



Quantum hooks and mirror symmetry for flag varieties

L. Chen¹ · E. Kalashnikov²

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Abstract

Given a flag variety $\mathrm{Fl}(n; r_1, \dots, r_\rho)$, there is natural ring morphism from the symmetric polynomial ring in r_1 variables to the quantum cohomology of the flag variety. In this paper, we show that for a large class of partitions λ , the image of s_λ under the ring homomorphism is a Schubert class which is described by partitioning λ into a quantum hook (or q -hook) and a tuple of smaller partitions. We use this result to show that the Plücker coordinate mirror of the flag variety describes quantum cohomology relations. This gives new insight into the structure of this superpotential, and the relation between superpotentials of flag varieties and those of Grassmannians (where the superpotential was introduced by Marsh–Rietsch).

1 Introduction

The extension of mirror symmetry for Fano varieties beyond the toric context, where it is well-understood due to foundational work by Hori and Vafa [12], Givental [10], Lian et al. [18] and others, is an area of active research. Grassmannians and flag varieties are central examples here, as Fano GIT quotients with a rich geometric and combinatorial structure.

One of the oldest proposals for a mirror, or superpotential, for the Grassmannian $\mathrm{Gr}(n, r)$ was given by Eguchi et al. [7]. This was later generalized to type A flag varieties by Batyrev et al. [1] (for simplicity, we refer to these mirrors as EHX mirrors). These proposals are motivated by taking toric degenerations of Grassmannians and flag varieties, and then applying toric methods to the singular fiber. However, the toric degeneration approach has not been successful in proving required properties of these mirrors—partial verification was completed for Grassmannian and flag varieties by Rietsch in [21] using the Lie theoretic superpotential, and a full verification for Grassmannians by Marsh and Rietsch [19] using the Plücker coordinate mirror.

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✉ E. Kalashnikov
e2kalash@uwaterloo.ca

L. Chen
lchen@swarthmore.edu

¹ Department of Mathematics and Statistics, Swarthmore College, Swarthmore, PA 19081, USA

² Department of Pure Mathematics, University of Waterloo, Waterloo N2L 3G1, Canada

The Plücker coordinate mirror for the Grassmannian is the most promising approach to mirror symmetry beyond the toric context. This remarkable construction connects the earlier, Lie theoretic proposals of the Grassmannian with the conjectures of the Fanosearch program and the toric degeneration approach. Extending the construction beyond the Grassmannian is thus an important problem. In [13], the second author introduces a conjectural Plücker coordinate mirror for type A flag varieties (see [23, 24] for recent progress on the subject in other types). As a first test of its validity, the second author proves in [13] that the Plücker coordinate mirror is compatible with the EHX mirror. More is required, however: a superpotential or mirror should compute quantum information about the variety—through determining both quantum relations as well as certain genus 0 Gromov–Witten invariants.

In this paper, we prove a theorem in this direction. We show that partial derivatives of the Plücker coordinate mirror of a type A flag variety give quantum cohomology relations. To state the result carefully, we need some more background.

The Plücker coordinate mirror of the Grassmannian is a rational function on the Grassmannian. As for toric varieties, there is a map from the Cox ring of the Grassmannian (the ring generated by Plücker coordinates) to the cohomology ring of the Grassmannian. Plücker coordinates of the Grassmannian of quotients $\mathrm{Gr}(n, r)$ are indexed by the same set as Schubert classes of the Grassmannian—i.e. by partitions fitting into an $r \times (n - r)$ box—and this map takes the Plücker coordinate p_λ to the Schubert class s_λ . Under this map, partial derivatives of the Plücker coordinate mirror give quantum cohomology relations [19].

For $n =: r_0$ and $\mathbf{r} = (r_1 > \cdots > r_\rho > r_{\rho+1} := 0)$, let $\mathrm{Fl}(n; \mathbf{r}) := \mathrm{Fl}(n; r_1, \dots, r_\rho)$ be the partial flag variety of successive quotients of \mathbb{C}^n of dimension r_i . The Plücker coordinate mirror W_P of $\mathrm{Fl}(n; \mathbf{r})$ was proposed in [13]. The mirror W_P is a rational function on a product of Grassmannians $Y = \prod_{i=1}^\rho \mathrm{Gr}(r_{i-1}, r_i)$, with the convention $r_0 := n$. See Definition 3.10 for the precise construction of W_P . More precisely, there are sets of algebraic independent Plücker coordinates on Y , called cluster charts, such that when W_P is expanded in a particular cluster chart C , the resulting function $W_{P,C}$ is a Laurent polynomial. See Sect. 3.1 for more details on cluster charts. We index Plücker coordinates on Y by p_λ^i , where $i = 1, \dots, \rho$ and λ is a partition that fits into an $r_i \times (r_{i-1} - r_i)$ box.

Schubert classes of the Grassmannian are indexed by partitions, and Schubert classes $s_{\vec{\lambda}}$ in a flag variety are indexed by tuples $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of partitions. To interpret partial derivatives of the Plücker coordinate mirror requires a map from the Cox ring of Y to the cohomology of the flag variety. This is not the natural map given by

$$p_\lambda^i \mapsto s_{\vec{\mu}},$$

where $\vec{\mu}_j$ is λ if $i = j$ and \emptyset otherwise. Instead, we require the Schubert map F (see definition 4.2). Our main result is then the following.

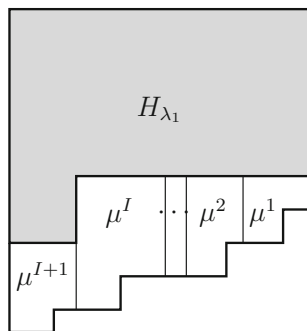
Theorem A *Let W_P be the Plücker coordinate mirror of a flag variety, and $W_{P,C}$ the expression of W_P in any choice of cluster charts. Then*

$$F\left(\frac{\partial}{\partial p_\lambda^i} W_{P,C}\right) = 0$$

in quantum cohomology, for any $i = 1, \dots, \rho$ and p_λ^i in the cluster chart.

This result represents a significant step towards a full verification of the Plücker coordinate mirror of flag varieties. If similar structure holds beyond the type A case, this result may also be important in extending candidate mirrors from cominuscule varieties to any

Fig. 1 A partition λ containing H_{λ_1} , the skew shape λ/H_{λ_1} , and the associated tuple of partitions $\vec{\mu} = (\mu^1, \dots, \mu^{I+1}, \emptyset, \dots, \emptyset)$



homogeneous space. It elucidates the increased complexity from the Grassmannian case. It also demonstrates a previously unobserved structure relating the mirrors of Grassmannians and flag varieties: although not at all obvious from the description of the Schubert map, we show that it precisely interpolates between the Plücker coordinate mirror of the flag variety $\text{Fl}(n; \mathbf{r})$ and containing Grassmannians $\text{Gr}(N, r_1)$, $N \gg 0$. It is this property of the Schubert map which is key to the proof of Theorem A, as it essentially allows us to reduce to the Grassmannian case. This interpolation result is a corollary of Theorem B below, a purely quantum cohomology statement.

Following the approach of [5, 8], we use a “quantization” approach for the quantum cohomology ring of the flag variety. This and other descriptions will be reviewed in Sects. 2 and 3. There is a natural ring homomorphism from the ring Λ_{r_1} of symmetric polynomials in r_1 variables to $\text{QH}^* \text{Fl}(n; \mathbf{r})$ defined sending elementary symmetric polynomials to certain quantum elementary polynomials. We write $s_\lambda^1 \in \text{QH}^* \text{Fl}(n; \mathbf{r})$ for the image of a Schur polynomial s_λ (see (7) for more details). The first part of Theorem B states that for certain partitions λ , the image is a Schubert class (up to multiplication by quantum parameters), and the second part of Theorem B states that for another class of partitions, the image is zero.

For $0 < b \leq n$, let $0 \leq I \leq \rho$ be such that $n - r_I < b \leq n - r_{I+1}$. In Sect. 5, we define the *quantum hook* or *q-hook* of width b to be the partition $H_b := (b^{b-n+r_1}, (b-n+r_1)^{n-r_{I+1}-b})$, and set $R_b := (b^{b-n+r_1})$ to be the maximal width rectangle contained in H_b , with $H_b = R_b = \emptyset$ if $b < n - r_1$. Set

$$q^{H_b} := q_1^{r_1-r_2} \dots (q_1 \dots q_{I-1})^{r_{I-1}-r_I} (q_1 \dots q_I)^{b-(n-r_I)}.$$

For a partition λ that contains the q -hook of width equal to the width of λ , we associate a tuple of partitions $\vec{\mu} = (\mu^1, \dots, \mu^{I+1}, \emptyset, \dots, \emptyset)$ by subdividing the skew shape λ/H_λ as in Fig. 1, where $\mu_i \in P(r_{i-1}, r_i)$ is of width $r_{i-1} - r_i$. (See Definition 5.6 for more details.)

Theorem B *Let $\lambda \subseteq r_1 \times n$ be a partition, and let I be such that $n - r_I < \lambda_1 \leq n - r_{I+1}$.*

(a) *If $H_{\lambda_1} \subseteq \lambda$, then*

$$s_\lambda^1 = q^{H_{\lambda_1}} \sigma_{\vec{\mu}} \text{ in } \text{QH}^* \text{Fl}(n; \mathbf{r}),$$

where $\vec{\mu} = (\mu^1, \dots, \mu^{I+1}, \emptyset, \dots, \emptyset)$ is the tuple of partitions associated to λ above.

(b) *If λ contains R_{λ_1} , but $H_{\lambda_1} \not\subseteq \lambda$, then*

$$s_\lambda^1 = 0 \text{ in } \text{QH}^* \text{Fl}(n; \mathbf{r}).$$

In particular, $s_{H_{\lambda_1}}^1 = q^{H_{\lambda_1}}$ since $H_{\lambda_1}/H_{\lambda_1} = \emptyset$, so $\mu^j = \emptyset$ for all j and $\sigma_{(\emptyset, \dots, \emptyset)} = 1$.

In the case of the Grassmannian $\text{Gr}(n, r)$, the rim-hook removal rule of Bertram, Ciocan-Fontanine, and Fulton [2, Main Lemma] recovers Theorem B: for a partition $\lambda \subseteq r \times n$ of width $\lambda_1 = n - r + a$, the quantity in part (a) of Theorem B is obtained after removing a n -hooks from λ , and the vanishing in part (b) of Theorem B corresponds to λ not containing a consecutive rim-hooks.

In Sect. 2, we review the necessary background on quantum cohomology of Grassmannians and flag varieties, and in Sect. 3, we discuss the EHX and Plücker coordinate mirror of the Grassmannian. In Sect. 4, we describe the Schubert map and prove Theorem A, and in Sect. 5, we study q -hooks and prove Theorem B.

2 Quantum cohomology of flag varieties

2.1 Permutations and Schubert classes

Fix an n -dimensional vector space V and a tuple of integers $\mathbf{r} = (n > r_1 > \dots > r_\rho > 0)$ and let $\text{Fl}(n; \mathbf{r}) = \text{Fl}(n; r_1, \dots, r_\rho)$ denote the partial flag variety parametrizing successive quotient flags of V of dimensions r_i . It comes equipped with a tautological sequence of quotient bundles $V_{\text{Fl}(n; \mathbf{r})} \rightarrow \mathcal{Q}_1 \rightarrow \dots \rightarrow \mathcal{Q}_\rho$ of ranks r_1, \dots, r_ρ .

The cohomology ring $H^* \text{Fl}(n; \mathbf{r})$ has a basis of Schubert classes commonly indexed by

$$S(n; \mathbf{r}) := \{w \in S_n : w(i) < w(i+1) \text{ if } i \notin \mathbf{r}\},$$

the set of permutations in S_n whose descent set is contained in $\{r_1, \dots, r_\rho\}$. If $S_{n, \mathbf{r}}$ is the parabolic subgroup of S_n generated by simple transpositions $(i, i+1)$ for $i \notin \mathbf{r}$, then $S(n; \mathbf{r})$ is a set of coset representations for $S_n/S_{n, \mathbf{r}}$. Given $w \in S(n; \mathbf{r})$, the corresponding Schubert class $\sigma_w \in H^{2\ell(w)} \text{Fl}(n; \mathbf{r})$, where the *length* of w is the number of inversions $\ell(w) = \#\{i < j \mid w(i) > w(j)\}$.

There is a unique permutation of longest length in $S(n; \mathbf{r})$ given explicitly by

$$w^\circ = [n - r_\rho + 1, \dots, n, \dots, n - r_1 + 1, \dots, n - r_2, \dots, 1, 2, \dots, n - r_1].$$

Its length is $\ell(w^\circ) = \dim \text{Fl}(n; \mathbf{r})$. Let w_\circ denote the unique permutation of greatest length in S_n given by $w_\circ(i) = n + 1 - i$. There is an involution on $S(n; \mathbf{r})$, using the longest element $w_\circ^\mathbf{r}$ of $S_{n, \mathbf{r}}$. (The permutation $w_\circ^\mathbf{r}$ can be defined by $w_\circ = w^\circ \cdot w_\circ^\mathbf{r}$.) Given $w \in S(n; \mathbf{r})$, define

$$w^\vee = w_\circ \cdot w \cdot w_\circ^\mathbf{r}.$$

This is an element of $S(n; \mathbf{r})$, with $\ell(w^\vee) = \dim \text{Fl}(n; \mathbf{r}) - \ell(w)$. The classes σ_{w^\vee} form a Poincaré dual basis: $\int_{\text{Fl}(n; \mathbf{r})} \sigma_w \cup \sigma_{w^\vee} = \delta_{w, w^\vee}$.

2.2 Another basis and tuples of partitions

We describe another basis for $H^* \text{Fl}(n; \mathbf{r})$ in terms of tuples of partitions. Consider the set

$$P(n, \mathbf{r}) := \prod_{i=1}^{\rho} P(r_{i-1}, r_i),$$

where we set $r_0 := n$ and $r_{\rho+1} := 0$, and where $P(a, b)$ denotes the partitions inside a $b \times (a - b)$ rectangle.

Remark 2.1 There is a bijection between permutations in $S(n, \mathbf{r})$ and tuples of partitions in $P(n, \mathbf{r})$. Given a tuple $\vec{\mu} = (\mu^1, \dots, \mu^\rho) \in P(n, \mathbf{r})$, for $1 \leq i \leq \rho$, denote by w^i the Grassmannian permutation in S_n with possible descent at r_i defined by the partition $\mu^i \subseteq P(r_{i-1}, r_i) \subseteq P(n, r_i)$, i.e. $\mu^i = (w^i(r_i) - r_i, \dots, w^i(1) - 1)$, so that $w^i = w_{(\emptyset, \dots, \mu^i, \dots, \emptyset)}$. Then the tuple $\vec{\mu} \in P(n, \mathbf{r})$ corresponds to the permutation

$$w_{\vec{\mu}} := w_{(\mu^1, \emptyset, \dots)} \dots w_{(\emptyset, \dots, \mu^\rho)} = w^1 w^2 \dots w^\rho.$$

On the other hand, given $w \in S(n, \mathbf{r})$, we can produce a tuple $\vec{\mu} = (\mu^1, \dots, \mu^\rho)$. (See also [4]).

If a tuple of partitions $\vec{\mu} = (\mu^1, \dots, \mu^\rho) \in P(n, \mathbf{r})$ corresponds to the permutation w under the bijection in Remark 2.1, we also write the Schubert class σ_w as $\sigma_{\vec{\mu}}$.

Example 2.2 Consider the flag variety $\text{Fl}(8; 6, 4, 3)$ with $n = 8$ and $\mathbf{r} = (6, 4, 3)$. For the tuple $(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \square, \emptyset)$ in $P(n, \mathbf{r})$, $w^1 = [123457|68]$, $w^2 = [1245|3678]$, $w^3 = id$ with descents marked at $r_1 = 6$ and $r_2 = 4$. The corresponding permutation in $S(n, \mathbf{r})$ is $w = w^1 w^2 w^3 = [1245|37|68]$. Similarly, $(\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix})$ corresponds to the permutation $[123468|57] \cdot [1356|2478] \cdot [234|15678] = [368|1|24|57]$, and $(\square, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \square)$ corresponds to $[123457|68] \cdot [1246|3578] \cdot [134|25678] = [147|2|35|68]$.

For a partition $\lambda \in P(r_{i-1}, r_i)$, we define the class s_λ^i to be the Schur polynomial associated to the partition λ in the Chern roots of Q_i , the rank r_i tautological quotient bundle on $\text{Fl}(n; \mathbf{r})$:

$$s_\lambda^i = \det(s_{1^{\lambda'_k + i - k}}^i). \quad (1)$$

Note that $s_{1^a}^i = c_a(Q_i)$ is the a th Chern class of the bundle Q_i so that $s_{1^a}^i = e_a(r_i)$. Via the bijection in Remark 2.1, s_λ^i is equal to the Schubert class

$$s_\lambda^i = \sigma_{(\emptyset, \dots, \lambda, \dots, \emptyset)}$$

associated to the tuple of partitions consisting i th partition equal to λ and the empty partition elsewhere.

Note that we can use (1) to define s_λ^i even when $\lambda \notin P(r_{i-1}, r_i)$, although it is no longer a Schubert class in general.

Remark 2.3 Given a tuple of partitions $\vec{\mu} = (\mu^1, \dots, \mu^\rho)$, we obtain another important class

$$s_{\vec{\mu}} := s_{\mu^1}^1 \dots s_{\mu^\rho}^\rho.$$

Running over all $\vec{\mu}$ we obtain another basis for the cohomology of the flag variety. The two bases $\{\sigma_{\vec{\mu}}\}$ and $\{s_{\vec{\mu}}\}$ are distinct, except in the case of the Grassmannian.

2.3 Quantum cohomology

The *quantum cohomology ring* $\text{QH}^* \text{Fl}(n; \mathbf{r})$ is a commutative and associative graded algebra over $\mathbb{Z}[q_1, \dots, q_\rho]$, where q_i is a parameter of degree $r_{i-1} - r_{i+1}$. As a module, $\text{QH}^* \text{Fl}(n; \mathbf{r})$ is simply $\mathbb{Z}[q] \otimes_{\mathbb{Z}} H^* \text{Fl}(n; \mathbf{r})$, so it has a $\mathbb{Z}[q]$ -basis of Schubert classes σ_w :

$$\text{QH}^* \text{Fl}(n; \mathbf{r}) = \bigoplus_{w \in S(n; \mathbf{r})} \mathbb{Z}[q] \cdot \sigma_w.$$

The quantum product is a deformation of the usual product. For permutations $u, v \in S(n; \mathbf{r})$, define a product by

$$\sigma_u * \sigma_v = \sum_{w, \mathbf{d}} \mathbf{q}^{\mathbf{d}} c_{u,v}^{w, \mathbf{d}} \sigma_w,$$

where \mathbf{d} ranges over $(n-1)$ -tuples of nonnegative integers, and the *three-pointed Gromov-Witten invariant* $c_{u,v}^{w, \mathbf{d}}$ is defined as follows.

Let $\overline{M}_{0,3}(\mathrm{Fl}(n; \mathbf{r}), \mathbf{d})$ be the Kontsevich moduli space of three-pointed genus-zero stable maps to $\mathrm{Fl}(n; \mathbf{r})$ of degree \mathbf{d} , parametrizing data $(f, C, (x_1, x_2, x_3))$, where C is a genus-zero curve with marked points x_i , $f: C \rightarrow \mathrm{Fl}(n; \mathbf{r})$ is a map of degree \mathbf{d} , and a certain stability condition is imposed [16]. The space of stable maps is of dimension $\dim \mathrm{Fl}(n; \mathbf{r}) + \sum_{i=1}^3 d_i(r_{i-1} - r_{i+1})$, and comes with natural evaluation morphisms

$$\mathrm{ev}_i: \overline{M}_{0,3}(\mathrm{Fl}(n; \mathbf{r}), \mathbf{d}) \rightarrow \mathrm{Fl}(n; \mathbf{r})$$

for $1 \leq i \leq 3$ that send $(f, C, (x_1, x_2, x_3))$ to $f(x_i)$. Now one defines $c_{u,v}^{w, \mathbf{d}} = \pi_*(\mathrm{ev}_1^* \sigma_u \cdot \mathrm{ev}_2^* \sigma_v \cdot \mathrm{ev}_3^* \sigma_{w^\vee})$. This defines an associative product. See [9] for more details on quantum cohomology.

2.4 Quantum cohomology of flag varieties

The Schubert polynomials of Lascoux and Schützenberger are defined inductively, starting from $\mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \dots x_{n-1}$ and moving down Bruhat order using divided difference operators [17]. For any $w \in S_n$, the polynomial $\mathfrak{S}_w(x)$ has a unique expansion in terms of elementary symmetric polynomials:

$$\mathfrak{S}_w(x) = \sum a_{k_1 \dots k_{n-1}} e_{k_1}(1) \dots e_{k_{n-1}}(n-1) \quad (2)$$

over sequences (k_1, \dots, k_{n-1}) with $0 \leq k_j \leq j$ and $\sum k_j = \ell(w)$, where the $a_{k_1 \dots k_{n-1}}$ are integers and $e_k(j) := e_k(x_1, \dots, x_j)$ is the k th elementary symmetric polynomial in the variables x_1, \dots, x_j .

Let

$$\sigma_1^\rho, \dots, \sigma_{r_\rho}^\rho, \sigma_1^{\rho-1}, \dots, \sigma_{r_{\rho-1}-r_\rho}^{\rho-1}, \dots, \sigma_1^0, \dots, \sigma_{n-r_1}^0$$

be n independent variables, with σ_i^j of degree i . To form quantum polynomials for $\mathrm{Fl}(n; \mathbf{r})$, one replaces $e_k(j)$ with *quantum elementary polynomials* $e_k^q(r_l)$, which are defined for $r_l \in \mathbf{r}$ and $r_0 = n$ recursively by

$$e_a^{\mathbf{r}, q}(r_{l-1}) = \sum_{m=0}^{r_{l-1}-r_l} \sigma_m^l e_{a-m}^{\mathbf{r}, q}(r_l) + (-1)^{r_{l-1}-r_l+1} q l e_{a-(r_{l-1}-r_l+1)}^{\mathbf{r}, q}(r_{l+1}), \quad (3)$$

where we set $e_0^q(r_l) = 1$ and $e_m^q(r_l) = 0$ if either $m < 0$ or $m > r_l$. When \mathbf{r} is understood, we simply write $e_k^q(r_l)$ for $e_k^{\mathbf{r}, q}(r_l)$. (Our conventions here differ from those found elsewhere in the literature, e.g. our r_l and σ_i^j correspond to $n_{\rho+1-l}$ and $\sigma_i^{\rho+1-j}$ in [5].)

From [5, 14], we know a presentation of the quantum cohomology ring and polynomial representatives of the quantum Schubert classes.

$$\mathrm{QH}^* \mathrm{Fl}(n; \mathbf{r}) \cong \mathbb{Z}[q][\sigma_1^\rho, \dots, \sigma_{r_\rho}^\rho, \dots, \sigma_1^0, \dots, \sigma_{n-r_1}^0]/I^q,$$

where I^q is the ideal $(e_1^{\mathbf{r},q}(n), \dots, e_n^{\mathbf{r},q}(n))$ generated by n relations $e_1^{\mathbf{r},q}(r_0) = 0$ which specialize to the known relations defining $H^* \text{Fl}(n; \mathbf{r})$ when $q \mapsto 0$, and

$$\sigma_w = \mathfrak{S}_w^{\mathbf{r},q}(\sigma)$$

for $w \in S(n; \mathbf{r})$, where the *quantum Schubert polynomial* $\mathfrak{S}_w^{\mathbf{r},q}(\sigma)$ is formed by substituting $e_k^{\mathbf{r},q}(r_l)$ for $e_k(j)$ on the RHS of (2) whenever $j \in [r_l, r_{l-1})$.

The quantum structure constants of the alternate basis, $s_{\tilde{\mu}}$, can be computed using rim-hook removals via the Abelian/non-Abelian correspondence [11].

2.5 Determinantal formulas

In Sect. 5, we will study certain skew shapes λ/μ along with a labeling $\omega(i, j) = r_1 + i - j$. By [3], associated to $(\lambda/\mu, \omega)$ is a 321-avoiding permutation w whose corresponding Schubert polynomial is equal to a *flagged skew Schur polynomial* that can be expressed as a determinant:

$$\mathfrak{S}_w(x) = \left| e_{\lambda'_i - \mu'_j + j - i}(f_j) \right|_{1 \leq i, j \leq t} \quad (4)$$

where $f_j = \omega(j, \lambda'_j) = r_1 + j - \lambda'_j$ is the “flagging” associated to w .

For a skew shape λ/μ and $\phi = (\phi_1, \dots, \phi_t)$ with $1 \leq \phi_i \leq \rho$, define

$$\Delta_{\lambda/\mu}(e^q(\phi)) := \left| e_{\lambda'_i - \mu'_j + j - i}^q(r_{\phi_j}) \right|_{1 \leq i, j \leq t}. \quad (5)$$

When ϕ_j is defined by $r_{\phi_j} \leq f_j < r_{\phi_j-1}$, substituting $e_k(j) = e_k^q(r_l)$ in (4) as in the discussion in Sect. 2.4, we obtain a determinantal expression for the quantum Schubert class:

$$\sigma_w = \Delta_{\lambda/\mu}(e^q(\phi)) \text{ in } QH^* \text{Fl}(n; \mathbf{r}). \quad (6)$$

We can also define quantum classes s_{λ}^i for partitions λ by computing the determinant (1) using the quantum product. When $\lambda \in P(r_{i-1}, r_i)$, this gives the quantum Schubert class $\sigma_{\emptyset, \dots, \lambda, \dots, \emptyset}$, but s_{λ}^i is also defined when $\lambda \notin P(r_{i-1}, r_i)$. In particular, since $s_{1^a}^i = e_a(r_i)$ classically, we have $s_{1^a}^i = e_a^q(r_i)$ in $\text{QH}(\text{Fl}(n; \mathbf{r}))$ and

$$s_{\lambda}^i = \left| s_{1^{\lambda'_k + l - k}}^i \right|_{1 \leq k, l \leq \lambda_1} = \Delta_{\lambda}(e^q(\phi)), \quad (7)$$

where $\phi = (i, \dots, i)$.

Remark 2.4 If $\mu \not\subseteq \lambda$, then $\lambda'_k < \mu'_k$ for some $1 \leq k \leq t$. If $i \geq k$ and $j \leq k$, the (i, j) th entry of the matrix in $\Delta_{\lambda/\mu}$ is indexed by $\lambda'_i - \mu'_j + j - i < \lambda'_k - \mu'_k < 0$, and so is zero. Since the matrix is block upper triangular with left upper block of determinant zero, $\Delta_{\lambda/\mu} = 0$.

3 Cluster structure and superpotentials

3.1 The cluster structure of the Grassmannian

In this section, some brief facts about the cluster structure of the Grassmannian are recalled. Good references include [20, 22]. Fix a Grassmannian of quotients $\text{Gr}(n, r)$. Plücker coordinates on the Grassmannian are indexed by partitions λ fitting in an $r \times (n - r)$ box, i.e.

by $\lambda \in P(n, r)$. The homogeneous coordinate ring of the Grassmannian is generated by p_λ , $\lambda \in P(n, r)$, and relations are given by the Plücker relations.

This ring, as well as certain localizations of it, has a cluster structure. Certain sets of algebraically independent Plücker coordinates are clusters. An important example of a cluster is the *rectangles cluster*.

Definition 3.1 The *rectangles cluster chart* is the set of Plücker coordinates indexed by all partitions $\lambda \in P(n, r)$ such that λ is a rectangle.

One cluster can be obtained from another via *mutation*. These mutations arise from three-term Plücker relations [22]. The three term quadratic Plücker relations are of the form

$$p_\lambda p_\mu = p_a p_b + p_c p_d$$

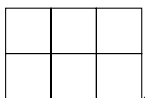
where $\lambda, \mu, a, b, c, d \in P(n, r)$ are six partitions related in a particular way. A cluster containing p_λ, p_a, p_b, p_c , and p_d can be mutated to one containing p_μ, p_a, p_b, p_c , and p_d . Any cluster is related to any other by a series of mutations of this form.

Let $M(n, r)$ be set of partitions that are either the empty set or a rectangular partition that is maximally wide or maximally tall.

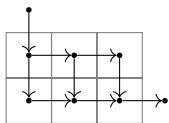
Remark 3.2 $M(n, r)$ is the set of *frozen variables* in the cluster structure of the Grassmannian: they appear in every cluster.

3.2 Superpotentials of the Grassmannian

The Eguchi–Hori–Xiong (EHX) superpotential for Grassmannians is described by building a ladder diagram for the Grassmannian, and super-imposing a dual quiver on the diagram. The ladder diagram for the Grassmannian $\text{Gr}(n, r)$ is an $r \times (n - r)$ grid. There is a toric degeneration of the Grassmannian to the quiver moduli space described by a quiver originating from the ladder diagram. The superpotential is given by a head-over-tails process on the dual quiver. We illustrate this briefly in the example $\text{Gr}(5, 2)$: the ladder diagram is a 2×3 grid:



The dual quiver is then:



In general, to form the dual quiver, place a vertex in each box of the ladder diagram, as well as one at the top left and bottom right of the diagram, and then add arrows oriented down and right.

To obtain the superpotential, assign to each of the vertices a variable z_{ij} , where i indicates the row (starting at 0) and j the column (starting at 1). We set $z_{10} = 1$ and $z_{r(n-r+1)} = q$. The EHX superpotential is then:

$$W_{EHX} = \sum_a \frac{z_{h(a)}}{z_{t(a)}}.$$

The sum is over the arrows of the quiver, and $h(a)$ and $t(a)$ indicate the head and tail of an arrow respectively.

Example 3.3 The EHX superpotential for $\text{Gr}(4, 2)$ is

$$z_{11} + \frac{z_{12}}{z_{11}} + \frac{z_{21}}{z_{11}} + \frac{z_{22}}{z_{12}} + \frac{z_{22}}{z_{21}} + \frac{q}{z_{22}}.$$

A superpotential is a mirror to a Fano manifold if information about the genus 0 Gromov–Witten invariants of the Fano manifold can be computed by the superpotential. More precisely, one or both of the following conditions might hold:

- (1) The period sequence of the superpotential is equal to the regularized quantum period of the Fano manifold (see [6] for definitions and details).
- (2) The Jacobi ring of the superpotential computes the quantum cohomology ring of the Fano manifold.

The first condition was the original conjecture of Eguchi–Hori–Xiong, later proved by Marsh–Rietsch [19] for the Grassmannian. This conjecture remains open for flag varieties.

The second condition—that the superpotential produces relations in the quantum cohomology ring—is the central focus of the paper. We first discuss the proof in the case of the Plücker coordinate mirror for the Grassmannian, introduced by Marsh–Rietsch in [19]; the same statement for the EHX mirror is obtained as a corollary.

To construct the Plücker coordinate mirror the Grassmannian $\text{Gr}(n, r)$, take n equations of the form

$$s_{\square} * s_{\lambda} = q^i s_{\mu} \quad (8)$$

where $i = 0, 1$ depending on the partition. Here $\lambda := (a, a, \dots, a)$ is a rectangular partition in $M(n, r)$.

Note that by manipulating the equations 8, we see that the sum

$$\sum_{\lambda \in M(n, r)} \frac{q^i s_{\mu}}{s_{\lambda}} \quad (9)$$

is equal to $n s_{\square} = -K_{\text{Gr}(n, r)}$. This is the anti-canonical class of the Grassmannian.

To transform the sum into a (rational) function, every Schubert class s_{λ} is replaced with the Plücker coordinate p_{λ} .

Remark 3.4 This construction is inspired by the Hori–Vafa superpotential of a Fano toric variety, which is also built out of a special breakdown of the anti-canonical class.

Example 3.5 The Marsh–Rietsch Plücker coordinate superpotential for $\text{Gr}(4, 2)$ is

$$\frac{p_{\square}}{p_{\emptyset}} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}{p_{\begin{smallmatrix} \square & \square \end{smallmatrix}}} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}{p_{\begin{smallmatrix} \square & \square \end{smallmatrix}}} + \frac{q p_{\square}}{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}.$$

Following [19], we denote the open subvariety on which the Plücker coordinate superpotential is a function (i.e. where $p_{\lambda} \neq 0$, $\lambda \in M(n, r)$) as $\text{Gr}(n, n-r)^{\circ}$.

Using Plücker relations, we can expand the Plücker coordinate mirror into a Laurent polynomial in each cluster chart in the cluster structure on the coordinate ring of the Grassmannian.

Example 3.6 We can use the three term Plücker relation

$$p_{\square} p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} p_{\square} + p_{\emptyset} p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$$

to find that in the rectangles cluster chart, the mirror for $\text{Gr}(4, 2)$ is

$$\frac{p_{\square}}{p_{\emptyset}} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}{p_{\square}} + \frac{p_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}}{p_{\square}} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}{p_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}} + \frac{qp_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}}{p_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}}.$$

In each cluster chart, one can compute the critical locus by setting the partial derivatives to zero: $\frac{\partial}{\partial W_C} W_C = 0$. It is clear how to interpret these equations as candidate relations in quantum cohomology: both Plücker coordinates and Schubert classes of the Grassmannian $\text{Gr}(n, r)$ are indexed by the same set of partitions, $\lambda \subset r \times (n - r)$.

Theorem 3.7 [19] *The Jacobi ring of the Plücker coordinate mirror is isomorphic to the quantum cohomology ring of the Grassmannian.*

The Plücker coordinate mirror is a compactification of the EHX mirror: that is,

Proposition 3.8 [19] *The Plücker coordinate mirror in the rectangles cluster chart is isomorphic to the EHX mirror under the map ϕ , where*

$$\phi(z_{ij}) = \frac{p_{i \times j}}{p_{(i-1) \times (j-1)}}.$$

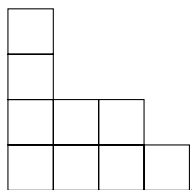
This proposition and theorem can be combined to show the following theorem:

Theorem 3.9 [19] . *Let $F : \mathbb{C}[p_{\lambda}] \rightarrow \text{QH}^* \text{Gr}(n, r)[s_{\lambda}^{-1} : \lambda \in R]$ be the map given by $p_{\lambda} \mapsto s_{\lambda}$. Then for any z_{ij} ,*

$$F \left(\phi \left(\frac{\partial}{\partial z_{ij}} W_{EHX} \right) \right) = 0.$$

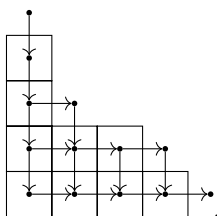
3.3 Superpotentials of flag varieties

We first recall the Batyrev–Ciocan-Fontanine–Kim–van Straten generalization of the EHX mirror to flag varieties [1]. Fixing $\text{Fl}(n; r_1, \dots, r_{\rho})$, for each Grassmannian step $\text{Gr}(r_{i-1}, r_i)$ draw an $r_i \times (r_i - r_{i-1})$ grid of boxes, placing them together. For example, the ladder diagram of $\text{Fl}(5; 4, 2, 1)$ is



The dual quiver is similar to the Grassmannian case. There are vertices inside each box, as well as at the top left and bottom right corners and in the inner corner of each step of the

diagram. In this example, it is



Assigning to each of the vertices a variable z_v , the EHX superpotential is:

$$W_{EHX} = \sum_a \frac{z_{h(a)}}{z_{t(a)}}.$$

In [13], the second author proposes a generalization of the Plücker coordinate mirror from Grassmannians to type A flag varieties. We recall the construction now. Fix a flag variety $\text{Fl}(n; r_1, \dots, r_\rho)$. For each $i = 1, \dots, \rho$, we can consider r_{i-1} equations

$$s_{\square}^i * s_{\lambda}^i = \tilde{G}_{\lambda}^i,$$

where $\lambda \in M(r_{i-1}, r_i)$, and \tilde{G}_{λ}^i is simply the expansion of the left hand side in quantum Schubert calculus. This can be described explicitly—see [13] for details. As in the Marsh–Rietsch construction, we can use this to obtain an expression of the anti-canonical class of the flag variety:

$$\sum_{i=1}^{\rho} \left(\left(\sum_{\lambda \in M(r_{i-1}, r_i)} \frac{\tilde{G}_{\lambda}^i}{s_{\lambda}^i} \right) - r_{i+1} s_{\square}^i \right).$$

The set $P(n, \mathbf{r})$ naturally indexes elements of the coordinate ring of the product of Grassmannians

$$Y(n, \mathbf{r}) := \prod_{i=1}^{\rho} \text{Gr}(r_{i-1}, r_i).$$

Let \mathcal{Q}_i be the tautological quotient bundle pulled back to $Y(n, \mathbf{r})$ from the i^{th} Grassmannian factor. Sections of $\det(\mathcal{Q}_i)$ are indexed by $\lambda \in P(r_{i-1}, r_i)$. We write p_{λ}^i for the Plücker coordinate associated to i and λ .

We denote $Y(n, \mathbf{r})^{\circ} := \prod_{i=1}^{\rho} \text{Gr}(r_{i-1}, r_{i-1} - r_i)^{\circ}$ the locus in $Y(n, \mathbf{r})$ where $p_{\lambda}^i \neq 0$ for all i and $\lambda \in M(r_{i-1}, r_i)$. This is the complement of an anti-canonical divisor on $Y(n, \mathbf{r})$.

To each Schubert class $s_{\tilde{\mu}}$ we associate the product

$$p_{\tilde{\mu}} := \prod_{i=1}^{\rho} p_{\mu_i}^i.$$

We denote the polynomial in the coordinate ring of $Y(n, \mathbf{r})$ and the q_1, \dots, q_{ρ} obtained by replacing the Schubert classes in \tilde{G}_{λ}^i with Plücker coordinates in this way as G_{λ}^i .

Definition 3.10 The Plücker coordinate superpotential W_P of the flag variety is

$$\sum_{i=1}^{\rho} \left(\left(\sum_{\lambda \in M(r_{i-1}, r_i)} \frac{G_{\lambda}^i}{p_{\lambda}^i} \right) - r_{i+1} p_{\square}^i \right).$$

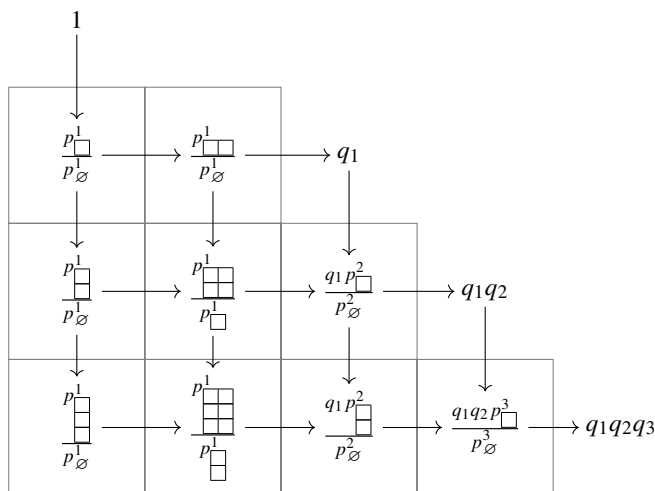
Example 3.11 Consider the flag variety $\text{Fl}(6; 4, 2, 1)$. The Plücker coordinate superpotential is

$$\begin{aligned} & \frac{p_{\square}^1}{p_{\emptyset}^1} + \frac{p_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}^1}{p_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}^1} + \frac{p_{\begin{smallmatrix} \square & \square \end{smallmatrix}}^1}{p_{\begin{smallmatrix} \square & \square \end{smallmatrix}}^1} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^1 + q_1 p_{\begin{smallmatrix} \square & \square \end{smallmatrix}}^1}{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^1} + \frac{p_{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}}^1 + q_1 p_{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}}^1 p_{\begin{smallmatrix} \square \end{smallmatrix}}^2}{p_{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}}^1} + \frac{q_1 p_{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}}^1 p_{\begin{smallmatrix} \square \end{smallmatrix}}^2}{p_{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}}^1} \\ & + \frac{p_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}^2}{p_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}^2} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^2}{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^2} + \frac{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^2 + q_2}{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^2} + \frac{q_2 p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^2 p_{\begin{smallmatrix} \square \end{smallmatrix}}^3}{p_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}^2} + \frac{p_{\begin{smallmatrix} \square \end{smallmatrix}}^3}{p_{\begin{smallmatrix} \square \end{smallmatrix}}^3} + \frac{q_3}{p_{\begin{smallmatrix} \square \end{smallmatrix}}^3}. \end{aligned}$$

By choosing a cluster chart for each Grassmannian factor of $Y(n, \mathbf{r})$, we can expand the Plücker coordinate mirror of the flag variety into algebraically independent sets of coordinates on $Y(n, \mathbf{r})$.

In [13], a first check of the validity of the Plücker coordinate mirror is carried out by demonstrating that the Plücker coordinate mirror is a compactification of the EHX mirror (that is, Proposition 3.8 in the flag case). Fix a flag variety $\text{Fl}(n; \mathbf{r})$. Recall that the ladder diagram is made up of the ladder diagrams of ρ Grassmannians, i.e. an $r_i \times (r_{i-1} - r_i)$ grid for each i . Recall the definition of the map ϕ from Proposition 3.8. Given a vertex v in the i^{th} block of the dual quiver, let $\phi(z_v)$ be as prescribed in the Grassmannian case for $\text{Gr}(r_{i-1}, r_i)$, and then scale by $q_1 \dots q_{i-1}$.

Example 3.12 To demonstrate, we label the vertices with $\phi(z_v)$ in the following example (where the flag variety is $\text{Fl}(5; 3, 2, 1)$):



Theorem 3.13 [13] For any type A flag variety, the Plücker coordinate mirror in the rectangles cluster chart is isomorphic to the EHX mirror under the isomorphism

$$z_v \mapsto \phi(z_v).$$

4 Quantum cohomology and mirrors of the flag variety

To summarize the situation for the Grassmannian, there are two mirrors—the Plücker coordinate mirror and the EHX mirror—the first of which is isomorphic with the second in a particular cluster chart. Because the same partitions index Plücker coordinates and Schubert classes, partial derivatives of the Plücker coordinate mirror can easily be interpreted—and indeed give—quantum cohomology relations.

Up until the last clause, the same is true for a multi-step flag variety: there are two mirrors—the Plücker coordinate mirror and the EHX mirror—the first of which is isomorphic with the second in a particular cluster chart. The same partitions index Plücker coordinates and Schubert classes—and indeed, the Abelian/non-Abelian basis of the cohomology as well. But consider the following example.

Example 4.1 The Plücker coordinate mirror of $\text{Fl}(4; 2, 1)$ is

$$\frac{p_{\square}^1}{p_{\emptyset}^1} + \frac{p_{\square}^1 + q_1}{p_{\square}^1} + \frac{p_{\square}^1}{p_{\square}^1} + \frac{q_1 p_{\square}^1 p_{\square}^2}{p_{\square}^1} + \frac{p_{\square}^2}{p_{\square}^2} + \frac{q_2}{p_{\square}^2}.$$

Expanding in the rectangles cluster and applying $p_{\square}^2 \frac{\partial}{\partial p_{\square}^2}$, we obtain

$$\frac{q_1 p_{\square}^1 p_{\square}^2}{p_{\square}^1} + p_{\square}^2 - \frac{q_2}{p_{\square}^2}.$$

The most natural way to interpret this as a quantum cohomology relation is as:

$$\frac{q_1 s_{\square}^1 s_{\square}^2}{s_{\square}^1} + s_{\square}^2 - \frac{q_2}{s_{\square}^2} = 0,$$

however, this relation does not hold. One could attempt to use Schubert classes instead, for example:

$$\frac{q_1 \sigma_{\square, \square}}{\sigma_{\square, \square}} + \sigma_{\emptyset, \square} - \frac{q_2}{\sigma_{\emptyset, \square}} = 0.$$

However, this relation also does not hold, and at any rate there will quickly be ambiguity with this approach with multi-step flag varieties.

The above example demonstrates the central difficulty in the flag case: the Plücker coordinate mirror is built out of quantum Schubert calculus, but is written in Plücker coordinates which have the same multiplicative structure of the $s_{\vec{\mu}}$ basis. By multiplicative structure, we mean the property that the basis element associated to a tuple $(\lambda_1, \dots, \lambda_{\rho})$ is the product of the ρ basis elements given by tuples with a single non-empty partition λ_i in the i^{th} spot, as i runs from 1 to ρ .

For the flag variety, we must instead use the *Schubert map*, which we introduce now. Fix a flag variety $\text{Fl}(n; \mathbf{r})$, where $\mathbf{r} := (r_1, \dots, r_{\rho})$ as usual. Recall that $P(n, \mathbf{r})$ is the set of Plücker coordinates p_{λ}^i on $Y(n, \mathbf{r})$, where λ is a rectangle. Let $U_{P(n, \mathbf{r})}$ be the open subvariety of $Y(n, \mathbf{r})$ where the $p_{\lambda}^i, \lambda \in P(n, \mathbf{r})$ do not vanish. Let $\widetilde{QH}^*(\text{Fl}(n; \mathbf{r}))$ denote the localization of the quantum cohomology ring at the rectangular Schubert classes.

The ring of functions $\mathbb{C}[U_{P(n,r)}]$ is generated (as an algebra) by $P(n, \mathbf{r})$, as every Plücker coordinate can be written as a Laurent polynomial in the rectangular Plücker coordinates using three term Plücker relations. We extend the coefficient field to the ring $R = \mathbb{C}[q_1, \dots, q_\rho]$. We define a map

$$F : R[U_{P(n,r)}] \rightarrow \widetilde{QH}^*(\mathrm{Fl}(n; \mathbf{r}))$$

a morphism of $\mathbb{C}[q_1, \dots, q_\rho]$ algebras—by specifying the images of the rectangular Plücker coordinates.

Fix some $p_{j \times k}^i$, where the rectangle $j \times k$ is an element of $P(r_{i-1}, r_i)$. We define two tuples of partitions. For $l = 1, \dots, i-1$, let R_l be the $(j-k+r_{i-1}-r_i) \times (r_{l-1}-r_l)$ rectangle, and set $R_i := j \times k$. Set $\vec{\mu}_1 := (R_1, \dots, R_i, \emptyset, \dots, \emptyset)$ and $\vec{\mu}_2 := (R_1, \dots, R_{i-1}, \emptyset, \emptyset, \dots, \emptyset)$.

Definition 4.2 The Schubert map

$$F : \mathbb{C}[U_{P(n,r)}][q_1, \dots, q_\rho] \rightarrow \widetilde{QH}^*(\mathrm{Fl}(n; \mathbf{r}))$$

is defined by setting

$$F(p_{j \times k}^i) = \frac{\sigma_{\vec{\mu}_1}}{\sigma_{\vec{\mu}_2}}.$$

Example 4.3 In $\mathrm{Fl}(4; 2, 1)$, $F(p_{j \times k}^1) = s_{j \times k}^1$, and

$$F(p_{\square}^2) = \frac{\sigma_{\square\square, \square}}{\sigma_{\square\square, \emptyset}}.$$

Remark 4.4 Note that the Schubert map in the Grassmannian case is just the map $p_\lambda \mapsto s_\lambda$, which agrees with the map defined in Theorem 3.9.

The Schubert map allows partial derivatives of the Plücker coordinate mirror to be interpreted as quantum relations. We are now ready to prove Theorem A as stated in the introduction, which we restate here.

Theorem A $C = (C_1, \dots, C_\rho)$ be a choice of clusters for each Grassmannian factor in Y , and let W_C be the expansion of the Plücker coordinate mirror in this chart. For all i and $p_\lambda^i \in C_i$,

$$F\left(\frac{\partial}{\partial p_\lambda^i} W_C\right) = 0.$$

To show this theorem will require two propositions.

Proposition 4.5 Let $C = (C_1, \dots, C_\rho)$ and $C' = (C'_1, \dots, C'_\rho)$ be two choices of clusters for Y connected by a mutation. Let W_C and $W_{C'}$ be the expansions of W in C and C' respectively. Suppose Theorem A holds for C . Then it holds for C' .

Proof For some $i = 1, \dots, \rho$ there is a $\lambda, \mu, a, b, c, d \in P(r_{i-1}, r_i)$ such that C'_i is obtain from C_i via the three term Plücker relation

$$p_\lambda^i p_\mu^i = p_a^i p_b^i + p_c^i p_d^i$$

That is, $p_\lambda^i \in C_i$ and $p_\mu^i \in C'_i$, and p_a^i, p_b^i, p_c^i and p_d^i are elements of both C_i and C'_i . The Laurent polynomial $W_{C'}$ is obtained from W_C by replacing p_λ^i with

$$\frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i}.$$

Note that by construction,

$$F(p_\lambda^i) = F\left(\frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i}\right).$$

For any C'_i , we can then compute using the multi-variable chain rule that

$$\frac{\partial}{\partial p_\alpha^i} W_{C'} = \frac{\partial}{\partial p_\alpha^i} \left(\frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i} \right) \frac{\partial}{\partial p_\lambda^i} W_C \Big|_{p_\lambda^i = \frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i}} + \frac{\partial}{\partial p_\alpha^i} W_C \Big|_{p_\lambda^i = \frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i}}.$$

It follows that

$$F\left(\frac{\partial}{\partial p_\alpha^i} W_{C'}\right) = 0,$$

as

$$F\left(\frac{\partial}{\partial p_\lambda^i} W_C \Big|_{p_\lambda^i = \frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i}}\right) = F\left(\frac{\partial}{\partial p_\lambda^i} W_C\right) = 0$$

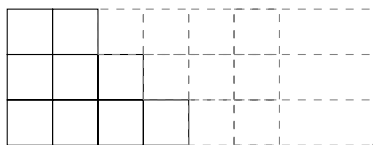
and

$$F\left(\frac{\partial}{\partial p_\alpha^i} W_C \Big|_{p_\lambda^i = \frac{p_a^i p_b^i + p_c^i p_d^i}{p_\mu^i}}\right) = F\left(\frac{\partial}{\partial p_\alpha^i} W_C\right) = 0.$$

□

The implication of this proposition is that we can reduce Theorem A to the statement for a single cluster, the rectangles cluster. The next proposition is the main ingredient in the proof of Theorem A, and is a corollary of the second theorem proved in this paper. This proposition uses the fact that the ladder diagram of a flag variety $\text{Fl}(n; \mathbf{r})$ can be viewed naturally as a subquiver of the ladder diagram of a Grassmannian $\text{Gr}(N, r_1)$, where $N \gg 0$ (or we can think of $\text{Gr}(\infty, r_1)$ if we wish).

For example, below, the ladder diagram of the flag variety $\text{Fl}(5; 3, 2, 1)$ is superimposed on that of $\text{Gr}(\infty, 3)$ (the second is drawn dashed in grey):



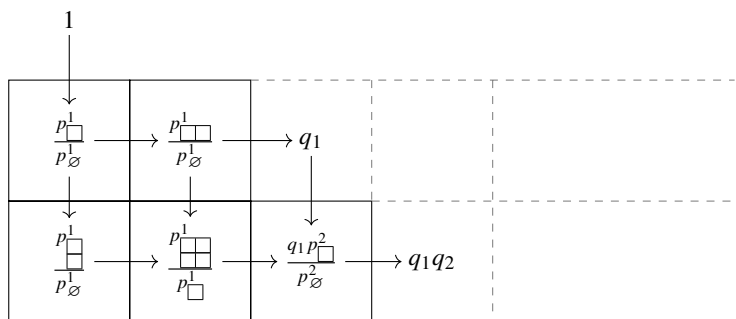
Recall that for a Grassmannian or flag variety, we have defined a map ϕ (see the discussion before Theorem 3.13) which maps coordinates on the ladder quiver mirror to Plücker coordinates. We also have defined the Schubert map, which takes Plücker coordinates to Schubert classes.

Viewing the ladder diagram of a flag variety either as a ladder diagram on its own right, or as a sub-quiver of the big Grassmannian's ladder diagram, we get two different ways of mapping the z_v to Plücker coordinates. That is, we now have two ϕ maps, both with domain $\mathbb{C}[z_v]$, where v ranges over the vertices of the dual ladder quiver of the flag variety. Let $\phi_{\text{Fl}} : \mathbb{C}[z_v] \rightarrow \mathbb{C}[U_{P(n,r)}]$ denote the homomorphism obtained by viewing vertices as vertices in the flag quiver. If we view a vertex z_v as a vertex of a Grassmannian quiver, then we obtain a map ϕ_{Gr} from $\mathbb{C}[z_v]$ to a localization of the coordinate ring of $\text{Gr}(\infty, r_1)$. More precisely, this is just the ring generated by minors of the infinite matrix

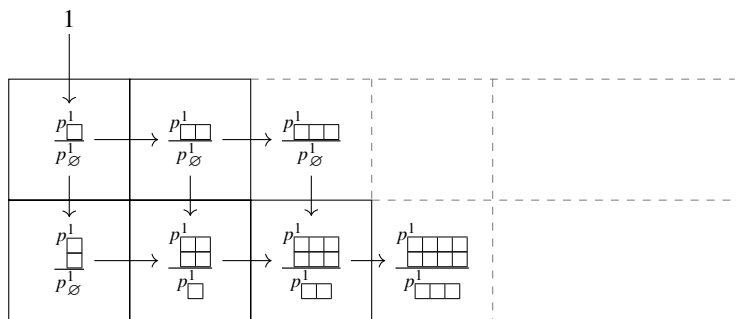
$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{12} & x_{13} & x_{14} & \cdots \\ \vdots & & & \vdots & \\ x_{r_1 1} & x_{r_1 2} & x_{r_1 3} & x_{r_1 4} & \cdots \end{bmatrix},$$

which we can index by all partitions of length at most r , localized at the rectangular partitions appearing in the flag quiver. Abusing notation, we call this ring $\mathbb{C}[U_{P(\infty, r_1)}]$.

Example 4.6 Labeling the vertices the dual ladder quiver for the flag variety $\text{Fl}(4; 2, 1)$ with $\phi_{\text{Fl}}(z_v)$ gives:



Labeling the same vertices with ϕ_{Gr} gives:



By taking limits, we can see that there is a well-defined map from the ring of minors of the infinite matrix above to the symmetric polynomial ring in r_1 variables, Λ_{r_1} , given by

$$p_\lambda \mapsto s_\lambda.$$

Let $\Lambda_{r_1}^\circ$ be the localization at the rectangular coordinates. The map above gives rise to a natural generalization of the Schubert map

$$F_{Gr} : \mathbb{C}[U_{P(\infty, r_1)}] \rightarrow \Lambda_{r_1}^\circ.$$

We also have the Schubert map for the flag variety:

$$F_{Fl} : \mathbb{C}[U_{P(n, \mathbf{r})}][q_1, \dots, q_\rho] \rightarrow \widetilde{QH}^*(Fl(n; \mathbf{r})).$$

Proposition 4.7 *Consider the natural map*

$$\pi : \Lambda_{r_1}^\circ \rightarrow \widetilde{QH}^*(Fl(n; \mathbf{r})), \quad s_\lambda \mapsto s_\lambda^1$$

discussed in the introduction and in (7). Then the following diagram commutes.

$$\begin{array}{ccc} & \mathbb{C}[U_{P(\infty, r_1)}] & \xrightarrow{F_{Gr}} \Lambda_{r_1}^\circ \\ \phi_{Gr} \nearrow & & \downarrow \pi \\ \mathbb{C}[z_v] & & \\ \phi_{Fl} \searrow & \mathbb{C}[U_{P(n, \mathbf{r})}][q_1, \dots, q_\rho] & \xrightarrow{F_{Fl}} \widetilde{QH}^*(Fl(n; \mathbf{r})) \end{array}$$

Example 4.8 (Example 4.6 continued) We illustrate the proposition in the case of $Fl(4; 2, 1)$. We have already described $\phi_{Gr}(z_v)$ and $\phi_{Fl}(z_v)$ in Example 4.6.

Proposition 4.7 states that if we apply the Schubert map to $\phi_{Gr}(z_v)$ and then apply π , we obtain the same cohomology class as applying the Schubert map to $\phi_{Fl}(z_v)$, for any vertex z_v appearing the dual ladder quiver. This is trivially true for the vertices in the first block.

Consider the vertex satisfying $\phi_{Gr}(z_v) = p_{\square\square\square}/p_\emptyset$. One can check using Theorem B (see Example 6.1) that

$$\pi \left(F_{Gr} \left(\frac{p_{\square\square\square}}{p_\emptyset} \right) \right) = \frac{s_{\square\square\square}^1}{s_\emptyset^1} = q_1,$$

which is indeed the image under F_{Fl} of the label corresponding to the same vertex in the flag diagram.

Similarly, from Example 6.1, we also have

$$\pi \left(F_{Gr} \left(\frac{p_{\square\square\square\square}}{p_{\square\square}} \right) \right) = \frac{s_{\square\square\square\square}^1}{s_{\square\square}^1} = \frac{q_1 \sigma_{\square\square, \emptyset}}{\sigma_{\square\square, \emptyset}} = F_{Fl} \left(\frac{q_1 p_{\square}^2}{p_\emptyset^2} \right),$$

and

$$\pi \left(F_{Gr} \left(\frac{p_{\square\square\square\square\square}}{p_{\square\square\square}} \right) \right) = \frac{s_{\square\square\square\square\square}^1}{s_{\square\square\square}^1} = \frac{q_1^2 q_2}{q_1} = q_1 q_2 = F_{Fl}(q_1 q_2).$$

To summarize, the ladder diagram of any flag variety is a subquiver of the ladder diagram of a sufficiently large Grassmannian. Using this inclusion of ladder diagrams, we can induce an inclusion of dual ladder quivers. For the Grassmannian, Theorem 3.9 gives a map from vertices of the Grassmannian ladder quiver to the cohomology of the Grassmannian. Theorem 3.13 together with the Schubert map gives a map from vertices of the flag variety to the quantum cohomology of the flag variety. There is a natural map from the cohomology of the

Grassmannian to the flag variety. Proposition 4.7 states that the Schubert map is precisely the map that makes this diagram commute. We'll delay the proof of Proposition 4.7 to the next section, where it will be an easy corollary of Theorem B.

Proof of Theorem A By Proposition 4.5, it suffices to show that for $C = (C_1, \dots, C_\rho)$ the rectangles cluster, and for all i and $p_\lambda^i \in C_i$,

$$F\left(\frac{\partial}{\partial p_\lambda^i} W_C\right) = 0.$$

Recall that Theorem 3.13 implies that W_C can be computed using the dual ladder quiver, together with the labels as in Example 3.12: that is,

$$W_C = \sum_a \frac{L(v_{t(a)})}{L(v_{s(a)})} \quad (10)$$

where a ranges over the arrows in the quiver, $v_{s(a)}$ and $v_{t(a)}$ are the vertices that are the source and target of the arrow a , and $L(v_{s(a)})$ and $L(v_{t(a)})$ the labels of these vertices.

Fixing a rectangle $j \times k$, in either the Grassmannian or the flag case, the partial derivative $p_{j \times k}^i \frac{\partial}{\partial p_{j \times k}^i}$ can be computed using the ladder diagram as well just as in (10). In this case, it is a signed sum involving the arrows with source or target at one of the two vertices where $p_{j \times k}^i$ appears in the numerator or denominator of the label. That is, the sum is over the following eight arrows, and it is a signed sum—dashed arrows have a negative sign and solid arrows a positive sign:

$$\begin{array}{c}
 \downarrow \\
 \begin{array}{ccccc}
 & & & & \\
 & & \downarrow & & \\
 \longrightarrow & \frac{p_{j \times k}^i}{p_{(j-1) \times (k-1)}^i} & \cdots \cdots \longrightarrow & & \\
 & \downarrow & & \downarrow & \\
 & \downarrow & & \downarrow & \\
 & \cdots \cdots \longrightarrow & \frac{p_{(j+1) \times (k+1)}^i}{p_{j \times k}^i} & \longrightarrow & \\
 & & \downarrow & & \\
 & & \downarrow & &
 \end{array}
 \end{array} \quad (11)$$

If a vertex is on the border of the diagram, some arrows do not appear. For example, a variable of the form $p_{r_i \times k}^i$ appears in the label of only one vertex, and that vertex is the in the bottom row. In this case, the diagram is simply

$$\begin{array}{c}
 \downarrow \\
 \begin{array}{ccc}
 \longrightarrow & \frac{p_{r_i \times k}^i}{p_{(r_i-1) \times (k-1)}^i} & \cdots \cdots \longrightarrow \\
 & \downarrow & \\
 & \downarrow &
 \end{array}
 \end{array}$$

Notice that if we consider two variables $p_{j \times k}^i$ and $p_{(j+1) \times (k+1)}^i$, the arrows involved overlap, and therefore the corresponding equations share half their terms in common. By

starting with a variable of the form $p_{r_1 \times k}^i$ and then consider consecutive variables

$$p_{r_1 \times k}, p_{(r_1-1) \times (k-1)}, p_{(r_1-2) \times (k-2)}, \dots$$

for some k , we can easily see the partial derivatives vanish under the Schubert map if and only if diagrams of the following form vanish:

$$\begin{array}{c} \downarrow \\ \rightarrow \frac{p_{j \times k}^i}{p_{(j-1) \times (k-1)}^i} \dashrightarrow \\ \vdots \\ \downarrow \end{array} \quad (12)$$

Again, some arrows may not appear depending on the position of the middle vertex in the quiver.

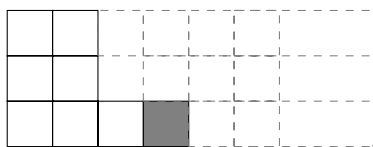
To summarize, it suffices to show for every internal vertex in the dual ladder quiver of the flag variety, the equation arising from (12) vanishes under the Schubert map. Let E_{Fl} be such an equation for a fixed vertex v . Let E_{Gr} be the corresponding equation for the Grassmannian for the same vertex. By Theorem 3.9,

$$F_{\text{Gr}}(E_{\text{Gr}}) = 0.$$

Our claim is that

$$0 = \pi(F_{\text{Gr}}(E_{\text{Gr}})) = F_{\text{Fl}}(E_{\text{Fl}}).$$

If the arrows with source or target at v as a vertex in the Grassmannian quiver are also arrows in the ladder quiver, this follows immediately from Proposition 4.7. For some vertices along the border, however, there may be extra arrows in the Grassmannian quiver that contribute extra terms to E_{Gr} . For example, the vertex in the gray box is such a vertex in the following diagram:



We claim, however, that these extra terms vanish under the Schubert map, and so the above equation still holds. In the example above, the extra term in E_{Gr} comes from the vertical arrow into the gray box, and is

$$\frac{p_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} p_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}}{p_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} p_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}}.$$

Note that $\pi(F_{\text{Gr}}(p_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}})) = s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^1 = 0$, so the whole term vanishes as required.

In general, these extra arrows come in two forms: vertical arrows along the top of a step in the ladder diagram and horizontal arrows along the side. Fixing a block or step $i \geq 1$ of the quiver, vertical arrows contribute the factor below to an extra term:

$$\frac{p_{(r_1-r_{i+1}-1) \times (n-r_i+k)}}{p_{(r_1-r_{i+1}) \times (n-r_i+1+k)}}$$

for $k = 1, \dots, r_i - r_{i+1} - 1$. Horizontal arrows contribute a factor of the form

$$\frac{P(r_1 - r_i + k) \times (n - r_i + 1)}{P(r_1 - r_i + k - 1) \times (n - r_i)},$$

for $k = 1, \dots, r_i - r_{i+1} - 1$. Since

$$s_{(r_1 - r_{i+1} - 1) \times (n - r_i + k)}^1 = 0, \quad k = 1, \dots, r_i - r_{i+1} - 1$$

and

$$s_{(r_1 - r_i + k) \times (n - r_i + 1)}^1 = 0, \quad k = 1, \dots, r_i - r_{i+1} - 1$$

in $\text{Fl}(n; \mathbf{r})$ by part (b) of Theorem B, the extra terms vanish as claimed. \square

5 Quantum hooks and quantum cohomology

In this section, we study a natural ring homomorphism from the ring Λ_{r_1} of symmetric polynomials in r_1 variables to $\text{QH}^* \text{Fl}(n; \mathbf{r})$ given by mapping the k th elementary symmetry polynomial in r_1 variables $e_k(r_1)$ to the k th quantum elementary polynomial $e_k^q(r_1)$, defined by the recursion (3) as in Sect. 2.

We have a \mathbb{Z} -basis of Λ_{r_1} given by Schur polynomials indexed by partitions λ of height at most r_1 . Using the identity $s_\lambda = \det(s_{\lambda'_i + j - i}) = \det(e_{\lambda'_i + j - i}(r_1))$, where λ' is the transpose of λ , we write s_λ^1 for the image of s_λ under the map $\Lambda_{r_1} \rightarrow \text{QH}^* \text{Fl}(n; \mathbf{r})$:

$$s_\lambda \mapsto s_\lambda^1 := \det(e_{\lambda'_i + j - i}^q(r_1)).$$

For $\lambda \in P(n, r_1)$, s_λ^1 represents a quantum Schubert class. When λ has width greater than $n - r_1$, s_λ^1 is still defined, and Theorem B states that for a particular class of partitions λ , s_λ^1 is equal to a Schubert class, up to power of q , and that for another class of partitions, $s_\lambda^1 = 0$.

We begin with some terminology. For $0 < b \leq n - r_\rho$, write $\bar{b} := b - (n - r_I)$ for I such that $n - r_I < b \leq n - r_{I+1}$. As in the introduction, set the quantum hook (or q -hook) of width b to be the partition

$$H_b := (b^{b-n+r_1}, (b-n+r_I)^{n-r_{I+1}-b}) = (b^{r_1-r_I+\bar{b}}, \bar{b}^{n-r_{I+1}-b}).$$

In the proof of our results, we will often consider the column heights of H_b , which we can read from the transpose of H_b :

$$H_b' = ((r_1 - r_{I+1})^{\bar{b}}, (r_1 - r_I + \bar{b})^{n-r_I}) = ((r_1 - r_{I+1})^{b-n+r_1}, (b-n+r_I)^{n-r_I}) \quad (13)$$

The q -hook H_b can also be described as the partition obtained from a $(r_1 - r_I) \times (n - r_I)$ rectangle after adding \bar{b} rim-hooks of length $n + r_1 - r_I - r_{I+1}$, each beginning in row $r_1 - r_{I+1}$ and ending in row 1 (see Fig. 2, also Fig. 3).

For a q -hook H_b of width b , set $q^{H_b} := q_1^{r_1-r_2} \dots (q_1 \dots q_{I-1})^{r_{I-1}-r_I} (q_1 \dots q_I)^{b-(n-r_I)}$. With this definition, note that

$$q^{H_b} = q^{H_{b-1}} \cdot q_1 \dots q_I. \quad (14)$$

Example 5.1 Consider $\text{QH}^* \text{Fl}(4; 2, 1)$ with $\deg q_1 = 3$ and $\deg q_2 = 2$. For the q -hooks $H_3 = (3, 0)$ and $H_4 = (4, 4)$ shown in Fig. 2, we have $q^{H_3} = q_1$ and $q^{H_4} = q_1(q_1 q_2) = q_1^2 q_2$.

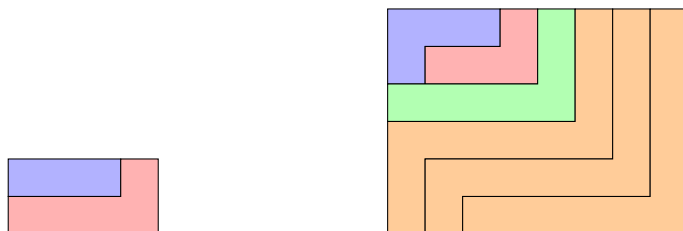


Fig. 2 q -hooks H_b of width $b = 3, 4$ for $\text{Fl}(4; 2, 1)$ and of width $3 \leq b \leq 8$ for $\text{Fl}(8; 6, 4, 3)$

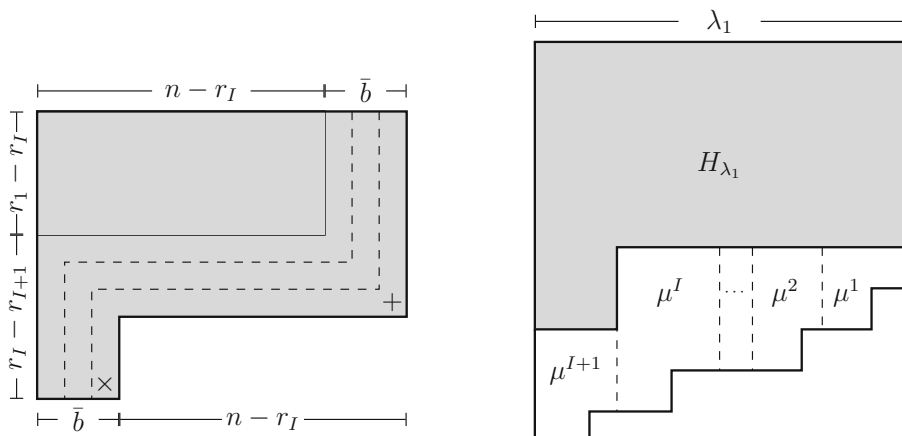


Fig. 3 A q -hook H_b of width b and a skew shape λ/H_{λ_1} with associated tuple of partitions $\bar{\mu}_\lambda = (\mu^1, \dots, \mu^{I+1}, \emptyset, \dots, \emptyset)$

Example 5.2 Consider $\text{QH}^*\text{Fl}(8; 6, 4, 3)$ with $\deg q_1 = 4$, $\deg q_2 = 3$, and $\deg q_3 = 4$. Let I be such that $n - r_I < b \leq n - r_{I+1}$. For the q -hooks of width $3 \leq b \leq 8$ (depicted in Fig. 2), we have:

b	I	H_b	q^{H_b}
3	1	(3, 1)	q_1
4	1	(4, 4)	q_1^2
5	2	(5, 5, 5)	$q_1^2(q_1q_2) = q_1^3q_2$
6	3	(6, 6, 6, 6, 1, 1)	$q_1^2(q_1q_2)(q_1q_2q_3) = q_1^4q_2^2q_3$
7	3	(7, 7, 7, 7, 7, 2)	$q_1^2(q_1q_2)(q_1q_2q_3)^2 = q_1^5q_2^3q_3^2$
8	3	(8, 8, 8, 8, 8, 8)	$q_1^2(q_1q_2)(q_1q_2q_3)^3 = q_1^6q_2^4q_3^3$

Let $R_b := (b^{r_1-r_I+\bar{b}}) = (b^{r_1-(n-b)})$ be the maximal rectangle of width b contained in H_b , with $H_b = R_b = \emptyset$ if $b < n - r_1$.

Remark 5.3 For a partition $\lambda \subseteq r_1 \times n$, let I be such that $n - r_I < \lambda_1 \leq n - r_{I+1}$. Then $R_{\lambda_1} \subseteq \lambda$ if condition (i) below holds, and $H_{\lambda_1} \subseteq \lambda$ if conditions (i) and (ii) below hold.

(i) $\lambda'_{\lambda_1} \geq \lambda_1 - (n - r_1)$

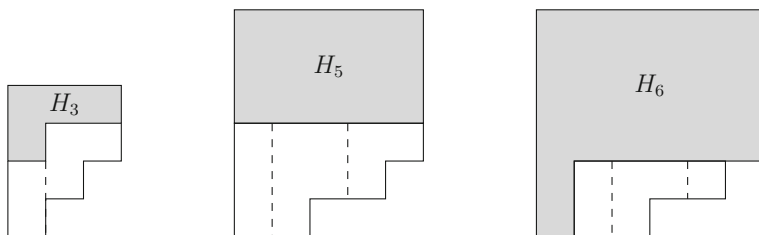


Fig. 4 Partitions η , λ , and ν , skew shapes η/H_3 , λ/H_5 and ν/H_6 , and their associated tuples of partitions $\vec{\mu}_\eta$, $\vec{\mu}_\lambda$ and $\vec{\mu}_\nu$ for $\text{Fl}(8; 6, 4, 3)$

$$(ii) \quad \lambda'_{\lambda_1 - (n - r_I)} \geq r_1 - r_{I+1}.$$

(Note that if $\lambda_1 = n - r_{I+1}$, then condition (ii) is redundant.)

Conditions (i) and (ii) are illustrated in the left diagram of Fig. 3 by λ containing the southeast corner boxes of the q -hook marked by + and \times , respectively. Here, $b = \lambda_1$, $\bar{b} := \lambda_1 - (n - r_I)$, and $r_1 - r_I + \bar{b} = \lambda_1 - (n - r_1)$.

Definition 5.4 A partition $\lambda \subseteq r_1 \times n$ is *compatible with a q -hook* if $H_{\lambda_1} \subseteq \lambda$, i.e. conditions (i) and (ii) of Remark 5.3 holds.

Remark 5.5 The partition H_b has height $r_1 - r_{I+1}$. By convention, $r_0 = n$, so when $0 = n - r_0 < b \leq n - r_1$, H_b is the empty partition, and so every partition λ of width at most $n - r_1$ is compatible with a q -hook.

For a partition $\lambda \subseteq r_1 \times n$ that is compatible with a q -hook, define partitions μ^1, \dots, μ^I by subdividing the skew shape λ/H_{λ_1} , where μ^1 is the partition consisting of the rightmost $n - r_1$ columns of H_{λ_1} , μ^2 is the partition consisting of the second rightmost $r_1 - r_2$ columns of H_{λ_1} , etc. If $I < \rho$, let μ^{I+1} be the partition consisting of the leftmost \bar{b} columns. (See Fig. 3.)

Definition 5.6 For a partition $\lambda \subseteq r_1 \times n$ that is compatible with a q -hook, define the *tuple of partitions associated to λ/H_{λ_1}* to be $\vec{\mu}_\lambda = (\mu^1, \dots, \mu^{I+1}, \emptyset, \dots, \emptyset)$ if $I < \rho$ and $\vec{\mu}_\lambda = (\mu^1, \dots, \mu^I)$ if $I = \rho$, as described above (see Figs. 3, 4). Here, $\vec{\mu}_\lambda \in P(n, \mathbf{r})$ since $\mu^\ell \subseteq r_\ell \times (r_{\ell-1} - r_\ell)$ for $1 \leq \ell \leq \rho$.

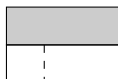
Lemma 5.7 For a partition λ that is compatible with a q -hook, let w be the (321-avoiding) permutation corresponding to $(\lambda/H_{\lambda_1}, \omega)$ with labeling $\omega(i, j) = r_1 + i - j$ under the bijection in [3]. Then w is equal to the permutation corresponding to the tuple $\vec{\mu}_\lambda$ via the bijection described in Remark 2.1. Moreover, w is either Grassmannian with descent at r_{I+1} or has descents at exactly r_I and r_{I+1} , where I is such that $n - r_I < \lambda_1 \leq n - r_{I+1}$.

Proof A reduced expression for the (321-avoiding) permutation w corresponding to $(\lambda/H_{\lambda_1}, \omega)$ is given by [3] as the product of simple transpositions obtained from reading the labeling from bottom to top, beginning with the rightmost column. This product respects the subdivision of λ/H_{λ_1} into the tuple of labeled partitions $\mu^1, \dots, \mu^I, \mu^{I+1}$ with labeling $\omega^\ell(i, j) = r_\ell + i - j$ for $1 \leq \ell \leq I + 1$. (From Definition 5.6, if $I = \rho$, then the tuple consists of only μ^1, \dots, μ^I .) Again, by [3] (see also [15]), a reduced word for μ^ℓ is the product of simple transpositions obtained by reading the labeling of μ^ℓ from bottom to top, beginning with the rightmost column. Concatenating these expressions recovers w . Moreover, define

the partition $\mu^{[I]}$ to be the partition obtained by appending the partitions μ^I, \dots, μ^1 together; this consists of the last $n - r_I$ columns of λ/H_{λ_1} . Let w^{I+1} and $w^{[I]} := w^1 \dots w^I$ be the Grassmannian permutations associated to μ^{I+1} and $\mu^{[I]}$; these have possible descents at r_{I+1} and r_I , respectively, and so their product has possible descents at only r_{I+1} and r_I . \square

Remark 5.8 For a partition λ that is compatible with a q -hook with corresponding tuple $\vec{\mu}_\lambda$ and permutation w , we denote the associated Schubert class by $\sigma_{\vec{\mu}_\lambda}$, σ_w , or simply $\sigma_{\lambda/H_{\lambda_1}}$.

Example 5.9 Consider $\text{Fl}(4; 2, 1)$ as in Example 5.1. The partition $(3, 3)$ is compatible with the q -hook $H_3 = (3, 0)$.



The associated tuple of partitions (\square, \square) is read from right to left from the skew shape $(3, 3)/H_3$.

Example 5.10 Consider $\text{Fl}(8; 6, 4, 3)$ as in Example 5.2 and partitions $\eta = (3, 3, 3, 2)$, $\lambda = (5, 5, 5, 5, 4, 2)$ and $\nu = (6, 6, 6, 6, 5, 3)$. Then η is compatible with the q -hook $H_3 = (3, 1)$, λ is compatible with the q -hook $H_5 = (5, 5, 5)$ and ν is compatible with the q -hook $H_6 = (6, 6, 6, 6, 1, 1)$. The associated tuples of partitions to η/H_3 , λ/H_5 and ν/H_6 are $\vec{\mu}_\eta = (\square, \square, \emptyset)$, $\vec{\mu}_\lambda = (\square, \square, \square)$ and $\vec{\mu}_\nu = (\square, \square, \square)$, as seen in Fig. 4 by reading

the associated tuple of partitions from right to left. Note that $I = \rho = 3$ in Definition 5.6 for ν since $n - r_3 = 5 < \nu_1$. Also note that as in Lemma 5.7 and Example 2.2, the first permutation has descents at $r_1 = 6$ and $r_2 = 4$ and the other two permutations are Grassmannian with descent at $r_3 = 3$.

Before proving Theorem B, we introduce and study the following auxiliary partitions.

Definition 5.11 Given a partition $\lambda \subseteq r_1 \times n$ and $1 \leq m \leq \lambda_1$, define $\lambda^{(m)}$ to be the partition obtained from λ by removing column m from λ and adding 1 to columns $1, \dots, m - 1$, i.e.

$$(\lambda^{(m)})' = (\lambda'_1 + 1, \dots, \lambda'_{m-1} + 1, \lambda'_{m+1} + 1, \dots, \lambda'_{\lambda_1}),$$

where λ' is the transpose of λ , i.e. $(\lambda^{(m)})'_i = \lambda'_i + 1$ for $i < m$ and $(\lambda^{(m)})'_i = \lambda_{i+1}$ for $i \geq m$. (See Fig. 5.)

Lemma 5.12 If a partition $\lambda \subseteq r_1 \times n$ is compatible with a q -hook, then $\lambda^{(m)}$ is compatible with a q -hook for $1 \leq m \leq \lambda_1$.

Proof This follows from Remark 5.3 and Definition 5.11, where conditions (i) and (ii) of Remark 5.3 for $\lambda^{(m)}$ are illustrated by \otimes and \oplus in Fig. 5. \square

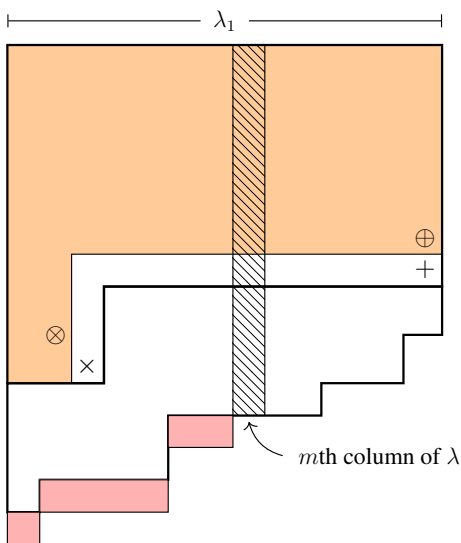
For a partition $\lambda \subseteq r_1 \times n$, consider $s_\lambda^1 := \det(s_{\lambda'_i + j - i}^1)$ and the determinants Δ as in (5) and (7).

Proposition 5.13 Given a partition $\lambda \subseteq r_1 \times n$ of width $b := \lambda_1$, let I be such that $n - r_I < b \leq n - r_{I+1}$ and let $\bar{b} = b - (n - r_I)$. Then

$$\sum_{m=1}^b (-1)^{m-1} s_{\lambda'_m - m + 1}^1 * \Delta_{\lambda^{(m)}/H_{b-1}}(\bar{\psi}) = q_1 \dots q_I \Delta_{\lambda/H_b}(e^q(\phi)) \text{ in } \text{QH}^* \text{Fl}(n; \mathbf{r}),$$

where $\phi = ((I+1)^{\bar{b}}, I^{r_{I-1}-r_I}, \dots, 1^{n-r_I})$ and $\bar{\psi} = ((I+1)^{\bar{b}-1}, I^{r_{I-1}-r_I}, \dots, 1^{n-r_I})$.

Fig. 5 The skew shape λ/H_{λ_1} and the skew shape $\lambda^{(m)}/H_{\lambda_1-1}$



Proof From (5), we have

$$\Delta_{\lambda/H_b}(\phi) = \det \left(e_{\lambda'_i - (H_b)'_j + j - i}^q(\phi_j) \right) =: \det[v_1, \dots, v_b] \quad (15)$$

$$\Delta_{\lambda^{(m)}/H_{b-1}}(\bar{\psi}) = \det \left(e_{(\lambda^{(m)})'_i - (H_{b-1})'_j + j - i}^q(\bar{\psi}_j) \right), \quad (16)$$

where ϕ and $\bar{\psi}$ are as in the statement of the proposition, and where we write v_j for the j th column of the matrix in (15). (Note that λ and $\lambda^{(m)}$ need not contain H_b and H_{b-1} , respectively.) Since $s_{1a}^1 = e_a^q(r_1)$ in $\text{QH}^* \text{Fl}(n; \mathbf{r})$, the left hand quantity of the proposition can be rewritten as the determinant:

$$\det \left(e_{\lambda'_i - (0, (H_{b-1})')_j + j - i}^q(1, \bar{\psi}_j) \right)_{i,j} \quad (17)$$

where $(1, \bar{\psi}) = (1, (I+1)^{\bar{b}-1}, I^{r_{I-1}-r_I}, \dots, 1^{n-r_1})$. We proceed by reordering the columns of this matrix and then comparing the resulting determinant to (15). More concretely, let $\tau \in S_b$ be the permutation defined by

$$\tau(j) = \begin{cases} \bar{b} + r_1 - r_I & \text{if } j = 1 \\ j & \text{if } 2 \leq j \leq \bar{b} \\ \bar{b} + r_{\ell+1} - r_I + 1 & \text{if } j = \bar{b} + r_{\ell-1} - r_I \text{ for } 1 \leq \ell < I \\ 1 & \text{if } j = \bar{b} + r_{I-1} - r_I \\ j + 1 & \text{otherwise.} \end{cases}$$

Reordering columns using the permutation τ , from the description of H'_b and H'_{b-1} in (13), (17) is equal to $\text{sgn}(\tau)$ times

$$\det \left(e_{\lambda'_i - \kappa_j + j - i}^q(\psi_j) \right)_{1 \leq i, j \leq b} =: \det[\bar{v}_1, \dots, \bar{v}_b], \quad (18)$$

where $\kappa = H'_b - (r_I - r_{I+1})\mathbf{e}_1 - \sum_{1 \leq \ell < I} (r_{\ell-1} - r_\ell)\mathbf{e}_{\bar{b}+1+r_\ell-r_I}$ and $\psi = \phi - \mathbf{e}_1 - \sum_{1 \leq \ell < I} \mathbf{e}_{\bar{b}+1+r_\ell-r_I}$. Here, \mathbf{e}_j denotes the sequence that is 1 in position j and 0 elsewhere,

and \bar{v}_j is the j th column of the determinant in (18). Thus, column v_j of (15) is equal to column \bar{v}_j of (18) except when $j = 1$ or $j = \bar{b} + 1 + (r_{\ell-1} - r_I)$ with $1 \leq \ell < I$.

We rewrite (3) as

$$e_a^q(r_\ell) = e_a^q(r_{\ell-1}) - \left(\sum_{m=1}^{r_{\ell-1}-r_\ell} \sigma_m^l e_{a-m}^q(r_\ell) \right) + (-1)^{r_{\ell-1}-r_\ell} q_\ell e_{a-(r_{\ell-1}-r_{\ell+1})}^q(r_{\ell+1}). \quad (19)$$

Note that for $l = 1$, the first term vanishes since $e_a^q(r_0) = e_a^q(n) = 0$ is a relation in the quantum cohomology ring for all a .

We now describe the transition matrix between vectors v_j and v'_j . Consider the $b \times b$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} -\sigma_{\bar{b}+r_{I-1}-r_I+1-i}^I & \text{if } j = 1 \\ -\sigma_{\bar{b}+r_{\ell-1}-r_I+1-i}^l & \text{if } j = \bar{b} + 1 + r_{\ell+1} - r_I \text{ for } 1 \leq \ell < I \\ 0 & \text{otherwise,} \end{cases} \quad (20)$$

with the convention that $\sigma_0^l = 1$ and $\sigma_i^l = 0$ for $i < 0$ and $i > r_{\ell-1} - r_\ell$. Then A is a lower triangular matrix with zeros along the diagonal. Let $D = (d_{ij})$ be the $b \times b$ diagonal matrix with entries

$$d_{jj} = \begin{cases} (-1)^{r_{I-1}-r_I} q_I & \text{if } j = 1 \\ (-1)^{r_{\ell-1}-r_\ell} q_\ell & \text{if } j = \bar{b} + 1 + r_{\ell+1} - r_I \text{ for } 1 \leq \ell < I \\ 1 & \text{otherwise.} \end{cases} \quad (21)$$

With this notation, the relation between the vectors v_j and v'_j is given by matrix multiplication

$$[\bar{v}_1, \dots, \bar{v}_b] = (A + D)[v_1, \dots, v_b].$$

Since $A + D$ is lower triangular with diagonal entries d_{jj} , $\det(A + D) = \prod_{\ell=1}^I (-1)^{r_{\ell-1}-r_\ell} q_\ell$, and so

$$\det[\bar{v}_1, \dots, \bar{v}_b] = (-1)^{n-r_1} q_1 \dots q_I \cdot \det[v_1, \dots, v_b]$$

and hence by (17) and (18), the left hand side of the proposition is equal to

$$(-1)^{n-r_1+I} q_1 \dots q_I \operatorname{sgn}(\tau) \det[v_1, \dots, v_b].$$

Since the signature $\operatorname{sgn}(\tau)$ of the permutation τ is $(-1)^{n-r_I}$, we conclude that (18) is equal to $q_1 \dots q_I$ times the determinant (15), as needed. \square

6 Proof of Theorem B

In this section, we use the set up and results from Sect. 5, including Proposition 5.13, to prove Theorem B, which we restate here.

Theorem B *Let $\lambda \subseteq r_1 \times n$ be a partition, and let I be such that $n - r_I < \lambda_1 \leq n - r_{I+1}$.*

(a) *If $H_{\lambda_1} \subseteq \lambda$, then*

$$s_\lambda^1 = q^{H_{\lambda_1}} \sigma_{\vec{\mu}} \text{ in } \operatorname{QH}^* \operatorname{Fl}(n; \mathbf{r}),$$

where $\vec{\mu} = (\mu^1, \dots, \mu^{I+1}, \emptyset, \dots, \emptyset)$ is the tuple of partitions associated to λ above. In particular, $s_{H_b}^1 = q^{H_b}$ since $H_b/H_b = \emptyset$, so $\mu^j = \emptyset$ for all j and $\sigma_{(\emptyset, \dots, \emptyset)} = 1$.

(b) If λ contains R_{λ_1} , but $H_{\lambda_1} \not\subseteq \lambda$, then

$$s_{\lambda}^1 = 0 \text{ in } \text{QH}^* \text{Fl}(n; \mathbf{r}).$$

Example 6.1 For $\text{QH}^* \text{Fl}(4; 2, 1)$, by part (a) of the theorem and Examples 5.1 and 5.9, we have

$$\begin{aligned} s_{\square}^1 &= \sigma_{\square, \emptyset} \\ s_{\square\square}^1 &= q_1 \\ s_{\square\square\square}^1 &= q_1 \sigma_{\square\square, \square} \\ s_{\square\square\square\square}^1 &= q_1^2 q_2. \end{aligned}$$

(See also Example 4.8.)

Example 6.2 Consider $\text{QH}^* \text{Fl}(8; 6, 3, 2)$ as in Examples 5.2 and 5.10. By part (a) of the theorem, for the partitions $\eta = (3, 3, 3, 2)$, $\lambda = (5, 5, 5, 5, 4, 2)$ and $\nu = (6, 6, 6, 6, 5, 3)$, we have

$$s_{\eta}^1 = q_1 \sigma_{\square\square\square, \square, \emptyset}, \quad s_{\lambda}^1 = q_1^3 q_2 \sigma_{\square\square\square, \square\square\square, \square}, \quad \text{and} \quad s_{\nu}^1 = q_1^4 q_2^2 q_3 \sigma_{\square\square\square, \square\square\square, \square}.$$

From Remark 2.1 and Example 2.2, we can also write this in terms of the indexing of Schubert classes by permutations as

$$s_{\eta}^1 = q_1 \sigma_{12453768}, \quad s_{\lambda}^1 = q_1^3 q_2 \sigma_{36812457} \quad \text{and} \quad s_{\nu}^1 = q_1^4 q_2^2 q_3 \sigma_{14723568}.$$

On the other hand, for the partition $\gamma = (6, 6, 6, 6, 3)$, we have $s_{\gamma}^1 = 0$ by part (b) of the theorem since γ contains $R_6 = (6, 6, 6, 6)$ but not $H_6 = (6, 6, 6, 6, 1, 1)$.

We now prove part (a) of Theorem B and then use part (a) to prove part (b).

Proof of part (a) of Theorem B We proceed by induction on the width $b := \lambda_1$ of λ . For the base cases, when $0 < b \leq n - r_1$, by Remark 5.5, H_b is the empty partition, and we have the equality $s_{\lambda}^1 = \sigma_{\lambda} = \sigma_{\lambda/\emptyset}$.

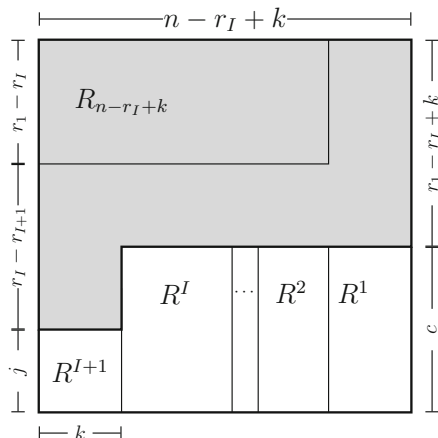
Now assume the result for partitions of width at most $b - 1$. Given a partition $\lambda \subseteq r_1 \times n$, expanding the determinant $s_{\lambda}^1 := \det(s_{\lambda'_i + j - i}^1)$ along the first column gives

$$s_{\lambda}^1 = \sum_{m=1}^b (-1)^{m-1} s_{1^{\lambda'_m - m + 1}}^1 * s_{\lambda^{(m)}}^1. \quad (22)$$

From Lemma 5.12, $\lambda^{(m)}$ is compatible with a q -hook, so by the induction hypothesis, $s_{\lambda^{(m)}}^1 = q^{H_{b-1} \sigma_{\lambda^{(m)}/H_{b-1}}}$, and (22) becomes

$$\begin{aligned} s_{\lambda}^1 &= \sum_{m=1}^b (-1)^{m-1} s_{1^{\lambda'_m - m + 1}}^1 * q^{H_{b-1} \sigma_{\lambda^{(m)}/H_{b-1}}} \\ &= q^{H_{b-1}} * (q_1 \dots q_l \sigma_{\lambda/H_b}) = q^{H_b} \sigma_{\lambda/H_b}, \end{aligned}$$

where the second and third equalities follow from Proposition 5.13, Lemma 5.7, (6), and (14). \square

Fig. 6 The partitioning of λ_a 

We can now prove Proposition 4.7.

Proof of Proposition 4.7 Choose a vertex in the $(I + 1)^{th}$ step of the ladder quiver (choosing this notation for compatibility with Theorem B), and let z_v be the associated variable in the EHX mirror. We need to show that

$$\pi(F_{\text{Gr}}(\phi_{\text{Gr}}(z_v))) = F_{\text{Fl}}(\phi_{\text{Fl}}(z_v)). \quad (23)$$

Suppose v is in the j^{th} row of the k^{th} column of the $(I + 1)^{th}$ block of the ladder quiver. Then

$$\phi_{\text{Fl}}(z_v) = q_1 \dots q_I \frac{p_{j \times k}^{I+1}}{p_{(j-1) \times (k-1)}^{I+1}}.$$

As in the definition of the Schubert map, for $\ell = 1, \dots, I$, let R_ℓ be the

$$c \times (r_{\ell-1} - r_\ell), \quad c := r_I - r_{I+1} + j - k$$

rectangle, and set $R_{I+1} := j \times k$. Set $\bar{R}_{I+1} := (j - 1) \times (k - 1)$. Set $\vec{\mu}_a := (R_1, \dots, R_{I+1}, \emptyset, \dots, \emptyset)$, $\vec{\mu}_b := (R_1, \dots, \bar{R}_{I+1}, \emptyset, \dots, \emptyset)$, and $\vec{\mu}_2 := (R_1, \dots, R_I, \emptyset, \emptyset, \dots, \emptyset)$. Then the right hand side of (23) is

$$q_1 \dots q_I \frac{\sigma_{\vec{\mu}_a}}{\sigma_{\vec{\mu}_2}} \frac{\sigma_{\vec{\mu}_2}}{\sigma_{\vec{\mu}_b}} = q_1 \dots q_I \frac{\sigma_{\vec{\mu}_a}}{\sigma_{\vec{\mu}_b}}.$$

Next, we compute the left hand side of (23). It isn't hard to see that the vertex under consideration is in the $n - r_I + k$ column and the $r_1 - r_{I+1} + j$ row of the Grassmannian quiver. Let $\lambda_a = (r_1 - r_{I+1} + j) \times (n - r_I + k)$, and let $\lambda_b = (r_1 - r_{I+1} + j - 1) \times (n - r_I + k - 1)$. The left hand side of (23) is therefore

$$\frac{s_{\lambda_a}^1}{s_{\lambda_b}^1}.$$

Both λ_a and λ_b are compatible with a q -hook. The partition λ_a is compatible with the q -hook H_{n-r_I+k} . Note that we can partition λ_a as in Fig. 6 (so, in the notation of Theorem B, $\bar{b} = k$):

Therefore

$$s_{\lambda_a}^1 = q^{H_{n-r_I+k}} \sigma_{\vec{\mu}_1}.$$

Similarly, λ_b is compatible with the q -hook H_{n-r_I+k-1} , and

$$s_{\lambda_b}^1 = q^{H_{n-r_I+k-1}} \sigma_{\bar{\mu}_b}.$$

It finally follows by comparing the q factors that

$$\frac{s_{\lambda_a}^1}{s_{\lambda_b}^1} = q_1 \dots q_I \frac{\sigma_{\bar{\mu}_a}}{\sigma_{\bar{\mu}_b}}$$

as required. \square

We now prove the second part of Theorem B, that for $\lambda \subseteq r_1 \times n$ such that $R_{\lambda_1} \subseteq \lambda$, but $H_{\lambda_1} \not\subseteq \lambda$, $s_{\lambda}^1 = 0$ in QH^*F .

Proof of part (b) of Theorem B We proceed by induction on the width of λ . If $\lambda_1 \leq n - r_1$, then λ is compatible with a q -hook since $H_{\lambda_1} = \emptyset$, so the proposition holds vacuously. Now assume the result holds for partitions of width $b - 1$, and consider a partition λ of width $b := \lambda_1$ that contains R_b but not H_b . Let I be such that $n - r_I < \lambda_1 \leq n - r_{I+1}$.

We use the expansion (22) of the determinant s_{λ}^1 . From Remark 5.3 and Definition 5.11, $\lambda^{(m)}$ contains R_{b-1} for $1 \leq m \leq \lambda_1$. If $\lambda_1 = n - r_{I+1}$, then $H_b = R_b$ and there is nothing to prove, so assume $n - r_I < \lambda_1 < n - r_{I+1}$ and write $\bar{b} = \lambda_1 - (n - r_I)$.

First consider the case where $\bar{b} > 1$ and $\lambda'_{\bar{b}-1} < r_1 - r_{I+1} - 1$. This corresponds to the cell marked \otimes in Fig. 5 not being contained in λ . In this case, $(\lambda^{(m)})'_{\bar{b}-1} < r_1 - r_{I+1}$ for $1 \leq m \leq \lambda_1$ by Remark 5.3 so that $\lambda^{(m)}$ does not contain H_{b-1} . Then by the inductive hypothesis, $s_{\lambda^{(m)}}^1 = 0$. Since all the summands in (22) are zero, $s_{\lambda}^1 = 0$.

Now if $\bar{b} > 1$ and $\lambda'_{\bar{b}-1} \geq r_1 - r_{I+1} - 1$, i.e. λ contains the cell marked \otimes in Fig. 5, then by Remark 5.3 and Definition 5.11, if $m < \bar{b}$, then $\lambda^{(m)}$ does not contain H_{b-1} , so by the inductive hypothesis, $s_{\lambda^{(m)}}^1 = 0$. On the other hand, if $m \geq \bar{b}$, then $\lambda^{(m)}$ contains H_{b-1} , and so by part (a) of Theorem B, the expansion (22) becomes

$$s_{\lambda}^1 = \sum_{m=\bar{b}}^b (-1)^{m-1} s_{\lambda^{(m-m+1)}}^1 * q^{H_{b-1}} \sigma_{\lambda^{(m)}/H_{b-1}}.$$

Since $H_{b-1} \not\subseteq \lambda^{(m)}$ for $m < \bar{b}$, by Remark 2.4 and (6), we have

$$s_{\lambda}^1 = \sum_{m=1}^b (-1)^{m-1} s_{\lambda^{(m-m+1)}}^1 * q^{H_{b-1}} \Delta_{\lambda^{(m)}/H_{b-1}}(e^q(\psi)), \quad (24)$$

where $\psi = ((i+1)^{\bar{b}-1}, i^{r_{I-1}-r_I}, \dots, 1^{n-r_1})$. Similarly, if $\bar{b} = 1$, then $H_{b-1} = R_{b-1}$ and $\lambda^{(m)}$ contains H_{b-1} for all $1 \leq m \leq b$, so by part (a) of Theorem B and (6), we have (24) as well. Moreover, by Proposition 5.13, (24) becomes

$$s_{\lambda}^1 = q_1 \dots q_I \Delta_{\lambda/H_b}(e^q(\phi)) \text{ in } \text{QH}^* \text{Fl}(n; \mathbf{r}),$$

where $\phi = ((i+1)^{\bar{b}}, i^{r_{I-1}-r_I}, \dots, 1^{n-r_1})$. Since $H_b \not\subseteq \lambda$, we conclude that $s_{\lambda}^1 = q_1 \dots q_I \cdot 0 = 0$ by Remark 2.4. \square

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