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Equilibrium states for non-uniformly expanding skew products

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Abstract. We study equilibrium states for a class of non-uniformly expanding skew products, and show how a family of fiberwise transfer operators can be used to define the conditional measures along fibers of the product. We prove that the pushforward of the equilibrium state onto the base of the product is itself an equilibrium state for a Hölder potential defined via these fiberwise transfer operators.

Key words: equilibrium states, RPF transfer operator, non-uniformly expanding, skew products
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1. *Introduction and main results*

The study of equilibrium states for uniformly hyperbolic dynamical systems via thermodynamic formalism dates back to Ruelle, Sinai, and Bowen in the mid-1970s. They proved the existence and uniqueness of an equilibrium state for Hölder continuous potentials by (semi)conjugating with two-sided subshifts of finite type (SFTs) via Markov partitions. The Ruelle–Perron–Frobenius (RPF) operator has been a key tool in proving the existence of equilibrium states. In particular, it is well known that given a Hölder potential $\varphi\colon X \to \mathbb{R}$ on an expanding dynamical system (X, f), the eigendata of the RPF operator

$$\mathcal{L}_\varphi \psi(x) = \sum_{\overline{x} \in f^{-1}(x)} e^{\varphi(\overline{x})} \psi(\overline{x}),$$

where $\psi\colon X \to X$ is a continuous function, uniquely determines the equilibrium state.

The purpose of this paper is to understand the construction of equilibrium states for higher-dimensional spaces. Let X and Y be compact connected manifold. We will consider equilibrium states for certain non-uniformly expanding skew products $F(x, y) = (fx, g_x y)$ on $X \times Y$. In the 1950s, Rohklin [10] proved that every measure can be disintegrated along measurable partitions into a unique family of conditional measures. Denote by Y_x the vertical fiber above $x \in X$. We will show that the conditional measures given by Rohklin can be obtained via a family of fiberwise transfer operators $\mathcal{L}_x \colon C(Y_x) \to C(Y_{fx})$ defined such that for any $\psi \in C(Y_x)$,

$$\mathcal{L}_x \psi(fx, y) = \sum_{\bar{y} \in g_x^{-1} y} e^{\varphi(x,\bar{y})} \psi(x, \bar{y}).$$

In the setting of random dynamics, Kifer [6] used these operators to prove the existence and uniqueness of equilibrium states that satisfy a fiberwise scaling property on almost every fiber if the system exhibits expansion on average. Denker and Gordin were able to strengthen this to results on every fiber in expanding systems. In their 1999 paper [4], they used these fiberwise transfer operators to show that if a fibred system, a class of systems including skew products, is uniformly expanding and topologically exact along fibers, then given a Hölder potential $\varphi\colon X \times Y \to \mathbb{R}$, there is a unique equilibrium state on $X \times Y$ that has conditionals defined by a fiberwise Gibbs property and whose transverse measure on X is a Gibbs measure for a certain Hölder potential on X.

Varandas and Viana [12] and Castro and Varandas [2] studied equilibrium states for non-uniformly expanding maps. In particular, [2] proved that for a certain class of non-uniformly expanding maps on compact, connected manifolds equipped with almost constant Hölder potentials, eigendata for the Ruelle operator acting on the space of Hölder potentials can be used to construct a unique equilibrium state μ on $X \times Y$. This gives us existence and uniqueness of an equilibrium state on $X \times Y$. Pollicott and Kempton [9] and Piraino [8] give conditions for when Gibbsianness is preserved for factors of SFTs. Given a Gibbs measure for a Hölder (or Walters) potential and a factor of the SFT, it has been shown that the potential

$$\Phi(x) = \lim_{n\to\infty} \log \frac{\langle \mathcal{L}_x^{n+1} \mathbb{1}, \sigma\rangle}{\langle \mathcal{L}_{fx}^{n} \mathbb{1}, \sigma\rangle} \tag{1}$$

exists independent of probability measure σ on Y, is Hölder (or Walters, respectively) and that the pushforward of the Gibbs measure is Φ-Gibbs.

In §2 we recall some background on skew products, describe our non-uniform expansion assumptions, and state our main result (see Theorem A). In §3, we use arguments similar to Piraino [8] to prove the existence and regularity of Φ using the Hilbert metric and the Birkhoff contraction theorem on convex cones. This requires new arguments since the non-uniform expansion in the fiberwise maps $\{g_x\}$ makes estimates on distances between preimages more complex. The proofs in this paper are similar to those in Stadlbauer, Suzuki and Varandas [11] and Hafouta [5]. In [11], Stadlbauer *et al* showed that hyperbolic potentials for random non-uniformly expanding dynamical systems admit an equilibrium state whose disintegration satisfies a weak-Gibbs property almost everywhere along the base. In §3.4 we describe a process of coding the orbits of points in $X \times Y$ that are similar to the ideas of hyperbolic time in [11]. We also show the existence of a family of Gibbs measures in the sense of [4] (see Theorem 3.6). In §4 we complete the proof of Theorem A by establishing that $\hat{\mu}$ is the equilibrium state for Φ and the existence of fiberwise measures that form the unique family of conditional measures for the equilibrium state μ.

We remark that the conditional measures constructed here should satisfy a weak Gibbs property similar to that in Stadlebauer, Suzuki and Varandas [11]. However, they assume that almost every fiber is uniformly topologically exact, which is key to their proofs. We do not assume this but note that a similar property should follow from topological exactness in the product. The author also expects that it can be shown that the family of conditionals measures we construct in §4 uniquely achieves the relative pressure of the system. The proofs in [**11**] rely on hyperbolic times so the proofs of a relative variational principle would need to be adapted in the setting of the current paper.

2. *Non-uniformly expanding skew products*

Let X and Y be compact, connected Riemannian manifolds. Consider the product space $X \times Y$. We will refer to X as the base and $\{Y_x = \{x\} \times Y\}_{x \in X}$ as the fibers of the product since

$$X \times Y = \bigcup_{x \in X} \{x\} \times Y.$$

Note that each fiber Y_x can be identified with Y. We will make the necessary distinctions as needed. Denote by d the L_1 distance on $X \times Y$ and by π_X and π_Y the natural projection maps from $X \times Y$ onto X and Y, respectively.

2.1. *Dynamics of skew products.*

Let F be a continuous skew product on $X \times Y$; that is, there are continuous maps $f\colon X \to X$ and $\{g_x\colon Y \to Y \mid x \in X\}$ such that

$$F(x, y) = (f(x), g_x(y)).$$

To understand the dynamics of F on $X \times Y$, define for any $n \geq 0$ and $x \in X$, the map

$$g_x^n := g_{f^{n-1}x} \circ \cdots \circ g_x\colon Y_x \to Y_{f^n x}.$$

Then for any $(x, y) \in X \times Y$, the behavior of the system can be investigated through the sequence $F^n(x, y) = (f^n(x), g_x^n(y))$.

For each $n \geq 0$, define the nth Bowen metric as

$$d_n((x, y), (x', y')) = \max_{0 \leq i \leq n} \{d(F^i(x, y), F^i(x', y'))\}.$$

Also denote the nth Bowen ball centered at (x, y) of radius $\delta > 0$ by

$$B_n((x, y), \delta) = \{(x', y')\colon d_n((x, y), (x', y')) < \delta\}.$$

2.2. *Non-uniform expansion along fibers.*

The following describes our assumptions of non-uniform expansion along the fibers on $X \times Y$. We shall assume that F is a topologically exact local homeomorphism and that there is a continuous function $(x, y) \mapsto L(x, y)$ such that the following statements hold.

(A1) There exists open $U \ni (x, y)$ such that $F|_U$ is invertible and

$$d(F^{-1}(u_1, u_2), F^{-1}(v_1, v_2)) \leq L(x, y) d((u_1, u_2), (v_1, v_2))$$

for all $(u_1, u_2), (v_1, v_2) \in F(U)$.

(A2) There exist constants $\gamma > 1$ and $L \geq 1$, and an open region $\mathcal{A} \subset X \times Y$ such that $L(x, y) \leq L$ for every $(x, y) \in \mathcal{A}$ and $L(x, y) < \gamma^{-1}$ for all $(x, y) \notin \mathcal{A}$, and L is close enough to 1 so that inequality (6) below is satisfied.

Assumption (A1) gives us control of preimages in the image of small balls for which the product map is invertible. Assumption (A2) says that F is uniformly expanding outside of some region $\mathcal{A}$ and not too contracting in $\mathcal{A}$. Thus, if $\mathcal{A}$ is empty, then everything is reduced to the uniformly expanding case.

We say that an open cover $\mathcal{U}$ of $X \times Y$ *separates curves* if, given a distance-minimizing geodesic segment c on $X \times Y$, each element of $\mathcal{U}$ can intersect at most one curve in $F^{-1}(c)$. Denote by $\mathbb{U}$ the collection of open covers $\mathcal{U}$ of $X \times Y$ that separates curves and each $U \in \mathcal{U}$ satisfies assumption (A1). We further assume the following.

(A3) There exists a finite covering $\mathcal{U} \in \mathbb{U}$ such that $\mathcal{A}$ can without loss of generality be covered by the first $q < \deg(F)$ elements of $\mathcal{U}$.

(A4) For every $x \in X$, there exists a finite covering $\mathcal{U}_x$ of Y_x which separates curves by sets in $\mathbb{U} \cap Y_x$ and $\mathcal{A} \cap Y_x$ can be covered by the first $q < d$ elements of $\mathcal{U}_x$.

Assumption (A3) ensures that every point has at least one preimage in the expanding region. Note that (A3) is a strengthened version of assumption H2 from Castro and Varandas [2]. Assumption (A4) guarantees that every point in a single fiber has at least preimage in the expanding region. This will be crucial to the arguments in §3 as discussed below.

A map $f: X \to X$ is *uniformly expanding* if there exist $C, \delta_f > 0$ and $\gamma > 1$ such that

$$d(f^n(x), f^n(x')) \geq C\gamma^n d(x, x')$$

whenever $d_n(x, x') \leq \delta_f$. One can assume without loss of generality that $C = 1$ by passing to an adapted metric. This reduces the expanding property to

$$d(f(x), f(x')) \geq \gamma d(x, x')$$

whenever $d(x, x') \leq \delta_f$.

LEMMA 2.1. *If $F: X \times Y \to X \times Y$ is a skew product that satisfies assumptions (A1)–(A4), then f is uniformly expanding.*

Proof. Fix $(x, y) \in X \times Y$. Choose $x' \in X$ such that $(f(x'), g_x(y)) \in F(U(x, y))$. Since F is a local homeomorphism, there exists $y' \in Y$ such that $(x', y') \in U(x, y)$ and $F(x', y') = (f(x'), g_x(y))$. Thus,

$$\begin{aligned} d(x, x') &\leq d((x, y), (x', y')) \\ &\leq L(x, y) d(F(x, y), F(x', y')) \\ &\leq L(x, y) d((f(x), g_x(y)), (f(x'), g_x(y))) \\ &\leq L(x, y) d(fx, fx'). \end{aligned}$$

This finishes the proof since (A4) implies $\inf_{y \in Y} L(x, y) < 1$. □

We remark that it would be interesting to see how the proofs in this paper would need to be changed if we removed assumption (A4). In such a setting, the base map could be non-uniformly expanding. It is known that the geometric potential for the Manneville–Pomeau map is Hölder continuous and has two equilibrium states. Uniqueness of the equilibrium state on the base is crucial to the proofs in §4 about the conditional measures of the equilibrium state μ. It is well known that Hölder potentials associated to expanding maps admit a unique equilibrium states. The uniform expansion in the base given by Lemma 2.1 is essential to proving that the potential Φ defined in equation (1) is Hölder continuous.

As a consequence of Lemma 2.1, we see that the number of preimages along fibers must be fixed. Indeed, since F is a local homeomorphism on a compact connected manifold, F is a covering map. Thus, we have that $\deg(F)$ is constant. Similarly, since f is expanding on a compact connected manifold onto itself, f is a covering map. Thus, $\hat{d} := \deg(f)$ is constant in x. Thus, $d := \deg(g_x)$ is constant for all $x \in X$ and $y \in Y_x$ and $\deg(F) = \hat{d}d$.

The following example shows that there is a robust class of systems that satisfies the given assumptions.

Example 2.2. The Manneville–Pomeau map $y \mapsto y + y^{p+1} \bmod \mathbb{Z}$ $(p > 0)$ on $\mathbb{S}^1$ is a classic example of a system that displays non-uniform expansion. Define a map $F\colon X \times Y \to X \times Y$ by taking the base map f to be the doubling map on $\mathbb{S}^1$ and Manneville–Pomeau maps $g_x(y) = y + y^{p(x)+1} \bmod \mathbb{Z}$ in the fibers where $p(x) > 0$ varies continuously in the base point. Each of these maps has two branches so $d = 2$. Note that $g'_x(y) > g'_x(0) = 1$ for all $y \neq 0$. Let $\mathcal{A}$ be any small neighborhood around $\mathbb{S}^1 \times \{0\} \subset \mathbb{T}^2$. Then on $\mathcal{A}^c$ the product map F does not decrease distances; that is, if $(x, y), (x', y') \in \mathcal{A}^c$, then

$$d((f(x), g_x(y)), (f(x'), g_{x'}(y'))) \geq \gamma d((x, y), (x', y')).$$

So $q = 1$. Then $F(x, y) = (f(x), g_x(y))$ satisfies assumptions (A1) and (A2) and thus Theorem A holds for this example.

LEMMA 2.3. *If F satisfies (A1) and (A2), then for any $x, x' \in X$ and $y, y' \in Y$, we can pair off the preimages of $g_x^{-1}(y) = \{y_1, \ldots, y_d\}$ and $g_{x'}^{-1}(y') = \{y'_1, \ldots, y'_d\}$ where for any $k = 1, 2, \ldots, q$,*

$$d((x, y_k), (x, y'_k)) \leq L d((fx, y), (fx', y'))$$

while for any $k = q + 1, \ldots, d$,

$$d((x, y_k), (x', y'_k)) \leq \gamma^{-1} d((fx, y), (fx', y')).$$

Proof. Let $(x, y), (x', y') \in X \times Y$ and c be a distance-minimizing geodesic segment between these points. Let $g_x^{-1}(y) = \{y_1, \ldots, y_d\}$. Since F is a covering map, we can uniquely lift c to curves $c_1, \ldots, c_d$ such that each c_k starts at y_i and $F(c_k) = c$ for all k. Then, letting y'_k be the other endpoint of c_k, we get a collection of preimages $g_{x'}^{-1}(y') = \{y'_1, \ldots, y'_d\}$. Cover each c_k by domains of injectivity as in (A2). Then at most q of these balls can intersect $\mathcal{A}$ and each one intersects at most one of the curves c_k. Thus there are at most q curves c_k that intersect $\mathcal{A}$. Without loss of generality, we can assume that these are the first q preimages. Applying (A1) gives the desired result. □

2.3. *Existence and uniqueness of equilibrium states.* We say $\varphi\colon X \times Y \to \mathbb{R}$ is *α-Hölder continuous* for some $\alpha > 0$ if

$$|\varphi|_\alpha := \sup_{(x,y)\neq(x',y')} \frac{|\varphi(x, y) - \varphi(x', y')|}{d((x, y), (x', y'))^\alpha} < \infty.$$

We denote by $C^\alpha(X \times Y)$ the Banach space of α-Hölder continuous functions on $X \times Y$. The nth Birkhoff sum is defined as $S_n\varphi(x, y) = \sum_{k=0}^{n} \varphi \circ F^k(x, y)$.

We denote by $\mathcal{M}(X \times Y)$ the space of Borel probability measures on $X \times Y$ and $\mathcal{M}(X \times Y, F)$ those that are F-invariant. Given a continuous map $F\colon X \times Y \to X \times Y$ and a potential $\varphi\colon X \times Y \to \mathbb{R}$, the variational principle asserts that

$$P(\varphi) = \sup\left\{h_\nu(F) + \int \varphi\, d\nu\colon \nu \in \mathcal{M}(X \times Y, F)\right\} \tag{2}$$

where $P(\varphi)$ denotes the topological pressure of F with respect to φ and $h_\mu(F)$ denotes the metric entropy of F. An *equilibrium state* for F with respect to φ is an invariant measure that achieves the supremum in the right-hand side of equation (2). For uniformly expanding maps, every equilibrium state μ satisfies the Gibbs property: for any $\varepsilon > 0$, there exists a $C > 0$ such that

$$C^{-1} \le \frac{\mu(B_n((x, y), \varepsilon))}{e^{-nP(\varphi)+S_n\varphi(x,y)}} \le C$$

for any $(x, y) \in X \times Y$ and $n \in \mathbb{N}$.

For our purposes in this paper, we fix a Holder potential $\varphi \in C^\alpha(X \times Y)$ satisfying

$$\sup \varphi - \inf \varphi < \varepsilon_\varphi \quad \text{and} \quad |e^\varphi|_\alpha < \varepsilon_\varphi e^{\inf \varphi} \tag{P}$$

for some $\varepsilon_\varphi > 0$ satisfying inequality (3) and equation (4) below (see §3). Almost constant potentials satisfy (P) so Theorem A applies to an open set of potentials. In particular, Theorem A holds for measures of maximal entropy. We assume that L is close enough to 1 and $0 < \varepsilon_\varphi < \log d - \log q$ so that

$$e^{\varepsilon_\varphi} \cdot \left(\frac{(d-q)\gamma^{-\alpha} + qL^\alpha}{d}\right) < 1. \tag{3}$$

Under these assumptions, it is known that there is a unique equilibrium state μ for φ on $X \times Y$.

LEMMA 2.4. *If F is topologically exact and satisfies (A1), (A2), and φ satisfies* $\sup \varphi - \inf \varphi < \log \deg(F) - \log q$*, then there exists an expanding conformal measure such that $\mathcal{L}_\varphi^* \nu = \lambda\nu$ and* $\operatorname{supp}(\nu) = \overline{X \times Y}$*, where the spectral radius of $\mathcal{L}_\varphi$, $\lambda := r(\mathcal{L}_\varphi) \ge$* $\deg(F)e^{\inf \varphi}$*. Moreover, ν is a non-lacunary Gibbs measure and has a Jacobian with respect to F given by $J_\nu F = \lambda e^{-\varphi}$.*

Proof. See Theorem 4.1 in Varandas and Viana [12]. □

We will not use the non-lacunary property of ν or $J_\nu F$. For more details, see [12].

THEOREM 2.5. *Let $F\colon X \times Y \to X \times Y$ be a local homeomorphism with Lipschitz continuous inverse and $\varphi\colon X \times Y \to \mathbb{R}$ be a Hölder continuous potential satisfying (A1), (A2), and (P). Then the RPF operator has a spectral gap property in the space of Hölder continuous observables, there exists a unique equilibrium state μ for F with respect to φ, and the density $d\mu/d\nu$ is Hölder continuous.*

Proof. See Theorem A in Castro and Varandas [2]. □

Denote by $\hat{\mu} = \mu \circ \pi_X^{-1}$ the pushforward of the equilibrium state μ onto the base X. Throughout this paper, we shall refer to this measure as the transverse measure for our skew product.

Our main result is the following theorem.

THEOREM A. *Let X and Y be compact connected Riemannian manifolds and $(X \times Y, F)$ be a Lipschitz skew product that satisfies assumptions (A1)–(A4). Let φ be a Hölder continuous potential on $X \times Y$ satisfying assumption (P) and μ be its corresponding equilibrium state. Then the following assertions are true.*

(1) *The potential Φ in equation (1) is independent of σ, is Hölder continuous, and satisfies $P(\varphi) = P(\Phi)$.*
(2) $\hat{\mu} = \mu \circ \pi_X^{-1}$ *is the unique equilibrium state for Φ.*
(3) *There is a unique family of measures $\{\nu_x : x \in X\}$ such that $\nu_x(Y_x) = 1$ and*

$$\mathcal{L}_x^* \nu_{fx} = e^{\Phi(x)} \nu_x.$$

(4) $x \mapsto \nu_x$ *is weak*-continuous.*
(5) *Let $\hat{h}$ and $\hat{\nu}$ be the eigendata of $\mathcal{L}_\Phi$, that is, $\mathcal{L}_\Phi^* \hat{\nu} = e^{P(\Phi)} \hat{\nu}$, $\mathcal{L}_\Phi \hat{h} = e^{P(\Phi)} \hat{h}$, and $\int \hat{h} d\hat{\nu} = 1$. Then the measures $\mu_x = (h(x,\cdot)/\hat{h}(x))\nu_x$ are probability measures on Y_x such that*

$$\mu = \int_X \mu_x \, d\hat{\mu}(x).$$

We remark that the existence in item (1) follows closely proofs from Piraino [8] on SFTs. However, the proof of the Hölder continuity of Φ took new ideas on compact, connected manifolds (see §3.4). It is worth noting that Stadlbauer, Varandas and Zhang proved a similar result to item (3) for conformal measure of Ruelle expanding iterated function systems.

2.4. *Fiberwise transfer operators for skew products.* As is common in the literature, we will utilize Ruelle operators to study the equilibrium state on $(X \times Y, F)$. Define the transfer operator $\mathcal{L}_\varphi$ acting on $C(X \times Y)$ by sending $\psi \in C(X \times Y)$ to

$$\mathcal{L}_\varphi \psi(x, y) = \sum_{(\bar{x},\bar{y}) \in F^{-1}(x,y)} e^{\varphi(\overline{x},\overline{y})} \psi(\overline{x}, \overline{y}).$$

Note that under the skew product representation of F, we may write

$$\sum_{(\overline{x},\overline{y}) \in F^{-1}(x,y)} e^{\varphi(\overline{x},\overline{y})} \psi(\overline{x}, \overline{y}) = \sum_{\overline{x} \in f^{-1}x} \sum_{\overline{y} \in g_{\overline{x}}^{-1} y} e^{\varphi(\overline{x},\overline{y})} \psi(\overline{x}, \overline{y}).$$

This gives rise to a fiberwise transfer operator on the fibers of $X \times Y$.

We disintegrate φ and get the family of fiberwise potentials $\{\varphi_x(\cdot) = \varphi(x, \cdot)\}_{x \in X}$. For every $x \in X$, let $\mathcal{L}_x : C(Y_x) \to C(Y_{fx})$ be defined by

$$\mathcal{L}_x \psi_x(y) = \sum_{\overline{y} \in g_x^{-1} y} e^{\varphi_x(\overline{y})} \psi_x(\overline{y})$$

for any $\psi \in C(X \times Y)$. We shall iterate the transfer operator by letting

$$\mathcal{L}_x^n = \mathcal{L}_{f^{n-1}x} \circ \cdots \circ \mathcal{L}_x : C(Y_x) \to C(Y_{f^n x}).$$

Along with each of these fiberwise operators, we define its dual $\mathcal{L}_x^*$ by sending a probability measure $\eta \in \mathcal{M}(Y_{fx})$ to the measure $\mathcal{L}_x^* \eta \in \mathcal{M}(Y_x)$ such that for any $\psi \in C(X \times Y)$,

$$\int \psi \, d(\mathcal{L}_x^* \eta) = \int \mathcal{L}_x \psi \, d\eta.$$

3. *A potential for the transverse measure*

Piraino [8] shows that for some factors of mixing SFTs, a Gibbs measure μ is pushed onto a Gibbs measure $\hat{\mu} = \mu \circ \pi_X^{-1}$ for the potential

$$\Phi(x) = \lim_{n\to\infty} \log \frac{\langle \mathcal{L}_x^{n+1} \mathbb{1}, \sigma \rangle}{\langle \mathcal{L}_{fx}^n \mathbb{1}, \sigma \rangle}$$

where σ is any probability measure supported on Y. We will show in Theorem 3.5 that this potential exists in our setting. Furthermore, in Theorem 3.15 we show that Φ is Hölder continuous.

3.1. *Birkhoff contraction theorem.*

It is not hard to check that the Ruelle operator preserves the Banach space of Hölder continuous potentials $C^\alpha(X \times Y)$, $0 < \alpha < 1$. A subset $\Lambda \subset C^\alpha(X \times Y)$ is called a *cone* if $a\Lambda = \Lambda$ for all $a > 0$. A cone Λ is *convex* if $\psi + \zeta \in \Lambda$ for all ψ, $\zeta \in \Lambda$. We say that Λ is a closed cone if $\Lambda \cup \{0\}$ is *closed* with respect to the Hölder norm. We assume our cones are closed, convex, and $\Lambda \cap (-\Lambda) = \emptyset$. For any probability measure η and Hölder potential ψ, let $\langle \psi, \eta \rangle = \int \psi \, d\eta$. Given a closed cone $\Lambda \subset C^\alpha(X \times Y)$, we can define the dual cone $\Lambda^* = \{\eta \in (C^\alpha(X \times Y))^* : \langle \psi, \eta \rangle \geq 0$ for all $\psi \in \Lambda\}$. For more on cones, see §4 in [7] or the appendices in [8].

Define a partial ordering $\preceq$ on $C^\alpha(X \times Y)$ by saying that $\phi \preceq \psi$ if and only if $\psi - \phi \in \Lambda \cup \{0\}$ for any $\phi, \psi \in C^\alpha(X \times Y)$. Let

$$A = A(\phi, \psi) = \sup\{t > 0 : t\phi \preceq \psi\} \quad \text{and} \quad B = B(\phi, \psi) = \inf\{t > 0 : \psi \preceq t\phi\}.$$

The *Hilbert projective metric* with respect to a closed cone Λ is defined as

$$\Theta(\phi, \psi) = \log \frac{B}{A}.$$

The following lemma is useful when calculating distances in the Hilbert metric. For a proof, see §4 in [7].

LEMMA 3.1. *Let Λ be a closed cone and Λ^* its dual. For any $\phi, \psi \in \Lambda$,*

$$\Theta(\phi, \psi) = \log\bigg(\sup\bigg\{\frac{\langle \phi, \sigma \rangle \langle \psi, \eta \rangle}{\langle \psi, \sigma \rangle \langle \phi, \eta \rangle} : \sigma, \eta \in \Lambda^* \text{ and } \langle \psi, \sigma \rangle \langle \phi, \eta \rangle \neq 0\bigg\}\bigg).$$

The main idea of the proof of Theorem A is to find a cone on which the fiberwise transfer operator is a contraction. To accomplish this, we will need the Birkhoff contraction theorem.

THEOREM 3.2. (Birkhoff [1]) *Let Λ_1, Λ_2 be closed cones and $\mathcal{L}\colon \Lambda_1 \to \Lambda_2$ a linear map such that $\mathcal{L}\Lambda_1 \subset \Lambda_2$. Then for all $\phi, \psi \in \Lambda_1$.*

$$\Theta_{\Lambda_2}(\mathcal{L}\phi, \mathcal{L}\psi) \le \tanh\left(\frac{\operatorname{diam}_{\Lambda_2}(\mathcal{L}\Lambda_1)}{4}\right)\Theta_{\Lambda_1}(\phi, \psi)$$

where $\operatorname{diam}_{\Lambda_2}(\mathcal{L}\Lambda_1) = \sup\{\Theta_{\Lambda_2}(\mathcal{L}\phi, \mathcal{L}\psi)\colon \phi, \psi \in \Lambda_1\}$ *and* $\tanh\infty = 1$.

3.2. *Existence of Φ.* We will use cones of the form

$$\Lambda_K = \Lambda_K(\alpha) = \{\psi \in C^\alpha(X \times Y)\colon \psi > 0 \text{ and } |\psi|_\alpha \le K \inf\psi\} \cup \{0\}.$$

It can be shown that Λ_K is a closed cone in $C^\alpha(X \times Y)$. For these cones, we get an alternate way of calculating distances in the Hilbert metric.

LEMMA 3.3. *For any $\phi, \psi \in \Lambda_K$,*

$$A(\phi, \psi) = \inf_{z_1,z_2,z_3 \in X\times Y} \frac{Kd(z_1, z_2)^\alpha\psi(z_3) - (\psi(z_1) - \psi(z_2))}{Kd(z_1, z_2)^\alpha\phi(z_3) - (\phi(z_1) - \phi(z_2))}$$

and

$$B(\phi, \psi) = \sup_{z_1,z_2,z_3 \in X\times Y} \frac{Kd(z_1, z_2)^\alpha\psi(z_3) - (\psi(z_1) - \psi(z_2))}{Kd(z_1, z_2)^\alpha\phi(z_3) - (\phi(z_1) - \phi(z_2))}.$$

Proof. See Lemma 4.2 in Castro and Varandas [2]. □

Denote by

$$\Lambda_K^x = \{\psi \in C^\alpha(X \times Y)\colon \psi_x(\cdot) > 0 \text{ and } |\psi|_\alpha \le K \inf\psi\} \cup \{0\}$$

the cross-section of Λ_K that lives on Y_x.

Let

$$s := e^{\varepsilon_\varphi} \cdot \left(\frac{(d-q)\gamma^{-\alpha} + qL^\alpha}{d}\right) < 1$$

as in inequality (3). We assume that $\varepsilon_\varphi > 0$ is small enough that

$$\zeta := s + 2s\varepsilon_\varphi \operatorname{diam}(Y)^\alpha < 1. \tag{4}$$

Then we have the following lemma based on similar arguments from Castro and Varandas [2] and Stadlbauer, Suzuki and Varandas [11].

LEMMA 3.4. *With ζ as in (4), for all K sufficiently large, we have $\mathcal{L}_x(\Lambda_K^x) \subset \Lambda_{\zeta K}^{fx}$ for all $x \in X$. Moreover, there is a constant $M = M(K) > 0$ such that for all $x \in X$,* $\operatorname{diam}(\mathcal{L}_x\Lambda_K^x) \le M < \infty$ *with respect to the Hilbert projective metric on Λ_K^{fx}.*

Proof. Fix $x \in X$ and $K > 0$. Denote by $\{y_k\}$ and $\{y'_k\}$ the preimages of y and y' in Y_x, respectively, as given by Lemma 2.3. Now fix $\psi \in \Lambda_K$. Since $\inf \mathcal{L}_x\psi \ge de^{\inf\varphi}\inf\psi$ and

$$\begin{aligned}&\mathcal{L}_x\psi_{fx}(y) - \mathcal{L}_x\psi_{fx}(y') \\ &\quad= \sum_{k=1}^{d}(e^{\varphi(x,y_k)}(\psi(x, y_k) - \psi(x, y'_k)) + (e^{\varphi(x,y_k)} - e^{\varphi(x,y'_k)})\psi(x, y'_k)),\end{aligned}$$

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we have

$$\frac{|\mathcal{L}_x\psi(fx,y)-\mathcal{L}_x\psi(fx,y')|}{\inf \mathcal{L}_x\psi} \le d^{-1}\sum_{k=1}^{d} e^{\varphi(x,y_k)-\inf\varphi}|\psi(x,y_k)-\psi(x,y_k')|(\inf\psi)^{-1}$$
$$+ d^{-1}\sum_{k=1}^{d}(\sup\psi/\inf\psi)e^{-\inf\varphi}|e^{\varphi(x,y_k)}-e^{\varphi(x,y_k')}|$$
$$=: I_1 + I_2.$$

Note that $|\psi(x,y_k)-\psi(x,y_k')| \le |\psi|_\alpha d(y_k,y_k')^\alpha \le K \inf\psi\ d(y_k,y_k')^\alpha$. By Lemma 2.3, $d(y_k,y_k') \le Ld(y,y')$ for any $1\le k\le q$ and $d(y_k,y_k') \le \gamma^{-1}d(y,y')$ for $q<k\le d$, so

$$\begin{aligned} I_1 &\le d^{-1}\sum_{k=1}^{d} e^{\varphi(x,y_k)-\inf\varphi}Kd(y_k,y_k')^\alpha \\ &\le d^{-1}e^{\varepsilon_\varphi}K\sum_{k=1}^{d} d(y_k,y_k')^\alpha \\ &\le Ke^{\varepsilon_\varphi}d^{-1}(L^\alpha q+(d-q)\gamma^{-\alpha})d(y,y')^\alpha \\ &\le sKd(y,y')^\alpha \end{aligned}$$

where the second inequality holds by (P).

To estimate I_2, note that $|e^{\varphi(x,y_k)}-e^{\varphi(x,y_k')}| \le |e^{\varphi_x}|_\alpha d(x,y_k),(x,y_k'))^\alpha$ and

$$\sup\psi \le \inf\psi + |\psi|_\alpha \operatorname{diam}(Y)^\alpha \le (1+K\operatorname{diam}(Y)^\alpha)\inf\psi$$

implies that

$$\sup\psi/\inf\psi \le 1+K\operatorname{diam}(Y)^\alpha \le 2K\operatorname{diam}(Y)^\alpha$$

provided that K is sufficiently large. Then (P) implies that

$$\begin{aligned} I_2 &\le 2K\operatorname{diam}(Y)^\alpha e^{-\inf\varphi}d^{-1}\sum_{k=1}^{d}|e^{\varphi_x}|_\alpha d(x,y_k),(x,y_k'))^\alpha \\ &\le 2K\operatorname{diam}(Y)^\alpha\varepsilon_\varphi d^{-1}\sum_{k=1}^{d}(L^\alpha q+(d-q)\gamma^{-\alpha})d(y,y')^\alpha \\ &\le 2K\operatorname{diam}(Y)^\alpha s\varepsilon_\varphi d(y,y')^\alpha \\ &\le 2s\varepsilon_\varphi\operatorname{diam}(Y)^\alpha Kd(y,y')^\alpha. \end{aligned}$$

Therefore, if we let $\zeta := s+2s\varepsilon_\varphi\operatorname{diam}(Y)^\alpha$, we have that

$$|\mathcal{L}_x\psi|_\alpha \le (s+2s\varepsilon_\varphi\operatorname{diam}(Y)^\alpha)K\inf\mathcal{L}_x\psi \le \zeta K\inf\mathcal{L}_x\psi$$

so $\mathcal{L}_x\psi \in \Lambda^{fx}_{\zeta K}$.

Note that $\sup\mathcal{L}_x\psi \le (1+\zeta K(\operatorname{diam} Y)^\alpha)\inf\mathcal{L}_x\psi$. Let $y_1,y_2,y_3\in Y$. Then since $|\mathcal{L}_x\psi|_\alpha \le \zeta K\inf\mathcal{L}_x\psi$, we have

$$\frac{Kd(y_1, y_2)^\alpha \mathcal{L}_x\psi(y_3) - (\mathcal{L}_x\psi(y_1) - \mathcal{L}_x\psi(y_2))}{Kd(y_1, y_2)^\alpha \mathcal{L}_x\phi(y_3) - (\mathcal{L}_x\phi(y_1) - \mathcal{L}_x\phi(y_2))} \leq \frac{(K \sup \mathcal{L}_x\psi + \zeta K \inf \mathcal{L}_x\psi)d(y_1, y_2)^\alpha}{(K \inf \mathcal{L}_x\phi - \zeta K \inf \mathcal{L}_x\phi)d(y_1, y_2)^\alpha}.$$

Thus, $B(\mathcal{L}_x\psi, \mathcal{L}_x\phi) \leq (K \sup \mathcal{L}_x\psi + \zeta K \inf \mathcal{L}_x\psi)/(K \inf \mathcal{L}_x\phi - \zeta K \inf \mathcal{L}_x\phi)$. A similar calculation gives a lower bound on $A(\mathcal{L}_x\psi, \mathcal{L}_x\phi)$. So by Lemma 3.3, we have

$$\begin{aligned}\Theta(\mathcal{L}_x\psi, \mathcal{L}_x\phi) &\leq \log\bigg(\frac{K \sup \mathcal{L}_x\phi + \zeta K \inf \mathcal{L}_x\phi}{K \inf \mathcal{L}_x\phi - \zeta K \inf \mathcal{L}_x\phi} \cdot \frac{K \sup \mathcal{L}_x\psi + \zeta K \inf \mathcal{L}_x\psi}{K \inf \mathcal{L}_x\psi - \zeta K \inf \mathcal{L}_x\psi}\bigg) \\ &\leq \log\bigg(\frac{K(1 + \zeta K \operatorname{diam}(Y)^\alpha)(1+\zeta) \inf \mathcal{L}_x\phi}{K(1-\zeta) \inf \mathcal{L}_x\phi}\bigg) \\ &\quad + \log\bigg(\frac{K(1 + \zeta K \operatorname{diam}(Y)^\alpha)(1+\zeta) \inf \mathcal{L}_x\psi}{K(1-\zeta) \inf \mathcal{L}_x\psi}\bigg) \\ &\leq 2\log\bigg(\frac{1+\zeta}{1-\zeta}\bigg) + 2\log(1 + \zeta K \operatorname{diam}(Y)^\alpha) < \infty.\end{aligned}$$

This proves the existence of M. □

THEOREM 3.5. *Let* $\Phi_n^\sigma(x) = \log \langle \mathcal{L}_x^{n+1}\mathbb{1}, \sigma\rangle / \langle \mathcal{L}_{fx}^n \mathbb{1}, \sigma\rangle$. *There exist* $0 < \tau < 1$ *and* $C_1 > 0$ *such that for all* $k \in \mathbb{N}$, $n, m \geq k$, $x \in X$, *and any probability measures* σ_n *on* $Y_{f^n x}$ *and* σ_m *on* $Y_{f^m x}$, *we have*

$$|\Phi_n^{\sigma_n}(x) - \Phi_m^{\sigma_m}(x)| \leq C_1\, \tau^k.$$

Thus, $\Phi(x) = \lim_{n\to\infty} \Phi_n^{\sigma_n}(x)$ *exists and* $|\Phi_n^{\sigma_n}(x) - \Phi(x)| \leq C_1 \tau^n$.

Proof. Fix $x \in X$. Suppose $n, m \geq k \geq 1$. Then

$$\begin{aligned}|\Phi_n^{\sigma_n}(x) - \Phi_m^{\sigma_m}(x)| &= \bigg|\log \frac{\langle \mathcal{L}_x^{n+1}\mathbb{1}, \sigma_n\rangle}{\langle \mathcal{L}_{fx}^n\mathbb{1}, \sigma_n\rangle} - \log \frac{\langle \mathcal{L}_x^{m+1}\mathbb{1}, \sigma_m\rangle}{\langle \mathcal{L}_{fx}^m\mathbb{1}, \sigma_m\rangle}\bigg| \\ &= \bigg|\log \frac{\langle \mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \sigma_{fx,n}\rangle \langle \mathcal{L}_{fx}^{k-1}\mathbb{1}, \sigma_{fx,m}\rangle}{\langle \mathcal{L}_{fx}^{k-1}\mathbb{1}, \sigma_{fx,n}\rangle \langle \mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \sigma_{fx,m}\rangle}\bigg|\end{aligned}$$

where $\sigma_{fx,n} = (\mathcal{L}_{f^{k+1}x})^* \cdots (\mathcal{L}_{f^n x})^* \sigma_n$. By Lemma 3.1, we see that

$$|\Phi_n^{\sigma_n}(x) - \Phi_m^{\sigma_m}(x)| \leq \Theta(\mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \mathcal{L}_{fx}^{k-1}\mathbb{1}).$$

Clearly, $\mathbb{1} \in \Lambda_K^x$ for any $K > 0$. Then $\mathcal{L}_x\mathbb{1} \in \Lambda_{\zeta K}^{fx}$ by Lemma 3.4. Fix K large and M as in Lemma 3.4. Set $\tau = \tanh(M/4)$. By Theorem 3.2, we have

$$\Theta(\mathcal{L}_{fx}^{k-1}(\mathcal{L}_x\mathbb{1}), \mathcal{L}_{fx}^{k-1}\mathbb{1}) \leq \tau^{k-1}\Theta((\mathcal{L}_x\mathbb{1}), \mathbb{1}) \leq \tau^{k-1}M.$$

Let $C_1 = M/\tau$. Hence, the sequence $\{\Phi_n\}_{n\geq 0}$ is Cauchy and the limit exists at every $x \in X$. □

This proves the existence of Φ.

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3.3. *Fiber measures.* To completely understand the equilibrium state μ on $(X \times Y, F)$, we need to understand how it gives weight to the fibers $\{Y_x\}_{x\in X}$. The first step is the following non-stationary RPF theorem adapted from [3], whose proof we include here for completeness (see Hafouta [5] for a similar result when the base is invertible).

THEOREM 3.6. *Let $F: X \times Y \to X \times Y$ satisfy (A1) and (A2). For any Hölder $\varphi: X \times Y \to \mathbb{R}$ satisfying (P) and its associated family of fiberwise transfer operators $\{\mathcal{L}_x\}_{x\in X}$, there exists a unique family of probability measures $\nu_x \in \mathcal{M}(Y_x)$ such that for all $x \in X$, $\mathcal{L}_x^*\nu_{fx} = \lambda_x\nu_x$, where $\lambda_x = \nu_{fx}(\mathcal{L}_x\mathbb{1}) = e^{\Phi(x)}$.*

Theorem 3.6 is a consequence of the following two propositions.

PROPOSITION 3.7. *Let $C_1 > 0$ and $0 < \tau < 1$ be as in Theorem 3.5. Given any $x \in X$, $k \in \mathbb{N}$, and $\sigma_k \in \mathcal{M}(Y_{f^kx})$, define $\nu_{x,k} \in \mathcal{M}(Y_x)$ by $\nu_{x,k} = (\mathcal{L}_x^k)^*\sigma_k/\langle \mathbb{1}, (\mathcal{L}_x^k)^*\sigma_k\rangle$. If $m, n \geq k$ and $\psi \in \Lambda_K$, we have*

$$\left| \int \psi \, d\nu_{x,n} - \int \psi \, d\nu_{x,m} \right| \leq C_1\|\psi\|\tau^k.$$

In particular, $\langle \psi, \nu_x\rangle := \lim_{n\to\infty}\langle \psi, \nu_{x,n}\rangle$ exists and defines a probability measure ν_x on Y_x with

$$\left| \int \psi \, d\nu_{x,n} - \int \psi \, d\nu_x \right| \leq C_1\|\psi\|\tau^n.$$

Proof. Let $b_k = \inf_y \mathcal{L}_x^k\psi(y)/\mathcal{L}_x^k\mathbb{1}(y)$ and $c_k = \sup_y \mathcal{L}_x^k\psi(y)/\mathcal{L}_x^k\mathbb{1}(y)$. Note that $\mathcal{L}_x^k\psi \leq c_k\mathcal{L}_x^k\mathbb{1}$. So

$$\langle \psi, \nu_{x,n}\rangle = \frac{\langle \psi, (\mathcal{L}_x^n)^*\sigma_n\rangle}{\langle \mathbb{1}, (\mathcal{L}_x^n)^*\sigma_n\rangle} = \frac{\langle \mathcal{L}_x^k\psi, (\mathcal{L}_x^{n-k})^*\sigma_n\rangle}{\langle \mathcal{L}_x^k\mathbb{1}, (\mathcal{L}_x^{n-k})^*\sigma_n\rangle} \leq \frac{c_k\langle \mathcal{L}_x^k\mathbb{1}, (\mathcal{L}_x^{n-k})^*\sigma_n\rangle}{\langle \mathcal{L}_x^k\mathbb{1}, (\mathcal{L}_x^{n-k})^*\sigma_n\rangle} = c_k.$$

A similar computation shows that $b_k \leq \langle \psi, \nu_{x,n}\rangle$. Then $b_k \leq \langle \psi, \nu_{x,n}\rangle \leq c_k$ for all $n \geq k$. Therefore, $|\langle \psi, \nu_{x,n}\rangle - \langle \psi, \nu_{x,m}\rangle| \leq c_k - b_k$ for all $n, m \geq k$. Lemma 3.4 implies that $\Theta(\mathcal{L}_x^k\psi, \mathcal{L}_x^k\mathbb{1}) \leq \operatorname{diam}(\mathcal{L}_x\Lambda_K)\tau^{k-1} \leq M\tau^{k-1}$. So $1 \leq c_k/b_k \leq e^{M\tau^{k-1}}$. Thus, $b_k \leq c_k \leq b_k e^{M\tau^{k-1}}$ which implies that $c_k - b_k \leq b_k(e^{M\tau^{k-1}} - 1)$. Moreover, for all $y \in Y$, we have

$$\mathcal{L}_x^k\psi(y) = \sum_{\overline{y}\in g_x^{-k}(y)} e^{S_k\varphi(x,\overline{y})}\psi(x, \overline{y}) \leq \sum_{\overline{y}\in g_x^{-k}(y)} e^{S_k\varphi(x,\overline{y})}\|\psi\| = \|\psi\|\mathcal{L}_x^k\mathbb{1}(y).$$

So $b_k \leq \|\psi\|$. Hence,

$$|\langle \psi, \nu_{x,n}\rangle - \langle \psi, \nu_{x,m}\rangle| \leq c_k - b_k \leq \|\psi\|(e^{M\tau^{k-1}} - 1).$$

Thus, $\{\nu_{x,n}\}$ is a Cauchy sequence. Then there is a constant $C_1 > 0$ such that

$$|\langle \psi, \nu_{x,n}\rangle - \langle \psi, \nu_x\rangle| \leq C_1\|\psi\|\tau^n$$

for all $n \geq 0$. □

PROPOSITION 3.8. *Let $\{\nu_x\}$ be as in Proposition 3.7. Then $\mathcal{L}_x^*\nu_{fx} = e^{\Phi(x)}\nu_x$.*

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Proof. For all $\psi \in C(Y_x)$, we have

$$\int \psi \, d(\mathcal{L}_x^* \nu_{fx}) = \lim_{n\to\infty} \frac{\langle \mathcal{L}_x \psi, (\mathcal{L}_{fx}^n)^* \sigma_{n+1} \rangle}{\langle \mathbb{1}, (\mathcal{L}_{fx}^n)^* \sigma_{n+1} \rangle}$$
$$= \lim_{n\to\infty} \frac{\langle \mathcal{L}_x^{n+1} \mathbb{1}, \sigma_{n+1} \rangle}{\langle \mathcal{L}_{fx}^n \mathbb{1}, \sigma_{n+1} \rangle} \cdot \frac{\langle \mathcal{L}_x^{n+1} \psi, \sigma_{n+1} \rangle}{\langle \mathcal{L}_x^{n+1} \mathbb{1}, \sigma_{n+1} \rangle} = e^{\Phi(x)} \int \psi \, d\nu_x. \qquad \square$$

Observe that

$$\nu_{fx}(\mathcal{L}_x \mathbb{1}) = \mathcal{L}_x^* \nu_{fx}(\mathbb{1}) = e^{\Phi(x)} \nu_x(\mathbb{1}) = e^{\Phi(x)} =: \lambda_x.$$

This completes the proof of Theorem 3.7.

As in Denker and Gordin [4], we call a system of $\{\nu_x : x \in X\}$ of conditional probabilities for $(X \times Y, F)$ a *family of Gibbs measures* for a continuous function $\varphi \colon X \times Y \to \mathbb{R}$ if there exists a positive measurable function $A \colon X \to \mathbb{R}$ with the following property: for all $x \in X$, the Jacobian of μ_x with respect to the map F is given by

$$\frac{d\nu_{f(x)} \circ g_x}{d\nu_x} = A(x) e^{-\varphi_x}.$$

COROLLARY 3.9. *The measures $\{\nu_x : x \in X\}$ form a family of Gibbs measures for φ in the sense of Denker and Gordin* [4] *with $A(x) = \lambda_x$.*

Proof. Choose $A \in Y_x$ such that $g_x|_A$ is invertible. This implies

$$\mathcal{L}_x(e^{-\varphi_x} \mathbb{1}_A)(y) = \sum_{\overline{y} \in g_x^{-1} y} e^{\varphi_x(\overline{y})} e^{-\varphi_x(\overline{y})} \mathbb{1}_A(\overline{y}) = \sum_{\overline{y} \in g_x^{-1} y} \mathbb{1}_A(\overline{y}) = \mathbb{1}_{g_x A}(y).$$

Therefore, $\int_A \lambda_x e^{-\varphi_x} \, d\nu_x = \int \mathcal{L}_x(e^{-\varphi_x} \mathbb{1}_A) \, d\nu_{f(x)} = \nu_{f(x)}(g_x A)$. Since this holds for $x \in X$, $\{\nu_x : x \in X\}$ forms a family of Gibbs measures. $\square$

3.4. *Regularity of* Φ. Now we will show that Φ is Hölder continuous. A direct consequence of Lemma 3.4 is the following lemma, which we will need to prove the Hölder continuity of Φ. For convenience, we write

$$\lambda_x^n = \lambda_x \lambda_{fx} \cdots \lambda_{f^{n-1}x} = e^{S_n \Phi(x)}.$$

LEMMA 3.10. *Let M be as in Lemma 3.4. Then $e^{-M} \lambda_x^n \le \mathcal{L}_x^n \mathbb{1}(y) \le e^M \lambda_x^n$ for all $n \in \mathbb{N}$ and $(x, y) \in X \times Y$.*

Proof. Let $\overline{\varphi}_x = \varphi_x - \log \lambda_x$ and write $\overline{\mathcal{L}}_x^n \mathbb{1} = \sum_{\overline{y} \in g_x^n(y)} e^{S_n \overline{\varphi}_x(y)}$. Theorem 3.7 gives $(\overline{\mathcal{L}}_x)^* \nu_{fx} = \nu_x$ for all $x \in X$. Inductively, we get that $(\overline{\mathcal{L}}_x^n)^* \nu_{f^n x} = \nu_x$. Then for any k, ℓ,

$$\int \overline{\mathcal{L}}_x^k \mathbb{1} \, d\nu_{f^k x} = \int \mathbb{1} \, d(\overline{\mathcal{L}}_x^k)^* \nu_{f^k x} = \nu_x(Y_x) = 1 = \int \mathbb{1} \, d\nu_{f^k x}.$$

Let Λ^+ be the cone of strictly positive continuous functions on $X \times Y$. Since $\Lambda_K \subset \Lambda^+$, the projective metrics of the two cones satisfy $\Theta^+(\phi, \psi) \le \Theta(\phi, \psi)$. Write $\psi_k = \overline{\mathcal{L}}_x^k \mathbb{1}$. Then $\inf \psi_k \le \mathbb{1} \le \sup \psi_k$, so $1 \le \sup \psi_k / \inf \psi_k \le e^M$. We know that $\Theta^+(\psi_k, \mathbb{1}) \le M$.

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This implies that $e^{-M} \leq \psi_k \leq e^M$ for all $k \in \mathbb{N}$. Thus,

$$e^{-M}\lambda_x^n \leq \mathcal{L}_x^n \mathbb{1}(y) \leq e^M \lambda_x^n. \qquad \square$$

Let $n \in \mathbb{N}$ and $x, x' \in X$, and $y \in Y$. Let $\mathcal{W}_n = \{1, \ldots, d\}^n$. By Lemma 2.3, we can write

$$g_{f^{n-1}x}^{-1}(y) = \{y_1, \ldots, y_d\} \quad \text{and} \quad g_{f^{n-1}x'}^{-1}(y) = \{y_1', \ldots, y_d'\}$$

such that

$$d((f^{n-1}x, y_k), (f^{n-1}x', y_k')) \leq L_k d_X(f^n x, f^n x')$$

where $L_k = L$ if $1 \leq k \leq q$ and $L_k = \gamma^{-1}$ if $q < k \leq d$. Continuing in this way, we get that

$$g_x^{-n}(y) = \{y_w \in Y_x : w \in \mathcal{W}_n\} \quad \text{and} \quad g_{x'}^{-n}(y) = \{y_w' \in Y_{x_w'} : w \in \mathcal{W}_n\}$$

such that for all $0 \leq k \leq n$,

$$d(F^k(x, y_w), F^k(x', y_w')) \leq L_{w_{k+1}} \cdots L_{w_n} d_X(f^n x, f^n x').$$

Let $m \leq \mathbb{N}$ and $0 < \iota < 1$. A pair of inverse branches for F of length n starting from $(f^n x, y)$ and $(f^n x', y)$ and labeled by $w \in \mathcal{W}_n$ is *good* if for all $j \in \mathbb{N}$ such that $jm \leq n$, we have

$$\#\{n - jm < i \leq n : w_i \leq q\} \leq \iota jm.$$

This means that the last jm iterates of an orbit segment of length n will be in the contraction region at most ιjm times. We will denote the collection of words corresponding to good trajectories by

$$\mathcal{W}_n^{\mathcal{G}} = \mathcal{W}_n^{\mathcal{G}}(m) = \{w \in \mathcal{W}_n : \text{ for all } j \leq n/m, \#\{n - jm < i \leq n : w_i \leq q\} \leq \iota jm\}$$

and the collection of words for bad trajectories by

$$\begin{aligned} \mathcal{W}_n^{\mathcal{B}} &= \mathcal{W}_n^{\mathcal{B}}(m) \\ &= \{w \in \mathcal{W}_n : \text{there exists } j \leq n/m \ni \#\{n - jm < i \leq n : w_i \leq q\} \geq \iota jm\}. \end{aligned}$$

The following lemma due to Varandas and Viana gives us a way to count the number of words that code bad trajectories of a given length. Let $I(\iota, n) = \{w \in \mathcal{W}_n : \#\{1 \leq k \leq n : w_i \leq q\} \geq \iota n\}$.

LEMMA 3.11. *Given $\varepsilon > 0$, there exists a $\iota_0 \in (0, 1)$ such that*

$$\limsup_{n \to \infty} \frac{1}{n} \log \#I(\iota, n) < \log q + \varepsilon$$

for all $\iota \in (\iota_0, 1)$. Therefore, there exists a $C > 0$ such that $\#I(\iota, n) \leq Cq^n e^{\varepsilon n}$ for all n.

Proof. See Lemma 3.1 in Varandas and Viana [12]. $\square$

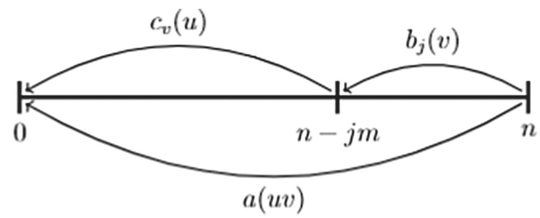

FIGURE 1. The maps a, b_j, and c_v on words of corresponding lengths.

Since it is assumed that $\varepsilon_\varphi < \log d - \log q$, it follows that $qe^{\varepsilon_\varphi}/d < 1$. Choose $\varepsilon > 0$ such that

$$\theta := \frac{qe^{\varepsilon}e^{\varepsilon_\varphi}}{d} < 1. \tag{5}$$

Let $\iota = \iota(\varepsilon, d, q) \in (0, 1)$ be given by Lemma 3.11. We now further our assumption on the constant L by assuming that it is close enough to 1 so that there is a $c > 0$ satisfying

$$0 < \gamma^{-(1-\iota)}L^{\iota} < e^{-2c} < 1. \tag{6}$$

LEMMA 3.12. *There is a $Q > 0$ such that for all $m \in \mathbb{N}$, if $(x, \overline{y})$ and $(x', \overline{y}')$ are preimages coded by a word in $\mathcal{W}_n^{\mathcal{G}}(m)$, then*

$$d(F^k(x, \overline{y}), F^k(x', \overline{y}')) \le Q^m e^{-2c(n-k)} d(f^n x, f^n x')$$

for all $0 \le k < n$.

Proof. Fix $m \in \mathbb{N}$. Write $n - k = jm + i$ for $0 \le i < m$. Since our preimage branches are assumed to be good, we get

$$\begin{aligned} d(F^k(x, \overline{y}), F^k(x', \overline{y}')) &\le L_{w_{k+1}} \cdots L_{w_n} d(F^{k+i}(x, \overline{y}), F^{k+i}(x', \overline{y}')) \\ &\le L_{w_{k+1}} \cdots L_{w_{k+i}} (L^{\iota jm} \gamma^{-(1-\iota)jm}) d(f^n x, f^n x'). \end{aligned}$$

Recall from (6) that we can choose $c > 0$ so that $0 < \gamma^{-(1-\iota)}L^{\iota} < e^{-2c} < 1$. Thus,

$$\begin{aligned} d(F^k(x, \overline{y}), F^k(x', \overline{y}')) &\le L^m e^{-2cjm} d(f^n x, f^n x') \\ &\le (Le^{2c})^m e^{-2c(n-k)} d(f^n x, f^n x'). \end{aligned}$$ □

In what follows it will be convenient to write $a\colon \mathcal{W}_n \to Y$ so that $a(w) = y_w$ and $a'\colon \mathcal{W}_n \to Y$ so that $a'(w) = y'_w$. Lemma 2.3 gives us bijections $b_j\colon \mathcal{W}_{jm} \to g^{-jm}_{f^{n-jm}x}(y)$ such that $b_j(v) = g_x^{n-jm}(a(uv))$ and $c_v\colon \mathcal{W}_{n-jm} \to g_x^{-(n-jm)}(b_j(v))$ such that $c_v(u) = a(uv)$ as well as their associated maps b'_j and c'_v. See Figure 1 for reference.

LEMMA 3.13. *Let θ as in (5) above. There exists $C_2 > 0$ such that*

$$\sum_{w \in \mathcal{W}_n^{\mathcal{B}}(m)} e^{S_n\varphi_x(a(w))} \le C_2\theta^m \sum_{w \in \mathcal{W}_n^{\mathcal{G}}(m)} e^{S_n\varphi_x(a(w))}$$

for all $m \in \mathbb{N}$, $n \in \mathbb{N}$, $x, x' \in X$, and $y \in Y$.

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Proof. Let $x \in X$ and $y \in Y$. For any $w \in \mathcal{W}_n^{\mathcal{B}}$, there is $1 \leq j \leq n/m$ such that $w = uv$ for some $u \in \mathcal{W}_{n-jm}$ and $v \in I(\iota, jm)$. Thus,

$$\sum_{w \in \mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))} = \sum_{j=1}^{\lfloor n/m \rfloor} \sum_{v \in I(\iota, jm)} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))} \sum_{u \in \mathcal{W}_{n-jm}} e^{S_{n-jm}\varphi_x(c_v(u))}$$
$$\leq \sum_{j=1}^{\lfloor n/m \rfloor} \sum_{v \in I(\iota, jm)} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))} e^M \lambda_x^{n-jm}$$

by Lemma 3.10. Note that for any $j \leq n/m$,

$$\sum_{w \in \mathcal{W}_n} e^{S_n\varphi_x(a(w))} = \sum_{v \in \mathcal{W}_{jm}} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))} \sum_{u \in \mathcal{W}_{n-jm}} e^{S_{n-jm}\varphi_x(c_v(u))}$$
$$\geq e^{-M} \lambda_x^{n-jm} \sum_{v \in \mathcal{W}_{jm}} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))}.$$

Lemma 3.11 implies that $\#I(\iota, n) \leq Cq^n e^{\varepsilon n}$ for all $n \geq 0$. Then since $\#\mathcal{W}_n$ is finite,

$$\frac{\sum_{w \in \mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))}}{\sum_{w \in \mathcal{W}_n} e^{S_n\varphi_x(a(w))}} \leq \sum_{j=1}^{\lfloor n/m \rfloor} \frac{e^M \lambda_x^{n-jm}}{\sum_{w \in \mathcal{W}_n} e^{S_n\varphi_x(a(w))}} \sum_{v \in I(\iota, jm)} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))}$$
$$\leq e^{2M} \sum_{j=1}^{\lfloor n/m \rfloor} \frac{\lambda_x^{n-jm} \sum_{v \in I(\iota, jm)} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))}}{\lambda_x^{n-jm} \sum_{v \in \mathcal{W}_{jm}} e^{S_{jm}\varphi_{f^{n-jm}x}(b_j(v))}}$$
$$\leq e^{2M} \sum_{j=1}^{\lfloor n/m \rfloor} \frac{\#I(\iota, jm)}{\#\mathcal{W}_{jm}} e^{jm(\sup\varphi - \inf\varphi)}$$
$$\leq e^{2M} \sum_{j=1}^{\lfloor n/m \rfloor} \frac{Cq^{jm} e^{jm\varepsilon}}{d^{jm}} e^{jm\varepsilon_\varphi}$$

where the last inequality holds by Lemma 3.11 and (P). Let $\theta = qe^{\varepsilon}e^{\varepsilon_\varphi}/d < 1$. Then

$$\frac{\sum_{w \in \mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))}}{\sum_{w \in \mathcal{W}_n} e^{S_n\varphi_x(a(w))}} \leq e^{2M} \sum_{j=1}^{\infty} \left(\frac{qe^{\varepsilon}e^{\varepsilon_\varphi}}{d}\right)^{jm} = e^{2M}\left(\frac{\theta^m}{1-\theta^m}\right).$$

But

$$\sum_{w \in \mathcal{W}_n} e^{S_n\varphi_x(a(w))} = \sum_{w \in \mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))} + \sum_{w \in \mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_x(a(w))}.$$

Choose m such that $1 - \theta^m \geq \frac{1}{2}$. Then

$$\sum_{w \in \mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))} \leq 2e^{2M} C\theta^m \left(\sum_{w \in \mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))} + \sum_{w \in \mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_x(a(w))} \right).$$

So if we increase m so that $2e^{2M}C\theta^m < \frac{1}{2}$, then

$$\sum_{w\in\mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))} \leq 4e^{2M}C\theta^m \sum_{w\in\mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_x(a(w))}.$$

This achieves the desired result. □

Now we apply the above with various values of m to prove the Hölder continuity of Φ. First, a bound on Φ_n.

LEMMA 3.14. *There exist $C_3 > 0$ and $\beta > 0$ such that*

$$|\Phi_n(x) - \Phi_n(x')| \leq C_3 d(f^n x, f^n x')^{\alpha\beta}$$

for all $x, x' \in X$ and $n \in \mathbb{N}$.

Proof. First note that along good orbit pairs, we have by Lemma 3.12 that

$$\begin{aligned}|S_n\varphi(x,\overline{y}) - S_n\varphi(x',\overline{y}')| &\leq \sum_{k=0}^{n-1} |\varphi|_\alpha d(F^k(x,\overline{y}), F^k(x',\overline{y}'))^\alpha \\ &\leq \sum_{k=0}^{n-1} |\varphi|_\alpha Q^{\alpha m} e^{-2c\alpha(n-k)} d(f^n x, f^n x')^\alpha \\ &\leq Q^{\alpha m} d(f^n x, f^n x')^\alpha \cdot \sum_{k=0}^{\infty} |\varphi|_\alpha e^{-2c\alpha(n-k)}.\end{aligned}$$

Let $V = \sum_{k=0}^{\infty} |\varphi|_\alpha e^{-2c\alpha(n-k)}$. Then

$$|S_n\varphi(x,\overline{y}) - S_n\varphi(x',\overline{y}')| \leq VQ^{\alpha m} d(f^n x, f^n x')^\alpha. \tag{7}$$

For convenience, we write $\Sigma_{\mathcal{G}} = \sum_{w\in\mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_x(a(w))}$ and $\Sigma_{\mathcal{B}} = \sum_{w\in\mathcal{W}_n^{\mathcal{B}}} e^{S_n\varphi_x(a(w))}$ as well as $\Sigma'_{\mathcal{G}}$ and $\Sigma'_{\mathcal{B}}$ for the sums of the preimages associated to $a'(w)$. By Lemma 3.13, we get that

$$\frac{\mathcal{L}_x^n \mathbb{1}(f^n x, y)}{\mathcal{L}_x^n \mathbb{1}(f^n x', y)} = \frac{\Sigma_{\mathcal{G}} + \Sigma_{\mathcal{B}}}{\Sigma'_{\mathcal{G}} + \Sigma'_{\mathcal{B}}} \leq \frac{\Sigma_{\mathcal{G}}(1 + C_2\theta^m)}{\Sigma'_{\mathcal{G}}(1 - C_2\theta^m)} \leq \frac{\Sigma_{\mathcal{G}}}{\Sigma'_{\mathcal{G}}} \cdot e^{C\theta^m}. \tag{8}$$

Note that by (7),

$$\begin{aligned}\frac{\Sigma_{\mathcal{G}}}{\Sigma'_{\mathcal{G}}} &= \frac{\sum_{w\in\mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_x(a(w))}}{\sum_{w'\in\mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_{x'}(a'(w))}} \\ &\leq \frac{\sum_{w'\in\mathcal{W}_n^{\mathcal{G}}} e^{VQ^{\alpha m} d(f^n x, f^n x')^\alpha} e^{S_n\varphi_{x'}(a'(w))}}{\sum_{w'\in\mathcal{W}_n^{\mathcal{G}}} e^{S_n\varphi_{x'}(a'(w))}} \\ &\leq e^{VQ^{\alpha m} d(f^n x, f^n x')^\alpha}.\end{aligned}$$

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Let Φ_n be as in Theorem 3.5 for the delta measure on Y, δ_y $(y \in Y)$. Then

$$\begin{aligned}|\Phi_n(x) - \Phi_n(x')| &= \left|\log\left(\frac{\mathcal{L}_x^{n+1}\mathbb{1}(f^n x, y)}{\mathcal{L}_{x'}^{n+1}\mathbb{1}(f^n x', y)} \cdot \frac{\mathcal{L}_{fx}^{n}\mathbb{1}(f^n x, y)}{\mathcal{L}_{fx'}^{n}\mathbb{1}(f^n x', y)}\right)\right| \\ &\leq 2\log\left(\frac{\Sigma_{\mathcal{G}}}{\Sigma'_{\mathcal{G}}} \cdot e^{C\theta^m}\right) \\ &\leq 2VQ^{\alpha m} d(f^n x, f^n x')^\alpha + 2C\theta^m\end{aligned}$$

where the second inequality holds due to (8).

Let $\rho_1 = \theta/Q^\alpha$ and note that $\rho_1 < 1$. Then there is a $k \in \mathbb{N}$ such that

$$\rho_1^{k+1} \leq d(f^n x, f^n x')^\alpha \leq \rho_1^k. \tag{9}$$

Now set $m = k$. Let $\beta = (\log\theta/\log\rho_1)$ and note that

$$\theta^m = e^{m\log\theta} = e^{\beta m\log\rho_1} = \rho_1^{\beta m} \leq \rho_1^{-\beta} d(f^n x, f^n x')^{\alpha\beta}.$$

Thus, $Q^{\alpha m} d(f^n x, f^n x')^\alpha \leq \theta^m \leq \rho_1^{-\beta} d(f^n x, f^n x')^{\alpha\beta}$. Hence, letting $C_3 = 2(V + C)\rho_1^{-\beta}$ yields

$$|\Phi_n(x) - \Phi_n(x')| \leq C_3 d(f^n x, f^n x')^{\alpha\beta}.$$ □

THEOREM 3.15. *The potential Φ constructed in Theorem 3.5 is Hölder continuous.*

Proof. Let $C = \max\{C_1, C_3\}$. For any $n \geq 0$,

$$\begin{aligned}|\Phi(x) - \Phi(x')| &\leq |\Phi(x) - \Phi_n(x)| + |\Phi_n(x) - \Phi_n(x')| + |\Phi_n(x') - \Phi(x')| \\ &\leq 2C\tau^n + Cd(f^n x, f^n x')^{\alpha\beta} \\ &\leq 2C\tau^n + C\Gamma^{\alpha\beta n} d(x, x')^{\alpha\beta}.\end{aligned}$$

where the second inequality follows from Theorem 3.5 and Lemma 3.14 and Γ is the inherited Lipschitz constant for f.

Similarly to the argument in the proof of Lemma 3.14, we need to adjust the Hölder exponent to establish our bound. Let $\rho_2 = \tau/\Gamma^{\alpha\beta}$. Then there is a k such that $\rho_2^{k+1} \leq d(x, x')^{\alpha\beta} \leq \rho_2^k$. Let $n = k$ and $\eta = \log\tau/\log\rho_2$. Then

$$\tau^n = e^{n\log\tau} = e^{\eta n\log\rho_2} = \rho_2^{\eta n} \leq \rho_2^{-\eta} d(x, x')^{\alpha\beta\eta}.$$

So $\Gamma^{\alpha\beta} d(x, x')^{\alpha\beta} \leq \tau^n \leq \rho_2^{-\eta} d(x, x')^{\alpha\beta\eta}$. Therefore,

$$|\Phi(x) - \Phi(x')| \leq 2C\rho_2^{-\eta} d(x, x')^{\alpha\beta\eta}.$$ □

4. *Conditional measures of equilibrium states*

Let $\mathcal{L}_\Phi : C(X) \to C(X)$ be defined by

$$\mathcal{L}_\Phi \xi(x) = \sum_{\bar{x} \in f^{-1}x} e^{\Phi(\bar{x})}\, \xi(\bar{x})$$

for any $\xi \in C(X)$. Since f is uniformly expanding on X, Theorem 3.15 implies that there is a unique equilibrium state that can be obtained via $\mathcal{L}_\Phi$. See Theorems 6 and 8 in [13] for details.

THEOREM 4.1. *Let X be a compact, connected manifold and $f: X \to X$ is uniformly expanding. For any Hölder $\Phi: X \to \mathbb{R}$, the following assertions hold.*

(1) *There is a unique probability measure $\hat{\nu} \in \mathcal{M}(X)$ with the property that $\mathcal{L}_\Phi^* \hat{\nu}$ is a scalar multiple of $\hat{\nu}$.*

(2) *There is a unique positive continuous function $\hat{h} \in C(X)$ with the property that $\mathcal{L}_\Phi \hat{h}$ is a scalar multiple of $\hat{h}$ and $\int_X \hat{h}(x) d\hat{\nu}(x) = 1$.*

(3) *The eigenvalues associated to $\hat{\nu}$ and $\hat{h}$ are the same.*

(4) *The unique equilibrium state for Φ is $\bar{\mu} = \hat{h}\hat{\nu}$.*

We shall show that $\bar{\mu} = \hat{\mu} = \mu \circ \pi_X^{-1}$ and construct the family of measures $\{\mu_x\}_{x \in X}$. To do this, we first prove the following lemmas. Place on $C(Y)$ the sup norm $\|\cdot\|_\infty$; that is, if $\psi \in C(Y)$, then $\|\psi\|_\infty = \sup\{\psi(y): y \in Y\}$.

LEMMA 4.2. *For any $\psi \in C(X \times Y)$, the map $x \mapsto \mathcal{L}_\varphi^x \psi_x$ is continuous with respect to the topology induced on $C(Y)$ by $\|\cdot\|_\infty$.*

Proof. Let $\psi \in C(X \times Y)$ and $y \in Y$. For any $x, x' \in X$,

$$|\mathcal{L}_x \psi_x(y) - \mathcal{L}_{x'} \psi_{x'}(y)| \leq \sum_{\overline{y} \in g_x^{-1} y} (e^{\varphi(x,\overline{y})} |\psi(x, \overline{y}) - \psi(x', \overline{y}')| + \|\psi\|_\infty \, |e^{\varphi(x,\overline{y})} - e^{\varphi(x',\overline{y}')}|)$$

where $\overline{y}'$ are the preimages of y under $g_{x'}^{-1}$ given by Lemma 2.3. Fix $\epsilon > 0$. Let $M_1 = \sup_{(x,y) \in X \times Y} \{\mathcal{L}_x \psi(y)\}$. Since ψ is continuous, there exists a $\delta_1 > 0$ such that $|\psi(u, v) - \psi(u', v')| < \epsilon / 2M_1$ whenever $d((u, v), (u'v')) < \delta_1$. Similarly, there is a $\delta_2 > 0$ such that $|e^{\varphi(u,v)} - e^{\varphi(u',v')}| < \epsilon / 2d \|\psi\|_\infty$ whenever $d((u, v), (u'v')) < \delta_2$.

Let $\delta_3 = \min\{\delta_1, \delta_2\}/L$. By the continuity of f, there exists $\delta > 0$ such that if $d(x, x') < \delta$, then $d((fx, y), (fx', y)) < \delta_3$. By Lemma 2.3, for all $(x, \overline{y}) \in F^{-1}(fx, y) \cap Y_x$, there exists $(x', \overline{y}') \in F^{-1}(fx', y) \cap Y_{x'}$ such that if $(x, \overline{y}) \in \mathcal{A}$, then

$$d((x, \overline{y}), (x', \overline{y}')) \leq L d(fx, fx') < L\delta_3 < \min\{\delta_1, \delta_2\}$$

and if $(x, \overline{y}) \notin \mathcal{A}$, then

$$d((x, \overline{y}), (x', \overline{y}')) \leq \gamma^{-1} d(fx, fx') < \delta_3.$$

Thus, $d((x, \overline{y}), (x', \overline{y}')) < \min\{\delta_1, \delta_2\}$. Hence,

$$\begin{aligned} |\mathcal{L}_x \psi_x(y) - \mathcal{L}_{x'} \psi_{x'}(y)| &\leq \frac{\epsilon}{2M_1} \sum_{\overline{y} \in g_x^{-1} y} e^{\varphi(x,\overline{y})} + \|\psi\|_\infty \sum_{\overline{y} \in g_x^{-1} y} \frac{\epsilon}{2d \|\psi\|_\infty} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since this is independent of y, we have $\|\mathcal{L}_x \psi_x - \mathcal{L}_{x'} \psi_{x'}\|_\infty < \epsilon$. □

Remark 1. This proof can be extended to hold for all iterates of the transfer operator.

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LEMMA 4.3. *For every continuous $\psi : X \times Y \to \mathbb{R}$, the map $x \mapsto \nu_x(\psi_x)$ is continuous with respect to the usual topology.*

Proof. Fix $y \in Y$ and let $\nu_{x,n} = (\mathcal{L}_x^n)^*\delta_y / \langle \mathbb{1}, (\mathcal{L}_x^n)^*\delta_y \rangle$ be as in Proposition 3.7. So as shown there, $\nu_{x,n} \xrightarrow{\text{weak}^*} \nu_x$. For any $x, x' \in X$,

$$\begin{aligned}\left| \int \psi \, d\nu_x - \int \psi \, d\nu_{x'} \right| &\leq \left| \int \psi \, d\nu_x - \int \psi \, d\nu_{x,n} \right| + \left| \int \psi \, d\nu_{x,n} - \int \psi \, d\nu_{x',n} \right| \\ &\quad + \left| \int \psi \, d\nu_{x',n} - \int \psi \, d\nu_{x'} \right| \\ &\leq 2C_1 \|\psi\| \tau^n + \left| \int \psi \, d\nu_{x,n} - \int \psi \, d\nu_{x',n} \right|\end{aligned}$$

where the last inequality holds by Proposition 3.7. Note that

$$\int \psi \, d\nu_{x,n} = \frac{\langle \psi, (\mathcal{L}_x^n)^*\delta_y \rangle}{\langle \mathbb{1}, (\mathcal{L}_x^n)^*\delta_y \rangle} = \frac{\mathcal{L}_x^n \psi(y)}{\mathcal{L}_x^n \mathbb{1}(y)}$$

is continuous in x by Lemma 4.2. Given $\epsilon > 0$, choose n sufficiently large so that $2C_1\|\psi\|\tau^n < \epsilon/2$ and $\delta > 0$ such that $d(x, x') < \delta$ implies $|\int \psi \, d\nu_{x,n} - \int \psi \, d\nu_{x',n}| < \epsilon/2$. Then

$$\left| \int \psi \, d\nu_x - \int \psi \, d\nu_{x'} \right| \leq 2C_1 \|\psi\| \tau^n + \left| \int \psi \, d\nu_{x,n} - \int \psi \, d\nu_{x',n} \right| < \epsilon.$$

This proves continuity of $x \mapsto \nu_x(\psi_x)$. □

Define $I : C(X \times Y) \to C(X)$ by $(I\psi)(x) = \int_{Y_x} \psi(x, y) \, d\nu_x(y)$. Observe that for any $\eta \in \mathcal{M}(X)$, we have

$$\langle \psi, I^*\eta \rangle = \int_X (I\psi)(x) \, d\eta(x) = \int_X \int_Y \psi(x, y) \, d\nu_x(y) \, d\eta(x).$$

So $\langle I\psi, \eta \rangle = \langle \psi, I^*\eta \rangle$ where $I^* : \mathcal{M}(X) \to \mathcal{M}(X \times Y)$ is defined by $I^*\eta = \int_X \nu_x \, d\eta(x)$.

THEOREM 4.4. *The operators I and I^* satisfy $I \circ \mathcal{L}_\varphi = \mathcal{L}_\Phi \circ I$ and $I^* \circ \mathcal{L}_\Phi^* = \mathcal{L}_\varphi^* \circ I^*$. That is, they make their respective diagrams below commute:*

$$\begin{array}{ccc} C(X \times Y) & \xrightarrow{\mathcal{L}_\varphi} & C(X \times Y) \\ \downarrow I & & \downarrow I \\ C(X) & \xrightarrow{\mathcal{L}_\Phi} & C(X) \end{array} \qquad \begin{array}{ccc} \mathcal{M}(X \times Y) & \xleftarrow[\mathcal{L}_\varphi^*]{} & \mathcal{M}(X \times Y) \\ I^* \uparrow & & I^* \uparrow \\ \mathcal{M}(X) & \xleftarrow[\mathcal{L}_\Phi^*]{} & \mathcal{M}(X) \end{array}$$

Proof. Given $\psi \in C(X \times Y)$, we have

$$\begin{aligned} I \circ \mathcal{L}_\varphi \psi(x, y) &= \int_{X \times Y} \mathcal{L}_\varphi \psi(x, y) \, d\nu_x(y) \\ &= \int_{X \times Y} \sum_{\overline{x} \in f^{-1}x} \sum_{\overline{y} \in g_{\overline{x}}^{-1} y} e^{\varphi(\overline{x}, \overline{y})} \psi(\overline{x}, \overline{y}) \, d\nu_x(y) \end{aligned}$$

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$$
\begin{aligned}
&= \int_{X\times Y} \sum_{\overline{x}\in f^{-1}x} (\mathcal{L}_{\overline{x}}\psi)(x,y)\, d\nu_x(y) \\
&= \sum_{\overline{x}\in f^{-1}x} \langle \mathcal{L}_{\overline{x}}\psi, \nu_x\rangle \\
&= \sum_{\overline{x}\in f^{-1}x} \langle \psi, (\mathcal{L}_{\overline{x}})^*\nu_x\rangle \\
&= \sum_{\overline{x}\in f^{-1}x} e^{\Phi(\overline{x})}\langle \psi, \nu_{\overline{x}}\rangle = \sum_{\overline{x}\in f^{-1}x} e^{\Phi(\overline{x})}(I\psi)(\overline{x}) \\
&= (\mathcal{L}_\Phi \circ I)\psi(x).
\end{aligned}
$$

Duality gives $I^* \circ \mathcal{L}_\Phi^* = \mathcal{L}_\varphi^* \circ I^*$. □

COROLLARY 4.5. *$P(\Phi) = P(\varphi)$. Moreover, ν, $\hat{\nu}$ and h, $\hat{h}$ satisfy $\nu = I^*\hat{\nu}$ and $\hat{h} = Ih$.*

Proof. By Theorem 4.4, we have

$$\mathcal{L}_\varphi^*(I^*\hat{\nu}) = I^*\mathcal{L}_\Phi^*\hat{\nu} = I^*e^{P(\Phi)}\hat{\nu} = e^{P(\Phi)}(I^*\hat{\nu}).$$

Then items (1) and (3) of Theorem 4.1 implies $I^*\hat{\nu} = \nu$ and $P(\varphi) = P(\Phi)$, respectively. Now we will show that Ih is the eigenfunction for $\mathcal{L}_\Phi$. We have

$$\mathcal{L}_\Phi(Ih) = I\mathcal{L}_\varphi h = Ie^{P(\varphi)}h = e^{P(\Phi)}(Ih)$$

where the first equality is by Theorem 4.4 and the last is by the paragraph above. Moreover, the paragraph above implies that $\int (Ih)d\hat{\nu} = \int h\, d\nu = 1$ where the last equality holds by item (2) of Theorem 4.1 for the potential $\varphi\colon X\times Y \to \mathbb{R}$. Thus, item (2) of Theorem 4.1 for $\Phi\colon X \to \mathbb{R}$ implies $Ih = \hat{h}$. □

Thus, given any $\psi \in C(X\times Y)$, we have

$$
\begin{aligned}
\int \psi\, d\mu = \int \psi h\, d\nu
&= \int_X \int_Y (\psi\cdot h)(x,y)\, d\nu_x(y)\, d\hat{\nu}(x) \\
&= \int_X \int_Y \psi(x,y)\cdot \frac{h(x,y)}{\hat{h}(x)}\, d\nu_x(y)\hat{h}\, d\hat{\nu}(x) \\
&= \int_X \int_{Y_x} \psi(x,y)\, d\mu_x(y)\, d\bar{\mu}(x)
\end{aligned}
$$

where μ_x is defined by $d\mu_x/d\nu_x = h(x,y)/\hat{h}(x)$ and $\overline{\mu} = \hat{h}\hat{\nu}$. Note that by Corollary 4.5

$$\mu_x(Y_x) = \int_{Y_x} \frac{h(x,y)}{\hat{h}(x)}\, d\nu_x(y) = \frac{(Ih)(x)}{\hat{h}(x)} = \frac{\hat{h}(x)}{\hat{h}(x)} = 1.$$

Therefore, $\bar{\mu} = \hat{\mu}$ and $\{\mu_x\}_{x\in X}$ is the unique family of conditional measures for μ.

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