Unpacking the Challenges and Predictors of Elementary-Middle School Students' Use of the Distributive Property

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Mathematics is a structured universal language for communicating patterns and solving problems. The distributive property, expressed mathematically as $a(b + c) = ab + ac$, is a foundational concept within this mathematical language. This property demonstrates how multiplication interacts with addition (or subtraction), stating that when multiplying a number by two or more added (or subtracted) numbers, you can "distribute" the multiplication to each number inside the group and combine the results (Brown, 2013). Understanding this property involves recognizing opportunities to "factor out" common elements from expressions. For instance, in $2x + 4$, one can factor out the common factor 2, rewriting it as $2(x + 2)$. This dual aspect of distributing and factoring makes the property powerful, facilitating flexible and thoughtful problem solving across arithmetic and algebra (National Mathematics Advisory Panel, 2008).

The goal of this study was to explore factors influencing fourth through eighth graders' use of the distributive property. Developing a deep understanding of multiplication depends on understanding the distributive property (Kinzer & Stanford, 2013). This property decomposes multiplication into manageable parts, facilitating the learning of new multiplication facts beyond rote learning (Carpenter et al., 2005). The distributive property is crucial for understanding multiplication complexities, including fractions, mixed numbers, and negative numbers (Kinzer & Standford, 2013). Robust knowledge of multiplication supports practical problem solving, such as calculating area and estimating travel times, while forming a foundation for advanced mathematical concepts. Together with the commutative and associative properties, the distributive property facilitates equation manipulation, supports algebraic simplification, and reinforces abstract mathematical thinking like proportional reasoning (Bruner, 1977; Kinzer & Standford, 2013).

Unfortunately, many students adopt procedural methods like long multiplication, successive addition, and order of operations, leading to a narrow view of multiplication (Zhang et al., 2017; Mehta & Gawali, 2009). These familiar methods lack adaptability, efficiency, and precision (Russel, 2000), leaving students with a fragmented understanding of foundational concepts (Hemi et al., 2021). Consequently, when entering their first elementary algebra course, students may be unprepared for required shifts in thinking (Vermeulen et al., 1996). To proactively mitigate these challenges, students may benefit from an early, comprehensive introduction to the distributive property.

In the evolving landscape of mathematics education, standards such as NCTM Principles and Standards (NCTM, 2000), Common Core State Standards for Mathematics (CCSS-M, [NGACBP, 2010]), and significant revisions in individual state standards, prompt teachers to reflect on their instructional practices and pinpoint critical content and instructional techniques possibly missing from their classrooms. Notably, these standards emphasize introducing the distributive property in third grade, reinforcing it in fourth, and expanding it into middle school (NGACBP, 2010; NCTM, 2000). Embedding these standards in lower grades ensures a robust understanding of multiplication and its associated properties (Matney & Daugherty, 2013), aligning with a learning trajectories approach (Clements & Sarama, 2004). Mastering the distributive property early serves as raw materials for future mathematical growth, facilitating a more successful understanding of the concepts in higher grades.

Recognizing the iterative relationship between procedural and conceptual understanding (Rittle-Johnson et al., 2015; Rittle-Johnson & Alibali, 1999), there is a shift from rote memorization and procedural drills to innovative pedagogical strategies. Ding and Li (2010) advocate for strategies that deepen students' comprehension and retention of principles like the distributive property. Their recommendations include techniques such as spacing learning, linking concrete and abstract representations, analyzing worked-out examples, promoting self-explanations by students, and posing deep questions to stimulate cognitive reflection (Ding & Li, 2010).

Ding and Li (2010) found that Chinese mathematics textbooks align well with this recommended approach to teaching the distributive property. Their comparative analysis of the distributive property usage in US and Chinese textbooks reveals that Chinese textbooks introduce the distributive property early and expand its application across grades. These textbooks incorporate problem contexts including multiple word problem types and strategies such as asking deep questions to enhance students' understanding of the property. This pedagogical approach mirrors the educational recommendations previously discussed: spaced learning, linking concrete and abstract representations, analyzing worked-out examples, promoting self-explanations by students, and posing deep questions for cognitive reflection. In contrast to Chinese textbooks, US textbooks present the distributive property narrowly, restricting students' use to mostly whole number contexts in a singular direction.

Adopting recommended teaching methods holds the promise of increased success for students. Yet, as we see in the U.S. textbooks, the inertia of older teaching methods and entrenched habits is often difficult to overcome (Barbieri et al., 2019; Ding & Li, 2010; McNeil, 2014; Silla et al., 2020). Consequently, a paradox emerges: students might excel in routine multiplication tasks that require mechanically executing multiplication, but not recognize opportunities to apply the distributive property, especially when problems are presented in less familiar symbolic formats (Vermeulen et al., 1996).

The emphasis on worked examples, explanation, and cognitive reflection found in Chinese textbooks aligns with the broader movement in mathematics education toward teacher preparation and instructional practices that encourage teachers' expertise in children's thinking, so they can anticipate errors and use students' mathematical intuitions as a springboard for teaching more advanced mathematics content (e.g., Carpenter et al. [2005, 2015] and Ball [1988]). Drawing from Vygotsky's (1978) constructs of scaffolding and the zone of proximal development (ZPD), teachers must guide children to articulate, reflect upon, and refine their mathematical thinking. Scaffolding describes the social interaction process between a teacher or advanced peer and a student less capable of solving a specific problem (Wood et al., 1976). ZPD is the distance between a student's independent performance and performance with scaffolding. As scaffolded practice and interaction accumulate, the student eventually starts solving the problems independently (Byrnes, 2008). For this social interaction to be fruitful, the teacher must be attuned to the students' developmental level, including their content knowledge and scaffolding needs to better assist them in their learning.

Educators can facilitate such scaffolds by prompting students to engage in cognitive reflection, encouraging them to reflect on their strategies, question intuitive responses in favor of logical ones, and engage in analytical reasoning (Pennycook et al., 2016). Another related approach is a problem-posing approach where students create multiplication problems to apply the distributive property (Chen and Cai, 2020). This approach stimulates cognitive reflection, promoting a deep understanding of the distributive property and encouraging students to think creatively (Chen & Cai, 2020). Chen and Cai's case study documented one teacher's challenges and triumphs with a problem-posing approach to teaching the distributive property. They found that despite initial challenges, this method fosters students' comprehension and clarification of math principles, encourages group work, and ignites thoughtful discussions among students. Further research is needed to determine the role of cognitive reflection scaffolding in teaching the distributive property.

Educators' Multiplication Use

Ball (1988) asserts that children's learning is shaped by the interaction between instruction and their existing knowledge. Emphasizing the critical role teachers play in this process, she highlights the need for their deep subject matter knowledge along with specialized knowledge for teaching mathematics to help students succeed (Ball et al., 2005). It is essential to recognize that teachers were also once students, shaping their teaching practices with the ideas and cognitive frameworks developed from their time as students. Furthermore, many children receive math instruction from teachers with a limited perspective on mathematical understanding and its acquisition (Lampert, 1986). Thus, Ball advocates for math teacher preparation programs to be oriented towards viewing teachers as learners to effectively help students. This perspective calls for educators to consider the knowledge and preconceptions that teacher candidates bring to figure out ways to challenge, transform, and expand upon their existing knowledge base, ultimately equipping them to become effective teachers in their subject matter.

Building on this foundation, Hecht's (1999) research on multiplication strategies employed by adults reveals a reliance on retrieval methods, which may be efficient for simpler problems. However, as multiplication problems increase in complexity, the limitations of relying solely on retrieval methods become apparent. Hecht's work underscores the need for diverse strategies, particularly highlighting the distributive property and the related strategy of decomposition. Therefore, incorporating diverse strategies into teaching and learning is essential, allowing educators to provide students with a comprehensive toolkit for solving complex multiplication problems. Rathouz (2011) stresses that fostering versatile problem-solving approaches is integral for mathematical proficiency and that integrating adults' multiplication strategies into teaching enhances students' problem-solving skills and overall mathematical thinking. Despite ample literature on supporting children's mathematical learning, less attention has been given to supporting future teachers in acquiring the necessary mathematical knowledge for effective teaching.

The Current Studies

Given the foundational role of the distributive property in mathematics, it is imperative to study students' understanding of it. Yet, few studies have focused on students' understanding of the distributive property (Izsák, 2004). To address this gap in the literature, we took advantage of two existing datasets containing mathematics problems involving the distributive property. The authors connected via the NSF-funded NUMBERS Workshop at Kent State University (Dunlosky et al., 2020) to discuss research questions of common interest. During these discussions, the first author expressed an interest in conducting a project on students' understanding of the distributive property. All researchers agreed this topic warranted further examination. Several team members recalled the presence of distributive property items on measures from past collaborative projects that could help the field gain a deeper understanding of students' thinking. We thus sought permission to explore the relevant data sets.

The primary objective was to gather as much information as we could about students' understanding and application of the distributive property. The first data set contained instructional interviews in which fourth to sixth-grade students explained worked examples involving the distributive property to an interviewer and then solved related problems on their own. The second data set contained middle school students' item-level responses on the brief assessment of mature number sense (Kirkland et al., 2022). Three items in this assessment focus on the distributive property. By conducting a fine-grained analysis of students' item-level responses on problems involving the distributive property, we hoped to gain information about students' use of this fundamental mathematical property to contribute to future theory building and instructional decision-making.

Study 1

Worked Example Interviews

For our first analysis, we accessed audio data of students being interviewed and instructed on worked examples, several of which used the distributive property. Each student worked with an interviewer who presented worked examples demonstrating how to correctly (or incorrectly) use the distributive property. For each worked example, interviewers were trained to go through the prompted questions with students and then allow the students time to complete an associated "Your Turn" practice problem on their own. If students solved this problem incorrectly, interviewers assisted until students were able to correctly solve the problem. We were interested to see if students used the distributive property on their practice problem as shown in the worked examples, or if they instead used an *intuitive correct approach.* We define the *intuitive correct approach* as common problem-solving approaches that students typically use to solve multiplication problems like long multiplication (also known as the standard multiplication algorithm [Hurst & Huntley, 2020]), successive addition, or simply following the order of operations (Mehta & Gawali, 2009). We draw a contrast between use of these strategies versus use of the intended correct approach, the *distributive property*, as this was the strategy that interviewers were trained to instruct within the worked examples.

In addition to examining the extent to which students opted to use the distributive property over intuitive correct approaches, we also coded the instructional interviewer's language. We wanted to know if interviewers used the distributive property prompts as instructed in the protocol, or if they allowed students to overlook the distributive property in favor of an intuitive correct approach. We asked the following two questions: (1) To what extent did students use the distributive property over the intuitive correct approach to solve the problems? (2) Did the interviewers consistently use the distributive property prompts, or did they ignore the protocol and revert to an intuitive correct approach?

Method

Participants

The sample was drawn from an existing dataset collected to study the types of worked examples that best scaffold mathematics learning (AUTHORS, 2023). Participants in the dataset ($N = 24$) were fourth to sixth-grade students recruited from a small private university school in the Northeast US located on a public university campus. This school serves students in grades 1-8. Many students attending this school have learning difficulties and disabilities, with 63% of the sample having reading difficulties and 42% having math difficulties. All participants with math difficulties also had reading difficulties. Of the 24 participants, five were fourth graders, seven were fifth graders, and twelve were sixth graders, comprising the entire fourth to sixth-grade population in the school. Gender distribution was equal, with 50% boys and 50% girls. The majority of students identified as white (79%), while 12.5% identified as Black, and 8.3% as Asian or multi-racial. No students identified as Hispanic or Latine.

Materials

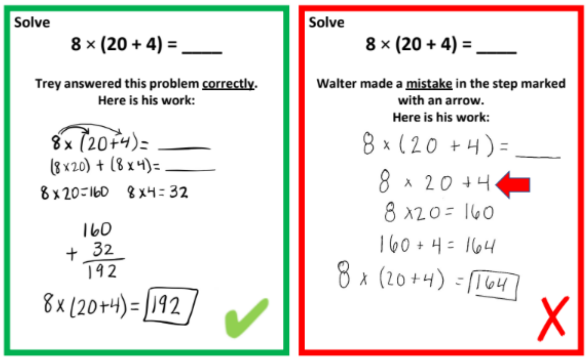
Materials in the larger study included instructional interview sessions, a grade-level math achievement assessment, and three sub-assessments examining students' understanding of equivalence, operations, and procedural flexibility. However, only the interviews were pertinent for our study focusing on the distributive property.

Instructional Interview Sessions. A structured interview protocol and student workbooks were used for the interviews. Interviews were conducted by five research assistants, including two graduate students (one a former teacher), and three preservice teachers. The workbooks contained 22 problems with varied worked examples (e.g., correct, incorrect, faded) in three sections: operations, equivalence, and procedural flexibility. We analyzed eight workbook questions involving the distributive property, drawn from the operations and procedural flexibility sections from all grades. Fourth graders had two such problems, fifth graders had four, and sixth graders had two. Each grade level included a counterexample, resulting in only using five problems to address our first research question concerning the use of the distributive property. Figure 1 demonstrates an example of one of these problems.

Figure 1

Worked Example Exercise

1. Initial Prompt: Both Trey and Walter tried to solve ___. Walter made a mistake in the step marked with an arrow. What was Walter's mistake?	**2. Your Turn!**



Now, solve this problem on your own.

$$9 \times (10 + 5) = \_\_\_\_$$

Sample Interview Prompts:

- What did Walter do wrong?
- Why did Walter multiply the 8 by 20?
- What do you know about distributive property?
- What should Walter have done after multiplying the 8 by 20?
- What should Walter have put in the box instead?

Note. Worked example exercise from a fifth-grade student's workbook.

Procedure

The larger study used a within-subjects pre-intervention-post design, where all the participants received the same instructional interview with no "business-as-usual" control. Following the pretest, instructional interviews forming the basis of our analysis took place in a quiet school setting, lasting approximately 20 minutes each. All sessions were audio-recorded.

During the interviews, students were introduced to the target area and presented with worked examples, including instances of incorrect examples only and, at times, both correct and incorrect examples. Interviewers were trained to adhere to a protocol, but the sessions were designed to be natural and responsive to students' answers, therefore interviewers did not read from a script verbatim. Interviewers prompted students to articulate errors in incorrect worked examples, suggest ways to rectify the mistakes (e.g., "This student's mistake is marked with an arrow? What is their mistake? How can they fix their work to make it correct?"), and determine the correct answer (e.g., "What should go in the blank to make the equation correct…?").

Interviewers were trained to reinforce targeted content knowledge (e.g., “The student should have used the distributive property. The student ought to distribute the number across the addition problem to use the distributive property correctly.”). If the student did not answer correctly, the interviewer used prompts to guide them toward the correct answer.

After discussing the worked examples, students completed “Your Turn” problems that aligned with those examples. These problems were designed for independent solving, with the interviewer offering assistance using the previously provided prompts if needed. If students initially solved the problem incorrectly, interviewers helped them until they got it correct. The goal was for interviewers to cover as many problems as possible within 20 minutes.

Distributive Property Interview Coding Scheme. We developed a coding scheme for problems focused on the distributive property (Table 1). Using the audio-recorded data transcripts and students’ written work, we extracted three key pieces of data. To answer our first question, we focused on the workbooks and written work that accompanied students’ answers to the “Your Turn” problem. We coded for whether students used the distributive property to solve the problem, or if they chose an intuitive method like long multiplication or PEMDAS. For this coding, we coded 38 problem interviews in total (5 4^{th} grade, 21 5^{th} grade, 12 6^{th} grade). To answer our second question, we focused on the audio-recorded interviews and coded whether the interviewer adhered to the prompts and mentioned the use of the distributive property as directed by the protocol. We then assessed whether students knew what the distributive property was when queried by the interviewer. For this, we computed proportions for the entire cohort and then by grade level. We coded 62 problem interviews in total (10 4^{th} grade, 28 5^{th} grade, 24 6^{th} grade). Each criterion was binary coded (yes=1, no=0).

Table 1

Coding Scheme for Study 1

Observations (N=62)	Targeted Questions		
	Did students use the distributive property to solve the practice problem?	Did interviewers mention the use of the DP or ask the students about their knowledge of the DP?	Of the students who were asked about their knowledge of the DP, were they familiar with this property?
	Example Coding		
4th (N=10) 5th (N=28) 6th (N=24)	1 = Students used the DP 0 = Students used an alternative method	1 = Interviewer mentioned the use of the DP. 0 = No mention of the DP	1 = Yes, students were familiar with the DP 0 = Students were not familiar with DP

Note. “DP” = distributive property. The number of observations stems from the number of questions observed for these questions and the number of students per grade. (i.e., two fourth-grade questions on the DP were addressed among five students, resulting in 10 total observations).

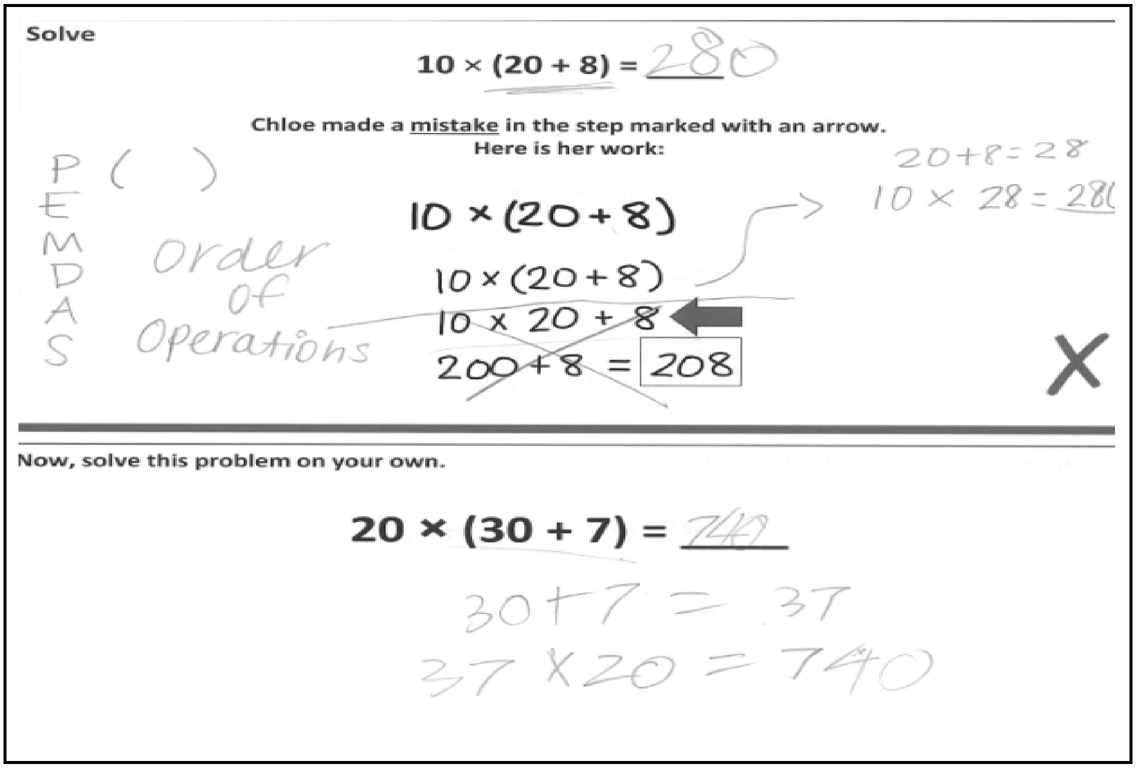
Results

To assess students' application of the distributive property in "Your Turn" problems after training with worked examples, we computed the proportion of total (*N*=38) observations in which students used the distributive property from the five problems. Across all five problems, 32% were solved using the distributive property, while 68% were solved using an intuitive correct approach. Examining grade levels, none (0%) of the fourth-grade observations used the distributive property, 24% of fifth graders did, and 58% of sixth graders did. At the student level, 48% of students employed the distributive property in their solutions (0% of fourth graders, 57%

of fifth graders, and 64% of sixth graders). Examples of student work are illustrated in Figures 2.1 and 2.2.

Figure 2.1

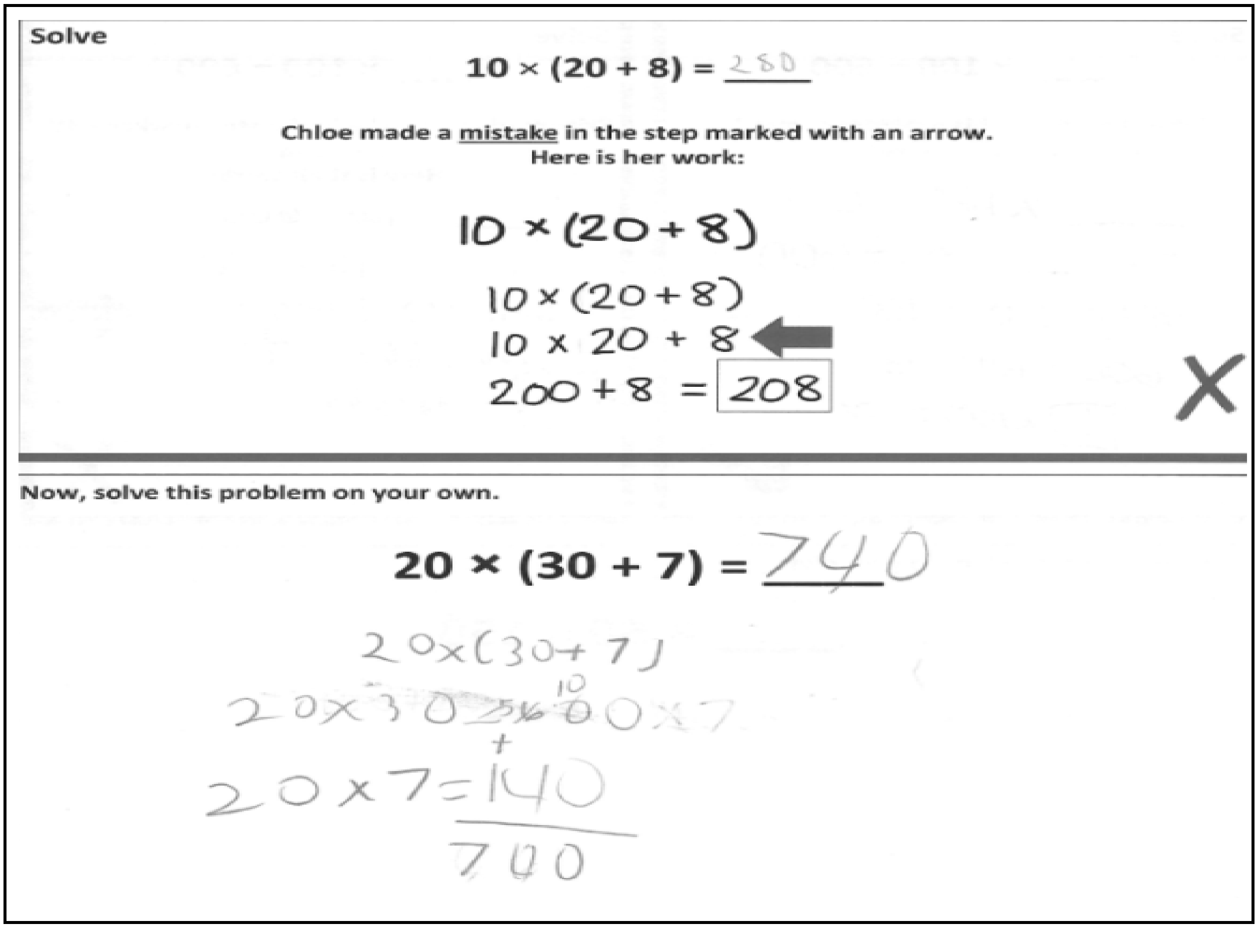
Student "A" Using an Intuitive Correct Approach to Solve the "Your Turn" Practice Problem



Note. This figure illustrates a student's reliance on an intuitive approach rather than the distributive property. After studying a worked example and explaining it with the interviewer, the student independently solved 20×(30+7) using the order of operations, as depicted.

Figure 2.2

Student "B" Using the Distributive Property to Solve the "Your Turn" Practice Problem

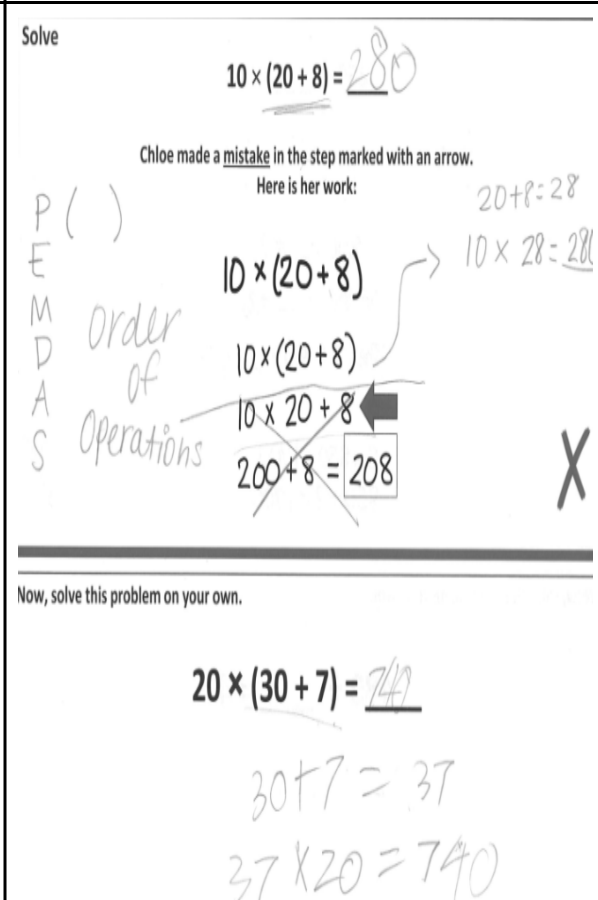


Note. This figure shows a student applying the distributive property to solve a practice problem. After studying and explaining a worked example with the interviewer, the student independently solved 20×(30+7) by multiplying 20 by 30 and 20 by 7, followed by adding the two products.

To assess if interviewers adhered to the distributive property prompts, we computed the proportion of total observations in which interviewers followed the protocol. Overall, 56% of observations revealed that interviewers did not inquire about the distributive property, despite it being part of the protocol and worked examples. Figures 3.1 and 3.2 illustrate an instance where an interviewer omitted mentioning the distributive property (Figure 3.1) and another where an interviewer prompted the student on it (Figure 3.2). When students were questioned about the distributive property, they indicated familiarity 64% of the time, with 57% of sixth-grade observations demonstrating awareness.

Figure 3.1

Student A's Interview Script

Student A's Interview	Problems
Interview: What is Chloe's mistake? **Student A**: Chloe's mistake was...10 x 20, plus 8. **Interviewer:** mhm **Student A**: They were supposed to do 20 plus 8. **Interviewer**: First! right? Awesome and have you ever heard of PEMDAS? **Student A**: No **Interviewer**: No? So PEMDAS is like a little phrase, right? And it helps us remember the order of operations. So first is parentheses, which are these things here. And what's in the parentheses? **Student A**: 20 plus 8. **Interviewer**: Awesome, so after the parentheses, you would do exponents, then multiplication, then division, then addition, then subtraction. **Student A**: mhm **Interviewer**: Alright. So you're right! The parentheses come first, so that's what Chloe should have done first. **Interviewer:** So if she followed the correct order of operations, what would she have done? **Student A**: Um, 30, wait no 20 plus 8 **Interviewer**: Yup, so 20 plus 8 equals **Student A**: 28!	

Notes. In this excerpt of Student *A*'s interview, we can see that the interviewer did not prompt the student about the distributive property, but rather focused on the order of operations.

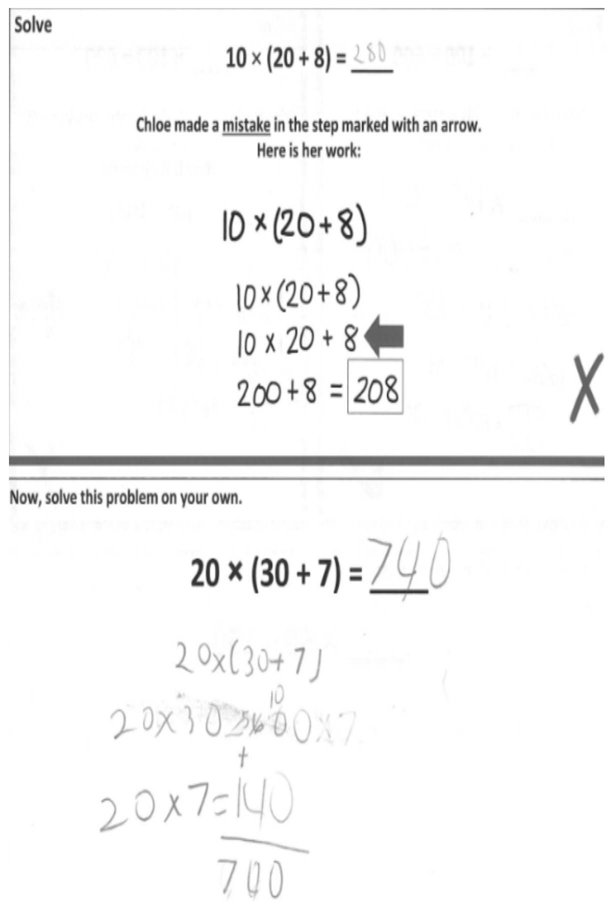
Figure 3.2

Student B's Interview Script

<table>
<tr><th>Student B's Interview</th><th>Problems</th></tr>
<tr><td>

Interviewer: What is Chloe's mistake?
Student B: Um, is there is not part of this? Like line? Over this? This line like over these two I think?
Interviewer: You said, wait can you repeat that please?
Student B: mhm, like this, like this kind of lines, she's supposed to put them over there, you have to put them on.
Interviewer: Okay, I see what you're saying. So today we're going to be focused on the distributive property. So the student, so we're working on distributive property, and it looks like this student did not use the distributive property correctly. Do you know what the distributive property is?
Student B: Um no.
Interviewer: So the distributive property is a property of multiplication when you're multiplying a number and an addition problem in parentheses, like this one, you can break up the addition problem and multiply the individual numbers. For example, if the problem is 2×(3+5), you can do 2×3, and then you can do 2×5.
Student B: Ok
Interviewer: Yes, so did they correctly use the distributive property in this case?
Student B: No

</td><td>



</td></tr>
</table>

Notes. In this excerpt of Student B's interview, we can see that the interviewer did prompt the student about the distributive property.

Discussion

We first examined if students used the distributive property to solve their "Your Turn" problems. Most of the students relied on intuitive approaches (e.g., long multiplication, successive addition, or following the order of operations [Mehta & Gawali, 2009]). The limited use of the distributive property, despite its prior introduction, aligns with Siegler's (1988) strategy choice model, wherein multiple strategies coexist and compete for selection across development. Students' reliance on intuitive approaches reflects the ongoing process of strategy evolution, where newer strategies compete with established ones. The gradual adoption of newer strategies, like the distributive property, hinges on increasing familiarity, effectiveness, and comprehension before implementation, aligning with self-regulation principles.

Additionally, we investigated if interviewers followed prompts to inquire about the distributive property. In most (56%) of the interviews, interviewers deviated from the script, neglecting to inquire about the distributive property. These results underscore the importance of reinforcing foundational concepts in teacher preparation. Moreover, it aligns with Ball (1988), Lampert (1986), and Rathouz (2011), emphasizing the importance of teachers' content knowledge. This also aligns with Siegler's (1988) notion that strategy choices become more adaptive with experience (Fazio et al., 2016). Given these findings, a future direction for both studies involves experimental manipulation to examine whether explicit prompts influence students' use of the distributive property.

Surprisingly, exposure to the distributive property did not appear to influence its use. Students often defaulted to familiar, intuitive approaches, highlighting the challenge of integrating new strategies. Siegler's (1996) overlapping waves theory suggests that children hold multiple problem-solving strategies simultaneously. The choice of a particular strategy depends on the nature of the problem, the learner's experience, and the specific context (Fazio et al., 2016; Siegler 1996), suggesting strategy improvement is highly context-dependent, with old strategies coexisting with new ones (McNeil & Alibali, 2005). The limited use of the distributive property in the present study might stem from insufficient grasp or exposure, a notion supported by Hurst and Hurrell (2018) and Hemi et al. (2021), who emphasize the role of comprehension in strategy adoption.

The study's limitations include a restricted sample of problems, which may not fully capture the range of students' problem-solving strategies across different problem types, and the potential influence of instructional interviewers' omissions on student choices. Furthermore, the variation in problems across grade levels limits the conclusions we can draw about developmental changes in understanding the distributive property. Nevertheless, the findings highlight the need for improved teacher education and suggest the utility of exploring metacognition-promoting prompts, questions eliciting reflection on problem-solving steps (Berthold et al., 2007), and direct measures of cognitive reflection in future research. Adding these prompts may provide a better rationale for making decisions to omit or include specific questions for specific students.

Study 2

The findings from Study 1 revealed a general overlooking of the distributive property by both students and instructional interviewers. Despite exposure to worked examples featuring the distributive property, students tended to rely on more familiar, intuitive strategies like long multiplication or PEMDAS. This tendency was further reinforced as interviewers often missed opportunities to highlight the distributive property. However, it is worth noting that students in that study were never *required* to use the distributive property, and all the “Your Turn” problems could be solved correctly using other established methods.

As mentioned above, conclusions about grade-level differences were challenging due to the grade-specific problems in the dataset. On some problems, there was not a clear advantage to using the distributive property over PEMDAS (e.g., 20×(30 + 7) = __ and 9×(10 + 5) = __), and on others it may have even been disadvantageous to use the distributive property over PEMDAS (e.g., 3×(6 + 4) and 3×(4 + 11)). Higher double-digit values like 20×(30 + 7) require additional steps in problem solving even with PEMDAS, possibly leading to the use of a backup strategy like the standard multiplication algorithm. Simpler problems like 3×(6 + 4) = 3×10 might be more straightforward with the order of operations, as students could complete the multiplication through direct retrieval (see Siegler[1988] for students’ strategy choice). The sixth-grade "Your Turn" problem 6×20.5 = __ was the only case where the distributive property was explicitly advantageous, and it is the problem that elicited the greatest use of the distributive property, perhaps suggesting that students may choose adaptively among strategies after exposure to a particular strategy (Siegler, 1996).

To gain more information about students’ use of the distributive property, we examined another dataset with distributive property items. This dataset contained item-level responses on Kirkland et al.’s (2024) brief assessment of mature number sense, encompassing three problems for grades 6-8 that require understanding the distributive property. Unlike the previous dataset, these problems were consistent across all three grades, enabling us to explore potential developmental changes in distributive property usage. Moreover, problems on the assessment have an incentive structure to guide students towards using the distributive property– an approach likened to offering “both a carrot and a stick” (cf. Siegler & Crowley, 1991). The "carrot" here is the relative ease with which problems can be solved with the distributive property. This approach simplifies the process making it more straightforward compared to computational methods. The "stick" is represented by the inherent difficulty in applying established strategies like PEMDAS and long multiplication within the given time constraints. Although long multiplication is technically an application of the distributive property, employing this standard algorithm without initially partitioning numbers through the distributive property might be more time-consuming.

For example, solving problems that involve the distributive property, like 6×24 = (6 ×__) + (6×4), using the standard algorithm may yield 144, but students could still be uncertain about the number for the blank. Although the standard algorithm isn't always lengthier, understanding the distributive property can simplify problems in specific circumstances. Hurst and Huntley (2020) note that the distributive property serves as a key to better comprehend the standard algorithm, especially since students inadvertently apply the standard algorithm but might benefit from a clearer understanding of number partitioning when using the distributive property. According to Kinzer and Stanford (2013), using the distributive property further enhances the understanding of multiplication. Thus, the existing dataset's problems provide an excellent source of data for investigating students' use of the distributive property across middle school grades. However, it is important to note that the data were not originally collected to study factors influencing students' use of the distributive property, prompting an exploratory approach to examining the potential factors based on the data available.

Method

Participants

The data set contained all assessments taken at the first time point of Kirkland's (2022) longitudinal study of middle schoolers' mature number sense. Participating students completed the assessments during two sessions in the fall of 2021 (August-October). Participants were 131 students from grades 6-8 (equivalent to ages 11-14 in the US). Two of the students did not complete all measures, so their responses are only included in analyses that include the measures they completed. Students in the data set were recruited from the greater South Bend community using invitation letters through schools and community partners, online advertisements, university listservs, and the research lab's email list. Participating students completed the assessments after school either at their middle school or in a room in a university research lab. Table 2 presents student demographics.

Table *2*

Student Demographics for Participants in the Data Set

Variable	*n*	%
Grade Level		
6th	46	35
7th	44	34
8th	41	31
Self-identified gender		
Boy	64	49
Girl	59	45
Prefer not to say	8	6
Race/ethnicity		
American Indian or Alaskan Native	1	1
Asian	4	3
Black or African American	10	8
Hispanic or Latine	7	5
White	86	66
Multiracial	17	13
Other races/ethnicities	6	5

Note. This table summarizes demographic statistics for Study 2 participants.

Data Collection

To maintain transparency, we included all measures used in the Kirkland (2022) longitudinal study in this data set (Weston et al., 2019). This included Kirkland et al.'s (2024) Brief Assessment of Mature Number Sense along with several additional measures of mathematics knowledge (grade-level mathematics achievement, a rational numbers measure, addition fluency), domain-general skills (cognitive reflection as measured by the developmental version of the cognitive reflection test [CRT-D], cognitive flexibility as measured by the Dimensional Change Card Sort [DCCS] Task), verbal fluency, and mathematical mindset variables from the Panorama (valuing of math, math learning strategies, and math mindset). A detailed description of all variables as well as their correlation matrixes and heatmaps are presented in Kirkland (2022). We provide a full correlation table of all variables in Table 6 for readers who may be interested. Here, we provide a brief description of each measure and then present the descriptive statistics for all variables in the data set in Table 3.

Brief Assessment of Mature Number Sense (Kirkland, 2022). The Brief Assessment of Mature Number Sense is an electronic, 24-item multiple-choice test designed to measure "individual students' tendency to *make sense of numerical situations* and use a rich conceptual understanding of numbers and operations to *flexibly* solve problems" (Kirkland, 2022). As mentioned above, this measure is useful for learning more about students' use of the distributive property because each item has a time limit of 60 seconds and students are not allowed to use paper or pencil. Kirkland (2022) and colleagues (2024) have provided evidence, including think-alouds and interviews, suggesting that the time limit precludes students from solving the items correctly using their more familiar procedural strategies. Problems included in the present study required students to apply the distributive property accurately for a successful solution. The brief assessment of mature number sense has validity evidence (Kirkland et al., 2024) and adequate reliability (α = .88).

Massachusetts Comprehensive Assessment System Grade-Level Mathematics Test (MCAS, 2019). Students completed the released 2019 MCAS paper test appropriate for their grade level. This is a freely available standardized test designed to assess student proficiency with grade-level mathematics standards. Student scores were converted to percent correct because it is an assessment of grade level standards and the maximum possible correct differs by grade level (19 for 6^{th}, 15 for 7^{th}, 16 for 8^{th}). Students had no time limit to complete each section of the test and could use scratch paper but not calculators.

Rational Numbers Measure (Powell, 2014). This is a 35-item paper and pencil test of students' skill at solving problems involving fractions, decimals, and percentages. Students are asked to perform the four operations with both fractions and decimals, find common denominators of fractions, generate equivalent fractions, and connect representations of rational numbers (e.g., "Convert 2.08 to a percentage"). Students worked for 20 minutes or until they finished. They could use scratch paper but not calculators. Students received one point for each correct response.

Addition Fluency (Geary et al., 1996). This measure included all combinations of single-digit addition facts with the numbers 1-9. The order of problems was randomized initially and then kept standard for all students. Students solved as many problems as they could in one minute on a computer. Students received one point for each correct response.

Cognitive Reflection Test - Developmental Version (CRT-D, Young et al. 2018). This test consists of eight non-numeric, free-response cognitive reflection items that measure an individual's "tendency to override an intuitive response that is incorrect and engage in reflection that leads to a correct response" (Young & Shtulman, 2020a, p. 1). Higher CRT-D scores have been shown to correspond to deeper conceptual understanding in both science and math (Young & Shtulman, 2020a, 2020b). Students were presented one question at a time and did not have a time limit to respond. They typed in their answer to each item. Scores were calculated according to the answer key provided by Young and Shtulman (2020a).

Dimensional Change Card Sort Task from the NIH Toolbox Cognition Battery (DCCS, Zelazo et al., 2013). This task measures cognitive flexibility, a component of executive function. Students are shown a series of cards and are told to match them based on either color or shape, with one practice block and a total of 30 mixed test trials. Scores were automatically generated by the program to incorporate accuracy and response time. This study uses standardized scores adjusted for age (M = 100, SD = 15).

Verbal Fluency (Snyder & Munakata, 2010; Weckerly et al., 2001). This task is a standard neuropsychology assessment that asks students to name as many words as they can in a minute that belonged to a specific category: first animals, then words that begin with the letter "F." Responses are recorded, transcribed, and then coded for the total number of unique items listed.

Survey of Students' Beliefs about Mathematics (Panorama Education, 2015). This survey instrument is a free and open-source resource for high-quality survey implementation with 3^{rd}-12^{th} grade students. The surveys went through a rigorous development process with large-scale administration and small-scale in-depth student interviews. Panorama (2015) provides evidence for the validity and reliability of scores from each of the 10 scales on the student survey. Students respond to a series of questions using a 5-point Likert scale (e.g., “not at all likely” to “extremely likely”). The data set included three scales: Valuing of Math (reflects how useful math is, how much the student identifies with math), Math Learning Strategies (reflects the student’s flexibility and persistence in mathematics problem solving), and Math Mindset (reflects how malleable the students believe math ability is).

Data Collection Procedures in Original Study

Participants completed the eight measures across two sessions. The measures were presented in a fixed order. The order was chosen so that the measures that require individual administration (e.g. verbal fluency and the Dimensional Change Card Sort Task) would be placed after measures with the greatest variability in completion time. “Filler tasks” (e.g. crosswords with no mathematical content) were inserted to allow for flexibility in pacing and timing between tasks. In the first session, participants completed the measures in the following order: MCAS part A, verbal fluency, the Brief Assessment of Mature Number Sense, and MCAS part B. The session lasted 45-60 minutes. Approximately one week later, participants returned for a second session in which they completed measures in the following order: the Survey of Students’ Beliefs about Mathematics, the Rational Numbers Measure, the Dimensional Change Card Sort Task, Addition Fluency, and Cognitive Reflection Test - Developmental Version. This session lasted 35-45 minutes.

Table 3

Descriptive Statistics for All Measures in the Existing Data Set

Measure	*M*	*SD*	Median	Min	Max	Range
Math Knowledge						

Mature Number Sense	14.23	5.68	14	4	24	20
MCAS	49.94	24.24	50	6.25	94.74	88.49
Rational Number	9.74	7.21	8	0	29	29
Addition Fluency	19.37	6.80	18	6	46	40
Cognitive Skills						
DCCS (Age-Adjusted)	106.22	19.44	106	63	146	83
CRT-D	5.32	1.68	6	1	8	7
Verbal Fluency	29.18	7.45	29	11	48	37
Math Attitudes						
Valuing of Math	3.46	0.79	3.40	1.60	5.00	3.40
Math Learning Strategies	3.60	0.62	3.60	2	5	3
Math Mindset	3.47	0.71	3.50	1.33	5.00	3.67

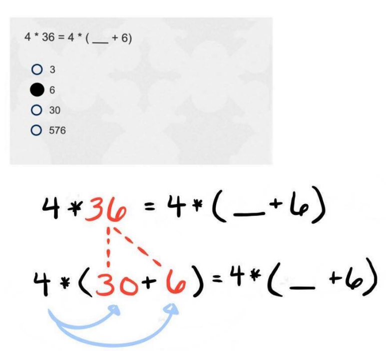
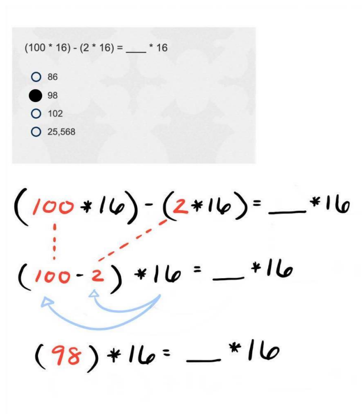
Note. Total sample size was 129 for Rational Number, Addition Fluency, CRTD, DCCS, Math Mindset; 130 for Valuing of Math and Math Learning Strategies; and 131 for MNS, MCAS, and Verbal Fluency.

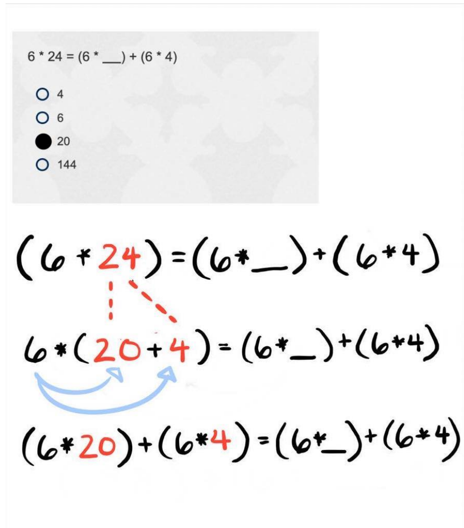
Procedure and Coding

Upon acquiring this data, we reviewed the item-level source code to identify problems pertaining to the distributive property. From the 24 items in the larger mature number sense assessment, three items necessitated an understanding of the distributive property, detailed in Table 4. The table also features worked examples illustrating potential problem-solving approaches. Notably, the presented solutions do not directly replicate students' work, as these items were presented as multiple-choice questions, hindering direct observation of individual responses.

Table 4

Items and Worked Examples from the Brief Assessment of Mature Number Sense Involving the Distributive Property

Item	Multiple Choice Answers (filled) and Worked Examples
4 * 36 = 4 * (___ + 6)	
(100 * 16) – (2 * 16) = ___ * 16	

6 * 24 = (6 * __) + (6 * 4)	

Next, we extracted responses to these items to determine students' solutions. Notably, the first item serves as a bridging item in the larger assessment to link the elementary and middle school versions of the assessment, and the other two items are exclusive to the middle school form. Given its bridging nature, we expected middle school students to perform best on that item relative to the other two items. Two independent researchers confirmed the descriptives and correlations of the variables within the data set. However, our primary interest remained to address our questions related to the distributive property.

Given that this was an existing data set with pre-reported demographic information, we controlled for demographics using available categories. Due to the dataset's limited size, we dichotomously coded race/ethnicity as historically underserved or not (1 = yes, including Black/African American, Hispanic/Latine, American Indian/Alaskan Native, and Multiracial; 0 = no, including Asian, White, and Other). Results remain consistent even when "Other races/ethnicities" is recategorized with the "yes" group. Considering literature that indicates some advantages for boys in STEM fields (e.g., Gallagher et al., 2000; Hornburg et al., 2017), gender was coded as identifying as a boy or not (1 = boy, 0 = not a boy), with stable results when extracting the "prefer not to say" category and establishing it as a third analytical category. We recognize the sensitivity and complexity of these categorizations, aiming to acknowledge potential sources of individual differences correlated with mathematics performance, as suggested by previous research.

Results

Table 5 presents the distribution of student responses on each item. The first item was easier than the second and third items across all grade levels, as confirmed by one-sample t-tests. The mean difference between the first item and the second item ($M = 0.40$) was significantly greater than zero, $t(131) = 7.23$, $p < .001$, suggesting the first item was easier than the second. Similarly, the mean difference between the first item and the third item ($M = 0.27$) was significantly greater than zero, $t(131) = 5.40$, $p < .001$, highlighting the first item's greater simplicity. Additionally, evidence suggested that the mean difference between the second and third items ($M = -.12$) was significantly less than zero, $t(131) = -2.17$, $p = .032$, indicating that the second item was significantly more difficult than the third item.

Table 5

Percent of Students Who Provided Each Response by Grade Level

Item	*Multiple Choice Answers **(correct)***	*Grade 6 (%)*	*Grade 7 (%)*	*Grade 8 (%)*
4 * 36 = 4 * (___ + 6)	6	11	7	7
	3	6.5	14	5
	30	**78**	**75**	**85**
	576	4	4.5	2
	NR, Timed Out	0	0	0
(100 * 16) – (2 * 16) = ___ * 16	86	13	14	15
	98	**50**	**32**	**37**

	102	28	50	42
	25,568	6.5	4.5	2
	NR, Timed Out	2	0	5
6 * 24 = (6 * __) + (6 * 4)	4	15	18	12
	6	24	20.5	15
	20	**52**	**43**	**61**
	144	4	14	10
	NR, Timed Out	4	4.5	2

Note: NR=no response.

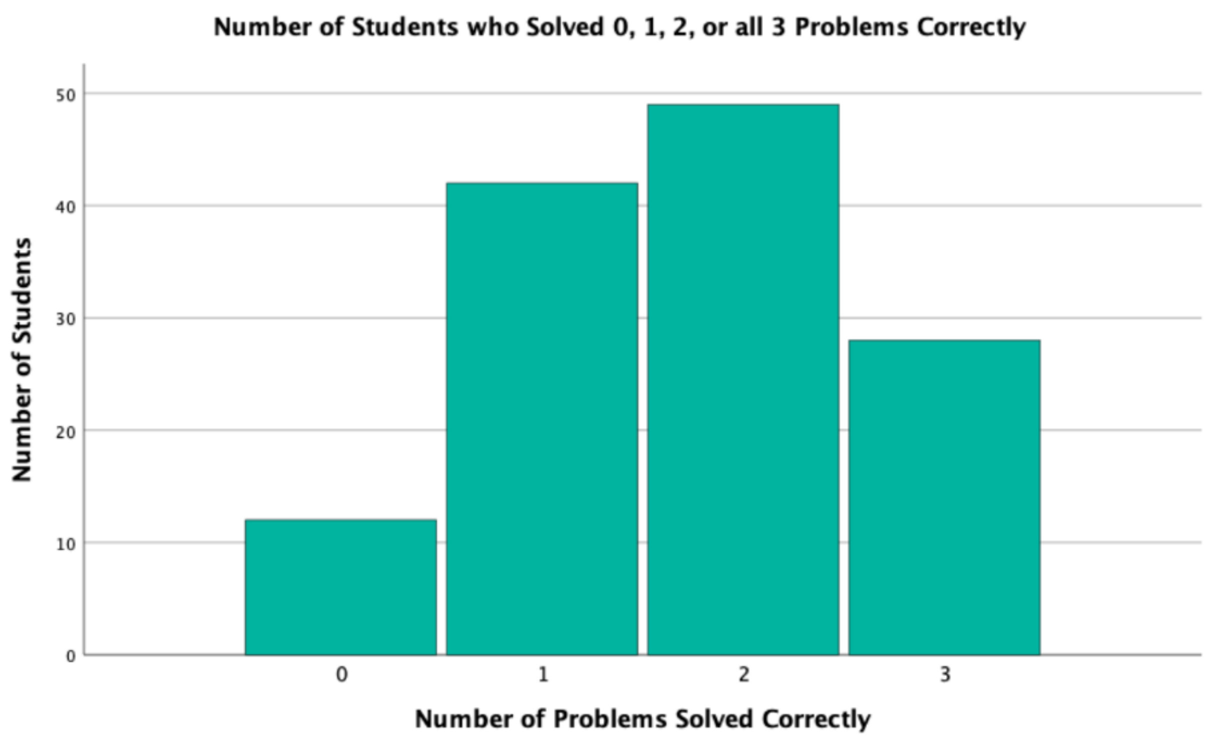
We initially considered using a composite "distributive property" score (number correct out of 3) to represent students' use of the distributive property. However, the reliability of this three-item measure was quite low (α = .30), which is not surprising given the limited number of items scored for correctness. We then examined correlations among the three items. Items 1 and 3 were significantly correlated, $r = .19$, $p = .03$, but correlations between Item 1 and Item 2, $r = .028$, $p = .754$, and Item 2 and Item 3, $r = .156$, $p = .074$ were not statistically significant, suggesting that Item 2 might involve a different underlying construct. This composite "distributive property" score did not correlate with grade level, $r = .006$, $p = .948$.

As the prerequisites for a continuous composite measure were unmet, we took a holistic approach, examining the three problems and contemplating the evidence needed to demonstrate a student's robust understanding and use of the distributive property. Each problem offers four possible responses, yielding a 25% chance of a correct answer. While a student might guess correctly on one (42.19% chance) or two problems (14.06% chance), the probability of a perfect score through guessing on all three plummets to 1.56%. Thus, a perfect score became a reliable indicator of a student's understanding and use of the distributive property, offering a conservative and credible benchmark for analysis.

Figure 4 presents a frequency plot of performance on the three distributive property items. Most students (91%) solved at least one correctly, and approximately 21% solved all three correctly. We did not see evidence that grade level correlates with achieving a perfect score on problems involving the distributive property, $r = -.067$, $p = .447$, suggesting that factors beyond typical grade-level mathematics knowledge or general development across ages 11-14 influenced distributive property use.

Figure 4

Number of Students Who Solved 0, 1, 2, or all 3 Problems Correctly



Next, we aimed to predict achieving a perfect score on the distributive property items using the variables available in the dataset. The dataset included demographics (grade level, race/ethnicity, self-reported gender), measures of mathematics knowledge (grade-level mathematics achievement, rational number assessment, addition fluency), domain-general cognitive skills (cognitive reflection, cognitive flexibility), verbal fluency, and mathematical attitude variables from the Panorama (valuing of math, math learning strategies, and math mindset). Table 6 provides zero-order and point-biserial correlations between achieving a perfect score on the distributive property items and each variable in the dataset.

Table 6

Correlations Between Achieving a Perfect Score on the Distributive Property Items and Each of the Variables Available in the Data Set

Variable	*r*	*p*
Grade level (6-8)	-.067	.447
Identifying as a boy (yes = 1, no = 0)	.083	.351
Identifying as a member of an underrepresented racial/ethnic group (yes = 1, no = 0)	-.272	.002
Grade-level mathematics achievement (MCAS)	.391	< .001
Rational number performance	.328	<.001
Addition fluency	.221	.012
Cognitive reflection (CRT-D)	.363	<.001
Cognitive flexibility (DCCS)	.290	<.001
Verbal fluency	.263	.002
Valuing of math	.094	.285
Math learning strategies	.189	.031
Mathematical mindset	-.066	.456

Note. Identifying as a boy and identifying as a member of an underrepresented racial/ethnic group are dichotomous predictors, so these *r* values are point bi-serial correlations (r_{pb}).

To identify variables uniquely related to achieving a perfect score on the distributive property items, we used binomial logistic regression to predict the log odds of solving all three problems correctly. Predictors included identifying as a member of an underrepresented racial/ethnic group, grade-level math achievement, rational number assessment score, addition fluency, cognitive reflection, cognitive flexibility, verbal fluency, and math learning strategies (Table 7). Due to the varying scales of the predictors, we standardized all continuous predictors to facilitate interpretation. Results showed a significant and positive association between grade-level mathematics achievement and a perfect score on the distributive property items ($B = 0.740$, *Wald* $(1, N = 129) = 4.172$, $OR = 2.096$, $p = 0.03$), indicating that each one standard deviation increase in grade-level math achievement produces, on average, a 0.740 increase in the log-odds of a perfect score on the distributive property items. Cognitive reflection was the only other variable that significantly predicted a higher likelihood of achieving a perfect score on distributive property items ($B = 0.963$, *Wald* $(1, N = 129) = 4.844$, $OR = 2.62$, $p = 0.030$), indicating that each standard deviation increase in cognitive reflection produces, on average, a 0.963 increase in the log-odds of obtaining a perfect score on the distributive property items. No other factors were uniquely related to a perfect score on distributive property items. Similar conclusions were held in a robustness check using multinomial logistic regression to predict membership in categories 0 correct, 1 correct, 2 correct, or 3 correct.

Table 7

Logistic Regression Predicting Achieving a Perfect Score on the Distributive Property Items

Variables in the Data Set	*B*	Wald	OR	*p*
Identifying as member of an underrepresented racial/ethnic group (yes = 1, no = 0)	-1.379	2.603	0.252	0.107
Grade-level mathematics achievement (MCAS)	0.740	4.712	2.096	0.03
Rational number performance	-0.257	0.496	0.774	0.481
Addition fluency	0.344	0.977	1.410	0.323
Cognitive reflection (CRT-D)	0.963	4.844	2.620	0.028
Cognitive flexibility (DCCS)	0.513	2.521	1.670	0.112

Verbal fluency	0.158	0.304	1.171	0.581
Math learning strategies	-0.207	.400	0.831	0.527

Notes. *B* = coefficient for the log odds change per one standard deviation increase in the predictor. *Wald* = significance test statistic for each coefficient. *OR* = odds ratio for a one standard deviation change in the predictor. *P* = probability of observing the data, or something more extreme, if null hypothesis is true.

Discussion

This analysis used three problems incentivizing distributive property use, exploring students' application over middle school years. Results indicated varying difficulty levels across the three problems of interest: 4×36=4× (_+6) being easiest, followed by 6×24=(6 × _)+(6×4), then (100×16) – (2×16) = __× 16. As anticipated, the first problem served as a linking item across the elementary and middle school forms, but no predictions were made about the difficulty of the others. Notably, the problem requiring the identification and extraction of a common factor posed the greatest challenge for students, possibly due to students prioritizing operational execution over perceiving the underlying problem structure.

Unexpectedly, distributive property use showed no improvement across grade levels, prompting us to question the constructs involved in the understanding and use of the property. Our analysis identified two positive predictors: grade-level math achievement and cognitive reflection. The latter, cognitive reflection, gauges students' tendency to reflect on their thoughts (Young & Shtulman, 2020a, 2020b). Stronger reflective thinking correlated with better distributive property problem performance, aligning with the identified challenging problem requiring conceptual understanding beyond computation and supporting prior research by Young and Shtulman (2020a, 2020b) on cognitive reflection predicting conceptual understanding in mathematics.

General Discussion

We investigated students' understanding and application of the distributive property in solving multiplication problems, employing diverse methodologies to provide a multi-faceted view of the challenges and strategies in teaching and learning this property. Study 1 highlighted a crucial aspect: exposure alone doesn't guarantee consistent use, with students often leaning on familiar intuitive approaches. This challenge in assimilating new strategies without sufficient fluency aligns with the concept of "fluency-differentiated domain knowledge" (Schwartz & Bransford, 1998) and emphasizes the importance of persistent reinforcement during late elementary and early middle school years. This supports guidelines for presenting the distributive property (Ozgun-Koca & Hagan, 2021), and strategies spaced out learning (Cepeda et al., 2006; Ding & Li, 2010; National Research Council, 2004). The findings echo existing research underscoring the necessity for students to fully understand strategies before successful application (Hemi et al., 2021; Hurst & Hurrell, 2018).

Notably, during Study 1, instructional interviewers, including graduate students and preservice teachers, frequently deviated from the intended script. This deviation highlights the important role of metacognition in educational settings and the necessity to weave cognitive reflection scaffolding into both instructional design and teacher formation programs. That is, these prompts would be based on prior knowledge of the content, thus providing a rationale for why specific student prompting is necessary versus other kinds of robust content knowledge and Ball et al.'s (2005) argument that content knowledge alone prompts for other students. Such findings amplify Rathouz's (2011) emphasis on teachers having sufficient knowledge for effective teaching, requiring an understanding of student thinking and instructional design. Thus, diverging from the script may be necessary in certain contexts depending on students' knowledge.

Study 2 contributed insights into the varying difficulty levels of distributive property problems, its use across middle school, and predictors of students' distributive property application. The analysis identified two predictors: students' grade-level math achievement and cognitive reflection. The significance of cognitive reflection aligns with Study 1, suggesting that higher levels of cognitive reflection may aid students in moving beyond their initial intuitive strategies. These results imply potential interventions promoting reflective thinking to enhance conceptual understanding and distributive property application.

Limitations

Several limitations impact the generalizability of our findings, including the small sample size and the limited number of distributive property problems in both studies. Additionally, support given to students (in instructional interviews) in case of incorrect solutions prevents a natural observation of the problem-solving evolution. Furthermore, omissions in explicitly mentioning the distributive property during interviews raise the possibility of influencing students' chosen solution methods. Despite these constraints, the smaller problem set allows for a detailed examination of process data, offering rich insights.

Future Directions

Future studies should address these limitations by diversifying the problems and participants while continuing to focus on detailed process data. This includes incorporating test items that specifically encourage distributive property use, such as multi-digit multiplication problems (e.g., 98×18), problems involving decimals, and problems in equation or open-ended formats. Open-ended questions can prompt explicit explanations of distributive property application, offering a valuable exploration avenue. Additionally, students must not only write expressions in expanded form but also be able to explain the conceptual equivalence behind them. Visual models, like the area model, can effectively illustrate this equivalence and reinforce understanding of problem structures.

As for diverse participants, it is necessary to emphasize the need for a comprehensive range of students with varying skills and backgrounds to understand the adaptability of distributive property application across different contexts, revealing challenges and deepening insights into students' conceptual grasp. Furthermore, this can enrich our understanding of mathematical cognition and problem-solving strategies that benefit different learners. This approach can provide insights into how students and teachers engage with the property, the effectiveness of teacher prompts in eliciting its use, and whether students readily adopt this approach.

Future research should explore diverse curricular approaches' impact on students' understanding and use of the distributive property. Study 1, in a small private school, featured a STEM-focused curriculum with smaller class sizes in a university setting, adding unique insights. In contrast, Study 2, with a broader recruitment from a Midwestern city, encompassed various educational backgrounds. This contrast between the intimate, specialized environment of interviews in Study 1 and the larger-scale assessment-focused setting of Study 2 allowed us to examine the use of the distributive property across different educational contexts. This comparison highlights the need for further investigation into how different curricular approaches and educational environments influence students' distributive property understanding and application.

Conclusion

These studies provide valuable insights into students' problem-solving strategies and their familiarity with the distributive property. They highlight the potential role of cognitive reflection in students' distributive property performance, aligning with prior research revealing cognitive reflection as a predictor of students' conceptual understanding in mathematics and science (Young & Shtulman, 2020a). Study 1 underscores the prevalence of students preferring familiar, intuitive methods over intended correct approaches, emphasizing the need to support metacognition and cognitive reflection in instructional design. The findings reveal gaps in interviewer training, as they did not sufficiently emphasize the distributive property during observations, underscoring the need for better preparing educators and research assistants in reinforcing fundamental math concepts.

In summary, these studies highlight the ongoing need for research and improvements in mathematics education, indicating the potential need to revisit the timing and depth of introducing mathematical concepts. The findings highlight the potential role of cognitive reflection in understanding and applying the distributive property, laying the groundwork for creating teaching resources to further investigate its causal influence and spur continued exploration of cognitive processes involved in understanding the distributive property, with implications for instructional approaches and teacher preparation programs.

References

AUTHORS. (under review). Leveraging common mathematical errors to support understandings of equivalence and operations.

Ball, D. L. (1988). Unlearning to teach mathematics. *For the learning of mathematics*, *8*(1), 40-48. https://www.jstor.org/stable/40248141

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(1), 14-17, 20-22, 43-46. http://hdl.handle.net/2027.42/65072

Barbieri, C. A., Miller-Cotto, D., & Booth, J. L. (2019). Lessening the load of misconceptions: Design-based principles for algebra learning. *Journal of the Learning Sciences, 28*(3), 381–417. https://doi.org/10.1080/10508406.2019.1573428

Berthold, K., Nückles, M., & Renkl, A. (2007). Do learning protocols support learning strategies and outcomes? The role of cognitive and metacognitive prompts. *Learning and Instruction, 17*(5), 564-577. https://doi.org/10.1016/j.learninstruc.2007.09.007

Brown, S.I. (2013). *Insights Into Mathematical Thought: Excursions with Distributivity.* National Council for Teachers of Mathematics.

Bruner, J. (1977). *The process of education*. Cambridge, MA: Harvard University Press. (Original work published in 1960)

Byrnes, J. P. (2008). Piaget's cognitive-developmental theory. *Encyclopedia of Infant and Early Childhood Development*, *87*, 543-552.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Heinemann.

Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *Zentralblatt für Didaktik der Mathematik*, *37*(1), 53-59. https://doi.org/10.1007/BF02655897

Cepeda, N. J., Pashler, H., Vul, E., Wixted, J. T., & Rohrer, D. (2006). Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin*, *132*(3), 354. https://doi.org/10.1037/0033-2909.132.3.354

Chen, T., & Cai, J. (2020). An elementary mathematics teacher learning to teach using problem posing: A case of the distributive property of multiplication over addition. *International Journal of Educational Research*, *102*, 101420. https://doi.org/10.1016/j.ijer.2019.03.004

Clements, D. H. & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6(2)*, 81-89.

Ding, M., & Li, X. (2010). A comparative analysis of the distributive property in US and Chinese elementary mathematics textbooks. *Cognition and Instruction*, *28*(2), 146-180. http://www.jstor.org/stable/27806347

Dunlosky, J., Morris, B., & Thompson, C. (2020). *Numerical understanding mentored by expert researchers (NUMBERs) workshop.* Kent State University, Kent, OH. https://www.nsf.gov/awardsearch/showAward?AWD_ID=1941132

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition,* *42*(1), 1. http://dx.doi.org/10.1037/xlm0000153

Gallagher, A. M., De Lisi, R., Holst, P. C., McGillicuddy-De Lisi, A. V., Morely, M., & Cahalan, C. (2000). Gender differences in advanced mathematical problem solving. *Journal of Experimental Child Psychology, 75*, 165–190. https://doi.org/10.1006/jecp.1999.2532

Geary, D. C., Bow-Thomas, C. C., Liu, F., & Siegler, R. S. (1996). Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development, 67*(5), 2022-2044. https://doi.org/10.2307/1131607

Hecht, S. A. (1999). Individual solution processes while solving addition and multiplication math facts in adults. *Memory & Cognition*, *27*, 1097-1107. https://doi.org/10.3758/BF03201239

Hemi, S. S., Sukidin, S., & Prastiti, T. D. (2021). The differences of the students mathematics learning outcomes under the implementation of the long multiplication and Chinese multiplication. *Pancaran Pendidikan*, *10*(4). 10.25037/pancaran.v10i4.367

Hornburg, C. B., Rieber, M. L., McNeil, N. M. (2017). An integrative data analysis of gender differences in children's understanding of mathematical equivalence. *Journal of Experimental Child Psychology, 163,* 140-150. https://doi.org/10.1016/j.jecp.2017.06.002

Hurst, C., & Huntley, R. (2020). Distributivity, Partitioning, and the Multiplication Algorithm. *Journal of Research and Advances in Mathematics Education, 5*(3), 231-246.

Hurst, C., & Hurrell, D. (2018). Algorithms are useful: Understanding them is even better! *Australian Primary Mathematics Classroom*, *23*(3), 17-21.

Izsák, A. (2004). Teaching and learning two-digit multiplication: Coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking and Learning*, *6*(1), 37-79. https://doi.org/10.1207/s15327833mtl0601_3

Kinzer, C. J., & Stanford, T. (2013). The distributive property: The core of multiplication. *Teaching Children Mathematics*, *20*(5), 302-309. https://doi.org/10.5951/teacchilmath.20.5.0302

Kirkland, P. K. (2022). *Characterizing mature number sense and its association to other constructs in middle school students*. [Doctoral dissertation, University of Notre Dame]. ProQuest Dissertations & Theses Global. https://doi.org/10.7274/vq27zk55h02

Kirkland, P. K., Cheng, Y., & McNeil, N. M. (2024). A Validity Argument for a Brief Assessment of Mature Number Sense. *Journal for Research in Mathematics Education, 55*(1), 51-67. https://doi.org/10.5951/jresematheduc-2022-0071

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction*, *3*(4), 305-342. https://www.jstor.org/stable/3233589

Massachusetts Department of Elementary and Secondary Education Assessment Development Committee Members (2019). "Massachusetts Comprehensive Assessment System." https://www.doe.mass.edu/mcas/testitems.html

Matney, G. T., & Daugherty, B. N. (2013). Seeing spots and developing multiplicative sense making. *Mathematics Teaching in the Middle School*, *19*(3), 148-155 . https://doi.org/10.5951/mathteacmiddscho.19.3.0148

Mehta, P., & Gawali, D. (2009). Conventional versus vedic mathematical method for hardware implementation of a multiplier. In *2009 International Conference on Advances in Computing, Control, and Telecommunication Technologies* (640-642). https://doi.org/10.1109/ACT.2009.162.

National Council of Teachers of Mathematics [NCTM] (2000*). Principles and standards for school mathematics*. Reston, VA: The National Council of Teachers of Mathematics.

McNeil, N. M. (2014). A change-resistance account of children's difficulties understanding mathematical equivalence*. Child Development Perspectives, 8*, 42-27.

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Washington DC: Authors.

National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

National Research Council. (2004). *On evaluating curricular effectiveness: Judging the quality of K-12 mathematics evaluations*. National Academies Press.

Ozgun-Koca, S. A., & Hagan, K. (2021). GPS: Distributive property throughout the grades. *Mathematics Teacher: Learning and Teaching PK-12*, *114*(6), 483-485. https://doi.org/10.5951/MTLT.2021.0087

Panorama Education. (2015). https://www.panoramaed.com/

Pennycook, G., Cheyne, J. A., Koehler, D. J., & Fugelsang, J. A. (2016). Is the cognitive reflection test a measure of both reflection and intuition? *Behavior Research Methods*, *48*, 341-348. https://doi.org/10.3758/s13428-015-0576-1

Powell, S. R. (2014). In S. R. Powell (Ed.), *Rational numbers measure*. The University of Texas at Austin.

Rathouz, M. (2011). 3 ways that promote student reasoning. *Teaching Children Mathematics*, *18*(3), 182-189. https://doi.org/10.5951/teacchilmath.18.3.0182

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175-189. https://doi.org/10.1037/0022-0663.91.1.175

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, 27, 587-597. https://doi.org/10.1007/s10648-015-9302-x

Russell, S. J. (2000). Developing computational fluency with whole numbers. *Teaching Children Mathematics, 7*(3), 154-158.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction, 16*(4), 475-522. https://doi.org/10.1207/s1532690xci1604_4

Siegler, R. S. (1988). Strategy choice procedures and the development of multiplication skill. *Journal of experimental psychology: General, 117*(3), 258. https://doi.org/10.1037/0096-3445.117.3.258

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking.* Oxford University Press.

Siegler, R. S., & Crowley, K. (1991). The microgenetic method: A direct means for studying cognitive development. *American Psychologist, 46*(6), 606 –620. https://doi.org/10.1037/0003-066X.46.6.606

Silla, E. M., Hornburg, C. B., Kloser, M., & McNeil, N. M. (2020). *Research-based teaching practices for improving students' understanding of mathematical equivalence have not made it into elementary classrooms.* In S. Denison, M. Mack, Y. Xu, & B. Armstrong (Eds.), Proceedings of the 42nd Annual Conference of the Cognitive Science Society (pp. 2937-2943). Austin, TX: Cognitive Science Society.

Snyder, H. R., & Munakata, Y. (2010). Becoming self-directed: Abstract representations support endogenous flexibility in children. *Cognition*, *116*(2), 155-167. https://doi.org/10.1016/j.cognition.2010.04.007

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, *36*(5), 404-411. https://doi.org/10.1016/j.cognition.2010.04.007

Vermeulen, N., Olivier, A., & Human, P. (1996, July). *Students' awareness of the distributive property.* In Proceedings of the 20th Conference for the Psychology of Mathematics Education (Vol. 4, pp. 4-379).

Vygotsky, L. S. (1978). *Mind in Society: the Development of Higher Psychological Processes.* Cambridge, MA: Harvard University Press.

Weckerly, J., Wulfeck, B., & Reilly, J. (2001). Verbal Fluency Deficits in Children With Specific Language Impairment: Slow Rapid Naming or Slow to Name? *Child Neuropsychology, 7*(3), 142–152. https://doi.org/10.1076/chin.7.3.142.8741

Weston, S. J., Ritchie, S. J., Rohrer, J. M., & Przybylski, A. K. (2019). Recommendations for Increasing the Transparency of Analysis of Preexisting Data Sets. *Advances in Methods and Practices in Psychological Science, 2*(3), 214–227. https://doi.org/10.1177/2515245919848684

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *17*(2), 89-100. https://doi.org/10.1111/j.1469-7610.1976.tb00381.x

Young, A. G., & Shtulman, A. (2020a). Children's cognitive reflection predicts conceptual understanding in science and mathematics. *Psychological Science, 31*(11), 1396–1408. https://doi.org/10.1177/0956797620954449

Young, A. G., & Shtulman, A. (2020b). How children's cognitive reflection shapes their science understanding. *Frontiers in Psychology,* 11, 1247. https://doi.org/10.3389/fpsyg.2020.01247

Young, A. G., Powers, A., Pilgrim, L., & Shtulman, A. (2018*). Developing a cognitive reflection test for school-age children.* In Proceedings of the 40th Conference of the Cognitive Science Society (1232–1237).

Zelazo, P. D., Anderson, J. E., Richler, J., Wallner-Allen, K., Beaumont, J. L., & Weintraub, S. (2013). II. NIH Toolbox Cognition Battery (CB): Measuring executive function and attention. *Monographs of the Society for Research in Child Development, 78*(4), 16–33. https://doi.org/10.1111/mono.12032

Zhang, D., Ding, Y., Lee, S., & Chen, J. (2017). Strategic development of multiplication problem solving: Patterns of students' strategy choices. *The Journal of Educational Research*, *110*(2), 159-170. https://doi.org/10.1080/00220671.2015.1060928