\big)^{1/12}\big\} \le p$. Suppose now that the opposing inequality holds. Choosing $q = \exp\big\{-g\big(\min\{a,1-a\}n\big)^{1/12}\big\}$, we find that

$$\begin{aligned}\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big) &\le \exp\big\{-g\big(\min\{a,1-a\}n\big)^{1/12}\big\} \cdot g^{-1} \exp\big\{4(G')^{11/6}\ell^6 g^{-5/6}\big\} \\ &\quad \times \exp\Big\{G'\ell g^{5/6}\big(\min\{a,1-a\}n\big)^{5/72}\Big\}.\end{aligned}$$

For positive constants H and h determined by the compact interval in $(0,1)$ in which a is supposed to lie, we obtain

$$\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big) \le \exp\big\{-hn^{1/12} + H\ell^{19} + H\ell^7 n^{5/72}\Big\};$$

or, more simply,

$$\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big) \le \exp\big\{-hn^{1/12} + H\ell^{19} n^{5/72}\Big\}.$$

Since we suppose that ℓ is at most a small constant multiple of $n^{1/1370}$, we also have, after a decrease in the value of $h > 0$,

$$\mathbb{P}\big(\mathsf{Mid}(Z) \cap \mathsf{NT}(Z)\big) \le \exp\big\{-hn^{1/12}\big\}.$$

This completes the proof of Corollary 2.13. □

Proof of Theorem 1.3. Corollary 2.13 implies that, when $a \in n^{-1}\mathbb{Z}$ lies in a compact interval in $(0,1)$, there exist constants $H, h > 0$ determined by this compact interval such that, for $\varepsilon \in (0, \ell/3)$,

$$\begin{aligned}&\mathbb{P}\Big(M \in [R-\ell/3, R+\ell/3]\,, \sup_{x\in\mathbb{R}:|x-M|\in[\varepsilon,2\varepsilon]} Z_n(x,a) \ge Z_n(M,a) - \sigma\varepsilon^{1/2}\Big) \\ \le\ &\max\Big\{\sigma\cdot\exp\big\{-hR^2\ell + H\ell^{19}\big(1+R^2+\log\sigma^{-1}\big)^{5/6}\big\}, \exp\big\{-hn^{1/12}\big\}\Big\}.\end{aligned}$$

Summing this bound over dyadic scales $[2^j\varepsilon, 2^{j+1}\varepsilon]$ from that indexed by $j=0$ until a final truncated scale of the form $[2^j\varepsilon, \ell']$, we learn that

$$\begin{aligned}&\mathbb{P}\Big(M \in [R-\ell/3, R+\ell/3]\,, \sup_{x\in\mathbb{R}:|x-M|\in[\varepsilon,\ell/3]} \big(Z_n(x,a) + \sigma 2^{-1/2}(x-M)^{1/2}\big) \ge Z_n(M,a)\Big) \\ \le\ &\lceil \log_2\big(\ell'\varepsilon^{-1}\big)\rceil \max\Big\{\sigma\cdot\exp\big\{-hR^2\ell + H\ell^{19}\big(1+R^2+\log\sigma^{-1}\big)^{5/6}\big\}, \exp\big\{-hn^{1/12}\big\}\Big\}.\end{aligned}$$

Relabelling σ and adjusting the values of H and h, we obtain Theorem 1.3. □

3 Fluctuation in polymer weight and geometry

Here we prove our robust modulus of continuity assertions Theorem 1.4, which concerns polymer geometry, and Theorem 1.6, which concerns polymer weight. As a consequence, we will prove Corollary 1.5, which addresses the maximum fluctuation of polymers. We further prove Theorem 1.7, a result which offers control on the fluctuation of polymers that is uniform in compact endpoint variation and in variation on a given dyadic scale for the polymer lifetime proportion at which fluctuation is measured.

There are eight subsections. The first introduces some basic tools needed on several later occasions in this article. Four sets of preliminaries that are needed for the proofs of Theorems 1.4 and 1.6 are respectively treated in the ensuing four subsections: the assertion of a strong form of invariance for parabolic weight in Brownian LPP; a

result, in the style of Corollary 1.5, concerning the maximum fluctuation of polymers; control on polymer weight that is uniform in compact endpoint variation; and control on large local fluctuations in unscaled geodesic energy. The sixth subsection proves Theorems 1.4 and 1.6 and Corollary 1.5. The seventh proves Theorem 1.7 and the eighth Proposition 1.8.

3.1 Some basics

3.1.1 The scaling principle

Write $\mathbb{R}^2_< = \{(x,y) \in \mathbb{R}^2 : x < y\}$. Let $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. The quantity $ns_{1,2}$ is a positive integer, in view of the defining property (1.4). The scaling map $R_k : \mathbb{R}^2 \to \mathbb{R}^2$ has been defined whenever $k \in \mathbb{N}^+$, and thus we may speak of R_n and $R_{ns_{1,2}}$. The map R_n is the composition of $R_{ns_{1,2}}$ and the transform $S_{s_{1,2}^{-1}}$ given by $\mathbb{R}^2 \to \mathbb{R}^2 : (a,b) \to \big(as_{1,2}^{-2/3}, bs_{1,2}^{-1}\big)$. That is, the system of $ns_{1,2}$-zigzags is transformed into the system of n-zigzags by an application of $S_{s_{1,2}^{-1}}$. Note that $\mathsf{Wgt}_n\big[(x,s_1) \to (y,s_2)\big] = s_{1,2}^{1/3}\mathsf{Wgt}_{ns_{1,2}}\big[(xs_{1,2}^{-2/3}, \kappa) \to (ys_{1,2}^{-2/3}, \kappa+1)\big]$, where $\kappa = s_1 s_{1,2}^{-1}$; indeed this weight transformation law is valid for all zigzags, rather than just polymers, in view of (1.6).

We may summarise these inferences by saying that the system of $ns_{1,2}$-zigzags, including their weight data, is transformed into the n-zigzag system, and its accompanying weight data, by the transformation $\big(a,b,c\big) \to \big(as_{1,2}^{-1/3}, bs_{1,2}^{-2/3}, cs_{1,2}^{-1}\big)$, where the components refer to the changes suffered in weight, and horizontal and vertical coordinates. This fact leads us to what we call the *scaling principle*.

The scaling principle. Let $(n, s_1, s_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. Any statement concerning the system of n-zigzags, including weight information, is equivalent to the corresponding statement concerning the system of $ns_{1,2}$-zigzags, provided that the following changes are made:

- the index n is replaced by $ns_{1,2}$;
- any time is multiplied by $s_{1,2}^{-1}$;
- any weight is multiplied by $s_{1,2}^{1/3}$;
- and any horizontal distance is multiplied by $s_{1,2}^{-2/3}$.

3.1.2 Tail bounds on one-point polymer weight

Recall from Section 1.9 that $\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big]$ denotes the parabolically adjusted weight $\mathsf{Wgt}_n\big[(x,h_1) \to (y,h_2)\big] + 2^{-1/2}(y-x)^2 h_{1,2}^{-1}$. We will have need on several occasions for control on the upper and lower tail of this random variable.

Lemma 3.1. *There exist positive constants C and c, and $n_0 \in \mathbb{N}$, such that the following holds. Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ satisfy $n \geq n_0$ and $|x-y| \leq cn^{1/9}$.*

1. *For $t \geq 0$,*
$$\mathbb{P}\Big(\mathsf{Wgt}_n^{\cup}\big[(x,0) \to (y,1)\big] \geq t\Big) \leq C\exp\big\{-ct^{3/2}\big\}.$$
2. *For $t \geq 0$,*
$$\mathbb{P}\Big(\mathsf{Wgt}_n^{\cup}\big[(x,0) \to (y,1)\big] \leq -t\Big) \leq C\exp\big\{-ct^{3/2}\big\}.$$

Proof. This result follows from [CHH19, Proposition 3.6] and translation invariance of Brownian LPP. □

3.1.3 Polymer uniqueness and ordering

A polymer with given endpoints is almost surely unique.

Lemma 3.2. *[Ham19b, Lemma 4.6(1)] Let $x, y \in \mathbb{R}$. There exists an n-zigzag from $(x,0)$ to $(y,1)$ if and only if $y \geq x - n^{1/3}/2$. When the last condition is satisfied, there is almost surely a unique n-polymer from $(x,0)$ to $(y,1)$.*

A rather simple sandwiching fact about polymers will also be needed. Let $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$ and consider a zigzag Z_1 from (x_1,s_1) to (y_1,s_2) and another Z_2 from (x_2,s_1) to (y_2,s_2). We declare that $Z_1 \preceq Z_2$ if 'Z_2 lies on or to the right of Z_1': formally, if Z_2 is contained in the union of the closed horizontal planar line segments whose left endpoints lie in Z_1.

Lemma 3.3. *[Ham20, Lemma 5.7] Let (n,s_1,s_2) be a compatible triple, and let (x_1,x_2) and (y_1,y_2) belong to $\mathbb{R}^2_{\leq}$. Suppose that there is a unique n-polymer from (x_i,s_1) to (y_i,s_2), both when $i=1$ and $i=2$. (This circumstance occurs almost surely, and the resulting polymers have been labelled $\rho_n\big[(x_1,s_1) \to (y_1,s_2)\big]$ and $\rho_n\big[(x_2,s_1) \to (y_2,s_2)\big]$.) Now let ρ denote any n-polymer that begins in $[x_1,x_2] \times \{s_1\}$ and ends in $[y_1,y_2] \times \{s_2\}$. Then*

$$\rho_n\big[(x_1,s_1) \to (y_1,s_2)\big] \preceq \rho \preceq \rho_n\big[(x_2,s_1) \to (y_2,s_2)\big].$$

3.1.4 Boldface notation for parameters in statement applications

Some later used outside results come equipped with parameters that must be set in any given application. When such applications are made, we employ a boldface notation to indicate the parameter labels of the results being applied. This device permits occasional reuse of symbols and disarms notational conflict.

3.2 Invariance of the polymer weight field

Given two collections of real-valued random variables M^1 and M^2 indexed by pairs $(x,h_1),(y,h_2) \in \mathbb{R} \times \mathbb{Z}$, we write $M^1 \equiv M^2$ to indicate that the two collections have the same law. This notation will be applied in the unscaled picture, when the fields of random variables are energies of geodesics, such as $M\big[(x,h_1) \to (y,h_2)\big]$. Thus, the conditions $x \leq y$ and $h_1 \leq h_2$ must be imposed, to ensure that these energies are well defined. We abuse notation by setting values such as $M\big[(x,h_1) \to (y,h_2)\big]$ equal to zero when they are not well defined, so that the indexing by $(x,h_1),(y,h_2) \in \mathbb{R} \times \mathbb{Z}$ is admissible. The $\equiv$ notation will also be adopted in the scaled picture, where two collections of real-valued random variables W^1 and W^2 indexed by pairs $(x,h_1),(y,h_2) \in \mathbb{R} \times n^{-1}\mathbb{Z}$ are said to satisfy $W^1 \equiv W^2$ when they have the same law, and where a similar extension of domain is applied to permit this choice of domain for the two collections.

For $K \in \mathbb{R}$, let $\tau_K : \mathbb{R}^2 \to \mathbb{R}^2$ denote the shear map

$$\tau(x,y) = (x + Ky, y) \tag{3.1}$$

The shear map will permit us to straighten sharply sloping corridors. Our invariance result asserts that parabolic weight is statistically almost unchanged under application of the shear map. Later, when we prove Proposition 3.16(2,3), it will permit us to propagate control on polymer weights from roughly square rectangles to much wider ones.

Proposition 3.4. *Suppose given two disjoint compact real intervals I_1 and I_2. For $n \in \mathbb{N}$ and $K > 0$, let $\mathcal{K}_n(K)$ denote the set of quadruples $(x,y,h_1,h_2) \in \mathbb{R}^4$, where the real values x and y vary over $[-2K,2K]$ and where h_1 and h_2 respectively vary over elements of $n^{-1}\mathbb{Z}$ lying in I_1 and I_2. Suppose that $n \geq \Theta(1)$ and $\Theta(1) \leq K \leq \Theta(1)n^{1/18}$. Then*

$$\mathsf{Wgt}_n^{\cup}\big[\tau_K(x,h_1) \to \tau_K(y,h_2)\big] \equiv \mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big] + \mathcal{E}(x,y,h_1,h_2)$$

where the error terms $\mathcal{E}(x,y,h_1,h_2)$ are random but small, satisfying the uniform tail bound

$$\begin{aligned}&\mathbb{P}\Big(\sup\Big\{\big|\mathcal{E}(x,y,h_1,h_2)\big| : (x,y,h_1,h_2)\in\mathcal{K}_n(K)\Big\}\ge\Theta(1)n^{-1/9}\Big)\\ \le\ &\Theta(1)n^{2+2/3}K^{-2}\exp\big\{-\Theta(1)n^{1/12}K^{-3/2}\big\}+\Theta(1)K^2\exp\big\{-\Theta(1)n^{1/12}\big\}.\end{aligned}\tag{3.2}$$

The $\Theta(1)$ terms may depend on the pair (I_1,I_2) and on no other parameter.

The parameters h_1 and h_2 vary subject to $h_{1,2}=\Theta(1)$ in this result. During the upcoming derivation, we will monitor dependence on $h_{1,2}$ more closely, with a view to the potential for applications where $h_{1,2}\ll 1$, and will impose $h_{1,2}=\Theta(1)$ as we close out the proof of Proposition 3.4.

The first element on our route to proving the proposition is the assertion of a strong form of energetic invariance that is enjoyed by Brownian LPP.

Lemma 3.5. *For $\kappa>0$, $M\big[(x,h_1)\to(y,h_2)\big]\equiv\kappa^{1/2}M\big[(x,h_1)\to(x+\kappa^{-1}(y-x),h_2)\big]$.*

Proof. The law of the Brownian motions $B:\mathbb{R}\times\mathbb{Z}\to\mathbb{R}$ that constituent the noise environment of Brownian LPP is invariant under the scaling $B(z,n)\to\kappa^{1/2}B(x+\kappa^{-1}(z-x),n)$. □

By the definition of the shear map τ_K and the specification (1.7) of weight in terms of its unscaled counterpart, energy, we see that $\mathsf{Wgt}_n\big[\tau_K(x,h_1)\to\tau_K(y,h_2)\big]$ is equal to

$$2^{-1/2}n^{-1/3}\Big(M\big[\big(nh_1+2n^{2/3}(x+Kh_1),h_1\big)\to\big(nh_1+2n^{2/3}(y+Kh_2),h_2\big)\big]-(2n+K)h_{1,2}-2n^{2/3}(y-x)\Big).\tag{3.3}$$

Set $x'=x\big(1+2n^{-1/3}K\big)^{-1}$ and $y'=y\big(1+2n^{-1/3}K\big)^{-1}$. Applying Lemma 3.5, we see that

$$\begin{aligned}&M\big[\big(nh_1+2n^{2/3}(x+Kh_1),h_1\big)\to\big(nh_1+2n^{2/3}(y+Kh_2),h_2\big)\big]\\ \equiv\ &\big(1+2n^{-1/3}K\big)^{1/2}M\big[\big(nh_1+2n^{2/3}x',h_1\big)\to\big(nh_1+2n^{2/3}y',h_2\big)\big],\end{aligned}$$

where note that, in the usage of the notation $\equiv$, the two fields are viewed as functions of the variables (x,h_1) and (y,h_2), with the dependence on x and y being communicated via x' and y' in the right-hand case. Translating to scaled coordinates, we see that $\mathsf{Wgt}_n\big[\tau_K(x,h_1)\to\tau_K(y,h_2)\big]$ takes the form

$$\big(1+2n^{-1/3}K\big)^{1/2}\mathsf{Wgt}_n\big[(x',h_1)\to(y',h_2)\big]+R_1+R_2+R_3,$$

where

$$\begin{aligned}R_1&=2^{1/2}n^{2/3}h_{1,2}\big(\big(1+2n^{-1/3}K\big)^{1/2}-1\big);\\ R_2&=2^{1/2}n^{1/3}(y-x)\big(\big(1+2n^{-1/3}K\big)^{-1/2}-1\big);\text{ and}\\ R_3&=-2^{1/2}n^{1/3}Kh_{1,2}.\end{aligned}$$

The term R_1 is seen to take the form $2^{1/2}n^{1/3}h_{1,2}K-2^{-1/2}h_{1,2}K^2+\Theta(n^{-1/6})h_{1,2}$, where the $\Theta(n^{-1/6})$ term is due to $|K|\le\Theta(1)n^{1/18}$. We have that

$$R_2=-2^{1/2}(y-x)K+2^{-1/2}3n^{-1/3}(y-x)K^2+\Theta\big(n^{-2/3}K^3\big)|y-x|.$$

Thus, $R_2=-2^{1/2}(y-x)K+\Theta(n^{-1/6})$, since $|x-y|\le 2K$ and $|K|\le\Theta(1)n^{1/18}$.

Considering now the parabolically adjusted weight $\mathsf{Wgt}_n^{\cup}\big[\tau_K(x,h_1)\to\tau_K(y,h_2)\big]$ that is the subject of Proposition 3.4, we see that it has the form

$$\mathsf{Wgt}_n\big[\tau_K(x,h_1)\to\tau_K(y,h_2)\big]+2^{-1/2}\big(y-x+Kh_{1,2}\big)^2h_{1,2}^{-1}.$$

We expand the right-hand square $(a+b)^2$, $a = y - x$ and $b = Kh_{1,2}$, and make use of the noted forms for R_1, R_2 and R_3, to find that $\mathsf{Wgt}_n^{\cup}\big[\tau_K(x,h_1) \to \tau_K(y,h_2)\big]$ equals (in the sense of the relation $\equiv$)

$$\big(1 + 2n^{-1/3}K\big)^{1/2}\mathsf{Wgt}_n\big[(x',h_1) \to (y',h_2)\big] + 2^{-1/2}(y-x)^2h_{1,2}^{-1} \,+\, \Theta(1)n^{-1/6}h_{1,2}\,.$$

Since x' and y' are small perturbations of x and y, we can already recognise the desired weight $\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big]$ in this display. We next summarise our progress by stating and proving a lemma that clarifies the form of the discrepancy between the obtained and desired terms. A second lemma offers control on the tail of the discrepancy, so that the two lemmas will directly yield Proposition 3.4. We employ the shorthand $\mathsf{Wgt}^{\cup}\Delta(x,y;x',y')$ to denote the parabolically adjusted weight difference

$$\mathsf{Wgt}_{nh_{1,2}}^{\cup}\big[\big(x'h_{1,2}^{-2/3},0\big) \to \big(y'h_{1,2}^{-2/3},1\big)\big] - \mathsf{Wgt}_{nh_{1,2}}^{\cup}\big[\big(xh_{1,2}^{-2/3},0\big) \to \big(yh_{1,2}^{-2/3},1\big)\big]\,.$$

Lemma 3.6. *We have that*

$$\begin{aligned} &\mathsf{Wgt}_n^{\cup}\big[\tau_K(x,h_1) \to \tau_K(y,h_2)\big] \\ \equiv\;\; &\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big] + \Theta(1)n^{-1/6}h_{1,2} + \mathrm{Error}_1(x,y) + \mathrm{Error}_2(x,y)\,, \end{aligned}$$

where

$$\mathrm{Error}_1(x,y) \equiv \big(1 + \Theta(n^{-5/18})\big)\Big(h_{1,2}^{1/3}\mathsf{Wgt}^{\cup}\Delta(x,y;x',y') \,+\, \Theta(1)h_{1,2}^{-4/3}K^3n^{-1/3}\Big)\,;$$

and

$$\mathrm{Error}_2(x,y) \equiv \Theta(n^{-5/18})\mathsf{Wgt}_n\big[(x,h_1) \to (y,h_2)\big]\,.$$

Proof. Setting $E = \mathsf{Wgt}_n\big[(x',h_1) \to (y',h_2)\big] - \mathsf{Wgt}_n\big[(x,h_1) \to (y,h_2)\big]$, and recalling that $|K|$ is at most $\Theta(1)n^{1/18}$, we find that

$$\begin{aligned} \mathsf{Wgt}_n^{\cup}\big[\tau_K(x,h_1) \to \tau_K(y,h_2)\big] \;&\equiv\; \big(1 + \Theta(n^{-5/18})\big)\Big(\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big] + E\Big) \\ &=\; \mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big] + \mathrm{Error}\,, \end{aligned}$$

with $\mathrm{Error} = \mathrm{Error}_1 + \mathrm{Error}_2 + \Theta(1)n^{-1/6}h_{1,2}$, where Error_2 has the form asserted in the lemma; and where $\mathrm{Error}_1 = \big(1 + \Theta(n^{-5/18})\big)E$, for a system E that by the scaling principle is seen to satisfy

$$E \,\equiv\, h_{1,2}^{1/3}\Big(\mathsf{Wgt}_{nh_{1,2}}\big[\big(x'h_{1,2}^{-2/3},0\big) \to \big(y'h_{1,2}^{-2/3},1\big)\big] - \mathsf{Wgt}_{nh_{1,2}}\big[\big(xh_{1,2}^{-2/3},0\big) \to \big(yh_{1,2}^{-2/3},1\big)\big]\Big)\,;$$

and thus

$$E \,\equiv\, h_{1,2}^{1/3}\mathsf{Wgt}^{\cup}\Delta(x,y;x',y') \,-\, 2^{-1/2}(x'-y')^2h_{1,2}^{-4/3} \,+\, 2^{-1/2}(x-y)^2h_{1,2}^{-4/3}\,.$$

Note that $2^{-1/2}(x'-y')^2h_{1,2}^{-4/3} - 2^{-1/2}(x-y)^2h_{1,2}^{-4/3}$ equals $2^{-1/2}h_{1,2}^{-4/3}\Theta\big(|y-x|(|x|+|y|)\big)n^{-1/3}K$, since $x' = x\big(1+\Theta(n^{-1/3}K)\big)$ and $y' = y\big(1+\Theta(n^{-1/3}K)\big)$. Note that $\Theta\big(|y-x|(|x|+|y|)\big) = \Theta(K^2)$ since $|x|$ and $|y|$ are at most $2K$; we have obtained the desired form for Error_1 and thus Lemma 3.6. □

Lemma 3.7. *Recall that $x, y \in \mathbb{R}$ are in absolute value at most $2K$.*

1. *Suppose that $h_{1,2} \le \Theta(1)$, $nh_{1,2} \ge \Theta(1)$ and $\Theta(1)h_{1,2}^{-1/18} \le K \le \Theta(1)n^{1/18}h_{1,2}^{13/18}$. Then*

$$\mathbb{P}\Big(\sup_{|x|,|y|\le K}\big|\mathrm{Error}_1(x,y)\big| \ge \Theta(n^{-1/9})\Big) \le \Theta(1)n^{2/3}K^{-2}\exp\big\{-\Theta(1)n^{1/12}K^{-3/2}\big\}\,.$$

2. Suppose that $n \geq \Theta(1)$, $K \leq \Theta(1)n^{1/18}$ and $h_{1,2} = \Theta(1)$. Then

$$\mathbb{P}\Big(\sup_{|x|,|y|\leq K} \big|\mathrm{Error}_2(x,y)\big| \geq \Theta(n^{-1/6})\Big) \leq \Theta(1)K^2 \exp\big\{ -\Theta(1)n^{1/12}\big\}.$$

Proof of Proposition 3.4. Following directly from the two preceding lemmas is a form of the sought result in which h_1 and h_2 are given; the factor of $\Theta(1)n^2$ is absent from line (3.2) in this version. When a sum indexed by $h_1 \in n^{-1}\mathbb{Z} \cap I_1$ and $h_2 \in n^{-1}\mathbb{Z} \cap I_2$ is performed, the factor of $\Theta(1)n^2$ enters the right-hand side. □

Proof of Lemma 3.7:(1). Our argument principally rests on establishing that

$$\begin{aligned} &\mathbb{P}\bigg(\sup\Big\{ \big|\mathsf{Wgt}^{\cup}\Delta(x,y;x',y')\big| : |x|,|y| \leq K\Big\} \geq \Theta(1)h_{1,2}^{-1/3}n^{-1/9}\bigg) \qquad (3.4)\\ \leq\ & \Theta(1)n^{2/3}K^{-2}\exp\big\{ -\Theta(1)n^{1/12}K^{-3/2}\big\}. \end{aligned}$$

Indeed, $\mathrm{Error}_1(x,y)$ is a sum of two terms, and the displayed bound controls the first. The second, $\Theta(1)h_{1,2}^{-4/3}K^3n^{-1/3}$, is at most $\Theta(n^{-1/9})$ because the needed condition $K \leq \Theta(1)h_{1,2}^{4/9}n^{2/27}$ is implied by the hypotheses of Lemma 3.7(1); thus, it indeed suffices to prove (3.4).

The difference $\mathsf{Wgt}^{\cup}\Delta(x,y;x',y')$ of parabolic weights will be addressed by [Ham19a, Theorem 1.1]. Indeed, if we denote by I and J two compact real intervals, and set, for $r > 0$, the quantity $p_{I,J}(r)$ equal to

$$\mathbb{P}\bigg(\sup\Big\{ \mathsf{Wgt}^{\cup}_{nh_1}\big[(u_1,0)\to(v_1,1)\big] - \mathsf{Wgt}^{\cup}_{nh_1}\big[(u_2,0)\to(v_2,1)\big] : u_1,u_2 \in I, v_1,v_2 \in J\Big\} \geq r\bigg),$$

then [Ham19a, Theorem 1.1] will shortly provide an upper bound on $p_{I,J}(r)$. To make a choice of the interval-pair (I,J) for which such a bound will aid our derivation of (3.4), we begin by noting that

$$\max\big\{|x'-x|,|y'-y|\big\} \leq 4K^2n^{-1/3}.$$

Indeed, $|x'-x| = |x|\big(1-(1+2n^{-1/3}K)^{-1}\big) \leq 8K^2n^{-1/3}$ since $|x| \leq 2K$ and $2n^{-1/3}K \leq 1/2$; and similarly for $|y'-y|$. In considering, as we do, $\mathsf{Wgt}^{\cup}\Delta(x,y;x',y')$, we see that the pair of starting endpoints $x'h_{1,2}^{-2/3}$ and $xh_{1,2}^{-2/3}$ are at distance at most $8K^2n^{-1/3}h_{1,2}^{-2/3}$ and have absolute value at most $2Kh_{1,2}^{-2/3}$. As such, we may find a set $\mathcal{I}$ of compact intervals, each contained in $\big[-2Kh_{1,2}^{-2/3}, 2Kh_{1,2}^{-2/3}\big]$ and of length $16K^2n^{-1/3}h_{1,2}^{-2/3}$, in such a way that $|\mathcal{I}| \leq \Theta(1)n^{1/3}K^{-1}$ while, for every $x \in [-K,K]$, there exists $I \in \mathcal{I}$ for which $x,x' \in I$.

This of course implies that, for any $x,y \in \mathbb{R}$ in absolute value at most $2K$, we may find $(I,J) \in \mathcal{I}^2$ such that $x,x' \in I$ and $y,y' \in J$. By a union bound indexed by $\mathcal{I}^2$, we see that the left-hand side of (3.4) is at most

$$\Theta(1)n^{2/3}K^{-2} \cdot p_{I,J}\Big(\Theta(1)h_{1,2}^{-1/3}n^{-1/9}\Big), \qquad (3.5)$$

where (I,J) is some element of $\mathcal{I}^2$. Thus we see that our usage of [Ham19a, Theorem 1.1] should be made so as to find an upper bound on $p_{I,J}\big(\Theta(1)h_{1,2}^{-1/3}n^{-1/9}\big)$. We set this result's parameters so that $\mathbf{n} = nh_{1,2}$; $\mathbf{x}$ and $\mathbf{y}$ are the left endpoints of the intervals I and J in (3.5); $\boldsymbol{\varepsilon}$ equals the interval length $8K^2n^{-1/3}h_{1,2}^{-2/3}$; and with $\mathbf{R}$ chosen so that $\boldsymbol{\varepsilon}^{1/2}\mathbf{R} = \Theta(1)h_{1,2}^{-1/3}n^{-1/9}$—which is to say, $\mathbf{R} = \Theta(1)n^{1/18}K^{-1}$. What we learn from this application of [Ham19a, Theorem 1.1] is that the $p_{I,J}$ term in (3.5) is at most $\exp\big\{-\Theta(1)n^{1/12}K^{-3/2}\big\}$. Thus are (3.4) and Lemma 3.7(1) obtained. It remains only

to verify that the hypotheses of Lemma 3.7(1) are adequate to permit the application of [Ham19a, Theorem 1.1]. The application requires five conditions:

$$[1]\ K^2n^{-1/3}h_{1,2}^{-2/3} \le \Theta(1)\,;[2]\ nh_{1,2} \ge \Theta(1),\ ;[3]\ Kh_{1,2}^{-2/3} \le \Theta(1)n^{1/18}h_{1,2}^{1/18}\,;$$
$$[4]\ n^{1/18}K^{-1} \ge \Theta(1)\,;\ \text{and}\ [5]\ n^{1/18}K^{-1} \le \Theta(1)n^{1/18}h_{1,2}^{1/18}\,.$$

We consider the further conditions

$$[6]\ K \le n^{1/6}h_{1,2}^{1/3}\Theta(1)\,;[7]\ K \le \Theta(1)n^{1/18}h_{1,2}^{13/18}\,;[8]\ h_{1,2} \le 1\,;\ \text{and}\ [9]\ K \ge \Theta(1)h_{1,2}^{-1/18}$$

Denoting equivalence and implication by $\leftrightarrow$ and $\rightarrow$, note that $[1] \leftrightarrow [6]$; $[3] \leftrightarrow [7]$; $[7] \rightarrow [6]$; $[7,8] \rightarrow 4$; and $[5] \leftrightarrow [9]$. Thus $[2,7,8,9] \rightarrow [1,2,3,4,5]$. Since $[2,7,8,9]$ are the hypotheses of Lemma 3.7(1) and $[1,2,3,4,5]$ permit its proof, this completes the needed hypothesis verification for this result. □

(2). The tail of the parabolically shifted weight $\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big]$ is addressed by [Ham19a, Proposition 1.5]. Indeed, when this result is applied via the scaling principle, and with parameter settings $\mathbf{n} = nh_{1,2}$, $\mathbf{x} = xh_{1,2}^{-2/3}$, $\mathbf{y} = yh_{1,2}^{-2/3}$ and $\mathbf{t} = R$, the outcome is the bound

$$\begin{aligned}&\mathbb{P}\Big(\sup\Big\{ \big|\mathsf{Wgt}_n^{\cup}\big[(x+u,h_1) \to (y+v,h_2)\big]\big| : u \in h_{1,2}^{2/3} \cdot [0,1], v \in h_{1,2}^{2/3} \cdot [0,1]\Big\} \ge h_{1,2}^{1/3}R\Big)\\ \le\ &\exp\big\{ -\Theta(1)R^{3/2}\big\}\,,\end{aligned}$$

where we should bear in mind for the upcoming selection of $R > 0$ that this application demands that $R \le \Theta(1)n^{1/18}h_{1,2}^{1/18}$. A union bound over a mesh of order $\big(Kh_{1,2}^{-2/3}\big)^2$ points $(x,y) \in [-K,K]^2$ then yields that

$$\begin{aligned}&\mathbb{P}\Big(\sup\Big\{ \big|\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big]\big| : x,y \in [-K,K]\Big\} \ge h_{1,2}^{1/3}R\Big)\\ \le\ &\Theta(1)K^2h_{1,2}^{-4/3}\exp\big\{ -\Theta(1)R^{3/2}\big\}\,.\end{aligned}$$

If we set $h_{1,2}^{1/3}R$ equal to $\Theta(1)n^{1/18}$—a choice made so that the needed upper bound on R is satisfied, provided that $h_{1,2}$ has unit order $\Theta(1)$, as we impose that it does—then this right-hand side takes the form $\Theta(1)K^2h_{1,2}^{-4/3}\exp\big\{ -\Theta(1)n^{1/12}h_{1,2}^{-1/2}\big\}$. Since $h_{1,2} = \Theta(1)$, this has the form of the right-hand quantity in the bound maintained by Lemma 3.7(2); thus, if the superscript $\cup$ for Wgt_n were omitted from the preceding display—so that weight without parabolic shift were instead addressed—we would obtain the result that we seek. This alteration is permitted because the parabolic term $(y-x)^2h_{1,2}^{-1}$, being at most $4K^2h_{1,2}^{-1}$, is at most $\Theta(1)n^{2/9}$, in view $K \le \Theta(1)n^{1/9}$ (which is implied by our hypothesis on K) and $h_{1,2} = \Theta(1)$. Thus, we obtain Lemma 3.7(2), subject to verifying that the hypotheses of this result are enough to permit usage of inputs during its proof. These inputs are the application of [Ham19a, Proposition 1.5], which requires the conditions

$$[1]\ nh_{1,2} \ge \Theta(1);\ [2]\ |x-y|h_{1,2}^{-2/3} \le \Theta(1)n^{1/18}h_{1,2}^{2/3};\ [3]\ R \ge \Theta(1);\ \text{and}\ [4]\ R \le \Theta(1)n^{1/18}h_{1,2}^{1/18}\,;$$

and control on a parabolic term, which requires $[5]\ K \le n^{1/9}h_{1,2}^{1/2}$. Recall that $|x-y| \le 2K$, since $|x|$ and $|y|$ are at most K. In the preceding proof, we have set $R = \Theta(1)n^{1/18}h_{1,2}^{-1/3}$. The hypotheses of Lemma 3.7(2), namely that $K \le \Theta(1)n^{1/18}$, $n \ge \Theta(1)$ and $h_{1,2} = \Theta(1)$, are readily seen to imply $[1,2,3,4,5]$; so the needed hypothesis verification has been carried out for the proof of Lemma 3.7(2). □

3.3 Local energetic fluctuation

In this short section, we offer control in Proposition 3.9 on the tail of large local fluctuation in the unscaled geodesic energy, relying on oscillation estimates for Brownian motion which constitutes the underlying noise.

Definition 3.8. *Let $n \in \mathbb{N}$ and $K, \sigma > 0$. Let $\Delta_n(K,\sigma) \subseteq \big(\mathbb{R}\times\mathbb{Z}\big)^2$ denote the set of pairs $(x_1,s_1),(x_2,s_2) \in \mathbb{R}\times\mathbb{Z}$ with $x_1 \le x_2 \le x_1+\sigma$; $s_1 \le s_2 \le s_1+\sigma$; and with each of x_1, x_2, s_1 and s_2 at most nK in absolute value.*

Proposition 3.9. *For $K,\sigma > 0$ and $r \ge 0$,*

$$\begin{aligned}&\mathbb{P}\Big(\sup\Big\{ \big| M\big[(x_1,s_1) \to (x_2,s_2)\big] \big| : \big((x_1,s_1),(x_2,s_2)\big) \in \Delta_n(K,\sigma) \Big\} \ge r \Big) \\ \le\ & n \cdot 2^5 \pi^{-1/2} K \sigma^{-1/2} (\sigma+1) r^{-1} \exp\big\{ -2^{-4} r^2 (\sigma+1)^{-2} \sigma^{-1} \big\} .\end{aligned}$$

Let $L > 0$, and let $f : [0,L] \to \mathbb{R}$. For $I \subseteq [0,L]$ and $r \in (0,L]$, set

$$\omega_I(f,r) \,=\, \sup\Big\{ |f(y)-f(x)| : 0 \le x \le y \le x+r \le L\,,\, x \in I \Big\} .$$

In this way, $(0,L] \to \mathbb{R} : r \to \omega_{[0,L]}(f,r)$ is the modulus of continuity of the function f.

Lemma 3.10. *For $L > 0$, let $B : [0,L] \to \mathbb{R}$ have the law of standard Brownian motion. For $r \in (0,L]$ and $x > 0$,*

$$\mathbb{P}\Big(\omega_{[0,L]}\big(B,r\big) \ge x \Big) \le 2^4 \pi^{-1/2} L r^{-1/2} x^{-1} \exp\big\{ -2^{-4} x^2 r^{-1} \big\} .$$

Proof. Set $I = r\mathbb{Z} \cap [0,L]$. It is readily verified that $\omega_I\big(B,2r\big) \ge \omega_{[0,L]}\big(B,r\big)/2$. This enables the first of the next displayed bounds,

$$\begin{aligned}&\mathbb{P}\Big(\omega_{[0,L]}\big(B,r\big) \ge x \Big) \le \mathbb{P}\Big(\omega_I\big(B,2r\big) \ge x/2 \Big) \\ \le\ & |I| \cdot \mathbb{P}\Big(\sup\big\{ |B(z)| : -1 \le z \le 1 \big\} \ge 2^{-3/2} x r^{-1/2} \Big) \\ \le\ & 2Lr^{-1} \cdot 2^3 \pi^{-1/2} L r^{-1/2} x^{-1} \exp\big\{ -2^{-4} x^2 r^{-1} \big\} ,\end{aligned}$$

where the second bound is due to Brownian translation and scaling symmetries and, in whose third, $L \ge r$ is used to bound $|I|$ above by $2L/r$, and the reasoning in (2.15) bounds above the probability term. □

Proof of Proposition 3.9. The concerned supremum is at most

$$\big(\sigma+1\big) \cdot \sup\Big\{ \omega_{[-nK,nK]}\big(x \to B(x,s),\sigma\big) : s \in \mathbb{Z} \cap [-K,K] \Big\} ,$$

where recall that the constituent curves $B : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ in the Brownian LPP noise environment are independent two-sided standard Brownian motions. The displayed supremum thus has probability of being at least $r/(\sigma+1)$ given by the bound offered by Lemma 3.10 with $\mathbf{L} = 2nK$, $\mathbf{r} = \sigma$ and $\mathbf{x} = r/(\sigma+1)$. The result is Proposition 3.9. □

3.4 Global polymer fluctuation

In this section, we present and prove the next result, which concerns the location at which the polymer $\rho_n = \rho_n\big[(0,0) \to (x,1)\big]$ departs a level $a \in n^{-1}\mathbb{Z} \cap (0,1)$. This theorem is in essence a weaker form of Corollary 1.5; it will be needed for our principal proofs.

In a shorthand usage, ρ_n^x will denote $\rho_n\big[(0,0) \to (x,1)\big]$ (a polymer that is almost surely unique for given $x \in \mathbb{R}$, by Lemma 3.2). By the convention of Subsection 1.7.8, $\rho_n^x(a)$ thus denotes the concerned location, at which a forward-in-time tracing of this polymer departs from the horizontal line $\mathbb{R}\times\{a\}$.

Theorem 3.11. *For K any compact interval of $(0,1)$, there exist positive constants $H = H(K)$ and $h = h(K)$ and an integer $n_0 = n_0(K)$ such that, if $n \in \mathbb{N}$, $R \in \mathbb{R}$, $a \in n^{-1}\mathbb{Z} \cap K$ and $x \in \mathbb{R}$ satisfy $n \geq n_0$, $|R| \leq hn^{1/9}$ and $|x| \leq n^{2/3}$, then*

$$\mathbb{P}\Big(\big|\rho_n^x(a) - xa\big| \geq R\Big) \leq H \exp\big\{-hR^3\big\}.$$

Next is a result that includes an important special case of Theorem 3.11. We write ρ_n for $\rho_n^0 = \rho_n\big[(0,0) \to (0,1)\big]$.

Proposition 3.12. *For K any compact interval of $(0,1)$, there exist positive constants $H = H(K)$ and $h = h(K)$ and an integer $n_0 = n_0(K)$ such that, if $n \in \mathbb{N}$, $R \in \mathbb{R}$ and $a \in n^{-1}\mathbb{Z} \cap K$ satisfy $n \geq n_0$ and $|R| \leq hn^{1/9}$, then*

$$\mathbb{P}\Big(\rho_n(a) \in [R-1, R+1]\Big) \leq H \exp\big\{-hR^3\big\}$$

and

$$\mathbb{P}\Big(\big|\rho_n(a)\big| \geq hn^{1/9}\Big) \leq H \exp\big\{-hn^{1/3}\big\}.$$

Proof. Recall from Lemma 2.1(2) that $\rho_n(a)$ is the almost surely unique maximizer of the random function $x \to Z_n(x,a)$, and recall the formula (2.4) for the latter process.

To prove the first bound, we first note from (2.4) that the event that $Z_n(x,a) \geq -r$ for given $x, r \in \mathbb{R}$ entails that either $a^{1/3}\mathsf{Nr}\mathcal{L}^{\uparrow;a}_{n;(0,0)}\big(a^{-2/3}x\big) \geq -2^{-1}r$ or

$$(1-a_+)^{1/3}\mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a_+}\big((1-a_+)^{-2/3}x^-\big) \geq -2^{-1}r\,.$$

Thus, the event $\big\{\sup_{x\in[R-1,R+1]} Z_n(x,a) \geq -R^2\big\}$ is contained in

$$\begin{aligned} &\left\{\sup_{x \in a^{-2/3}\cdot[R-1,R+1]} \mathsf{Nr}\mathcal{L}^{\uparrow;a}_{n;(0,0)}(x) \geq -2^{-1}a^{-1/3}R^2\right\} \\ \cup\ &\left\{\sup_{x \in (1-a_+)^{-2/3}\cdot[R-1,R+1]} \mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a_+}(x^-) \geq -2^{-1}(1-a_+)^{-1/3}R^2\right\}. \end{aligned}$$

We now apply the regularity of normalized ensembles expressed in Proposition 2.3(1,2), and a tail bound [Ham22, Proposition 2.28] for the deviation of regular ensemble top curves relative to parabolic curvature. Indeed, using the boldface notation, we choose the latter result's parameter $\mathbf{n}$ equal to na and to $n(1-a_+)$, and thus find that the probability of each event in the above union is at most

$$H \max\Big\{\exp\big\{-hR^3\big\}, \exp\big\{-hn^{1/2}\big\}\Big\},$$

where the existence of positive constants H and h is ensured because we may select the lower bound $n_0(K)$ so that na and $n(1-a_+)$ are at least the lower bound on $\mathbf{n}$ demanded in [Ham22, Proposition 2.28].

Since $|R| \leq hn^{1/9}$, we find that

$$\mathbb{P}\left(\sup_{x\in[R-1,R+1]} Z_n(x,a) \geq -R^2\right) \leq 2H \exp\big\{-hR^3\big\}.$$

However, $\sup_{x\in\mathbb{R}} Z_n(x,a)$ equals $\mathsf{Wgt}_n\big[(0,0) \to (0,1)\big]$ and thus has probability to exceed $-R^2$ which is seen to be least $1 - C\exp\big\{-cR^3\big\}$ provided that $R \leq hn^{1/3}$ (with $h = c^{1/2}$) by Lemma 3.1(2).

From these inferences, the former assertion of Proposition 3.12 follows after a relabelling of H and h.

To prove the latter bound, we first note from (2.4) that the event that $Z_n(x,a) \geq -gn^{2/9}$ for given $x \in \mathbb{R}$ entails that either $\mathsf{Nr}\mathcal{L}^{\uparrow;a}_{n;(0,0)}\big(a^{-2/3}x\big) \geq -2^{-1}gn^{2/9}$ or

$$(1-a_+)^{1/3}\mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a_+}\big((1-a_+)^{-2/3}x^-\big) \geq -2^{-1}gn^{2/9}\,.$$

By Proposition 2.3(1,2), and [Ham22, Proposition 2.30], for any $g' > 0$, there exist positive G and $g > 0$, such that, for $a \in n^{-1}\mathbb{Z} \cap K$, the probabilities

$$\mathbb{P}\Big(\exists\, y \in \mathbb{R} : |y| \geq a^{-2/3}g'n^{1/9}\,,\, \mathsf{Nr}\mathcal{L}^{\uparrow;a}_{n;(0,0)}(y) \geq -2^{-1}gn^{2/9}\Big)$$

and

$$\mathbb{P}\Big(\exists\, y \in \mathbb{R} : |y| \geq (1-a_+)^{-2/3}g'n^{1/9}\,,\, \mathsf{Nr}\mathcal{L}^{\downarrow;(0,1)}_{n;a_+}(y) \geq -2^{-1}gn^{2/9}\Big)$$

are at most $G\exp\big\{-gn^{1/3}\big\}$. Thus, the probability that $Z_n(x,a)$ attains a value of at least $-gn^{2/9}$ for $x \in \mathbb{R}$ satisfying $|x| \geq g'n^{1/9}$ is at most $2G\exp\big\{-gn^{1/3}\big\}$. However, $\sup_{x\in\mathbb{R}} Z_n(x,a)$ equals $\mathsf{Wgt}_n\big[(0,0) \to (0,1)\big]$ and thus has probability to exceed $-gn^{2/9}$ which is seen by Lemma 3.1(2) to be at least $1 - C\exp\big\{-cg^{3/2}n^{1/3}\big\}$. From these inferences, the latter assertion of Proposition 3.12 follows. □

To obtain Theorem 3.11 from Proposition 3.12, we need to reach a comparable conclusion about polymers of route $(0,0) \to (x,1)$ as we have in the case $x = 0$. In this task, we benefit from the strong invariance property of Brownian LPP that we indicated in Lemma 3.5, but which is now presented in a scaled guise in the next lemma. Recall that $\rho^x_n(a)$ denotes the location of departure $\rho_n\big[(0,0) \to (x,1)\big](a)$ of this polymer from the horizontal line $\mathbb{R} \times \{a\}$.

Lemma 3.13.

1. *Let $x > -2n^{2/3}$. The random processes $n^{-1}\mathbb{Z} \cap [0,1] \to \mathbb{R}$ that map a to $\rho^x_n(a) - xa$ and $\big(1 + 2^{-1}n^{-2/3}x\big)\rho_n(a)$ are equal in law.*
2. *Let $b > -2n^{2/3}$. Let $\tau_b : \mathbb{R}^2 \to \mathbb{R}^2$ denote the shear map $\tau(x,y) = (x + yb, y)$. The field of polymers ρ_n is indexed by starting and ending points in $\mathbb{R} \times n^{-1}\mathbb{Z}$ (with a formal empty-set value for inadmissible choices of index). The image under τ_b of this field is equal in law to the field of polymers $\big(1 + 2^{-1}n^{-2/3}b\big)\rho_n$.*

Proof: (1). The geodesics $\Gamma_n\big[(0,0) \to (n + 2n^{2/3}x, n)\big]$ and $\Gamma_n\big[(0,0) \to (n,n)\big]$ are mapped to ρ^x_n and ρ_n by the scaling map R_n in (1.3). We *claim* that the former geodesic is mapped to a distributional copy of the latter by applying the transformation $\mathbb{R} \times \mathbb{Z} \to \mathbb{R} \times \mathbb{Z}$, $(z,h) \to \big(z, (1 + 2^{-1}n^{-2/3}x)^{-1}h\big)$. Indeed, the image of the noise environment of static Brownian LPP under this transformation is a copy of Brownian LPP in which the constituent Brownian motions have rate $1 + 2^{-1}n^{-2/3}x$. The energy of any given staircase is thus simply the product of $\big(1 + 2^{-1}n^{-2/3}x\big)^{1/2}$ and the energy of the staircase in Brownian LPP where the motions have the standard rate of one. Since this factor is present in computing the energy of all staircases, its presence does not affect the status of the geodesic, so that the claim is confirmed. Given the claim, we apply the scaling map R_n to obtain Lemma 3.13(1).

(2). The above inferences apply equally when we instead consider the (z_1, z_2)-indexed fields of geodesics $\Gamma_n\big[(2n^{2/3}z_1, 0) \to (n + 2n^{2/3}(z_2 + x), n)\big]$ and $\Gamma_n\big[(2n^{2/3}z_1, 0) \to (2n^{2/3}n + z_2, n)\big]$ where $z_1, z_2 \in \mathbb{R}$. □

Allying Proposition 3.12 with Lemma 3.13 leads to the next result, which equips us to give a short proof of Theorem 3.11.

Corollary 3.14. *For K any compact interval of $(0,1)$, there exist positive constants $H = H(K)$ and $h = h(K)$ and an integer $n_0 = n_0(K)$ such that, if $n \in \mathbb{N}$, $R \in \mathbb{R}$, $a \in n^{-1}\mathbb{Z} \cap K$ and $x \in \mathbb{R}$ satisfy $n \geq n_0$, $|R| \leq hn^{1/9}$ and $|x| \leq n^{2/3}$, then*

$$\mathbb{P}\Big(\rho^x_n(a) - xa \in [R-1, R+1]\Big) \;\leq\; H\exp\big\{-hR^3\big\}\,,$$

and

$$\mathbb{P}\Big(\big|\rho_n^x(a) - xa\big| \geq hn^{1/9}\Big) \leq H\exp\big\{-hn^{1/3}\big\}.$$

Proof. The dilation factor $1 + 2^{-1}n^{-2/3}x$ in Lemma 3.13 lies on the interval $[1/2, 3/2]$ under the present hypothesis that $|x| \leq n^{2/3}$. Denote this factor by d. When the probabilities in Corollary 3.14 are addressed by Proposition 3.12 via Lemma 3.13, the proposition is applied with $\mathbf{R} = d^{-1}R, d^{-1}R - 1$ and $d^{-1}R + 1$ in alliance with a simple union bound. Thus we obtain the corollary by suitably adjusting the values of the constants H and h. □

Proof of Theorem 3.11. Note that the set $\mathbb{R} \setminus (-R, R)$ is a subset of the union of intervals:

$$\big(-\infty, -hn^{1/9}\big] \cup \big[hn^{1/9}, \infty\big) \cup \bigcup_{i=\lceil R\rceil}^{\lfloor hn^{1/9}\rfloor} \Big(\big[-i-1, -i+1\big] \cup \big[i-1, i+1\big]\Big).$$

By the latter assertion of Corollary 3.14, the probability that $\rho_n^x(a) - xa$ takes a value in the union of the first two displayed intervals is bounded above by $H\exp\big\{-hn^{1/3}\big\}$. Let $i \in \mathbb{N}$ satisfy $i \leq hn^{1/9}$. By this corollary's former assertion applied with $\mathbf{R}$ equal to either i or $-i$, the probability that $\rho_n^x(a) - xa$ assumes a value in $[i-1, i+1]$, or in $[-i-1, -i+1]$, is at most $H\exp\big\{-hi^3\big\}$. Summing these bounds over $i \in [\![\lceil R\rceil, \lfloor hn^{1/9}\rfloor]\!]$, and employing the first inference alongside $|R| \leq hn^{1/9}$, we obtain Theorem 3.11 after a relabelling of H and h. □

3.5 Compact uniform control on polymer weight

In this section, we present a further tool needed for the proofs of Theorems 1.4 and 1.6. Control is gained on the tail of the maximum absolute value of the parabolic weight of polymers whose endpoints are varied over compact regions.

Proposition 3.15. *For $n \geq \Theta(1)$, $\Theta(1) \leq R \leq n^{1/30}$ and $0 < K \leq n^{1/46}$,*

$$\mathbb{P}\left(\sup_{\substack{(x,h_1)\in[-K,K]\times[-3,-1]\\(y,h_2)\in[-K,K]\times[1,3]}} \Big|\mathsf{Wgt}_n^{\cup}\big[(x,h_1)\to(y,h_2)\big]\Big| \geq R\right) \leq \Theta(1)K^2\exp\big\{-\Theta(1)R^{3/2}\big\}.$$

The argument that we will give for Proposition 3.15 mimics that of [BSS16, Propositions 10.1 and 10.5].

Proof of Proposition 3.15. The next presented proposition is sufficient to prove this assertion. Indeed, Proposition 3.16(2) and (3) imply it. □

Proposition 3.16. *For $n \geq \Theta(1)$ and $\Theta(1) \leq R \leq \Theta(1)n^{4/9}(\log n)^{-2}$, the following hold.*

1. *The probability that*

$$\inf\Big\{\mathsf{Wgt}_n^{\cup}\big[(0,0)\to(y,h)\big] : y \in [-1,1], h \in [1,3]\Big\} \leq -R \tag{3.6}$$

   is at most $\Theta(1)\exp\big\{-\Theta(1)R^{3/2}\big\}$.
2. *For $K \leq \Theta(1)n^{1/18}$, the probability that the condition*

$$\inf\Big\{\mathsf{Wgt}_n^{\cup}\big[(x,h_1)\to(y,h_2)\big] : x, y \in [-K,K], h_1 \in [-3,-1], h_2 \in [1,3]\Big\} \leq -R$$

   holds is at most $\Theta(1)K^2\exp\big\{-\Theta(1)R^{3/2}\big\} + \Theta(1)n^3K^4\exp\big\{-\Theta(1)n^{1/12}K^{-3/2}\big\}$.
3. *Likewise for the condition*

$$\sup\Big\{\mathsf{Wgt}_n^{\cup}\big[(x,h_1)\to(y,h_2)\big] : x, y \in [-K,K], h_1 \in [-3,-1], h_2 \in [1,3]\Big\} \geq R.$$

Proof. We will starting by proving the proposition's first part; and then the later parts when $K = 1$. Note that the parabolic discrepancy $\mathsf{Wgt}_n^{\cup}\big[(x,h_1) \to (y,h_2)\big] - \mathsf{Wgt}_n\big[(x,h_1) \to (y,h_2)\big]$ is uniformly bounded in absolute value over all concerned routes $(0,0) \to (y,h)$ or $(x,h_1) \to (y,h_2)$ in these cases. Thus, we may prove these assertions with Wgt_n in place of $\mathsf{Wgt}_n^{\cup}$.

(1). We start by defining an infinite tree $T = (V,E)$, embedded in the plane and rooted at $(0,0)$, each of whose vertices has four offspring. The children of each vertex will be called left-low, left-high, right-low and right-high. The root is the unique vertex in generation zero. Its left-low child is at $(-1/2,1/2)$; its left-high child at $(-1/2,1/2+1)$; its right-low child at $(1/2,1/2)$; and its right-high child at $(1/2,1/2+1)$. Let w denote the four-vector of planar points whose coordinates are these respective locations. The offspring of any child c of the root lie at $c + w/2$. Iteratively, suppose that the locations of any vertex in T of generation at most $m \in \mathbb{N}$ have been determined. Vertices in generation $m+1$ are placed in locations in the set $\big\{v + 2^{-m}w : v \in V_n\big\}$, with an edge running from parent to child.

It is straightforward that the closure of the vertex set V contains $[-1,1] \times [1,2]$. Let (y,h) be an element in the latter set for which $h \in n^{-1}\mathbb{Z}$. In order to find a lower bound on $\mathsf{Wgt}_n\big[(0,0) \to (y,h)\big]$, we aim to consider the sum of the weights of polymers that interpolate the endpoints of the edges in T along the end of the tree that runs from $(0,0)$ to the element (y,h) of the closure of V. However, such endpoints lie in the plane, rather than in $\mathbb{R} \times n^{-1}\mathbb{Z}$; we begin by finding a nearby path through the latter index set. This path P will follow the tree end until its distance from the destination (y,h) is a large multiple (of order $\log n$) of the microscopic spacing n^{-1}; then it will jump directly to (y,h), so that the approximating path is of finite length.

To any planar point $(z,s) \in \mathbb{R}^2$, we associate $(z,s)^{\downarrow}$, the element of $\mathbb{R} \times n^{-1}\mathbb{Z}$ that is first encountered on a journey due south that commences at (z,s).

We now specify $j \in \mathbb{N}$ to denote the smallest integer such that $2^{-j} \le K_0 n^{-1} \log n$, where K_0 is a large constant.

Let $\big\{(x_i,s_i) : i \in \mathbb{N}\big\}$ denote a sequence of adjacent elements of V with $(x_0,s_0) = 0$ whose limit equals (y,h). Set $\big(x_i(n),s_i(n)\big) = (x_i,s_i)^{\downarrow}$. Respecify $\big(x_j(n),s_j(n)\big)$ to equal (y,h). The path P has elements $\big(x_i(n),s_i(n)\big)$ for $i \in [\![0,j]\!]$, with edges between consecutively indexed elements.

The path P indeed offers a lower bound on the polymer weight from $(0,0)$ to (y,h). Namely,

$$\mathsf{Wgt}_n\big[(0,0) \to (y,h)\big] \ge \sum_{i=0}^{j-1} \mathsf{Wgt}_n\big[\big(x_i(n),s_i(n)\big) \to \big(x_{i+1}(n),s_{i+1}(n)\big)\big]\,. \tag{3.7}$$

An edge in T that connects a vertex (u,h_1) in generation $m \in \mathbb{N}$ to one of its children (v,h_2) satisfies $h_{1,2} \in 2^{1-m} \cdot \{1,3\}$ and $|v-u| = 2^{-1-m}$. Let $R > 0$. The edge is called R-typical if

$$2^{(m+1)/3}\Big|\mathsf{Wgt}_n\big[(u,h_1)^{\downarrow} \to (v,h_2)^{\downarrow}\big]\Big| \le R(m+1)^{2/3}\,. \tag{3.8}$$

The left-hand quantity is a normalized weight—it is random but of unit order, satisfying tail bounds that are uniform over edges in the tree T. Indeed, the scaling principle from Section 3.1.1 and Lemma 3.1 imply that any given edge crossing between generations m and $m+1$ such that $n2^{-(m+1)} \ge n_0$—a condition verified when $m \le j$—fails to be R-typical with probability that is at most $C\exp\big\{-cR^{3/2}(m+1)\big\}$. The parabolic curvature term in Lemma 3.1 has been discarded because it takes the form $h_{1,2}^{-4/3}(u-v)^2 = h_{1,2}^{-4/3} \cdot h_{1,2}^2 = h_{1,2}^{2/3} \le 1$.

We now specify the event $\mathsf{Typical} = \mathsf{Typical}(R)$ that every edge in the tree between generations m and $m+1$ with $m \in [\![0, j-1]\!]$ is R-typical. We see that

$$\mathbb{P}\big(\neg\, \mathsf{Typical}\big) \,\le\, \sum_{m=0}^{j-1} 4^m C \exp\big\{ -cR^{3/2}(m+1)\big\}\,,$$

whose right-hand side is at most $2C\exp\big\{-2^{-1}cR^{3/2}\big\}$ provided that $R \ge \Theta(1)$.

In its final step $\big(x_{j-1}(n), s_{j-1}(n)\big) \to \big(x_j(n), s_j(n)\big)$, P departs from the path beaten along T. In this regard, we *claim* that

$$\begin{aligned} &\mathbb{P}\Big(\Big| \mathsf{Wgt}_n\big[\big(x_{j-1}(n), s_{j-1}(n)\big) \to \big(x_j(n), s_j(n)\big)\big] \Big| \ge 1 \Big) \qquad (3.9)\\ \le\;& \Theta(1) n^{2/3} (\log n)^{1/2} K_0^{1/2} \exp\big\{ -2^{-5} n^{2/3} (K_0 \log n + 2)^{-3} \big\}\,. \end{aligned}$$

The differences $x_j(n) - x_{j-1}(n)$ and $s_j(n) - s_{j-1}(n)$ are at most $K_0 n^{-1} \log n + n^{-1}$ in absolute value; and the four coordinates $x_{j-1}(n)$, $s_{j-1}(n)$, $x_j(n)$ and $s_j(n)$ are at most two in this sense. Merely an order of $\log n$ microscopic levels separate the vertical coordinates $s_{j-1}(n)$ and $s_j(n)$, making the unscaled picture a suitable context for proving the claim. Indeed, specifying weight in terms of energy via (1.7), and applying Proposition 3.9 with $\mathbf{K} = 3$, $\boldsymbol{\sigma} = K_0 \log n + 1$ and $\mathbf{r} = 2^{-1/2} n^{1/3}$, we obtain the claimed (3.9).

Applying (3.8) and (3.9) to (3.7), we find that, when $\Theta(1) \le R \le \Theta(1) K_0^{-2} n^{4/9} (\log n)^{-2}$,

$$\mathbb{P}\left(\inf_{\substack{y \in [-1,1] \\ h \in [1,2]}} \mathsf{Wgt}_n\big[(0,0) \to (y,h)\big] \le -\kappa R \right) \le \Theta(1) \exp\big\{ -\Theta(1) R^{3/2} \big\}$$

for suitable positive d, H and h; here, we set $\kappa = 1 + \sum_{m=0}^{\infty} (m+1)^{2/3} 2^{-(m+1)/3}$. The appearance of κ may be absorbed by the usage of $\Theta(1)$ notation. The result is Proposition 3.16(1) with the instance of $[1,3]$ in (3.6) replaced by $[1,2]$. We need to obtain the counterpart statement where the interval in question is $[2,3]$. We move the tree T upwards by one unit and add to it the edge that connects $(0,0)$ to $(0,1)$. We treat this edge in the preceding analysis as if it connects vertices of generation zero and one, and follow the rest of the analysis unchanged. This completes the proof of Proposition 3.16(1).

(2) for $K = 1$. We prove this assertion in the *stronger form* where the probability upper bound is $\Theta(1) K^2 \exp\big\{ -\Theta(1) R^{3/2} \big\}$.

Let $x, y \in [-1,1]$ and note that $\mathsf{Wgt}_n\big[(x,h_1) \to (y,h_2)\big]$ is at least the sum of $\mathsf{Wgt}_n\big[(x,h_1) \to (0,0)\big]$ and $\mathsf{Wgt}_n\big[(0,0) \to (y,h_2)\big]$. Since $\big\{\mathsf{Wgt}_n\big[(x,h_1) \to (0,0)\big] : (x,h_1) \in [-K,K] \times [-3,-1]\big\}$ has the law of $\big\{\mathsf{Wgt}_n\big[(0,0) \to (y,h_2)\big] : (y,h_2) \in [-K,K] \times [1,3]\big\}$ when the identification of (x,h_1) with $(-x,-h_1)$ is made, two applications of Proposition 3.16(1) and a union bound yield Proposition 3.16(2), up to a relabelling of the constants H and h.

(3) for $K = 1$. We also prove this assertion in the above mentioned stronger form.

Note that the occurrence of the condition

$$\sup\Big\{ \mathsf{Wgt}_n\big[(x,-1) \to (y,1)\big] : (x,h_1) \in [-K,K] \times [-3,-1]\,,\; (y,h_2) \in [-K,K] \times [1,3] \Big\} > R\,,$$

alongside the conditions

$$\min\left\{ \inf_{\substack{x \in [-K,K] \\ h_1 \in [-3,1]}} \mathsf{Wgt}_n\big[(0,-4) \to (x,-1)\big]\,,\; \inf_{\substack{y \in [-K,K] \\ h_2 \in [1,3]}} \mathsf{Wgt}_n\big[(y,h_2) \to (0,4)\big] \right\} \,\ge\, -R/4\,, \qquad (3.10)$$

entails that

$$\mathsf{Wgt}_{n;(0,-4)}^{(0,4)} > R/2\,. \tag{3.11}$$

Bounds on the failure probabilities of the two conditions (3.10) arise by applying Proposition 3.16(2) in light of simple scaling properties. The one-point upper tail control offered by Lemma 3.1(1) via the scaling principle provides an upper bound on the probability of (3.11). Thus we obtain Proposition 3.16(3). □

(2,3) for general K. Proposition 3.4 permits us to reduce this derivation to the just obtained stronger form of the special case where $K = 1$, at the expense of the appearance of the additive term $\Theta(1)n^3K^4 \exp\big\{-\Theta(1)n^{1/12}K^{-3/2}\big\}$. Indeed, this further term takes the form of a multiple $\Theta(1)K^2$ of an upper bound on the probability appearing in Proposition 3.4. The reason for this form is that the indexing set-pair $\big([-K,-K]\times[-3,-1],[-K,K]\times[1,3]\big)$ may be covered by a union of at most $\Theta(1)K^2$ sets that are *distortions* of the *standard* set-pair $\big([-1,-1]\times[-3,-1],[-1,1]\times[1,3]\big)$. By distortion, we mean that both elements in the pair are the image of their standard counterpart by an application of a given shear map of the form τ_κ, where $|\kappa| \le \Theta(K)$, composed with a horizontal translation. Proposition 3.4 with $\mathbf{K} = \kappa$ offers control over weights indexed by pairs (x,h_1) and (y,h_2) lying in a given image in this sense, and a union bound is then taken over the $\Theta(1)K^2$ distortions, so that the general K result is obtained with the indicated additive term. Note that it is these applications of Proposition 3.4 which are responsible for the imposition of the hypothesis that $K = \Theta(1)n^{1/18}$ in Proposition 3.16. This completes the proof of this proposition. □

3.6 Modulus of continuity for polymer weight and geometry

Here we prove Theorems 1.4 and 1.6, and Corollary 1.5. To survey the route ahead, it is perhaps helpful to recall first that the two theorems are expressed via a parameter $k \in \mathbb{N}$, and that they control polymer fluctuation and subpath weights uniformly along stretches of vertical extension at most 2^{-k}. Theorem 3.11 offers control on the mid-life fluctuation of a polymer that traverses between given endpoints. By use of polymer ordering, a uniform form of this control may be gained for the fluctuation at height one-half of all polymers on routes $(x,0) \to (y,1)$ for $x,y \in \mathbb{R}$ of absolute value at most r. A mesh of points in a rectangular pattern with heights zero, one-half and one—and with horizontal distance between adjacent points shrunk in essence by a factor of $2^{2/3}$ from the level zero value of order r—may thus be constructed such that polymers progress without sudden left or right movements between consecutive levels in the mesh. Theorem 3.11 may be applied to mesh endpoints at adjacent vertical levels, so that, in view of polymer sandwiching, control on polymer fluctuation is gained at heights one-quarter and three-quarters. The construction proceeds iteratively, down to a dyadic scale that we will label $j \in \mathbb{N}$. The outcome is that Lemmas 3.18 and 3.19 assert that, with a high probability indicated in Lemma 3.17(3), polymer fluctuation between consecutive mesh heights at level j—at distance 2^{-j}—is controlled as Theorem 1.4 asserts, with an upper bound of the form $\Theta(1)2^{-2j/3}(\log j)^{1/3}$. For Theorem 1.4 to be obtained, what remains is to give the Kolmogorov continuity criterion argument, in which a sum over dyadic scales j at least k is performed to find a similar fluctuation upper bound between generic heights at displacement of order 2^{-k}. There is a slight twist: the mesh construction becomes unmanageable at close to the microscopic scale, and a separate but simple device, Lemma 3.21, is used to handle the very short scale. This apparatus also delivers Theorem 1.6, which concerns polymer subpath weight. The aspect ratio of rectangles whose consecutive corners are vertically or horizontally adjacent in the mesh at scale j respects the two-thirds KPZ exponent up to a factor that is polylogarithmic in j. We may thus apply (via the scaling principle) the uniform control gained in Proposition 3.15 on

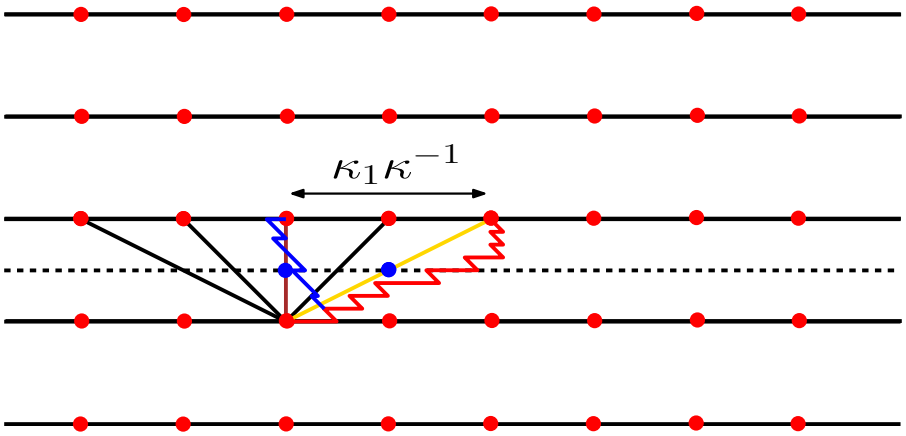

Figure 4: The graph structure G_i on vertex set V_i, in which edges in E_i connect elements that are vertically adjacent and horizontally close. The rightmost depicted edge is closed on account of fluctuation of the polymer away from the midpoint of the straight line representing the edge, while the vertical edge is open.

weights for polymers crossing rectangles to derive Theorem 1.6.

We begin, then, by constructing the meshes. Let $n \in \mathbb{N}$. For each $i \in \mathbb{N}$ for which $2^i \leq n-1$, let Y_i denote a subset of $n^{-1}[\![0,n]\!]$ of cardinality 2^i+1 such that, for each $k \in [\![0,2^i]\!]$, there exists $y \in Y_i$ satisfying $\bigl|y-k2^{-i}\bigr| \leq 2^{-1}n^{-1}$. For example, we take $Y_0 = \{0,1\}$ and $Y_1 = \{0, n^{-1}\lfloor n/2 \rfloor, 1\}$. Note further that this set sequence may be constructed so that each set Y_i for $i \in \mathbb{N}^+$ contains its predecessor, with the set $Y_i \setminus Y_{i-1}$ of newcomers interlacing the set Y_{i-1} of existing members.

Let r and κ be positive. Set $X_0 = \{-r, r\}$. For $i \in \mathbb{N}^+$ and $j \in \mathbb{N}$, set $X_i = X_i(r)$ according to

$$X_i(r) = 2^{-2i/3}(j+1)^{1/3}\kappa r\mathbb{Z} \cap (-2r, 2r)\,.$$

The parameter r appears in Theorem 1.4. The just introduced κ is presently unspecified; it will shortly be joined by two further positive parameters κ_1 and κ_2, with the triple being specified when the relation that is demanded of them is derived.

Suppose given $j \in \mathbb{N}$. Let $i \in [\![0,j]\!]$. We consider the graph $G_i = G_i(j) = \bigl(V_i, E_i\bigr)$ whose vertex set V_i equals $X_i \times Y_i$. Note that two dyadic scale parameters i and j (with $i \leq j$) are implicated in this definition, because, although we omit reference to j from the notation X_i, this parameter enters via this object. What we called the mesh of scale j in overview is $G_j = G_j(j)$—in our inductive gaining of control on polymer fluctuation, we will treat the parameter $j \in \mathbb{N}$ as fixed, with a view to controlling fluctuation at vertical displacement $\Theta(1)2^{-j}$; we will descend through the scales 2^{-i}, considering the $G_i(j)$ as i increases to its final value of j.

Two elements $v, w \in V_i$ are *vertically adjacent* if their vertical coordinates v_2 and w_2 differ, and no element of Y_i lies in the open interval delimited by u_2 and v_2. The elements v and w are *horizontally close* if their horizontal coordinates differ by at most $\kappa_1 2^{-2i/3}(j+1)^{1/3}r$, where κ_1 is a parameter that is at least κ. The edge between v and w belongs to the edge-set E_i precisely when the vertices v and w are vertically adjacent and horizontally close. Note that V_i may be identified with a rectangle in the lattice $\mathbb{Z}^2$ and that, when this identification is made, an element of E_i is an edge between vertices that differ by one vertical unit and by at most $\kappa_1\kappa^{-1}$ horizontal units. See Figure 4.

We now assign a status of open, or closed, to each edge in each of the graphs G_i. The assignation of this status will be random, and determined by a common realization of

static Brownian LPP, governed by a law labelled $\mathbb{P}$ in a manner we now specify.

Let $e \in E_i$ be an edge of the graph G_i that connects the vertices (x_1, h_1) and (x_2, h_2), with $h_1 < h_2$. The distance $h_{1,2} = h_2 - h_1$ satisfies $|h_{1,2} - 2^{-i}| \le n^{-1}$. There is a unique element h^+ of Y_{i+1} in (h_1, h_2), and its distance from both h_1 and h_2 differs from 2^{-i-1} by at most n^{-1}.

Let $\ell\big[(x_1, h_1) \to (x_2, h_2)\big]$ denote the planar line segment with endpoints (x_1, h_1) and (x_2, h_2). Extending the notational abuse introduced in Subsection 1.7.8, we write $\ell\big[(x_1, h_1) \to (x_2, h_2)\big](h)$ for the horizontal coordinate at which this line segment visits the vertical coordinate $h \in [h_1, h_2]$.

Let $\kappa_2 > 0$ be a further parameter. The edge $e \in E_i$ will be declared to be r-*open* if

$$\Big| \rho_n\big[(x_1, h_1) \to (x_2, h_2)\big](h^+) - \ell\big[(x_1, h_1) \to (x_2, h_2)\big](h^+) \Big| \le \kappa_2 2^{-2i/3} \big(j+1\big)^{1/3} r\,. \tag{3.12}$$

An edge $e \in E_i$ that is not r-open is r-*closed*.

Lemma 3.17. *There exist positive h, g_1, g_2, g_3 and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfy $n \ge n_0$ and $2^j \le hn$, and $r \in \mathbb{R}$ satisfies $r_0 \le r \le n^{1/10}$, the following hold.*

1. *Let $i \in [\![0, j]\!]$. The $\mathbb{P}$-probability that a given edge $e \in E_i$ is $(j+1)^{1/3}r$-closed is at most $\exp\big\{ - g_1 r^3 (j+1)\big\}$.*
2. *Let $i \in [\![0, j]\!]$. The $\mathbb{P}$-probability that a $(j+1)^{1/3}r$-closed edge in E_i exists is at most $\exp\big\{ - g_2 r^3 (j+1)\big\}$.*
3. *The $\mathbb{P}$-probability that there exist $i \in [\![0, j]\!]$ and an edge in E_i that is $\big(j+1\big)^{1/3}r$-closed is at most $\exp\big\{ - g_3 r^3 (j+1)\big\}$.*

The various constants in this result may depend on κ, κ_1 and κ_2.

Proof of Lemma 3.17: (1). By the scaling principle from Section 3.1.1, the probability $\mathbb{P}\big(e \,\text{is closed}\big)$ takes the form

$$\mathbb{P}\Big(\big| \rho_m\big[(0,0) \to (x,1)\big](a) - xa \big| \ge \big(j+1\big)^{1/3} r \big(1 + \varepsilon\big) \Big)$$

where $m \in \mathbb{N}$ satisfies $\big| m - 2^{-i} n \big| < 1$; $a \in m^{-1}\mathbb{Z} \cap (0,1)$ satisfies $|a - 1/2| \le m^{-1}$; $x \in \mathbb{R}$ satisfies $|x| \le \big(j+1\big)^{1/3} r$; and ε is a small error term, satisfying $|\varepsilon| \le m^{-1}$.

Since $|\varepsilon| \le 1/2$, the displayed probability is bounded above by Theorem 3.11 with $\mathbf{n} = m$ and $\mathbf{R} = 3/2 \cdot \big(j+1\big)^{1/3} r$. The theorem implies that the probability in question is at most the quantity $\exp\big\{ - h(3/2)^3 r^3 (j+1)\big\}$. Setting $g_1 = h(3/2)^3$ yields the lemma's first part.

(2). By the preceding part and a union bound, the probability in question is found to be at most $2^{5i/3} r \exp\big\{ - g_1 r^3 (j+1)\big\}$. Since $i \le j$ and $r \ge r_0$, the desired bound results by making suitable positive choices for g_2 and r_0.

(3). The second part of the lemma is summed over $i \in [\![0, j]\!]$ to obtain this result. □

Let $i \in [\![0, j]\!]$. A *horizontal piece* of scale i is a closed horizontal planar interval whose endpoints are consecutive elements in $X_i \times Y_i$. (This means that the concerned elements of X_i differ by $2^{-2i/3}(j+1)^{1/3}r$.) If the vertical coordinates of two horizontal pieces are *vertically adjacent*, we apply the latter term to the pair of pieces.

Let ϕ be an n-zigzag from $(x, 0)$ to $(y, 1)$ where $|x|$ and $|y|$ are at most r. Let P denote the set of horizontal pieces of scale i that contain a point of departure of ϕ from a horizontal planar line segment (of the form $\mathbb{R} \times \{h\}$ for some $h \in n^{-1}\mathbb{Z} \cap [0,1]$). Consider any pair $\chi = \big\{ [x_1, x_2] \times \{h_1\}, [y_1, y_2] \times \{h_2\} \big\}$ of vertically adjacent horizontal pieces in P. The pair χ is called *good* if two conditions are met. First, $|y_1 - x_1|$, which equals $|y_2 - x_2|$, must be at most $\kappa_1 \cdot 2^{-2i/3}(j+1)^{1/3} r$. Second, $|x_1|$ and $|y_1|$ must be at most $(j+1)^{1/3} r \big(4 - 2^{-2i/3} (1 - 2^{-2/3})^{-1} \kappa \big)$.

We call ϕ viable at scale i if every pair of vertically adjacent horizontal pieces of scale i in P is good. In a viable zigzag, fluctuation on a vertical mesh of scale 2^{-i} is consistently controlled; and the global horizontal location is also controlled via the upper bounds on $|x_1|$ and $|y_1|$, which contain a negative term with a factor of $2^{-2i/3}$ in order to facilitate the induction on i that will deliver the next result.

Lemma 3.18. *Let $j \in \mathbb{N}$ satisfy $2^j \le hn$, where the constant $h > 0$ is furnished by Lemma 3.17. Suppose that every edge in E_i is open for all $i \in [\![0, j]\!]$. Let x and y be two reals of absolute value at most r. When $n \ge 2^{39}$, the polymer $\rho_n\big[(x,0) \to (y,1)\big]$ is viable at scale j.*

Proof. We will prove by induction on $i \in [\![0, j]\!]$ that, under the hypothesis of the lemma, any polymer of the given form is viable at scale i. First take $i = 0$. The edges with endpoint pairs $\big\{(-r,0),(-r,1)\big\}$ and $\big\{(r,0),(r,1)\big\}$ have elements whose horizontal coordinates are shared by members of a pair and which are in absolute value equal to r. Thus these pairs are good, and the concerned polymer is viable at scale zero.

Now consider $i \in [\![1, j]\!]$, and assume that the inductive hypothesis holds for values of the index that are lower than i. Write $\rho = \rho_n\big[(x,0) \to (y,1)\big]$. Let $[x_1, x_2] \times \{h_1\}$ and $[y_1, y_2] \times \{h_2\}$ be two vertically adjacent horizontal pieces of scale i with $h_1 < h_2$ that contain the point of departure of $\rho(h)$ at the respective heights $h \in \{h_1, h_2\}$. Thus h_1 and h_2 are elements of Y_i.

In order to demonstrate that this pair of vertically adjacent horizontal pieces is good, and thus complete the proof of the inductive step, we must show that

$$\big|y_1 - x_1\big| \le \kappa_1 \cdot 2^{-2i/3}(j+1)^{1/3} r\,; \tag{3.13}$$

$$\big|y_2 - x_2\big| \le \kappa_1 \cdot 2^{-2i/3}(j+1)^{1/3} r\,; \tag{3.14}$$

and

$$\max\big\{|x_2|, |y_2|\big\} \le (j+1)^{1/3} r\big(4 - 2^{-2i/3}\lambda\kappa\big)\,, \tag{3.15}$$

where we set $\lambda = (1 - 2^{-2/3})^{-1}$.

We will argue that $y_1 - x_1 \ge -\kappa_1 \cdot 2^{-2i/3}(j+1)^{1/3} r$. Indeed, the bound $y_2 - x_2 \le \kappa_1 \cdot 2^{-2i/3}(j+1)^{1/3} r$ follows from an almost identical argument to the one that we are about to give. These two bounds imply (3.13) and (3.14) because $y_2 - x_2$ is in fact equal to $y_1 - x_1$.

Since a horizontal piece of scale i has length $2^{-2i/3}(j+1)^{1/3}\kappa r$, we see that

$$y_1 \ge \rho(h_2) - 2^{-2i/3}(j+1)^{1/3}\kappa r\,. \tag{3.16}$$

One or other of h_1 and h_2 also belongs to Y_{i-1}. Suppose that $h_1 \in Y_{i-1}$; the other case entails no further complication. Let h_3 denote the lowest element of Y_{i-1} that exceeds h_1.

Let $[u_1, u_2] \times \{h_1\}$ be the horizontal piece of scale $i-1$ that contains $\rho(h_1)$. Let $[v_1, v_2] \times \{h_3\}$ be a horizontal piece of scale $i-1$ that contains $\rho(h_3)$.

By the inductive hypothesis, $|u_1| \le (j+1)^{1/3} r\big(4 - 2^{-2(i-1)/3}\lambda\kappa\big)$. Note that

$$|u_1 - x_1| \le 2^{-2(i-1)/3}(j+1)^{1/3}\kappa r \tag{3.17}$$

because $\rho(h_1)$ lies in $[u_1, u_2]$ and $[x_1, x_2]$, so that the distance between x_1 and u_1 may be at most the length $u_2 - u_1$, which is the longer of these two intervals. Thus,

$$|x_1| \le (j+1)^{1/3} r\big(4 - 2^{-2(i-1)/3}\lambda\kappa + 2^{-2(i-1)/3}\kappa\big)\,.$$

This implies that $|x_1| \le (j+1)^{1/3} r\big(4 - 2^{-2i/3}\lambda\kappa\big)$ because $2^{2/3}(\lambda - 1)$ is equal to λ. A symmetric argument furnishes the same bound for $|x_2|$. Thus do we obtain (3.15).

The planar intervals $[u_1, u_2] \times \{h_1\}$ and $[v_1, v_2] \times \{h_3\}$ are vertically consecutive horizontal intervals of scale $i-1$ that ϕ intersects. By the inductive hypothesis, the edge with endpoints (u_1, h_1) and (v_1, h_3) is thus seen to belong to E_{j-1}. By the lemma's hypothesis, this edge is open. By (3.12), we learn that

$$\rho(h_2) - \ell\big[(u_1,h_1) \to (v_1,h_3)\big](h_2) \,\ge\, -\kappa_2 2^{-2(i-1)/3}(j+1)^{1/3} r\,. \tag{3.18}$$

The levels h_1, h_2 and h_3 each differ by at most n^{-1} from the respective elements of a three-term arithmetic progression of real numbers with consecutive difference 2^{-j}. Thus, $\big|h_2 - (h_1+h_3)/2\big| \le 2^{1-j}n^{-1} \le 2|h_3 - h_1|n^{-1}$. We know by the inductive hypothesis that u_1 and v_1 have absolute value at most $2(j+1)^{1/3}r$. We thus see that

$$\ell\big[(u_1,h_1) \to (v_1,h_3)\big](h_2) \,\ge\, \tfrac{u_1+v_1}{2} \,-\, 2n^{-1}\cdot 4(j+1)^{1/3} r\,. \tag{3.19}$$

By (3.16), (3.18) and (3.19),

$$y_1 \ge u_1 - \tfrac{u_1-v_1}{2} - 8n^{-1}(j+1)^{1/3}r - 2^{-2(i-1)/3}(j+1)^{1/3}\kappa_2 r - 2^{-2i/3}(j+1)^{1/3}\kappa r\,.$$

Note that $v_1 - u_1 \ge -\kappa_1 \cdot 2^{-2(i-1)/3}(i+1)^{1/3}r$ because, as we have noted, the edge with endpoints (u_1, h_1) and (v_1, h_3) lies in E_{i-1}. Also using (3.17), we find that

$$\begin{aligned} y_1 \;\ge\; & x_1 - 2^{-2i/3}(j+1)^{1/3}2^{2/3}\kappa r - 2^{-1}\kappa_1 \cdot 2^{-2(i-1)/3}(j+1)^{1/3}r \\ & \quad - 8(j+1)^{1/3}n^{-1}r \,-\, 2^{-2(i-1)/3}(j+1)^{1/3}\kappa_2 r \,-\, 2^{-2i/3}(j+1)^{1/3}\kappa r \end{aligned}$$

and, since $2^i \le n$,

$$y_1 - x_1 \ge -2^{-2i/3}(j+1)^{1/3} r \Big((2^{2/3}+1)\kappa + 2^{-1/3}\kappa_1 + 2^{2/3}\kappa_2 + 8n^{-1/3} \Big)\,.$$

We now choose κ, κ_1 and κ_2 positive so that $(2^{2/3}+1)\kappa + 2^{-1/3}\kappa_1 + 2^{2/3}\kappa_2 < \kappa_1 - 2^{-10}$, alongside the already supposed $\kappa_1 \ge \kappa$. We find then that, for $n \ge 2^{39}$,

$$y_1 - x_1 \ge -\kappa_1 2^{-2i/3}(j+1)^{1/3} r\,.$$

This is the bound that we sought to show in order to verify (3.13) and (3.14). Since we already obtained (3.15), the proof of the inductive step in deriving Lemma 3.18 is complete. □

Lemma 3.19. *Let $j \in \mathbb{N}$, and let $x, y \in \mathbb{R}$ be such that $|x|$ and $|y|$ are at most r. For $i \in [\![0, j]\!]$, let ϕ be an n-zigzag between $(x,0)$ and $(y,1)$ that is viable at scale i for each $i \in [\![0,j]\!]$. For any such i, let $s_1, s_2 \in n^{-1}\mathbb{Z} \cap (0,1)$ denote* consecutive *elements of Y_i. Then*

$$\big|\phi(s_2) - \phi(s_1)\big| \le \big(\kappa_1 + \kappa\big) 2^{-2i/3}(j+1)^{1/3} r\,. \tag{3.20}$$

Proof. The horizontal pieces of scale i to which $\big(\phi(s_1), s_1\big)$ and $\big(\phi(s_2), s_2\big)$ belong have length $2^{-2i/3}(j+1)^{1/3}\kappa r$ and are bordered on the left by a pair of points forming the endpoints of an edge in E_i; thus, these left-hand endpoints have horizontal separation of at $\kappa_1 \cdot 2^{-2i/3}(j+1)^{1/3}r$. From this, Lemma 3.19 is seen to hold. □

Corollary 3.20. *There exist positive H, h and r_0, and $n_0 \in \mathbb{N}$, such that, when $n \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfy $n \ge n_0$ and $2^j \le hn$, and $r \in \mathbb{R}$ satisfies $r_0 \le r \le n^{1/10}$, it is with probability at least $1 - H\exp\big\{-hr^3(j+1)\big\}$ that, for every $x, y \in \mathbb{R}$ of absolute value at most r, and for any* consecutive *elements $h_1, h_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ of Y_j,*

$$\Big|\rho_n\big[(x,0) \to (y,1)\big](h_1) - \rho_n\big[(x,0) \to (y,1)\big](h_2)\Big| \le H h_{1,2}^{2/3}(j+1)^{1/3} r\,.$$

Proof. This statement follows from Lemma 3.17(3), and Lemmas 3.18 and 3.19. □

Lemma 3.21. *Let ϕ be an n-zigzag. Let (x, s_1) and (y, s_2) be elements in $\big(\mathbb{R} \times n^{-1}\mathbb{Z}\big) \cap \phi$ with $s_1 < s_2$. Let (z, s) denote an element of ϕ that is encountered after (x, s_1) but before (y, s_2) as ϕ is traced in the sense of increasing height. Then $|z - x|$ and $|z - y|$ are at most $|y - x| + 2^{-1}n^{1/3}s_{1,2}$.*

Proof. By the deterministic properties of an n-zigzag outlined in Section 1.7.2, the leftmost position that z may adopt is $x - 2^{-1}n^{1/3}s_{1,2}$. The rightmost such position is $y + 2^{-1}n^{1/3}s_{1,2}$. □

Proof of Theorem 1.4:(1). This argument is in the style of the derivation of the Kolmogorov continuity criterion.

Recall that the statement we seek to verify comes equipped with a parameter $k \in \mathbb{N}$ that satisfies $2^k \leq hn$ for a small constant $h > 0$. The statement claims the existence of an event of probability at least $1 - H \exp\big\{ - hr^3k\big\}$ on which the conclusion holds. We choose this event to be the intersection over indices $j \in \mathbb{N}$ satisfying $k \leq j \leq \log_2(hn)$ of the event in Corollary 3.20, modifying the values of H and h so that this lower bound on probability holds. In deriving the inequality in the conclusion of Theorem 1.4(1), we are thus permitted to invoke the conclusion of this corollary for any such index j.

Recall further that we suppose $h_{1,2} \in (2^{-k-1}, 2^{-k}]$. Let $[s_1, s_2]$ be an interval of maximum length among those that are contained in $[h_1, h_2]$ and whose endpoints s_1 and s_2 are consecutive elements of Y_i for some index i. We denote by k^* the index i thus selected; note that k^* is one among k, $k + 1$ or $k + 2$. We select in the interval $[h_1, s_1]$ an interval of maximum length delimited by a pair (s_3, s_4) of consecutive elements of some mesh Y_i; necessarily, $s_4 = s_1$, with the concerned value of i being at least k^*. Likewise, an interval is selected within $[s_2, h_2]$, with the resulting endpoint pair (s_5, s_6) satisfying $s_5 = s_2$. We write K^* for the maximum index $i \in \mathbb{N}$ for which $2^{-i} \geq hn$, where the positive constant h is contributed by Corollary 3.20. The selection of intervals is iterated, both to the left and to the right. It runs upwards through dyadic scales, and is stopped when all intervals of scale i at most K^* have been selected.

Among the closed intervals obtained in the procedure, the represented scales i satisfy $k^* \leq i \leq K^*$, with $K^* = \lfloor \log_2(hn) \rfloor$, and with each such scale appearing at most twice; the union $[s_1, s_2]$ of the intervals satisfies $0 \leq s_1 - h_1 \leq 2h^{-1}n^{-1}$ and $0 \leq h_2 - s_2 \leq 2h^{-1}n^{-1}$.

Recall that we consider elements (u, h_1) and (v, h_2) of ρ, where here we denote $\rho = \rho_n\big[(x, 0) \to (y, 1)\big]$ for given $x, y \in \mathbb{R}$ with $\max\{|x|, |y|\} \leq r$. To obtain the sought upper bound on $|u - v|$, we write

$$|u - v| \leq |u - \rho(s_1)| + |\rho(s_2) - \rho(s_1)| + |\rho(s_2) - v|\,, \tag{3.21}$$

whose middle right-hand term is seen to be at most $2H \sum_{i=k^*}^{K^*} 2^{-2i/3}(i+1)^{1/3}$, and thus at most $H2^{-2k/3}(k+1)^{1/3}$ after increase of H, by the form of the procedure that constructed $[s_1, s_2]$ alongside Corollary 3.20. After an increase of H, this upper bound is seen to take the form $H2^{-2k/3}(k+1)^{1/3}$; or equally the form $Hh_{1,2}^{2/3}\big(\log h_{1,2}^{-1}\big)^{1/3}$. Thus, Theorem 1.4(1) will be obtained, provided that we verify that the first and third right-hand terms in (3.21) are smaller than the middle term. Seeking to prove this, we let s_0 denote the greatest element of Y_{K^*} that is less than s_1, and let s_3 be the least element of this set that exceeds s_2. Since $s_0 < h_1 \leq s_1$, (u, h_1) lies on the subpath of ρ between $\big(\rho(s_0), s_0\big)$ and $\big(\rho(s_1), s_1\big)$. Similarly, (v, h_2) lies on the subpath of ρ between $\big(\rho(s_2), s_2\big)$ and $\big(\rho(s_3), s_3\big)$. Lemma 3.21 thus implies that $|u - \rho(s_1)|$ is at most $|\rho(s_0) - \rho(s_1)| + 2^{-1}n^{1/3}s_{0,1}$; and that $|\rho(s_2) - v|$ is at most $|\rho(s_2) - \rho(s_3)| + 2^{-1}n^{1/3}s_{2,3}$. The pairs (s_0, s_1) and (s_2, s_3) comprise consecutive elements of Y_{K^*}, where recall that $K^* = \lfloor \log_2(hn) \rfloor$; so that $s_{0,1}$ and $s_{2,3}$ are at most $(2h^{-1} + 1)n^{-1}$. By the conclusion of Corollary 3.20, $|\rho(s_0) - \rho(s_1)|$ and $|\rho(s_2) - \rho(s_3)|$ are thus seen to be at most of order $n^{-2/3}\big(\log n\big)^{1/3}r$ after suitable adjustment to H. The sought upper bound, of order $2^{-2k/3}(k+1)^{1/3}$, holds because $2^k \leq n$. This completes the

proof of Theorem 1.4(1).

(2). Recall that instead we suppose that $h_{1,2} < Hn^{-1}$. Let $[s_0, s_3]$ denote an interval containing $[h_1, h_2]$ whose endpoints are consecutive elements of the mesh Y_{K^*}; here, K^* continues to denote $\lfloor \log_2(hn) \rfloor$, with h now decreased from its value in Corollary 3.20 suitably to ensure the existence of such s_0 and s_3. The notation s_0 and s_3 is used to indicate the similarity of these quantities with the usage in the preceding case. Now, however, $s_{0,3}$ is at most $2n^{-1}$. Thus, Corollary 3.20 with j chosen so that $2^j = \Theta(n)$ yields that, on an event of probability at least 1 - n^{-hr^3}, the bound $|\rho(s_0) - \rho(s_3)| \le Gn^{-2/3}\big(\log n\big)^{1/3} r$ for suitably high G and for all concerned choices of h_1 and h_2. Since (u, h_1) and (v, h_2) lie on the subpath of ρ between $\big(\rho(s_0), s_0\big)$ and $\big(\rho(s_3), s_3\big)$, Lemma 3.21 thus implies that $|u - \rho(s_0)|$ and $|v - \rho(s_0)|$ are at most $Gn^{-2/3}\big(\log n\big)^{1/3} r + n^{-2/3}H$. Applying the triangle inequality, we learn that Theorem 1.4(2) holds with a suitably increased value of G. □

Proof of Corollary 1.5. Take $\mathbf{j} = 0$, $\mathbf{u} = x$, $\mathbf{h_1} = 0$, $\mathbf{v} = u$ and $\mathbf{h_2} = h$ in Theorem 1.4. Since $|x| \le r$, the corollary follows by increasing $H > 0$. □

Proof of Theorem 1.6: (1). The mooted event of probability at least $1 - H \exp\big\{ - hr^3 k\big\}$ will be chosen to ensure that the conclusions of Theorem 1.4 and Corollary 1.5 hold.

We have $h_{1,2} \in (2^{-k-1}, 2^{-k}]$. By the conclusion of Theorem 1.4,

$$|v - u| \le Hh_{1,2}^{2/3}\big(\log(1 + h_{1,2}^{-1})\big)^{1/3} r\,. \tag{3.22}$$

By the conclusion of Corollary 1.5, $|u| \le Hr$.

Let $R \subset \mathbb{R}^2$ denote the rectangle $\big[0, Y\big] \times \big[0, 2^{-k}\big]$, where Y denotes $H2^{-2k/3}k^{1/3}r$. The lower-third R^- of R is $[0, Y] \times \big[0, 2^{-k}/3\big]$; its upper-third R^+ is $[0, Y] \times \big[2/3 \cdot 2^{-k}, 2^{-k}\big]$. To any translate $T = R + (x, y)$ indexed by (x, y) is evidently associated a lower-third T^- and an upper-third T^+.

In light of the noted bounds, we may find a collection $\mathcal{C}$ of translates of R by vectors in $\mathbb{R} \times n^{-1}\mathbb{Z}$ such that $|\mathcal{C}|$ is at most a constant multiple of $2^{5k/3}k^{-1/3}$ such that $(u, h_1) \in T^-$ and $(v, h_2) \in T^+$ for some element T of $\mathcal{C}$.

We apply Proposition 3.15 with $\mathbf{K}$ of order $k^{1/3}r$, $\mathbf{R} = K$ and $\mathbf{n} = 6n2^{-k}$ via the scaling principle, and use a union bound, to find that, for K sufficiently high,

$$\begin{aligned} & \mathbb{P}\Big(\sup \big| \mathsf{Wgt}_n\big[(u, h_1) \to (v, h_2)\big] \big| \ge \Theta(1)(K + H^2k^{2/3}r^2)2^{-k/3} \Big) \\ \le \quad & |\mathcal{C}| \cdot k^{2/3}r^2 G \exp\big\{ - dK^{3/2} \big\}\,, \end{aligned}$$

where the supremum is taken over all choices (u, h_1) and (v, h_2) with $h_1, h_2 \in n^{-1}\mathbb{Z} \cap [0, 1]$ and $h_{1,2} \in (2^{-k-1}, 2^{-k}]$ that belong to $\rho_n\big[(x, 0) \to (y, 1)\big]$ where x and y vary over real values of absolute value at most r. Because Proposition 3.15 treats parabolically adjusted weight, we need to take account of parabolic curvature, and, in view of (3.22), we do so by means of the above term $H^2k^{2/3}r^2 2^{-k/3}$. The upper bounds on $\mathbf{K}$ and $\mathbf{R}$ hypothesised by the proposition are satisfied in view of the assumption that $r \le (nh_{1,2})^{1/50}$.

By choosing K to be a large multiple of $\big(\log(r2^k)\big)^{2/3}$, the right-hand factors of $|\mathcal{C}|k^{2/3}$, which grows as a power of 2^k, and of r^2, may be removed, at the expense of a decrease in the value of $d > 0$. The form of Theorem 1.6(1) entails that we desire this failure probability upper bound $\exp\big\{ - dK^{3/2} \big\}$ be at most $H \exp\big\{ - hr^3 k\big\}$. This condition is ensured if we adjust K so that it equals a large constant multiple of $r^2k^{2/3}$. This adjustment is an increase to the value of K: indeed, since r is supposed to be high, and $h_{1,2} \le 1$, the adjusted value of K is at least the maximum of r^2 and the given large multiple of $\big(\log(r2^k)\big)^{2/3}$. (The choice $K = \Theta(1)r^2k^{2/3}$ imposes the constraint $rk^{1/3} \le \Theta(1)(n2^{-k})^{1/60}$ to admit the above usage of Proposition 3.15. The upper bound

on r in Theorem 1.6(1) implies this constraint.) Since the adjusted value of K may be absorbed into the preceding display's term $H^2k^{2/3}r^2$ that arose from parabolic curvature, we complete the derivation of Theorem 1.6(1) by decreasing $h > 0$ if need be.

(2). Suppose instead that $h_{1,2} < Hn^{-1}$. The points (u, h_1) and (v, h_2) lie on a polymer of the form $\rho\big[(x,0) \to (y,1)\big]$ where $|x|$ and $|y|$ are at most r. In this case, we will rely on control on Brownian oscillation and adopt the unscaled perspective to finish the proof. By Theorem 1.4(2), with probability at least $1 - Hn^{-hr^3}$,

$$\big|v - u\big| \le G_1 n^{-2/3} (\log n)^{1/3} r\,.$$

Thus, recalling (1.7), it suffices to prove that, with probability at least $1 - Hn^{-hr^3}$,

$$M\big[(u,i) \to (v,j)\big] \le \frac{G}{10} r^2 \cdot (\log n)^{2/3}$$

for all $1 \le i \le j \le n$, with $j - i \le H$, $u \le v$, $|u - v| \le G_1(\log n)^{1/3} r$, and $|u| \le O\big(n(\log n)^{1/3} r\big)$. To prove this bound, we will simply bound the probability that

$$M\big[(u,i) \to (v,i+1)\big] \ge \frac{G}{H} r^2 \cdot (\log n)^{2/3}\,,$$

for some u, v as above. Since we may suppose that $G \ge 2H$, simple Brownian oscillation estimates, relying on the reflection principle, yield that this probability is at most

$$O\Big(n \exp\big\{ -r^4 (\log n)^{4/3} / r (\log n)^{1/3} \big\}\Big) \le \exp\big\{ -2^{-1} r^3 \log n \big\} \;=\; n^{-2^{-1} r^3}\,,$$

for all large enough r. Thus the proof of Theorem 1.6(2) is complete. □

3.7 Polymer fluctuation tails, uniform in variation of endpoints and lifetime fraction

In this section, we prove Theorem 1.7.

Let $a \in (0,1)$ and $r > 0$. Define the *lower zone* $Z_n^-(a,r)$ to be the product of $[-1,1] \cdot ra^{2/3}\big(\log a^{-1}\big)^{1/3}$ and $n^{-1}\mathbb{Z} \cap [0, a/4]$; and the *upper zone* $Z_n^+(a,r)$ to be the product of $[-1,1] \cdot ra^{2/3}\big(\log a^{-1}\big)^{1/3}$ and $n^{-1}\mathbb{Z} \cap [1 - a/4, 1]$.

Recall $\mathsf{Fluc}_n\big[\cdot,\cdot\big]$ from (1.9).

Lemma 3.22. *There exist positive constants C_1, C_2, C_3 and C_4 such that, for $r \ge C_1$, $a \in (0, 1/16]$ and $n \in \mathbb{N}$ for which $\min\big\{na, n(1-a)\big\} \ge C_2$,*

$$\mathbb{P}\Big(\sup \mathsf{Fluc}_n\big[(x,h_1) \to (y,h_2); h\big] \ge 3ra^{2/3}\big(\log a^{-1}\big)^{1/3} \Big) \le C_4 a^{C_3 r^3}\,,$$

where the supremum is taken over (x,h_1) in the lower zone $Z_n^-(a,r)$; (y,h_2) in the upper zone $Z_n^+(a,r)$; and $h \in n^{-1}\mathbb{Z} \cap \big([a/2, 4a] \cup [1-4a, 1-a/2]\big)$.

Proof. Let $z \in \mathbb{R}$. Define the event $\mathsf{Narrow}_n(z,a,r)$ that

$$\begin{aligned} &\rho_n\big[(z,0) \to (z,1)\big] \cap \Big(\mathbb{R} \times \Big(n^{-1}\mathbb{Z} \cap \big(\big[0, 4a\big] \cup \big[1-4a, 1\big] \big) \Big) \Big) \qquad (3.23) \\ \subseteq\;& \big[z - \alpha(a,r), z + \alpha(a,r)\big] \times \big[0,1\big]\,, \end{aligned}$$

where $\alpha(a,r) = 4^{-1} ra^{2/3}\big(\log a^{-1}\big)^{1/3}$.

To argue that this event is typical, we make two sets of applications of Theorem 1.4, first with $\mathbf{h_1} = 0$ and then with $\mathbf{h_2} = 1$. In each, we apply Theorem 1.4(1) with $\mathbf{k}$ ranging upwards from the maximal $i \in \mathbb{N}$ for which $2^{-i} \ge a/4$; and then use Theorem 1.4(2) to treat the smallest scale. We take $\mathbf{r} = r$ in these applications. Since $a \ge \Theta(1) n^{-1}$, what

we learn by doing so is that it is with probability at least $1 - \Theta(1)a^{\Theta(1)r^3}$ that (3.23) holds when $\alpha(a,r)$ is multiplied by a large constant. By replacing r by a small constant multiple of this quantity, and relabelling, we find that, for $z \in \mathbb{R}$,

$$\mathbb{P}\Big(\neg\, \mathsf{Narrow}_n(z,a,r)\Big) \le \Theta(1)a^{\Theta(1)r^3}\,. \tag{3.24}$$

Set $z^- = -5/4 \cdot ra^{2/3}\big(\log a^{-1}\big)^{1/3}$ and $z^+ = 5/4 \cdot ra^{2/3}\big(\log a^{-1}\big)^{1/3}$. When $\mathsf{Narrow}_n\big(z^-,a,r\big)$ and $\mathsf{Narrow}_n\big(z^+,a,r\big)$ occur, every polymer of the form $\rho_n\big[(x,h_1) \to (y,h_2)\big]$, with (x,h_1) in the lower zone $Z_n^-(a,r)$ and (y,h_2) in the upper zone $Z_n^+(a,r)$, has the property that its endpoint locations (x,h_1) and (y,h_2) are bounded on the left by $\rho_n\big[(z^-,0) \to (z^-,1)\big]$, and on the right by $\rho_n\big[(z^+,0) \to (z^+,1)\big]$. By polymer ordering Lemma 3.3, every point in $\rho_n\big[(x,h_1) \to (y,h_2)\big]$ is also thus bounded. However, $\rho_n\big[(z^-,0) \to (z^-,1)\big]$ and $\rho_n\big[(z^+,0) \to (z^+,1)\big]$ remain in vertical strips

$$\big[z - \alpha(a,r), z + \alpha(a,r)\big] \times \big[0,1\big]\,,$$

with z equal to z^- or z^+, during $\big[0,4a\big] \cup \big[1-4a,1\big]$ as indicated in (3.23). We learn that $\rho_n\big[(x,h_1) \to (y,h_2)\big]$ remains in the strip $\big[-\frac{3}{2}ra^{2/3}\big(\log a^{-1}\big)^{1/3}, \frac{3}{2}ra^{2/3}\big(\log a^{-1}\big)^{1/3}\big]$ during $\big[0,4a\big] \cup \big[1-4a,1\big]$. This implies that the quantity $\mathsf{Fluc}_n\big[(x,h_1) \to (y,h_2); h\big]$ appearing in Lemma 3.22 is for $h \in n^{-1}\mathbb{Z} \cap \big([a/2,4a] \cup [1-4a, 1-a/2]\big)$ at most $3ra^{2/3}\big(\log a^{-1}\big)^{1/3}$, since $(x,h_1) \in Z_n^-(a,r)$ and $(y,h_2) \in Z_n^+(a,r)$.

Thus the proof of Lemma 3.22 is complete, because, the upper bound $\Theta(1)a^{\Theta(1)r^3}$ in this result is in light of (3.24) a bound on the probability that $\mathsf{Narrow}_n\big(z^-,a,r\big) \cup \mathsf{Narrow}_n\big(z^+,a,r\big)$ fails to occur. □

Proof of Theorem 1.7. By the scaling principle, it suffices to treat that case that $s_1 = 0$ and $s_2 = 1$. Specify the starting region S equal to the product of $[-K,K]$ and $n^{-1}\mathbb{Z} \cap [0,1/3]$; and the ending region E equal to the product of $[-K,K]$ and $n^{-1}\mathbb{Z} \cap [2/3,1]$. We are concerned with journeys between (x,h_1) and (y,h_2), where (x,h_1) lies in the starting region S and (y,h_2) in the ending region E. We plan to argue that typically such journeys simultaneously have the desired property of fluctuation by applying Lemma 3.22 via the scaling principle and using a union bound. We first construct a family of maps which will map any pair of points in $S \times E$ to a pair of points in $Z_n^-(a,r) \times Z_n^+(a,r)$ as in Lemma 3.22, which then will allow us to finish the proof by applying the lemma.

For $K \in \mathbb{R}$, recall from (3.1) the shear map $\tau_K : \mathbb{R}^2 \to \mathbb{R}^2$, $\tau(x,y) = (x + Ky, y)$. Further, define a *KPZ dilation* to be a map of the form $\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to (\zeta^{2/3}x, \zeta y)$ which sends the line $\mathbb{R} \times \{1\}$ to a line of the form $\mathbb{R} \times \{y\}$ for $y \in n^{-1}\mathbb{N}$ with $y > 0$; that is, we ask that $\zeta \in n^{-1}\mathbb{N}^+$. A *vertical shift* is a map of the form $\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to (x, y+h)$, where $h \in n^{-1}\mathbb{Z}$. A *horizontal shift* is a map of the form $\mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \to (x+u, y)$, where $u \in \mathbb{R}$.

Let Θ denote the class of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ that take the form $\phi_h \circ \phi_v \circ \phi_d \circ \phi_s$, where ϕ_s is a shear map τ_κ with $|\kappa| \le n^{2/3}$; ϕ_d is a KPZ dilation; ϕ_v is a vertical shift; and ϕ_h is a horizontal shift.

A basic covering pair for the product set $S \times E$ is a pair (B^-, B^+), where there exists an element $\theta \in \Theta$ for which B^- is the image under θ of the lower zone $Z_n^-(a,r)$; B^+ is the image under θ of the upper zone $Z_n^+(a,r)$ where a and r are as in the statement of Theorem 1.7, with $B^- \cap S \ne \emptyset$; and $B^+ \cap E \ne \emptyset$. The *covering number* is defined to be the minimum cardinality of a set of basic covering pairs for $S \times E$ such that $S \times E \subseteq \bigcup \big(B^- \times B^+\big)$ where the union ranges over pairs (B^-, B^+) in the set.

Lemma 3.23. *Suppose that $Ka^{1/3}$ is bounded above. The covering number is at most a constant multiple of $a^{-10/3}K^2$.*

Proof. Let $(x, h_1) \in S$ and $(y, h_2) \in E$. We want to locate a basic covering pair (B^-, B^+) with $(x, h_1) \in B^-$ and $(y, h_2) \in B^+$. Associated to (B^-, B^+) is the composition of a shear, a KPZ dilation, a vertical shift and a horizontal shift. We attempt to recover these operations by undoing them.

1. *The inverse horizontal shift.* Shift the two points horizontally by a common displacement that is a multiple of $3^{-3/2}a^{2/3}$ so that the resulting points (x', h_1) and (y', h_2) satisfy the condition that $|x'| \le 3^{-3/2}a^{2/3}$.
2. *Next is the inverse vertical shift.* Shift the two new points down by the maximum multiple of $n^{-1}\lfloor na/12 \rfloor$ so that the resulting points (x', h_1') and (y', h_2') lie in the upper half-plane. Note that $0 \le h_1' < a/12$ and $1/3 < h_2' \le 1$.
3. *Next the inverse KPZ dilation.* We map (x', h_1') and (y', h_2') to $(\hat{x}, \hat{h}_1)$ and $(\hat{y}, \hat{h}_2)$ via an inverse KPZ dilation $(x, y) \to (\zeta^{-2/3}x, \zeta^{-1}y)$, where $\zeta^{-1} \in [1, 3]$ is chosen to ensure that $0 \le \hat{h}_1 \le a/4$ and $1 - a/4 \le \hat{h}_2 \le 1$. This choice of ζ may be made from a set of cardinality of order a^{-1}, where this entropy factor is adequate to ensure the desired bounds on $\hat{y}$.
4. *Now, we undo the shear map.* Let $(\hat{x}_1, \hat{h}_1)$ and $(\hat{y}_1, \hat{h}_2)$ denote the image of $(\hat{x}, \hat{h}_1)$ and $(\hat{y}, \hat{h}_2)$ under an inverse shear map $\tau_{-\kappa}$, selected to ensure that $|\hat{y}_1 - \hat{x}_1| \le a^{2/3}$. Since $0 \le \hat{x} \le a^{2/3} \le 1 \le K$ and $|\hat{y}| \le 3K$, the value of κ may be chosen among a constant multiple of $Ka^{-2/3}$ options to guarantee this outcome.
5. *A final shift.* Having undone the several maps, we hoped to obtain $(\hat{x}_1, \hat{h}_1) \in Z_n^-(a, r)$ and $(\hat{y}_1, \hat{h}_2) \in Z_n^+(a, r)$. The vertical coordinates satisfy the desired conditions $\hat{h}_1 \in [0, a/4]$ and $\hat{h}_2 \in [1 - a/4, 1]$; and the horizontal displacement $|\hat{y}_1 - \hat{x}_1|$, being at most $a^{2/3}$, is consistent with our aim. But a final horizontal shift is needed, to ensure that the horizontal coordinates are both at most $ra^{2/3}\bigl(\log a^{-1}\bigr)^{1/3}$. Since $|\hat{x}_1|$ is readily seen to have order at most Ka, and $r \ge 1$ as well as $a \le e^{-1}$, we see that this final shift may be chosen from among an order of $Ka^{1/3}$ choices. However, since we hypothesise that $Ka^{1/3}$ is most a large constant, the entropic term associated to this final step is bounded.

The product of upper bounds neglecting bounded factors on the number of choices for the maps employed in the respective steps is equal to

$$Ka^{-2/3} \times a^{-1} \times a^{-1} \times Ka^{-2/3} \times 1 = K^2 a^{-10/3}\,.$$

If we apply the inverses of our inverse maps—in reverse order!—to the lower and upper zones $Z_n^-(a, r)$ and $Z_n^+(a, r)$, we will obtain B^- and B^+, elements of a basic covering pair that respectively contain the given points $(x, h_1) \in S$ and $(y, h_2) \in E$. Since the constructed map is one among a collection whose cardinality is at most the displayed quantity up to a bounded factor, we have completed the proof of Lemma 3.23. □

In seeking to obtain Theorem 1.7 in the case that $s_1 = 0$ and $s_2 = 1$, it is enough, in view of $h_{1,2} \le 1$, to bound above the probability of the event—that we will denote by A—that there exist $(x, h_1) \in S$ and $(y, h_2) \in E$ for which

> there exists a moment h at which a fraction lying in $[a, 2a] \cup [1 - 2a, 1 - a]$
> of the lifetime $[h_1, h_2]$ has elapsed such that
> $h_{1,2}^{-2/3}\bigl|\rho_n\bigl[(x, h_1) \to (y, h_2)\bigr](h) - \ell\bigl[(x, h_1) \to (y, h_2)\bigr](h)\bigr|$ is at least $Ra^{2/3}\bigl(\log a^{-1}\bigr)^{1/3}$,

when $R = 5r$, where the positive parameter R is displayed to permit convenient later reference to the display, and where the choice of a multiple of five—or, indeed, of any given multiple, but five will work for our purpose—is admissible via the absorptive proclivity of $\Theta(1)$ terms in the theorem. We cover $E \times S$ by a union indexed by $i \in \mathcal{C}$ of

basic covering boxes (B_i^-, B_i^+) which by Lemma 3.23 we know may be chosen so that $|\mathcal{C}|$ is at most a constant multiple of $K^2 a^{-10/3}$.

Fix a given index $i \in \mathcal{C}$ and consider the event that there exist $(x, h_1) \in B_i^-$ and $(y, h_2) \in B_i^+$ such that the displayed circumstance takes place with $R = r$. We have constructed a composite function that maps B_i^- into $Z_n^-(a, r)$ and B_i^+ into $Z_n^+(a, r)$. The composite is the outcome of a five-step composition. After each step, the given points $(x, h_1) \in B_i^-$ and $(y, h_1) \in B_i^+$ have been mapped to certain locations; we may consider the probability that there exist $(x, h_1) \in B_i^-$ and $(y, h_2) \in B_i^+$ such that the last displayed event occurs when (x, h_1) and (y, h_2) are replaced by these locations. For the step indexed by $j \in [\![0, 5]\!]$, we denote this probability by $p_j(R, i)$.

Note that the probability $\mathbb{P}(A)$ that we seek to bound in order to obtain Theorem 1.7 when $s_1 = 0$ and $s_2 = 1$ is bounded above by a constant multiple of $K^2 a^{-10/3} \sup_{i \in \mathcal{C}} p_0(5r, i)$. Thus, if we can show that $p_0(5r, i)$ is at most a^{dr^3} for each $i \in \mathcal{C}$, we will have proved this theorem.

On the other hand, $p_5(3^{4/3}r, i)$ for any given $i \in \mathcal{C}$ is at most the probability that the last displayed event occurs with $R = 3^{4/3}r$ and when (x, h_1) and (y, h_2) are respectively replaced by certain given elements of $Z_n^-(a, r)$ and $Z_n^+(a, r)$. As such, $p_5(3^{4/3}r, i)$ is bounded above by Lemma 3.22: indeed, this result implies that $p_5\big(3^{4/3}r, i\big) \le \Theta(1) a^{\Theta(1) r^3}$ because $h_{1,2} \ge 1/3$ (since $h_1 \le 1/3$ and $h_2 \ge 2/3$) and because, if $h \in [0, 1]$ satisfies $\frac{h - h_1}{h_{1,2}} \in [a, 2a] \cup [1-2a, 1-a]$ for $h_1 \in [0, a/4]$ and $h_2 \in [1-a/4, 1]$, then $h \in [a/2, 4a] \cup [1-4a, 1-a/2]$.

To close out the proof of Theorem 1.7 , it suffices to show that $p_0(5r, i) \le p_5\big(3^{4/3}r, i\big)$. Indeed, this bound proves this result when the left-hand instance of r in (1.10) is replaced by $5r$; but, as we have noted, the stated form may then be obtained since $\Theta(1)$ notation is used. We seek then to obtain the just stated bound. Of the five maps involved in the composition, four are shifts or a KPZ dilation. For each index advance $j \to j + 1$ in which one of these maps is involved, the scaling principle shows that $p_j(s, i)$ equals $p_{j+1}(s, i)$. The remaining map is the fourth—the inverse shear map $\tau_{-\kappa}$—involved in the index change $3 \to 4$. Since the value of the parameter κ specifying this map is at most a constant multiple of K, and $|K|$ is at most a small constant multiple of $n^{2/3}$, κ itself is at most a small multiple of $n^{2/3}$; thus, Lemma 3.13(2) and $5 > 3^{4/3}$ imply that $p_3\big(5r, i\big)$ is at most $p_4\big(3^{4/3}r, i\big)$. This confirms that $p_0(5r, i) \le p_5\big(3^{4/3}r, i\big)$ and completes the proof of Theorem 1.7. □

3.8 Polymer weight tails, uniform in variation of endpoints

In this section, we prove Proposition 1.8.

Proof of Proposition 1.8. Proposition 3.15 is concerned with the tail of parabolic weight for polymers that begin in the lower third and end in the upper third of the rectangle $[-K, K] \times n^{-1}\mathbb{Z} \cap [-3, 3]$. With $\nu = 6^{-1} \cdot 5/4 \cdot 2^{-\ell}$, this rectangle is mapped under the transformation $(x, y) \to (\nu^{2/3}x, \nu y)$ and a vertical shift to the rectangle

$$[-K, K] \cdot 6^{-2/3}(5/4)^{2/3} 2^{-2\ell/3} \times m^{-1}\mathbb{Z} \cap \left[0, 5/4 \cdot 2^{-\ell}\right],$$

where $m = n \cdot 4/5 \cdot 2^{\ell}$. Consider the collection of translates of the displayed rectangle by vectors of the form $\big(K \cdot 6^{-2/3}(5/4)^{2/3} 2^{-2\ell/3} \cdot j, 2^{-\ell} \cdot 1/12 \cdot k\big)$, with $(j, k) \in \mathbb{Z}^2$. Let Ψ denote the subcollection indexed by those (j, k) for which the translation vector has horizontal component lying in $[-M, M]$ and vertical component lying in $[0, 1]$. Set $K > 0$ so that $L = K \cdot 6^{-2/3}(5/4)^{2/3}$. Then Ψ is a set of cardinality $\Theta(1) M K^{-1} 2^{5\ell/3}$ such that any pair of elements (x, s_1) and (y, s_2) implicated in the definition in the events $\mathsf{Low}_n\big(\zeta, \ell, L, M\big)$ and $\mathsf{High}_n\big(\zeta, \ell, L, M\big)$ belongs to some element of Ψ. What we learn from this is that we may apply Proposition 3.15 with $\mathbf{n} = n \cdot 5/4 \cdot 2^{-\ell}$, $\mathbf{K} = K$ and $\mathbf{R} = (4/5)^{1/3} 2^{-1/3} \zeta$ via translation invariance and the scaling principle, and use a union bound, to conclude that

the probability in Proposition 1.8 is at most $\Theta(1)MK^{-1}2^{5\ell/3}\Theta(1)K^2\exp\big\{-\Theta(1)\zeta^{3/2}\big\}$. (That the selection of $\mathbf{R}$ is satisfactory depends on $s_{1,2} \geq 2^{-\ell-1}$.) Since the obtained bound takes the desired form, and the hypotheses on n, L and ζ in Proposition 1.8 enable this application of Proposition 3.15, the proof of Proposition 1.8 is complete. □

4 Slim pickings for slender excursions

Here we prove our result Theorem 1.10 asserting the significant shortfall in weight accrued by zigzags that are constrained to follow excursions relative to a given zigzag that are narrower than the width dictated by the KPZ scaling exponent of two-thirds.

More precisely, but still in summary, this theorem concerns the maximum weight that may be accrued by an n-zigzag ψ that is constrained to pursue a slender excursion relative to a given zigzag ϕ of duration of order $2^{-\ell}$. By slender, we mean that, at a high but fixed proportion $1-\chi$ of heights along the excursion, the width between ψ and ϕ is at most a small multiple θ of the characteristic separation $2^{-2\ell/3}$. There is a degree of choice in the levels, of proportion χ, at which ψ is not bound by this slenderness constraint. Proposition 4.5 is a result *en route* to Theorem 1.10 in which a counterpart conclusion is reached when this set of levels is instead fixed.

There are four subsections. In the first, we give a brief LPP-based proof of a result that we will need: the mean of the GUE Tracy-Widom distribution is negative. (See [BGHH20, Lemma $A.4$] for another proof pointed out by Ivan Corwin.) In the second section, we record two needed results, including a form of [Ham19a, Theorem 1.1] that concerns variation of polymer weight under endpoint perturbation. In the third section, we prove Proposition 4.5. In the fourth, we sum out over the levels fixed in this proposition in order to obtain Theorem 1.10.

4.1 The negative mean of the GUE Tracy-Widom distribution

Proposition 4.1. *There exist $d > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$ and $|x| \leq 2^{-1}cn^{1/19}$,*

$$\mathbb{E}\,\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] \leq -d\,.$$

It is Proposition 4.1 that we will later employ, but this result has the following interesting consequence.

Corollary 4.2. *The mean of the GUE Tracy-Widom distribution is negative.*

Proof. The limit in law as $n \to \infty$ of $\mathsf{Wgt}_n\big[(0,0) \to (0,1)\big]$ has the distribution of $2^{1/3}X$, where, by [TW94, Bar01], X has the GUE Tracy-Widom law. Thus the result follows from Proposition 4.1. □

The next result is the principal component of Proposition 4.1.

Lemma 4.3. *There exist $d_1 > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$ and $|x| \leq 2^{-1}cn^{1/19}$,*

$$\mathbb{E}\Big[\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] \vee \mathsf{Wgt}_n\big[(0,0) \to (x+1,1)\big]\Big] \geq \mathbb{E}\,\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] + d_1 \quad (4.1)$$

provided that $x \leq 0$ while, if $x > 0$, the same inference holds when $x+1$ is replaced by $x-1$.

Proof. We will give a Brownian Gibbs argument. The random profile $x \to \mathsf{Wgt}_n\big[(0,0) \to (x,1)\big]$ is $(c, C, n+1)$-regular in the sense of Subsection 2.2.3 by Proposition 2.3(1) (with $\mathbf{a} = 1$) therein. We will write $\mathcal{L}$ for the $(n+1)$-curve regular ensemble whose uppermost curve $\mathcal{L}(1,x)$ equals $\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big]$.

Suppose first that $x \in [-1/2, 1/2]$. For $D > 0$, let $N = N(D)$ denote the event that the weight $\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big]$ is at most D, and that $\mathsf{Wgt}_n\big[(0,0) \to (x+2,1)\big]$ is at least $-D$. We make two claims.

Claim 1. There exist $D_1 > 0$, $d_2 \in (0,1)$ and $n_1 \in \mathbb{N}$ such that, when $n \geq n_1$, it is with probability at least d_2 that $N(D_1)$ occurs.

Claim 2. For $n \in \mathbb{N}$ and $D > 0$, the conditional probability of

$$\mathsf{Wgt}_n\big[(0,0) \to (x+1,1)\big] \geq \mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] + 1$$

given $N(D)$ is at least $\nu_{0,1/2}\big(D+1,\infty\big)$, where $\nu_{0,1/2}$ denotes the Gaussian law of mean zero and variance one-half. The same statement holds when the displayed left-hand side is replaced by $\mathsf{Wgt}_n\big[(0,0) \to (x-1,1)\big]$.

Proof of Claim 1. The ensemble $\mathcal{L}$ satisfies $\mathrm{Reg}(2)$ and $\mathrm{Reg}(3)$ in Definition 2.2, and from this, the result follows.

Proof of Claim 2. The event on which we condition in this claim is that $\mathcal{L}(1,x) \leq D$ and $\mathcal{L}(1,x+2) \geq -D$. If we further condition on the value (u,v) of $\big(\mathcal{L}(1,x), \mathcal{L}(1,x+2)\big)$, and on the form f of $\mathcal{L}(2,\cdot) : [x,x+2] \to \mathbb{R}$, then the conditional distribution of $\mathcal{L}(1,\cdot) : [x,x+2] \to \mathbb{R}$ is given by Brownian bridge $B : [x,x+2] \to \mathbb{R}$, with $B(x) = u$ and $B(x+2) = v$—whose law we label $\mathcal{B}_{u,v}^{[x,x+2]}$—conditioned on $B(z) \geq f(z)$ for $z \in [x,x+2]$. Note that

$$\mathcal{B}_{u,v}^{[0,2]}\big(B(1) \geq u{+}1 \big| B > f\big) \geq \mathcal{B}_{u,v}^{[0,2]}\big(B(1) \geq u{+}1\big) \geq \mathcal{B}_{0,0}^{[0,2]}\big(B(1) \geq D{+}1\big) = \nu_{0,1/2}\big(D{+}1,\infty\big)\,,$$

the first inequality by the monotonicity offered in [Ham22, Lemma 2.18] (a result originally proved in [CH14]); the second by the affine scaling property of Brownian bridge; and the third by the law of the midpoint value of standard Brownian bridge. This completes the proof of the first assertion of Claim 2. The second assertion has an almost identical proof.

The two claims show that, for $x \in [-1/2, 1/2]$,

$$\mathbb{E}\Big[\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] \vee \mathsf{Wgt}_n\big[(0,0) \to (x{+}1,1)\big]\Big] \geq \mathbb{E}\,\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] + d_2\,, \tag{4.2}$$

as well as the bound after we replace $x+1$ by $x-1$. This proves Lemma 4.3 in the case that $x \in [-1/2, 1/2]$. By [Ham22, Lemma 2.26]—a tool of near parabolic invariance that propagates spatial information from unit-order to much broader intervals—we learn from (4.2) that, for $|x| \leq 2^{-1}cn^{1/19}$,

$$\mathbb{E}\Big[\mathsf{Wgt}_n^{\cup}\big[(0,0) \to (x,1)\big] \vee \mathsf{Wgt}_n^{\cup}\big[(0,0) \to (x+1,1)\big]\Big] \geq \mathbb{E}\,\mathsf{Wgt}_n^{\cup}\big[(0,0) \to (x,1)\big] + d_2\,,$$

where recall that $\mathsf{Wgt}_n^{\cup}\big[(0,0) \to (x,1)\big]$ is a shorthand that denotes the parabolically adjusted weight $\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big] + 2^{-1/2}x^2$. Subtracting $2^{-1/2}x^2$ yields (4.1) for $x \leq -1/2$. We obtain (4.1) for $x \geq 1/2$ by the same argument, with the role of (4.2) played by its counterpart where $x-1$ replaces $x+1$. This completes the proof of Lemma 4.3. □

Proof of Proposition 4.1. We will prove this result by showing that $\mathbb{E}\,\mathsf{Wgt}_n\big[(0,0) \to (x,1)\big]$ is at most $-d_1$, where Lemma 4.3 furnishes $d_1 > 0$. To verify this, suppose first that $x \leq 0$. We claim that

$$\liminf \sup_{z \in \mathbb{R}} K^{-1}\mathsf{Wgt}_n\big[(0,0) \to (z,K)\big] \;\geq\; \frac{n}{n+1}\Big(\mathbb{E}\,\mathsf{Wgt}_n\big[(0,0) \to (z,1)\big] + d_1\Big)\,, \tag{4.3}$$

where the limit infimum is taken as $K \to \infty$ through $K \in \mathbb{N}$. This claim is substantiated by constructing an n-zigzag that begins at $(0,0)$. It travels first either to $(x,1)$ or to $(x+1,1)$, the choice being made so that more weight is captured along the way. After arrival, the zigzag makes an immediate microscopic jump, moving by $\big(-2n^{-2/3}, n^{-1}\big)$. Zigzag formation continues as if the point of arrival plays the role that $(0,0)$ did at the

outset. That is, the zigzag continues by travelling to one of the points whose displacement from its present location is $(x,1)$ or $(x+1,1)$, the selection made to maximize weight; then a further microscopic jump is made; and the process iterates indefinitely. If an arbitrarily small constant is subtracted from the right-hand side of (4.3), the left-hand supremum is seen to exceed the right-hand side for all sufficiently high K. Thus, we obtain (4.3).

However, the left-hand side of (4.3) is at most zero almost surely. Indeed, by the scaling principle, $\sup_{z\in\mathbb{R}} K^{-1}\mathsf{Wgt}_n\big[(0,0)\to(z,K)\big]$ equals $K^{-2/3}\sup_{z\in\mathbb{R}}\mathsf{Wgt}_{nK}\big[(0,0)\to(z,1)\big]$ in law; and the latter supremum converges as $K\to\infty$ in law to the Tracy-Widom GOE distribution ν, because the process $z\to\mathsf{Wgt}_m\big[(0,0)\to(z,1)\big]$ converges in law as $m\to\infty$ in a compact uniform topology to the parabolic Airy process, whose maximum has the law ν (see e.g. [BFPS07]).

Thus, the mean of $\mathsf{Wgt}_n\big[(0,0)\to(z,1)\big]$ is at most $-d_1$. This completes the proof of Proposition 4.1. □

4.2 Two tools

4.2.1 Gaussian increments for weight profiles

For a shortly upcoming use, we record a result bounding the tail of increments for the weight of polymers subject to horizontal endpoint perturbation. The result is better expressed using parabolically adjusted weight, so that a slope arising from a difference of parabolas is eliminated and much higher choices of horizontal endpoint discrepancy may be treated. The parabolic weight notation $\mathsf{Wgt}_n^{\cup}$ was specified in Subsection 3.1.2. We now specify a variant notation in order to describe *differences* in parabolic weight. Let (x_1,x_2) and (y_1,y_2) belong to $\mathbb{R}^2_<$. The *parabolically adjusted* weight difference

$$\Delta^{\cup}\mathsf{Wgt}_n\big[(\{x_1,x_2\},s_1)\to(\{y_1,y_2\},s_2)\big]$$

denotes

$$\begin{aligned}&\left(\mathsf{Wgt}_n\big[(x_2,s_1)\to(y_2,s_2)\big]+2^{-1/2}\frac{(y_2-x_2)^2}{s_2-s_1}\right)\\ -\quad&\left(\mathsf{Wgt}_n\big[(x_1,s_1)\to(y_1,s_2)\big]+2^{-1/2}\frac{(y_1-x_1)^2}{s_2-s_1}\right),\end{aligned}$$

Proposition 4.4. *Positive constants C and c exist for which the following holds. Let $a\in(0,2^{-4}]$. Let $(n,s_1,s_2)\in\mathbb{N}\times\mathbb{R}^2_<$ be a compatible triple for which $ns_{1,2}\geq 10^{32}c^{-18}$ and let $x,y\in\mathbb{R}$ satisfy $\big|x-y\big|s_{1,2}^{-2/3}\leq 2^{-2}3^{-1}c(ns_{1,2})^{1/18}$. Let $K\in\big[10^4\,,\,10^3(ns_{1,2})^{1/18}\big]$. Then*

$$\mathbb{P}\left(\sup_{\substack{x_1,x_2\in[x,x+as_{1,2}^{2/3}]\,,\,x_1<x_2\\ y_1,y_2\in[y,y+as_{1,2}^{2/3}]\,,\,y_1<y_2}}\Big|\Delta^{\cup}\mathsf{Wgt}_n\big[(\{x_1,x_2\},s_1)\to(\{y_1,y_2\},s_2)\big]\Big|\;\geq\;Ka^{1/2}s_{1,2}^{1/3}\right)$$

is at most $C\exp\big\{-c2^{-24}K^2\big\}$.

Proof. The special case that $s_1=0$ and $s_2=1$ is implied by [Ham19a, Theorem 1.1]. (The upper bound in the latter result is $10032\,C\exp\big\{-c_1 2^{-21}R^{3/2}\big\}$. But $c_1=2^{-5/2}c\wedge 1/8$, where $c>0$ is a constant that is at most one, so that we obtain the upper bound in Proposition 4.4. We also relabel $10032C$ to C to obtain the form stated above in the special case.) The scaling principle from Section 3.1.1 then yields the proposition from this special case. □

4.2.2 A control on weight that is uniform as endpoints vary.

Here we record a consequence of Proposition 1.8. Recall that we used $\mathsf{Low}_n(\zeta,\ell,L,M)$ to denote the event that

$$s_{1,2}^{-1/3}\mathsf{Wgt}_n^{\cup}\big[(x,s_1)\to(y,s_2)\big]$$

is less than $-\zeta$ for some pair $(x,s_1),(y,s_2)\in\mathbb{R}\times n^{-1}\mathbb{Z}\cap[0,1]$ with $|x|\vee|y|\le M$, $|x-y|\le 2^{-2\ell/3}L$ and $s_{1,2}\in(2^{-\ell-1},2^{-\ell}]$. Similarly, $\mathsf{High}_n(\zeta,\ell,L,M)$ denotes the event that the displayed quantity exceeds ζ for some such pair. For $M=n^{1/20}$ and $L=(n2^{-\ell})^{1/47}$, specify the *uniform boundedness* event

$$\mathsf{UnifBdd}_n(\zeta)\,=\bigcap_{\ell}\Big(\neg\mathsf{Low}_n\big(\zeta,\ell,L,M\big)\cap\neg\mathsf{High}_n\big(\zeta,\ell,L,M\big)\Big)\,,$$

where the intersection ranges over ℓ such that $\zeta^{20}n^{-1}\le 2^{-\ell}\le 1$.

From Proposition 1.8 and a union bound, it follows that, for $\zeta\ge\Theta\big((\log n)^{2/3}\big)$,

$$\mathbb{P}\big(\mathsf{UnifBdd}_n(\zeta)\big)\ge 1-e^{-\Theta(1)\zeta^{3/2}}. \tag{4.4}$$

4.3 Excursions constrained at given heights are uncompetitive

Let $r\in\mathbb{N}$, and let $\kappa>0$ satisfy

$$\kappa^{3/2}\in r^{-1}\mathbb{N}\text{ and }\kappa^{-3/2}\in\mathbb{N}\,. \tag{4.5}$$

The parameter κ will be positive but small, and these conditions are then ensured if need be by slight adjustment to its value. They ensure that an r-zigzag ϕ of lifetime $[0,1]$ begins and ends at moments that are multiples of $\kappa^{3/2}$, and that every intervening such multiple, being an element of $r^{-1}\mathbb{N}$, is the vertical coordinate of a horizontal interval in ϕ. Further let $b\in(0,1)$. A *segment* is a horizontal planar line segment of length $b\kappa$ whose height is an integer multiple of $\kappa^{3/2}$. The role of b, which will be taken to be a small enough absolute constant, is elucidated in the discussion following the statement of Proposition 4.5.

Let $\chi\in(0,1)$. A *plentiful segment collection* is a set of segments that numbers at least $(1-\chi)\kappa^{-3/2}$ whose elements have distinct heights that include 0 and 1 and that belong to $[-r^{1/20},r^{1/20}]\times[0,1]$. Let $\mathcal{C}$ denote the set of plentiful segment collections.

Let $\mathsf{c}\in\mathcal{C}$. A c-path is an r-zigzag from an element $(z_1,0)$ in the lowest of c's segments to an element $(z_2,1)$ in its highest that intersects every segment in c. Let $\mathsf{Wgt}_r\big[\text{c-path}\big]$ denote the supremum of the weights of c-paths.

Proposition 4.5. *There exist positive parameters κ_0, b, d_1, d_2, and $\chi_0\in(0,1/2]$ such that, if $r\in\mathbb{N}$ and $\kappa\in(0,\kappa_0)$ satisfy (4.5); if $\chi\in(0,\chi_0)$ satisfies $\chi\ge 2\kappa^{3/2}$ and $2\chi\kappa^{-3/2}\in\mathbb{N}$; and if $\mathsf{c}\in\mathcal{C}$; then*

$$\mathbb{P}\Big(\mathsf{Wgt}_r\big[\text{c-path}\big]\ge -d_1\kappa^{-1}\Big)\le\exp\big\{-d_2\kappa^{-3/2}\big\}\,.$$

To derive this result, our task is to show that typically $\mathsf{Wgt}_r\big[\text{c-path}\big]$ is a large negative number. To argue this, let ψ be a c-path. A value of the form $j\kappa^{3/2}$ for $j\in[\![0,\kappa^{-3/2}-1]\!]$ is said to be ψ-*useful*—but we will simply say 'useful'—if ψ intersects a segment in c of height $j\kappa^{3/2}$ and another of height $(j+1)\kappa^{3/2}$. We will write $\mathsf{Wgt}_r(\psi)$ in the form $W_u(\psi)+W_o(\psi)$ in a sense that we now explain. We divide ψ into sub-zigzags by splitting at points of departure of ψ from levels that are integer multiples of $\kappa^{3/2}$ lying in $(0,1)$. Each sub-zigzag is called *useful*, or *otherwise*, according to whether the height of its starting point is useful or not. Then $W_u(\psi)$ and $W_o(\psi)$ are the respective sums of the

weights of the useful, or otherwise, sub-zigzags. We aim to carry out the needed task by finding bounds on the upper tail of the two right-hand terms in the inequality

$$\mathsf{Wgt}_r\big[\text{c-path}\big] \le \sup\big\{W_u(\psi) : \psi \text{ a c-path}\big\} + \sup\big\{W_o(\psi) : \psi \text{ a c-path}\big\}. \tag{4.6}$$

We will first analyse the useful weight sum supremum $\sup\big\{W_u(\psi) : \psi \text{ a c-path}\big\}$; and then do likewise for the otherwise counterpart. The resulting bounds will then permit a quick proof of Proposition 4.5.

Analysing the useful sum is the principal component in the proof of this proposition, and a few words in summary of this analysis will, we hope, be helpful. There are two elements: we will show that weights of the sub-zigzags that contribute to $W_u(\psi)$ have negative mean; and we will then appeal to concentration inequalities for sums of independent random variables.

Regarding the first element, it follows from Proposition 4.1 and the scaling principle that the weight of a useful sub-zigzag with *fixed* endpoints has mean at most $-d_1\kappa^{1/2}$. However, the endpoints of useful sub-zigzags are not fixed, but in fact vary over horizontal segments of length $b\kappa$. An effect of Brownian oscillation for polymer weight that will be controlled in Lemma 4.12 causes our upper bound on the mean to rise by an order of $(b\kappa)^{1/2}$. It is at this moment that we will select the value of $b > 0$. By choosing this parameter to be small enough, the mean supremum weight of useful sub-zigzags traversing between vertically consecutive elements of c will be shown to be at most $-\frac{d_1}{2}\kappa^{1/2}$.

In consecutive subsections, we analyse the useful sum; and the otherwise sum; and give the proof of Proposition 4.5.

4.3.1 The useful sum

Definition 4.6. *Let I and J be compact intervals in $[-r^{1/20}, r^{1/20}]$ of length $b\kappa$. Set*

$$Y_{I,J} = \kappa^{-1/2} \sup_{u\in I, v\in J} \mathsf{Wgt}_r\big[(u,0) \to (v, \kappa^{3/2})\big].$$

Let $\mathcal{U}$ denote the set of useful values. A useful sub-zigzag starts in a segment belonging to c and intersects that segment only at this starting point. Its ending point is $\kappa^{3/2}$ higher than its starting point. The law of the weight of a useful sub-zigzag with given starting height is at most the supremum of weights of r-zigzags that begin and end in two given segments and whose lifetime has duration $\kappa^{3/2}$. Since the sub-zigzag immediately departs from its starting height, the weights of distinct sub-zigzags in our partition are independent. Thus, we find that the useful weight sum supremum

$$\sup\big\{W_u(\psi) : \psi \text{ a c-path}\big\} \text{ is stochastically dominated by } \sum_{i=1}^{|\mathcal{U}|} U_i\,, \tag{4.7}$$

where the latter quantity is a sum of independent random variables, U_i having the law of $\kappa^{1/2}Y_{I,J}$, where $Y_{I,J}$ has been specified in Definition 4.6, and where the pair (I,J) satisfies the hypothesis in that definition.

The next result is our conclusion regarding the useful sum; indeed, with $U_i = \kappa^{1/2}X_i$, it will permit analysis of the right-hand sum in (4.7).

Proposition 4.7. *Let $j \in \mathbb{N}$ satisfy $\kappa^{-3/2}/2 \le j \le \kappa^{-3/2}$. Let $\big\{X_i : i \in [\![1,j]\!]\big\}$ be an independent sequence of random variables, where X_i has the law of $Y_{I,J}$ for a possibly i-dependent pair (I,J) satisfying the hypothesis in Definition 4.6. Then*

$$\mathbb{P}\bigg(\sum_{i=1}^{j} X_i \ge -\Theta(1)\kappa^{-3/2}\bigg) \le \exp\big\{-\Theta(1)\kappa^{-3/2}\big\}.$$

To derive this result, we will need the second element to which we alluded in summary of the useful sum analysis: a concentration result for independent random variables.

Proposition 4.8. *[Ver18, Theorem 2.8.1] Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of independent real-valued random variables of zero mean. For any $C > 0$, there exist positive c_1 and c_2 such that, for $k \in \mathbb{N}$ and $t \geq 0$,*

1. *if $\mathbb{E}\exp\{|X_i|/C\} \leq 2$ then*

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{k} X_i\Big| \geq t\Big) \leq \exp\Big\{-\min\{c_1 t^2 k^{-1}, c_2 t\}\Big\};$$

2. *and if $\mathbb{E}\exp\{X_i^2/C\} \leq 2$, then*

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{k} X_i\Big| \geq t\Big) \leq \exp\big\{-c_1 t^2 k^{-1}\big\}.$$

The variables satisfying the hypotheses in (1) and (2) are known to be sub-exponential and sub-Gaussian variables respectively. Now let $(u', v') \in I \times J$ satisfy

$$|v' - u'| = \inf\big\{|v - u| : u \in I, v \in J\big\}. \tag{4.8}$$

Write $Y_{I,J} = Z_{I,J} + E_{I,J}$, where $Z_{I,J} = \kappa^{-1/2}\mathsf{Wgt}_r\big[(u',0) \to (v', \kappa^{3/2})\big]$ and

$$E_{I,J} = \kappa^{-1/2} \sup_{u \in I, v \in J} \Big(\mathsf{Wgt}_r\big[(u,0) \to (v,\kappa^{3/2})\big] - \mathsf{Wgt}_r\big[(u',0) \to (v',\kappa^{3/2})\big]\Big).$$

This decomposition reflects the argument promised in the first element in our summary: $Z_{I,J}$ is a point-to-point weight (normalized to be of unit-order by the factor $\kappa^{-1/2}$), and the *error* term $E_{I,J}$ is a weight difference due to horizontal endpoint perturbation. We offer a Gaussian form of control on the latter next. In a usage also found later in this section, C_0 and c_0 denote positive constants whose value may change from line to line.

Lemma 4.9. *There exist positive constants C_0 and c_0 such that, for $r \geq C_0\kappa^{-195}$ and $0 < b \leq 2^{-4}$, the following holds. For all I and J as above, there exists an event $\mathsf{E}_{I,J}$ such that*

$$\mathbb{P}\big(\mathsf{E}_{I,J}\big) \leq C_0 \exp\big\{-c_0 r^{1/9}\kappa^{1/6}\big\} \tag{4.9}$$

and that

$$\mathbb{P}\Big(E_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} \geq b^{1/2}h\Big) \leq C_0 \exp\big\{-c_0 h^2\big\} \tag{4.10}$$

for $h \geq 0$.

Proof. Note that, for $u \in I$ and $v \in J$,

$$\begin{aligned} & \mathsf{Wgt}_r\big[(u,0) \to (v,\kappa^{3/2})\big] - \mathsf{Wgt}_r\big[(u',0) \to (v',\kappa^{3/2})\big] \\ = \;& \Delta^{\cup}\mathsf{Wgt}_r\big[(\{u',u\},0) \to (\{v',v\},\kappa^{3/2})\big] + 2^{-1/2}\frac{(v'-u')^2}{\kappa^{3/2}} - 2^{-1/2}\frac{(v-u)^2}{\kappa^{3/2}} \\ \leq \;& \Delta^{\cup}\mathsf{Wgt}_r\big[(\{u',u\},0) \to (\{v',v\},\kappa^{3/2})\big], \end{aligned}$$

where the inequality is due to (4.8). Thus, $E_{I,J} \leq \kappa^{-1/2}\Delta^{\cup}\mathsf{Wgt}_r\big[(\{u',u\},0) \to (\{v',v\},\kappa^{3/2})\big]$. We now apply Proposition 4.4 with parameter settings $\mathbf{n} = r$, $\mathbf{s_{1,2}} = \kappa^{3/2}$, $\mathbf{a} = b$ and $\mathbf{K} = h$, and with $\mathbf{x}$ and $\mathbf{y}$ equal to the left endpoints of I and J. Note that the hypothesis $\mathbf{a} \leq 2^{-4}$ holds due to $b \leq 2^{-4}$. The hypothesis $\big|\mathbf{x}-\mathbf{y}\big|s_{1,2}^{-2/3} \leq 2^{-2}3^{-1}c(rs_{1,2})^{1/18}$ holds due to $|\mathbf{x}|, |\mathbf{y}| \leq r^{1/20}$, which follows by hypothesis on I and J; and to the hypothesised lower bound on r.

The hypothesis $\mathbf{n}\,\mathbf{s}_{\mathbf{1},\mathbf{2}} \geq 10^{32}c^{-18}$ is due to $r\kappa^{3/2} \geq 10^{32}c^{-18}$, a consequence of the hypothesised lower bound on r alongside $c < 1$ and $\kappa < 1$. The hypothesis $\mathbf{K} \in \big[10^4\,,\, 10^3(r\kappa^{3/2})^{1/18}\big]$ holds provided that we impose this condition on h. This application of Proposition 4.4 yields that

$$\mathbb{P}\big(E_{I,J} \geq b^{1/2}h\big) \leq C\exp\big\{-c2^{-24}h^2\big\}$$

for $h \in \big[10^4\,,\, 10^3 r^{1/18}\kappa^{1/12}\big]$. Define the error event $\mathsf{E}_{I,J} = \big\{E_{I,J} \geq h_0(b\kappa)^{1/2}\big\}$ where h_0 equals the maximal value $10^3 r^{1/18}\kappa^{1/12}$ for the range of h. We obtain (4.9), and (4.10) for $h \geq 0$. □

To address the point-to-point normalized weight $Z_{I,J}$, we introduce a parabolically adjusted version:

$$\bar{Z}_{I,J} = Z_{I,J} + 2^{-1/2}\frac{(v'-u')^2}{\kappa^2}\,. \tag{4.11}$$

By Lemma 3.1 with parameter setting $\mathbf{n} = r\kappa^{3/2}$, and the scaling principle, it follows that, for $t \geq 0$,

$$\mathbb{P}\Big(|\bar{Z}_{I,J}| \geq t\Big) \leq C\exp\big\{-ct^{3/2}\big\}\,,\ \text{ and hence } \mathbb{P}\Big(|\bar{Z}_{I,J}|\mathbf{1}_{\mathsf{E}^c_{I,J}} \geq t\Big) \leq C\exp\big\{-ct^{3/2}\big\}\,. \tag{4.12}$$

This implies that

$$\mathbb{P}\Big(\Big|\bar{Z}_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}[\bar{Z}_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}}]\Big| \geq t\Big) \leq C\exp\big\{-ct^{3/2}\big\}\,. \tag{4.13}$$

Concentration of the sums $\sum_{i=1}^{j}\bar{Z}_{I,J}$ and $\sum_{i=1}^{j}Z_{I,J}$ will be related by means of

$$\begin{aligned}&\sum_{i=1}^{j}\bar{Z}_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}[\bar{Z}_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}}] \qquad (4.14)\\ =\ &\sum_{i=1}^{j}\big[Z_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}}]\big] + \sum_{i=1}^{j}2^{-1/2}\frac{(v'-u')^2}{\kappa^2}[\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}(\mathbf{1}_{\mathsf{E}^c_{I,J}})]\end{aligned}$$

with the next result offering control on the latter parabolic term.

Lemma 4.10. *With probability at least* $1 - C_0\kappa^{-3/2}\exp\big\{-c_0 r^{1/9}\kappa^{1/6}\big\}$,

$$\sum_{i=1}^{j}2^{-1/2}\frac{(v'-u')^2}{\kappa^2}[\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}(\mathbf{1}_{\mathsf{E}^c_{I,J}})] \leq C_0 r^{1/10}\kappa^{-4}\exp\big\{-c_0 r^{1/9}\kappa^{1/6}\big\}\,,$$

Note that, by our hypothesis on κ, the right-hand side is less than one when r is large enough.

Proof. Since by (4.9), $\mathbb{E}(\mathbf{1}_{\mathsf{E}^c_{I,J}}) \geq 1 - C_0\exp\big\{-c_0 r^{1/9}\kappa^{1/6}\big\}$, on the event $\mathbf{1}_{\mathsf{E}^c_{I,J}} = 1$ we have

$$2^{-1/2}\frac{(v'-u')^2}{\kappa^2}[\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}(\mathbf{1}_{\mathsf{E}^c_{I,J}})] \leq C_0 r^{1/10}\kappa^{-2}\exp\big\{-c_0 r^{1/9}\kappa^{1/6}\big\}\,,$$

since $|v'-u'| \leq r^{1/20}$. A union bound over $i \in [\![1,j]\!]$ now implies the lemma. □

We now note that the tail bound (4.13) allows us to invoke Proposition 4.8(1). For brevity's sake, let $W_{I,J} = \bar{Z}_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}} - \mathbb{E}[\bar{Z}_{I,J}\mathbf{1}_{\mathsf{E}^c_{I,J}}]$. By this proposition, we may find $d > 0$ so that

$$\mathbb{P}\bigg(\sum_{i=1}^{j}W_{I,J} \geq \frac{d}{10}\kappa^{-3/2}\bigg) \leq \exp\big\{-c_0 d\kappa^{-3/2}\big\}\,.$$

Now, by our assumptions on r and κ, (4.14) and Lemma 4.10, it follows that

$$\mathbb{P}\Big(\sum_{i=1}^{j} Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} - \mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c}] \geq \frac{d}{8}\kappa^{-3/2} \Big) \leq \exp\big\{ - c_0 d\kappa^{-3/2} \big\} . \tag{4.15}$$

Here we used $C_0\kappa^{-3/2}\exp\big\{-c_0 r^{1/9}\kappa^{1/6}\big\} \leq e^{-c_0\kappa^{-3/2}}$, which follows from our hypotheses on r and κ.

In the last display, we see, in a mildly truncated form, the point-to-point mean weight that was the mainstay of our overview of the first element of useful sum analysis. Indeed, we next argue that this truncated mean is suitably negative.

Lemma 4.11. *There exists $d > 0$ such that, for $r \geq \Theta(1)$,*

$$\sum_{i=1}^{j} \mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c}] \leq -\frac{d}{4}\kappa^{-3/2} .$$

Proof. Since Proposition 4.1 and the scaling principle imply that $\mathbb{E}[Z_{I,J}] \leq -d$, it suffices to show that $\sum_{i=1}^{j} \mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}}] \leq \frac{d}{10}\kappa^{-3/2}$. Now note that $\mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}}] \leq \big(\mathbb{E}(Z_{I,J}^2)\mathbb{P}(\mathsf{E}_{I,J})\big)^{1/2}$. Since r is assumed to be large enough, we are now done by (4.9), and $\sqrt{\mathbb{E}(Z_{I,J}^2)} \leq O(\frac{r^{1/10}}{\kappa^2})$, the latter due to Lemma 3.1, which implies that, in the decomposition (4.11), the first term is $O(1)$, and hence the above bound arises from the maximum value of the parabolic term. □

A final result needed to deliver Proposition 4.7 concerns the error terms $E_{I,J}$.

Lemma 4.12. *There exist positive constants C_1 and c_1 such that, for small enough b and κ,*

$$\mathbb{P}\Big(\sum_{i=1}^{j} E_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} \geq b^{1/2}C_1\kappa^{-3/2} \Big) \leq \exp\big\{ - c_1\kappa^{-3/2} \big\} .$$

Proof. Note that, by definition, the random variables $b^{-1/2}E_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c}$ are independent. Further, by (4.10), they are sub-Gaussian. Thus, Proposition 4.8(2)'s hypotheses are satisfied, and the lemma is obtained. □

Proof of Proposition 4.7. By (4.15) and the two just stated lemmas, we find that, for a suitably small choice of $b > 0$, it is with probability at least $1 - e^{-c_0\kappa^{-3/2}}$ that

$$\begin{aligned}
\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} X_i\mathbf{1}_{\mathsf{E}_{I,J}^c} &= \sum_{i=1}^{j} Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} + \sum_{i=1}^{j} E_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} \\
&= \sum_{i=1}^{j} \mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c}] + \sum_{i=1}^{j} [Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} - \mathbb{E}[Z_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c}] + \sum_{i=1}^{j} E_{I,J}\mathbf{1}_{\mathsf{E}_{I,J}^c} \\
&\leq \frac{-d}{4}\kappa^{-3/2} + \frac{d}{8}\kappa^{-3/2} + b^{1/2}C_1\kappa^{-3/2} \leq \frac{-d}{16}\kappa^{-3/2} .
\end{aligned}$$

The first equality follows from (4.9) and a union bound. □

4.3.2 The otherwise sum

Our upper bound on the otherwise sum, namely on the latter right-hand term in (4.6), will depend partly on there being few otherwise summands. To this end, we begin by stating and proving a simple claim giving a lower bound on the number of useful summands. Recall that it is hypothesised in Proposition 4.5 that $2\chi\kappa^{-3/2} \in \mathbb{N}$ and $2\chi < 1$.

We *claim* that,

$$\text{for at least } (1-2\chi)\kappa^{-3/2}-2 \text{ indices } j\in[\![0,\kappa^{-3/2}-1]\!], \text{ the value } j\kappa^{3/2} \text{ is useful}\,, \tag{4.16}$$

where the notion of usefulness was specified after Proposition 4.5. In verifying this, we will describe two segments in c as being *vertically consecutive* if their heights differ by $\kappa^{3/2}$. Consider the set formed from $\mathbb{N}\kappa^{3/2}\cap[0,1]$ by the removal of those elements that are the heights of members of c. This set has cardinality at most $\chi\kappa^{-3/2}+1$. Any element $j\kappa^{3/2}$ in the set forbids the values $(j-1)\kappa^{3/2}$ and $j\kappa^{3/2}$ from being useful. Cumulatively, at most $2\chi\kappa^{3/2}+2$ elements of $\kappa^{3/2}[\![0,\kappa^{-3/2}-1]\!]$ are thus forbidden. The remainder, numbering at least $\kappa^{-3/2}-2\chi\kappa^{-3/2}-2$, are useful. This is as we claimed.

We now present a bound on the upper tail of the otherwise weight sum supremum $\sup\big\{W_o(\psi):\psi \text{ a c-path}\big\}$. Let ψ again denote a given c-path. Let $O\in\mathbb{N}$ denote the number of otherwise sub-zigzags of ψ. Recalling that $\mathcal{U}$ denotes the set of useful values, with $U=|\mathcal{U}|$, we have $(U+O)\kappa^{3/2}\le 1$, because every sub-zigzag, useful or otherwise, has height at least $\kappa^{3/2}$; so that (4.16) implies that

$$O\le 2\chi\kappa^{-3/2}+2\,. \tag{4.17}$$

The expression $W_o(\psi)$ is the sum of weights of the otherwise sub-zigzags of ψ. Each otherwise sub-zigzag begins, but immediately leaves, a given segment in c, and ends in another such segment. The two segments may be called the *starting* and *finishing* segments of the sub-zigzag. Labelling the otherwise sub-zigzags $\big\{Z_i:i\in[\![1,O]\!]\big\}$ in order of increasing height, we denote by S_i and F_i the starting and ending segments of Z_i. The respective heights of S_i and F_i will be denoted by s_i and f_i. Note that $s_i<f_i\le s_{i+1}<f_{i+1}$ for $i\in[\![1,O-1]\!]$.

For $i\in[\![1,O]\!]$, let $(u_i,s_i)\in S_i$ and $(v_i,f_i)\in F_i$ be chosen so that the gradient of the line segment connecting (u_i,s_i) and (v_i,f_i) is maximal given that these endpoints lie in S_i and F_i. This implies that

$$|v_i-u_i|=\inf\Big\{|v-u|:u,v\in\mathbb{R},(u,s_i)\in S_i,(v,s_{i+1})\in F_i\Big\}\,. \tag{4.18}$$

We find then that the otherwise weight sum supremum

$$\sup\big\{W_o(\psi):\psi \text{ a c-path}\big\} \text{ is stochastically dominated by } \sum_{i=1}^{O}\big(\mathsf{W}_i+E_i\big)\,, \tag{4.19}$$

where $\big\{\mathsf{W}_i:i\in[\![1,O]\!]\big\}$ is an independent sequence whose term W_i has the distribution of $\mathsf{Wgt}_r\big[(u_i,s_i)\to(v_i,f_i)\big]$; and where $\big\{E_i:i\in[\![1,O]\!]\big\}$ is an independent sequence of error terms given by

$$E_i=\sup_{\substack{u,v\in\mathbb{R}:\\(u,s_i)\in S_i\\(v,f_i)\in F_i}}\Big(\mathsf{Wgt}_r\big[(u,s_i)\to(v,f_i)\big]-\mathsf{Wgt}_r\big[(u_i,s_i)\to(v_i,f_i)\big]\Big)\,.$$

Reminiscently of useful sum analysis, the right-hand quantity in (4.19) is a sum of a point-to-point W-sum and an error E-sum. The next two lemmas, which provide tail bounds on these two sums, are the outcomes of otherwise sum analysis that will be needed to prove Proposition 4.5.

Lemma 4.13. *There exist $y\in\mathbb{R}$ and $s\in[0,1]$ that satisfy $|y|\le r^{1/20}+2\chi b\kappa^{-1/2}$ and $s\le 2\chi$ such that the random variable $\sum_{i=1}^{O}\mathsf{W}_i$ is stochastically dominated by $\mathsf{Wgt}_r\big[(0,0)\to(y,s)\big]$.*

Proof. Set $s = \sum_{i=1}^{O}(f_i - s_i)$ and $y = \sum_{i=1}^{O}(v_i - u_i)$. Set S equal to the set of planar points $p_j = \big(\sum_{i=1}^{j}(v_i - u_i), \sum_{i=1}^{j}(f_i - s_i)\big)$ where j varies over $[\![1, O]\!]$. The value $\mathsf{Wgt}_r\big[(0,0) \to (y,s)\big]$ is the supremum of the weights of r-zigzags from $(0,0)$ to (y,s). This value is at least the supremum W of weights of r-zigzags with these endpoints but that also contain the set S. Note that W has the law of $\sum_{i=1}^{O}\mathsf{W}_i$, because, if Z denotes the maximizer zigzag in the optimization that specifies W, the weight of the sub-zigzag of Z between p_i and p_{i+1} equals W_i in law, and the various sub-zigzags have independent weights. □

Lemma 4.14. *There exist $C_0, c_0 > 0$ such that the following holds. Suppose that $r \geq C_0\kappa^{-195}$, $h \in \big[10^4\,,\, 10^3(rs_{i,i+1})^{1/18}\big]$ and $b \leq 2^{-4}$.*

1. *For $i \in [\![1, O]\!]$,*
$$\mathbb{P}\Big(E_i \geq h(b\kappa)^{1/2}\Big) \leq C_0 \exp\big\{-c_0h^2\big\}.$$
2. *Suppose that $\chi \geq 2\kappa^{3/2}$. There exist $h_0 > 0$ and an error event E satisfying*
$$\mathbb{P}\big(\mathsf{E}\big) \leq C_0\chi\kappa^{-3/2}\exp\big\{-c_0r^{1/9}\kappa^{1/6}\big\} \tag{4.20}$$
such that, for $h \geq h_0$,
$$\mathbb{P}\Big(\sum_{i=1}^{O}E_i\mathbf{1}_{\mathsf{E}^c} \geq h\chi b^{1/2}\kappa^{-1}\Big) \leq C_0\exp\big\{-c_0\chi\kappa^{-3/2}h^2\big\}. \tag{4.21}$$

Proof: (1). Note that, for $u, v \in \mathbb{R}$ with $(u, s_i) \in S_i$ and $(v, s_{i+1}) \in S_{i+1}$,
$$\begin{aligned}
&\mathsf{Wgt}_r\big[(u,s_i) \to (v,s_{i+1})\big] - \mathsf{Wgt}_r\big[(u_i,s_i) \to (v_{i+1},s_{i+1})\big]\\
=\;& \Delta^{\cup}\mathsf{Wgt}_n\big[(\{u_i,u\},s_1) \to (\{v_i,v\},s_2)\big] + 2^{-1/2}\frac{(v_i-u_i)^2}{s_2-s_1} - 2^{-1/2}\frac{(v-u)^2}{s_2-s_1}\\
\leq\;& \Delta^{\cup}\mathsf{Wgt}_n\big[(\{u_i,u\},s_1) \to (\{v_i,v\},s_2)\big],
\end{aligned}$$
where the inequality is due to (4.18). Thus, $E_i \leq \Delta^{\cup}\mathsf{Wgt}_n\big[(\{u_i,u\},s_1) \to (\{v_i,v\},s_2)\big]$. We now apply Proposition 4.4 with parameter settings $\mathbf{n} = r$, $\mathbf{s_{1,2}} = s_{i,i+1}$, $\mathbf{a} = s_{i,i+1}^{-2/3}b\kappa$ and $\mathbf{K} = h$, and with $\mathbf{x}$ and $\mathbf{y}$ equal to the left endpoints of S_i and S_{i+1}. Note that the hypothesis $\mathbf{a} \leq 2^{-4}$ holds due to $b \leq 2^{-4}$ and $s_{i,i+1} \geq \kappa^{3/2}$. The hypothesis $\big|\mathbf{x}-\mathbf{y}\big|s_{i,i+1}^{-2/3} \leq 2^{-2}3^{-1}c(rs_{i,i+1})^{1/18}$ holds due to $|\mathbf{x}|, |\mathbf{y}| \leq r^{1/20}$, which follows from $S_1, S_2 \in \mathsf{c}$; to $s_{i,i+1} \geq \kappa^{3/2}$; and to the hypothesised lower bound on r. The hypothesis $\mathbf{n}\,\mathbf{s_{1,2}} \geq 10^{32}c^{-18}$ is due to $r\kappa^{3/2} \geq 10^{32}c^{-18}$, a consequence of the hypothesised lower bound on r alongside $c < 1$ and $\kappa < 1$. The hypothesis $\mathbf{K} \in \big[10^4\,,\,10^3(ns_{1,2})^{1/18}\big]$ holds because this condition is imposed on h. Lemma 4.14(1) follows from this application of Proposition 4.4.

(2). Set the error event E by defining $\mathsf{E} = \cap_{i\in[\![0,O-1]\!]}\mathsf{E}_i$, where
$$\mathsf{E}_i = \Big\{E_i \geq (b\kappa)^{1/2}10^3\big(rs_{i,i+1}\big)^{1/18}\Big\}.$$
Since $\chi \geq 2\kappa^{3/2}$, (4.17) implies that $|O| \leq 3\chi\kappa^{-3/2}$. From this bound, and since $s_{i,i+1} \geq \kappa^{3/2}$ for all concerned indices i, we obtain (4.20), from the conclusion of Lemma 4.14(1) and a union bound.

The random variables $E_i\mathbf{1}_{\mathsf{E}_i^c}$ verify the conclusion of Lemma 4.14(1) for all $h \geq 10^4$; that is, even after the removal of the upper bound on h hypothesised in that result. Hence, they are, after scaling by the factor $(b\kappa)^{-1/2}$, sub-Gaussian variables as in Proposition 4.8(2). Since (4.17) holds, (4.21) is implied by the bound $\mathbb{P}\big(\sum_{i=1}^{K}E_i' \geq hK\big) \leq e^{-\Theta(1)Kh^2}$ for all large h, where $K = \lceil 2\chi\kappa^{-2/3}\rceil + 2$ and $E_i' = (b\kappa)^{-1/2}E_i\mathbf{1}_{\mathsf{E}_i^c}$. Since the E_i' are by definition independent, the desired bound follows from Proposition 4.8(2). □

4.3.3 Proof of Proposition 4.5

We are ready to return to the bound (4.6) in order to prove the upper tail bound stated by this proposition. Indeed, by (4.6), (4.7) and (4.19),

$$\mathbb{P}\Big(\mathsf{Wgt}_r\big[\text{c-path}\big] \geq -d_1\kappa^{-1}\Big) \leq A_1 + A_2 + A_3\,,$$

where here we set $A_1 = \mathbb{P}\big(\sum_{i=1}^{|\mathcal{U}|} U_i \geq -2d_1\kappa^{-1}\big)$, $A_2 = \mathbb{P}\big(\sum_{i=1}^{O} \mathsf{W}_i \geq 2^{-1}d_1\kappa^{-1}\big)$ and $A_3 = \mathbb{P}\big(\sum_{i=1}^{O} E_i \geq 2^{-1}d_1\kappa^{-1}\big)$.

To find an upper bound on A_1, note that $\sum_{i=1}^{|\mathcal{U}|} U_i \geq -2d_1\kappa^{-1}$ entails that $\sum_{i=1}^{\kappa^{-3/2}} U_i \geq -2d_1\kappa^{-1}$, where on the right-hand side, further U_i-terms have been introduced consistently with the conditions on this sequence. Provided that $b \leq b_0$ and $d_1 \leq d/8$, we may apply Proposition 4.7 with $j = \kappa^{-3/2}$ to find that

$$A_1 \leq \exp\big\{-d\kappa^{-3/2}\big\}\,.$$

Let the parameters $y \in \mathbb{R}$ and $s \in [0,1]$ satisfy the hypotheses of Lemma 4.13. This result implies that $A_2 \leq \mathbb{P}\big(\mathsf{Wgt}_r\big[(0,0) \to (y,s)\big] \geq 2^{-1}d_1\kappa^{-1}\big)$. From $s \leq 2\chi$, and the one-point upper tail bound $\mathbb{P}\big(\mathsf{Wgt}_r\big[(0,0) \to (y,s)\big] \geq hs^{1/3}\big) \leq C\exp\big\{-ch^{3/2}\big\}$ offered by Lemma 3.1(1) via the scaling principle, we see that

$$A_2 \leq C\exp\big\{-2^{-2}c\chi^{-1/2}d_1^{3/2}\kappa^{-3/2}\big\}\,.$$

In the notation of Lemma 4.14(2), $A_3 \leq \mathbb{P}\big(\sum_{i=1}^{O} E_i\mathbf{1}_{\mathsf{E}^c} \geq 2^{-1}d_1\kappa^{-1}\big) + \mathbb{P}\big(\mathsf{E}\big)$. Choose h in Lemma 4.14(2) so that $h\chi b^{1/2} = 2^{-1}d_1$; we ensure the needed condition that $h \geq h_0$ by insisting that $\chi > 0$ be small enough (as we do by demanding that $\chi \leq \chi_0$ in Proposition 4.5). From Lemma 4.14(2), we thus learn that

$$A_3 \leq C_0\exp\big\{-c_0\kappa^{-3/2}\chi^{-1}b^{-1}2^{-2}d_1^2\big\} + C_0\chi\kappa^{-3/2}\exp\big\{-c_0r^{1/9}\kappa^{1/6}\big\}\,.$$

Applying $r \geq \kappa^{-15/2}$ in the guise $r^{1/9}\kappa^{1/6} \geq \kappa^{-2/3}$, we obtain Proposition 4.5 by choosing (or adjusting) the positive parameters κ_0, d_1 and d_2 to be suitably small. □

4.4 Deriving Theorem 1.10

Throughout this section, we suppose that $\theta^{-1/4} > C\log n$ and that $\ell \in \mathbb{N}$ satisfies $2^\ell \leq n\theta^{40}$, since these conditions are hypothesised by the result that we seek to show.

To apply Proposition 4.5, let $\kappa > 0$ satisfy $\kappa^{3/2} \in n^{-1}\mathbb{Z}$, $\kappa^{-3/2} \in \mathbb{N}$ and $b\kappa/8 \in 2^{-2\ell/3}\theta\cdot[1,2]$, where b appears in the statement of the proposition. Let κ also satisfy $\kappa^{3/2} \leq 2^{-1-\ell}\chi$.

As we derive Theorem 1.10, we will define the terms segments, plentiful segment collection and c-path. In doing so, we abuse the notation employed in Proposition 4.5. However, as we will explain shortly, the different usages coincide when suitable parameters are specified and a simple change of coordinates is made.

Suppose given an n-zigzag ϕ from $(0,0)$ to $(0,1)$. For now, we take an arbitrary such ϕ, though we later impose the condition, seen in Theorem 1.10, that ϕ be $\theta^{-1/40}$ regular.

Further suppose given $s_1, s_2 \in n^{-1}\mathbb{Z} \cap [0,1]$ for which $2^{-1-\ell} \leq s_{1,2} \leq 2^{-\ell}$; Recall that our choice of κ ensures that $\kappa^{3/2} \leq \chi s_{1,2}$. Let $r_0 \in n^{-1}[\![0, n\kappa^{3/2} - 1]\!]$. An r_0-*segment* is a horizontal planar interval of the form

$$I_{j/n+r_0} := \big[\phi\big(j/n + r_0\big) - b\kappa/2, \phi\big(j/n + r_0\big) + b\kappa/2\big] \times \{jn^{-1} + r_0\}$$

where $j/n \in \kappa^{3/2}\mathbb{Z}$ and $j/n + r_0 \in n^{-1}[\![ns_1, ns_2]\!]$: see Figure 5.

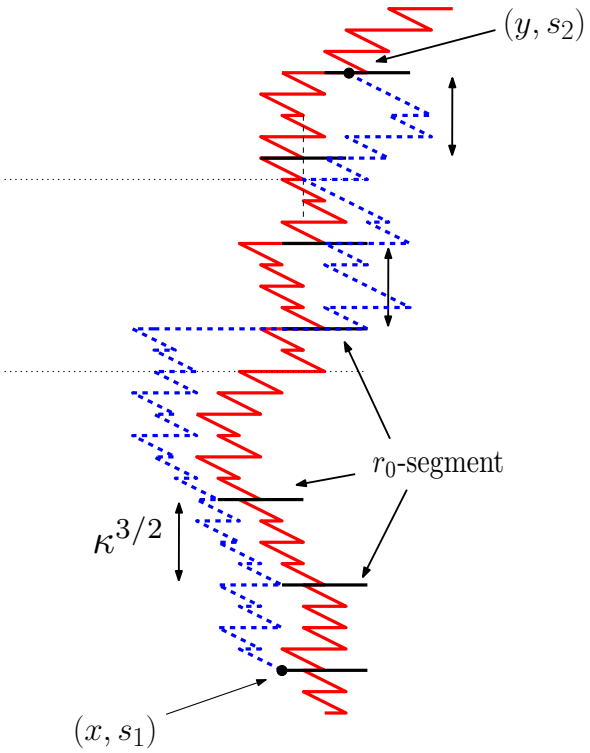

Figure 5: The solid zigzag is ϕ and the dashed zigzag is a c-path which passes through the elements of c, denoted by the black horizontal planar lines of length $b\kappa$, a plentiful r_0-segment collection. Such segments occur regularly at vertical separation being integer multiples of $\kappa^{3/2}$.

A plentiful r_0-segment collection is a subset of $\{I_{j/n+r_0} : j/n \in \kappa^{3/2}\mathbb{Z}, j/n + r_0 \in n^{-1}[\![ns_1, ns_2]\!]\}$ whose cardinality is at least $(1-\chi)s_{1,2}\kappa^{-3/2}$, where note that the quantity $s_{1,2}\kappa^{-3/2}$ differs from the cardinality of the set $\{j/n \in \kappa^{3/2}\mathbb{Z} : j/n + r_0 \in n^{-1}[\![ns_1, ns_2]\!]\}$ by at most one. If v^- and v^+ denote the lowest and highest vertical coordinates assumed by elements in a plentiful r_0-segment collection, note that

$$\big|v^- - s_1\big| \vee \big|v^+ - s_2\big| \le 2\chi s_{1,2}\,. \tag{4.22}$$

Indeed, the left-hand side is at most $\chi s_{1,2} + \kappa^{3/2}$, and $\kappa^{3/2} \le \chi s_{1,2}$.

Let $\mathcal{C}_{r_0}$ denote the set of plentiful r_0-segment collections. Set $\mathcal{C}$ equal to the union of $\mathcal{C}_{r_0}$ as r_0 varies over $n^{-1}[\![0, n\kappa^{3/2} - 1]\!]$. For given such r_0, let $\mathsf{c} \in \mathcal{C}_{r_0}$. A c-path is an n-zigzag whose starting moment is the lowest height of an element of c; whose ending moment is the greatest such height; and that intersects every element of c. Let $\mathsf{Wgt}_n\big[\text{c-path}\big]$ denote the supremum of the weights of c-paths.

Define

$$\mathsf{HighSlenderWeight}\big(s_1, s_2, 1-\chi; \phi\big) := \bigcup_{\mathsf{c}\in\mathcal{C}} \Big\{ \mathsf{Wgt}_n\big[\text{c-path}\big] \ge -2^{-1}d_0\kappa^{-1}s_{1,2} \Big\}\,.$$

Recall the event $\mathsf{UnifBdd}_n(\cdot)$ from Subsection 4.2.2. The argument to prove Theorem 1.10 proceeds by showing that, on the event $\mathsf{UnifBdd}_n(\cdot)$—with a choice of this event's parameter that will make it typical—the non-occurrence of $\mathsf{LowSlenderWeight}^*(\ell, \theta, 1-\chi; \phi)$ implies the occurrence of $\mathsf{HighSlenderWeight}\big(s_1, s_2, 1-\chi; \phi\big)$ for some s_1 and s_2. The latter event is then shown to be rare, implying the desired rarity of $\mathsf{LowSlenderWeight}^*(\ell, \theta, 1-\chi; \phi)$.

Recall the notion of regularity from (1.11). and that the definition of the quantity $\mathsf{LowSlenderWeight}^*(\ell,\theta,1-\chi;\phi)$ involves a constant d_0.

Lemma 4.15. *Suppose that $\chi^{1/3}\le 2^{-11/3}d_0$. Then, for any zigzag ϕ from $(0,0)$ to $(0,1)$ that is $\theta^{-1/4}$-regular,*

$$\begin{aligned}&\mathsf{UnifBdd}\bigl(\frac{b}{10}\theta^{-1}\bigr)\cap\neg\,\mathsf{LowSlenderWeight}^*(\ell,\theta,1-\chi;\phi)\\ \subseteq\;&\bigcup_{\substack{0\le s_1\le s_2\le 1\\ s_{1,2}\in[2^{-1-\ell},2^{-\ell}]}}\mathsf{HighSlenderWeight}(s_1,s_2,1-\chi;\phi)\,.\end{aligned}$$

Proof. When $\neg\,\mathsf{LowSlenderWeight}^*(\ell,\theta,1-\chi;\phi)$ occurs, there exist $(x,s_1),(y,s_2)\in\mathbb{R}\times n^{-1}\mathbb{Z}$ for which $2^{-1-\ell}\le s_{1,2}\le 2^{-\ell}$, and a $(\phi,\theta,1-\chi)$-close zigzag ψ from (x,s_1) to (y,s_2) that satisfies

$$\mathsf{Wgt}_n(\psi)>-d_0\kappa^{-1}s_{1,2}\,. \tag{4.23}$$

By our choice of b and κ, the zigzag ψ satisfies the bound

$$\bigl|\psi(s)-\phi(s)\bigr|\le b\kappa/2 \tag{4.24}$$

for at least $(1-\chi)\bigl|[s_1,s_2]\cap n^{-1}\mathbb{Z}\bigr|$ values of $s\in[s_1,s_2]\cap n^{-1}\mathbb{Z}$. When elements of $[s_1,s_2]\cap n^{-1}\mathbb{Z}$ are identified if they differ by a multiple of $\kappa^{3/2}$, they are partitioned into classes which are naturally indexed by r_0. Since $s_{1,2}\ge\kappa^{3/2}$, the number of classes equals $n\kappa^{3/2}$. At least one of these classes—call it $\mathcal{D}_{r_0}$—contains at least $n^{-1}\kappa^{-3/2}(1-\chi)\bigl|[s_1,s_2]\cap n^{-1}\mathbb{Z}\bigr|=\kappa^{-3/2}(1-\chi)\bigl(s_{1,2}+n^{-1}\bigr)$ elements s that satisfy (4.24). Let c denote the set of $I_{j/n+r_0}$ indexed by those $j/n+r_0\in\mathcal{D}_{r_0}$ such that (4.24) is satisfied with $s=j/n+r_0$ (where note that $j/n\in\kappa^{3/2}\mathbb{Z}$ and $j/n+r_0\in n^{-1}[\![ns_1,ns_2]\!]$).

Note that $|\mathsf{c}|\ge\kappa^{-3/2}(1-\chi)s_{1,2}$. Thus, c is a plentiful r_0-segment collection; so that $\mathsf{c}\in\mathcal{C}$. We claim that

$$\mathsf{Wgt}_n\bigl[\mathsf{c}\text{-path}\bigr]\;\ge\;-2^{-1}s_{1,2}d_0\kappa^{-1}\,. \tag{4.25}$$

To verify this, let ψ^0 denote the sub-zigzag of ψ from the entry of ψ to the lowest vertical coordinate v^- in $\mathcal{D}_{r_0}$ to its departure from the highest such coordinate v^+. Note that $\mathsf{Wgt}_n\bigl[\mathsf{c}\text{-path}\bigr]$ is at least $\mathsf{Wgt}_n(\psi^0)$. To bound below the latter weight, we write ψ as a concatenation $\psi^-\circ\psi^0\circ\psi^+$.

We now claim that, since ϕ is regular, the occurrence of the event $\mathsf{UnifBdd}_n\bigl(b\theta^{-1}\bigr)$ entails that $\mathsf{Wgt}_n(\psi^-)$ and $\mathsf{Wgt}_n(\psi^+)$ are at most

$$2\bigl(4\chi s_{1,2}\bigr)^{1/3}s_{1,2}^{2/3}\kappa^{-1}=2^{5/3}\chi^{1/3}s_{1,2}\kappa^{-1}.$$

Admitting the claim, note that, by weight additivity, $\mathsf{Wgt}_n(\psi^0)$ equals $\mathsf{Wgt}_n(\psi)-\mathsf{Wgt}_n(\psi^-)-\mathsf{Wgt}_n(\psi^+)$; by (4.23), and the claim, this weight is thus seen to be at least $-s_{1,2}d_0\kappa^{-1}+2^{8/3}\chi^{1/3}s_{1,2}\kappa^{-1}$. Since $2^{8/3}\chi^{1/3}\le 2^{-1}d_0$, we have verified (4.25) for all n high enough. In view of the definition of the $\mathsf{HighSlenderWeight}$ event, this reduces the proof of Lemma 4.15 to deriving the claim.

To this end, note first that bounds on $\mathsf{Wgt}_n(\psi^-)$ and $\mathsf{Wgt}_n(\psi^+)$ cannot be obtained directly from Proposition 1.8, since the duration of these zigzags may be too small for this result to offer a meaningful bound. This said, the claimed bounds follow easily from superadditivity. We will provide only the argument for ψ^-. Let (x_1,s_1) and (x_2,v_-) denote this zigzag's endpoints (recall from (4.22), that $v_--s_1\le 2\chi s_{1,2}$). Now consider the point $(x_2,v_-+\chi s_{1,2})$. We then have

$$\mathsf{Wgt}_n(\psi_-)\le\mathsf{Wgt}_n\bigl[(x_1,s_1)\to(x_2,v_-+\chi s_{1,2})\bigr]-\mathsf{Wgt}_n\bigl[(x_2,v_-)\to(x_2,v_-+\chi s_{1,2})\bigr]\,.$$

Now, recall that, when ϕ is $\theta^{-1/4}$-regular, we have that, for any $(u,h_1),(v,h_2)\in\phi$,

$$\big|v-u\big| \le h_{1,2}^{2/3}\theta^{-1/4} \le h_{1,2}^{2/3}\max(\theta^{-50/4}, nh_{1,2})^{1/50}.$$

In particular, this means that x_1 and x_2 are at most $n^{1/50}$ in absolute value, and $|x_1-x_2|\le (4\chi s_{1,2})^{2/3}(n4\chi s_{1,2})^{1/50}$. Note that this last deduction needs $4\chi s_{1,2}\ge \frac{\theta^{-50/4}}{n}$ for all fixed χ and all large enough n, which condition is implied by our stronger standing assumption that $2^{-\ell}\ge\frac{\theta^{-40}}{n}$.

We can now conclude that the event $\mathsf{UnifBdd}_n\big(\frac{b}{10}\theta^{-1}\big)$ implies that the quantities $\mathsf{Wgt}_n\big[(x_1,s_1)\to(x_2,v_-+\chi s_{1,2})\big]$ and $\mathsf{Wgt}_n\big[(x_2,v_-)\to(x_2,v_-+\chi s_{1,2})\big]$ are at most $\big(4\chi s_{1,2}\big)^{1/3}(\frac{b}{8}\theta^{-1})$. We prove only the *claim* that this bound holds for the first term, because a similar argument works for the second. The occurrence of $\mathsf{UnifBdd}_n\big(\frac{b}{10}\theta^{-1}\big)$ entails that the parabolically adjusted weight satisfies

$$\mathsf{Wgt}_n^{\cup}\big[(x_1,s_1)\to(x_2,v_-+\chi s_{1,2})\big] \le \frac{b}{10}\theta^{-1},$$

while the parabolic correction term $2^{-1/2}(x_1-x_2)^2(v_-+\chi s_{1,2}-s_1)^{-1}$ is bounded above by $O(\big(4\chi s_{1,2}\big)^{1/3}\theta^{-1/2})\le\big(4\chi s_{1,2}\big)^{1/3}\frac{b}{100}\theta^{-1}$ for all large enough n, since θ is assumed to satisfy $\theta^{-1/4}\ge C\log n$ throughout this section.

Since $b\kappa\le 8s_{1,2}^{2/3}\theta$, we find that the above bound is at most $\big(4\chi s_{1,2}\big)^{1/3}s_{1,2}^{2/3}\kappa^{-1}$, finishing the proof of the claim that we sought to show. □

Lemma 4.16. *There exists $\chi_0\in(0,1)$ such that, when $\chi\in(0,\chi_0)$, s_1,s_2 satisfy $0\le s_1\le s_2\le 1$, and $s_{1,2}\in[2^{-1-\ell},2^{-\ell}]$ and ϕ is a $\theta^{-1/4}$-regular n-zigzag from $(0,0)$ to $(0,1)$,*

$$\mathbb{P}\Big(\mathsf{HighSlenderWeight}\big(s_1,s_2,1-\chi;\phi\big)\Big)\ \le\ \exp\big\{-d_2\kappa^{-3/2}2^{-\ell}\big\}.$$

Proof. Note that

$$|\mathcal{C}|\le n\kappa^{3/2}\sum_{k=0}^{\lceil\chi s_{1,2}\kappa^{-3/2}\rceil}\binom{\lceil s_{1,2}\kappa^{-3/2}\rceil}{k}\le n\kappa^{3/2}\cdot s_{1,2}\kappa^{-3/2}\binom{\lceil s_{1,2}\kappa^{-3/2}\rceil}{\lceil\chi s_{1,2}\kappa^{-3/2}\rceil},$$

where the latter bound invokes $\chi\le 1/4$.

Now let $\mathsf{c}\in\mathcal{C}$. We now proceed to express our present circumstance in the notation of Proposition 4.5. Given c, let $r=n(v^+-v^-)$, where v^- and v^+ are the lowest and highest vertical coordinates assumed by elements in the plentiful r_0-segment collection given by c. Note that, when χ_0 is small enough, $\frac{1}{2}ns_{1,2}\le r\le ns_{1,2}$. We will now apply Proposition 4.5 with $\mathbf{r}=r$ and $\boldsymbol{\kappa}=\kappa\big(\frac{n}{r}\big)^{2/3}\in b^{-1}\theta\cdot[8,16]$. In a detail to ensure formal accuracy of the application, we apply a vertical translation that sends v^- to zero. Furthermore, given the collection of horizontal segments of length $b\kappa$ forming c, let c_* be the collection obtained by multiplying each of them by the factor $\big(\frac{n}{\mathbf{r}}\big)^{2/3}$, and their vertical heights by $\frac{n}{r}$. Now, by the scaling principle, we conclude that $\big(\frac{r}{n}\big)^{1/3}\mathsf{Wgt}_n\big[\mathsf{c}\text{-path}\big]$ is equal in law to $\mathsf{Wgt}_r\big[\mathsf{c}_*\text{-path}\big]$. Note that, by the assumed regularity of ϕ, the horizontal segments of c are confined in an interval of length ${s_{1,2}}^{2/3}(ns_{1,2})^{1/30}\le\big(rn^{-1}\big)^{2/3}r^{1/20}$, and hence the elements of c_* are contained in a horizontal interval of length $r^{1/20}$. Thus, by Proposition 4.5, we learn that

$$\mathbb{P}\Big(\mathsf{Wgt}_n\big[\mathsf{c}\text{-path}\big]\ge -s_{1,2}d_1\kappa^{-1}\Big)\le\exp\big\{-d_2\kappa^{-3/2}2^{-\ell}\big\}.$$

Hence, if $2^{-1}d_0\ge d_1$,

$$\mathbb{P}\Big(\mathsf{HighSlenderWeight}\big(s_1,s_2,1-\chi;\phi\big)\Big)\le n\kappa^{3/2}\cdot s_{1,2}\kappa^{-3/2}\binom{\lceil s_{1,2}\kappa^{-3/2}\rceil}{\lceil\chi s_{1,2}\kappa^{-3/2}\rceil}\exp\big\{-d_2\kappa^{-3/2}2^{-\ell}\big\}.$$

Since $\kappa \in 2^{-2\ell/3} b^{-1}\theta \cdot [8,16]$, $s_{1,2} \in [2^{-1-\ell}, 2^{-\ell}]$, $\theta^{-1/4} \geq C(\log n)$ and $b > 0$ is given, we may choose $\chi_0 \in (0,1)$ small enough that this right-hand side is at most the quantity $\exp\{-2^{-1} d_2 \kappa^{-3/2}\}$. Lemma 4.16 follows by relabelling $d_2 > 0$. □

Proof of Theorem 1.10. By Lemmas 4.15 and 4.16,

$$\mathbb{P}\Big(\mathsf{UnifBdd}\big(\frac{b}{10}\theta^{-1}\big) \cap \neg\, \mathsf{LowSlenderWeight}^*(\ell,\theta,1-\chi;\phi)\Big) \leq (n+1)^2 \exp\big\{-d_2(b^{-1}\theta)^{-3/2}\big\}$$
$$\leq \exp\big\{-\frac{d_2}{2}(b^{-1}\theta)^{-3/2}\big\}.$$

The factor $(n+1)^2$ arises from the union bound taken over values of s_1 and s_2 in Lemma 4.15; it is absorbed during the second inequality in view of $\theta^{-1/4} \geq C(\log n)$. Further, by (4.4),

$$\mathbb{P}\Big(\neg\mathsf{UnifBdd}\big(\frac{b}{10}\theta^{-1}\big)\Big) \leq \exp\big\{-C\theta^{-3/2}\big\}.$$

For a suitably high choice of $n_0 \in \mathbb{N}$, the theorem is obtained from Proposition 1.8 by assembling the preceding estimates and by adjusting the constant $d_2 > 0$. □

We are ready to prove Theorem 1.9. Alongside the result just proved, the main ingredients are Theorem 1.4, which asserts that the polymer ρ_n is typically regular with high probability; and the FKG inequality. Indeed, from the latter, we will learn that conditioning on ρ_n has a negative effect, so that the proof will be completed by invoking Theorem 1.10 and the observation that $\mathsf{LowSlenderExcursion}(\ell,\theta,1-\chi;\rho_n)$ is a decreasing event in the remaining environment.

Proof of Theorem 1.9. We start by recording some notation in order to state a stochastic domination lemma. A noise field will be viewed as a random function sending $\mathbb{R} \times n^{-1}\mathbb{Z}$ to $\mathbb{R}$. Let X and Y denote two such. For any subset A of $\mathbb{R} \times n^{-1}\mathbb{Z}$, Y stochastically dominates X on A if there exists a coupling of X and Y such that, whenever $(j,u,v) \in \mathbb{Z} \times \mathbb{R}^2$ satisfies $u < v$ and $\{j/n\} \times [u,v] \subset A$, the bound $Y(v,j/n) - Y(u,j/n) \geq X(v,j/n) - X(u,j/n)$ holds. An event E that is measurable with respect to the natural σ-algebra generated by the increments of the Brownian motion on A is called decreasing on A if $\mathbb{P}(Y \in E) \leq \mathbb{P}(X \in E)$ whenever Y stochastically dominates X on A.

For a given n-zigzag ϕ, let the *exterior* $\mathrm{Ext}(\phi)$ of ϕ denote $\big(\mathbb{R} \times n^{-1}\mathbb{Z}\big) \setminus \phi$. Recall from Sections 1.6 and 1.7 that our scaled noise environment is given by an ensemble of independent two-sided Brownian motions, thought of as a function $\mathbb{R} \times \frac{1}{n}\mathbb{Z} \to \mathbb{R}$.

Lemma 4.17. *Given a zigzag ϕ, and two independent noise environments Ω and $\tilde{\Omega}$, the restriction of $\widetilde{\Omega}$ to $\mathrm{Ext}(\rho_n)$ stochastically dominates Ω on this set.*

Proof. Consider the noise environment that is given by Ω on ρ_n and by $\widetilde{\Omega}$ on $\mathrm{Ext}(\rho_n)$. When this environment is conditioned on the event that there exists no n-zigzag from $(0,0)$ to $(0,1)$ whose weight determined by this environment exceeds that of ρ_n, the result is a distributional copy of Ω. The event in the conditioning is negative for Ω on $\mathrm{Ext}(\rho_n)$. The system Ω on $\mathrm{Ext}(\rho_n)$ is a countable collection of Brownian motions whose domains are either copies of the real line or semi-infinite real intervals; indeed, to each height in $y \in n^{-1}\mathbb{Z}$ are associated one or two intervals, formed by the sometimes vacuous removal from $\mathbb{R} \times \{y\}$ of this set's intersection with ρ_n. The FKG inequality for products of independent Brownian motions is implied by [Bar05, Theorems 3 and 4]. Applying it, we obtain the lemma. □

The next lemma says that the polymer is typically regular.

Lemma 4.18. *Given $C > 0$, there exists $c > 0$ such that, for all large n, and for θ with $C\log n < \theta^{-1/4} < Cn^{\frac{1}{10}}$, with probability at least $1 - \exp(-c\theta^{-1/2})$, the polymer ρ_n is $\theta^{-1/4}$-regular.*

Proof. By Theorem 1.4 and a union bound over all $h_1, h_2 \in \frac{1}{n}\mathbb{Z} \cap [0,1]$, we get

$$\begin{aligned}\mathbb{P}\big(\rho_n \text{ is } \theta^{-1/4}\text{-regular}\big) &\geq 1 - \exp\big\{ - c\theta^{-3/4}/\log n)\big\} \\ &\geq 1 - \exp\big\{ - c\theta^{-1/2}\big\},\end{aligned}$$

using the upper bound on θ. □

Note that θ in the hypothesis of Theorem 1.9 satisfies $C\log n < \theta^{-1/4} < n^{\frac{1}{160}}$ since $2^\ell \geq 1$. Theorem 1.9 now follows from Theorem 1.10; the just noted bound; Lemma 4.17; the event $\mathsf{LowSlenderExcursion}(\ell, \theta, 1-\chi; \rho_n)$ being decreasing on $\mathrm{Ext}(\rho_n)$; and that $\mathsf{LowSlenderExcursion}(\ell, \theta, 1-\chi; \rho_n)$ implies $\mathsf{LowSlenderExcursion}^*(\ell, \theta, 1-\chi; \rho_n)$.

We finish with a brief discussion regarding the point that Theorem 1.10 bounds the probability of $\mathsf{LowSlenderExcursion}^*(\ell, \theta, 1-\chi; \phi)$ for a fixed zigzag ϕ while Theorem 1.9 only bounds the probability of $\mathsf{LowSlenderExcursion}(\ell, \theta, 1-\chi; \rho_n)$. In short, this is because ϕ is deterministic and hence independent of the noise environment, while ρ_n is highly correlated with the latter. Namely, notice that the proof of Theorem 1.9 uses Theorem 1.10, along with an FKG inequality; the latter implies that the noise environment off ρ_n is stochastically smaller than a typical environment, this rendering $\mathsf{LowSlenderExcursion}(\ell, \theta, 1-\chi; \rho_n)$ more likely. However, the same cannot be said for $\mathsf{LowSlenderExcursion}^*(\ell, \theta, 1-\chi; \rho_n)$, since the environment on ρ_n is, in fact, stochastically larger than a typical one—indeed, it is easily seen that the path ρ_n itself obstructs the event $\mathsf{LowSlenderExcursion}^*(\ell, \theta, 1-\chi; \rho_n)$ from occurring.

□

5 There are few cliffs along the geodesic

Here we derive Theorem 1.11. In a first subsection, we reduce to a principal component, Proposition 5.1; and, in a second, we prove this proposition. Theorem 1.11 will find application in the investigation of Brownian LPP under dynamical perturbation in [GH20a]. This study is undertaken in scaled coordinates, and uses a scaled counterpart to Theorem 1.11. In the third and final subsection, the counterpart, Proposition 5.6, is presented and proved.

5.1 Proving Theorem 1.11, a main component admitted

Let $\gamma \subset [0,n]^2$ denote any staircase between $(0,0)$ and (n,n). We may associate to γ the index set $\mathcal{I}(\gamma)$ (that is specified before the theorem), just as we did to the geodesic staircase Γ_n.

For now, let α be any given value in $(1/2, 1)$ for which $\alpha m \in \mathbb{N}$; the lower bound $\alpha_0 > 1/2$ on this parameter's value will be set later in the proof of Theorem 1.11. Consider the class Θ of difference functions $\Psi : [\![0,m]\!] \to [\![0,n]\!]$ that are associated to staircases $\gamma \subset [0,n]^2$ with $(0,0), (n,n) \in \gamma$ and $\big|\mathcal{I}(\gamma)\big| \geq \alpha m$.

Let $\Gamma(\Psi) \subset [0,n]^2$ denote the staircase of maximum energy that contains $(0,0)$ and (n,n) and whose Z-difference function as specified by (1.12) is equal to Ψ.

Our approach to proving Theorem 1.11 is governed by the bound

$$\mathbb{P}\big(\big|\mathcal{I}\big| \geq \alpha m\big) \leq \sum_{\Psi \in \Theta} \mathbb{P}\Big(E\big(\Gamma(\Psi)\big) \geq E(\Gamma_n)\Big)\,.$$

This right-hand side is in fact equal to $\sum_{\Psi\in\Theta} \mathbb{P}\Big(E\big(\Gamma(\Psi)\big) = E(\Gamma_n)\Big)$.

The next two results form the backbone of the proof of Theorem 1.11.

Proposition 5.1. *There exist positive constants H and h such that, for A high enough and $\Psi \in \Theta$,*

$$\mathbb{P}\Big(E\big(\Gamma(\Psi)\big) \geq E(\Gamma_n)\Big) \leq H\exp\big\{ - hn\big\}\,. \tag{5.1}$$

The quantity Θ grows at a rate that is exponential in n/A when α is close to one.

Lemma 5.2.

$$\big|\Theta\big| \,\le\, 3^{\alpha m}\binom{m+1}{\alpha m}\binom{n+1}{(1-\alpha)m+1}. \tag{5.2}$$

We close out the proof of Theorem 1.11 before deriving these two inputs.

Proof of Theorem 1.11. By Proposition 5.1 and Lemma 5.2, we see that

$$\mathbb{P}\big(\big|\mathcal{I}\big| \ge \alpha m\big) \,\le\, 3^{\alpha m}\binom{m+1}{\alpha m}\binom{n+1}{(1-\alpha)m+1}\cdot H\exp\big\{-hn\big\}.$$

Since $m \in [n/A, n/A+1)$, the right-hand factor arising from Lemma 5.2 is at most

$$3^{\alpha(A^{-1}n+1)}\cdot(n/A+1)\cdot\exp\big\{(nA^{-1}+1)K(\alpha)\big\}\cdot(n+1)\exp\big\{(n+1)K\big((1-\alpha)A^{-1}\big)\big\},$$

where $K : (0,1) \to \mathbb{R}$ denotes the entropy rate $K(p) = p\log p + (1-p)\log(1-p)$. Our choice of $A \in \mathbb{N}$ made so that (5.1) holds as well as $3^{\alpha(A^{-1}n+1)} \le e^{hn/4}$, we specify $\alpha_0 \in (1/2,1)$ to be high enough that the last display with any $\alpha \in [\alpha_0, 1]$ is less than $e^{hn/2}$ when n is supposed to be sufficiently high. We obtain Theorem 1.11 by further relabelling the parameter $h > 0$ to be one-half of its present value. □

Proof of Lemma 5.2. For any staircase γ that offers a function Ψ belonging to Θ, let $J(\gamma)$ denote the set of the αm lowest elements of $\mathcal{I}(\gamma)$. Further set $J^c(\gamma) = [\![0,m]\!] \setminus J(\gamma)$. The element Ψ of Θ associated to γ may be surmised from three pieces of data:

- the set $J(\gamma)$;
- the values $\Psi(j)$ indexed by $j \in J(\gamma)$;
- and the remaining values, namely $\Psi(i)$ for $i \in J^c(\gamma)$.

The number of choices for $J(\gamma)$ is equal to $\binom{m+1}{\alpha m}$. For each index $j \in J(\gamma)$, $\Psi(j)$ is valued in $\{0,1,2\}$, so that there are at most $3^{\alpha m}$ choices for the second piece of data. The remaining values, in the third piece of data, are indexed by $i \in J^c(\gamma)$; to each such index i is associated the partial sum $p(i)$ of $\Psi(j)$ over $j \in J^c(\gamma)$ with $j \le i$. The index set $J^c(\gamma)$ has cardinality $m+1-\alpha m$ and may be identified with the integer interval $[\![0,(1-\alpha)m]\!]$ via an increasing map $I : [\![0,(1-\alpha)m]\!] \to J^c(\gamma)$. The third item data is specified by the function mapping $[\![0,(1-\alpha)m]\!]$ to $[\![0,n]\!]$ given by $i \to (p\circ I)(i)$. This function is increasing, so that it is determined by its values; thus, the number of such functions is at most $\binom{n+1}{(1-\alpha)m+1}$.

The right-hand side of (5.2) is a product of three factors. These factors have been verified to offer respective upper bounds on the cardinality of the set of choices for the second, first and third pieces of displayed data. Thus, the proof of Lemma 5.2 is complete. □

5.2 Energy near a given cliff-strewn route is unlikely to attain the maximum

To complete the proof of Theorem 1.11, it remains to give the next derivation.

Proof of Proposition 5.1. Let $\Psi \in \Theta$. Let P denote the set of points of the form $\big(\sum_{j=0}^{i}\Psi(j), iA\big)$ for $i \in [\![0,m]\!]$. Let $\Gamma_0(\Psi)$ denote the almost surely unique staircase between $(0,0)$ and (n,n) that has maximum energy among those that visit every element in P and that, on arrival at any such element, immediately jump upwards by one unit. The staircase $\Gamma_0(\Psi)$ offers a coarse-grained description of any staircase specifying Ψ, including the staircase $\Gamma(\Psi)$ among these of maximum energy.

The plan of attack for proving Proposition 5.1 has three steps:

1. We wish to argue that the energies of $\Gamma(\Psi)$ and its coarse-grained cousin $\Gamma_0(\Psi)$ are typically similar. Indeed, we will find positive constants H and h such that, for $r \geq 0$,
$$\mathbb{P}\Big(E\big(\Gamma(\Psi)\big) - E\big(\Gamma_0(\Psi)\big) \geq (5+r)nA^{-1/2}\Big) \leq H\exp\big\{-hnr^2\big\}. \tag{5.3}$$
2. We will then need to analyse $E\big(\Gamma_0(\Psi)\big)$. This is the maximum energy of a staircase from $(0,0)$ to (n,n) that visits every point $\big(\sum_{j=0}^{i}\Psi(j), iA\big)$ for $i \in [\![0,m]\!]$. We will argue that this maximum energy is unchanged in law if the vector of differences between consecutively visited points is reordered so that these vectors are presented in decreasing order of gradient. Because $\Psi \in \Theta$, the reordered collection of points-to-be-visited contains an element whose distance from the diagonal has order n.
3. Thus, $E\big(\Gamma_0(\Psi)\big)$ has the law of the maximum energy of a staircase from $(0,0)$ and (n,n) that visits a given point at a distance of order n from the diagonal. We will exploit this information to argue that there exist positive constants K and κ such that
$$\mathbb{P}\Big(E\big(\Gamma_0(\Psi)\big) \geq (2-\kappa)n\Big) \leq K\exp\big\{-\kappa n\big\}. \tag{5.4}$$
The sought bound (5.1) will then emerge directly from (5.3) and (5.4).

In three subsections, we accomplish these respective steps.

5.2.1 The coarse-grained energetic approximation: deriving (5.3)

To derive (5.3), we split the staircases $\Gamma(\Psi)$ and $\Gamma_0(\Psi)$ into pieces that traverse consecutive strips of height A. The staircase $\Gamma(\Psi)$ is divided into pieces by splittling at its points of entry to the levels $\{iA\}\times\mathbb{R}$ indexed by $i \in [\![0,m]\!]$. The coarse-grained counterpart $\Gamma_0(\Psi)$ is partitioned by splitting at the points $\big(\sum_{j=0}^{i}\Psi(j), iA\big)$ indexed by the same set. The elements of the two partitions may be paired according to which strip of height A they cross. If the difference in energy between the fragment of $\Gamma(\Psi)$ crossing the i^{th} strip and its counterpart for $\Gamma_0(\Psi)$ is denoted by E_i, then $E\big(\Gamma(\Psi)\big) - E\big(\Gamma_0(\Psi)\big) = \sum_{i=1}^{\lceil n/A\rceil} E_i$. The pair of fragments involved in specifying E_i each begin with a unit vertical movement, from level $(i-1)A$ to level $(i-1)A+1$. With $h_1 = (i-1)A+1$ and $h_2 = iA$, note thus that, for $i \in [\![1, \lfloor n/A\rfloor]\!]$, E_i takes the form $M\big[(u,h_1)\to(v,h_2)\big] - M\big[(x,h_1)\to(y,h_2)\big]$ for a choice of (u,v,x,y) that satisfy the hypotheses of the next result.

Lemma 5.3. *Let $u,v \in \mathbb{N}$ and $x,y \in \mathbb{R}$ satisfy $u \leq v$, $x \leq y$, $x \in [u,u+1]$ and $y \in [v,v+1]$. Let $h_1, h_1 \in \mathbb{N}$ satisfy $h_1 \leq h_2$. Writing $h_{1,2} = h_2 - h_1$, we have that*
$$\mathbb{P}\left(\sup_{\substack{x\in[u,u+1]\\ y\in[v,v+1]}}\Big|M\big[(u,h_1)\to(v,h_2)\big] - M\big[(x,h_1)\to(y,h_2)\big]\Big| \geq 4\big(h_{1,2}+1\big)^{1/2} + \big(h_{1,2}+1\big)^{-1/6}r\right) \tag{5.5}$$
is at most $C\exp\Big\{-c\frac{r^2}{(h_{1,2}+1)^{1/3}}\Big\}$.

The tail of the remainder term indexed by $i = \lceil n/A\rceil$ is also treated by Lemma 5.3 with $h_{1,2}$ assuming a value in $[\![0, A-1]\!]$. We derive (5.3) and then prove this lemma.

Proof of (5.3)**.** We rely on the bound
$$\begin{aligned} E\big(\Gamma(\Psi)\big) - E\big(\Gamma_0(\Psi)\big) &\leq \sum_{i=1}^{\lfloor n/A\rfloor} E_i + R \leq (R - 4A^{1/2}) + 4A^{1/2} + \sum_{i=1}^{\lfloor n/A\rfloor}\big[4A^{1/2} + (E_i - 4A^{1/2})_+\big] \\ &\leq (4n+A)A^{-1/2} + (R-4A^{1/2})_+ \sum_{i=1}^{\lfloor n/A\rfloor}(E_i - 4A^{1/2})_+, \end{aligned}$$

where R is the remainder term (and $a = \max(a, 0)$). By Lemma 5.3, the random variables $X_i = (E_i - 4A^{1/2})_+$ are independent and satisfy the uniform tail bound $\mathbb{P}(X_i \geq r) \leq Ce^{-cr^2}$. This tail bound is also satisfied by $\hat{R} = (R - 4A^{1/2})_+$. By Proposition 4.8(2), we thus see that, for $t \geq c\lceil n/A \rceil$,

$$\mathbb{P}\Big(\sum_{i=1}^{\lceil n/A \rceil} X_i + \hat{R} \geq \sum_{i=1}^{\lceil n/A \rceil} \mathbb{E}(X_i) + t \Big) \leq e^{-Ct^2 An^{-1}}.$$

Observing that $\mathbb{E}(X_i) = O(1)$, we see that, for a large enough A, and for all large n,

$$\mathbb{P}\Big(E\big(\Gamma(\Psi)\big) - E\big(\Gamma_0(\Psi)\big) \geq (5 + r)nA^{-1/2} \Big) \leq e^{-hnr^2},$$

for some constant $h > 0$. □

Proof of Lemma 5.3. The argument is simple and relies on using superadditivity to bound the expression on the left-hand side of (5.5) by a linear combination of passage times between deterministic points, which is then easy to bound using the well-known correspondence between passage time in Brownian LPP between fixed points and the largest eigenvalue of a matrix drawn from a suitable Gaussian unitary ensemble.

We now carry out the first part of the above strategy. Suppose first that $v \geq u + 1$. Note that

$$\begin{aligned} & M\big[(x, h_1) \to (u + 1, h_1)\big] + M\big[(u + 1, h_1) \to (v, h_2)\big] + M\big[(v, h_2) \to (y, h_2)\big] \\ \leq\ & M\big[(x, h_1) \to (y, h_2)\big] \\ \leq\ & M\big[(x, h_1) \to (u + 1, h_2)\big] + M\big[(u + 1, h_1) \to (v, h_2)\big] + M\big[(v, h_1) \to (y, h_2)\big]. \end{aligned}$$

These bounds hold also when the replacements $x \to u$ and $y \to v$ are made. Thus,

$$\begin{aligned} & \Big| M\big[(x, h_1) \to (y, h_2)\big] - M\big[(u, h_1) \to (v, h_2)\big] \Big| \\ \leq\ & M\big[(x, h_1) \to (u + 1, h_2)\big] - M\big[(x, h_1) \to (u + 1, h_1)\big] \\ & \quad + M\big[(v, h_1) \to (y, h_2)\big] - M\big[(v, h_2) \to (y, h_2)\big] \\ \leq\ & M\big[(u, h_1) \to (u + 1, h_2)\big] - M\big[(u, h_1) \to (u + 1, h_1)\big] \\ & \quad + M\big[(v, h_1) \to (v + 1, h_2)\big] - M\big[(v, h_2) \to (v + 1, h_2)\big]. \end{aligned} \tag{5.6}$$

Here, the latter inequality depended on

$$M\big[(u, h_1) \to (u + 1, h_2)\big] \geq M\big[(u, h_1) \to (x, h_1)\big] + M\big[(x, h_1) \to (u + 1, h_2)\big]$$

and

$$M\big[(u, h_1) \to (u + 1, h_1)\big] = M\big[(u, h_1) \to (x, h_1)\big] + M\big[(x, h_1) \to (u + 1, h_1)\big]$$

as well as

$$M\big[(v, h_1) \to (v + 1, h_2)\big] \geq M\big[(v, h_1) \to (y, h_2)\big] + M\big[(y, h_2) \to (v + 1, h_2)\big]$$

and

$$M\big[(v, h_2) \to (v + 1, h_2)\big] = M\big[(v, h_2) \to (y, h_2)\big] + M\big[(y, h_2) \to (v + 1, h_2)\big].$$

Note that the right-hand expression in (5.6) does not depend on x or y, so that the upper bound on the left-hand term is valid when the supremum over $x \in [u, u + 1]$ and $y \in [v, v + 1]$ is taken. For $n \in \mathbb{N}$ and $s \geq 0$, let $G_n(s)$ denote the uppermost eigenvalue of an $n \times n$ random matrix drawn from the Gaussian unitary ensemble with entry variance

s. Viewing the right-hand expression in (5.6) as a sum of four terms, we see that the first and third are independent and have the law of $G_{h_{12}+1}(1)$, and the second and fourth are independent and have the law of $G_1(1)$. The latter two clearly have Gaussian tails. We also use the following tail estimate of $G_{h_{12}+1}(1)$, which is the content of [Aub05, (5),(6)]. For all $r \geq 0$,

$$\mathbb{P}\Big(G_{h_{12}+1}(1) - 2(h_{12}+1)^{1/2} \geq \frac{r}{(h_{12}+1)^{1/6}}\Big) \leq C \exp\Big\{ - c \max\Big(r^{3/2}, \frac{r^2}{(h_{1,2}+1)^{1/3}}\Big)\Big\}. \tag{5.7}$$

The lemma in the case that $v \geq u+1$ now follows by a simple union bound over the possibilities that one of the four quantities is bigger than $\frac{r}{4(h_{12}+1)^{1/6}}$.

Suppose then that $v < u+1$. Since $u, v \in \mathbb{N}$ and $v \geq u$, we have $v = u$. Thus, $M\big[(u,h_1) \to (v,h_2)\big] = 0$. Note that

$$\begin{aligned} & M\big[(u,h_1) \to (u+1,h_2)\big] \\ \geq \; & M\big[(u,h_1) \to (x,h_1)\big] + M\big[(x,h_1) \to (y,h_2)\big] + M\big[(y,h_2) \to (u+1,h_2)\big] \end{aligned} \tag{5.8}$$

and that

$$\begin{aligned} & M\big[(u,h_1) \to (u+1,h_2)\big] \\ \leq \; & M\big[(u,h_1) \to (x,h_2)\big] + M\big[(x,h_1) \to (y,h_2)\big] + M\big[(y,h_1) \to (u+1,h_2)\big]. \end{aligned} \tag{5.9}$$

Hence, by rearranging, we obtain

$$\begin{aligned} & M\big[(x,h_1) \to (y,h_2)\big] \\ \geq \; & M\big[(u,h_1) \to (u+1,h_2)\big] - M\big[(u,h_1) \to (x,h_2)\big] - M\big[(y,h_1) \to (u+1,h_2)\big]. \end{aligned}$$

Note further that

$$M\big[(u,h_1) \to (x,h_2)\big] + M\big[(x,h_2) \to (u+1,h_2)\big] \leq M\big[(u,h_1) \to (u+1,h_2)\big],$$

and that

$$M\big[(u,h_1) \to (y,h_1)\big] + M\big[(y,h_1) \to (u+1,h_2)\big] \leq M\big[(u,h_1) \to (u+1,h_2)\big].$$

Applying the two preceding bounds, we find that

$$\begin{aligned} & M\big[(x,h_1) \to (y,h_2)\big] \\ \geq \; & -M\big[(u,h_1) \to (u+1,h_2)\big] + M\big[(x,h_2) \to (u+1,h_2)\big] + M\big[(u,h_1) \to (y,h_1)\big]. \end{aligned}$$

From this bound, and (5.8), we see that

$$\begin{aligned} & -M\big[(u,h_1) \to (u+1,h_2)\big] + M\big[(x,h_2) \to (u+1,h_2)\big] + M\big[(u,h_1) \to (y,h_1)\big] \\ \leq \; & M\big[(x,h_1) \to (y,h_2)\big] \\ \leq \; & M\big[(u,h_1) \to (u+1,h_2)\big] - M\big[(u,h_1) \to (x,h_1)\big] - M\big[(y,h_2) \to (u+1,h_2)\big]. \end{aligned}$$

If the first line takes the form $A_1 + A_2 + A_3$, then A_1 has the law of $G_{h_{1,2}+1}(1)$, while the suprema of $|A_2|$ and $|A_3|$ over $x \in [u, u+1]$ and $y \in [v, v+1]$ are stochastically dominated by the supremum of standard Brownian motion on the interval $[0,1]$. This statement is equally true of the third line. Since $M\big[(u,h_1) \to (v,h_2)\big] = 0$, Lemma 5.3 when $v = u$ follows from (5.7) and an upper tail bound on the supremum of Brownian motion obtained from the reflection principle and a standard bound on the Gaussian tail. This completes the proof of Lemma 5.3. □

5.2.2 Reordering the trajectory of the coarse-grained cousin

We now analyse $E\big(\Gamma_0(\Psi)\big)$. We first argue that the path $\Gamma_0(\Psi)$ may be reordered in order to visit a point far away from the diagonal without a change to the law of its energy. We will present notation and a general form Lemma 5.4 for such a rearrangement result before discussing the energy $E\big(\Gamma_0(\Psi)\big)$.

Let P denote a collection of points in $[\![0,n]\!]^2$ with distinct y-coordinates, which lie in the range of a staircase that begins at $(0,0)$ and ends at (n,n). We write $M_n[P]$ for the supremum of the energies of staircases that begin at $(0,0)$; end at (n,n); that visit every element of P; and that, on visiting any element (m_1,m_2) of P for which $m_2<n$, immediately jump upwards by one step, to (m_1,m_2+1). When P is a singleton set whose element is (m_1,m_2), we abuse notation, and denote $M_n[P]$ by $M_n[m_1,m_2]$.

Suppose that $(u,0)\in P$ for some $u\in[\![0,n]\!]$. Thus, the staircases involved in specifying $M_n[P]$ begin at $(0,0)$; remain on the x-axis until $(u,0)$; and end at (n,n).

To the collection P, we may add the point (n,n). The resulting set of points may be ordered so that each successive element lies strictly upwards, and to the right, of its predecessor. To each consecutive pair in this sequence, we may associate the rectangle whose lower-left corner is the former element in the pair and whose upper-right corner is the latter. Any staircase from $(u,0)$ to (n,n) that visits every element of P crosses all of these rectangles, passing out of the upper-right corner of one into the lower-left corner of the next.

The rectangles may be placed in increasing order of width, and translated so that the lower-left corner of the lowest rectangle equals $(0,0)$, and the upper-right corner of one rectangle is the lower-left corner of the next one. Note that the upper-right corner of the last rectangle is $(n-u,n)$.

Let $P^{\rightarrow}$ denote the collection of upper-right corners of the rectangles when so placed.

Lemma 5.4. *Let $u\in[\![0,n]\!]$. Let P denote a collection of points in $[\![0,n]\!]^2$ which satisfies the condition in the second paragraph of this section and which contains $(u,0)$. Then $M_n[P]$ and $M_n[P^{\rightarrow}]$ are equal in law.*

Proof. The quantity $M_n[P]$ equals A_1+A_2, where $A_1=B(0,u)-B(0,0)$ is the energy accrued along $(0,0)\rightarrow(u,0)$ and where A_2 is the sum over the rectangles associated to P of the maximum energy available in a staircase that crosses the rectangle from its lower-left to its upper-right corner. The quantity $M_n[P^{\rightarrow}]$ equals A_1+B_2, where B_2 is the sum, counterpart to A_2, over rectangles associated to $P^{\rightarrow}$. The two collections of rectangles are disjoint, except for endpoint intersections, and one is a rearrangement of the other given by translations applied to the elements. So A_2 and B_2 are equal in law, conditionally on the value of A_1. This proves the lemma. □

We now employ Lemma 5.4 to find an upper bound on the energy $E\big(\Gamma_0(\Psi)\big)$ of the coarse-grained staircase. In the notation of Lemma 5.4, this energy takes the form

$$E\big(\Gamma_0(\Psi)\big)=M_n[P]\,, \tag{5.10}$$

where $P=\big\{\big(\sum_{j=0}^{i}\Psi(j),iA\big):i\in[\![0,m]\!]\big\}\cup\big\{(n,n)\big\}$. By this lemma, $M_n[P]$ and $M_n[P^{\rightarrow}]$ are equal in law. Note that if $\alpha\geq 1/2$ then $P^{\rightarrow}$ contains a point of the form $\big(\kappa m/2,Am/2\big)$ where $\kappa\leq 2$ is determined by the given element $\Psi\in\Theta$ that we are considering. Thus,

$$M_n[P^{\rightarrow}]\leq M_n\big[\big(\kappa m/2,Am/2\big)\big]. \tag{5.11}$$

5.2.3 The energetic penalty for highly off-diagonal travel: deriving (5.3)

In this third step, we present a tool indicating how $M_n\big[(\kappa\alpha m,A\alpha m)\big]$ typically falls far below the typical energy maximum $2n$ for the route $(0,0)\rightarrow(n,n)$. We will then promptly

be able to obtain (5.4). Finally, from (5.3) and (5.4), we will obtain (5.1), the derivation of which is the final step in proving Theorem 1.11.

Lemma 5.5. *Let $\mu \in (0,1)$. Let $m_1, m_2 \in [\![0,n]\!]$ satisfy $m_2 \in \big[\mu n, (1-\mu)n\big]$ and $\big|m_2 - m_1\big| \geq \mu n$. Then there exist positive μ-dependent K and κ such that*

$$\mathbb{P}\big(M_n[m_1,m_2] \geq (2-\kappa)n\big) \leq Ke^{-\kappa n}\,.$$

Proof. Note that $M_n[m_1,m_2] = M_n\big[(0,0) \to (m_1,m_2)\big] + M_n\big[(m_1,m_2) \to (n,n)\big]$ is a sum of two independent terms having the respective distributions $m_1^{1/2}G_{m_2+1}(1)$ and $(n-m_1)^{1/2}G_{n-m_2+1}(1)$. In an expression for this sum arising by writing the two G terms as a sum of a leading order term and a random fluctuation, the deterministic part is

$$2\big(m_1^{1/2}(m_2+1)^{1/2} + (n-m_1)^{1/2}(n-m_2+1)^{1/2}\big)\,.$$

This term falls short of $2n$ by a quantity that grows linearly in n under the hypotheses of the lemma. The fluctuation term in the sum equals $m_1^{1/2}(m_2+1)^{-1/6}R_1 + (n-m_1)^{1/2}(n-m_2+1)^{-1/6}R_2$ where the R-terms are independent random variables that satisfy $\mathbb{P}\big(R \geq r\big) \leq Ce^{-cr^{3/2}}$ for $r \geq 0$. By our hypothesis on m_2, the fluctuation term is at most a constant multiple of $n^{1/3}(R_1+R_2)$. Thus the fluctuation term exceeds one-half of the shortfall of the leading term with probability at most $Ke^{-\kappa n}$ for suitable positive K and κ. Decreasing the value of $\kappa > 0$ completes the proof of Lemma 5.5. □

From Lemma 5.5, we learn that for κ, A as in (5.11),

$$\mathbb{P}\Big(M_n\big[(\kappa m/2, Am/2)\big] \geq (2-\kappa)n\Big) \leq K\exp\big\{-\kappa n\big\}$$

for suitable positive K and κ not depending on A. Thus, we confirm (5.4) via (5.10).

From (5.3) and (5.4), we find that

$$\mathbb{P}\Big(E\big(\Gamma(\Psi)\big) \geq \big((2-\kappa) + (5+r)A^{-1/2}\big)n\Big) \leq H\exp\big\{-hnr^2\big\} + K\exp\big\{-\kappa n\big\}\,.$$

Setting $r=1$ and choosing A high enough that $6A^{-1/2} \leq \kappa/2$, we learn that

$$\mathbb{P}\Big(E\big(\Gamma(\Psi)\big) \geq (2-\kappa/2)n\Big) \leq H\exp\big\{-hn\big\}\,,$$

where we have relabelled the positive constants H and h.

Using the bound Lemma 3.1(2) on the lower tail of the uppermost GUE eigenvalue, we have

$$\mathbb{P}\Big(E(\Gamma_n) \leq 2n - xn^{1/3}\Big) \leq C\exp\big\{-cx^{3/2}\big\}$$

for $x \geq 0$. Setting x equal to $(c \wedge \kappa/4)n^{2/3}$, we confirm from the two preceding displays that (5.1) holds, after suitable adjustment to the positive values of H and h. This bound derived, the proof of Proposition 5.1 is complete, and, with it, the derivation of the elements needed for Theorem 1.11. □

5.3 Few cliffs in scaled coordinates

In [GH20a], use is made of a scaled counterpart to Theorem 1.11. We finish by presenting the notation needed to express this result; by stating it as Proposition 5.6; and by deriving it from Theorem 1.11.

We start by recalling some notation: ρ_n denotes the polymer $\rho_n\big[(0,0) \to (0,1)\big]$. For $i \in [\![0,n]\!]$, $\rho_n(i/n)$ equals the supremum of the set $\big\{x \in \mathbb{R} : (x, i/n) \in \rho_n\big\}$. The sequence $\big\{\rho_n(i/n) : i \in [\![0,n]\!]\big\}$ records the horizontal coordinates of departures of the polymer ρ_n from the consecutive horizontal intervals that it traverses. Indeed, the

projections to $\mathbb{R}$ of the horizontal intervals of ρ_n take the form $[0, \rho_n(0)]$ and $\big[\rho_n\big((i-1)/n\big) - 2^{-1}n^{-2/3}, \rho_n(i/n)\big]$ for $i \in [\![1, n]\!]$. Thus, by writing $\omega_0 = \rho_n(0)$ and $\omega_i = \rho_n(i/n) - \rho_n\big((i-1)/n\big) + 2^{-1}n^{-2/3}$ for $i \in [\![1, n]\!]$, the lengths of the consecutive horizontal intervals of ρ_n are recorded in the sequence $\big\{\omega_i : i \in [\![0, n]\!]\big\}$. The unscaled preimage $R_n^{-1}(\rho_n)$ of ρ_n has endpoints with horizontal coordinates zero and n, so the form (1.3) of the scaling map $R_n : \mathbb{R}^2 \to \mathbb{R}^2$ implies that $\sum_{i=0}^{n} \omega_i = 2^{-1}n^{1/3}$.

Let $\beta_1 > 0$ and $\beta_2 \in (0, 1)$. The polymer ρ_n is said to advance horizontally with (β_1, β_2)-steadiness if the cardinality of the set of $i \in [\![0, n]\!]$ for which $\omega_i \geq \beta_1 n^{-2/3}$ is at least $\beta_2 n$.

Proposition 5.6. *There exist $\beta_1 > 0$, $\beta_2 \in (0, 1)$, $h > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, the probability that ρ_n fails to advance horizontally with (β_1, β_2)-steadiness is at most e^{-hn}.*

Proof. For $i \in [\![0, n]\!]$, let W_i denote the length of the horizontal interval at which Γ_n intersects $\mathbb{R} \times \{i\}$. The geodesic Γ_n maps to the polymer ρ_n under the scaling map R_n from Subsection 1.7.1. Recall that ω_i denotes the length of the horizontal interval at which ρ_n intersects $\mathbb{R} \times \{i/n\}$. The form of the scaling map thus dictates that $\omega_i = 2^{-1}n^{-2/3}W_i$.

Suppose that, for some $i \in [\![0, m-1]\!]$, $\Psi(i) \geq 2$. The sum $\sum_{j=iA}^{(i+1)A-1} W_j$ is readily seen to be at least two. Thus, there is at least one $j \in [\![iA, (i+1)A - 1]\!]$ for which $W_j \geq 2A^{-1}$.

If $\big|\mathcal{I}(\Gamma_n)\big| \leq \alpha m$, then the cardinality of the set of $i \in [\![0, mA]\!]$ for which $\omega_i \geq 2A^{-1}n^{-2/3}$ is thus seen to be at least $(1-\alpha)m$. In the language of Theorem 1.11, the event that $\big|\mathcal{I}(\Gamma_n)\big| \leq \alpha m$ entails that ρ_n advances horizontally with (β_1, β_2)-steadiness, for $\beta_1 = 2A^{-1}$, $\beta_2 = \frac{(1-\alpha)}{A}$, and for n at least a level n_0 determined by β_2.

Thus Theorem 1.11 implies Proposition 5.6. □

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