WEYL FORMULAE FOR SCHRÖDINGER OPERATORS WITH CRITICALLY SINGULAR POTENTIALS

XIAOQI HUANG AND CHRISTOPHER D. SOGGE

Abstract. We obtain generalizations of classical versions of the Weyl formula involving Schrödinger operators $H_V = -\Delta_g + V(x)$ on compact boundaryless Riemannian manifolds with critically singular potentials V. In particular, we extend the classical results of Avakumović [1], Levitan [13] and Hörmander [8] by obtaining $O(\lambda^{n-1})$ bounds for the error term in the Weyl formula in the universal case when we merely assume that V belongs to the Kato class, $\mathcal{K}(M)$, which is the minimal assumption to ensure that H_V is essentially self-adjoint and bounded from below or has favorable heat kernel bounds. In this case, we can also obtain extensions of the Duistermaat-Guillemin [4] theorem yielding $o(\lambda^{n-1})$ bounds for the error term under generic conditions on the geodesic flow, and we can also extend Bérard's [2] theorem yielding $O(\lambda^{n-1}/\log\lambda)$ error bounds under the assumption that the principal curvatures are non-positive everywhere. We can obtain further improvements for tori, which are essentially optimal, if we strengthen the assumption on the potential to $V \in L^p(M) \cap \mathcal{K}(M)$ for appropriate exponents $p = p_n$.

1. **Introduction.**

The purpose of this paper is to prove Weyl formulae for Schrödinger operators

$$H_V = -\Delta_g + V(x) \tag{1.1}$$

on smooth compact n-dimensional Riemannian manifolds (M, g). We shall assume throughout that the potentials V are real-valued. Moreover, we shall assume that

$$V \in \mathcal{K}(M), \tag{1.2}$$

where $\mathcal{K}(M)$ denotes the Kato class. Recall that $\mathcal{K}(M)$ is all V satisfying

$$\lim_{\delta\to 0}\Big(\sup_{x\in M}\int_{B(x,\delta)} |V(y)|\, h_n(d_g(x,y))\, dy\Big) = 0, \tag{1.3}$$

where d_g, dy and $B(x,\delta)$ denote geodesic distance, the volume element and the geodesic ball of radius δ about x associated with the metric g on M, respectively, and

$$h_n(r) = \begin{cases} r^{2-n}, & n \ge 3 \\ \log(2 + 1/r), & n = 2. \end{cases} \tag{1.4}$$

For later use, note that $\mathcal{K}(M) \subset L^1(M)$.

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As was shown in [3] (see also [15]) the assumption that V is in the Kato class is needed to ensure that the eigenfunctions of H_V are bounded. If H_V has unbounded eigenfunctions, then its spectral projection kernels will be unbounded for large enough λ, and obtaining spectral bounds in this situation seems far-fetched. The assumption that $V \in \mathcal{K}(M)$ ensures that this is not the case.

Moreover, if V is as in (1.2) then the Schrödinger operator H_V in (1.1) is self-adjoint and bounded from below. Additionally, in this case, since M is compact, the spectrum of H_V is discrete. Also, (see [15]) the associated eigenfunctions are continuous. Assuming, as we may, that H_V is a positive operator, we shall write the spectrum of $\sqrt{H_V}$ as

$$\{\tau_k\}_{k=1}^{\infty}, \tag{1.5}$$

where the eigenvalues, $\tau_1 \le \tau_2 \le \cdots$, are arranged in increasing order and we account for multiplicity. For each τ_k there is an eigenfunction $e_{\tau_k} \in \mathrm{Dom}\,(H_V)$ (the domain of H_V) so that

$$H_V e_{\tau_k} = \tau_k^2 e_{\tau_k}. \tag{1.6}$$

We shall always assume that the eigenfunctions are L^2-normalized, i.e.,

$$\int_M |e_{\tau_k}(x)|^2\,dx = 1.$$

After possibly adding a constant to V we may, and shall, assume throughout that H_V is bounded below by one, i.e.,

$$\|f\|_2^2 \le \langle\, H_V f,\, f\,\rangle, \quad f \in \mathrm{Dom}\,(H_V). \tag{1.7}$$

Also, to be consistent, we shall let

$$H^0 = -\Delta_g + 1 \tag{1.8}$$

be the unperturbed operator also enjoying this lower bound. The corresponding eigenvalues and associated L^2-normalized eigenfunctions are denoted by $\{\lambda_j\}_{j=1}^{\infty}$ and $\{e_j^0\}_{j=1}^{\infty}$, respectively so that

$$H^0 e_j^0 = \lambda_j^2 e_j^0, \quad \text{and} \quad \int_M |e_j^0(x)|^2\,dx = 1. \tag{1.9}$$

Both $\{e_{\tau_k}\}_{k=1}^{\infty}$ and $\{e_j^0\}_{j=1}^{\infty}$ are orthonormal bases for $L^2(M)$. Recall (see e.g. [16]) that if $N^0(\lambda)$ denotes the Weyl counting function for H^0 then one has the "sharp Weyl formula"

$$N^0(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\,\lambda^n \,+\, O(\lambda^{n-1}), \quad N^0(\lambda) = \#\{j : \lambda_j \le \lambda\}, \tag{1.10}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $\mathrm{Vol}_g(M)$ denotes the Riemannian volume of M. This result is due to Avakumović [1] and Levitan [13], and it was generalized to general self-adjoint elliptic pseudo-differential operators by Hörmander [8]. The bound in (1.10) cannot be improved for the standard round sphere, which accounts for the nomenclature "sharp Weyl formula".

The main goal of this paper is to show that this sharp Weyl formula also holds for the operators H_V in (1.1) involving critically singular potentials V as in (1.2). Specifically we have the following.

Theorem 1.1. *Let $V \in \mathcal{K}(M)$ and let H_V as above and set*

$$N_V(\lambda) = \#\{k : \tau_k \le \lambda\}. \tag{1.11}$$

We then have

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\, \lambda^n + O(\lambda^{n-1}). \tag{1.12}$$

We shall also be able to obtain improved counting estimates under certain geometric assumptions.

The first such result is an extension of the Duistermaat-Guillemin theorem [4]. Recall the assumption in this theorem is that the set $\mathcal{C} \subset S^*M$ of all (x, ξ) lying on a periodic geodesic in S^*M have measure zero (see [16]). Here, S^*M denotes the unit cotangent bundle of (M, g). In this case Duistermaat and Guillemin [4] showed that one can improve the bounds for the error term Weyl law (1.10) (assuming that $V = 0$ or V is smooth) to be $o(\lambda^{n-1})$. The proof of this relies on Hörmander's theory of propagation of singularities for smooth pseudo-differential operators. Even though this theory does not apply to our situation involving very singular potentials, we can extend the theorem of Duistermaat and Guillemin to include the above operators.

Theorem 1.2. *Let $V \in \mathcal{K}(M)$ and let H_V be as above, and assume that the set $\mathcal{C}$ of directions of periodic geodesics has measure zero in S^*M. Then*

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\, \lambda^n + o(\lambda^{n-1}). \tag{1.13}$$

We also can extend the classical theorem of Bérard [2].

Theorem 1.3. *Assume that the sectional curvatures of (M, g) are non-positive. Then, if $V \in \mathcal{K}(M)$,*

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\, \lambda^n + O(\lambda^{n-1}/\log\lambda). \tag{1.14}$$

In the special case of the torus, we can do much better.

Theorem 1.4. *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric, and assume that $V \in \mathcal{K}(M)$ when $n = 2$ and $V \in L^p(M) \cap \mathcal{K}(M)$ for some $p > \frac{2n}{n+2}$ if $n \ge 3$. Then*

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\, \lambda^n + O(\lambda^{n-2+2/(n+1)}). \tag{1.15}$$

Moreover, if $V \in L^2(M) \cap \mathcal{K}(M)$ and $n \ge 4$, we have

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\, \lambda^n + O(\lambda^{n-2+\varepsilon}). \tag{1.16}$$

If $V \equiv 1$, the bounds in (1.15) are the classical results of Hlwaka [7]. The same bounds hold for irrational tori. Also, with the stronger condition on the potential V in the second part of the Theorem, if we use more recent improved bounds for the error term in the Weyl formula for $V \equiv 1$ and related bounds for the trace of certain spectral projection operators, we obtain the improved bounds in (1.16) involving singular potentials.

2. **Preliminaries.**

We first recall that, if as above, $\{e_{\tau_k}\}$ is an orthonormal basis of eigefunctions of H_V then

$$N_V(\lambda) = \#\{k : \tau_k \le \lambda\} = \int_M \sum_{\tau_k \le \lambda} |e_{\tau_k}(x)|^2 \, dx. \tag{2.1}$$

Thus, $N_V(\lambda)$ is the *trace* of the spectral function

$$E^V_\lambda(x, y) = \sum_{\tau_k \le \lambda} e_{\tau_k}(x) e_{\tau_k}(y). \tag{2.2}$$

Here, we are assuming, as we may, that all the eigenfunctions of H_V in our orthonormal basis are real-valued. To simplify the notation, as we may, we shall assume the same for those of H^0, i.e., the $\{e^0_j\}$.

To prove the Weyl formula (1.12), we shall need the fact that if we consider the kernels of the unit band spectral projection operators

$$\chi^V_\lambda(x, y) = \sum_{\tau_k \in [\lambda, \lambda+1)} e_{\tau_k}(x) e_{\tau_k}(y) \tag{2.3}$$

for H_V, then we have

$$\int_M \chi^V_\lambda(x, x) \, dx = O(\lambda^{n-1}), \;\; \lambda \ge 1. \tag{2.4}$$

We postpone the proof of (2.4) until the last section. Also observe that if we assume additionally $V \in L^{\frac{n}{2}}(M) \cap \mathcal{K}(M)$, (2.4) would be a consequence of the kernel bound

$$\chi^V_\lambda(x, y) = O(\lambda^{n-1})$$

which is proved in [3].

The general strategy behind the proof of Theorem 1.1 will be to exploit the classical results of Avakumović [1] and Levitan [13] telling us that if,

$$E^0_\lambda(x, y) = \sum_{\lambda_j \le \lambda} e^0_j(x) e^0_j(y),$$

is the spectral function corresponding to $V \equiv 1$ then we have the following estimate for its trace

$$\int_M \sum_{\lambda_j \le \lambda} |e^0_j(x)|^2 \, dx = N^0(\lambda) = (2\pi)^{-n} \omega_n \mathrm{Vol}_g(M) \lambda^n + O(\lambda^{n-1}), \tag{2.1$'$}$$

and then compare the trace of E^0_λ with that of the one E^V_λ for the perturbed operators.

To do this, we shall follow the classical approach of rewriting these traces using the wave equation. To this end, let $P^0 = \sqrt{H^0}$ and $P_V = \sqrt{H_V}$ be the square roots of the two Hamiltonians. Then since the Fourier transform of the indicator function $\mathbb{1}_\lambda(\tau)$ is $2\frac{\sin \lambda t}{t}$, we have for λ not in the spectrum of P^0

$$N^0(\lambda) = \frac{1}{\pi} \int_M \int_{-\infty}^{\infty} \frac{\sin t\lambda}{t} \big(\cos tP^0\big)(x, x) \, dt dx, \tag{2.5}$$

if

$$\big(\cos(tP^0)\big)(x,y) = \sum_j \cos t\lambda_j e_j^0(x) e_j^0(y) \tag{2.6}$$

is the kernel of the solution operator for $f \to (\cos tP^0)f = u^0(t,x)$, where u^0 solves the wave equation

$$(\partial_t^2 + H^0)u^0(x,t) = 0,\ (x,t) \in M \times \mathbb{R},\ u^0|_{t=0} = f,\ \partial_t u^0|_{t=0} = 0. \tag{2.7}$$

Note that (2.6) is the kernel of a bounded operator on $L^2(M)$, and when we check that (2.7) is valid, it suffices to do so when f is a finite linear combination of the $\{e_j^0\}$ since such functions are dense in $L^2(M)$. We shall use similar facts in what follows. See [16] for more details.

Similarly, for λ not in the spectrum of P_V

$$N_V(\lambda) = \frac{1}{\pi} \int_M \int_{-\infty}^{\infty} \frac{\sin t\lambda}{t} \big(\cos(tP_V)\big)(x,x)\, dtdx, \tag{2.8}$$

if

$$\big(\cos(tP_V)\big)(x,y) = \sum_k \cos t\tau_k\, e_{\tau_k}(x) e_{\tau_k}(y) \tag{2.9}$$

is the kernel of $f \to \cos(tP_V)f = u_V(x,t)$, where u_V solve the wave equation

$$(\partial_t^2 + H_V)u_V(x,t) = 0,\ (x,t) \in M \times \mathbb{R},\ u_V|_{t=0} = f,\ \partial_t u_V|_{t=0} = 0. \tag{2.10}$$

To exploit (2.1′) and prove its more general version (1.11), in view of (2.5)–(2.10), it will be useful to relate the kernels in (2.6) and (2.9). To do so we shall make use of the following simple calculus lemma.

Lemma 2.1. *If $\mu \neq \tau$ we have*

$$\int_0^t \frac{\sin(t-s)\mu}{\mu} \cos s\tau\, ds = \frac{\cos t\tau - \cos t\mu}{\mu^2 - \tau^2}. \tag{2.11}$$

Similarly,

$$\int_0^t \frac{\sin(t-s)\tau}{\tau} \cos s\tau\, ds = \frac{t \sin t\tau}{2\tau}. \tag{2.12}$$

Proof. To prove (2.11) we make use of the identity

$$\sin\big(s(\tau-\mu)+t\mu\big) = \sin\big((t-s)\mu + s\tau\big) = \sin((t-s)\mu)\cos s\tau + \cos((t-s)\mu)\sin s\tau,$$

and, similarly,

$$-\sin\big((\tau+\mu)s - t\mu\big) = \sin\big((t-s)\mu - s\tau\big) = \sin((t-s)\mu)\cos s\tau - \cos((t-s)\mu)\sin s\tau.$$

Thus,

$$\sin\big(s(\tau-\mu)+t\mu\big) - \sin\big((\tau+\mu)s - t\mu\big) = 2\sin((t-s)\mu)\cos s\tau.$$

Consequently, the left side of (2.11) equals

$$\frac{1}{2\mu}\cdot\left[\frac{\cos\bigl(s(\tau-\mu)+t\mu\bigr)}{\mu-\tau}+\frac{\cos\bigl(s(\tau+\mu)-t\mu\bigr)}{\mu+\tau}\right]_0^t$$
$$=\frac{1}{2\mu}\Bigl[\cos t\tau\cdot\Bigl(\frac{1}{\mu-\tau}+\frac{1}{\mu+\tau}\Bigr)-\cos t\mu\cdot\Bigl(\frac{1}{\mu-\tau}+\frac{1}{\mu+\tau}\Bigr)\Bigr]$$
$$=\frac{1}{2\mu}\cdot\Bigl(\frac{2\mu\cos t\tau}{\mu^2-\tau^2}-\frac{2\mu\cos t\mu}{\mu^2-\tau^2}\Bigr)=\frac{\cos t\tau-\cos t\mu}{\mu^2-\tau^2},$$

as desired.

The proof of (2.12) is similar. □

Let us now describe how we shall use (2.1) and Lemma 2.1 to prove the Weyl formula (1.12). If, as above, $\mathbb{1}_\lambda(\tau)$ is the indicator function of $[-\lambda,\lambda]$, by (2.1), proving this amounts to showing that the trace of $\mathbb{1}_\lambda(P_v)$ satisfies the bounds in (1.12). As is the custom (cf. [16]), we shall do this indirectly by showing that an approximation $\tilde{\mathbb{1}}_\lambda(P_V)$ also enjoys these bounds, and, separately showing that the difference between the trace of $\mathbb{1}_\lambda(P_V)$ and $\tilde{\mathbb{1}}_\lambda(P_V)$ is $O(\lambda^{n-1})$.

To this end, fix an even real-valued function $\rho\in C^\infty(\mathbb{R})$ satisfying

$$\rho(t)=1 \text{ on } [-1/2,1/2] \text{ and } \operatorname{supp}\rho\subset(-1,1). \tag{2.13}$$

We then define

$$\tilde{\mathbb{1}}_\lambda(\tau)=\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(t)\frac{\sin\lambda t}{t}\,\cos t\tau\,dt. \tag{2.14}$$

Then since the Fourier tranform of $\mathbb{1}_\lambda(\tau)$ is $2\frac{\sin\lambda t}{t}$ it is not difficult to see that for $\tau>0$ and large λ we have

$$\mathbb{1}_\lambda(\tau)-\tilde{\mathbb{1}}_\lambda(\tau)=O\bigl((1+|\lambda-\tau|)^{-N}\bigr)\ \forall N. \tag{2.15}$$

Also, for later use, for $\tau>0$ we have

$$\bigl(\tfrac{d}{d\tau}\bigr)^j\,\tilde{\mathbb{1}}_\lambda(\tau)=O\bigl((1+|\lambda-\tau|)^{-N}\bigr)\ \forall N,\quad \text{if } j=1,2,3,\dots. \tag{2.16}$$

If we use (2.4) we can estimate the difference between the trace of $\mathbb{1}_\lambda(P_V)-\tilde{\mathbb{1}}_\lambda(P_V)$. Indeed, by (2.15) we have

$$\Bigl|\int_M\bigl(\mathbb{1}_\lambda(P_V)(x,x)-\tilde{\mathbb{1}}_\lambda(P_V)(x,x)\bigr)\,dx\Bigr|=\Bigl|\int_M\sum_k\bigl(\mathbb{1}_\lambda(\tau_k)-\tilde{\mathbb{1}}_\lambda(\tau_k)\bigr)\,|e_{\tau_k}(x)|^2\,dx\Bigr| \tag{2.17}$$
$$\lesssim\sum_k\int_M\bigl(1+|\lambda-\tau_k|\bigr)^{-2n}\,|e_{\tau_k}(x)|^2\,dx\lesssim\lambda^{n-1},$$

using (2.4) in the last inequality. Here, and in what follows, we are using the notation that $A\lesssim B$ means that A is less than or equal to a constant times B where the constant may change at each occurrence.

The fact that (2.17) holds when $V\equiv 1$ is due to Avakumović [1] and Levitan [13]. Since they also showed that the Weyl formula (1.12) is also valid when $V\equiv 1$, in view of

(2.1) and (2.8), in order to prove Theorem 1.1, it suffices to prove our main estimate

$$\int_M \Big(\tilde{\mathbb{1}}_\lambda(P_V)(x,x) - \tilde{\mathbb{1}}_\lambda(P^0)(x,x)\Big)\,dx = O(\lambda^{n-1}). \tag{2.18}$$

We shall actually be able to prove better bounds, namely,

$$\int_M \Big(\tilde{\mathbb{1}}_\lambda(P_V)(x,x) - \tilde{\mathbb{1}}_\lambda(P^0)(x,x)\Big)\,dx = O(\lambda^{n-3/2}). \tag{2.18$'$}$$

The implicit constants here of course depend on our V as in Theorem 1.1.

To prove this, we shall use the fact that, by (2.9) and (2.14) the kernel of $\tilde{\mathbb{1}}_\lambda(P_V)$ is

$$\tilde{\mathbb{1}}_\lambda(P_V)(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \rho(t)\frac{\sin\lambda t}{t} \sum_k \cos t\tau_k\, e_{\tau_k}(x)e_{\tau_k}(y)\,dt. \tag{2.19}$$

To use this formula, we note that, by (2.10) if f is a finite combination of the $\{e_{\tau_k}\}$, then

$$\begin{aligned}\big(\partial_t^2 + H^0\big)&\int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y)\,f(y)\,dy \\ &= -V(x)\cdot \int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y)\,f(y)\,dy = -V(x)\cdot\big(\cos tP_V\big)(f)(x).\end{aligned}$$

Also, since

$$\Big(\frac{d}{dt}\Big)^j\Big(\int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y)\,f(y)\,dy - \int_M \sum_j \cos t\lambda_j e_j^0(x)e_j^0(y)\,f(y)\,dy\Big)\Big|_{t=0} = 0, \quad j=0,1,$$

by Duhamel's principle we have

$$\begin{aligned}\int_M \sum_k &\cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y)\,f(y)\,dy - \int_M \sum_j \cos t\lambda_j e_j^0(x)e_j^0(y)\,f(y)\,dy \\ &= -\int_0^t \big(\tfrac{\sin(t-s)P^0}{t-s}\big(V\cos(sP_V)f\big)\big)(x)\,ds \\ &= -\int_0^t\int_M\int_M \sum_j \tfrac{\sin(t-s)\lambda_j}{t-s} e_j^0(x)e_j^0(z)V(z)\sum_k \cos s\tau_k e_{\tau_k}(z)e_{\tau_k}(y)f(y)\,dzdyds.\end{aligned}$$

By (2.14) or (2.19) if we integrate this against $\pi^{-1}\rho(t)\frac{\sin\lambda t}{t}$ we obtain $\mathbb{1}_\lambda(P_V)f(x) - \mathbb{1}_\lambda(P^0)f(x)$. Therefore, by Lemma 2.1 the kernel of $\tilde{\mathbb{1}}_\lambda(P_V) - \tilde{\mathbb{1}}_\lambda(P^0)$ is

$$\begin{aligned}\big(\tilde{\mathbb{1}}_\lambda(P_V) - \tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y) &= \\ \frac{1}{\pi}\sum_{j,k}\int_M\int_{-\infty}^{\infty} &\rho(t)\frac{\sin\lambda t}{t}\, m(\tau_k,\lambda_j)\, e_j^0(x)e_j^0(z)V(z)e_{\tau_k}(z)e_{\tau_k}(y)\,dzdt,\end{aligned} \tag{2.20}$$

where

$$m(\tau,\mu) = \begin{cases} \frac{\cos t\tau - \cos t\mu}{\tau^2-\mu^2}, & \text{if } \tau\neq\mu \\ -\frac{t\sin t\tau}{2\tau}, & \text{if } \tau=\mu. \end{cases} \tag{2.21}$$

Thus, by (2.19)–(2.20) we have

$$(2.20') \quad \big(\tilde{\mathbb{1}}_\lambda(P_V)-\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)= \sum_{j,k}\int_M \frac{\tilde{\mathbb{1}}_\lambda(\tau_k)-\tilde{\mathbb{1}}_\lambda(\lambda_j)}{\tau_k^2-\lambda_j^2} e_j^0(x)e_j^0(z)V(z)e_{\tau_k}(z)e_{\tau_k}(y)\,dz,$$

if, by the second part of (2.21) we interpret

$$\frac{\tilde{\mathbb{1}}_\lambda(\tau)-\tilde{\mathbb{1}}_\lambda(\mu)}{\tau^2-\mu^2}=\tilde{\mathbb{1}}_\lambda'(\tau)/2\tau, \quad \text{if} \quad \tau=\mu. \tag{2.22}$$

Thus, we would have (2.18′) and consequently Theorem 1.1 if we could prove the following:

Proposition 2.2. *As in Theorem 1.1 fix $V\in\mathcal{K}(M)$. Then*

$$(2.23) \quad \Big|\sum_{j,k}\int_M\int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V\|V\|_{L^1(M)}\,\lambda^{n-\frac32},$$

for some constant C_V depending on V.

We remark that even though the right side, curiously, involves the L^1-norm of V, we need that V is in $\mathcal{K}(M)$ since, among other things, the proof of Proposition 2.2 will use heat kernel estimates involving H_V. Steps like this will contribute to the constant C_V in (2.23).

Note that the kernel in (2.20′) involves an amalgamation of the kernels of $\tilde{\mathbb{1}}_\lambda(P^0)$, $\tilde{\mathbb{1}}_\lambda(P_V)$ and the resolvent kernels $(H_V-\mu^2)^{-1}$ and $(H^0-\mu^2)^{-1}$. To prove (2.23) we shall attempt to separate the contributions of the various components by using the following simple lemma.

Lemma 2.3. *Let $I\subset\mathbb{R}_+$ and for eigenvalues $\tau_k\in I$ assume that $\delta_{\tau_k}\in[0,\delta]$. Then if $m\in C^1(\mathbb{R}_+\times M)$*

$$(2.24) \quad \int_M\Big|\sum_{\tau_k\in I} m(\delta_{\tau_k},x)\,a_k e_{\tau_k}(x)\Big|\,dx \le \Big(\|m(0,\,\cdot\,)\|_{L^2(M)}+\int_0^\delta \big\|\tfrac{\partial}{\partial s}m(s,\,\cdot\,)\big\|_{L^2(M)}\,ds\Big)\times\Big(\sum_{\tau_k\in I}|a_k|^2\Big)^{1/2}.$$

Proof. We shall use the fact that $m(\delta_{\tau_k},x)=m(0,x)+\int_0^\delta \mathbb{1}_{[0,\delta_{\tau_k}]}(s)\,\frac{\partial}{\partial s}m(s,x)\,ds$, where $\mathbb{1}_{[0,\delta_{\tau_k}]}(s)$ is the indicator function of the the interval $[0,\delta_{\tau_k}]\subset[0,\delta]$. Therefore, by

Minkowski's inequality, the left side of (2.24) is dominated by

$$\int_M \Big| m(0,x)\cdot \sum_{\tau_k\in I} a_k e_{\tau_k}(x)\Big|\,dx + \int_M \Big|\sum_{\tau_k\in I}\int_0^\delta \mathbb{1}_{[0,\delta_{\tau_k}]}(s)\tfrac{\partial}{\partial s}m(s,x)a_k e_{\tau_k}(x)\,ds\Big|\,dx$$

$$\le \int_M \Big| m(0,x)\cdot \sum_{\tau_k\in I} a_k e_{\tau_k}(x)\Big|\,dx + \int_0^\delta \Big(\int_M \big|\tfrac{\partial}{\partial s}m(s,x)\big|\cdot\Big|\sum_{\tau_k\in I} a_k e_{\tau_k}(x)\Big|\,dx\Big)\,ds$$

$$\le \|m(0,\,\cdot\,)\|_2\cdot \|\sum_{\tau_k\in I} a_k e_{\tau_k}\|_2 + \int_0^\delta \big(\,\|\tfrac{\partial}{\partial s}m(s,\,\cdot\,)\|_2\cdot \big\|\mathbb{1}_{[0,\delta_{\tau_k}]}(s)a_k e_{\tau_k}\big\|_2\big)\,ds$$

$$= \|m(0,\,\cdot\,)\|_2\cdot (\sum_{\tau_k\in I}|a_k|^2)^{1/2} + \int_0^\delta \|\tfrac{\partial}{\partial s}m(s,\,\cdot\,)\|_2\cdot \big(\sum_{\tau_k\in I}|\mathbb{1}_{[0,\delta_{\tau_k}]}(s)a_k|^2\big)^{1/2}\,ds$$

$$\le \Big(\|m(0,\,\cdot\,)\|_{L^2(M)} + \int_0^\delta \big\|\tfrac{\partial}{\partial s}m(s,\,\cdot\,)\big\|_{L^2(M)}\,ds\Big)\times\big(\sum_{\tau_k\in I}|a_k|^2\big)^{1/2},$$

as desired. □

Next, recall that we mentioned that the kernel in (2.23) is a juxtaposition of the kernels of $\tilde{\mathbb{1}}_\lambda(P^0)$ as well as resolvent-type kernels. To handle the former, we shall appeal to the following straightforward result.

Lemma 2.4. *Let $\tilde{\mathbb{1}}_\lambda(P^0)$ be defined by* (2.14) *and the analog of* (2.19) *involving P^0. Then the kernel of $(P^0)^\mu\tilde{\mathbb{1}}_\lambda(P^0)$, $\mu=0,1,2,\dots$ satisfies*

$$\big((P^0)^\mu\,\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y) = \sum_j \lambda_j^\mu\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y) = O(\lambda^{n+\mu}), \tag{2.25}$$

and, moreover,

$$\big\|\big((P^0)^\mu\,\tilde{\mathbb{1}}_\lambda(P^0)\big(\,\cdot\,,y)\big\|_{L^2(M)} = O(\lambda^{n/2+\mu}). \tag{2.26}$$

The proof of the lemma is very simple. First, by the pointwise Weyl formula of Avakumović [1], Levitan [13] and Hörmander [8] (see also [16]),

$$\sum_{\lambda_j\in[\ell,\ell+1]} |e_j^0(x)e_j^0(y)| = O(\ell^{n-1}),\quad \ell\in\mathbb{N}. \tag{2.27}$$

If we use this and (2.15), we obtain (2.25). To prove the other inequality, (2.26), we note that, by orthogonality

$$\big\|(P^0)^\mu\tilde{\mathbb{1}}_\lambda(P^0)(\,\cdot\,,y)\big\|^2_{L^2(M)} = \sum_j \lambda_j^{2\mu}\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)\big)^2|e_j^0(y)|^2 = O(\lambda^{n+2\mu}),$$

by this argument, which is (2.26).

To deal with the contributions of resolvent type operators in the mixture (2.23) we shall need a couple more results. The first is bounds for cutoff resolvent operators for the free operator H^0.

Lemma 2.5. *Fix $\eta \in C^\infty(\mathbb{R}_+)$ satisfying $\eta(s)=0$ on $s\le 2$ and $\eta(s)=1$, $s>4$. Then if we set for $\tau \gg 1$*

$$R_\tau(x,y)=\sum_j \frac{\eta(\lambda_j/\tau)}{\lambda_j^2-\tau^2}\, e_j^0(x)e_j^0(y). \tag{2.28}$$

we have

$$|R_\tau(x,y)|\le C_N\tau^{n-2}h_n\big(\tau\, d_g(x,y)\big)\,\big(1+\tau\, d_g(x,y)\big)^{-N}, \tag{2.29}$$

for any $N=1,2,3,\dots$, if $h_n(r)$ is as in (1.4). The constant C_N depends on N, (M,g) and finitely many derivatives of η.

Here we are abusing the notation a bit. In (2.29) we mean that the inequality holds near the diagonal (so that $d_g(x,y)$ is well-defined) and that outside of this neighborhood of the diagonal $R_\tau(x,y)$ is $O(\tau^{-N})$ for all N. We shall state certain inequalities in this manner in what follows.

To verify (2.29), we note that the integral operator R_τ arising from the kernel $R_\tau(x,y)$ is

$$\tau^{-2}m(P^0/\tau),$$

where

$$m(\mu)=\frac{\eta(|\mu|)}{\mu^2-1}.$$

Thus, m is a symbol of order -2, i.e.,

$$\partial_\mu^j m(\mu)=O((1+\mu)^{-2-j}),\ j=0,1,2,\dots.$$

As a result, one can use the arguments in [17, §4.3] to see that (2.29) is valid. Indeed, modulo lower order terms, $R_\tau(x,y)$ equals

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\tau^{-2}\frac{\eta(|\xi|/\tau)}{(|\xi|/\tau)^2-1}\, e^{id_g(x,y)\xi_1}\, d\xi,$$

near the diagonal, which satisfies the bounds in (2.29), while outside of a fixed neighborhood of the diagonal $R_\tau(x,y)=O(\tau^{-N})$ for all N.

We also need bounds for the kernels of $(H_V)^{-j}$.

Lemma 2.6. *Let $(H_V)^{-j}(x,y)=\sum_k \tau_k^{-2j}e_{\tau_k}(x)e_{\tau_k}(y)$ be the kernel of $(H_V)^{-j}$, $j=1,2,\dots$. Then if $h_n(r)$ is as in (1.4)*

$$\big(H_V\big)^{-1}(x,y)\lesssim\begin{cases}h_n(d_g(x,y)), & \text{if } d_g(x,y)\le \mathit{Inj}\,(M)/2,\\ 1, & \text{otherwise.}\end{cases} \tag{2.30}$$

Furthermore, if $n\ge 5$ and $j<n/2$, $j\in\mathbb{N}$ we have

$$\big(H_V\big)^{-j}(x,y)\lesssim\begin{cases}\big(d_g(x,y)\big)^{-n+2j}, & \text{if } d_g(x,y)\le \mathit{Inj}\,(M)/2,\\ 1 & \text{otherwise.}\end{cases} \tag{2.31}$$

To prove (2.30) or (2.31), we note that

$$\big(H_V\big)^{-j}(x,y)=\int_0^\infty t^j\,\big(e^{-tH_V}\big)(x,y)\,dt. \tag{2.32}$$

We then use the heat kernel estimates of Li and Yau [14] ($V \in C^\infty$) and Sturm [18, (4.14) Corollary] ($V \in \mathcal{K}(M)$), which say that for $0 < t \le 1$ there is a uniform constant $c = c_{M,V} > 0$ so that

$$(e^{-tH_V})(x,y) \lesssim \begin{cases} t^{-n/2}\exp(-c(d_g(x,y))^2/t), & \text{if } \ d_g(x,y) \le \text{Inj } (M)/2, \\ 1 \quad \text{otherwise.} \end{cases} \tag{2.33}$$

As a consequence of (2.33), we have for $0 < t \le 1$

$$\int_M |(e^{-tH_V})(x,y)|^2 dy \lesssim t^{-\frac{n}{2}}.$$

By Schwarz's inequality, we have $\|e^{-tH_V}\|_{L^2 \to L^\infty} \lesssim t^{-\frac{n}{4}}$. If we consider the kernels of the dyadic spectral projection operators

$$\tilde{\chi}_\lambda^V(x,y) = \sum_{\tau_k \in [\lambda, 2\lambda)} e_{\tau_k}(x) e_{\tau_k}(y), \tag{2.34}$$

for H_V, then, by the spectral theorem, we have

$$\|\tilde{\chi}_\lambda^V\|_{L^2 \to L^\infty} \lesssim \|e^{-\lambda^{-2}H_V}\|_{L^2 \to L^\infty} \lesssim \lambda^{\frac{n}{2}},$$

which, along with the Cauchy-Schwarz inequality, implies

$$\sup_{x,y\in M} \Big| \sum_{\tau_k \in [\lambda, 2\lambda)} e_{\tau_k}(x) e_{\tau_k}(y) \Big| \le \sup_{x \in M} \sum_{\tau_k \in [\lambda, 2\lambda)} |e_{\tau_k}(x)|^2 = \big\|\tilde{\chi}_\lambda^V\big\|^2_{L^2 \to L^\infty} \lesssim \lambda^n. \tag{2.35}$$

Since the eigenvalues of H^V are all ≥ 1, by (2.35) we have

$$(e^{-tH_V})(x,y) \lesssim e^{-t/2}, \quad t > 1. \tag{2.36}$$

If we use (2.33), (2.36) along with (2.32), we obtain (2.30) and (2.31).

3. **Proof of the universal Weyl law involving singular potentials.**

To prove Proposition 2.2, which, as noted, implies our main result, Theorem 1.1, we shall split things into three different cases that require slightly different arguments. Specifically, we shall first handle the contribution of frequencies τ_k which are comparable to λ, and then those that are relatively small followed by ones that are relatively large.

Handling the contribution of comparable frequencies. In this subsection we shall handle frequencies τ_k which are comparable to λ, which one would expect to be the main contribution to the Weyl error term in (1.12). Specifically, we shall prove the following.

Proposition 3.1. *As in Theorem 1.1 fix $V \in \mathcal{K}(M)$. Then*

$$\Big| \sum_j \sum_{\{k:\, \tau_k \in [\lambda/2, 10\lambda]\}} \int_M \int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j) - \tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx dy \Big| \le C_V \|V\|_{L^1(M)} \, \lambda^{n-\frac{3}{2}}, \tag{3.1}$$

for some constant C_V depending on V.

To prove Proposition 3.1, let us fix a Littlewood-Paley bump function $\beta \in C_0^\infty((1/2,2))$ satisfying

$$\sum_{\ell=-\infty}^{\infty} \beta(2^{-\ell}s)=1, \quad s>0.$$

Let us then set

$$\beta_0(s)=\sum_{\ell\le 0}\beta(2^{-\ell}|s|)\in C_0^\infty((-2,2)),$$

and

$$\tilde\beta(s)=s^{-1}\beta(|s|)\in C_0^\infty\big(\{|s|\in(1/2,2)\}\big).$$

We then write for $\lambda/2\le\tau\le 10\lambda$

$$\begin{aligned}
K_\tau(x,y)&=\sum_j \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j^2-\tau^2}e_j^0(x)e_j^0(y) \\
&=\sum_j \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j-\tau}\,\frac{\beta_0(\lambda_j-\tau)}{\lambda_j+\tau}\,e_j^0(x)e_j^0(y) \\
&\quad+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda/100\}}\Big(\sum_j \frac{2^{-\ell}\tilde\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\,(\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau))\,e_j^0(x)e_j^0(y)\Big) \\
&\quad+\sum_j\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)\big)\,e_j^0(x)e_j^0(y).
\end{aligned}\tag{3.2}$$

Next, let

$$K_{\tau,0}(x,y)=\sum_j \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j-\tau}\,\frac{\beta_0(\lambda_j-\tau)}{\lambda_j+\tau}\,e_j^0(x)e_j^0(y),$$

$$R_{\tau,\ell}(x,y)=\sum_j \frac{2^{-\ell}\tilde\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\,e_j^0(x)e_j^0(y), \quad \text{if } 2^\ell\le\lambda/100,$$

and

$$R_{\tau,\infty}(x,y)=\sum_j\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\,e_j^0(x)e_j^0(y).$$

Also, for $2^\ell\le\lambda/100$ let

$$K^-_{\tau,\ell}(x,y)=\sum_j \frac{2^{-\ell}\tilde\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\big)\,e_j^0(x)e_j^0(y)$$

$$K^+_{\tau,\ell}(x,y)=\sum_j \frac{2^{-\ell}\tilde\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\tilde{\mathbb{1}}_\lambda(\lambda_j)\,e_j^0(x)e_j^0(y),$$

and, finally,

$$K^-_{\tau,\infty}(x,y)=\sum_j\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\big)\,e_j^0(x)e_j^0(y)$$

$$K^+_{\tau,\infty}(x,y)=\sum_j\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\tilde{\mathbb{1}}_\lambda(\lambda_j)\,e_j^0(x)e_j^0(y).$$

If K_τ is as in (3.2), our current task, (3.1), is to show that

$$\Big|\sum_{\tau_k\in[\lambda/2,10\lambda]}\iint K_{\tau_k}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim\lambda^{n-\frac32}\,\|V\|_{L^1(M)}. \tag{3.1$'$}$$

To prove this, we note that we can write

$$\begin{aligned}K_\tau(x,y)=K_{\tau,0}(x,y)&+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda/100\}}K^-_{\tau,\ell}(x,y)+K^-_{\tau,\infty}(x,y)\\&+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda/100\}}R_{\tau,\ell}(x,y)\big(1-\tilde{\mathbb{1}}_\lambda(\tau)\big)+R_{\tau,\infty}(x,y)\big(1-\tilde{\mathbb{1}}_\lambda(\tau)\big),\end{aligned}\tag{3.3}$$

or

$$\begin{aligned}K_\tau(x,y)=K_{\tau,0}(x,y)&+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda/100\}}K^+_{\tau,\ell}(x,y)+K^+_{\tau,\infty}(x,y)\\&-\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda/100\}}R_{\tau,\ell}(x,y)\tilde{\mathbb{1}}_\lambda(\tau)-R_{\tau,\infty}(x,y)\tilde{\mathbb{1}}_\lambda(\tau),\end{aligned}\tag{3.4}$$

We shall use (3.3) to handle the summands in (3.1$'$) with $\tau=\tau_k\in[\lambda/2,\lambda]$ and (3.4) to handle those with $\tau=\tau_k\in(\lambda,10\lambda]$.

For $\ell\in\mathbb{N}$ with $2^\ell\le\lambda/100$, let for $j=0,1,2,\dots$

$$I^-_{\ell,j}=\big(\lambda-(j+1)2^\ell,\,\lambda-j2^\ell\big]\quad\text{and}\quad I^+_{\ell,j}=\big(\lambda+j2^\ell,\,\lambda+(j+1)2^\ell\,\big].\tag{3.5}$$

Then to use the δ_τ–Lemma (Lemma 2.3), we shall use the following result whose proof we momentarily postpone.

Lemma 3.2. *If $\ell\in\mathbb{Z}_+$, $2^\ell\le\lambda/100$, and $j=0,1,2,\dots$, we have for each $N\in\mathbb{N}$*

$$\begin{aligned}&\|K^\pm_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)},\ \|2^\ell\tfrac{\partial}{\partial\tau}K^\pm_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)}\\&\qquad\lesssim\lambda^{\frac{n-1}{2}-1}2^{-\ell/2}(1+j)^{-N},\quad\tau\in I^\pm_{\ell,j}\cap[\lambda/2,10\lambda].\end{aligned}\tag{3.6}$$

Also,

$$\begin{aligned}&\|K_{\tau,0}(\,\cdot\,,y)\|_{L^2(M)},\ \|\tfrac{\partial}{\partial\tau}K_{\tau,0}(\,\cdot\,,y)\|_{L^2(M)}\\&\qquad\lesssim\lambda^{\frac{n-1}{2}-1}(1+j)^{-N},\quad\tau\in I^\pm_{0,j}\cap[\lambda/2,10\lambda],\end{aligned}\tag{3.7}$$

$$\|K^+_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda\tfrac{\partial}{\partial\tau}K^+_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)}\lesssim\lambda^{\frac n2-2},\quad\tau\in[\lambda,10\lambda],\tag{3.8}$$

and we can write

$$K^-_{\tau,\infty}(x,y)=\tilde K^-_{\tau,\infty}(x,y)+H^-_{\tau,\infty}(x,y),$$

where for $\tau \in [\lambda/2, \lambda]$

$$\|\tilde{K}^-_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda\tfrac{\partial}{\partial\tau}\tilde{K}^-_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)} \lesssim \lambda^{\frac{n}{2}-2} \tag{3.9}$$

$$|H^-_{\tau,\infty}(x,y)| \lesssim \lambda^{n-2}h_n(\lambda d_g(x-y))(1+\lambda d_g(x,y))^{-N},$$

where h_n *is as in* (1.4)*. Finally, we also have for* $1 \le 2^\ell \le \lambda/100$ *and* $\tau \in [\lambda/2, 10\lambda]$

$$\|R_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)},\ \|2^\ell\tfrac{\partial}{\partial\tau}R_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)} \lesssim \lambda^{\frac{n-1}{2}-1}2^{-\ell/2}, \tag{3.10}$$

and

$$|R_{\tau,\infty}(x,y)| \lesssim \lambda^{n-2}h_n(\lambda d_g(x,y))\,(1+\lambda d_g(x,y))^{-N}. \tag{3.11}$$

As before, we are abusing notation a bit. First, in (3.6) we mean that if $K_{\tau,\ell}$ equals $K^+_{\tau,\ell}$ or $K^-_{\tau,\ell}$ then the bounds in (3.6) for τ in $I^+_{\ell,j}\cap[\lambda,10\lambda]$ or $I^-_{\ell,j}\cap[\lambda/2,\lambda]$, respectively. Also, in both the second inequality in (3.9) and in (3.11) we mean that the kernels satisfy the bounds when x is sufficiently close to y (so that $d_g(x,y)$ is well-defined) and that they are $O(\lambda^{-N})$ away from the diagonal.

Before proving this result let us see how we can use it along with Lemma 2.3 to prove Proposition 3.1.

Proof of Proposition 3.1. First, by (3.6) and Lemma 2.3 with $\delta = 2^\ell$, we have

$$\begin{aligned}
&\Big|\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}\iint K^{\pm}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dydx\Big| \\
&\le \|V\|_{L^1}\cdot\sup_y\Big\|\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}\iint K^{\pm}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\Big\|_{L^1(dx)} \\
&\lesssim \|V\|_{L^1}\cdot\sup_y\Big(\|K^{\pm}_{\lambda+j2^\ell,\ell}(\,\cdot\,,y)\|_{L^2(M)} + \int_{I^{\pm}_{\ell,j}}\big\|\tfrac{\partial}{\partial\tau}K^{\pm}_{s,\ell}(\,\cdot\,y)\|_{L^2(M)}\,ds\Big) \\
&\qquad\qquad\times\Big(\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2\Big)^{1/2} \\
&\lesssim \lambda^{\frac{n-1}{2}-1}2^{-\ell/2}(1+j)^{-N}\Big(\sum_{\tau_k\in[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2\Big)^{1/2} \\
&\lesssim \lambda^{n-\frac{3}{2}}2^{-\ell/2}(1+j)^{-N}.
\end{aligned} \tag{3.12}$$

In the second to last inequality we used the fact that, by (2.35),

$$\sum_{\tau_k\in[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2 \lesssim \lambda^n, \quad \lambda\ge 1. \tag{3.13}$$

If we sum over $j = 0, 1, 2, \dots$, we see that (3.12) yields

$$\Big| \sum_{\lambda<\tau_k\le 10\lambda} \iint K^+_{\tau_k,\ell}(x,y) e_{\tau_k}(x) e_{\tau_k}(y)\, V(y)\, dxdy \Big| + \Big| \sum_{\lambda/2\le\tau_k\le\lambda} \iint K^-_{\tau_k,\ell}(x,y) e_{\tau_k}(x) e_{\tau_k}(y)\, V(y)\, dxdy \Big| \lesssim \lambda^{n-\frac32} 2^{-\ell/2} \|V\|_{L^1(M)}, \quad 1 \le 2^\ell \le \lambda/100. \tag{3.14}$$

If we take $\delta = 1$ in Lemma 2.3, this argument also gives

$$\Big| \sum_{\lambda/2\le\tau_k\le\lambda} \iint K_{\tau_k,0}(x,y) e_{\tau_k}(x) e_{\tau_k}(y)\, V(y)\, dxdy \Big| + \Big| \sum_{\lambda<\tau_k\le 10\lambda} \iint K_{\tau_k,0}(x,y) e_{\tau_k}(x) e_{\tau_k}(y)\, V(y)\, dxdy \Big| \lesssim \lambda^{n-\frac32} \|V\|_{L^1(M)}. \tag{3.15}$$

Similarly, if we use Lemma 2.3 with $\delta = \lambda$ along with (3.8) we find that

$$\Big| \sum_{\lambda<\tau_k\le 10\lambda} \iint K^+_{\tau_k,\infty}(x,y) e_{\tau_k}(x) e_{\tau_k}(y)\, V(y)\, dxdy \Big| \lesssim \lambda^{\frac n2 -2} \|V\|_{L^1} \Big(\sum_{\tau_k\in[\lambda/2,10\lambda]} |e_{\tau_k}(y)|^2 \Big)^{1/2} \lesssim \lambda^{n-2} \|V\|_{L^1}, \tag{3.16}$$

using (3.13) for the last inequality.

Next, since $R_{\tau,\ell}$ enjoys the bounds in (3.10), we can repeat the arguments in (3.12) to see that for $1 \le 2^\ell \le \lambda/100$ we have

$$\begin{aligned} &\Big| \sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]} \iint R_{\tau_k,\ell}(x,y) \tilde{\mathbb{1}}_\lambda(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y)\, dxdy \Big| \\ &\quad \lesssim \|V\|_{L^1} \cdot 2^{-\ell/2} \lambda^{\frac{n-1}{2}-1} \sup_y \Big(\sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]} |\tilde{\mathbb{1}}_\lambda(\tau_k) e_{\tau_k}(y)|^2 \Big)^{1/2} \\ &\quad \lesssim \lambda^{n-\frac32} 2^{-\ell/2} \|V\|_{L^1} \cdot (1+j)^{-N}, \end{aligned}$$

since $\tilde{\mathbb{1}}_\lambda(\tau_k) = O((1+j)^{-N})$ if $\tau_k \in I^+_{\ell,j}$. Summing over this bound over j of course yields

$$\Big| \sum_{\lambda<\tau_k\le 10\lambda} \iint R_{\tau_k,\ell}(x,y) \tilde{\mathbb{1}}_\lambda(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y)\, dxdy \Big| \lesssim \lambda^{n-\frac32} 2^{-\ell/2} \|V\|_{L^1}. \tag{3.17}$$

The same argument gives

$$\Big| \sum_{\lambda/2\le\tau_k\le\lambda} \iint R_{\tau_k,\ell}(x,y) \big(1-\tilde{\mathbb{1}}_\lambda(\tau_k)\big) e_{\tau_k}(x) e_{\tau_k}(y) V(y)\, dxdy \Big| \lesssim \lambda^{n-\frac32} 2^{-\ell/2} \|V\|_{L^1}. \tag{3.18}$$

Also, by (3.11) we have

$$\sup_y \int \sup_{\lambda/2\le\tau\le 10\lambda} |R_{\tau,\infty}(x,y)|\, dx \lesssim \lambda^{-2},$$

and since (3.13) yields $\sum_{\tau_k\le 10\lambda}|e_{\tau_k}(x)e_{\tau_k}(y)|\lesssim \lambda^n$, we have

$$
\begin{aligned}
&\Big|\sum_{\tau_k\in(\lambda,10\lambda]}\iint R_{\tau_k,\infty}(x,y)\tilde{\mathbb{1}}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}\\
&\Big|\sum_{\tau_k\in[\lambda/2,\lambda]}\iint R_{\tau_k,\infty}(x,y)\big(1-\tilde{\mathbb{1}}_\lambda(\tau_k)\big)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}
\end{aligned}
\tag{3.19}
$$

If $H^-_{\tau,\infty}$ is as in (3.9) this argument also gives us

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint H^-_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1},$$

while the proof of (3.16) along with the first part of (3.9) yields

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint \tilde{K}^-_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}.$$

Since $K_{\tau,\infty}=\tilde{K}_{\tau,\infty}+H^-_{\tau,\infty}$, we deduce

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint K^-_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}. \tag{3.20}$$

We now have assembled all the ingredients for the proof of (3.1′). If we use (3.14), (3.15), (3.18), (3.19) and (3.20) along with (3.3), we conclude that the analog of (3.1′) must be valid where the sum is taken over $\tau_k\in[\lambda/2,\lambda]$. We similarly obtain the analog of (3.1′) where the sum is taken over $\tau_k\in(\lambda,10\lambda]$ from (3.4) along with (3.14), (3.15), (3.17) and (3.19).

From this, we deduce that (3.1′) must be valid, which finishes the proof that Lemma 2.3 and Lemma 3.2 yield Proposition 3.1. □

To finish the present task we need to prove this lemma.

Proof of Lemma 3.2. To prove the first inequality we note that if $\tau\in I^\pm_{\ell,j}\cap[\lambda/2,10\lambda]$ then $|\lambda_i-\tau|\le 2^{\ell+1}$, $\lambda_i,\tau\approx\lambda$ if $\beta(2^{-\ell}(\lambda_i-\tau))\neq 0$, and, in this case, we also have $\tilde{\mathbb{1}}_\lambda(\lambda_i)-1=O((1+|j|)^{-N})$ if $\tau\in I^-_{\ell,j}$ and $\tilde{\mathbb{1}}_\lambda(\lambda_i)=O((1+|j|)^{-N})$ if $\tau\in I^+_{\ell,j}$. Therefore, by orthogonality and (2.27), we have

$$
\begin{aligned}
\big\|K^\pm_{\tau,\ell}(\,\cdot\,,y)\big\|_{L^2(M)}&\lesssim(1+|j|)^{-N}2^{-\ell}\lambda^{-1}\Big(\sum_{\{i:\,|\lambda_i-\tau|\le 2^{\ell+1}\}}|e_i^0(y)|^2\Big)^{1/2}\\
&\lesssim(1+|j|)^{-N}2^{-\ell}\lambda^{-1}\Big(\sum_{\{\mu\in\{\mathbb{N}:\,|\mu-\tau|\le 2^{\ell+1}\}}\mu^{n-1}\Big)^{1/2}\\
&\le(1+|j|)^{-N}2^{-\ell/2}\lambda^{\frac{n-1}{2}-1},
\end{aligned}
$$

which is the first part of (3.6). In the second inequality, we used (2.27). The other inequality in (3.6) follows from this argument since

$$\frac{\partial}{\partial\tau}\frac{\tilde{\beta}(2^{-\ell}(\lambda_i-\tau))}{\lambda_i+\tau}=O(2^{-\ell}\lambda^{-1}),$$

due to the fact that we are assuming that $2^\ell \le \lambda/100$.

This argument also gives us (3.7) if we use the fact that $\tau \to (\tilde{\mathbb{1}}_\lambda(\tau) - \tilde{\mathbb{1}}_\lambda(\mu))/(\tau^2-\mu^2)$ is smooth if we define it as in (2.22) when $\tau=\mu$ (which is consistent with (2.20′)) and use the fact that

$$\partial_\tau^k\big(\beta_0(\lambda_i-\tau)(\tilde{\mathbb{1}}_\lambda(\lambda_i)-\tilde{\mathbb{1}}_\lambda(\tau))/(\lambda_i-\tau)\big) = O((1+|j|)^{-N}),\ k=0,1,\ \tau\in I^\pm_{0,j},$$

and the fact that, if this expression is nonzero, we must have $|\lambda_i-\tau|\le 2$.

To prove (3.8) we use the fact that for $k=0,1$ we have for $\tau\in(\lambda,10\lambda]$

$$\Big|\Big(\frac{\partial}{\partial\tau}\Big)^k\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\tilde{\mathbb{1}}_\lambda(\lambda_i)\Big| \lesssim \begin{cases}\lambda^{-2-k} & \text{if } \lambda_i\le\lambda\\ \lambda^{-2-k}(1+\lambda_i-\lambda)^{-N} & \text{if } \lambda_i>\lambda.\end{cases}$$

Thus for $k=0,1$

$$\begin{aligned}\big\|(\lambda\partial_\tau)^k\,K^+_{\tau,\infty}(\,\cdot\,,y)\big\|_{L^2(M)} &\lesssim \lambda^{-2}\Big(\sum_{\lambda_i\le\lambda}|e_i^0(y)|^2+\sum_{\lambda_i>\lambda}(1+\lambda_i-\lambda)^{-N}|e_i^0(y)|^2\Big)^{1/2}\\ &\lesssim \lambda^{-2+\frac{n}{2}},\end{aligned}$$

as desired if $N>2n$, using (2.27) again.

Next we turn to the bounds in (3.9) for $K^-_{\tau,\infty}$. To handle this, let η be as in Lemma 2.5 and put

$$\begin{aligned}H^-_{\tau,\infty}(x,y) &= -\sum_i\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\eta(\lambda_i/\tau)\,e_i^0(x)e_i^0(y)\\ &= -\sum_i\frac{\eta(\lambda_i/\tau)}{\lambda_i^2-\tau^2}e_i^0(x)e_i^0(y),\end{aligned}$$

assuming, as we may, that $\lambda\gg1$. The last equality comes from the properties of our Littlewood-Paley bump function, β. We then conclude from Lemma 2.5 that $H^-_{\tau,\infty}$ satisfies the bounds in (3.9). If we then set

$$\begin{aligned}\tilde{K}^-_{\tau,\infty}(x,y) &= \sum_i\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\tilde{\mathbb{1}}_\lambda(\lambda_i)\,e_i^0(x)e_i^0(y)\\ &\quad-\sum_i\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\big(1-\eta(\lambda_i/\tau)\big)\,e_i^0(x)e_i^0(y),\end{aligned}$$

we have $K^-_{\tau,\infty}=\tilde{K}^-_{\tau,\infty}+H^-_{\tau,\infty}$, and, also, by the proof of (3.8), $\tilde{K}^-_{\tau,\infty}$ satisfies the bounds in (3.9).

It just remains to prove the bounds in (3.10) for the $R_{\tau,\ell}(x,y)$ and that in (3.11) for $R_{\tau,\infty}(x,y)$. The former just follows from the proof of (3.6).

To prove the remaining inequality, (3.11), we note that if η is as above and we set

$$\tilde{R}_{\tau,\infty}(x,y)=\sum_i\frac{\eta(\lambda_i/\tau)}{\lambda_i^2-\tau^2}\,e_i^0(x)e_i^0(y),$$

then, by Lemma 2.5, $\tilde{R}_{\tau,\infty}$ satisfies the bounds in (3.11). Also, we have

$$R_{\tau,\infty}(x,y)=R^0_{\tau,\infty}(x,y)+\tilde{R}_{\tau,\infty}(x,y),$$

if

$$R^0_{\tau,\infty}(x,y)=\sum_i\bigl(1-\eta(\lambda_i/\tau)\bigr)\Bigl(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda/100\}}\beta(2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Bigr)\,e^0_i(x)e^0_i(y),$$

(again using the properties of β), and, since the proof of Lemma 2.5 shows that for $\tau\in[\lambda/2,10\lambda]$ we have

$$|R^0_{\tau,\infty}(x,y)|\lesssim\tau^{n-2}\bigl(1+\tau d_g(x,y)\bigr)^{-N}\lesssim\lambda^{n-2}\bigl(1+\lambda d_g(x,y)\bigr)^{-N},$$

we conclude that (3.11) must be valid, which completes the proof. □

Handling the contribution of relatively large frequencies of H_V. In this section we shall handle relatively large frequencies of H_V by proving the following.

Proposition 3.3. *As in Theorem 1.1 fix $V\in\mathcal{K}(M)$. Then*

$$\Bigl|\sum_j\sum_{\{k:\,\tau_k>10\lambda\}}\int_M\int_M\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}e^0_j(x)e^0_j(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Bigr| \le C_V\|V\|_{L^1(M)}\,\lambda^{n-2}(\log\lambda)^{1/2}, \tag{3.21}$$

for some constant C_V depending on V.

To prove (3.21) fix

$$\Psi\in C_0^\infty((1/2,2)),\quad\text{with}\ \ \Psi(s)=1,\ s\in[3/4,5/4]. \tag{3.22}$$

To proceed, assume that $\tau_k>10\lambda$. Since, by the mean value theorem and (2.16)

$$\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j-\tau_k}=O(\tau_k^{-\sigma})\ \forall\,\sigma,\ \text{if}\ \ \lambda_j\in(\tau_k/2,2\tau_k),\ \tau_k>10\lambda,$$

by (2.27) and (3.13), to prove (3.21) it suffices to show that

$$\Bigl|\sum_j\sum_{\{k:\,\tau_k>10\lambda\}}\iint\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}\bigl(1-\Psi(\lambda_j/\tau_k)\bigr)e^0_j(x)e^0_j(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Bigr| \lesssim\|V\|_{L^1(M)}\,\lambda^{n-2}(\log\lambda)^{1/2}, \tag{3.21$'$}$$

since

$$\Bigl|\sum_j\sum_{\{k:\,\tau_k>10\lambda\}}\iint\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}\,\Psi(\lambda_j/\tau_k)e^0_j(x)e^0_j(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Bigr| \lesssim\lambda^{-\sigma}\|V\|_{L^1(M)},\quad\forall\,\sigma.$$

As $\tilde{\mathbb{1}}_\lambda(\tau_k)=O(\tau_k^{-\sigma})$ for all $\sigma\in\mathbb{N}$ for $\tau_k>10\lambda$ and, by Lemma 2.5,

$$\Bigl|\sum_j\frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}e^0_j(x)e^0_j(y)\Bigr|\lesssim\begin{cases}\tau_k^{n-2}+(d_g(x,y))^{2-n}, & n\ge3\\ \log\bigl(2+1/(\tau_kd_g(x,y))\bigr), & n=2,\end{cases}$$

the analog of (3.21′) where we replace $(\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k))$ by $\tilde{\mathbb{1}}_\lambda(\tau_k)$ is trivial. Consequently, we would have (3.21′) and consequently Proposition 3.3 if we could show that

$$(3.21'') \quad \Big|\sum_j \sum_{\{k:\,\tau_k>10\lambda\}} \iint \frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}\,\tilde{\mathbb{1}}_\lambda(\lambda_j)\,e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\,\Big| \lesssim \|V\|_{L^1(M)}\,\lambda^{n-2}(\log\lambda)^{1/2}.$$

If $1-\Psi(\lambda_j/\tau_k)\neq 0$ we have $\lambda_j\neq\tau_k$, and then can write

$$\frac{1}{\tau_k^2-\lambda_j^2}=\tau_k^{-2}+\tau_k^{-2}\big(\lambda_j/\tau_k\big)^2+\cdots+\tau_k^{-2}\big(\lambda_j/\tau_k\big)^{2N-2}+(\lambda_j/\tau_k)^{2N}\frac{1}{\tau_k^2-\lambda_j^2}.$$

As a result, we would have (3.21″) if we could choose $N\in\mathbb{N}$ so that we have

$$(3.23)\quad \Big|\iint\sum_j \lambda_j^{2\ell}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)\Big(\sum_{\tau_k>10\lambda}\tau_k^{-2-2\ell}(1-\Psi(\lambda_j/\tau_k))\,e_{\tau_k}(x)e_{\tau_k}(y)\Big)\,V(y)\,dxdy\Big| \lesssim \|V\|_{L^1(M)}\lambda^{n-2}(\log\lambda)^{1/2},\quad \ell=0,\dots,N-1,$$

as well as

$$(3.24)\quad \Big|\sum_j\sum_{\tau_k>10\lambda}\iint\frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}(\lambda_j)^{2N}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N}e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \lesssim \lambda^{n-2}\|V\|_{L^1(M)}.$$

To handle (3.23) we start with a trivial reduction. We note that if $\tau_k>10\lambda$, then by (2.15), (3.13) and (3.22)

$$\begin{aligned}\big|\tilde{\mathbb{1}}_\lambda(\lambda_j)\sum_{\tau_k>10\lambda}\Psi(\lambda_j/\tau)\tau_k^{-2-2\ell}e_{\tau_k}(x)e_{\tau_k}(y)\big| &\lesssim \big|\tilde{\mathbb{1}}_\lambda(\lambda_j)\sum_{\tau_k\in(\lambda_j/2,2\lambda_j)}\tau_k^{-2-2\ell}e_{\tau_k}(x)e_{\tau_k}(y)\big|\\ &\lesssim \lambda_j^{-\sigma}\sum_{\tau_k\approx\lambda_j}\tau_k^{-\sigma}|\tau_k^{-2-2\ell}e_{\tau_k}(x)e_{\tau_k}(y)|\\ &\lesssim \lambda_j^{n-2\ell-2\sigma},\end{aligned}$$

for any σ. If $\sigma>n$, by (2.27) this yields

$$\Big|\iint\sum_j\lambda_j^{2\ell}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)\Big(\sum_{\tau_k>10\lambda}\tau_k^{-2-2\ell}\Psi(\lambda_j/\tau_k)\,e_{\tau_k}(x)e_{\tau_k}(y)\Big)\,V(y)\,dxdy\Big| \lesssim \|V\|_{L^1(M)},$$

which means that in order to prove (3.23) it suffices to show that

$$(3.23')\quad \Big|\iint\big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)\Big(\sum_{\tau_k>10\lambda}\tau_k^{-2-2\ell}\,e_{\tau_k}(x)e_{\tau_k}(y)\Big)\,V(y)\,dxdy\Big| \lesssim \|V\|_{L^1(M)}\lambda^{n-2}(\log\lambda)^{1/2},\quad \ell=0,\dots,N-1,$$

since

$$\sum_j \lambda_j^{2\ell}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y) = \big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y).$$

To prove this, in certain cases, we shall rewrite the expression inside the absolute value in the left side of (3.23′) slightly. Specifically, we can split it into the following two terms

$$\tag{3.25} \begin{aligned} &\iint \big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)\Big(\sum_{\tau_k\ge 1}\tau_k^{-2-2\ell}\, e_{\tau_k}(x)e_{\tau_k}(y)\Big)\, V(y)\,dxdy \\ &\quad - \iint \big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)\Big(\sum_{\tau_k\le 10\lambda}\tau_k^{-2-2\ell}\, e_{\tau_k}(x)e_{\tau_k}(y)\Big)\, V(y)\,dxdy \\ &\qquad = I + II, \quad \text{if } \ell\le (n-4)/4 \text{ and } n\ge 4. \end{aligned}$$

If $n\le 3$ we shall not split things up in this way, and, instead, just deal with the expression in the left side of (3.23′) directly.

Note that if $n\ge 5$ and $\ell\le (n-4)/4$

$$\begin{aligned} |I| &= \Big|\iint \big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)\,\big(H_V\big)^{-1-\ell}(x,y)\,V(y)\,dxdy\Big| \\ &\le \iint_{d_g(x,y)\le\lambda^{-1}} + \iint_{d_g(x,y)\ge\lambda^{-1}} \Big(\big|\big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)\big|\; \big|\big(H_V\big)^{-1-\ell}(x,y)\big|\, |V(y)|\,dxdy \\ &\lesssim \iint_{d_g(x,y)\le\lambda^{-1}} \lambda^{n+2\ell}\,(d_g(x,y))^{-n+2+2\ell}\,|V(y)|\,dxdy \\ &\qquad + \|V\|_{L^1}\cdot \sup_y \Big(\int_{d_g(x,y)\ge\lambda^{-1}} \big|\big((P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)\big)(x,y)\big|^2\,dx\Big)^{1/2} \\ &\qquad\qquad \times \Big(\int_{d_g(x,y)\ge\lambda^{-1}} \big|\big(d_g(x,y)\big)^{-2(n-2-2\ell)}\,dx\Big)^{1/2} \\ &\lesssim \|V\|_{L^1}\cdot\lambda^{n+2\ell-(2-2\ell)} + \|V\|_{L^1}\cdot\big(\lambda^{\frac n2+2\ell}\cdot\lambda^{\frac n2-2-2\ell}\big) \\ &= \lambda^{n-2}\|V\|_{L^1}, \end{aligned}$$

which is better than the bounds in (3.23′). Here we used Lemma 2.6 to bound $(H_V^{-1-\ell})(x,y)$ (and our momentary assumption $\ell\le (n-4)/4$). In the second inequality we also used Schwarz's inequality, while in the second inequality and the second to last step we also used Lemma 2.4.

If $n=4$ than the requirement in (3.25) forces $\ell=0$. In this case, if we repeat the above arguments we obtain slightly worse bounds, i.e.,

$$|I|\lesssim \lambda^{n-2}(\log\lambda)^{1/2}\|V\|_{L^1},$$

with the $\log\lambda$ factor coming from the fact that when $n=4$ we have

$$\int_{d_g(x,y)\ge\lambda^{-1}} (d_g(x,y))^{-4}\,dx \approx \log\lambda.$$

On the other hand, this bound is in agreement with the one posited in (3.23′).

We still need to handle the second term, II, in (3.25). To do this we shall again use Lemma 2.4 and (3.13) along with Schwarz's inequality to deduce that

$$\begin{aligned}|II| &\le \|V\|_{L^1}\cdot\sup_y\Big(\|(P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)(\,\cdot\,,y)\|_{L^2}\cdot\|\sum_{\tau_k\le 10\lambda}\tau_k^{-2-2\ell}e_{\tau_k}(\,\cdot\,)e_{\tau_k}(y)\|_{L^2}\Big)\\ &\lesssim \|V\|_{L^1}\cdot\lambda^{\frac n2+2\ell}\cdot\big(\sum_{\tau_k\le 10\lambda}\tau_k^{-4-4\ell}|e_{\tau_k}(y)|^2\big)^{1/2}\\ &\lesssim \|V\|_{L^1}\cdot\lambda^{\frac n2+2\ell}\cdot\big(\sum_{\{j\in\mathbb{N}:\,2^j\le 10\lambda\}}2^{-j(4+4\ell)}2^{nj}\big)^{1/2}\\ &\lesssim \|V\|_{L^1}\cdot\lambda^{\frac n2+2\ell}\cdot\lambda^{-2-2\ell+\frac n2}\\ &=\|V\|_{L^1}\cdot\lambda^{n-2},\end{aligned}$$

assuming in the last step $\ell<(n-4)/4$. In the remaining case covered in (3.25) where $n\ge 4$ and $\ell=(n-4)/4$ (forcing n to be a multiple of 4), as was the case for $\ell=0$ and $n=4$, the bound is somewhat worse and we instead get, in this case,

$$|II|\lesssim\lambda^{n-2}(\log\lambda)^{1/2}\|V\|_{L^1},$$

which still better than that of our current goal, (3.21).

Since we have obtained favorable estimates for I and II in (3.25), we have shown that (3.23′) is valid when $n\ge 4$ and $\ell\le(n-4)/4$. For the remaining cases where $n=2,3$ and $0\le\ell\le N-1$ is arbitrary or $(n-4)/4<\ell\le N-1$ for $n\ge 4$, we shall just repeat the argument that we used to control II. We have not specified N; however, to get the other inequality, (3.24), that is needed to obtain our current goal (3.21), N will have to be chosen to be larger than $(n-4)/4$.

In these remaining cases for (3.23′) if we argue as above we find that the left side of (3.23′) is dominated by

$$\begin{aligned}&\|V\|_{L^1}\cdot\sup_y\Big(\|(P^0)^{2\ell}\tilde{\mathbb{1}}_\lambda(P^0)(\,\cdot\,,y)\|_{L^2}\cdot\|\sum_{\tau_k>10\lambda}\tau_k^{-2-2\ell}e_{\tau_k}(\,\cdot\,)e_{\tau_k}(y)\|_{L^2}\Big)\\ &\quad\lesssim \|V\|_{L^1}\cdot\lambda^{\frac n2+2\ell}\cdot\big(\sum_{\tau_k>10\lambda}\tau_k^{-4-4\ell}|e_{\tau_k}(y)|^2\big)^{1/2}\\ &\quad\lesssim \|V\|_{L^1}\cdot\lambda^{\frac n2+2\ell}\cdot\big(\sum_{\{j\in\mathbb{N}:\,2^j>10\lambda\}}2^{-j(4+4\ell)}2^{nj}\big)^{1/2}\\ &\quad\lesssim \|V\|_{L^1}\cdot\lambda^{\frac n2+2\ell}\cdot\lambda^{-2-2\ell+\frac n2}\\ &\quad=\lambda^{n-2}\|V\|_{L^1},\end{aligned}$$

using the fact that our current conditions ensure that $4+4\ell>n$. This completes the proof of (3.23′) and hence (3.23).

To finish this subsection we need to show that we can fix $N\in\mathbb{N}$ sufficiently large so that (3.24) is valid. As we mentioned before we shall specify our $N>(n-4)/4$ in a moment.

To prove (3.24) for large enough N, here too it will be convenient to split matters into two cases. First, let us deal with the sum in (3.24) where $\tau_k\ge\lambda^2$. We can handle this case using trivial methods if N is large enough. In fact, by using Schwarz's inequality, Lemma 2.4 and orthogonality, we see that for $\tau_k \ge \lambda^2$ we have the uniform bounds

$$\begin{aligned}\int \Big| \sum_j \frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}(\lambda_j)^{2N}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)\Big|\, dx \\ \lesssim \Big\|\sum_j \frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}(\lambda_j)^{2N}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(\,\cdot\,)e_j^0(y)\Big\|_{L^2} \\ \lesssim \big\|(P^0)^{2N}\tilde{\mathbb{1}}_\lambda(P^0)(\,\cdot,y)\big\|_{L^2} \lesssim \lambda^{\frac n2+2N}.\end{aligned}$$

This does not present a problem since if N is large, since, by (3.13) and the Cauchy-Schwartz inequality,

$$\sum_{\tau_k\ge\lambda^2}\tau_k^{-2N}|e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \sum_{\{j\in\mathbb{N}:\, 2^j\ge\lambda^2\}} 2^{-2Nj}2^{nj} \lesssim \lambda^{-4N}\lambda^{2n},$$

if $N>n$. Using these two inequalities we deduce that

$$\begin{aligned}\Big|\sum_j\sum_{\tau_k\ge\lambda^2}\iint \frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}(\lambda_j)^{2N}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N}e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \\ \lesssim \lambda^{\frac n2+2N}\lambda^{-4N+2n}\|V\|_{L^1(M)} < \|V\|_{L^1},\end{aligned}$$

if we assume, as we may, that $N=2n$.

Based on this, we would be done with handling relatively large frequencies of H_V if we could show that

(3.24′)
$$\begin{aligned}\Big|\sum_j\sum_{10\lambda<\tau_k<\lambda^2}\iint \frac{(1-\Psi(\lambda_j/\tau_k))}{\lambda_j^2-\tau_k^2}(\lambda_j)^{2N}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N}e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \\ \lesssim \lambda^{n-2}\|V\|_{L^1(M)}, \quad \text{if} \quad N=2n.\end{aligned}$$

To this end, let $\Phi\in C^\infty(\mathbb{R}_+)$ satisfy $\Phi(s)=1$, $s\le 3/2$ and $\Phi(s)=0$, $s\ge 2$. Then if $\tau_k\ge 10\lambda$, it follows that

$$\Phi(\lambda_j/\lambda)\big(1-\Psi(\lambda_j/\tau_k)\big)=\Phi(\lambda_j/\lambda),$$

and also for all $\sigma\in\mathbb{N}$

$$\tilde{\mathbb{1}}_\lambda(\lambda_j)\frac{(1-\Phi(\lambda_j/\lambda))}{\lambda_j^2-\tau_k^2}\big(1-\Psi(\lambda_j/\tau_k)\big)=O(\tau_k^{-\sigma}), \ \text{if} \ \ 10\lambda\le\tau_k\le\lambda^2.$$

Thus, by an earlier argument, modulo $O(\lambda^{-\sigma}\|V\|_{L^1})\ \forall\,\sigma\in\mathbb{N}$, the left side of (3.24′) agrees with the expression where we replace $(1-\Psi(\lambda_j/\tau_k))$ with $\Phi(\lambda_j/\lambda)$. Therefore since $(\lambda_j^2-\tau_k^2)^{-1}=-(1-(\lambda_j/\tau_k)^2)^{-1}\cdot\tau_k^{-2}$, we would have (3.24′) if we could show that

(3.24″)
$$\begin{aligned}\Big|\sum_j\sum_{10\lambda<\tau_k<\lambda^2}\iint \frac{\Phi(\lambda_j/\lambda)}{1-(\lambda_j^2/\tau_k)^2}(\lambda_j)^{2N}\tilde{\mathbb{1}}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N-2}e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \\ \lesssim \lambda^{n-2}\|V\|_{L^1(M)}, \quad N=2n.\end{aligned}$$

Since the left side is dominated by $\|V\|_{L^1}$ times

$$\sup_y \Big| \int \Big(\sum_j \frac{\Phi(\lambda_j/\lambda)}{1-\tau_k^{-2}\lambda_j^2} (\lambda_j)^{2N} \tilde{\mathbb{1}}_\lambda(\lambda_j) e_j^0(x) e_j^0(y) \Big) \times \Big(\sum_{10\lambda<\tau_k<\lambda^2} \tau_k^{-2N-2} e_{\tau_k}(x) e_{\tau_k}(y) \Big) \, dx \Big|, \tag{3.26}$$

it suffices to show that this expression is $O(\lambda^{n-2})$.

To do so, we shall appeal to the δ_τ–Lemma, Lemma 2.3. We set for a given $y \in M$

$$m(s,x) = \sum_j \frac{\Phi(\lambda_j/\lambda)}{1-s^2\lambda_j^2} (\lambda_j)^{2N} \tilde{\mathbb{1}}_\lambda(\lambda_j) e_j^0(x) e_j^0(y), \quad s \in [0, 1/10\lambda].$$

Then, since

$$\Big| \frac{\Phi(\lambda_j/\lambda)}{1-s^2\lambda_j^2} \Big| \lesssim 1, \tag{3.27}$$

and

$$\Big| \frac{\partial}{\partial s} \Big(\frac{\Phi(\lambda_j/\lambda)}{1-s^2\lambda_j^2} \Big) \Big| \lesssim s\lambda^2, \tag{3.28}$$

if $s \in [0, 1/10\lambda]$, it follows from orthogonality and Lemma 2.4 that

$$\|m(0,\,\cdot\,)\|_{L^2(M)} + \int_0^{1/10\lambda} \big\| \tfrac{\partial}{\partial s} m(s,\,\cdot\,) \big\|_{L^2(M)} \, ds = O(\lambda^{\frac{n}{2}+2N}). \tag{3.29}$$

Consequently, by Lemma 2.3, (3.26) is dominated by

$$\begin{aligned} \lambda^{\frac{n}{2}+2N} \Big\| \sum_{10\lambda<\tau_k<\lambda^2} \tau_k^{-2N-2} e_{\tau_k}(\,\cdot\,) e_{\tau_k}(y) \Big\|_{L^2} &= \lambda^{\frac{n}{2}+2N} \Big(| \sum_{10\lambda<\tau_k<\lambda^2} \tau_k^{-4N-4} |e_{\tau_k}(y)|^2 \Big)^{1/2} \\ &\lesssim \lambda^{\frac{n}{2}+2N} \Big(\sum_{\{j\in\mathbb{N}:\, 2^j>10\lambda\}} 2^{-(4N+4)j} 2^{nj} \Big)^{1/2} \\ &\lesssim \lambda^{\frac{n}{2}+2N} \cdot \lambda^{-2N-2} \cdot \lambda^{\frac{n}{2}} = \lambda^{n-2}, \end{aligned}$$

using (3.13) in the second to last step and the fact that $N = 2n$ in the final one. Thus, the quantity in (3.26) is $O(\lambda^{n-2})$, which, by the above, yields (3.24″) and finishes the proof of Proposition 3.3.

Handling the contribution of relatively small frequencies of H_V. In this subsection we shall handle relatively small frequencies of H_V and prove the following result.

Proposition 3.4. *As in Theorem 1.1 fix $V \in \mathcal{K}(M)$. Then*

$$\Big| \sum_j \sum_{\tau_k<\lambda/2} \int_M \int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j) - \tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dxdy \Big| \le C_V \lambda^{n-2} \|V\|_{L^1(M)}, \tag{3.30}$$

for some constant C_V depending on V.

If we combine this with Proposition 3.1 and Proposition 3.3 which handle frequencies which are comparable to λ and large compared to λ, respectively, we obtain Proposition 2.2, which by the arguments in §2, yield our main result, Theorem 1.1.

Proof of Proposition 3.4. As in the earlier cases, we shall first handle a trivial case. To do so, we note that, by (2.16) and the mean value theorem,

$$\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j^2-\tau^2}=O(\lambda^{-\sigma}),\ \forall\,\sigma\in\mathbb{N},\quad \text{if } \ 1\le\tau\le\lambda/2 \text{ and } \lambda_j\le 7\lambda/8.$$

Also, by (2.27) and (3.13)

$$\sum_{\lambda_j\le\lambda}|e_j^0(x)e_j^0(y)|,\ \sum_{\tau_k<\lambda/2}|e_{\tau_k}(x)e_{\tau_k}(y)|\lesssim\lambda^n. \tag{3.31}$$

To use these and make our first reduction fix $a\in C^\infty(\mathbb{R}_+)$ satisfying

$$a(s)=0,\quad s\le 3/4\quad\text{and}\quad a(s)=1,\ s\ge 7/8.$$

Using the preceding inequalities we see that in order to prove (3.30) it suffices to show that

$$\Big|\sum_j\sum_{\tau_k<\lambda/2}\int_M\int_M\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}a(\lambda_j/\lambda)e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V\lambda^{n-2}\|V\|_{L^1(M)}, \tag{3.30$'$}$$

due to the fact that the difference between the quantities inside the absolute values in the left side of (3.30) and that of (3.30′) is $O(\lambda^{-\sigma}\|V\|_{L^1})$ for all σ.

For the next reduction, note that the proof of Lemma 2.5 yields

$$\Big|\sum_j\frac{a(\lambda_j/\lambda)}{\lambda_j^2-\tau_k^2}e_j^0(x)e_j^0(y)\Big|\lesssim\begin{cases}(d_g(x,y))^{2-n}, & n\ge 3,\\ \log\big(2+(d_g(x,y))^{-1}\big), & n=2,\end{cases}$$

if $1\le\tau_k\le\lambda/2$. Based on this and the second part of (3.31) and the fact that $1-\tilde{\mathbb{1}}_\lambda(\tau_k)=O(\lambda^{-\sigma})$ for all σ when $1\le\tau_k\le\lambda/2$, we easily see that

$$\Big|\sum_j\sum_{\tau_k<\lambda/2}\int_M\int_M\frac{1-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}a(\lambda_j/\lambda)e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le\lambda^{-\sigma}\|V\|_{L^1(M)},\quad\forall\,\sigma\in\mathbb{N}.$$

Consequently, we would have (3.30′) if we could show that

$$\Big|\sum_j\sum_{\tau_k<\lambda/2}\int_M\int_M\frac{a(\lambda_j/\lambda)}{\lambda_j^2-\tau_k^2}\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\big)\,e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V\lambda^{n-2}\|V\|_{L^1(M)}. \tag{3.30$''$}$$

We need to make one final reduction before we can appeal to the δ_τ–Lemma, Lemma 2.3. For this, let η be as in Lemma 2.5, i.e., $\eta\in C^\infty(R_+)$ with $\eta(s)=0$ on $s\le 2$ and $\eta(s)=1$, $s>4$. It then follows that

$$\eta(s)a(s)=\eta(s).$$

Consequently, we can write the quanity inside the absolute value in the left side of (3.30″) as

$$\sum_j \sum_{\tau_k<\lambda/2} \int_M \int_M \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2-\tau_k^2} \left(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\right) e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy$$

$$+\sum_j \sum_{\tau_k<\lambda/2} \int_M \int_M \frac{a(\lambda_j/\lambda)(1-\eta(\lambda_j/\lambda))}{\lambda_j^2-\tau_k^2} \left(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\right) e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy$$

$$=I+II.$$

Therefore, in order to prove (3.30″), it suffices to show that both $|I|$ and $|II|$ are dominated by the right side of (3.30″).

We can easily handle I without appealing to the δ_τ–lemma. Indeed since $\tilde{\mathbb{1}}_\lambda(\lambda_j) = O(\lambda_j^{-\sigma})$ for all σ if $\eta(\lambda_j/\lambda)\neq 0$, we see that (2.27) yields

$$\sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2-\tau_k^2}\left(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\right) e_j^0(x)e_j^0(y) = -\sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2-\tau_k^2} e_j^0(x)e_j^0(y) + O(\lambda^{-\sigma}), \quad \forall\,\sigma.$$

Consequently, by the second part of (3.31) we would have the desired bounds for I if we could show that

$$\Big|\iint \sum_{\tau_k<\lambda/2} R_{\tau_k}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \lesssim \lambda^{n-2}\|V\|_{L^1}, \tag{3.32}$$

where

$$R_{\tau_k}(x,y)=\sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2-\tau_k^2} e_j^0(x)e_j^0(y).$$

To use this we note that the proof of Lemma 2.5 implies that

$$\sup_{1\le\tau_k<\lambda/2} |R_{\tau_k}(x,y)| \le C_0\lambda^{n-2}h_n(\lambda d_g(x,y))(1+\lambda d_g(x,y))^{-\sigma}, \quad \forall\,\sigma,$$

and, therefore,

$$\sup_y \int \sup_{1\le\tau_k<\lambda/2} |R_{\tau_k}(x,y)|\,dx \lesssim \lambda^{-2}.$$

Since, we always have $\tau_k\ge 1$ by (1.7), by the second part of (3.31) we have

$$\sup_y \int \Big|\sum_{\tau_k<\lambda/2} R_{\tau_k}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\Big|\,dx \lesssim \lambda^n\cdot \sup_y \int \sup_{1\le\tau_k<\lambda/2} |R_{\tau_k}(x,y)|\,dx \lesssim \lambda^{n-2},$$

which clearly yields (3.32).

Since we have the desired estimate for I above, it only remains to prove the corresponding estimate for II. For this, let

$$m(s,x,y)=\sum_j \frac{b(\lambda_j/\lambda)}{\lambda_j^2-s^2}\left(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\right) e_j^0(x)e_j^0(y),$$

$$\text{with } b(s)=a(s)(1-\eta(s))\in C^\infty((3/4,4)).$$

We then can rewrite this desired bound for II as follows

$$\Big|\iint \sum_{\tau_k<\lambda/2} m(\tau_k,x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \lesssim \lambda^{n-2}\|V\|_{L^1}.$$

Just as the last step in the proof of Proposition 3.3 was to establish (3.26), the final step here would be to show that

$$\sup_y \int \big| \sum_{\tau_k<\lambda/2} m(\tau_k, x, y) e_{\tau_k}(x) e_{\tau_k}(y) \big| \, dx \lesssim \lambda^{n-2}. \tag{3.33}$$

To prove this we shall argue as in the very end of the last subsection and appeal to the δ_τ–Lemma 2.3 with the δ there equal to $\lambda/2$. We first note that by (2.27) and the fact that $b(\lambda_j/\lambda) \neq 0$ implies $\lambda_j/\lambda \in [3/4, 4]$. Consequently,

$$\Big| \Big(\frac{\partial}{\partial s}\Big)^\ell \Big(\frac{b(\lambda_j/\lambda)}{\lambda_j^2 - s^2}\Big) \Big| \le C\lambda^{-2}(s\lambda^{-2})^\ell, \ \ell = 0, 1, \quad \text{if } 1 \le s \le \lambda/2.$$

Using this and the support properties b we can easily see that by the proof of Lemma 2.4 that (2.27) and orthogonality yields for $\ell = 0, 1$

$$\big\|(\tfrac{\partial}{\partial s})^\ell m(s, \, \cdot \, , y)\big\|_{L^2(M)} \le C_0 \lambda^{\frac{n}{2}-2}(s\lambda^{-2})^\ell, \quad \text{if } \ y \in M, \ 0 \le s \le \lambda/2.$$

Consequently,

$$\sup_y \Big(\|m(1, \, \cdot \, , y)\|_{L^2(M)} + \int_1^{\lambda/2} \big\|(\tfrac{\partial}{\partial s}) m(s, \, \cdot \, , y)\big\|_{L^2(M)} \Big) \lesssim \lambda^{\frac{n}{2}-2}.$$

By Lemma 2.3 and the second part of (3.31) we deduce from this that the left side of (3.33) is dominated by

$$\lambda^{\frac{n}{2}-2} \sup_y \big(\sum_{\tau_k<\lambda/2} |e_{\tau_k}(y)|^2 \big)^{1/2} \lesssim \lambda^{n-2},$$

which completes the proof. □

4. **Improved Weyl formulae under geometric assumptions.**

Generic improvements: Extension of the Duistermaat-Guillemin theorem.

To prove (1.13), we shall first establish the following proposition concerning the error term in Weyl law estimates.

Proposition 4.1. *Assume that the set $\mathcal{C}$ of directions of periodic geodesics has measure zero in S^*M and that $V \in \mathcal{K}(M)$. Then, for a constant C_V independent of $\varepsilon \in (0, 1)$, we have*

$$\int_M \sum_{\tau_k \in [\lambda, \lambda+\varepsilon]} |e_{\tau_k}(x)|^2 dx \le C_V \varepsilon \lambda^{n-1}, \ \ \lambda \ge \Lambda(\varepsilon), \tag{4.1}$$

where $\Lambda(\varepsilon) < +\infty$ depends on ε. Here $\{e_{\tau_k}\}$ are eigenfunctions of the operator H_V defined in the first section.

Proof. To prove (4.1), let us fix a non-negative function $\chi \in \mathcal{S}(\mathbb{R})$ satisfying:

$$\chi(\tau) > 1, \ |\tau| \le 1 \ \text{ and } \ \hat{\chi}(t) = 0, \ |t| \ge 1/2. \tag{4.2}$$

Then it suffice to show that for any fixed constant $T \gg 1$

$$\int_M \sum_{k=1}^{\infty} \chi(T(\lambda-\tau_k))|e_{\tau_k}(x)|^2 dx \le C_V T^{-1}\lambda^{n-1}, \quad \lambda \ge \Lambda(T), \tag{4.3}$$

where $\Lambda(T) < +\infty$ depends on T.

By Euler's formula we can rewrite the left side of (4.3) as

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_M \frac{1}{T}\hat{\chi}(t/T)e^{it\lambda}\sum_{k=1}^{\infty}\cos t\tau_k |e_{\tau_k}(x)|^2 dxdt \tag{4.3$'$}$$

plus

$$\int_M \sum_{k=1}^{\infty} \chi(T(\lambda+\tau_k))|e_{\tau_k}(x)|^2 dx.$$

Since $\chi \in \mathcal{S}(\mathbb{R})$, the last term is rapidly decreasing in λ with bounds independent of $T \ge 1$. Thus we just need to show that the expression in (4.3$'$) is bounded by the right side of (4.3).

On the other hand, under the assumption of the proposition, it is known that when $V \equiv 1$ (see e.g [16])

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_M \frac{1}{T}\hat{\chi}(t/T)e^{it\lambda}\sum_{j=1}^{\infty}\cos t\lambda_j |e_j^0(x)|^2 dxdt \le CT^{-1}\lambda^{n-1}, \quad \lambda \ge \Lambda(T). \tag{4.4}$$

Again, by using Lemma 2.1 and Duhamel's principle, we can rewrite the difference of (4.3$'$) and (4.4) as

$$\sum_{j,k}\int_M\int_M \frac{\tilde{\chi}_\lambda(\lambda_j)-\tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy \tag{4.5}$$

where

$$\tilde{\chi}_\lambda(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{T}\hat{\chi}(t/T)e^{it\lambda}\cos\tau\, dt = \chi(T(\lambda-\tau)) + \chi(T(\lambda+\tau)) \tag{4.5$'$}$$

and similarly, we interpret

$$\frac{\tilde{\chi}_\lambda(\tau)-\tilde{\chi}_\lambda(\mu)}{\tau^2-\mu^2} = \frac{\tilde{\chi}'_\lambda(\tau)}{2\tau}, \quad \text{if} \quad \tau = \mu \tag{4.6}$$

Since $\chi \in \mathcal{S}(\mathbb{R})$, we have

$$\big(\tfrac{d}{d\tau}\big)^j\, \tilde{\chi}_\lambda(\tau) = O\big(T^j(1+T|\lambda-\tau|)^{-N}\big) \ \forall N, \quad \text{if} \ j = 0,1,2,3,\dots. \tag{4.7}$$

Given (4.7), we can use the same arguments as in the proof of Proposition 2.2 with minor modifications to get the following:

$$\Big|\sum_{j,k}\int_M\int_M \frac{\tilde{\chi}_\lambda(\lambda_j)-\tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V T^2 \|V\|_{L^1(M)}\,\lambda^{n-\frac{3}{2}}, \tag{4.8}$$

where the constant T^2 comes from the fact that

$$\Big|\frac{\partial}{\partial\tau}\frac{\tilde\chi_\lambda(\tau)-\tilde\chi_\lambda(\mu)}{\tau-\mu}\Big|\lesssim T^2,$$

which arises when we apply lemma 2.3. Since the right side of (4.8) is better than the one in (4.1), the proof is complete. Note that if we take $T=1$ in the above argument, we obtain (2.4). □

Proof of Theorem 1.2. Fix an even real-valued function $\rho\in C^\infty(\mathbb{R})$ satisfying (2.13), and for any fixed constant T, define

$$\tilde{\mathbb{1}}_\lambda(\tau)=\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(t/T)\frac{\sin\lambda t}{t}\,\cos t\tau\,dt. \tag{4.9}$$

By a change of variable argument, it follows from (2.15) that for $\tau>0$ and large λ we have

$$\mathbb{1}_\lambda(\tau)-\tilde{\mathbb{1}}_\lambda(\tau)=O\big((1+T|\lambda-\tau|)^{-N}\big)\ \ \forall N. \tag{4.10}$$

Also, for $\tau>0$ we have the analog of (2.16)

$$\big(\tfrac{d}{d\tau}\big)^j\,\tilde{\mathbb{1}}_\lambda(\tau)\le CT^j\big((1+|\lambda-\tau|)^{-N}\big)\ \ \forall N,\quad \text{if }\ j=1,2,3,\dots. \tag{4.11}$$

Furthermore, if $\tilde{\mathbb{1}}_\lambda(P_0)$ is defined as in (4.9), by (4.10), (2.25) and (2.26) in Lemma (2.4) still hold.

Now if we use (4.1) , we can estimate the difference between the trace of $\mathbb{1}_\lambda(P_V)-\tilde{\mathbb{1}}_\lambda(P_V)$. Indeed, by (2.15) or (4.10) we have

$$\begin{aligned}\Big|\int_M\big(\mathbb{1}_\lambda(P_V)(x,x)-\tilde{\mathbb{1}}_\lambda(P_V)(x,x)\big)\,dx\Big| &= \Big|\int_M\sum_k\big(\mathbb{1}_\lambda(\tau_k)-\tilde{\mathbb{1}}_\lambda(\tau_k)\big)\,|e_{\tau_k}(x)|^2\,dx\Big| \\ &\lesssim\sum_k\int_M\big(1+T|\lambda-\tau_k|\big)^{-2n}\,|e_{\tau_k}(x)|^2\,dx\lesssim\frac{1}{T}\lambda^{n-1},\ \ \lambda\ge\Lambda(T),\end{aligned} \tag{4.12}$$

using the (4.1) in the last inequality.

Since it is known that when $V\equiv 1$ (see e.g [16])

$$\Big|\int_M\tilde{\mathbb{1}}_\lambda(P^0)(x,x)\,dx-(2\pi)^{-n}\omega_n\mathrm{Vol}_g(M)\,\lambda^n\Big|\lesssim\frac{1}{T}\lambda^{n-1},\ \lambda\ge\Lambda(T), \tag{4.13}$$

in order to prove Theorem 1.2, it suffices to prove the following:

$$\Big|\int_M\big(\tilde{\mathbb{1}}_\lambda(P_V)(x,x)-\tilde{\mathbb{1}}_\lambda(P^0)(x,x)\big)\,dx\Big|\lesssim\frac{1}{T}\lambda^{n-1},\ \lambda\ge\Lambda(T). \tag{4.14}$$

By Lemma 2.1 and Duhamel's principle, this equivalent to

$$\Big|\sum_{j,k}\int_M\int_M\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big|\lesssim\frac{1}{T}\lambda^{n-1}. \tag{4.15}$$

By (4.10), (4.11), as well as (2.25) and (2.26), the properties of the function $\tilde{\mathbb{1}}_\lambda(\tau)$ here is similar to the one defined in (2.14). It follows from the proof of Proposition 2.2 and Proposition 4.1 that, the left side of (4.15) is bounded by $C_VT^2\|V\|_{L^1(M)}\,\lambda^{n-\frac{3}{2}}$, which is better than the right side of (4.14), which completes the proof.

□

Improvements for manifolds with non-positive curvature.

To prove (1.14), we need the following analog of (4.1).

Proposition 4.2. *Assume that the sectional curvatures of (M,g) are non-positive and that $V \in \mathcal{K}(M)$. Then for $\lambda \geq 1$*

$$\int_M \sum_{\tau_k \in [\lambda, \lambda + \frac{1}{\log \lambda}]} |e_{\tau_k}(x)|^2 dx \leq C_V \lambda^{n-1}/\log\lambda, \tag{4.16}$$

where $\{e_{\tau_k}\}$ are eigenfunctions of the operator H_V defined in the first section.

If we use the same arguments as in the proof of (4.1), we conclude that (4.16) follows from

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_M \frac{1}{T}\hat{\chi}(t/T)e^{it\lambda}\sum_{j=1}^{\infty}\cos t\lambda_j |e_j^0(x)|^2 dxdt \leq CT^{-1}\lambda^{n-1}, \;\; T = \log\lambda. \tag{4.17}$$

Also, given (4.16), by repeating the arguments in (4.9)-(4.15), we see that (1.14) would be a consequence of

$$\Big| \int_M \tilde{\mathbb{1}}_\lambda(P^0)(x,x)\,dx - (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\,\lambda^n \Big| \lesssim \frac{1}{T}\lambda^{n-1}, \quad T = \log\lambda, \tag{4.18}$$

if $\tilde{\mathbb{1}}_\lambda(\tau)$ is defined as in (4.9).

Since both (4.17) and (4.18) follow from the classical theorem of Bérard [2], the proof is complete.

Improvements for tori.

To prove Theorem 1.4, we will first establish a simpler variant of the first part of the theorem, (1.15), under the stronger assumption that $V \in L^2(M) \cap \mathcal{K}(M)$. After presenting this model argument, we shall see how we can modify the argument to prove Theorem 1.4. In all cases, the main strategy is the same as in the proof of (1.13) and (1.14). That is, we need to utilize the standard known result when $V \equiv 1$. To that end, let us recall the following:

Proposition 4.3. *If $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the standard torus with the flat metric and $N^0(\lambda)$ denotes the Weyl counting function for H^0, then*

$$N^0(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\,\lambda^n + O(\lambda^{n-1-\frac{n-1}{n+1}}). \tag{4.19}$$

The result in (4.19) is due to Hlwaka [7] for any $n \geq 2$. As a consequence of (4.19), we claim that we have the following two inequalities:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_M \frac{1}{\lambda^a}\hat{\chi}(t/\lambda^a)e^{it\lambda}\sum_{j=1}^{\infty}\cos t\lambda_j |e_j^0(x)|^2 dxdt \leq C\lambda^{n-1-a}, \tag{4.20}$$

and

$$(4.21)\quad \Big|\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(t/\lambda^a)\frac{\sin\lambda t}{t}\,\big(\cos tP^0\big)(x,x)\,dtdx-(2\pi)^{-n}\omega_n\mathrm{Vol}_g(M)\,\lambda^n\,\Big|\le C\lambda^{n-1-a}.$$

where $\chi(\tau),\rho(t)$ are the Schwartz functions defined in (4.2), (2.13) respectively, and $a=\frac{n-1}{n+1}$.

To see this, we first note that the left side of (4.20) equals

$$\int_M\sum_{j=1}^{\infty}\chi(\lambda^a(\lambda-\lambda_j))|e_j^0(x)|^2dx+\int_M\sum_{j=1}^{\infty}\chi(\lambda^a(\lambda+\lambda_j))|e_j^0(x)|^2dx,$$

so by a direct calculation, (4.20) is a consequence of

$$(4.22)\qquad N^0(\lambda+\lambda^{-a})-N^0(\lambda)=O(\lambda^{n-1-a}),$$

which is a corollary of (4.19). For the other inequality note that

(4.23)

$$\begin{aligned}\Big|\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(t/\lambda^a)\frac{\sin\lambda t}{t}\,\big(\cos tP^0\big)(x,x)\,dtdx-N^0(\lambda)\Big|&\\ =\Big|\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\rho(t/\lambda^a))\frac{\sin\lambda t}{t}\,\big(\cos tP^0\big)(x,x)\,dtdx\Big|&\\ \lesssim\int_M\sum_{j=1}^{\infty}r_\lambda(\lambda_j)|e_j^0(x)|^2dx,&\end{aligned}$$

where

$$|r_\lambda(\tau)|=\Big|\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\rho(t/\lambda^a))\frac{\sin\lambda t}{t}\,\cos t\tau\,dtd\tau\Big|=O\big((1+\lambda^a|\lambda-\tau|)^{-N}\big)\ \ \forall N.$$

Therefore, (4.21) is also a consequence of (4.19) and (4.22).

To obtain the desired bounds for the torus, we need to modify the earlier arguments since the right sides of (4.8) and (4.15) are too large to obtain the desired bounds for the improved Weyl formula on the torus. To do this, we begin with the following Proposition.

Proposition 4.4. *Let $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric and $\{e_{\tau_k}\}$ be eigenfunctions of the operator H_V, with $V\in L^2(M)\cap\mathcal{K}(M)$. Given (4.19), if we assume, for all $\lambda>1$ that*

$$(4.24)\qquad \int_M\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-a}]}|e_{\tau_k}(x)|^2dx\le C_V\lambda^{n-1-b},\ \ \text{for some}\quad -1\le b\le a,$$

then,

$$(4.25)\qquad \int_M\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-a}]}|e_{\tau_k}(x)|^2dx\le C_V\|V\|_{L^2(M)}\lambda^{n-2+\frac{a-b}{2}}\log\lambda+C\lambda^{n-1-a},$$

where $a=\frac{n-1}{n+1}$.

To put this in perspective, note that the conclusion here is reminiscent to how we used (2.35) to prove the $O(\lambda^{n-\frac{3}{2}})$ error bounds in (3.1). Indeed, by (2.34) we have (4.24) with $a=0$ and $b=-1$, and, in this case, the first λ-factor in the right side of (4.25) is $\lambda^{n-2+\frac{a-b}{2}}=\lambda^{n-\frac{3}{2}}$.

Before proving this Proposition, let us present a simple but useful corollary. We note that, by (2.35), if $V \in \mathcal{K}(M)$, then for all $x \in M$

$$\sum_{\tau_k \in [\lambda, \lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 \le \sum_{\tau_k \in [\lambda, 2\lambda]} |e_{\tau_k}(x)|^2 \le C_V \lambda^n. \tag{4.26}$$

So (4.24) is true for $b = -1$, and note that every time we apply the Proposition, we would have (4.24) for a larger value of b. Consequently, we can obtain the following:

Corollary 4.5. *Let $V \in L^2(M) \cap \mathcal{K}(M)$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the standard torus with the flat metric, and $\{e_{\tau_k}\}$ are eigenfunctions of the operator H_V. Then given* (4.20)*, we have for all $\lambda > 1$*

$$\int_M \sum_{\tau_k \in [\lambda, \lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 dx \le C_V \lambda^{n-1-a}, \tag{4.27}$$

where C_V is a constant depend on V, and $a = \frac{n-1}{n+1}$.

Proof. To prove (4.27), let us first ignore the $\log\lambda$ factor on the right side of (4.25). Define b_m to be the best exponent such that

$$\int_M \sum_{\tau_k \in [\lambda, \lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 dx \le C_V \lambda^{n-1-b_m},$$

after applying Proposition 4.4 m times. We have

$$n-1-b_{m+1} = \max\{n-2+\frac{n-1}{2(n+1)} - \frac{b_m}{2},\ n-1-\frac{n-1}{n+1}\}, \quad m = 0,1,2... \tag{4.27$'$}$$

with $b_0 = -1$.

Now if $b_m \le \frac{n-5}{n+1}$, we have $b_{m+1} = \frac{b_m}{2} + \frac{n+3}{2(n+1)}$. In this case, $b_{m+1} - b_m = \frac{n+3}{2(n+1)} - \frac{b_m}{2} \ge \frac{4}{n+1}$, which means the sequence is strictly increasing in this case. Let $N = [\frac{\frac{n-5}{n+1}+1}{\frac{4}{n+1}}] + 1$, we have $b_N > \frac{n-5}{n+1}$. Thus by (4.27$'$), $b_m \equiv \frac{n-1}{n+1}$ for all $m > N$.

Since $\log\lambda \lesssim \lambda^\varepsilon$ for all $\varepsilon > 0$, by this argument, we have $b_{N+1} \ge \frac{n-1}{n+1} - \varepsilon$. However, if ε is small enough,

$$\max\{n-2+\frac{n-1}{2(n+1)} - \frac{\frac{n-1}{n+1}-\varepsilon}{2},\ n-1-\frac{n-1}{n+1}\} = n-1-\frac{n-1}{n+1}.$$

So we have in this case $b_m \equiv \frac{n-1}{n+1}$ for all $m > N+1$. The proof of (4.27) is complete. □

By using Corollary 4.5, along with the arguments in Section 3, we have the following result on torus.

Theorem 4.6. *If $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the standard torus with the flat metric and $V \in L^2(M) \cap \mathcal{K}(M)$, then given* (4.19)*, we have*

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \mathrm{Vol}_g(M)\,\lambda^n + O(\lambda^{n-1-\frac{n-1}{n+1}}). \tag{4.28}$$

Note that (4.28) is a variant of (1.15) with a stronger condition on the potential V. The condition $V \in L^2(M)$ arises when we try to get an improvement over the main terms in (3.1), which are (3.12), (3.15), etc. For more details, see (4.45) in the argument below, and we postpone the proof of the stronger version of (1.15) to the end of the section.

Proof of Proposition 4.4 and Theorem 4.6. We shall first give the proof of (4.28), then prove (4.25) by modifying the argument.

Similar to (4.9), let $a = \frac{n-1}{n+1}$, and

$$\tilde{\mathbb{1}}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(t/\lambda^a) \frac{\sin \lambda t}{t} \cos t\tau \, dt. \tag{4.29}$$

By a change of scale argument, it follows from (2.15) that for $\tau > 0$ and large λ we have

$$\mathbb{1}_\lambda(\tau) - \tilde{\mathbb{1}}_\lambda(\tau) = O\big((1 + \lambda^a|\lambda - \tau|)^{-N}\big) \ \ \forall N. \tag{4.30}$$

Additionally, for $\tau > 0$ we have the analog of (2.16)

$$\big(\tfrac{d}{d\tau}\big)^j \, \tilde{\mathbb{1}}_\lambda(\tau) \le C\lambda^{aj}\big((1 + \lambda^a|\lambda - \tau|)^{-N}\big) \ \ \forall N, \quad \text{if } j = 1, 2, 3, \dots. \tag{4.31}$$

Furthermore, by (4.30), the bounds in Lemma 2.4 still hold if we let $\tilde{\mathbb{1}}_\lambda(P^0)$ be defined as in (4.29).

In view of (4.21), after repeating the arguments in (4.9)-(4.15), we see that (4.28) would be a consequence of

$$\Big| \sum_{j,k} \int_M \int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j) - \tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dxdy \Big| \le C_V \|V\|_{L^2(M)} \lambda^{n-2} \log \lambda, \tag{4.32}$$

where we interpret

$$\frac{\tilde{\mathbb{1}}_\lambda(\tau) - \tilde{\mathbb{1}}_\lambda(\mu)}{\tau^2 - \mu^2} = \frac{\tilde{\mathbb{1}}_\lambda'(\tau)}{2\tau}, \ \text{ if } \ \tau = \mu. \tag{4.33}$$

The proof of (4.32) requires a bound on the trace of certain spectral projection operators, which is (4.27). The fact that we rely on trace inequalities, rather than pointwise ones as was done in the past, accounts for our assumption here that $V \in L^2(M)$.

As before, we shall split things into three different cases that require slightly different arguments. The main contribution still comes from frequencies τ_k which are comparable to λ. We shall skip the proof for large or small frequencies τ_k, since, by the earlier arguments, these two cases only contribute terms which are $O(\|V\|_{L^1(M)} \lambda^{n-2} (\log \lambda)^{1/2})$.

Consequently, we would obtain (4.32) if we could prove the following.

Proposition 4.7. *As in Theorem 4.6, fix $V \in L^2(M) \cap \mathcal{K}(M)$. If $\tilde{\mathbb{1}}_\lambda(\tau)$ is defined as in (4.29), then*

$$\Big| \sum_j \sum_{\{k:\, \tau_k \in [\lambda/2, 10\lambda]\}} \int_M \int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j) - \tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dxdy \Big| \le C_V \|V\|_{L^2(M)} \lambda^{n-2} \log \lambda. \tag{4.34}$$

To prove Proposition 4.7, we shall make appropriate modifications of the arguments in the beginning of §3. So, as before, let us fix a Littlewood-Paley bump function $\beta \in C_0^\infty((1/2,2))$ satisfying

$$\sum_{\ell=-\infty}^{\infty} \beta(2^{-\ell}s) = 1, \quad s>0,$$

and then set

$$\beta_0(s) = \sum_{\ell\le 0} \beta(2^{-\ell}|s|) \in C_0^\infty((-2,2)),$$

and

$$\tilde\beta(s) = s^{-1}\beta(|s|) \in C_0^\infty\big(\{|s|\in(1/2,2)\}\big).$$

We then now write for $\lambda/2 \le \tau \le 10\lambda$, $a=\frac{n-1}{n+1}$

(4.35)

$$\begin{aligned}
K_\tau(x,y) &= \sum_j \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j^2-\tau^2} e_j^0(x)e_j^0(y) \\
&= \sum_j \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j-\tau}\,\frac{\beta_0\big(\lambda^a(\lambda_j-\tau)\big)}{\lambda_j+\tau}\, e_j^0(x)e_j^0(y) \\
&\quad + \sum_{\{\ell\in\mathbb{N}:\, 2^\ell\le\lambda\cdot\lambda^a/100\}} \Big(\sum_j \frac{\lambda^a 2^{-\ell}\tilde\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\,(\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau))\, e_j^0(x)e_j^0(y)\Big) \\
&\quad + \sum_j \Big(\frac{\sum_{\{\ell\in\mathbb{N}:\, 2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)\big)\, e_j^0(x)e_j^0(y).
\end{aligned}$$

Next, let

$$K_{\tau,0}(x,y) = \sum_j \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau)}{\lambda_j-\tau}\,\frac{\beta_0(\lambda^a(\lambda_j-\tau))}{\lambda_j+\tau}\, e_j^0(x)e_j^0(y),$$

$$R_{\tau,\ell}(x,y) = \sum_j \frac{\lambda^a2^{-\ell}\tilde\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\, e_j^0(x)e_j^0(y), \quad \text{if } 2^\ell\le\lambda\cdot\lambda^a/100,$$

and

$$R_{\tau,\infty}(x,y) = \sum_j \Big(\frac{\sum_{\{\ell\in\mathbb{N}:\, 2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\, e_j^0(x)e_j^0(y).$$

Also, for $2^\ell\le\lambda\cdot\lambda^a/100$ let

$$K^-_{\tau,\ell}(x,y) = \sum_j \frac{\lambda^a2^{-\ell}\tilde\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\big)\, e_j^0(x)e_j^0(y)$$

$$K^+_{\tau,\ell}(x,y) = \sum_j \frac{\lambda^a2^{-\ell}\tilde\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j+\tau}\tilde{\mathbb{1}}_\lambda(\lambda_j)\, e_j^0(x)e_j^0(y),$$

and, finally,

$$K^-_{\tau,\infty}(x,y)=\sum_j\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\big(\tilde{\mathbb{1}}_\lambda(\lambda_j)-1\big)\,e^0_j(x)e^0_j(y)$$

$$K^+_{\tau,\infty}(x,y)=\sum_j\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_j-\tau))}{\lambda_j^2-\tau^2}\Big)\tilde{\mathbb{1}}_\lambda(\lambda_j)\,e^0_j(x)e^0_j(y).$$

If K_τ is as in (4.35), our current task, (4.34), is to show that

$$\Big|\sum_{\tau_k\in[\lambda/2,10\lambda]}\iint K_{\tau_k}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim\|V\|_{L^2(M)}\lambda^{n-2}\log\lambda. \tag{4.34′}$$

To prove this, similar to before, we note that we can write

$$\begin{aligned}K_\tau(x,y)=K_{\tau,0}(x,y)&+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda\cdot\lambda^a/100\}}K^-_{\tau,\ell}(x,y)+K^-_{\tau,\infty}(x,y)\\&+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda\cdot\lambda^a/100\}}R_{\tau,\ell}(x,y)\big(1-\tilde{\mathbb{1}}_\lambda(\tau)\big)+R_{\tau,\infty}(x,y)\big(1-\tilde{\mathbb{1}}_\lambda(\tau)\big),\end{aligned}\tag{4.36}$$

or

$$\begin{aligned}K_\tau(x,y)=K_{\tau,0}(x,y)&+\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda\cdot\lambda^a/100\}}K^+_{\tau,\ell}(x,y)+K^+_{\tau,\infty}(x,y)\\&-\sum_{\{\ell\in\mathbb{N}:\,2^\ell\le\lambda\cdot\lambda^a/100\}}R_{\tau,\ell}(x,y)\tilde{\mathbb{1}}_\lambda(\tau)-R_{\tau,\infty}(x,y)\tilde{\mathbb{1}}_\lambda(\tau).\end{aligned}\tag{4.37}$$

We shall use (4.36) to handle the summands in (4.34′) with $\tau=\tau_k\in[\lambda/2,\lambda]$ and (4.37) to handle those with $\tau=\tau_k\in(\lambda,10\lambda]$.

For $\ell\in\mathbb{N}$ with $2^\ell\le\lambda\cdot\lambda^a/100$, let for $j=0,1,2,\dots$

$$I^-_{\ell,j}=\big(\lambda-(j+1)\lambda^{-a}2^\ell,\,\lambda-j\lambda^{-a}2^\ell\big]\quad\text{and}\quad I^+_{\ell,j}=\big(\lambda+j\lambda^{-a}2^\ell,\,\lambda+(j+1)\lambda^{-a}2^\ell\,\big].\tag{4.38}$$

Then to use the δ_τ–Lemma (Lemma 2.3), we shall use the following result whose proof we momentarily postpone.

Lemma 4.8. *If $a=\frac{n-1}{n+1}$, $\ell\in\mathbb{Z}_+$, $2^\ell\le\lambda\cdot\lambda^a/100$, and $j=0,1,2,\dots$, we have for each $N\in\mathbb{N}$*

$$\begin{aligned}&\|K^\pm_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda^{-a}2^\ell\tfrac{\partial}{\partial\tau}K^\pm_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)}\\&\qquad\lesssim\lambda^{\frac{n-1+a}{2}-1}2^{-\ell/2}(1+j)^{-N},\quad\tau\in I^\pm_{\ell,j}\cap[\lambda/2,10\lambda].\end{aligned}\tag{4.39}$$

Also,

$$\begin{aligned}&\|K_{\tau,0}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda^{-a}\tfrac{\partial}{\partial\tau}K_{\tau,0}(\,\cdot\,,y)\|_{L^2(M)}\\&\qquad\lesssim\lambda^{\frac{n-1+a}{2}-1}(1+j)^{-N},\quad\tau\in I^\pm_{0,j}\cap[\lambda/2,10\lambda],\end{aligned}\tag{4.40}$$

$$\|K^+_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda\tfrac{\partial}{\partial\tau}K^+_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)}\lesssim\lambda^{\frac{n}{2}-2},\quad\tau\in[\lambda,10\lambda],\tag{4.41}$$

and we can write

$$K^-_{\tau,\infty}(x,y)=\tilde K^-_{\tau,\infty}(x,y)+H^-_{\tau,\infty}(x,y),$$

where for $\tau \in [\lambda/2, \lambda]$

$$\|\tilde{K}^-_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda\tfrac{\partial}{\partial\tau}\tilde{K}^-_{\tau,\infty}(\,\cdot\,,y)\|_{L^2(M)} \lesssim \lambda^{\frac{n}{2}-2} \tag{4.42}$$

$$|H^-_{\tau,\infty}(x,y)| \lesssim \lambda^{n-2}h_n(\lambda d_g(x-y))(1+\lambda d_g(x,y))^{-N},$$

with h_n *as in* (1.4)*. Finally, we also have for* $1 \le 2^\ell \le \lambda \cdot \lambda^a/100$ *and* $\tau \in [\lambda/2, 10\lambda]$

$$\|R_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)},\ \|\lambda^{-a}2^\ell\tfrac{\partial}{\partial\tau}R_{\tau,\ell}(\,\cdot\,,y)\|_{L^2(M)} \lesssim \lambda^{\frac{n-1+a}{2}-1}2^{-\ell/2}, \tag{4.43}$$

and

$$|R_{\tau,\infty}(x,y)| \lesssim \lambda^{n-2}h_n(\lambda d_g(x,y))\,(1+\lambda d_g(x,y))^{-N}. \tag{4.44}$$

As before, we are abusing notation a bit. First, in (4.39) we mean that if $K_{\tau,\ell}$ equals $K^+_{\tau,\ell}$ or $K^-_{\tau,\ell}$ then the bounds in (4.39) for τ in $I^+_{\ell,j}\cap[\lambda,10\lambda]$ or $I^-_{\ell,j}\cap[\lambda/2,\lambda]$, respectively. Also, in both the second inequality in (4.42) and in (4.44) we mean that the kernels satisfy the bounds when x is sufficiently close to y (so that $d_g(x,y)$ is well-defined) and that they are $O(\lambda^{-N})$ away from the diagonal.

Before proving this result let us see how we can use it along with Lemma 2.3 to prove Proposition 4.7.

Proof of Proposition 4.7. First, by (4.39) and Lemma 2.3 with $\delta = \lambda^{-a}2^\ell$, we have

$$\begin{aligned}
&\Big|\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}\iint K^{\pm}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dydx\Big| \\
&\le \|V\|_{L^2}\cdot\Big\|\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}\iint K^{\pm}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\Big\|_{L^2\left(dy;L^1(dx)\right)} \\
&\lesssim \|V\|_{L^2}\cdot\sup_y\Big(\|K^{\pm}_{\lambda+j\lambda^{-a}2^\ell,\ell}(\,\cdot\,,y)\|_{L^2(M)}+\int_{I^{\pm}_{\ell,j}}\big\|\tfrac{\partial}{\partial\tau}K^{\pm}_{s,\ell}(\,\cdot\,y)\|_{L^2(M)}\,ds\Big) \\
&\qquad\qquad \times\Big(\int_M\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2dy\Big)^{1/2} \\
&\lesssim \|V\|_{L^2}\lambda^{\frac{n-1+a}{2}-1}2^{-\ell/2}(1+j)^{-N}\Big(\int_M\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2dy\Big)^{1/2} \\
&\lesssim \lambda^{n-2}(1+j)^{-N}\|V\|_{L^2}.
\end{aligned} \tag{4.45}$$

In the second to last inequality we used Corollary 4.5 and the fact that $|I^{\pm}_{\ell,j}| = \lambda^{-a}2^\ell$. As we alluded to before, since Corollary 4.5 only affords us trace bounds, the preceding inequality involves $\|V\|_{L^2(M)}$ in the right, as opposed to L^1-norms of the potential as was the case in the past.

If we sum over $j=0,1,2,\dots$, we see that (4.45) yields that for $1\le 2^\ell\le \lambda\cdot\lambda^a/100$

$$\Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint K^+_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| + \Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint K^-_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \lesssim \lambda^{n-2}\|V\|_{L^2(M)}. \tag{4.46}$$

If we take $\delta=\lambda^{-a}$ in Lemma 2.3, this argument also gives

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint K_{\tau_k,0}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| + \Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint K_{\tau_k,0}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \lesssim \lambda^{n-2}\|V\|_{L^2(M)}. \tag{4.47}$$

Similarly, if we use Lemma 2.3 with $\delta=\lambda$ along with (4.41) we find that

$$\Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint K^+_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \lesssim \lambda^{\frac n2-2}\|V\|_{L^1}\big(\sum_{\tau_k\in[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2\big)^{1/2}\lesssim \lambda^{n-2}\|V\|_{L^1}, \tag{4.48}$$

using (4.26) for the last inequality.

Next, since $R_{\tau,\ell}$ enjoys the bounds in (4.43), we can repeat the arguments in (4.45) to see that for $1\le 2^\ell\le\lambda\cdot\lambda^a/100$ we have

$$\begin{aligned}\Big|\sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]}&\iint R_{\tau_k,\ell}(x,y)\tilde{\mathbb 1}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\\ &\lesssim \|V\|_{L^2}\cdot 2^{-\ell/2}\lambda^{\frac{n-1+a}{2}-1}\big(\int_M\sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]}|\tilde{\mathbb 1}_\lambda(\tau_k)e_{\tau_k}(y)|^2dy\big)^{1/2}\\ &\lesssim \lambda^{n-2}(1+j)^{-N}\|V\|_{L^2},\end{aligned}$$

since $\tilde{\mathbb 1}_\lambda(\tau_k)=O((1+j)^{-N})$ if $\tau_k\in I^+_{\ell,j}$. Summing over this bound over j of course yields

$$\Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint R_{\tau_k,\ell}(x,y)\tilde{\mathbb 1}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\lesssim\lambda^{n-2}\|V\|_{L^2(M)}. \tag{4.49}$$

The same argument gives

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint R_{\tau_k,\ell}(x,y)\big(1-\tilde{\mathbb 1}_\lambda(\tau_k)\big)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\lesssim\lambda^{n-2}\|V\|_{L^2(M)}. \tag{4.50}$$

Also, by (4.44) we have

$$\sup_y\int\sup_{\lambda/2\le\tau\le 10\lambda}|R_{\tau,\infty}(x,y)|\,dx\lesssim\lambda^{-2},$$

and since (4.26) yields $\sum_{\tau_k\le 10\lambda}|e_{\tau_k}(x)e_{\tau_k}(y)|\lesssim \lambda^n$, we have

$$\begin{gathered}\Big|\sum_{\tau_k\in(\lambda,10\lambda]}\iint R_{\tau_k,\infty}(x,y)\tilde{1\!\!1}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}\\ \Big|\sum_{\tau_k\in[\lambda/2,\lambda]}\iint R_{\tau_k,\infty}(x,y)\big(1-\tilde{1\!\!1}_\lambda(\tau_k)\big)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}.\end{gathered}\tag{4.51}$$

If $H^-_{\tau,\infty}$ is as in (4.42) this argument also gives us

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint H^-_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1},$$

while the proof of (4.48) along with the first part of (4.42) yields

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint \tilde K^-_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}.$$

Since $K_{\tau,\infty}=\tilde K_{\tau,\infty}+H^-_{\tau,\infty}$, we deduce

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint K^-_{\tau_k,\infty}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim \lambda^{n-2}\|V\|_{L^1}.\tag{4.52}$$

We now have assembled all the ingredients for the proof of (4.34′). If we use (4.46), (4.47), (4.50), (4.51) and (4.52) along with (4.36), we conclude that the analog of (4.34′) must be valid where the sum is taken over $\tau_k\in[\lambda/2,\lambda]$. The log-loss comes from the fact that there are $\approx\log\lambda$ terms $K^-_{\tau,\ell}$ and $R_{\tau,\ell}$. We similarly obtain the analog of (4.34′) where the sum is taken over $\tau_k\in(\lambda,10\lambda]$ from (4.37) along with (4.46), (4.47), (4.49) and (4.51).

From this, we deduce that (4.34′) must be valid, which finishes the proof that Lemma 2.3 and Lemma 4.8 yield Proposition 4.7. □

To finish the present task we need to prove Lemma 4.8.

Proof of Lemma 4.8. To prove the first inequality we note that if $\tau\in I^\pm_{\ell,j}\cap[\lambda/2,10\lambda]$ then $|\lambda_i-\tau|\le\lambda^{-a}2^{\ell+1}$ and $\lambda_i,\tau\approx\lambda$ if $\beta(\lambda^a2^{-\ell}(\lambda_i-\tau))\ne0$, and, in this case, we also have $\tilde{1\!\!1}_\lambda(\lambda_i)-1=O((1+|j|)^{-N})$ if $\tau\in I^-_{\ell,j}$ and $\tilde{1\!\!1}_\lambda(\lambda_i)=O((1+|j|)^{-N})$ if $\tau\in I^+_{\ell,j}$. Therefore, by orthogonality, we have

$$\begin{aligned}\big\|K^\pm_{\tau,\ell}(\,\cdot\,,y)\big\|_{L^2(M)}&\lesssim(1+|j|)^{-N}\lambda^{a-1}2^{-\ell}\big(\sum_{\{i:\,|\lambda_i-\tau|\le\lambda^{-a}2^{\ell+1}\}}|e^0_i(y)|^2\big)^{1/2}\\&\le(1+|j|)^{-N}2^{-\ell/2}\lambda^{\frac{n-1+a}{2}-1},\end{aligned}$$

which is the first part of (4.39). In the second inequality, we used the fact that

$$\sum_{\lambda_i\in[\lambda,\lambda+\lambda^{-a}]}|e^0_i(x)e^0_i(y)|\lesssim\lambda^{n-1-a},\tag{4.53}$$

which is a consequence of (4.22) if we choose$\{e^0_i\}$ to be the standard orthonormal basis, $\{\exp(2\pi i j\cdot x,\ j\in \mathbb{Z}_n)\}$ for the Laplacian on the torus. For we then have that the left side of (4.53) equals the number of eigenvalues of P^0 in $[\lambda,\lambda+\lambda^{-a}]$.

The other inequality in (4.39) follows from this argument since

$$\frac{\partial}{\partial\tau}\frac{\tilde\beta(\lambda^a 2^{-\ell}(\lambda_i-\tau))}{\lambda_i+\tau}=O(\lambda^{a-1}2^{-\ell}),$$

due to the fact that we are assuming that $2^\ell\le \lambda\cdot\lambda^a/100$.

This argument also gives us (4.40) if we use the fact that $\tau\to(\tilde{\mathbb{1}}_\lambda(\tau)-\tilde{\mathbb{1}}_\lambda(\mu))/(\tau^2-\mu^2)$ is smooth if we define it as in (4.33) when $\tau=\mu$ and use the fact that

$$\partial^k_\tau\big(\beta_0(\lambda^a(\lambda_i-\tau))(\tilde{\mathbb{1}}_\lambda(\lambda_i)-\tilde{\mathbb{1}}_\lambda(\tau))/(\lambda_i-\tau)\big)=O(\lambda^{a(k+1)}(1+|j|)^{-N}),\ k=0,1,\ \tau\in I^{\pm}_{0,j},$$

and the fact that, if this expression is nonzero, we must have $|\lambda_i-\tau|\le 2\lambda^{-a}$.

To prove (4.41) we use the fact that for $k=0,1$ we have for $\tau\in(\lambda,10\lambda]$

$$\Big|\Big(\frac{\partial}{\partial\tau}\Big)^k\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\tilde{\mathbb{1}}_\lambda(\lambda_i)\Big|$$
$$\lesssim\begin{cases}\lambda^{-2-k} & \text{if } \lambda_i\le\lambda\\ \lambda^{-2-k}(1+\lambda^a(\lambda_i-\lambda))^{-N} & \text{if } \lambda_i>\lambda.\end{cases}$$

Thus for $k=0,1$

$$\begin{aligned}\big\|(\lambda\partial_\tau)^k\,K^+_{\tau,\infty}(\,\cdot\,,y)\big\|_{L^2(M)}&\lesssim\lambda^{-2}\Big(\sum_{\lambda_i\le\lambda}|e^0_i(y)|^2+\sum_{\lambda_i>\lambda}(1+\lambda^a(\lambda_i-\lambda))^{-N}|e^0_i(y)|^2\Big)^{1/2}\\ &\lesssim\lambda^{-2+\frac n2},\end{aligned}$$

as desired if $N>2n$, using (4.53) again.

Next we turn to the bounds in (4.42) for $K^-_{\tau,\infty}$. To handle this, let η be as in Lemma 2.5 and put

$$\begin{aligned}H^-_{\tau,\infty}(x,y)&=-\sum_i\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\eta(\lambda_i/\tau)\,e^0_i(x)e^0_i(y)\\ &=-\sum_i\frac{\eta(\lambda_i/\tau)}{\lambda_i^2-\tau^2}e^0_i(x)e^0_i(y),\end{aligned}$$

assuming, as we may, that $\lambda\gg1$. The last equality comes from the properties of our Littlewood-Paley bump function, β. We then conclude from Lemma 2.5 that $H^-_{\tau,\infty}$ satisfies the bounds in (4.42). If we then set

$$\begin{aligned}\tilde K^-_{\tau,\infty}(x,y)&=\sum_i\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\tilde{\mathbb{1}}_\lambda(\lambda_i)\,e^0_i(x)e^0_i(y)\\ &\quad-\sum_i\Big(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Big)\big(1-\eta(\lambda_i/\tau)\big)\,e^0_i(x)e^0_i(y),\end{aligned}$$

we have $K^-_{\tau,\infty}=\tilde K^-_{\tau,\infty}+H^-_{\tau,\infty}$, and, also, by the proof of (4.41), $\tilde K^-_{\tau,\infty}$ satisfies the bounds in (4.42).

It just remains to prove the bounds in (4.43) for the $R_{\tau,\ell}(x,y)$ and that in (4.44) for $R_{\tau,\infty}(x,y)$. The former just follows from the proof of (4.39).

To prove the remaining inequality, (4.44), we note that if η is as above and we set

$$\tilde{R}_{\tau,\infty}(x,y)=\sum_i \frac{\eta(\lambda_i/\tau)}{\lambda_i^2-\tau^2}\, e_i^0(x)e_i^0(y),$$

then, by Lemma 2.5, $\tilde{R}_{\tau,\infty}$ satisfies the bounds in (4.44). Also, we have

$$R_{\tau,\infty}(x,y)=R^0_{\tau,\infty}(x,y)+\tilde{R}_{\tau,\infty}(x,y),$$

if

$$R^0_{\tau,\infty}(x,y)=\sum_i\bigl(1-\eta(\lambda_i/\tau)\bigr)\Bigl(\frac{\sum_{\{\ell\in\mathbb{N}:\,2^\ell>\lambda\cdot\lambda^a/100\}}\beta(\lambda^a2^{-\ell}(\lambda_i-\tau))}{\lambda_i^2-\tau^2}\Bigr)\, e_i^0(x)e_i^0(y),$$

(again using the properties of β), and, since the proof of Lemma 2.5 shows that for $\tau\in[\lambda/2,10\lambda]$ we have

$$|R^0_{\tau,\infty}(x,y)|\lesssim \tau^{n-2}\bigl(1+\tau d_g(x,y)\bigr)^{-N}\lesssim \lambda^{n-2}\bigl(1+\lambda d_g(x,y)\bigr)^{-N},$$

we conclude that (4.44) must be valid, which completes the proof.

□

Now we give the proof of Propostion 4.4. Let $a=\frac{n-1}{n+1}$, $\tilde{\chi}_\lambda(\tau)=\chi(\lambda^a(\lambda-\tau))+\chi(\lambda^a(\lambda+\tau))$. Since $\chi\in\mathcal{S}(\mathbb{R})$, we have

$$\bigl(\tfrac{d}{d\tau}\bigr)^j\,\tilde{\chi}_\lambda(\tau)=O\bigl(\lambda^{aj}(1+\lambda^a|\lambda-\tau|)^{-N}\bigr)\ \forall N,\quad \text{if } j=0,1,2,3,\dots. \tag{4.54}$$

By using (4.54) and (2.27), it is not hard to prove that the kernel of $(P^0)^\mu\tilde{\chi}_\lambda(P^0)$, $\mu=0,1,2...$ satisfies

$$\bigl((P^0)^\mu\tilde{\chi}_\lambda(P^0)\bigr)(x,y)=\sum_j\lambda_j^\mu\tilde{\chi}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)=O(\lambda^{n+\mu}), \tag{4.55}$$

and, moreover,

$$\bigl\|\bigl((P^0)^\mu\,\tilde{\chi}_\lambda(P^0)\bigl(\,\cdot\,,y)\bigr\|_{L^2(M)}=O(\lambda^{n/2+\mu}). \tag{4.56}$$

Both (4.55) and (4.56) are analogs of inequalities in Lemma 2.4. Given (4.20), by the arguments in (4.2)-(4.8), (4.25) would be a consequence of

$$\Bigl|\sum_{j,k}\int_M\int_M\frac{\tilde{\chi}_\lambda(\lambda_j)-\tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Bigr| \le C_V\|V\|_{L^2(M)}\,\lambda^{n-2+\frac{a-b}{2}}\log\lambda, \tag{4.57}$$

where we interpret

$$\frac{\tilde{\chi}_\lambda(\tau)-\tilde{\chi}_\lambda(\mu)}{\tau^2-\mu^2}=\frac{\tilde{\chi}'_\lambda(\tau)}{2\tau},\ \text{ if }\ \tau=\mu.$$

As before, we shall split things into three different cases that require slightly different arguments. The main contribution still comes from frequencies τ_k which are comparable to λ. For large or small frequencies τ_k, by (4.54), (4.55), and (4.56), it follows from earlier arguments in Section 3 that the left side of (4.57) is $O(\|V\|_{L^1(M)}\lambda^{n-2}(\log\lambda)^{1/2})$.

Conequently, we would obtain (4.57) if we could prove the following.

(4.58)
$$\Big|\sum_j \sum_{\{k:\, \tau_k\in[\lambda/2,10\lambda]\}} \sum_{j,k} \int_M \int_M \frac{\tilde{\chi}_\lambda(\lambda_j)-\tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V\|V\|_{L^2(M)}\,\lambda^{n-2+\frac{a-b}{2}}\log\lambda,$$

The proof of (4.58) is similar to (4.34). After replacing $\tilde{\mathbb{1}}_\lambda(\tau)$ by $\tilde{\chi}_\lambda(\tau)$ in the proof of (4.34), the main contribution, which gives the right side of (4.58), still comes from terms like (4.45), (4.47), etc. The difference is we use (4.24) instead of Corollary 4.5 to bound (4.45) in this case. Moreover, in view of (4.54), we do not need to divide the proof into two cases as in (4.36) and (4.37), since $\tilde{\chi}_\lambda(\tau)$ is rapidly decreasing away from λ on both regions $\tau_k\in[\lambda/2,\lambda]$ and $\tau_k\in[\lambda,10\lambda]$. This completes the proof of (4.58).

Having presented the model argument, let us now prove Theorem 1.4.

Proof of (1.16). To get an improvement over the error term as in (1.16), we need the following

Proposition 4.9. *If $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ denotes the standard torus with the flat metric and $N^0(\lambda)$ denotes the Weyl counting function for H^0, then*

$$N^0(\lambda)=(2\pi)^{-n}\omega_n\mathrm{Vol}_g(M)\,\lambda^n+r_n(\lambda), \tag{4.59}$$

where

$$r_n(\lambda)\lesssim\begin{cases}\lambda^{n-2}, & \text{if } n\ge 5\\ \lambda^2(\log\lambda)^{2/3}, & \text{if } n=4\\ \lambda^{\frac{21}{16}+\varepsilon}, & \text{if } n=3\\ \lambda^{\frac{131}{208}}(\log\lambda)^{\frac{18627}{8320}}, & \text{if } n=2.\end{cases}$$

There has been a lot of research related to the Weyl formula on the torus, which is equivalent to counting the lattice points inside the ball of radius λ. Currently, the exact order of the error term is only known when $n\ge 5$. See e.g. E. Landau [12], A. Walfisz [19], and E. Krätzel [11]. The above best known results in lower dimensions are due to A. Walfisz [20] (n=4), D. R. Health-Brown [6] (n=3), and M. N. Huxley [9] (n=2). For more details and a discussion of recent progress on the problem, see e.g. the survey paper [10], and W. Freeden [5].

For simplicity, we will only give the proof of (1.16) for $n\ge 5$. The proof for $n=4$ follows from the same argument due to the fact that the extra $(\log\lambda)^{2/3}$-factor is harmless in the presence of the λ^ε-factor in (1.16). Also, for the $n=2,3$ cases, if we use the improved results in (4.59), by the same argument as in the proof of Theorem 4.6, we can recover the improved bound without a λ^ε-loss.

As a consequence of (4.59), if we repeat the arguments in (4.20)-(4.23), we have the following two inequalities

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_M \frac{1}{\lambda}\hat{\chi}(t/\lambda)e^{it\lambda}\sum_{j=1}^{\infty}\cos t\lambda_j|e_j^0(x)|^2dxdt \le C\lambda^{n-2} \tag{4.60}$$

$$\Big|\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(t/\lambda)\frac{\sin\lambda t}{t}\big(\cos tP^0\big)(x,x)\,dtdx-(2\pi)^{-n}\omega_n\mathrm{Vol}_g(M)\,\lambda^n\Big| \le C\lambda^{n-2}, \tag{4.61}$$

where $\chi(\tau),\rho(t)$ are the Schwartz functions defined in (4.2), (2.13) respectively.

As before, we first prove a Proposition which allows us to do iterations.

Proposition 4.10. *Let $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric and $\{e_{\tau_k}\}$ be eigenfunctions of the operator H_V. Given (4.59), when $n\ge 5$, if we assume, for all $\lambda>1$*

$$\int_M\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-1}]}|e_{\tau_k}(x)|^2dx\le C_V\lambda^{n-1-b},\ \textit{ for some }\quad -1\le b\le 1, \tag{4.62}$$

then,

$$\int_M\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-1}]}|e_{\tau_k}(x)|^2dx\le C_V\|V\|_{L^2(M)}\lambda^{n-2+\frac{1-b}{2}}\log\lambda+C\lambda^{n-2}. \tag{4.63}$$

Note that, by (4.26), (4.62) holds for $b=-1$, and that every time we apply the Proposition, we would have (4.62) for a larger value of b. Consequently, just as before, after finitely many iterations, we will obtain the following:

Corollary 4.11. *Let $V\in L^2(M)\cap\mathcal{K}(M)$, $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric, and $\{e_{\tau_k}\}$ be eigenfunctions of the operator H_V. Then given (4.59), when $n\ge 5$, we have for all $\lambda>1$*

$$\int_M\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-1}]}|e_{\tau_k}(x)|^2dx\le C_{V,\varepsilon}\lambda^{n-2+\varepsilon},\quad\forall\,\varepsilon>0, \tag{4.64}$$

where $C_{V,\varepsilon}$ is a constant depending on V and ε.

Proof. To prove (4.64), let us first ignore the $\log\lambda$ factor on the right side of (4.25). Define b_m to be the best exponent such that

$$\int_M\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-1}]}|e_{\tau_k}(x)|^2dx\le C_V\lambda^{n-1-b_m},$$

after applying Proposition 4.4 m times. We have

$$b_{m+1}=\frac{1+b_m}{2}\quad m=0,1,2...$$

with $b_0=-1$.

By solving the arithmetic sequences explicitly, we have $b_m=1-\frac{1}{2^{m-1}}\quad m=0,1,2....$ So (4.64) follows by letting $m\to\infty$. And since $\log\lambda\le C_\varepsilon\lambda^\varepsilon$ for all ε, (4.64) follows from the same argument if we consider the $\log\lambda$-factor.

∎

Let $\tilde{\chi}_\lambda(\tau)=\chi(\lambda(\lambda-\tau))+\chi(\lambda(\lambda+\tau))$, since $\chi\in\mathcal{S}(\mathbb{R})$, we have

$$\left(\tfrac{d}{d\tau}\right)^j\tilde{\chi}_\lambda(\tau)=O\big(\lambda^j(1+\lambda|\lambda-\tau|)^{-N}\big)\ \forall N,\quad \text{if } j=0,1,2,3,\dots. \tag{4.65}$$

Given (4.60), by the arguments in (4.2)-(4.8), (4.63) would be a consequence of

$$\Big|\sum_{j,k}\int_M\int_M\frac{\tilde{\chi}_\lambda(\lambda_j)-\tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V\|V\|_{L^2(M)}\,\lambda^{n-2+\frac{1-b}{2}}\log\lambda, \tag{4.66}$$

where we interpret

$$\frac{\tilde{\chi}_\lambda(\tau)-\tilde{\chi}_\lambda(\mu)}{\tau^2-\mu^2}=\frac{\tilde{\chi}_\lambda'(\tau)}{2\tau},\ \text{ if }\ \tau=\mu.$$

Also, similar to (4.9), if we let

$$\tilde{\mathbb{1}}_\lambda(\tau)=\frac{1}{\pi}\int_{-\infty}^{\infty}\rho(t/\lambda)\frac{\sin\lambda t}{t}\,\cos t\tau\,dt, \tag{4.67}$$

then by a change of scale argument, it follows from (2.15) that for $\tau>0$ and large λ we have

$$\mathbb{1}_\lambda(\tau)-\tilde{\mathbb{1}}_\lambda(\tau)=O\big((1+\lambda|\lambda-\tau|)^{-N}\big)\ \forall N. \tag{4.68}$$

Additionally, for $\tau>0$ we have the analog of (2.16)

$$\left(\tfrac{d}{d\tau}\right)^j\tilde{\mathbb{1}}_\lambda(\tau)\le C\lambda^j\big((1+\lambda|\lambda-\tau|)^{-N}\big)\ \forall N,\quad \text{if } j=1,2,3,\dots. \tag{4.69}$$

In view of (4.61), after repeating the arguments in (4.9)-(4.15), we see that (1.15) would be a consequence of

$$\Big|\sum_{j,k}\int_M\int_M\frac{\tilde{\mathbb{1}}_\lambda(\lambda_j)-\tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2-\tau_k^2}e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)\,dxdy\Big| \le C_V\|V\|_{L^2(M)}\lambda^{n-2+\varepsilon/2}\log\lambda, \tag{4.70}$$

where ε is the same constant as in Corollary 4.11, and we interpret

$$\frac{\tilde{\mathbb{1}}_\lambda(\tau)-\tilde{\mathbb{1}}_\lambda(\mu)}{\tau^2-\mu^2}=\frac{\tilde{\mathbb{1}}_\lambda'(\tau)}{2\tau},\ \text{ if }\ \tau=\mu. \tag{4.71}$$

So Proposition 4.10 and (1.16) follow from (4.66) and (4.70), respectively, and both require a bound on the trace of certain spectral projection operators, which are (4.62) and (4.64) respectively.

Similarly, we shall split things into three different cases. The main contribution still comes from frequencies τ_k which are comparable to λ. For large or small frequencies τ_k, by slightly modifying the corresponding arguments in Section 3, we see that the left side of (4.66) and (4.70) would be bounded by $C_V\|V\|_{L^1(M)}\,\lambda^{n-2}(\log\lambda)^{1/2}$, which is better than the the right side of (4.66) and (4.70).

Finally, for frequencies τ_k which are comparable to λ, if we let $a=1$ in the proof of Proposition 4.7, and use (4.62) or (4.64) correspondingly for the main terms (e.g. (4.45), (4.47), etc.), it follows from the same arguments as in (4.45)-(4.52) that the left side of (4.66) and (4.70) are controlled by their right sides. The proof of (1.16) is complete. □

Proof of (1.15). To recover to Hlawka bound [7] under the weaker conditions on V in Theorem 1.4, the strategy is similar to previous cases. That is, to get improvements for the main terms in (3.1), which are (3.12), (3.15), etc., we begin with the following Proposition.

Proposition 4.12. *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric, and $\{e_{\tau_k}\}$ be the eigenfunctions of the operator H_V. Then given* (4.19)*, if we assume, for all $\lambda > 1$ that*

$$\int_M \sum_{\tau_k\in[\lambda,\lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 dx \le C_V \lambda^{n-1-b}, \text{ for some } \quad -1 \le b \le a, \tag{4.72}$$

then it follows that for any $1 \le p \le 2$ we have

$$\int_M \sum_{\tau_k\in[\lambda,\lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 dx \le \begin{cases} C_V\|V\|_{L^1(M)}\lambda^{\frac{1}{6}-\frac{b}{2}}\log\lambda + C\lambda^{2/3} & \text{if } n=2 \\ C_V\|V\|_{L^p(M)}\lambda^{k(b,p)}\log\lambda + C\lambda^{n-1-a} & \text{if } n\ge 3, \end{cases} \tag{4.73}$$

where $a = \frac{n-1}{n+1}$, and $k(b,p) = \frac{n-1+a}{2} - 1 + \frac{n-1-b}{2}\cdot(2-\frac{2}{p}) + \frac{n}{2}\cdot(\frac{2}{p}-1)$.

If $V \in \mathcal{K}(M)$, by (4.26), (4.72) is true for $b=-1$. In particular, when $n=2$, by applying the spectral projection bounds in [3], we have

$$\sum_{\tau_k\in[\lambda,\lambda+\lambda^{-\frac{1}{3}}]} |e_{\tau_k}(x)|^2 \le \sum_{\tau_k\in[\lambda,\lambda+1]} |e_{\tau_k}(x)|^2 \le C_V\lambda. \tag{4.74}$$

So (4.72) is true for $b=0$ when $n=2$. As before, after finitely many iterations, we have:

Corollary 4.13. *Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the standard torus with the flat metric, and $\{e_{\tau_k}\}$ be the eigenfunctions of the operator H_V. Assume also that $V \in L^p(M)\cap\mathcal{K}(M)$ for some $p > \frac{2n}{n+2}$ for $n\ge 3$ and $V\in\mathcal{K}(M)$ for $n=2$. Then given* (4.20)*, we have for all $\lambda > 1$*

$$\int_M \sum_{\tau_k\in[\lambda,\lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 dx \le C_V \lambda^{n-1-a}, \tag{4.75}$$

where C_V is a constant depending on V, and $a = \frac{n-1}{n+1}$.

Proof. For $n=2$, by using (4.74), which is (4.72) corresponding to $b=0$, (4.75) follows from (4.73) directly since $C_V\|V\|_{L^1(M)}\lambda^{\frac{1}{6}-\frac{b}{2}}\log\lambda$ is better than the right side of (4.75).

To prove (4.75) for $n\ge 3$, let us first ignore the $\log\lambda$ factor on the right side of (4.73). Define b_m to be the best exponent such that

$$\int_M \sum_{\tau_k\in[\lambda,\lambda+\lambda^{-a}]} |e_{\tau_k}(x)|^2 dx \le C_V \lambda^{n-1-b_m},$$

after applying Proposition 4.12 m times. We have

$$n-1-b_{m+1} = \max\{k(b_m,p),\ n-1-\frac{n-1}{n+1}\}, \quad m=0,1,2... \tag{4.75$'$}$$

with, as before, $b_0 = -1$.

Now by a straightforward calculation, if $b_m \le \frac{2n-(n+2)p}{(n+1)(p-1)} + \frac{n-1}{n+1}$, we have $b_{m+1} = n-1-k(b_m,p)$. In this case, $b_{m+1} - b_m = \frac{n+3}{2(n+1)} - \frac{1}{p} + \frac{1}{2} - \frac{b_m}{p} \ge \mu(p)$, where

$$\mu(p) = \frac{n+3}{2(n+1)} - \frac{1}{p} + \frac{1}{2} - \frac{2n-(n+2)p}{(n+1)(p-1)p} - \frac{n-1}{(n+1)p} > 0, \quad \text{if} \;\; p > \frac{2n}{n+2}.$$

So the sequence is strictly increasing in this case. If $N = [\frac{\frac{2n-(n+2)p}{(n+1)(p-1)} + \frac{n-1}{n+1} + 1}{\mu(p)}] + 1$, we have $b_N > \frac{2n-(n+2)p}{(n+1)(p-1)} + \frac{n-1}{n+1}$. Thus by (4.75′), $b_m \equiv \frac{n-1}{n+1}$ for all $m > N$.

Since $\log\lambda \lesssim \lambda^\varepsilon$ for all ε, by the same argument, we have $b_{N+1} \ge \frac{n-1}{n+1} - \varepsilon$. However, if ε is small enough,

$$\max\{k(\frac{n-1}{n+1} - \varepsilon, p),\; n-1-\frac{n-1}{n+1}\} = n-1-\frac{n-1}{n+1}, \quad \text{if} \;\; p > \frac{2n}{n+2}.$$

So we have in this case $b_m \equiv \frac{n-1}{n+1}$ for all $m > N+1$. The proof of (4.75) is complete.

□

To obtain (4.73), if we repeat the arguments in the proof of Proposition 4.4, this inequality would be a consequence of

$$\Big|\sum_{j,k} \int_M \int_M \frac{\tilde{\chi}_\lambda(\lambda_j) - \tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y)\, dxdy\Big| \le \begin{cases} C_V \|V\|_{L^1(M)} \lambda^{\frac{1}{6} - \frac{b}{2}} \log\lambda & \text{if} \;\; n = 2 \\ C_V \|V\|_{L^p(M)} \lambda^{k(b,p)} \log\lambda & \text{if} \;\; n \ge 3, \end{cases} \tag{4.76}$$

where $\tilde{\chi}_\lambda(\tau) = \chi(\lambda^a(\lambda - \tau)) + \chi(\lambda^a(\lambda + \tau))$ and $a = \frac{n-1}{n+1}$.

And similarly, to obtain (4.15), if we repeat the arguments in the proof of Theorem 4.6, it suffices to show that

$$\Big|\sum_{j,k} \int_M \int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j) - \tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y)\, dxdy\,\Big| \le \begin{cases} C_V \|V\|_{L^1(M)} \lambda^{\frac{1}{6}} \log\lambda & \text{if} \;\; n = 2 \\ C_V \|V\|_{L^p(M)} \lambda^{k(p)} \log\lambda & \text{if} \;\; n \ge 3, \end{cases} \tag{4.77}$$

where $\tilde{\mathbb{1}}_\lambda(\tau)$ is defined as in (4.29), and $k(p) = k(a,p) = \frac{n-1+a}{2} - 1 + \frac{n-1-a}{2} \cdot (2 - \frac{2}{p}) + \frac{n}{2} \cdot (\frac{2}{p} - 1)$, if $a = \frac{n-1}{n+1}$. Note that by a straightforward calculation, when $p > \frac{2n}{n+2}$, the right side of (4.77) is controlled by the right side of (1.15).

As before, since the proofs of (4.76) and (4.77) are similar, we shall only give the details of (4.77) here. By using the same argument as in Section 3, the terms for large or small frequencies τ_k in (4.77) will only contribute $C_V \|V\|_{L^1(M)}\, \lambda^{n-2} (\log\lambda)^{1/2}$ to the right side. So the proof of (4.77) would be complete if we could establish the following.

Proposition 4.14. *As in Theorem 1.4, fix $p > \frac{2n}{n+2}$ and assume that $V \in L^p(M) \cap \mathcal{K}(M)$ for $n \geq 3$, and $V \in \mathcal{K}(M)$ for $n = 2$. If $\tilde{\mathbb{1}}_\lambda(\tau)$ is defined as in (4.29), then*

$$\Big|\sum_j \sum_{\{k:\, \tau_k \in [\lambda/2, 10\lambda]\}} \int_M \int_M \frac{\tilde{\mathbb{1}}_\lambda(\lambda_j) - \tilde{\mathbb{1}}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y)\, dx dy\Big| \leq \begin{cases} C_V \|V\|_{L^1(M)} \lambda^{\frac{1}{6}} \log\lambda & \text{if } n = 2 \\ C_V \|V\|_{L^p(M)} \lambda^{k(p)} \log\lambda & \text{if } n \geq 3. \end{cases} \tag{4.78}$$

We prove Proposition 4.14 directly by using the same setup as in (4.35)-(4.38), as well as Lemma 4.8. Let K_τ be as in (4.35), our current task, (4.78), then is to show that

$$\Big|\sum_{\tau_k \in [\lambda/2, 10\lambda]} \iint K_{\tau_k}(x, y) e_{\tau_k}(x) e_{\tau_k}(y)\, V(y)\, dx dy\Big| \lesssim \begin{cases} \|V\|_{L^1(M)} \lambda^{\frac{1}{6}} \log\lambda & \text{if } n = 2 \\ \|V\|_{L^p(M)} \lambda^{k(p)} \log\lambda & \text{if } n \geq 3. \end{cases} \tag{4.78$'$}$$

First, by (4.39) and Lemma 2.3 with $\delta = \lambda^{-a} 2^\ell$, we have for $n = 2$

$$\begin{aligned} &\Big|\sum_{\tau_k \in I^\pm_{\ell,j} \cap [\lambda/2, 10\lambda]} \iint K^\pm_{\tau_k,\ell}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y)\, dy dx\Big| \\ &\leq \|V\|_{L^1} \cdot \sup_y \Big\|\sum_{\tau_k \in I^\pm_{\ell,j} \cap [\lambda/2, 10\lambda]} \iint K^\pm_{\tau_k,\ell}(x, y) e_{\tau_k}(x) e_{\tau_k}(y)\Big\|_{L^1(dx)} \\ &\lesssim \|V\|_{L^1} \cdot \sup_y \Big(\|K^\pm_{\lambda + j\lambda^{-a}2^\ell, \ell}(\,\cdot\,, y)\|_{L^2(M)} + \int_{I^\pm_{\ell,j}} \|\tfrac{\partial}{\partial\tau} K^\pm_{s,\ell}(\,\cdot\, y)\|_{L^2(M)}\, ds\Big) \\ &\qquad\qquad \times \big(\sum_{\tau_k \in I^\pm_{\ell,j} \cap [\lambda/2, 10\lambda]} |e_{\tau_k}(y)|^2\big)^{1/2} \\ &\lesssim \|V\|_{L^1} \lambda^{\frac{2-1+a}{2} - 1} 2^{-\ell/2} (1 + j)^{-N} \big(\sum_{\tau_k \in I^\pm_{\ell,j} \cap [\lambda/2, 10\lambda]} |e_{\tau_k}(y)|^2\big)^{1/2} \\ &\lesssim \|V\|_{L^1} \lambda^{-1/3} 2^{-\ell/2} (1 + j)^{-N} \big(\sum_{\mu \in \mathbb{N} \cap (I^\pm_{\ell,j} \cap [\lambda/2, 10\lambda]} \mu^{n-1}\big)^{1/2} \\ &\lesssim \|V\|_{L^1} \lambda^{\frac{1}{6}} (1 + j)^{-N}. \end{aligned} \tag{4.79}$$

In the second to last inequality we used (4.74) and the fact that $a = \frac{1}{3}$, $|I^\pm_{\ell,j}| = \lambda^{-a} 2^\ell \leq 2^\ell$.

Similarly, for $n \ge 3$

$$
\begin{aligned}
(4.80)\qquad & \Big|\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}\iint K^{\pm}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dydx\Big| \\
&\le \|V\|_{L^p}\cdot\Big\|\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}\iint K^{\pm}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\Big\|_{L^{p'}\big(dy;L^1(dx)\big)} \\
&\lesssim \|V\|_{L^p}\cdot\sup_y\Big(\|K^{\pm}_{\lambda+j\lambda^{-a}2^\ell,\ell}(\,\cdot\,,y)\|_{L^2(M)}+\int_{I^{\pm}_{\ell,j}}\|\tfrac{\partial}{\partial\tau}K^{\pm}_{s,\ell}(\,\cdot\,y)\|_{L^2(M)}\,ds\Big) \\
&\qquad\qquad\times\Big(\int_M\Big(\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2\Big)^{p'/2}dy\Big)^{1/p'} \\
&\lesssim \|V\|_{L^p}\lambda^{\frac{n-1+a}{2}-1}2^{-\ell/2}(1+j)^{-N}\|\Big(\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2\Big)^{1/2}\|_{L^{p'}(M)},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Now in view of Corollary 4.13 and (4.26), since $\frac{1}{p'}=\frac{1}{2}\cdot\frac{2}{p'}+\frac{1}{\infty}\cdot(1-\frac{2}{p'})$, by Hölder's inequality, we have for all $\lambda>1$ and $2\le p'\le\infty$

$$\|\Big(\sum_{\tau_k[\lambda,\lambda+\lambda^{-a}]}|e_{\tau_k}(y)|^2\Big)^{1/2}\|_{L^{p'}(M)}\lesssim\lambda^{\frac{n-1-a}{2}\frac{2}{p'}+\frac{n}{2}(1-\frac{2}{p'})}.$$

Since the number of intervals in $I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]$ with length comparable to λ^{-a} is about 2^ℓ, by Minkowski's inequality

$$\|\Big(\sum_{\tau_k\in I^{\pm}_{\ell,j}\cap[\lambda/2,10\lambda]}|e_{\tau_k}(y)|^2\Big)^{1/2}\|_{L^{p'}(M)}\lesssim 2^{\ell/2}\lambda^{\frac{n-1-a}{2}\frac{2}{p'}+\frac{n}{2}(1-\frac{2}{p'})}.$$

So the right side of (4.80) is bounded by $\lambda^{k(p)}(1+j)^{-N}\|V\|_{L^p}$.

If we sum over $j=0,1,2,\dots,$ we see that (4.79) and (4.80) yields that for $1\le 2^\ell\le\lambda\cdot\lambda^a/100$

$$
\begin{aligned}
(4.81)\quad & \Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint K^{+}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \\
&+\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint K^{-}_{\tau_k,\ell}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim\begin{cases}\|V\|_{L^1(M)}\lambda^{\frac16} & \text{if } n=2\\ \|V\|_{L^p(M)}\lambda^{k(p)} & \text{if } n\ge 3.\end{cases}
\end{aligned}
$$

If we take $\delta=\lambda^{-a}$ in Lemma 2.3, this argument also gives

$$
\begin{aligned}
(4.82)\quad & \Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint K_{\tau_k,0}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big| \\
&+\Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint K_{\tau_k,0}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)\,V(y)\,dxdy\Big|\lesssim\begin{cases}\|V\|_{L^1(M)}\lambda^{\frac16} & \text{if } n=2\\ \|V\|_{L^p(M)}\lambda^{k(p)} & \text{if } n\ge 3.\end{cases}
\end{aligned}
$$

Next, since $R_{\tau,\ell}$ enjoys the bounds in (4.43), we can repeat the arguments in (4.79) and (4.80) to see that for $1 \le 2^\ell \le \lambda \cdot \lambda^a/100$ we have for $n=2$,

$$\Big|\sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]}\iint R_{\tau_k,\ell}(x,y)\tilde{\mathbb{1}}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|$$
$$\lesssim \|V\|_{L^1}\cdot 2^{-\ell/2}\lambda^{\frac{2-1+a}{2}-1}\sup_y\Big(\sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]}|\tilde{\mathbb{1}}_\lambda(\tau_k)e_{\tau_k}(y)|^2\Big)^{1/2}$$
$$\lesssim \lambda^{\frac16}(1+j)^{-N}\|V\|_{L^1},$$

since $\tilde{\mathbb{1}}_\lambda(\tau_k)=O((1+j)^{-N})$ if $\tau_k\in I^+_{\ell,j}$.

Similarly for $n\ge 3$, we have

$$\Big|\sum_{\tau_k\in I^+_{\ell,j}\cap(\lambda,10\lambda]}\iint R_{\tau_k,\ell}(x,y)\tilde{\mathbb{1}}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big|$$
$$\lesssim \|V\|_{L^p}\cdot 2^{-\ell/2}\lambda^{\frac{n-1+a}{2}-1}\|\Big(\sum_{\tau_k\in I^\pm_{\ell,j}\cap[\lambda/2,10\lambda]}|\tilde{\mathbb{1}}_\lambda(\tau_k)e_{\tau_k}(y)|^2\Big)^{1/2}\|_{L^{p'}(M)}$$
$$\lesssim \lambda^{k(p)}(1+j)^{-N}\|V\|_{L^p}.$$

Summing over this bound over j of course yields

$$\Big|\sum_{\lambda<\tau_k\le 10\lambda}\iint R_{\tau_k,\ell}(x,y)\tilde{\mathbb{1}}_\lambda(\tau_k)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big| \lesssim \begin{cases}\|V\|_{L^1(M)}\lambda^{\frac16} & \text{if } n=2\\ \|V\|_{L^p(M)}\lambda^{k(p)} & \text{if } n\ge 3.\end{cases} \tag{4.83}$$

The same argument gives

$$\Big|\sum_{\lambda/2\le\tau_k\le\lambda}\iint R_{\tau_k,\ell}(x,y)\big(1-\tilde{\mathbb{1}}_\lambda(\tau_k)\big)e_{\tau_k}(x)e_{\tau_k}(y)V(y)\,dxdy\Big| \lesssim \begin{cases}\|V\|_{L^1(M)}\lambda^{\frac16} & \text{if } n=2\\ \|V\|_{L^p(M)}\lambda^{k(p)} & \text{if } n\ge 3.\end{cases} \tag{4.84}$$

We now have assembled all the ingredients for the proof of (4.78′). If we use (4.81), (4.82), (4.84), (4.51) and (4.52) along with (4.36), we conclude that the analog of (4.78′) must be valid where the sum is taken over $\tau_k\in[\lambda/2,\lambda]$. The log-loss comes from the fact that there are $\approx\log\lambda$ terms $K^-_{\tau,\ell}$ and $R_{\tau,\ell}$. We similarly obtain the analog of (4.78′) where the sum is taken over $\tau_k\in(\lambda,10\lambda]$ from (4.37) along with (4.81), (4.82), (4.83) and (4.51). So the proof of (4.78) is complete.

□

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(X.H.) Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: `xhuang49@math.jhu.edu`

(C.D.S.) Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: `sogge@jhu.edu`