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Practical implementation of a scalable discrete Fourier transform using logical phi-bits: nonlinear acoustic qubit analogues

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Abstract It is shown that multiple logical phi-bit large-scale unitary operations analogous to quantum circuits can be realized by design. Logical phi-bits are nonlinear acoustic analogues of qubits which arise when elastic waveguides are coupled and driven at multiple frequencies in the presence of non-linearities. The contribution presents an approach that maps both the state of multiple phi-bits in their supporting nonlinear acoustic metastructure and their representations as complex state vectors in exponentially scaling Hilbert spaces. Upon physically actuating π changes in phi-bit phases and by engineering appropriate multiple phi-bits representations, one can realize a scalable phi-bit-based quantum Fourier transform.

Keywords Acoustic qubit analogues · Phi-bit-based quantum Fourier transform · Acoustic metastructure · Unitary operations

1 Introduction

Quantum computing harnesses quantum superposition and quantum correlation. The first phenomenon provides the support for encoding massive amounts of information in coherent superpositions of states of a composite quantum system constituted of subsystems, e.g., qubits [1,2]. The second phenomenon provides the correlation between the qubits and the capability of processing information in a parallel manner. The ultimate goal of quantum computing is to realize large-scale unitary operations or gates on multiple qubits. An N-qubit gate operates on states supported by an exponentially scaling basis of dimension 2^N [3]. It is this exponential scaling that potentially gives the advantage of quantum computing over classical computers. However, the fragility of quantum superpositions of large number of qubits means this task is achieved using decomposition of large-scale unitary matrices into quantum circuits

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involving sequences of elementary single- and two-qubit gates which may form a universal set of gates [4]. The challenge is then to determine the minimal quantum circuit capable of realizing a desired large-scale operation. The discrete Fourier transform has many applications (see Refs. [5–8], for example). This contribution focuses on the quantum Fourier transform (QFT), i.e., the discrete Fourier transform implemented on qubits [9], is a canonical large-scale operation of interest in many applications. The QFT can be realized with a circuit utilizing the single qubit Hadamard gate and the phase gate. The QFT is also an important component of quantum algorithms such as Shor's algorithm for finding prime factors [10].

While some analogies between sound and quantum properties have previously been observed [11], engineered superpositions of states of acoustic waves and quantum waves potentially offer a decoherence-free, acoustic, phase-based computing alternative to quantum systems for some quantum information processing applications [12,13]. A phi-bit [14] is a two-state degree of freedom of an acoustic wave, which can be in a coherent superposition of states with complex amplitude coefficients. A phi-bit is a classical acoustic analogue of a qubit. Phi-bits have been used to experimentally demonstrated the single phi-bit quantum-like phase and Hadamard gates [15], and the two phi-bit C-NOT gate [16]. These gates are elementary components of a universal set of gates that can be used to generate multiple phi-bit quantum-like circuits. The effort to develop large-scale phi-bit gates, which do not need to be decomposed into circuits of elementary gates, spurred the realization of a non-trivial three phi-bit unitary transformation that swaps components of a 2^3 vector about a pivot [17]. It has been demonstrated this operation can scale to increasing numbers of phi-bits [18]. However, so far, this research has constituted of demonstrating the existence of phi-bit-based gates. The primary focus of the present work is to show that multiple phi-bit large-scale unitary operations can be realized by design. This design is accomplished by engineering the correspondence between the state of the logical phi-bits in their supporting physical space and a representation in an exponentially scaling Hilbert space that transforms according to the desired gate upon changes in the experimental driving conditions. This approach is applied to the design of scalable multiple phi-bit QFT operations.

2 Logical phi-bits

A logical phi-bit is a two-state degree of freedom in the spectral domain of nonlinear acoustic modes supported by an externally driven array of nonlinearly coupled waveguides [13,14]. In these experiments, the array of waveguides is a metastructure composed of three acoustic waveguides, taking the form of aluminum rods, coupled with epoxy resin. The experimental aluminum rod (McMaster-Carr 1615T172: diameter = 1/2-inch, length = 0.6096 m, and density = 2660 kg/m^3) produces longitudinal waves with a wavelength close to 10 cm at the chosen frequency, see below. This means the waves travel almost one dimensionally along the rod-like waveguides. The external driving is achieved via piezoelectric transducers attached to the ends of the rods. Each transducer is actuated by separate signal generators and amplifiers. Detecting transducers are attached to the ends of the rods opposite to the driving transducers. Transmission from the transducers to the rods is enhanced by a thin layer of honey which serves as ultrasonic coupling agent. The transducer/couplant/rod assemblies are maintained under the constant pressure of three independent rubber bands. The entire array of waveguides is suspended by thin threads for isolation. Further details of this experimental setup can be found in Ref. [14].

By exciting the waveguides using different frequencies, e.g., f_1 and f_2, the displacement field measured at the waveguide's detection ends is the Fourier sum of modes with the primary frequencies f_1 and f_2 as well as secondary nonlinear modes. In the experiment, rod 1 operated at a frequency of f_1 = 62 kHz, while rod 2 operated at f_2 = 66 kHz. The frequency f_1 was adjusted in increments of 50 Hz. The nonlinear modes have frequencies which are linear combinations of the driving frequencies: $pf_1 + qf_2$, where p and q are integer. Each nonlinear mode represents a logical phi-bit. The displacement field of a logical phi-bit at the end of three waveguides can be referred and renormalized to that of a waveguide at the edge of the array, defined as waveguide 1, to obtain a 2×1 vector:

$$U^{(p,q)} = \begin{pmatrix} \hat{c}_2 \exp(i\varphi_{12}) \\ \hat{c}_3 \exp(i\varphi_{13}) \end{pmatrix} \exp(i\left(p\omega_1 + q\omega_2\right)t), \tag{1}$$

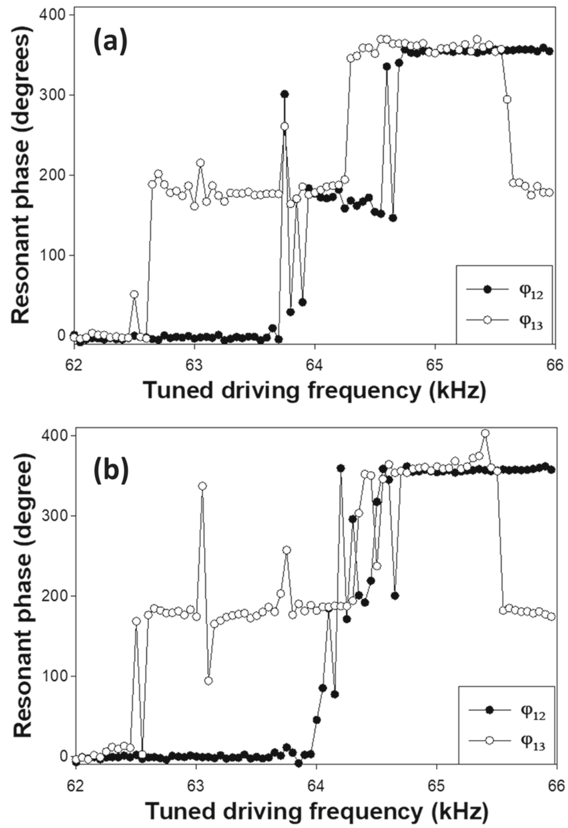

Fig. 1 Experimentally measured behavior of the resonant phases φ_{12} and φ_{13} for phi-bits **a** $\{p = 4, q = -2\}$ and **b** $\{p = 3, q = -1\}$ as functions of the driving frequency f_1 (see text for details). Initially, rod 1 is driven at the frequency f_1 = 62 kHz, and rod 2 is driven at f_2 = 66 kHz f_1 is tuned by increasing its value by increments of 50 Hz

where $\omega_i = 2\pi f_i$; $i = 1, 2$. The phase difference between the displacement field at waveguides 2 and 3 relative to waveguide 1 are $\varphi_{12} = \varphi_2 - \varphi_1$ and $\varphi_{13} = \varphi_3 - \varphi_1$. The phases φ_1, φ_2, and φ_3 arise from the complex nature of the resonant amplitudes of the driven system with dissipation as well as any nonlinear effect combining these resonant amplitudes. $\hat{c}_2$ and $\hat{c}_3$ are the magnitudes of the acoustic field at the ends of the waveguides 2 and 3, normalized to the magnitude at the end of waveguide 1. The state of the logical phi-bit $\{p, q\}$ can be redefined in terms of phase differences of the displacement field only, by constructing the nontemporal part of the field as the normalized complex amplitude state vector:

$$\boldsymbol{u}^{(p,q)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(i\varphi_{12}) \\ \exp(i\varphi_{13}) \end{pmatrix} \tag{2}$$

A single logical phi-bit state, represented in this form, spans the Bloch sphere and is analogous to a quantum bit (qubit). By manipulating the frequency, phase, and/or magnitude of the driving signals, one can rotate the logical phi-bit state vector, $\boldsymbol{u}^{(p,q)}$. Importantly, the state of each logical phi-bit (i.e., nonlinear mode) is strongly correlated to all other logical phi-bits through the nonlinearity of the physical system. This means that a single change in the driving conditions of the physical system will lead to simultaneous changes in the state vectors of all individual logical phi-bits. The single physical action will subsequently rotate a state of N logical phi-bits, represented in the tensor product space of individual phi-bits. This space scales exponentially as 2^N. A change in the conditions of the physical systems changes all 2^N components of a N phi-bit representation. Logical phi-bit correlations and manipulation of the supporting physical system can therefore be used to operate in a massively parallel manner on the components of exponentially scaling multiple phi-bit state representations.

It has been experimentally observed that changes in the driving conditions of the array of waveguides can lead to π jumps in the logical phi-bit phases φ_{12} and φ_{13}. A possible origin of these phase jumps in terms of nonlinear interactions of driven acoustic waves in the metastructure was discussed in detail previously [14,19]. The phases as functions of driving conditions can be decomposed into a monotonously varying contribution and sharp π jumps. For the sake of brevity, these are referred to these as non-resonant and resonant contributions to the phi-bit phase. Considering a phi-bit $\{p,q\}$ the non-resonant phases can be expressed as linear combinations $p\varphi_{12}\ (f_1)+q\varphi_{12}\ (f_2)$ and $p\varphi_{13}\ (f_1)+q\varphi_{13}\ (f_2)$ of the phase differences associated with the linear displacement of the array of waveguides at the frequencies f_1 and f_2. That is, one can easily subtract these phases for each phi-bit to obtain resonant contribution to φ_{12} and φ_{13}. Figure 1 illustrates the experimentally measured behavior of the resonant phases for two logical phi-bits. Here, rods 1 and 2 are driven at two different initial frequencies. The frequency f_1 is then tuned and the phases φ_{12} and φ_{13} for each phi-bit are obtained from the complex components of the Fourier spectrum at the ends of the rods. Figure 1 shows that both phi-bits exhibit simultaneous π jumps in their corresponding phases φ_{12} and φ_{13}.

It is this type of simultaneous phase jumps that will be exploited to design single and multiple phi-bit representations that will enable QFT operations.

3 Single phi-bit QFT

Recall that the QFT for N bits is associated with the unitary matrix:

$$F_Q=\frac{1}{\sqrt{Q}}\begin{pmatrix}1 & 1 & 1 & 1 & \dots & 1\\ 1 & w & w^2 & w^3 & \dots & w^{Q-1}\\ 1 & w^2 & w^4 & w^6 & \dots & w^{2(Q-1)}\\ 1 & w^3 & w^6 & w^9 & \dots & w^{3(Q-1)}\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 1 & w^{Q-1} & w^{2(Q-1)} & w^{3(Q-1)} & \dots & w^{(Q-1)(Q-1)}\end{pmatrix} \tag{3}$$

with $Q=2^N$ and $w=e^{i\frac{2\pi}{Q}}$.

Consider a variation on the single logical phi-bit analytical representation of Eq. (2) that could be achieved by some unitary transformation in the single phi-bit Hilbert space. Define the new single logical phi-bit representation:

$$V_0=e^{i\phi_2}\begin{pmatrix}\sin\phi_1\\ \cos\phi_1\end{pmatrix}, \tag{4}$$

where $\phi_1=\varphi_{12}/2$ and $\phi_2=\varphi_{13}/2$.

Any physical action of the supporting metastructure leading to a π jump in both φ_{12} and φ_{13} will produce a rotation of V_0 to a new state $V_0^{'}$ such that

$$V_0^{'}=M_{2\times 2}V_0 \tag{5}$$

where the unitary matrix $M_{2\times 2}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ is the σ_y Pauli matrix. $M_{2\times 2}$ has been obtained using the standard trigonometric relations: $\sin(a+b)=\sin a\cos b+\cos a\sin b$ and $\cos(a+b)=-\sin a\sin b+\cos a\cos b$.

Now design a new representation, V, that would transform (upon π jumps) according to the single phi-bit QFT (or Hadamard gate):

$$F_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{6}$$

This new representation is written analytically in the form:

$$V = X_{2\times 2}V_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} e^{i\phi_2}\begin{pmatrix} \sin\phi_1 \\ \cos\phi_1 \end{pmatrix} \tag{7}$$

When applying the physical action that produces π jumps, the vector V transforms as:

$$V_\pi^{'} = X_{2\times 2}V_0^{'} = \quad X_{2\times 2}M_{2\times 2}V_0 \tag{8}$$

Upon the application of the QFT, the vector V transform according to

$$V_F^{'} = F_2 V = F_2 X_{2\times 2}V_0 \tag{9}$$

One therefore seeks a matrix $X_{2\times 2}$ such that $V_F^{'} = V_\pi^{'}$. Using Eqs. (8) and (9), this condition is satisfied if

$$F_2 X_{2\times 2} - X_{2\times 2}M_{2\times 2} = 0 \tag{10}$$

Equation (10) is a special form of Sylvester equation [20]. Its solution is obtained as follows. Rewrite the matrix $X_{2\times 2}$ as the vector composed of the columns of the matrix:

$$\mathrm{vec}(X_{2\times 2}) = (a_{11}, a_{21},\ a_{12},\ a_{22})^T \tag{11}$$

Here, the superscript T denotes the transpose, that is $\mathrm{vec}(X_{2\times 2})$ is a column vector.

Equation (10) is then rewritten as:

$$C\mathrm{vec}(X_{2\times 2}) = 0, \tag{12}$$

where the matrix C is given by:

$$C = I_{2\times 2} \otimes F_2 - M_{2\times 2}^T \otimes I_{2\times 2}. \tag{13}$$

In Eq. (13), $I_{2\times 2}$ is the identity matrix. The condition for the existence of non-trivial solutions for $\mathrm{vec}(X_{2\times 2})$ (or $X_{2\times 2}$) is that the determinant of C be equal to zero.

Inserting Eq. (6) and the σ_y Pauli matrix into Eq. (13) yields:

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & -i\sqrt{2} & 0 \\ 1 & -1 & 0 & -i\sqrt{2} \\ i\sqrt{2} & 0 & 1 & 1 \\ 0 & i\sqrt{2} & 1 & -1 \end{pmatrix}$$

It is straightforward to show that $\det(C) = 0$. There are two vectors $\mathrm{vec}(X_{2\times 2})$ forming an orthonormal basis for the null space of C, namely

$$(\mathrm{vec}(X_{2\times 2}))^{(1)} = (-0.205,\ -0.6613,\ 0.6447i,\ -0.2905i)^T$$

and

$$\left(\mathrm{vec}(X_{2\times 2})\right)^{(2)} = (0.6613,\ -0.2505,\ -0.2905i,\ -0.6447i)^T.$$

Any linear combination of these two vectors will also be a solution to Eq. (12). Note that these vectors (or a linear combination of these vectors) can be recast in matrix form and inserted into Eq. (7) to obtain a representation of the phi-bit state which upon physically actuating π phase jumps will transform according to the one-bit QFT. Further, the transformation will be independent of the initial value of the phases φ_{12} and φ_{13} as well as of the coefficient in a linear combination of the null space basis. These coefficients and initial phase values can be used to define the input to the QFT.

4 Two phi-bit QFT

Redefine the representation of a single phi-bit "(i)":

$$V_0^{(i)} = e^{i\phi_2^{(i)}} \begin{pmatrix} \sin\phi_1^{(i)} \\ \cos\phi_1^{(i)} \end{pmatrix}, \tag{14}$$

where $\phi_1^{(i)} = \varphi_{12}^{(i)}/4$ and $\phi_2^{(i)} = \varphi_{13}^{(i)}/4$. The superscript (i) =1,2 refers to the phi-bits. Note that now the phi-bit phases are divided by 4 instead of 2 as was the case for a single phi-bit QFT. Define the two phi-bit state as the tensor product of single phi-bit states, namely

$$V_0 = V_0^{(1)} \otimes V_0^{(2)}. \tag{15}$$

Expanding Eq. (15),

$$V_0 = e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)} \begin{pmatrix} \sin\phi_1^{(1)}\sin\phi_1^{(2)} \\ \sin\phi_1^{(1)}\cos\phi_1^{(2)} \\ \cos\phi_1^{(1)}\sin\phi_1^{(2)} \\ \cos\phi_1^{(1)}\cos\phi_1^{(2)} \end{pmatrix}. \tag{16}$$

Using trigonometric relations, upon a π jump for the four phi-bit phases $\varphi_{12}^{(1)}$, $\varphi_{13}^{(1)}$ and $\varphi_{12}^{(2)}$, $\varphi_{13}^{(2)}$, the vector V_0 transforms to

$$V_0^{'} = M_{4\times 4} V_0. \tag{17}$$

In anticipation of scaling the approach to designing a representation that results in the QFT, note that $M_{4\times 4} = \frac{i}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = i\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ where $M_{2\times 2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is now the transformation for the single phi-bit representation given by Eq. (14).

Again, introduce a new representation V:

$$V = X_{4\times 4} V_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} V_0. \tag{18}$$

While the representation V_0 is separable (it is a tensor product), the representation V may not be factorizable into a tensor product, that is, it may be non-separable or classically entangled [13]. In contrast to quantum systems where entanglement is necessary to correlate qubit states, classical entanglement of phi-bits is only needed to expand the accessible regions of multiple phi-bit Hilbert spaces and subsequently increase the number of possible states and the range of information that can be encoded and processed in those states. Recall that logical phi-bits are correlated via the nonlinearity of the elasticity of the physical system.

When acting physically on the system to produce π jumps, the vector V transforms as

$$V_\pi^{'} = X_{4\times4} V_0^{'} = \quad X_{4\times4} M_{4\times4} V_0. \tag{19}$$

Using Eq. (3), the QFT, F_4, for two bits is written as

$$F_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}. \tag{20}$$

Upon application of the QFT to the vector V, one gets

$$V_F^{'} = F_4 V = F_4 X_{4\times4} V_0. \tag{21}$$

One, therefore, seeks a matrix $X_{4\times4}$ such that $V_F^{'} = V_\pi^{'}$. This condition is satisfied when:

$$F_4 X_{4\times4} - X_{4\times4} M_{4\times4} = 0. \tag{22}$$

Following the previous section, rewrite the matrix $X_{4\times4}$ as the vector composed of the columns of the matrix:

$$\mathrm{vec}(X_{4\times4}) = (a_{11}, a_{21}, a_{31},\ a_{41}, \ldots, a_{14},\ a_{24}, a_{34}, a_{44})^T. \tag{23}$$

Equation (22) is then rewritten as

$$C\mathrm{vec}(X_{4\times4}) =, \tag{24}$$

where the matrix C is given by

$$C = I_{4\times4} \otimes F_4 - M_{4\times4}^T \otimes I_{4\times4}. \tag{25}$$

Inserting the expression for $M_{4\times4}$ and Eq. (20) into Eq. (24), Matlab© has been used to verify that $\det(C) = 0$. There exist 5 solutions that form an orthogonal basis for the null space of C (see Sect. 6). Similar to the single phi-bit case, a linear combination of these solutions will transform upon π jumps according to the two-bit QFT. Again, this transformation will be independent of the initial value of the phases $\varphi_{12}^{(i)}$ and $\varphi_{13}^{(i)}$ as well as of the coefficient in a linear combination of the null space basis. These coefficients and initial phase values can be used to define the input to the two phi-bit QFT.

5 Multiple phi-bit QFT

Consider a system composed of N logical phi-bits and define the representation of a single phi-bit "(i)" among the N's as

$$V_0^{(i)} = e^{i\phi_2^{(i)}} \begin{pmatrix} \sin\phi_1^{(i)} \\ \cos\phi_1^{(i)} \end{pmatrix}, \tag{26}$$

where $\phi_1^{(i)} = \varphi_{12}^{(i)}/2^N$ and $\phi_2^{(i)} = \varphi_{13}^{(i)}/2^N$. The superscript (i) =1,…, N refers to the phi-bits.

The N phi-bit state is represented as the tensor product of single phi-bit states, namely

$$V_0 = V_0^{(1)} \otimes V_0^{(2)} \otimes \cdots \otimes V_0^{(N)}. \tag{27}$$

Upon a π jump for all the phi-bit phases $\varphi_{12}^{(i)}, \varphi_{13}^{(i)}$, the vector V_0 transforms to

$$V_0^{'} = M_{2^N \times 2^N} V_0, \tag{28}$$

where $M_{2^N\times 2^N} = e^{iN\frac{\pi}{2^N}} M_{2\times 2} \otimes \ldots \otimes M_{2\times 2}$ with $M_{2\times 2} = \begin{pmatrix} \beta & \alpha \\ -\alpha & \beta \end{pmatrix}$. The quantities, $\alpha = \sin\frac{\pi}{2^N}$ and $\beta = \cos\frac{\pi}{2^N}$.

Define the target representation as

$$V = X_{2^N\times 2^N} V_0 \tag{29}$$

The components of the matrix $X_{2^N\times 2^N}$ are a_{kl} with the indices $k = 1, \ldots, 2^N$ and $l = 1, \ldots, 2^N$.

The physical action applies the transformation:

$$V_\pi^{'} = X_{2^N\times 2^N} V_0^{'} = X_{2^N\times 2^N} M_{2^N\times 2^N} V_0. \tag{30}$$

The N phi-bit QFT, F_{2^N}, yields

$$V_F^{'} = F_{2^N} V = F_{2^N} X_{2^N\times 2^N} V_0 \tag{31}$$

$V_\pi^{'}$ will be equal to $V_F^{'}$ when

$$F_{2^N} X_{2^N\times 2^N} - X_{2^N\times 2^N} M_{2^N\times 2^N} = 0. \tag{32}$$

Following the preceding sections, the sought matrix $X_{2^N\times 2^N}$ is reformulated as the vector $\mathrm{vec}(X_{2^N\times 2^N})$. The solutions of

$$C\mathrm{vec}(X_{2^N\times 2^N}) = 0, \tag{33}$$

where the matrix C is now given by

$$C = I_{2^N\times 2^N} \otimes F_{2^N} - M_{2^N\times 2^N}^T \otimes I_{2^N\times 2^N}. \tag{34}$$

The solutions of Eq. (34) that form an orthonormal basis can be used to construct the representation V (Eq. (29)) that will transform according to the Nphi-bit QFT when operations on the physical system lead to π jumps in the logical phi-bit phases.

It is conjectured that Eq. (33) possesses solutions for all number of phi-bits. The existence of 8 solutions forming the basis of the null space of C for a three phi-bit system has been verified. The basis of the null space for four phi-bits contains 9 vectors. This conjecture is supported by the fact that the representations V and V_0 involve, upon π jumps, powers of terms of the form $e^{i\frac{2\pi}{2^N}}$ which are also the components of the QFT matrix (Eq. (3)).

This general approach enables us to design representations for inputting data into a multiple phi-bit QFT that can be physically realized by actuating an acoustic metastructure. This approach establishes a link between multiple phi-bit gates in exponentially complex Hilbert spaces and operations on the physical system scaling linearly with the number of phi-bits. In this example, the QFT of the input data is achieved through the π jumps in the phi-bit phases. It is worth stressing that the phi-bit-based QFT can be defined and executed for any input.

6 A two phi-bit example

The concept of phi-bit-based QFT is illustrated with a simple two phi-bit case. The phi-bits of Fig. 1 may serve as the physical platform. Here, instead of determining an orthonormal basis for the null space of the matrix C, a rational basis for that null space is calculated using Matlab©. That basis is not usually orthonormal. However, linear combinations of these basis vectors will also provide a means of defining inputs for the QFT. The five matrices below form the rational basis:

$$X^{[1]}_{4\times4}=\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&0&0&0\\0&-1&1&0\end{pmatrix};\quad X^{[2]}_{4\times4}=\begin{pmatrix}-1&-i/2&-i/2&1\\0&0&-i&0\\0&-i/2&-i/2&0\\0&-i&0&0\end{pmatrix};\quad X^{[3]}_{4\times4}=\begin{pmatrix}0&-i/2&-i/2&0\\0&0&0&1\\0&i/2&i/2&0\\-1&0&0&0\end{pmatrix};$$

$$X^{[4]}_{4\times4}=\begin{pmatrix}0&-i/2&-i/2&0\\0&0&i&0\\-1&-i/2&-i/2&1\\0&i&0&0\end{pmatrix};\quad X^{[5]}_{4\times4}=\begin{pmatrix}0&-i/2&-i/2&0\\-1&0&0&0\\0&i/2&i/2&0\\0&0&0&1\end{pmatrix}.$$

Consider calculating the Fourier transform of the following input vector $V_I=\begin{pmatrix}:0\\1\\0\\-1\end{pmatrix}$. For this, choose the representation $V=X^{[1]}_{4\times4}V_0$. This vector takes the form:

$$V=e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)}\begin{pmatrix}0\\ \sin\phi_1^{(1)}\cos\phi_1^{(2)}-\cos\phi_1^{(1)}\sin\phi_1^{(2)}\\0\\-\sin\phi_1^{(1)}\cos\phi_1^{(2)}+\cos\phi_1^{(1)}\sin\phi_1^{(2)}\end{pmatrix}\tag{35}$$

or

$$V=e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)}\left(\sin\phi_1^{(1)}\cos\phi_1^{(2)}-\cos\phi_1^{(1)}\sin\phi_1^{(2)}\right)\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}.$$

By changing the driving conditions applied on the acoustic metastructure such that all phi-bit phases change by π, (for instance by changing the driving frequency f_1 of Fig. 1 from 62 to 64.2 kHz), the vector V transforms into

$$V'=e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)}\left(\sin\phi_1^{(1)}\cos\phi_1^{(2)}-\cos\phi_1^{(1)}\sin\phi_1^{(2)}\right)\begin{pmatrix}0\\2i\\0\\-2i\end{pmatrix}$$

To within a general phase, the output vector $V_O = \begin{pmatrix} 0 \\ 2i \\ 0 \\ -2i \end{pmatrix}$ is the Fourier transform of V_I.

One may be interested in determining if some function possesses some specific period. Represent the function as the input vector: $V_I = \begin{pmatrix} -i \\ 1 \\ i \\ -1 \end{pmatrix}$. To realize that vector, use the representation $V = X_{4\times4}^{[3]} V_0$. This vector takes the form:

$$V = e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)} \begin{pmatrix} -\frac{i}{2}\sin\phi_1^{(1)}\cos\phi_1^{(2)} - \frac{i}{2}\cos\phi_1^{(1)}\sin\phi_1^{(2)} \\ \cos\phi_1^{(1)}\cos\phi_1^{(2)} \\ \frac{i}{2}\sin\phi_1^{(1)}\cos\phi_1^{(2)} + \frac{i}{2}\cos\phi_1^{(1)}\sin\phi_1^{(2)} \\ -\sin\phi_1^{(1)}\sin\phi_1^{(2)} \end{pmatrix} \tag{36}$$

By initializing the two phi-bit system such that $\varphi_{12}^{(1)} = \varphi_{12}^{(2)} = \pi$, the sines and cosines are equal to each other (i.e., $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$) and V reduces to

$$V = e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)} \frac{1}{2} \begin{pmatrix} -i \\ 1 \\ i \\ -1 \end{pmatrix}$$

This initial state can be achieved by driving the two phi-bits of Fig. 1 with a frequency f_1 ~64.2 kHz. A change of driving conditions by raising f_1 to 65 kHz adds π to the phi-bit phases leading to the following transformations:

$$\begin{aligned}
&\sin\phi_1^{(1)}\sin\phi_1^{(2)} \to \sin\phi_1^{(1)}\sin\phi_1^{(2)} + \sin\phi_1^{(1)}\cos\phi_1^{(2)} + \cos\phi_1^{(1)}\sin\phi_1^{(2)} + \cos\phi_1^{(1)}\cos\phi_1^{(2)} \\
&\sin\phi_1^{(1)}\cos\phi_1^{(2)} \to -sin\phi_1^{(1)}\sin\phi_1^{(2)} + \sin\phi_1^{(1)}\cos\phi_1^{(2)} - \cos\phi_1^{(1)}\sin\phi_1^{(2)} + \cos\phi_1^{(1)}\cos\phi_1^{(2)} \\
&\cos\phi_1^{(1)}\sin\phi_1^{(2)} \to -\sin\phi_1^{(1)}\sin\phi_1^{(2)} - \sin\phi_1^{(1)}\cos\phi_1^{(2)} + \cos\phi_1^{(1)}\sin\phi_1^{(2)} + \cos\phi_1^{(1)}\cos\phi_1^{(2)} \\
&\cos\phi_1^{(1)}\cos\phi_1^{(2)} \to \sin\phi_1^{(1)}\sin\phi_1^{(2)} - \sin\phi_1^{(1)}\cos\phi_1^{(2)} - \cos\phi_1^{(1)}\sin\phi_1^{(2)} + \cos\phi_1^{(1)}\cos\phi_1^{(2)}.
\end{aligned}$$

With $\varphi_{12}^{(1)} = \varphi_{12}^{(2)} = \pi$, the first three components of the representation given in Eq. (36) become zero. The fourth component becomes $-4/2 = -2$. Upon π jumps, the prefactor, $e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)}$, transforms into $ie^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)}$, so

$$V' = V = e^{i\left(\phi_2^{(1)}+\phi_2^{(2)}\right)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2i \end{pmatrix}.$$

To within a general phase, the Fourier transform output is obtained: $V_O = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2i \end{pmatrix}$.

After the steps of initialization and physical operation that leads to π jumps in the phi-bit phases, by calculating the fourth component of the chosen two phi-bit representation, it can be determined if the input function possesses a specific period. For instance, a non-zero value of the first component of the output vector indicates a constant function over the domain of four discrete points of the function. A non-zero value of the third component of the output vector corresponds to a period extending over half of the domain of the function. A non-zero value of the fourth component corresponds to a function of the form $e^{i\frac{\pi}{2}n}$ with the domain of the function defined through the integer $n = 1, 2, 3, 4$. The QFT approach implemented on multiple phi-bit can, therefore, be used to extract the periods of sequences of real but also complex numbers.

Note that after a QFT on a qubit-based computing platform, there is no effective way to recover the Fourier coefficients which are stored in quantum superpositions of probability amplitudes (i.e., quantum wave functions). Because of quantum wave function collapse, a measurement of the state of qubits will only return the computation basis and not the superposition. In contrast, a logical phi-bit-based QFT operates on complex amplitudes of nonlinear acoustic modes which are not subjected to wave function collapse. Retrieving all Fourier coefficients on N phi-bits would require 2^N calculations, thus creating significant computational overhead. However, phi-bit-based QFT will realize its full potential when one needs the calculation of only one or a few Fourier components enabling the measurement of the period(s) of a function. Phi-bit-based QFT will also demonstrate its full power when combined with other multiple phi-bit gates and/or algorithms.

7 Conclusion

An approach that establishes a correspondence between the state of multiple logical phi-bits, acoustic qubit analogues, in their supporting physical space and their representations as complex state vectors in exponentially scaling Hilbert spaces has been introduced. The physical response of the correlated logical phi-bits to changes in the experimental conditions is mapped onto a multiple phi-bit discrete Fourier transformation unitary matrix analogous to the quantum Fourier transform (QFT). QFT is a ubiquitous operation in quantum information science [21,22] and a common component of quantum algorithms [23–25]. The scalable QFT operates in parallel on the components of multiple phi-bit complex state vectors requiring only a single physical action on the nonlinear acoustic metastructure that supports the phi-bits. It has been demonstrated that the approach is scalable by showing that there exists a solution for the QFT matrix for any number of phi-bits in Sect. 5. This approach forms the foundation for the design and development of other multiple phi-bit unitary operations and/or algorithms that, in contrast to qubits, are unaffected by measurements and decoherence. The ingredients for applying this approach to the design of useful multiple phi-bit unitary matrices and gates include (a) a knowledge of phi-bit phase response to changes in driving conditions of the supporting acoustic metastructure, (b) analytical or numerical representations of single phi-bit and multiple phi-bits in exponentially scaling Hilbert space initializable into all (or as many possible) separable or non-separable states, and (c) an algebraic or numerical mapping between the effect of the desired unitary matrix and the change in driving conditions on the representations. The strong nonlinear correlation between phi-bits enables bypassing the need of qubit-based computing platforms for decomposition of large-scale unitary matrices into quantum circuits of elementary gates. Finally, a significant limitation in the design of a quantum algorithm is related to its mapping to a specific quantum processor. Indeed, current quantum computers are constituted of qubits occupying different parts of the processor imposing physical constraints on the possible coupling between qubits [26]. Therefore, the design of an efficient quantum algorithm needs to be adapted to a specific quantum processor map of qubit coupling, that is, its coupling map. As nonlinear acoustic modes, logical phi-bits live within the same physical space of the acoustic metastructure. Strongly correlated logical phi-bits occupy the same real estate, and thus the design of phi-bit-based gates and algorithms will not suffer from spatial physical constraints.

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Author contributions PAD wrote the main manuscript text with contributions from all authors. PAD developed the theoretical models. MAH and TDL conducted the experiments and acquired the data. PAD, KR, MAH, TDL, and JAL contributed equally to the analysis and interpretation of the data.

Data availability The data that support the findings of the present study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors have no conflicts to disclose.

References

1. Akhavan, O., Rezskhani, A.T., Golshani, M.: Quantum dense coding by spatial state entanglement. Phys. Lett. A **313**, 261 (2003)
2. Feix, A., Araújo, M., Brukner, Č: Quantum superposition of the order of parties as a communication resource. Phys. Rev. A **92**, 052326 (2015)
3. Gilyén, A., Su, Y., Low, G.H., Wiebe, N.: Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics. In: Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory of Computing (STOC '19), June 23–26, 2019, Phoenix, AZ, USA. ACM, New York, NY, USA, 12 pages (2019)
4. Alimirzaei, F., Kieslich, C.A.: Machine learning models for predicting membranolytic anticancer peptides. In: Kokossis, A., Georgiadis, M.C., Pistikopoulos, E.N. (eds.) Proceedings of the 33rd European Symposium on Computer Aided Process Engineering (ESCAPE33), June 18–21, 2023, Athens, Greece, p. 2691
5. Barenco, A., Bennett, C.H., Cleve, R., DiVincenzo, D.P., Margolus, N., Shor, P., Sleator, T., Smolin, J.A., Weinfurter, H.: Elementary gates for quantum computation. Phys. Rev. A **52**, 3457 (1995)
6. Ezeamaku, U.L., Ezekannagha, C., Eze, O.I., Odimegwu, N., Nwakaudu, A., Okafor, A., Ekuma, I., Onukwuli, O.D.: The impact of potassium permanganate (KMnO4) treatment on the tensile strength of pineapple leaf fiber reinforced with tapioca-based bio resin. Arab Gulf J. Sci. Res. **41**, 416 (2023)
7. Bhattacharjee, S., Satpathi, U., Sinha, S.: Quantum Langevin dynamics of a charged particle in a magnetic field: response function, position-velocity and velocity autocorrelation functions. Pramana-J. Phys. **96**, 53 (2022)
8. Al-Raeei, Marwan: Applying fractional quantum mechanics to systems with electrical screening effects. Chaos Solitons Fractals **150**, 111209 (2021)
9. Coppersmith, D.: An approximate Fourier transform useful in quantum factoring. Technical Report RC19642, IBM (1994). arXiv:quant-ph/0201067
10. Shor, P.W.: Algorithms for quantum computation: discrete logarithms and factoring. In: Proceedings 35th Annual Symposium on Foundations of Computer Science. IEEE Computer Society Press, p. 124 (1994)
11. Mannone, M., Rocchesso, D.: Sing and measure: sound as voice as quanta. In: Miranda, E.R. (ed.) Quantum Computer Music, p. 297. Springer, Cham (2022)
12. Arif Hasan, M., Calderin, L., Lata, T., Lucas, P., Runge, K., Deymier, P.A.: The sound of Bell states. Commun. Phys. **2**, 106 (2019)
13. Hasan, M.A., Runge, K., Deymier, P.A.: Experimental classical entanglement in a 16 acoustic qubit-analogue. Sci. Rep. **11**, 24248 (2021)
14. Deymier, P.A., Runge, K., Hasan, M.A., Lata, T.D., Levine, J.A., Cutillas, P.: Realizing acoustic qubit analogues with nonlinearity tunable phi-bits in externally driven coupled acoustic waveguides. Sci. Rep. **13**, 635 (2023)
15. Deymier, P.A., Runge, K., Hasan, M.A., Levine, J.A., Cutillas, P.: Setting the stage for materials simulation using acoustic metamaterials digital quantum analogue computing platforms. Model. Simul. Mater. Sci. Eng. **30**, 084003 (2022)
16. Runge, K., Hasan, M.A., Levine, J.A., Deymier, P.A.: Demonstration of a two-bit controlled-NOT quantum-like gate using classical acoustic qubit-analogues. Sci. Rep. **12**, 14066 (2022)
17. Deymier, P.A., Runge, K., Cutillas, P., Hasan, M.A., Lata, T.D., Levine, J.A.: Scalable exponentially complex representations of logical phi-bit states and experimental demonstration of an operable three phi-bit gate using an acoustic metastructure. Appl. Phys. Lett. **122**, 141701 (2023)
18. Runge, K., Deymier, P.A., Hasan, M.A., Lata, T.D., Levine, J.A.: Scalability of a multiple phi-bit unitary transformation. J. Appl. Phys. (in review)
19. Deymier, P.A., Runge, K., Hasan, M.A., Lata, T.D., Levine, J.A.: Tuning logical phi-bit state vectors in an externally driven nonlinear array of acoustic waveguides via drivers' phase. Quantum Rep. **5**(2), 325 (2023)
20. Sylvester, J.: Sur l'equation en matrices px=xq. C. R. Acad. Sci. Paris **99**(67–71), 115–116 (1884)
21. Pachuau, J.L., Roy, A., Saha, A.K.: Integer numeric multiplication using quantum Fourier transform. Quantum Stud. Math. Found. **9**, 155–164 (2022)
22. Kurgalin, S., Borzunov, S.: Concise Guide to Quantum Computing. Springer, Dordrecht (2021)
23. Trivedi, D., Saharia, A., Choure, K., Tiwari, M., Maddila, R.K., Singh, G.: Three-qubit implementation of quantum Fourier transform for Shor's algorithm. In: Optical and Wireless Technologies: Proceedings of OWT 2020, pp. 115–121. Springer, Singapore (2022)
24. Giorda, P., Iorio, A., Sen, S., Vitiello, G.: Searching for a semiclassical Shor's algorithm. In: Quantum Computing and Quantum Bits in Mesoscopic Systems, pp. 123–131. Springer, Boston (2004)

25. de Boutray, H., Jaffali, H., Holweck, F., Giorgetti, A., Masson, P.A.: Mermin polynomials for non-locality and entanglement detection in Grover's algorithm and Quantum Fourier Transform. Quantum Inf. Process. **20**, 1–29 (2021)
26. Acampora, G., Schiattarella, R.: Deep neural networks for quantum circuit mapping. Neural Comput. Appl. **33**, 13723 (2021)