Steady-State-Aware Model Predictive Control for Tracking in Systems with Limited Computing Capacity

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***Abstract*—Model Predictive Control (MPC) determines the control input by solving a receding horizon optimal control problem at each time instant, which may be computationally challenging for systems with limited computing capacity. One possible approach to address this issue in tracking problems is to reduce the prediction horizon length and modify the conventional MPC formulation so as to enlarge the region of attraction. Prior work assumes that the desired *admissible* steady-state configuration is known for each sequence of the reference, which is unrealistic when sequences of the reference are unknown *a priori*. This letter develops a steady-state-aware MPC that guarantees tracking of piecewise constant references and satisfaction of constraints, without requiring the desired admissible steady-state configuration and without adding extra computational load. Stability, recursive feasibility, and local infinite-horizon optimality of the proposed MPC are proven analytically. The effectiveness of the proposed MPC is investigated in comparison with prior work.**

***Index Terms*—Model Predictive Control, Steady-State Configuration, Limited Computing Capacity, Output Tracking.**

I. Introduction

Model Predictive Control (MPC) is a widely used method to control systems that are subject to state and/or input constraints [1]. At each time instant, MPC determines the control input by solving a receding horizon optimal control problem, which can be computationally challenging [2].

Reducing the computational complexity of MPC has been widely investigated in the literature. One method to reduce the computational cost of MPC, employed in explicit MPC [3], [4], is to pre-compute the optimal laws offline and store them for future online use. Making use of triggering mechanisms [5], [6] is another way to reduce computational load. Anytime MPC [7] ensures stability with minimal iterations, though does not ensure constraint satisfaction. Converting MPC problem into the evolution of a continuous-time system has been discussed in [8], [9], without addressing its discrete-time implementation.

The most intuitive approach to reduce the computational cost of MPC is to shorten the prediction horizon length, which has been considered in [10], [11]. Although this approach maintains *stability*, it may degrade *feasibility* by reducing the Region of Attraction (RoA), defined as the set of all initial conditions at which MPC is feasible.

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One possible way to enlarge the RoA of MPC problem is to modify the structure of the conventional MPC. In this context, [12] characterizes the set of steady-state configurations (a.k.a. operation points) by a characterizing vector and considers that as a decision variable in addition to control sequence; it has been shown that adding a penalty term to the cost function ensures output tracking without violating the constraints. However, when the desired reference can be tracked with multiple steady-state configurations, [12] has no control over the achieved steady-state configuration. One possible approach to address this issue, which is pursued in [13], is to determine the desired *admissible* steady-state configuration for the given reference and penalize the difference between the steady-state configuration and the desired one. Unfortunately, [13] does not address how to compute the desired admissible steady-state configuration, in particular, for piecewise constant references where the sequences are not known *a priori*; note that computing the desired admissible steady-state configuration is not trivial and its online computation can add extra computational burden [14]–[16]. Given a desired steady-state configuration, [17] develops a MPC that tracks the desired configuration if it is admissible, or else, tracks the "best" admissible configuration; however, [17] does not guarantee output tracking.

This letter addresses the above-mentioned issues by proposing a steady-state-aware MPC for tracking piecewise constant references. Given a desired steady-state configuration, the term steady-state-aware indicates that the proposed MPC guarantees that the steady-state configuration of the system converges to the desired configuration if it is admissible, or else, to the *best* admissible configuration, while ensuring output tracking and constraint satisfaction at all times. Thus, despite prior work, the proposed MPC addresses the problem of tracking piecewise constant references and of steady-state configuration convergence simultaneously, without requiring any prior knowledge on the reference.

II. Problem Statement

Consider the following discrete-time system:

$$x(t+1) =Ax(t)+Bu(t),\quad y(t)=Cx(t)+Du(t), \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^p$ is the control input $y(t)\in\mathbb{R}^m$ is the output. At any t, system (1) should satisfy the following constraints:

$$x(t)\in\mathcal{X},\quad u(t)\in\mathcal{U}, \tag{2}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^p$ are convex and compact sets.

Assumption 2.1: The pair (A, B) is stabilizable.

Let $r \in \mathbb{R}^m$ be the desired reference. Assumption 2.1 implies that there exists at least one steady-state configuration $(\mathbf{x_s}, \mathbf{u_s})$ such that

$$\mathbf{x_s} = A\mathbf{x_s} + B\mathbf{u_s}, \quad r = C\mathbf{x_s} + D\mathbf{u_s}, \tag{3}$$

where $\mathbf{x_s} \in \text{Int}(\mathcal{X})$ and $\mathbf{u_s} \in \text{Int}(\mathcal{U})$. Such a reference is called a steady-state admissible reference; we denote the set of all such references by $\mathcal{R} \subseteq \mathbb{R}^m$. For any $r \in \mathcal{R}$, we denote the set of admissible steady-state configurations by $\mathcal{Z}_r$, i.e., $\mathcal{Z}_r = \{(\mathbf{x_s}, \mathbf{u_s}) \in \mathbb{R}^{n+p} | \mathbf{x_s} = A\mathbf{x_s} + B\mathbf{u_s},\ C\mathbf{x_s} + D\mathbf{u_s} = r,\ \mathbf{x_s} \in \text{Int}(\mathcal{X}), \text{ and } \mathbf{u_s} \in \text{Int}(\mathcal{U})\}$. According to (3), given $r \in \mathcal{R}$, elements of $\mathcal{Z}_r$ must satisfy the following equation:

$$\overbrace{\begin{bmatrix} A - I_n & B & \mathbf{0} \\ C & D & -I_m \end{bmatrix}}^{\mathcal{A}} \begin{bmatrix} \mathbf{x_s} \\ \mathbf{u_s} \\ r \end{bmatrix} = \mathbf{0}, \tag{4}$$

implying that $\begin{bmatrix} \mathbf{x_s}^\top & \mathbf{u_s}^\top & r^\top \end{bmatrix}^\top$ belongs to the null space of $\mathcal{A}$. Assumption 2.1 ensures [15] null space of $\mathcal{A}$ is non-trivial; thus, elements of $\mathcal{Z}_r$ can be characterized [13] as:

$$\begin{bmatrix} \mathbf{x_s}^\top & \mathbf{u_s}^\top & r^\top \end{bmatrix}^\top = \mathcal{M}\theta, \tag{5}$$

where columns of $\mathcal{M} \in \mathbb{R}^{(n+p+m)\times n_\theta}$ form a basis for the null space of $\mathcal{A}$, and $\theta \in \mathbb{R}^{n_\theta}$ is the characterizing vector with n_θ being equal to the nullity of $\mathcal{A}$. Partitioning $\mathcal{M}$ as $\mathcal{M} = \begin{bmatrix} M_1^\top & M_2^\top & L^\top \end{bmatrix}^\top$, where $M_1 \in \mathbb{R}^{n\times n_\theta}$, $M_2 \in \mathbb{R}^{p\times n_\theta}$, and $L \in \mathbb{R}^{m\times n_\theta}$, elements of $\mathcal{Z}_r$ can be expressed as $\mathbf{x_s} = M_1\theta$, $\mathbf{u_s} = M_2\theta$, and $r = L\theta$.

Remark 2.2: According to the above-mentioned discussion, to determine the characterizing matrices, one can follow the following procedure: i) find a basis for the null space of $\mathcal{A}$ given in (4); ii) construct a matrix with basis vectors as its columns; iii) partition the matrix into three row blocks, where the number of rows of the first, second, and third blocks are n, p, and m, respectively.

Problem 2.3: Given $r \in \mathcal{R}$, the desired steady-state configuration $(x_{des}, u_{des}) \in \mathbb{R}^n \times \mathbb{R}^p$, and the initial condition $x(0) \in \mathcal{X}$, obtain an optimal control input that drives the output of system (1) to r and its steady-state configuration to (x_{des}, u_{des}), without violating constraints (2).

III. Steady-State-Aware MPC for Tracking

Let $N \in \mathbb{Z}_{>0}$ be the prediction horizon length. At time instant t, the proposed MPC computes the optimal characterizing vector $\theta^*(t) \in \mathbb{R}^{n_\theta}$ and the optimal control sequence $\mathbf{u}^*(t) := \Big[(u^*(0|t))^\top \ \ \cdots \ \ (u^*(N-1|t))^\top\Big]^\top \in \mathbb{R}^{Np}$ as:

$$\begin{aligned}\theta^*(t), \mathbf{u}^*(t) = \arg\min_{\mathbf{u},\theta} \Bigg(&\sum_{k=0}^{N-1} \|\hat{x}(k|t) - M_1\theta\|_{Q_x}^2 \\ &+ \sum_{k=0}^{N-1} \|u(k|t) - M_2\theta\|_{Q_u}^2 + \|\hat{x}(N|t) - M_1\theta\|_{Q_N}^2 \\ &+ \|r - L\theta\|_{Q_r}^2 + f(\theta)\Bigg),\end{aligned} \tag{6a}$$

subject to the following constraints:

$$\hat{x}(k+1|t) = A\hat{x}(k|t) + Bu(k),\ \hat{x}(0|t) = x(t), \tag{6b}$$

$$\hat{x}(k|t) \in \mathcal{X},\ u(k|t) \in \mathcal{U},\ k \in \{0, \cdots, N-1\}, \tag{6c}$$

$$(\hat{x}(N|t), \theta) \in \Omega. \tag{6d}$$

In (6), $\hat{x}(k|t)$ represents state prediction at instant k, $Q_x = Q_x^\top \succeq 0$, $Q_u = Q_u^\top \succ 0$, $Q_N \succeq 0$, $Q_r = Q_r^\top \succ 0$, and $f : \mathbb{R}^{n_\theta} \to \mathbb{R}_{\geq 0}$ is convex, μ-Lipschitz, and twice-differentiable, such that $f(\theta) = 0 \Leftrightarrow (\mathbf{x_s}, \mathbf{u_s}) = (x_{des}, u_{des})$. Given a terminal control law $u(k) = \kappa\big(\hat{x}(k|t), \theta\big)$ with $\kappa : \mathbb{R}^n \times \mathbb{R}^{n_\theta} \to \mathbb{R}^p$, the terminal constraint set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{n_\theta}$ is designed to ensure that if $\big(\hat{x}(N|t), \theta\big) \in \Omega$, then $\Big(\hat{x}(k|t), \kappa\big(\hat{x}(k|t), \theta\big)\Big) \in \mathcal{X} \times \mathcal{U}$ for all $k \geq N$.

Remark 3.1: Similar to conventional MPC (see, e.g., [8], [9]), it is convenient to select Q_N in (6a) as the solution of the algebraic Riccati equation $Q_N = A^\top Q_N A - (A^\top Q_N B)(Q_u + B^\top Q_N B)^{-1}(B^\top Q_N A) + Q_x$, and use the terminal control law $\kappa(x(t), \theta) = M_2\theta + K(x(t) - M_1\theta)$ with $K = -(Q_u + B^\top Q_N B)^{-1}(B^\top Q_N A)$.

Remark 3.2: The most intuitive choice for the function $f(\theta)$ is the quadratic form $f(\theta) = \|\mathbf{x_s} - x_{des}\|_{Q_{sx}}^2 + \|\mathbf{u_s} - u_{des}\|_{Q_{su}}^2$, where $Q_{sx} = Q_{sx}^\top \succeq 0$ ($Q_{sx} \in \mathbb{R}^{n\times n}$), $Q_{su} = Q_{su}^\top \succeq 0$ ($Q_{su} \in \mathbb{R}^{p\times p}$), and $x_{des} \in \mathbb{R}^n$ and $u_{des} \in \mathbb{R}^p$ are desired steady state and control input determined by the designer. Note that in many applications (see, e.g., [15]), it is desired to minimize the steady-state control effort without any particular preference on the steady state (i.e., $u_{des} = \mathbf{0}$, without any preference on x_{des}); selecting $f(\theta) = \|\mathbf{u_s}\|_{Q_{su}}^2$ would steer the system to a steady-state configuration at which the control effort is minimized.

A. Determining the Terminal Constraint Set Ω

System (1) when controlled by the terminal control law takes the following form:

$$x(t+1) = (A + BK)x(t) + (BM_2 - BKM_1)\theta. \tag{7}$$

Let define the maximal output admissible set as the set of all states x and characterizing vector θ such that the predicted states from the initial state x and with the characterizing vector θ kept constant satisfies the constraints (2), i.e.,

$$\mathcal{O}_\infty = \{(x, \theta) | \hat{x}(\omega|x, \theta) \in \mathcal{X}, \hat{u}(\omega|x, \theta) \in \mathcal{U}, \omega = 0, 1, \cdots\}, \tag{8}$$

where $\hat{x}(\omega|x, \theta)$ is the predicted state at the prediction instant k, which, according to (7), can be computed as follows:

$$\begin{aligned}\hat{x}(\omega|x, \theta) =& (A + BK)^\omega x \\ &+ \sum_{j=1}^{\omega} (A + BK)^{j-1}(BM_2 - BKM_1)\theta,\end{aligned} \tag{9}$$

and $\hat{u}(\omega|x, \theta)$ is the control input at the prediction instant k:

$$\hat{u}(\omega|x, \theta) = M_2\theta + K\left(\hat{x}(\omega|x, \theta) - M_1\theta\right). \tag{10}$$

Therefore, constraint (6d) with $\Omega \subseteq \mathcal{O}_\infty$ ensures that if the characterizing vector θ is kept constant from the time instant

$N+t$ onward, the ensuing state when the terminal control law is applied will always satisfy constraints (2).

Although $\Omega = \mathcal{O}_\infty$ is an acceptable choice, it is usually avoided as the set $\mathcal{O}_\infty$ is not finitely determined (i.e., it cannot be described by a finite set of constraints). However, Appendix A implies that [2], [13], [18] the terminal constraint set defined as $\Omega = \mathcal{O}_\infty \cap \mathcal{O}^\epsilon$, where $\mathcal{O}^\epsilon = \{\theta | M_1\theta \in (1-\epsilon)\mathcal{X}, M_2\theta \in (1-\epsilon)\mathcal{U}\}$ for some $\epsilon \in (0,1)$, is finitely determined and positively invariant. That is, there exists a finite index ω^* such that Ω can be defined as $\Omega = \mathcal{O}_{\omega^*} \cap \mathcal{O}^\epsilon$. The value of index ω^* can be obtained by solving a sequence of mathematical programming problems detailed in [18].

Remark 3.3: According to the above-mentioned discussion, and equations (9) and (10), constraint (6d) can be implemented by $2\omega^*+2$ constraints given in (11), where $\epsilon \in (0,1)$ (typically small) is a design parameter.

IV. Theoretical Analysis

Theorem 4.1 (Recursive Feasibility): Consider system (1) which is subject to constraints (2). Suppose that (6) is feasible at $t=0$. Then, it remains feasible for all $t>0$.

Proof: Suppose that MPC (6) is feasible at t, where the optimal characterizing vector and the optimal control sequence are denoted by $\theta^*(t)$ and $\mathbf{u}^*(t) = \begin{bmatrix}(u^*(0|t))^\top & \cdots & (u^*(N-1|t))^\top\end{bmatrix}^\top$, respectively. Also, $(\hat{x}(N|t), \theta^*(t)) \in \Omega$, where $\hat{x}(N|t)$ is the terminal state given the optimal control sequence $\mathbf{u}^*(t)$.

Since the terminal constraint set Ω is positively invariant (see Subsection III-A and Equation (11)), it is concluded that $\kappa\big(\hat{x}(N|t), \theta^*(t)\big) = K(A+BK)\hat{x}(N|t) + K(BM_2 - BKM_1)\theta^*(t) + (BM_2 - BKM_1)\theta^*(t) \in \mathcal{U}$, $\hat{x}(N+1|t) = A\hat{x}(N|t) + B\kappa\big(\hat{x}(N|t), \theta^*(t)\big) \in \mathcal{X}$, and $(\hat{x}(N+1|t), \theta^*(t)) \in \Omega$.Thus, the characterizing vector $\theta^*(t)$ and the control sequence $\big[\,(u^*(1|t))^\top \;\; \cdots \;\; (u^*(N-1|t))^\top \;\; \big(\kappa\big(\hat{x}(N|t), \theta^*(t)\big)\big)^\top\,\big]^\top$ construct a feasible solution for the proposed steady-state-aware MPC at time instant $t+1$. Therefore, feasibility at time instant t implies feasibility at time instant $t+1$, meaning that the proposed MPC is recursively feasible. ∎

Theorem 4.2 (Closed-Loop Stability): Suppose that the MPC given in (6) is used to address Problem 2.3. Then,

- if $(x_{des}, u_{des}) \in \mathcal{Z}_r$, $y(t) \to r$ and $(\mathbf{x_s}, \mathbf{u_s}) \to (x_{des}, u_{des})$ as $t \to \infty$.
- if $(x_{des}, u_{des}) \notin \mathcal{Z}_r$, $y(t) \to L\bar{\theta}$ and $(\mathbf{x_s}, \mathbf{u_s}) \to \big(M_1\bar{\theta}, M_2\bar{\theta}\big)$ as $t \to \infty$, where $\bar{\theta}$ satisfies:

$$\big\|\theta^\diamond - \bar{\theta}\big\| \le \mu / \underline{\lambda}(L^\top Q_r L), \tag{12}$$

with $\theta^\diamond$ being the solution of the following problem:

$$\theta^\diamond = \arg\min_\theta f(\theta), \text{ s.t. } (M_1\theta, M_2\theta) \in \mathcal{Z}_r. \tag{13}$$

Proof: Let $J\big(\theta, \mathbf{u}|x(t)\big)$ and $J\big(\theta, \mathbf{u}|x(t+1)\big)$ be the cost functions of the MPC given in (6) at time instants t and $t+1$, respectively. Also, let $\big(\theta^*(t), \mathbf{u}^*(t)\big)$ and $\big(\theta^*(t+1), \mathbf{u}^*(t+1)\big)$ be the optimal solutions at time instants t and $t+1$, respectively. First, we show that $J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \le 0$.

According to the optimality of the solution $\big(\theta^*(t+1), \mathbf{u}^*(t+1)\big) \in \mathbb{R}^{n_\theta} \times \mathbb{R}^{Np}$ at time instant $t+1$, we have:

$$\begin{aligned} &J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) \\ &\qquad \le J\big(\theta^*(t), \mathbf{u}^*(t+1)|x(t+1)\big). \end{aligned} \tag{14}$$

where $\theta^*(t)$ is the optimal characterizing vector at t. Subtracting $J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$ from both sides of (14) yields:

$$\begin{aligned} &J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \\ &\quad \le J\big(\theta^*(t), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big). \end{aligned} \tag{15}$$

From (6a), and since $u^*(k+1|t) = u^*(k|t+1)$ and $\hat{x}(k+1|t) = \hat{x}(k|t+1)$, $k = 0, \cdots, N-2$, (15) implies that:

$$\begin{aligned} &J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \\ &\le \|\hat{x}(N|t+1) - M_1\theta^*(t)\|^2_{Q_N} - \|\hat{x}(N|t) - M_1\theta^*(t)\|^2_{Q_N} \\ &+ \|\hat{x}(N-1|t+1) - M_1\theta^*(t)\|^2_{Q_x} \\ &+ \|u(N-1|t+1) - M_2\theta^*(t)\|^2_{Q_u} \\ &- \|\hat{x}(0|t) - M_1\theta^*(t)\|^2_{Q_x} \\ &- \|u^*(0|t) - M_2\theta^*(t)\|^2_{Q_u}. \end{aligned} \tag{16}$$

As shown in [19], when the matrix Q_N is determined via the algebraic Riccati equation discussed in Remark 3.1, for any $\theta^*(t) \in \mathbb{R}^{n_\theta}$, we have $\big\|\hat{x}(N|t+1) - M_1\theta^*(t)\big\|^2_{Q_N} - \big\|\hat{x}(N|t) - M_1\theta^*(t)\big\|^2_{Q_N} + \big\|\hat{x}(N-1|t+1) - M_1\theta^*(t)\big\|^2_{Q_x} + \big\|u(N-1|t+1) - M_2\theta^*(t)\big\|^2_{Q_u} \le 0$. Thus, (16) implies that:

$$\begin{aligned} &J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \\ &\le -\big\|\hat{x}(0|t) - M_1\theta^*(t)\big\|^2_{Q_x} - \big\|u^*(0|t) - M_2\theta^*(t)\big\|^2_{Q_u} \le 0 \end{aligned} \tag{17}$$

At this stage, we use LaSalle invariance principle [20] to show that the only entire trajectory that satisfies $J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \equiv 0$ is the desired steady-state configuration (x_{des}, u_{des}) if $(x_{des}, u_{des}) \in \mathcal{Z}_r$, and is $\big(M_1\bar{\theta}, M_2\bar{\theta}\big)$ if $(x_{des}, u_{des}) \notin \mathcal{Z}_r$.

$$(A+BK)^\omega \hat{x}(N|t) + \sum_{j=1}^{\omega} (A+BK)^{j-1}(BM_2 - BKM_1)\theta \in \mathcal{X}, \; \omega \in \{0, 1, \cdots \omega^*\}, \tag{11a}$$

$$K(A+BK)^\omega \hat{x}(N|t) + K\sum_{j=1}^{\omega} (A+BK)^{j-1}(BM_2 - BKM_1)\theta + (BM_2 - BKM_1)\theta \in \mathcal{U}, \; \omega \in \{0, 1, \cdots \omega^*\}, \tag{11b}$$

$$M_1\theta \in (1-\epsilon)\mathcal{X}, \quad M_2\theta \in (1-\epsilon)\mathcal{U}. \tag{11c}$$

On the one hand, as shown in Appendix B, imposing $J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = 0$ for $t \geq t^\dagger$ implies that the optimal cost function $J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$ takes the following form for $t \geq t^\dagger$:

$$J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = \|r - L\theta^*(t)\|_{Q_r}^2 + f\left(\theta^*(t)\right). \quad (18)$$

On the other hand, from optimality of the solution $\big(\theta^*(t), \mathbf{u}^*(t)\big)$ for $t \geq t^\dagger$, we have:

$$J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \leq J\big(\theta^\diamond, \mathbf{u}^*(t)|x(t)\big),\ t \geq t^\dagger, \quad (19)$$

where $\theta^\diamond$ is as in (13). Since $J\big(\theta^\diamond, \mathbf{u}^*(t)|x(t)\big) = f\left(\theta^\diamond\right)$, (18) and (19) imply that:

$$\|r - L\theta^*(t)\|_{Q_r}^2 + f\left(\theta^*(t)\right) \leq f\left(\theta^\diamond\right),\ t \geq t^\dagger. \quad (20)$$

Now, consider the following two cases.

- Case I—$(x_{des}, u_{des}) \in \mathcal{Z}_r$: We have $\theta^\diamond = \theta_{des}$, where $\theta_{des} \in \mathbb{R}^{n_\theta}$ is the characterizing vector corresponding to the desired steady-state configuration (x_{des}, u_{des}). Since $f\left(\theta^\diamond = \theta_{des}\right) = 0$, (20) implies that $\|r - L\theta^*(t)\|_{Q_r}^2 = f\left(\theta^*(t)\right) = 0$ for $t \geq t^\dagger$; consequently $L\theta^*(t) = r$, $M_1\theta^*(t) = x_{des}$, and $M_2\theta^*(t) = u_{des}$ for $t \geq t^\dagger$.
- Case II—$(x_{des}, u_{des}) \notin \mathcal{Z}_r$: From (13), we have $L\theta^\diamond = r$. Furthermore, we have $\underline{\lambda}(L^\top Q_r L)\left\|\theta^\diamond - \theta^*(t)\right\|^2 \leq \|L\theta^\diamond - L\theta^*(t)\|_{Q_r}^2$. Thus, according to Lipschitz continuity of the function $f(\theta)$, (20) implies that

$$\begin{aligned}&\underline{\lambda}(L^\top Q_r L)\left\|\theta^\diamond - \theta^*(t)\right\|^2 \leq f\left(\theta^\diamond\right) - f\left(\theta^*(t)\right)\\ &\leq \left\|f\left(\theta^\diamond\right) - f\left(\theta^*(t)\right)\right\| \leq \mu\left\|\theta^\diamond - \theta^*(t)\right\|, \quad (21)\end{aligned}$$

for $t \geq t^\dagger$, which yields the inequality (12). ∎

Remark 4.3: Since Q_r and the function $f(\theta)$ are design parameters, the upper-bound in (12) can be made arbitrarily small; thus, the output tracks r and the steady-state configuration converges to the best admissible one.

Remark 4.4: Consider the reference r, desired steady state x_{des}, and desired steady input u_{des}. It is not unusual that the steady-state configuration (x_{des}, u_{des}) is not admissible associated with the desired reference r. To deal with such situations in predictive controllers, prior work [21], [22] suggests to add an upper level steady-state optimizer to decide the best admissible steady-state configuration; note that this approach can add extra computational burden [14]–[16]. According to Theorem 4.2, the proposed MPC scheme steers the system to the optimal steady-state configuration according to the offset cost function $f(\theta)$, while ensuring output tracking. Then, it can be concluded that the proposed MPC scheme has a built-in steady-state configuration optimizer, and $f(\theta)$ defines the function to be optimized.

V. Local Infinite-Horizon Optimality

Let $r \in \mathcal{R}$ be the desired reference and $(x_{des}, u_{des}) \in \mathcal{Z}_r$ be the desired steady-state configuration. Let $\theta_{des} \in \mathbb{R}^{n_\theta}$ be the characterizing vector corresponding to the desired steady-state configuration, i.e., $x_{des} = M_1\theta_{des}$ and $u_{des} = M_2\theta_{des}$. Furthermore, let $f(\theta) = \|M_1\theta - x_{des}\|_{Q_{sx}}^2 + \|M_2\theta - u_{des}\|_{Q_{su}}^2$ (see Remark 3.2). Thus, we have

$$\underline{\lambda}(L^\top Q_r L)\left\|\theta - \theta_{des}\right\|^2 \leq \lambda_M\left\|\theta - \theta_{des}\right\|^2 \quad (22)$$

where $\lambda_M := \overline{\lambda}(L^\top Q_r L) + \overline{\lambda}(M_1^\top Q_{sx} M_1) + \overline{\lambda}(M_2^\top Q_{su} M_2)$. It is obvious that $\lambda_M \geq \underline{\lambda}(L^\top Q_r L)$.

The standard MPC control law to regulate the output of the system to r can be derived from the solution of the optimization problem (6) subject to the constraint $\left\|\theta - \theta_{des}\right\|^2 = 0$; we use $J_s\big(\theta, \mathbf{u}|x(t)\big)$ to denote the cost function of the resulting optimization problem and $\mathbb{X}_N^{MPC}$ to denote its RoA.

Theorem 5.1: For sufficiently large $\underline{\lambda}(L^\top Q_r L)$, we have:

- The proposed MPC is equal to the MPC for regulation.
- If the terminal control gain K is chosen as in Remark 3.1, then the control law generated by the proposed MPC is equal to the constrained Linear Quadratic Programming (LQR) control law.

Proof: Consider the optimization problem (6) where the term $\|r - L\theta\|_{Q_r}^2 + f(\theta)$ in (6a) is replaced with $\underline{\lambda}(L^\top Q_r L)\left\|\theta - \theta_{des}\right\|^2$; we denote the resulting cost function by $J_1\big(\theta, \mathbf{u}|x(t)\big)$. Also, we use $J_2\big(\theta, \mathbf{u}|x(t)\big)$ to denote the cost function in which the term $\|r - L\theta\|_{Q_r}^2 + f(\theta)$ is replaced with $\lambda_M\left\|\theta - \theta_{des}\right\|^2$. It is obvious that $J_1\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \leq J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \leq J_2\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$, where $\big(\theta^*(t), \mathbf{u}^*(t)\big) \in \mathbb{R}^{n_\theta} \times \mathbb{R}^{Np}$ is the optimal solution at time instant t.

In virtue of the exact penalty method, there exist $\sigma \in \mathbb{R}_{>0}$ such that for $\underline{\lambda}(L^\top Q_r L) \geq \sigma$, we have $J_1\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = J_s\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$. Also, since $\lambda_M \geq \underline{\lambda}(L^\top Q_r L) \geq \sigma$, we have $J_2\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = J_s\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$. Therefore, according to the above-mentioned discussion, we have $J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = J_s\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$, which implies the first statement.

For what regards the second statement, let $\theta = \theta_{des}$ and let $\mathbb{X}_N^{IH}$ be the set of all states such that the infinite-horizon optimal trajectories enter the terminal constraint set Ω after N steps. Then, the second statement can be derived according to the fact [12], [17] that $\mathbb{X}_N^{IH} \subseteq \mathbb{X}_N^{MPC}$. ∎

VI. Comparison Study

Consider a system with the following matrices [13]:

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\ B = \begin{bmatrix}0.0 & 0.5\\ 1.0 & 0.5\end{bmatrix},\ C = \begin{bmatrix}1 & 0\end{bmatrix}, \quad (23)$$

which is subject to the constraints $|x_i(t)| \leq 5$ and $|u_i(t)| \leq 0.3,\ i \in \{1, 2\}$. The characterizing matrices are:

$$M_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ M_2 = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix},\ N = \begin{bmatrix}1 & 0\end{bmatrix}. \quad (24)$$

Let $r = 4.95$, and the weighting matrices be chosen as $Q_x = \text{diag}\{1, 1\}$ and $Q_u = \text{diag}\{1, 1\}$. We use the method described in [9] to determine the terminal control law $\kappa(\cdot)$, matrix Q_N, and the terminal constraint set Ω. To implement the proposed steady-state-aware MPC, we set $Q_r = 1$. We want to minimize the steady-state control effort, without any particular preference on the steady state; thus, according to Remark 3.2, we select $f(\theta) = \|\mathbf{u_s}\|^2$.

Simulations are run on an 13th Gen Intel® Core™ i9-13900K processor with 64GB RAM. We use YALMIP toolbox to implement the computations of the methods.

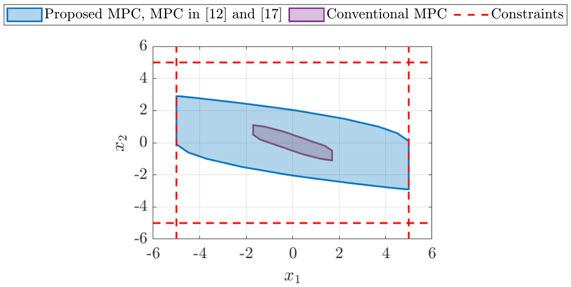

Fig. 1. RoA of the conventional MPC, of the proposed steady-state-aware MPC, and of MPC schemes presented in [12] and [17] for $N = 3$.

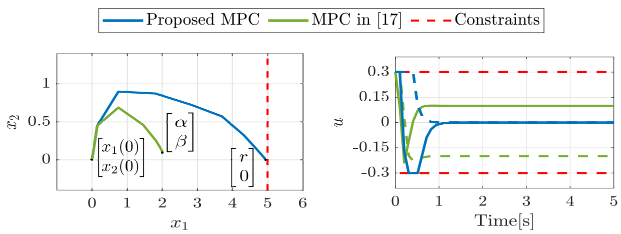

Fig. 2. Achieved steady-state configuration with the proposed steady-state-aware MPC and MPC scheme described in [17]; in the right figure, solid line represents $u_1(t)$ and dashed line represents $u_2(t)$.

A. Comparison with respect to the RoA

The RoA of the conventional MPC, of the proposed MPC, and of the MPC in [12] and [17] are shown in Fig. 1 for $N = 3$. As seen in this figure, the proposed MPC and those described in [12] and [17] provide the same RoA.

B. Comparison with respect to computing time

To provide a quantitative comparison, we consider 2,000 experiments with initial condition $x(0) = [\alpha\ 0]^\top$, where in each experiment, α is uniformly selected from the interval [-5,5]. TABLE I reports the computing time of the proposed MPC and MPC schemes described in [12] and [17]; for any N, the methods have a comparable computing time.

C. Comparison with respect to output tracking performance

TABLE II reports the statistics of 2,000 experiments on the achieved performance with the proposed MPC and the one described in [12], where Performance Index (PI) $:= \sum_{t=0}^{50} \|y(t) - r\|^2$. From this table, both methods yield a comparable performance index.

For what regards MPC in [17], let $x_{des} = [\alpha\ \beta]^\top$ and $u_{des} = [0\ 0]^\top$, where $\alpha, \beta \in \mathbb{R}$. When $\alpha \in [-5, 5]$ and $\beta \in [-0.15, 0.15]$, MPC in [17] steers the steady state and input to $\mathbf{x_s} = [\alpha\ \beta]^\top$ and $\mathbf{u_s} = [\beta, -2\beta]^\top$, respectively, implying that the output converges to α; thus, if $\alpha \neq r$, [17] does not guarantee output tracking; see Fig. 2.

D. Comparison with respect to steady-state configuration

It can be shown that for the system given in (23), we have $\mathcal{Z}_r = \{(\mathbf{x_s}, \mathbf{u_s}) | \mathbf{x_s} = [r\ \theta]^\top, \mathbf{u_s} = [\theta\ -2\theta]^\top, \theta \in [-0.15, 0.15]\}$. Suppose that $x_{des} = [r\ \beta]^\top$ and $u_{des} = [0\ 0]^\top$.

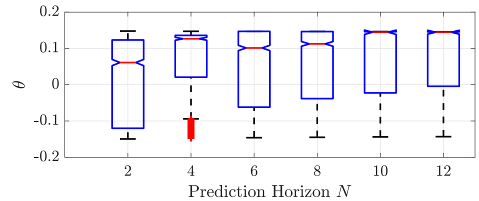

Fig. 3. Distribution of the achieved steady-state configuration with the MPC scheme described in [12].

The proposed MPC steers the characterizing vector θ to zero, which implies that $\mathbf{u_s}$ converges to $[0\ 0]^\top$.

When the MPC in [17] is applied, even though the steady input $\mathbf{u_s}$ converges to a single value, it does not converge to u_{des}; using the MPC in [17], the characterizing vector θ converges to β implying that $\mathbf{u_s}$ converges to $[\beta\ \ -2\beta]^\top$.

When the MPC in [12] is employed, θ may converge to any value between -0.15 and 0.15, which implies that the steady input $\mathbf{u_s}$ does not converge to a single value. Fig. 3 presents the distribution of θ for 2,000 experiments (details are provided above) with MPC in [12].

VII. Conclusion

This letter proposed a steady-state-aware MPC which is capable of addressing tracking objectives and guaranteeing constraint satisfaction with any prediction horizon length, without adding any extra computational burden to the system. Closed-loop stability, recursive feasibility, and local infinite-horizon optimality of the proposed MPC have been proven analytically. The effectiveness of the proposed MPC has been assessed in comparison with prior work.

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TABLE I

MEAN COMPUTING TIME (CT) OF THE PROPOSED STEADY-STATE-AWARE MPC AND MPC SCHEMES DESCRIBED IN [12] AND [17].

CT	$N=2$	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$
Proposed MPC	0.0426 ± 0.001	0.0474 ± 0.001	0.0536 ± 0.001	0.0618 ± 0.001	0.0720 ± 0.001	0.0862 ± 0.001
MPC in [12]	0.0453 ± 0.0024	0.0494 ± 0.0017	0.0554 ± 0.0009	0.0632 ± 0.0011	0.0737 ± 0.0016	0.0879 ± 0.0018
MPC in [17]	0.0436 ± 0.0008	0.0482 ± 0.0009	0.0540 ± 0.0004	0.0614 ± 0.0003	0.0705 ± 0.0005	0.0814 ± 0.0007

TABLE II

MEAN PERFORMANCE INDEX (PI) WITH THE PROPOSED STEADY-STATE-AWARE MPC AND MPC SCHEME DESCRIBED IN [12].

PI	$N=2$	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$
Proposed MPC	27.4791 ± 19.68	22.3833 ± 16.10	23.0800 ± 16.03	22.7131 ± 15.98	22.8107 ± 16.17	22.3675 ± 15.75
MPC in [12]	24.9635 ± 18.11	22.3875 ± 15.81	23.1144 ± 16.03	22.7245 ± 15.97	22.8141 ± 16.16	22.3608 ± 15.73

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APPENDIX A
SUPPLEMENTARY DISCUSSION FOR SET Ω

For a fixed θ, consider the extended state $\phi(t) = \begin{bmatrix} x(t)^\top & \theta^\top \end{bmatrix}^\top$. System (7) can be expressed as follows:

$$\phi(t+1) = \Psi\phi(t), \quad \mathcal{Y}(t) = x(t) = \mathcal{C}\phi(t), \tag{A.1}$$

where $\Psi = \begin{bmatrix} A+BK & BM_2 - BKM_1 \\ \mathbf{0} & I_{n_\theta} \end{bmatrix}$, and $\mathcal{C} = \begin{bmatrix} I_n & \mathbf{0} \end{bmatrix}$. Since $A+BK$ is Schur (see Remark 3.1), it can be easily shown that $|\lambda_i(\Psi)| \leq 1,\ \forall i$, and $|\lambda_i(\Psi)| = 1$ implies that $\lambda_i(\Psi)$ is simple, where $\lambda_i(\Psi)$ indicates the ith eigenvalue of matrix Ψ. Also, it is obvious that the pair $(\mathcal{C}, \Psi)$ is observable, implying [18] that $\mathcal{O}_\infty$ given in (8) is bounded.

APPENDIX B
SUPPLEMENTARY DISCUSSION FOR THEOREM 4.2

According to (17), imposing $J\big(\theta^*(t+1), \mathbf{u}^*(t+1)|x(t+1)\big) - J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = 0$ for $t \geq t^\dagger$ implies that:

$$\hat{x}(0|t) = M_1\theta^*(t),\ u^*(0|t) = M_2\theta^*(t), \quad t \geq t^\dagger. \tag{A.2}$$

Since $\hat{x}(0|t) = x(t)$ and $\hat{x}(1|t) = \hat{x}(0|t+1) = x(t+1)$, it follows from (1) and (A.2) that for any $t \geq t^\dagger$, we have:

$$M_1\theta^*(t+1) = AM_1\theta^*(t) + BM_2\theta^*(t). \tag{A.3}$$

Since $M_1\theta^*(t) = AM_1\theta^*(t) + BM_2\theta^*(t)$ for all t (see Eq. (4)), it follows from (A.3) that $M_1\theta^*(t+1) = M_1\theta^*(t),\ t \geq t^\dagger$, which implies that $\hat{x}(1|t) = \hat{x}(0|t) = M_1\theta^*(t),\ t \geq t^\dagger$.

Following the same procedure, it can be easily shown that $\hat{x}(k|t) = \hat{x}(0|t) = M_1\theta^*(t),\ t \geq t^\dagger,\ k \in \mathbb{Z}_{\geq 0}$. Hence,

$$\begin{cases} \sum_{k=0}^{N-1} \|\hat{x}(k|t) - M_1\theta^*(t)\|_{Q_x}^2 = 0 \\ \|\hat{x}(N|t) - M_1\theta^*(t)\|_{Q_N}^2 = 0 \end{cases},\ \forall t \geq t^\dagger. \tag{A.4}$$

As a result, the optimal cost function $J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big)$ takes the following form for $t \geq t^\dagger$:

$$J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = \sum_{k=0}^{N-1} \|u^*(k|t) - M_2\theta^*(t)\|_{Q_u}^2 + \|r - L\theta^*(t)\|_{Q_r}^2 + f\left(\theta^*(t)\right). \tag{A.5}$$

Let $\tilde{\mathbf{u}}(t) := \Big[\big(u^*(0|t)\big)^\top \cdots \big(u^*(0|t)\big)^\top\Big]^\top \in \mathbb{R}^{Np}$. From the optimality of the solution at any $t \geq t^\dagger$, we have

$$J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) \leq J\big(\theta^*(t), \tilde{\mathbf{u}}(t)|x(t)\big), \tag{A.6}$$

which implies that $\sum_{k=0}^{N-1} \|u^*(k|t) - M_2\theta^*(t)\|_{Q_u}^2 \leq \sum_{k=0}^{N-1} \|u^*(0|t) - M_2\theta^*(t)\|_{Q_u}^2$. According to (A.2), the right-hand side of this last inequality is zero, which implies that $\sum_{k=0}^{N-1} \|u^*(k|t) - M_2\theta^*(t)\|_{Q_u}^2 = 0$. Thus, it follows from (A.6) that the optimal cost function takes the following form for $t \geq t^\dagger$:

$$J\big(\theta^*(t), \mathbf{u}^*(t)|x(t)\big) = \|r - L\theta^*(t)\|_{Q_r}^2 + f\left(\theta^*(t)\right),\ t \geq t^\dagger. \tag{A.7}$$