THE HOMOTOPY LIE ALGEBRA OF A TOR-INDEPENDENT TENSOR PRODUCT

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Abstract. In this article we investigate a pair of surjective local ring maps $S_1 \leftarrow R \to S_2$ and their relation to the canonical projection $R \to S_1 \otimes_R S_2$, where S_1, S_2 are Tor-independent over R. Our main result asserts a structural connection between the homotopy Lie algebra of S, denoted $\pi(S)$, in terms of those of R, S_1 and S_2, where $S = S_1 \otimes_R S_2$. Namely, $\pi(S)$ is the pullback of (adjusted) Lie algebras along the maps $\pi(S_i) \to \pi(R)$ in a wide variety cases, including when the maps above have residual characteristic zero. Consequences to the main theorem include structural results on André–Quillen cohomology, stable cohomology, and Tor algebras, as well as an equality relating the Poincaré series of the common residue field of R, S_1, S_2 and S.

Introduction

Given a pair of Tor-independent modules over a commutative ring, there are strong connections between homological properties of the modules and their tensor product. Building upon work in [2, 4, 23, 33, 34], we focus on surjective local ring maps whose targets are Tor-independent over their common source; the minimal intersection rings introduced in [23] served as the starting place for the present investigation. The main result of this article establishes an especially strong connection between homotopical aspects of the ring maps and their tensor product.

Let R be a commutative noetherian local ring with residue field k. Let S_1, S_2 be quotients of R which are Tor-independent, that is, $\operatorname{Tor}_i^R(S_1, S_2) = 0$ for $i > 0$. With inspiration from [2] and setting $S = S_1 \otimes_R S_2$, this paper examines relationships between R, S_1, S_2 and S through the lens of their homotopy Lie algebras. We write $\pi(R)$ for the homotopy Lie algebra of R, which is a graded Lie k-algebra naturally associated to the ring R. We recall its construction in 1.5. These homotopy Lie algebras have been adopted from rational homotopy theory to local algebra by Avramov [5], and their importance is, in part, due to the ring-theoretic properties they encode; see, for example, [6, 10, 11, 15, 16, 17] as well as Corollary 5.2.

Our main result is Theorem A, below; before stating it we set notation and terminology. We write $\mathfrak{g} \times_{\mathfrak{h}} \mathfrak{g}'$ for the pullback along Lie algebra maps $\mathfrak{g} \to \mathfrak{h} \leftarrow \mathfrak{g}'$. Our theorem applies to surjective local maps of residual characteristic zero, as well as two large classes of ring maps having slightly technical definitions: *almost small* and *Gulliksen minimal* maps recalled in 3.1 and Definition 4.2, respectively. The former generalizes the small homomorphisms of Avramov [4] while the latter have

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appeared in some form in [3, 15, 20]; both contain the class of maps whose source is regular.

Theorem A. *Suppose $S_1 \xleftarrow{\varphi_1} R \xrightarrow{\varphi_2} S_2$ is a Tor-independent pair of surjective local ring maps with residue field k, and set $S = S_1 \otimes_R S_2$. Consider the following conditions:*

1. *k has characteristic zero;*
2. *at least one φ_i is almost small;*
3. *each φ_i is Gulliksen minimal.*

If any of the conditions above hold, then there is a naturally induced isomorphism of graded Lie algebras

$$\pi(S) \cong \pi(S_1) \times_{\pi(R)} \pi(S_2)\,.$$

Under more restrictive hypotheses, the conclusion of Theorem A can be asserted from two results of Avramov: first, when R is regular and a further assumption is imposed on the kernels of the φ_i in [2], and second, when one of the φ_i is assumed to be small in [4]. Both arguments require a careful analysis of certain spectral sequences while further relying on celebrated theorems of André, Milnor–Moore, and Sjödin in [1, 26, 32], respectively. Besides the greater generality of Theorem A, its proof is a direct, mostly self-contained one given by an examination of certain DG R-algebra resolutions of S_1, S_2 and S. The arguments of Avramov rely on the (anti-)equivalence of André, Milnor–Moore, and Sjödin, and invoke machinery that was developed in [2] and further refined in [4]. We direct the reader to Theorem 3.3 and Theorem 4.7, which in conjunction establish Theorem A; see also Remark 3.4.

The paper is organized as follows. Section 1 recalls preliminaries regarding DG algebras, especially minimal models and homotopy Lie algebras. In Section 2 we provide general consequences of Tor-independence and shared properties of the projections $R \to S_i$ and $S_i \to S$ by examining the fibers on the induced maps on minimal models; see Propositions 2.7, 2.8 and 2.10.

Sections 3 and 4 are the heart of the present article, as they contain the proofs of the main result above. Section 5 is the final section, which provides several applications of Theorem A including results regarding the structure of stable cohomology modules, Tor algebras and André–Quillen cohomology modules for Tor-independent maps; see Remark 5.4 and Corollaries 5.7 and 5.12, respectively. An application in a different direction is a numerical relationship on the Poincaré series of R, S_1, S_2 and S which is presented in Corollary 5.9, and stated below.

Theorem B. *Suppose $S_1 \xleftarrow{\varphi_1} R \xrightarrow{\varphi_2} S_2$ is a pair of surjective local ring maps with residue field k and Tor-independent targets, and let $S = S_1 \otimes_R S_2$. If at least one φ_i is almost small, then there is an equality of formal power series*

$$\mathrm{P}_k^S(t) = \frac{\mathrm{P}_k^{S_1}(t) \cdot \mathrm{P}_k^{S_2}(t)}{\mathrm{P}_k^R(t)}\,.$$

This theorem is of particular interest as it generalizes results from [2, 4], while its conclusion is, interesting enough, only established under one of the three specified conditions in Theorem A; cf. Question 5.10.

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1. DG algebras and homotopy Lie algebras

Throughout this section, let $(R, \mathfrak{m}, k)$ be a local ring and fix a surjective local homomorphism $\varphi\colon R \to S$. We recall the necessary background regarding certain semifree DG algebra resolutions as well as the homotopy Lie algebra; suitable references include [5, 7, 15].

By convention all DG algebras will be nonnegatively graded, strictly graded commutative and local. That last condition means that for a DG algebra A, the base ring A_0 is a commutative noetherian local ring, which we assume also has residue field k, and each $\mathrm{H}_i(A)$ is a finitely generated $\mathrm{H}_0(A)$-module; throughout A will denote such a DG algebra.

1.1. A *semifree extension* of R is a DG R-algebra $R[X]$ where $X = X_{\geqslant 1}$ is a graded set of variables consisting of exterior variables in each odd degree and polynomial variables in each even degree. By a slight abuse of notation we write $k[X]$ for the DG k-algebra $k \otimes_R R[X]$. We say $R[X]$ is a *minimal semifree extension* of R provided there is the containment $\partial(k[X]) \subseteq (X)^2$.

1.2. A *minimal model for* φ is a semifree extension $R[X]$ fitting into a factorization of φ as $R \to R[X] \xrightarrow{\tilde{\varphi}} S$ satisfying the following:

(1) $R[X]$ is a minimal semifree extension of R;
(2) $\tilde{\varphi}\colon R[X] \to S$ is a surjective quasi-isomorphism.

Moreover, minimal models always exist and are unique up to homotopy equivalence of DG R-algebras; see [7, Section 7.2] as well as [15, Section 2.7].

1.3. A *semifree* Γ*-extension of* A is a DG algebra $A\langle Y\rangle$ where $Y = Y_{\geqslant 1}$ is a graded set of variables consisting of exterior variables in each odd degree and divided powers variables in each positive even degree; see, for example, [7, Section 6.1].

Assume there is a surjection $A \to S$ and let $J = \mathrm{Ker}(A \to S)$. Following [7, Construction 6.3.1], an *acyclic closure of* S *over* A is a semifree Γ-extension $A\langle Y\rangle$ of A with $\mathrm{H}(A\langle Y\rangle) = S$ such that

(1) $\partial(Y_1)$ minimally generates $J_0 \mod \partial(A_1)$;
(2) $\{\mathrm{cls}(\partial(y)) \mid y \in Y_{n+1}\}$ minimally generates $\mathrm{H}_n(A\langle Y_{\leqslant n}\rangle)$ for $n \geqslant 1$.

Existence of an acyclic closure is the content of [20, Proposition 1.9.3], while uniqueness up to DG Γ-algebra isomorphism is established in [20, Theorem 1.9.5].

1.4. Let $A\langle Y\rangle$ be a semifree Γ-extension and M be a DG $A\langle Y\rangle$-module. An A-linear map $d\colon A\langle Y\rangle \to M$ is called an Γ*-derivation* if it satisfies

(1) $d(bb') = d(b)b' + (-1)^{|b||d|}bd(b')$ for all $b, b' \in A\langle Y\rangle$;
(2) $d(y^{(i)}) = d(y)y^{(i-1)}$ for all $y \in Y_{2i}$ and all $i \geqslant 1$.

We denote the collection of A-linear Γ-derivations from $A\langle Y\rangle$ to M by $\mathrm{Der}^{\gamma}_A(A\langle Y\rangle, M)$.

1.5. A graded Lie algebra over k is a graded k-vector space equipped with a k-bilinear pairing, called its Lie bracket, and a squaring operation satisfying a list of axioms specified in [7, Remark 10.1.2].

The *homotopy Lie algebra of* A, denoted by $\pi(A)$, is defined to be

$$\pi(A) = \mathrm{H}\left(\mathrm{Der}^{\gamma}_{A}(A\langle Y\rangle, A\langle Y\rangle)\right) ,$$

where $A\langle Y\rangle$ is an acyclic closure of k over A. The Lie bracket is the graded commutator and the square of an element is the composition of that element with itself. As any two acyclic closures are isomorphic as semifree Γ-extensions, see 1.3, $\pi(A)$ is independent of choice of acyclic closure for k over A. Furthermore, $\pi(A)$ is naturally identified as a k-subspace of $\mathrm{Ext}_A(k, k)$ with the latter being the universal enveloping algebra of $\pi(A)$; see [7, Theorem 10.2.1]. Finally, given a quasi-isomorphism of local DG algebras $A \xrightarrow{\simeq} B$ it follows from [7, Lemma 7.2.10] that $\pi(B) \xrightarrow{\cong} \pi(A)$ as graded Lie algebras over k.

When A is the local ring $(R, \mathfrak{m}, k)$ and $R\langle Y\rangle$ is an acyclic closure of k over R, it is well-known that the quasi-isomorphism

$$R\langle Y\rangle \to \widehat{R} \otimes_R R\langle Y\rangle$$

induces an isomorphism of graded Lie algebras $\pi(\widehat{R}) \xrightarrow{\cong} \pi(R)$, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R.

Moreover, the surjective map $\varphi\colon R \to S$ induces a map of graded Lie algebras over k:

$$\pi(\varphi)\colon \pi(S) \to \pi(R)\,.$$

See, for example, [7, Remark 10.2.4] or [20] for more details on the construction of this map.

1.6. Let $B = R[X]$ be a semifree extension of R. Its indecomposable complex, denoted $\mathrm{ind}_R\, R[X]$, is the complex of free R-modules

$$\frac{B}{B_0 + (X)^2} \cong \ldots \to RX_n \to RX_{n-1} \to \ldots \to RX_2 \to RX_1 \to 0\,.$$

Minimality of B is detected using $\mathrm{ind}_k\, k[X]$. Namely, B is a minimal semifree extension of R if and only if $\mathrm{ind}_k\, k[X]$ has trivial differential.

On the other hand, given a semifree Γ-extension $A\langle Y\rangle$ of A with $S = \mathrm{H}_0(A\langle Y\rangle)$, its complex of Γ-indecomposables (with respect to A), denoted $\mathrm{ind}^{\gamma}_A\, A\langle Y\rangle$, is the complex

$$\frac{A\langle Y\rangle}{A + JY + (Y)^{(2)}} \cong \ldots \to SY_n \to SY_{n-1} \to \ldots \to SY_2 \to SY_1 \to 0$$

where $J = \mathrm{Ker}(A\langle Y\rangle \to S)$ and $(Y)^{(2)}$ consists of the DG ideal of $A\langle Y\rangle$ consisting of all decomposable Γ-monomials on the set Y; see [7, Construction 6.2.5] for more details. Finally, $A\langle Y\rangle$ is an acyclic closure of S if and only if $\mathrm{ind}^{\gamma}_A\, A\langle Y\rangle$ is minimal as a complex of (free) S-modules; cf. [7, Lemma 6.3.2].

We write Σ for the shift functor on a graded object. Namely, for a graded object V, ΣV is the graded object with $(\Sigma V)_n = V_{n-1}$.

1.7. Given any minimal semifree extensions $k[X]$ and $k[X']$ over k, a local DG algebra map $\psi\colon k[X] \to k[X']$ induces a well-defined map on the graded k-vector spaces

$$\mathrm{ind}_k\, \psi\colon\ \mathrm{ind}_k\, k[X] \to \mathrm{ind}_k\, k[X']. \tag{1}$$

Consider the commutative diagram of local rings

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ R \arrow[r, "\varphi"] & S \end{tikzcd}$$

where the vertical maps are minimal Cohen presentations, with φ from above. Fix minimal models $P[X] \xrightarrow{\simeq} R$ and $Q[X'] \xrightarrow{\simeq} S$, and let $\tilde{\varphi}\colon P[X] \to Q[X']$ be the induced map by φ; cf. [7, Proposition 2.1.9]. Set ψ to be the map

$$k \otimes \tilde{\varphi}\colon k[X] \to k[X'].$$

By [13, Theorem 3.4], there is the following commutative diagram of graded k-spaces:

$$\begin{tikzcd} (\Sigma \operatorname{ind}_k k[X'])^\vee \arrow[r, "(\Sigma \operatorname{ind}_k \psi)^\vee"] \arrow[d, "\cong"] & (\Sigma \operatorname{ind}_k k[X])^\vee \arrow[d, "\cong"] \\ \pi^{\geqslant 2}(S) \arrow[r, "\pi^{\geqslant 2}(\varphi)"] & \pi^{\geqslant 2}(R) \end{tikzcd}$$

where $(-)^\vee$ denotes graded k-space duality. In particular, we conclude that $\pi^{\geqslant 2}(\varphi)$ is exactly the k-linear dual of $\operatorname{ind}_k \psi$, defined in (1), up to a shift.

1.8. We write F^φ for the (derived) fiber of φ. Namely, F^φ is the DG k-algebra $k \otimes_R R[X]$ where $R[X]$ is a minimal model of φ; it is clear that $\mathrm{F}^\varphi \simeq k \otimes^{\mathbf{L}}_R S$, justifying the terminology. As any two minimal models are isomorphic, F^φ is an invariant of φ.

In the construction above, if one replaces the minimal model $R[X]$ with an acyclic closure $R\langle Y\rangle$ for φ one obtains a semifree Γ-extension $k \otimes_R R\langle Y\rangle$, which will be denoted by $\mathrm{F}^\varphi_\gamma$.

2. Tor-independent quotients

Throughout the rest of the article we fix the following notation.

Notation 2.1. Consider the following diagram of surjective local homomorphisms with common residue field k:

$$\begin{tikzcd} R \arrow[r, two heads, "\varphi_1"] \arrow[d, two heads, "\varphi_2"'] \arrow[dr, two heads, "\varphi"] & S_1 \arrow[d, two heads, "\psi_2"] \\ S_2 \arrow[r, two heads, "\psi_1"'] & S \end{tikzcd}$$

We denote by $\mathfrak{m}$ the maximal ideal of R and assume that the kernel of φ_i, denoted by I_i, is a nontrivial proper ideal of R.

Remark 2.2. By a slight abuse of terminology, we will refer to the pair of maps φ_1, φ_2 as being Tor-independent provided their targets are Tor-independent over R. This is equivalent to requiring that S is both the ordinary and derived pushout in the category of R-algebras along the maps φ_1 and φ_2; the derived pushout condition simply says that $S \simeq S_1 \otimes^{\mathbf{L}}_R S_2$.

2.3. When R is regular, φ_1, φ_2 are Tor-independent if and only if $I_1 I_2 = I_1 \cap I_2$. In this case, if each of the I_i are contained in $\mathfrak{m}^2$, the ring S is referred to as a minimal intersection ring in [23]. In [34, Definition 5.5.1], the ideals I_1 and I_2 are called *transversal*; such ideals, and their corresponding surjective ring maps, have been studied in *loc. cit.* as well as [2, 4, 22, 33].

Example 2.4. Anytime I_1 is generated by an R-regular sequence that is also S_2-regular the pair φ_1, φ_2 are Tor-independent. In light of this point, Tor-independent maps generalize complete intersection maps of codimension at least two.

Lemma 2.5. *Whenever $I_1 \cap I_2 \subseteq \mathfrak{m}_R^2$, there is the equality*

$$\dim_k \frac{\mathfrak{m}_S}{\mathfrak{m}_S^2} = \dim_k \frac{\mathfrak{m}_{S_1}}{\mathfrak{m}_{S_1}^2} + \dim_k \frac{\mathfrak{m}_{S_2}}{\mathfrak{m}_{S_2}^2} - \dim_k \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}\,.$$

Proof. Let I be $I_1 + I_2$ and

$$\bar{\varphi}\colon \frac{I}{\mathfrak{m}_R I} \longrightarrow \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}$$

be the map induced by $\varphi\colon R \to S$; its codimension is exactly the embedding dimension of S, i.e.,

$$\dim_k \frac{\mathfrak{m}_S}{\mathfrak{m}_S^2} = \dim_k \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} - \operatorname{rank}\bar{\varphi}\,. \tag{2}$$

We define $\bar{\varphi}_i$ similarly and note the analogous equalities to (2) also hold.

From the commutative diagram

$$\begin{tikzcd} \dfrac{I}{\mathfrak{m}_R I} \arrow[r, "\bar{\varphi}"] \arrow[d, "\cong"'] & \dfrac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \\ \dfrac{I_1}{\mathfrak{m}_R I_1} \oplus \dfrac{I_2}{\mathfrak{m}_R I_2} \arrow[ur, "\bar{\varphi}_1 + \bar{\varphi}_2"'] & \end{tikzcd}$$

and the fact that $\operatorname{Im}\bar{\varphi}_1$ and $\operatorname{Im}\bar{\varphi}_2$ intersect trivially in $\mathfrak{m}_R/\mathfrak{m}_R^2$, it follows that

$$\operatorname{rank}\bar{\varphi} = \operatorname{rank}\bar{\varphi}_1 + \operatorname{rank}\bar{\varphi}_2\,; \tag{3}$$

the vertical isomorphism in the diagram and the fact $\operatorname{Im}\bar{\varphi}_1$ and $\operatorname{Im}\bar{\varphi}_2$ intersect at 0 are the only times the hypothesis $I_1 \cap I_2 \subseteq \mathfrak{m}_R^2$ is used. Now substituting (3) into (2) and using the analogs to (2) for each $\bar{\varphi}_i$, the desired result follows readily. □

The following lemma will be applied in Theorems 3.3 and 4.7. It is likely well known but we include the proof here for convenience of the reader. The hypothesis of Lemma 2.6 is satisfied if φ_1, φ_2 are Tor-independent, since $I_1 \cap I_2 = I_1 I_2 \subseteq \mathfrak{m}^2$ is forced by the hypothesis that $\operatorname{Tor}_1^R(S_1, S_2) = 0$.

Lemma 2.6. *Adopting Notation 2.1, if $I_1 \cap I_2 \subseteq \mathfrak{m}_R^2$, then there is the following isomorphism of k-vector spaces*

$$\pi^1(S) \cong \pi^1(S_1) \times_{\pi^1(R)} \pi^1(S_2)\,.$$

Proof. Recall that for a local ring $(A, \mathfrak{m}_A, k)$ there is the following natural isomorphism of k-spaces

$$\pi^1(A) \cong \operatorname{Hom}_k\left(\frac{\mathfrak{m}_A}{\mathfrak{m}_A^2}, k\right).$$

Hence, by k-vector space duality it suffices to show the following diagram

$$\begin{tikzcd} \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \arrow[r, two heads, "\overline{\varphi}_1"] \arrow[d, two heads, "\overline{\varphi}_2"'] & \frac{\mathfrak{m}_{S_1}}{\mathfrak{m}_{S_1}^2} \arrow[d, two heads] \\ \frac{\mathfrak{m}_{S_2}}{\mathfrak{m}_{S_2}^2} \arrow[r, two heads] & \frac{\mathfrak{m}_S}{\mathfrak{m}_S^2} \end{tikzcd}$$

is a pushout diagram of k-vector spaces. Let P denote the pushout along $\overline{\varphi}_1$ and $\overline{\varphi}_2$. From the commutative diagram above the uniquely induced map $\psi\colon P \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective; so it suffices to show their k-space dimensions coincide. From the assumption $I_1 \cap I_2 \subseteq \mathfrak{m}_R^2$, it follows, by Lemma 2.5, that

$$\dim_k \frac{\mathfrak{m}_S}{\mathfrak{m}_S^2} = \dim_k \frac{\mathfrak{m}_{S_1}}{\mathfrak{m}_{S_1}^2} + \dim_k \frac{\mathfrak{m}_{S_2}}{\mathfrak{m}_{S_2}^2} - \dim_k \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}.$$

Moreover,

$$\dim_k P = \dim_k \frac{\mathfrak{m}_{S_1}}{\mathfrak{m}_{S_1}^2} + \dim_k \frac{\mathfrak{m}_{S_2}}{\mathfrak{m}_{S_2}^2} - \dim_k \left\{ (\overline{\varphi}_1(x), -\overline{\varphi}_2(x)) \colon x \in \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \right\}$$

and so to complete the proof it is enough to show

$$\dim_k \left\{ (\overline{\varphi}_1(x), -\overline{\varphi}_2(x)) \colon x \in \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \right\} = \dim_k \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}\,.$$

The left-hand side is exactly the dimension of the image of the following surjective map

$$\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \xrightarrow{(\overline{\varphi}_1, -\overline{\varphi}_2)} \frac{\mathfrak{m}_{S_1}}{\mathfrak{m}_{S_1}^2} \oplus \frac{\mathfrak{m}_{S_2}}{\mathfrak{m}_{S_2}^2}\,;$$

note that x is in the kernel of $(\overline{\varphi}_1, -\overline{\varphi}_2)$ if and only if $x = y + \mathfrak{m}_R^2$ for some $y \in I_1 \cap I_2$, and so the assumption that $I_1 \cap I_2 \subseteq \mathfrak{m}_R^2$ implies $(\overline{\varphi}_1, -\overline{\varphi}_2)$ is injective, as needed. □

The rest of this section provides results illustrating how properties of φ determine and are determined by properties of φ_1 and φ_2, provided the latter pair are Tor-independent. We also show that in this setting φ_i and ψ_i exhibit properties simultaneously.

Recall the notions of Golod, Gorenstein, and quasi-complete intersection homomorphisms discussed in [6, 8, 12], respectively, the last two generalizing the notions of Gorenstein and complete intersection local rings. Parts (1) and (2) of the next result generalize their ring theoretic versions in [23, Proposition 3.3] with simpler proofs not relying on the celebrated *(new) intersection theorem* [28, 31].

Proposition 2.7. *Suppose $\varphi_i\colon R \to S_i$ is a pair of Tor-independent surjective local ring maps for $i = 1, 2$, and set $\varphi = \varphi_1 \otimes \varphi_2$.*

1. *φ is Gorenstein if and only if each φ_i is;*
2. *φ is complete intersection if and only if each φ_i is;*
3. *more generally, φ is quasi-complete intersection if and only if each φ_i is.*

Proof. First, we point out that S has finite projective dimension over R if and only if each S_i does since φ_1, φ_2 are Tor-independent. From this, (2) follows from (3); see [12, Theorem 2.5].

For the first statement, observe that there is the following isomorphism of DG k-algebras

$$\mathrm{F}^{\varphi} \cong \mathrm{F}^{\varphi_1} \otimes_k \mathrm{F}^{\varphi_2} .$$

From this it follows that the homology of F^{φ} is finite dimensional if and only if each F^{φ_i} has finite dimensional homology. Now a calculation involving k-duality and the Künneth formula yields the following isomorphism of graded k-spaces

$$\mathrm{Ext}_{\mathrm{F}^{\varphi}}(k, \mathrm{F}^{\varphi}) \cong \mathrm{Ext}_{\mathrm{F}^{\varphi_1}}(k, \mathrm{F}^{\varphi_1}) \otimes_k \mathrm{Ext}_{\mathrm{F}^{\varphi_2}}(k, \mathrm{F}^{\varphi_2})$$

provided the homology of F^{φ} is finite. Therefore the desired result holds from [8, Theorem 4.4], and the observation at the beginning of the proof.

The Tor-independence of φ_1, φ_2 implies that given acyclic closures $A_i \xrightarrow{\simeq} S_i$ for φ_i, the DG R-algebra $A := A_1 \otimes_R A_2$ is an acyclic closure for φ. As a direct consequence we obtain the following isomorphism of k-spaces

$$\mathrm{ind}_k^{\gamma}(A_1 \otimes_R k) \oplus \mathrm{ind}_k^{\gamma}(A_2 \otimes_R k) \cong \mathrm{ind}_k^{\gamma}(A \otimes_R k) .$$

Now (3) follows as the quasi-complete intersection property is characterized by the k-space above being concentrated in degrees one and two. □

Whenever φ_1, φ_2 are Tor-independent, we have isomorphisms of DG k-algebras $\mathrm{F}^{\varphi_i} \cong \mathrm{F}^{\psi_i}$ for $i = 1, 2$. Thus, whenever a property of a map can be characterized in terms of the fiber, the parallel map enjoys the same property. This is demonstrated in Proposition 2.8 below.

Proposition 2.8. *When φ_1, φ_2 are Tor-independent, with Notation 2.1, the following hold:*

1. *φ_i is Golod if and only ψ_i is Golod;*
2. *φ_i is Gorenstein if and only if ψ_i is Gorenstein;*
3. *φ_i is (quasi-)complete intersection if and only ψ_i is (quasi-)complete intersection.*

Proof. First, using that φ_1, φ_2 are Tor-independent, the isomorphism of DG k-algebras $\mathrm{F}^{\varphi_i} \cong \mathrm{F}^{\psi_i}$ yields one of graded Lie algebras over k

$$\pi(\mathrm{F}^{\varphi_i}) \cong \pi(\mathrm{F}^{\psi_i}) .$$

Now (1) holds as the homotopy Lie algebra of a map being free characterizes the Golod property of that map; see [6, Theorem 3.4].

The second assertion is immediate from $\mathrm{F}^{\varphi_i} \cong \mathrm{F}^{\psi_i}$ and [8, Theorem 4.4].

Finally, for (3) the assumption that φ_1, φ_2 are Tor-independent forces $A \otimes_R k$ to be isomorphic to $B \otimes_{S_j} k$ where A and B are acyclic closures over φ_i and ψ_i, respectively, where $i \neq j$. This determines the isomorphism of k-spaces

$$\mathrm{ind}_k^{\gamma}(A \otimes_R k) \cong \mathrm{ind}_k^{\gamma}(B \otimes_{S_j} k) .$$

Now the desired result holds as the quasi-complete intersection property is detected by these k-spaces being concentrated in degrees one and two. The statement for complete intersection homomorphisms now follows using that S_i has finite projective dimension over R if and only if S has finite projective dimension over S_j, since φ_1, φ_2 are Tor-independent; again one is also appealing to [12, Theorem 2.5]. □

We end this section by abandoning the local situation of Notation 2.1, and instead work in the graded setting.

2.9. Let R be a standard graded connected k-algebra. One can adapt the definitions from Section 1 by requiring all cycles, ideals, maps, etc. to be homogeneous with respect to the internal degree determined by the grading on R. Furthermore, in this setting, $\pi(S)$ is naturally bigraded with respect to the usual cohomological degree, as well as the internal degree acquired from R; hence we will write $\pi^{*,*}(S)$ to indicate that the homotopy Lie algebra of S is being considered as a bigraded object. Finally, recall that S is a Koszul algebra if k has a linear minimal free resolution over S; this resolution being the acyclic closure forces $\pi^{i,j}(S) = 0$ for all $i \neq j$ with $i \geqslant 2$.

Proposition 2.10. *Let R be a standard graded polynomial ring over the field k and consider quotients S_1, S_2 by homogeneous ideals of R. If S_1, S_2 are Tor-independent over R, then S is Koszul if and only if each S_i is Koszul.*

Proof. In the graded setting, as in the proof of Proposition 2.7, it follows directly from the assumption that S_1, S_2 are Tor-independent that $A := A_1 \otimes_R A_2$ is a minimal model for S where A_i is a minimal model for S_i. As a consequence there is an isomorphism of bigraded vector spaces

$$\operatorname{ind}_k(A_1 \otimes_R k) \oplus \operatorname{ind}_k(A_2 \otimes_R k) \cong \operatorname{ind}_k(A \otimes_R k)\,.$$

Shifting and taking k-linear dual induces an isomorphism of bigraded k-spaces

$$\pi^{\geqslant 2,*}(S) \cong \pi^{\geqslant 2,*}(S_1) \times \pi^{\geqslant 2,*}(S_2)\,;$$

this is analogous to the local case explained in 1.7. The desired result now follows immediately from 2.9. $\square$

3. Almost small homomorphisms

In this section we still adopt Notation 2.1 and prove the conclusion of Theorem A holds when at least one φ_i is almost small.

3.1. A surjective local map $\varphi\colon R \to S$ is *almost small* if the morphism of graded Lie algebras

$$\pi^{\geqslant 2}(\varphi)\colon \pi^{\geqslant 2}(S) \to \pi^{\geqslant 2}(R)$$

is surjective. In [13] these are defined in terms of Tor algebras. Namely, the map on Tor algebras

$$\operatorname{Tor}^{\varphi}(k,k)\colon\ \operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k)$$

has its kernel generated in degree one; these definitions coincide by [13, Proposition 4.3].

We first prove a result that identifies pullbacks, also known as fiber products, of graded Lie algebras.

Lemma 3.2. *Given the following commutative diagram of graded Lie k-algebras*

$$\begin{tikzcd} \mathfrak{g} & \mathfrak{h}_1 \arrow[l, "\beta"'] \\ \mathfrak{h}_2 \arrow[u, "\delta"] & \mathfrak{k}. \arrow[l, two heads, "\gamma"] \arrow[u, "\alpha"'] \end{tikzcd}$$

If $\alpha|_{\mathrm{Ker}\gamma}\colon \mathrm{Ker}\gamma \to \mathrm{Ker}\beta$ *is an isomorphism of Lie ideals, then* $\mathfrak{f} \cong \mathfrak{h}_1 \times_{\mathfrak{g}} \mathfrak{h}_2$.

Proof. The commutative diagram defines a graded Lie k-algebra map $\sigma\colon \mathfrak{f} \to \mathfrak{h}_1 \times_{\mathfrak{g}} \mathfrak{h}_2$ given by $x \mapsto (\alpha(x), \gamma(x))$. We show σ is an isomorphism.

Let $x \in \mathrm{Ker}\sigma$, then $\alpha(x) = 0$ and $\gamma(x) = 0$. The latter says $x \in \mathrm{Ker}\gamma$ and so the former implies $x = 0$ since $\alpha|_{\mathrm{Ker}\gamma}$ is injective by assumption; thus, σ is injective.

Let $(x_1, x_2) \in \mathfrak{h}_1 \times_{\mathfrak{g}} \mathfrak{h}_2$, then by the assumption that γ is surjective there exists $y \in \mathfrak{f}$ such that $\gamma(y) = x_2$. As $\delta(x_2) = \beta(x_1)$ and the diagram above commutes we conclude that $\alpha(y) - x_1 \in \mathrm{Ker}\beta$. Now using that $\alpha(\mathrm{Ker}\gamma) = \mathrm{Ker}\beta$, there exists $y' \in \mathrm{Ker}\gamma$ such that $\alpha(y') = \alpha(y) - x_1$. Now observe that

$$\sigma(y - y') = (\alpha(y - y'), \gamma(y - y')) = (x_1, \gamma(y)) = (x_1, x_2),$$

as desired, justifying that σ is surjective. $\square$

Theorem 3.3. *Suppose* $S_1 \xleftarrow{\varphi_1} R \xrightarrow{\varphi_2} S_2$ *is a pair of Tor-independent surjective local ring maps with residue field* k*, and set* $S = S_1 \otimes_R S_2$*. If at least one* φ_i *is almost small, then* π *induces the following isomorphism of graded Lie* k*-algebras*

$$\pi(S) \cong \pi(S_1) \times_{\pi(R)} \pi(S_2)\,.$$

Proof. We assume that φ_1 is almost small. As homotopy Lie algebras and the Tor-independence of the maps are both invariant under completion, see 1.5 for the former point, we can assume R is complete. Now let $\rho\colon P \to R$ be a minimal Cohen presentation of R.

Let $P[W] \to R$ be a minimal model of ρ. Since φ_1 is almost small, by [13, Theorem 4.11], we have the following commutative diagram of differential graded P-algebras:

$$\begin{tikzcd} P[W] \arrow[r, "\tilde{\varphi}_1"] \arrow[d] & Q_1[X,W] \arrow[d] \\ R \arrow[r, "\varphi_1"] & S_1 \end{tikzcd}$$

where Q_1 is a regular quotient of P with $Q_1 \to S_1$ a minimal Cohen presentation; the vertical map on the right $Q_1[X,W] \to S_1$ is a minimal model for S_1 over Q_1, and $\tilde{\varphi}_1(w) = w$ for all $w \in W$. Let $Q_2[Y] \to S_2$ be minimal model of S_2 where Q_2 is a regular quotient of P and $Q_2 \to S_2$ is a minimal Cohen presentation. Finally, set Q to be the regular quotient $Q_1 \otimes_P Q_2$ of P.

Since φ_1, φ_2 are Tor-independent, the canonical map

$$Q_1[X,W] \otimes_{P[W]} Q_2[Y] \to S$$

is a quasi-isomorphism where $Q_2[Y]$ is regarded as a DG $P[W]$-module by lifting φ_2. The following isomorphisms of graded algebras

$$\begin{aligned} Q_1[X,W] \otimes_{P[W]} Q_2[Y] &\cong Q_1[X] \otimes_P \left(P[W] \otimes_{P[W]} Q_2[Y]\right) \\ &\cong Q[X,Y] \end{aligned}$$

induce a differential on the semifree extension $Q[X,Y]$. Moreover, $Q[X,Y] \to S$ is a minimal model over Q. Indeed, the isomorphisms above induce an isomorphism

$$\alpha\colon k[X,W] \otimes_{k[W]} k[Y] \xrightarrow{\cong} k[X,Y]$$

where $\alpha(x \otimes 1) = x$ and $\alpha(1 \otimes y) = y$ for each $x \in X, y \in Y$, and $\alpha(W \otimes 1)$ is contained in the DG ideal generated by Y in $k[X,Y]$. As a consequence the induced differential on $k[X,Y]$ satisfies

$$\partial(k[X,Y]) \subseteq (X,Y)^2$$

since $k[X,W]$ and $k[Y]$ are minimal semifree extensions over k. Finally,

$$Q \cong Q_1 \otimes_P Q_2 \to S_1 \otimes_P S_2 \cong S$$

is a minimal Cohen presentation as $Q_i \to S_i$ are each minimal Cohen presentations. Thus, we have the following commutative diagram of semifree extensions

$$\begin{tikzcd} k[W] \arrow[r, hook] \arrow[d] & k[X,W] \arrow[d] \\ k[Y] \arrow[r, hook] & k[X,Y] \end{tikzcd} \tag{4}$$

where the horizontal maps are the canonical inclusions and the vertical map on the right being the identity when restricted to X.

Since the minimal models were taken with respect to *minimal* Cohen presentations, by 1.7 there exists the following corresponding commutative diagram of graded Lie algebras:

$$\begin{tikzcd} \pi^{\geqslant 2}(R) & \pi^{\geqslant 2}(S_1) \arrow[l, two heads, "\pi^{\geqslant 2}(\varphi_1)"'] \\ \pi^{\geqslant 2}(S_2) \arrow[u, "\pi^{\geqslant 2}(\varphi_2)"] & \pi^{\geqslant 2}(S) \arrow[l, two heads, "\pi^{\geqslant 2}(\psi_1)"] \arrow[u, "\pi^{\geqslant 2}(\psi_2)"'] \end{tikzcd} \tag{5}$$

where the horizontal maps in (5) are natural projections whose kernels have a k-basis of derivations indexed by the variables X. Also, as the vertical map on the right in (4) is the identity when restricted to X, the morphism $\pi^{\geqslant 2}(\psi_2)$ maps the derivations corresponding to X in $\pi^{\geqslant 2}(S)$ bijectively to the derivations corresponding to X in $\pi^{\geqslant 2}(S_1)$. Applying Lemma 3.2 we have established the following isomorphism of graded Lie algebras

$$\pi^{\geqslant 2}(S) \cong \pi^{\geqslant 2}(S_1) \times_{\pi^{\geqslant 2}(R)} \pi^{\geqslant 2}(S_2)\,. \tag{6}$$

From Lemma 2.6, we obtain the following isomorphism of k-spaces

$$\pi^1(S) \cong \pi^1(S_1) \times_{\pi^1(R)} \pi^1(S_2)\,. \tag{7}$$

The isomorphisms in (6) and (7) are restrictions of the unique map of graded Lie algebras

$$\pi(S) \to \pi(S_1) \times_{\pi(R)} \pi(S_2)\,,$$

completing the proof of the theorem. □

Remark 3.4. With the stronger assumption that one of the φ_i is assumed to be small, the assertion of Theorem 3.3 was established in [4]. The argument in *loc. cit.* proceeds by establishing a spectral sequence of Hopf algebras whose E^2-page is determined by the Tor Hopf algebras over R and the S_i; the E^∞-page of this spectral sequence is the Tor algebra over S. Tor-independence of the S_i over R, forces the spectral sequence to degenerate from which the isomorphism asserted in Corollary 5.7 can be established. Finally, one can now use a celebrated (anti-)equivalence of categories of André, Milnor–Moore, and Sjödin in [1, 26, 32] to prove the assertion of Theorem 3.3 holds—still with the assumption that one of the φ_i is assumed to be small.

It seems likely that one can follow a similar line of arguing to prove Theorem 3.3; however, besides involving a wealth of machinery in [2, 4], there are still a few sticking points we wish to highlight. A hurdle is establishing an analog of the spectral sequence in [4, Theorem 5.5]. One can use [13, Theorem 4.11] in lieu of [4, Corollary 5.4] when trying to extend [4, Theorem 5.5] to when one of the φ_i is small. However it is not clear to the present authors for how to adapt [4, Lemma 5.2], and the references therein, to work in this generality.

Remark 3.5. When R is regular the assumption that one of the φ_i is almost small is vacuous. In particular, the conclusion of Theorem 3.3 holds for any minimal intersection ring in the terminology of [23]; cf. 2.3. As a consequence, such a ring S whose completion $\widehat{S}$ can be written as $Q/(J_1 + J_2)$ where Q is a regular local ring and J_i are nontrivial ideals with $\operatorname{Tor}^Q(Q/J_1, Q/J_2) \cong \widehat{S}$, has $\pi^{\geqslant 2}(S)$ decomposable as

$$\pi^{\geqslant 2}(Q/J_1) \times \pi^{\geqslant 2}(Q/J_2)$$

where we use the isomorphism $\pi^{\geqslant 2}(S) \cong \pi^{\geqslant 2}(\widehat{S})$; see 1.5. This observation can be used to deduce the well known fact that the homotopy Lie algebra of a complete intersection ring of codimension $c \geqslant 2$ in degrees strictly larger than one is an abelian Lie algebra with c generators in degree two; cf. Example 2.4.

Remark 3.6. Let $\varphi\colon R \to S$ be a not necessarily surjective map of local rings. Take a Cohen factorization of (the completion of) φ

$$\widehat{R} \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S},$$

as described in [9]; here $\dot{\varphi}$ is weakly regular—in the sense it is a flat local homomorphism with regular fiber—and φ' is surjective. If $\widehat{S}$ can be written as $S_1 \otimes_{R'} S_2 \cong \widehat{S}$, where S_1, S_2 are Tor-independent quotients of R and at least one of $R' \to S_i$ is almost small, then from Theorem 3.3 one can deduce

$$\pi^{\geqslant 2}(S) \cong \pi^{\geqslant 2}(S_1) \times_{\pi^{\geqslant 2}(R)} \pi^{\geqslant 2}(S_2),$$

using the facts that $\pi^{\geqslant 2}(R) \cong \pi^{\geqslant 2}(R')$ since $\dot{\varphi}$ is weakly regular and $\pi^{\geqslant 2}(S) \cong \pi^{\geqslant 2}(\widehat{S})$.

If φ is assumed to be almost small to begin with then, by [13, Proposition 4.8], the map φ' is also almost small. It follows from [13, Corollary 4.5(b)], that when $\widehat{S}$ is a tensor product of Tor-independent quotients S_1, S_2 of R' that both maps $R' \to S_i$ are always almost small.

4. Gulliksen minimality and residual characteristic zero

We keep the notation set in Notation 2.1.

4.1. Given a surjective quasi-isomorphism of DG algebras $\alpha\colon A \to B$ we recall a lifting property of semifree extensions. For a set of cycles Z in B, upon choosing a set of lifts $\tilde{Z}$ for Z in A, there exists a surjective quasi-isomorphism

$$A[X] \to B[X]$$

extending α and sending each x to the corresponding x where $X = \{x_z : |x_z| = |z|+1\}_{z\in Z}$ and
$$\partial^{A[X]}(x_z) = \tilde{z} \quad\text{and}\quad \partial^{B[X]}(x_z) = z\,.$$
This will be applied specifically in the following setting:
$$Q \xrightarrow{\rho} R \xrightarrow{\varphi} S$$
is a composition of surjective local maps with minimal models $Q[W]$ and $R[X]$ for ρ and φ, respectively. The surjective quasi-isomorphism $Q[W] \xrightarrow{\simeq} R$ extends to a surjective quasi-isomorphism
$$Q[W,X] \xrightarrow{\simeq} R[X]$$
as described above. Note as the map above is a quasi-isomorphism, we have $Q[W,X] \xrightarrow{\simeq} S$. We will freely use this construction in the sequel.

Definition 4.2. Let $R \xrightarrow{\varphi} S$ be a surjective map of local rings. Fix a minimal Cohen presentation $\rho\colon P \to \widehat{R}$, and minimal models $P[W] \to \widehat{R}$ and $\widehat{R}[Y] \to \widehat{S}$. The map φ is *ρ-Gulliksen minimal* if the semifree extension $P[W,Y]$ satisfies
$$\partial_{2i}(P[W,Y]) \subseteq \mathfrak{m}_P[W,Y] + (W,Y)^2 \tag{8}$$
for all $i \geqslant 0$, equivalently $\partial_{2i}(\operatorname{ind}_k k[W,Y]) = 0$ for all $i \geqslant 0$, where $\widehat{(-)}$ denotes completion at the maximal ideal.

Remark 4.3. The definition above was introduced in [15, Section 3.5]. In the case of semifree Γ-extensions, a similar condition to (8) is identified in [20, Chapter 3] and notably put to use in the proof of [3, Theorem 1.1]. As any two minimal models are isomorphic, the definition is independent of choice of minimal models (yet, explicitly dependent on minimal Cohen presentation).

4.4. There exist Gulliksen minimal maps that are not almost small. For example, whenever φ is a map with finite weak category, as defined in [13, Definition 5.3], φ is Gulliksen minimal; see [15, Theorem 25]. By [13, Theorem 5.6], the weak category of a map with finite projective dimension is finite and in turn provides a class of maps that are Gulliksen minimal.

Lemma 4.5. *Let $\varphi\colon R \to S$ be a surjective map of local rings. If φ is Gulliksen minimal or k has characteristic zero then there is a commutative diagram*
$$\begin{tikzcd} P[W] \arrow[r, "\tilde{\varphi}"] \arrow[d, "\simeq"'] & Q[U,X] \arrow[d, "\simeq"] \\ \widehat{R} \arrow[r] & \widehat{S} \end{tikzcd}$$
satisfying the following conditions:

1. *the vertical maps are minimal models with respect to minimal Cohen presentations $P \to \widehat{R}$ and $Q \to \widehat{S}$;*
2. *$\tilde{\varphi}$ extends a surjective map $P \to Q$, with $U \subseteq W$ and* $\tilde{\varphi}(w) = \begin{cases} w & \textit{if } w \in U \\ 0 & \textit{otherwise} \end{cases}$ *for $w \in W$.*

Proof. Assume R and S are complete and let $P \to R$ be a minimal Cohen presentation. In the case that φ is Gulliksen minimal, further assume that (8) is satisfied where $P[W] \to R$ and $R[Y] \to S$ are minimal models. There exists a minimal semifree extension $P[U, X']$ with $U \subseteq W$ and $X' \subseteq Y$ that fit into a commutative diagram

$$\begin{tikzcd} P[W,Y] \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & P[U,X'] \arrow[d, "\simeq"] \\ & S \end{tikzcd}$$

where the horizontal map is determined by $t \mapsto \begin{cases} t & t \in U \cup X' \\ 0 & \text{otherwise} \end{cases}$ for $t \in W \cup Y$ and $X'_{2i+1} = Y_{2i+1}$ for all nonnegative integers i. The existence of this diagrams is contained in the proof of [20, Lemma 3.2.1] in the case that the characteristic of k is zero, and it holds when φ is Gulliksen minimal by [15, Theorem 24].

Next take the ideal J in P generated by the elements in

$$(\mathfrak{m}_P \backslash \mathfrak{m}_P^2) \cap \ker(P \to S)$$

and set $Q = P/J$; note that by construction $Q \to S$ is a minimal Cohen presentation. The quotient map $P \to Q$ induces the quasi-isomorphism of DG algebras $P[X'_1] \xrightarrow{\simeq} Q[X_1]$ where $X'_1 \backslash X_1$ corresponds to a minimal generating set for J and for $x' \in X'_1$ we have

$$x' \mapsto \begin{cases} x' & x' \notin X_1 \\ 0 & x' \in X_1 \,. \end{cases}$$

By 4.1, this extends to a quasi-isomorphism of DG algebras $P[U, X'] \xrightarrow{\simeq} Q[U, X]$ with $X_i = X'_i$ for $i > 1$ satisfying $x \mapsto x$ for each $x \in X_{\geqslant 2}$. Setting $\tilde{\varphi}$ to be the composition of the maps

$$P[W] \to P[W, Y] \xrightarrow{\simeq} P[U, X'] \xrightarrow{\simeq} Q[U, X]\,,$$

where the maps are constructed above, has the desired properties. Finally, by the construction of $P[U, X'] \xrightarrow{\simeq} Q[U, X]$ it follows that $Q[U, X]$ is a minimal model for S over Q since $P[U, X']$ is a minimal model for S over P. □

Lemma 4.6. *Let $Q \xrightarrow{\rho} R \xrightarrow{\varphi_i} S_i$ be surjective ring maps with residue field k for $i = 1, 2$. Set $\sigma_i = \varphi_i \circ \rho$ and $\sigma = \varphi \circ \rho$, where φ is the canonical quotient $R \to S_1 \otimes_R S_2$.*

If φ_1, φ_2 are Tor-independent, and in addition the characteristic of k is zero or each φ_i is ρ-Gulliksen minimal, then the functor π induces the following isomorphism of graded Lie algebras

$$\pi(\mathrm{F}^{\sigma}) \cong \pi(\mathrm{F}^{\sigma_1}) \times_{\pi(\mathrm{F}^{\rho})} \pi(\mathrm{F}^{\sigma_2})\,.$$

Proof. Let $Q[W] \to R$, $R[X'] \to S_1$ and $R[Y'] \to S_2$ be minimal models of ρ, φ_1 and φ_2 respectively. They determine quasi-isomorphisms of DG Q-algebras:

$$Q[W, X'] \to S_1 \quad \text{and} \quad Q[W, Y'] \to S_2.$$

There exist minimal semifree extensions $Q[U, X]$ and $Q[V, Y]$ where $U, V \subseteq W$, $X \subseteq X'$ and $Y \subseteq Y'$ that fit into commutative diagrams

$$\begin{tikzcd} Q[W,X'] \arrow[r, two heads, "\simeq"] \arrow[dr, "\simeq"'] & Q[U,X] \arrow[d, "\simeq"] \\ & S_1 \end{tikzcd} \qquad \begin{tikzcd} Q[W,Y'] \arrow[r, two heads, "\simeq"] \arrow[dr, "\simeq"'] & Q[V,Y] \arrow[d, "\simeq"] \\ & S_2 \end{tikzcd}$$

where the horizontal maps are the DG algebra maps determined by

$$t \mapsto \begin{cases} t & t \in U \cup X \\ 0 & t \in (W \cup X')\backslash(U \cup X) \end{cases} \quad \text{and} \quad t \mapsto \begin{cases} t & t \in V \cup Y \\ 0 & t \in (W \cup Y')\backslash(V \cup Y)\,, \end{cases}$$

respectively, and the vertical maps are the augmentation maps. The existence of these diagrams is contained in the proof of [20, Lemma 3.2.1] in the case that the characteristic of k is zero, and it holds when each φ_i is ρ-Gulliksen minimal by [15, Theorem 24]. Because φ_1, φ_2 are Tor-independent we have the following quasi-isomorphism

$$Q[U,X] \otimes_{Q[W]} Q[V,Y] \xrightarrow{\simeq} S\,.$$

Furthermore, as the kernels of the maps $Q[W] \to Q[U,X]$ and $Q[W] \to Q[V,Y]$ are generated by subsets of W, it follows that the DG algebra $Q[U,X] \otimes_{Q[W]} Q[V,Y]$ is a semifree extension of Q. Finally, since $Q[U,X]$ and $Q[V,Y]$ are minimal models, it follows that $Q[U,X] \otimes_{Q[W]} Q[V,Y]$ is a minimal model over Q as well.

Therefore, taking indecomposables of the following diagram on the DG fibers

$$\begin{tikzcd} \mathrm{F}^{\rho} \arrow[r] \arrow[d] & \mathrm{F}^{\sigma_1} \arrow[d] \\ \mathrm{F}^{\sigma_2} \arrow[r] & \mathrm{F}^{\sigma} \end{tikzcd}$$

induces the following commutative diagram:

$$\begin{tikzcd} kW \arrow[r] \arrow[d] & kU \oplus kX \arrow[d] \\ kV \oplus kY \arrow[r] & (kU \cap kV) \oplus kX \oplus kY \end{tikzcd} \tag{9}$$

where maps are the identity when restricted to kX or kY and the obvious projection onto the first component when restricted to kW, kU, or kV. Following 1.7, upon shifting and taking k-linear duals one obtains the desired pullback diagram of Lie algebras since (9) is a pushout diagram of k-vector spaces. This finishes the proof of the theorem. □

The main result of this section, see Theorem 4.7 below, shows the conclusion of Theorem A is satisfied when the residue field has characteristic zero or when each φ_i is Gulliksen minimal. Theorem 4.7 requires a refinement of 1.8 to discuss the relation between homotopy Lie algebras of the fiber and the target ring of a not necessarily minimal Cohen presentation; this discussion is what follows directly below.

Theorem 4.7. *If, in addition to the hypotheses of Lemma 4.6, ρ is a minimal Cohen presentation, then there is an isomorphism of graded Lie algebras over k*

$$\pi(S) \cong \pi(S_1) \times_{\pi(R)} \pi(S_2).$$

Proof. We are in the context for which Lemma 4.6 and the observations in **??** apply. Hence, there is the following pullback diagram of graded Lie algebras over k

$$\begin{tikzcd}
\pi(K^R) & \pi(K^{S_1}) \times L_1 \arrow[l] \\
\pi(K^{S_2}) \times L_2 \arrow[u] & \pi(K^S) \times (L_1 \times L_2) \arrow[l] \arrow[u]
\end{tikzcd}$$

where the L_i are Lie algebras over k concentrated in degree two. The diagram above decomposes as a product of the pullback diagrams

$$\begin{tikzcd}
\pi(K^R) & \pi(K^{S_1}) \arrow[l] \\
\pi(K^{S_2}) \arrow[u] & \pi(K^S) \arrow[l] \arrow[u]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
0 & L_1 \arrow[l] \\
L_2 \arrow[u] & L_1 \times L_2 \arrow[l] \arrow[u]
\end{tikzcd}$$

Now by referring to 1.8, the first diagram establishes the isomorphism of graded Lie algebras over k:

$$\pi^{\geqslant 2}(S) \cong \pi^{\geqslant 2}(S_1) \times_{\pi^{\geqslant 2}(R)} \pi^{\geqslant 2}(S_2)\,. \tag{10}$$

Again using that φ_1, φ_2 are Tor-independent we have that $I_1 \cap I_2 = I_1 I_2$. Therefore one can apply Lemma 2.6, finishing the proof of the desired result when combined with (10). □

By 4.4 and [13, Theorem 5.6], if φ_i is almost small, then it is Gulliksen minimal. So in light of this point and Theorem 4.7, the next question is prompted:

Question 4.8. *Can one replace the assumption that at least one of φ_i is almost small with the assumption that at least one φ_i is Gulliksen minimal in Theorem 3.3?*

Remark 4.9. Let $\varphi\colon R \to S$ be a not necessarily surjective map of local rings. Consider a Cohen factorization of (the completion of) φ

$$\widehat{R} \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}\,.$$

Assume there is an isomorphism $S_1 \otimes_{R'} S_2 \cong \widehat{S}$ where S_1, S_2 are Tor-independent quotients of R', and that one of the following hold:

(1) The characteristic of the residue field of S is zero;
(2) The maps $R' \to S_i$ are Gulliksen minimal for $i = 1, 2$.

From 1.5 and Theorem 4.7 it follows that

$$\pi^{\geqslant 2}(S) \cong \pi^{\geqslant 2}(S_1) \times_{\pi^{\geqslant 2}(R)} \pi^{\geqslant 2}(S_2).$$

5. Applications

Corollary 5.1. *Suppose $S_1 \xleftarrow{\varphi_1} R \xrightarrow{\varphi_2} S_2$ is a pair of Tor-independent surjective local ring maps with residue field k. Then is a natural isomorphism of graded Lie algebras*

$$\pi(\mathrm{F}^{\varphi_1 \otimes \varphi_2}) \cong \pi(\mathrm{F}^{\varphi_1}) \times \pi(\mathrm{F}^{\varphi_2})$$

where $\varphi_1 \otimes \varphi_2$ are the canonical quotient of R whose target is $S_1 \otimes_R S_2$.

Proof. This is an immediate consequence of Lemma 4.6 with $\rho = \mathrm{id}^R$. □

Corollary 5.2. *Suppose $S_1 \leftarrow R \rightarrow S_2$ is a pair of Tor-independent quotient maps of the local ring R with nontrivial kernels. The canonical quotient $R \rightarrow S_1 \otimes_R S_2$ is never Golod.*

Proof. This follows from Corollary 5.1, [6, Theorem 3.4(4)], and the fact that a product of nonzero Lie algebras cannot be free; the fact that each $R \rightarrow S_i$ has nontrivial kernel, forces the Lie algebras in question to be nonzero. □

Before stating the next Corollary, we recall the definition of depth for modules over a (not necessarily commutative) algebra. Let $\mathcal{A}$ be a nonnegatively graded connected k-algebra and $\mathcal{M}$ a graded left $\mathcal{A}$-module. The *depth* of $\mathcal{M}$ over $\mathcal{A}$ is

$$\operatorname{depth}_{\mathcal{A}} \mathcal{M} = \inf\{n \geqslant 0 \mid \operatorname{Ext}^n_{\mathcal{A}}(k, \mathcal{M}) \neq 0\}.$$

Corollary 5.3. *Let $S_1 \leftarrow R \rightarrow S_2$ be a pair of Tor-independent minimal Cohen presentations, and set $S = S_1 \otimes_R S_2$. If S_1, S_2 are singular local rings, then*

$$\operatorname{depth}\ \operatorname{Ext}_S(k,k) = \operatorname{depth}\ \operatorname{Ext}_{S_1}(k,k) + \operatorname{depth}\ \operatorname{Ext}_{S_2}(k,k) \geqslant 2.$$

Proof. For a singular local ring A, the depth of $\operatorname{Ext}_A(k,k)$ is at least one; see, for example, [14, Lemma 5.1.7]. Now the desired result follows from the equalities below

$$\begin{aligned}
\operatorname{depth}\ \operatorname{Ext}_S(k,k) &= \operatorname{depth}\ \mathrm{U}\pi^{\geqslant 2}(S) \\
&= \operatorname{depth}\ \pi^{\geqslant 2}(S) \\
&= \operatorname{depth}\ \pi^{\geqslant 2}(S_1) + \operatorname{depth}\ \pi^{\geqslant 2}(S_2) \\
&= \operatorname{depth}\ \operatorname{Ext}_{S_1}(k,k) + \operatorname{depth}\ \operatorname{Ext}_{S_2}(k,k).
\end{aligned}$$

The first equality comes from [14, Lemma 5.1.6] where $\mathrm{U}\pi^{\geqslant 2}(R)$ denotes the universal enveloping algebra of $\pi^{\geqslant 2}(R)$; the third equality comes from Corollary 5.1—combined with 1.8—and [18, Proposition 36.2]. The final equality follows again from [14, Lemma 5.1.6]. Since S_1 and S_2 are both singular, it follows that

$$\operatorname{depth}\ \operatorname{Ext}_S(k,k) \geqslant 2.$$ □

Remark 5.4. An important consequence of Corollary 5.3 is due to [14, Theorem 7.2(3)], which asserts if the depth of the Ext algebra of a local ring is at least two, then the stable cohomology algebra of that ring has a particularly simple structure. Namely, the stable cohomology of a local ring is a direct sum of its Ext algebra and a specific torsion submodule; see the reference above for more details. Furthermore, if the ring S is also assumed to be Gorenstein then its stable cohomology algebra is a trivial extension of its Ext algebra and a shift of the k-dual of its Ext algebra; cf. [19, Theorem 5.2].

5.5. We denote by $\mathcal{H}$ the category of nonnegatively graded cocommutative Hopf k-algebras with codivided powers. That is, those Hopf algebras A whose dual $\operatorname{Hom}_k(A,k)$ is a nonnegatively graded commutative Hopf algebra with divided powers; see [1] for more details. This category becomes relevant to the present discussion as the universal enveloping algebra functor takes values in $\mathcal{H}$ and restricts to an equivalence of categories when specializing to positively graded finite type (adjusted) Lie algebras over k; cf. [1, 26, 32].

The next result is now an immediate consequence of Theorem A and 5.5.

Corollary 5.6. *Let $S_1 \leftarrow R \to S_2$ be a pair of Tor-independent surjective local ring maps, and set $S = S_1 \otimes_R S_2$. If at least one of the conditions in Theorem A, then $\operatorname{Ext}_S(k,k)$ is the pullback in $\mathcal{H}$ along the naturally induced maps $\operatorname{Ext}_{S_1}(k,k) \to \operatorname{Ext}_R(k,k)$ for $i = 1,2$.*

Corollary 5.7. *With the notation and assumptions in Corollary 5.6, there is an isomorphism of graded commutative Hopf algebras with divided powers*

$$\operatorname{Tor}^S(k,k) \cong \operatorname{Tor}^{S_1}(k,k) \otimes_{\operatorname{Tor}^R(k,k)} \operatorname{Tor}^{S_2}(k,k)\,.$$

Proof. Consider the functor $\operatorname{Hom}_k(-,k)$ from $\mathcal{H}$ to the category of graded commutative Hopf algebras with divided powers. This functor takes pullbacks to pushouts. Moreover Tor algebras and Ext algebras are Hopf k-dual of each other, see [20, Theorem 2.3.3]. Therefore the corollary follows from Corollary 5.6. $\square$

Corollary 5.7 generalizes results of Avramov in [2, Theorem 2] and [4, Corollary 5.6(a)], while the next corollary, which is Theorem B from the introduction, specializes to [4, Corollary 5.6(b)]. First, a bit of notation.

5.8. For a local ring A with residue field k, recall the Poincaré series of k over A is the formal power series

$$\mathrm{P}_k^A(t) = \sum_{i\geqslant 0} \dim_k(\operatorname{Ext}_A^i(k,k))t^i\,.$$

The number $\varepsilon_i(A) := \dim_k \pi^i(A)$ is called the i^{th} deviation of A. The sequence of deviations are encoded in the Poincaré series of k over A according to the equality

$$\mathrm{P}_k^A(t) = \prod_{i=1}^{\infty} \frac{(1+t^{2i-1})^{\varepsilon_{2i-1}(A)}}{(1-t^{2i})^{\varepsilon_{2i}(A)}}\,;$$

for the equality see [7, 7.1]

Corollary 5.9. *Suppose $S_1 \xleftarrow{\varphi_1} R \xrightarrow{\varphi_2} S_2$ is a pair of Tor-independent surjective local ring maps with residue field k, and set $S = S_1 \otimes_R S_2$. If at least one φ_i is almost small, then*

$$\mathrm{P}_k^S(t) = \frac{\mathrm{P}_k^{S_1}(t)\cdot \mathrm{P}_k^{S_2}(t)}{\mathrm{P}_k^R(t)}.$$

Proof. By Theorem 3.3, there is the following exact sequence of graded Lie algebras over k:

$$0 \to \pi^{\geqslant 2}(S_1) \times_{\pi^{\geqslant 2}(R)} \pi^{\geqslant 2}(S_2) \to \pi^{\geqslant 2}(S_1) \times \pi^{\geqslant 2}(S_2) \to \pi^{\geqslant 2}(R)$$

where the map on the right is the difference of the induced maps on homotopy Lie algebras. Under the assumptions of the corollary, we deduce that the map on the right is surjective and therefore

$$\varepsilon_i(S) = \varepsilon_i(S_2) + \varepsilon_i(S_2) - \varepsilon_i(R)$$

for all $i \geqslant 2$. Combining this with Lemma 2.6 we obtain the desired result on Poincaré series. $\square$

Notice the conclusion of Corollary 5.9 is only asserted under one of the three conditions in Theorem A, while other corollaries like Corollary 5.7 hold under any of the conditions listed in Theorem A. In light of this point we pose the following question.

Question 5.10. *In the notation of Corollary 5.9, does the conclusion of Corollary 5.9 hold if the characteristic of k is zero, or if both φ_1, φ_2 are Gulliksen minimal?*

Remark 5.11. The conclusion of Corollary 5.9 also holds if at least one of the maps ψ_i is large, under the standing assumption that S_1, S_2 are Tor-independent.

Indeed, if ψ_1 is large, then the induced map $\mathrm{Tor}^{S_1}(S,k) \to \mathrm{Tor}^{S}(k,k)$ is injective; see [24, Theorem 1.1]. Furthermore, from following commutative diagram

$$\begin{tikzcd}
\mathrm{Tor}^R(S_2,k) \arrow[r, "\cong"] \arrow[d] & \mathrm{Tor}^{S_1}(S,k) \arrow[d] \\
\mathrm{Tor}^R(k,k) \arrow[r] & \mathrm{Tor}^{S_1}(k,k),
\end{tikzcd}$$

it follows that the map $\mathrm{Tor}^R(S_2,k) \to \mathrm{Tor}^R(k,k)$ is injective. Therefore φ_1 is also large. Finally by [24, Theorem 1.1], one has the following equalities on Poincaré series

$$\mathrm{P}_k^R(t) = \mathrm{P}_{S_2}^R(t) \cdot \mathrm{P}_k^{S_2}(t) \quad \text{and} \quad \mathrm{P}_k^{S_1}(t) = \mathrm{P}_S^{S_1}(t) \cdot \mathrm{P}_k^{S}(t)\,,$$

from which the desired equality is easily deduced.

We end this paper with one final application generalizing results of Quillen in [29, Corollary 4.9 & Remark 11.13], which can also be interpreted as a generalized "dual" statement to a result of Quillen [30, Corollary 7.5] (as well as [30, Theorem 11.10]) involving André–Quillen cohomology. The latter result can also be deduced by an analogous result from local algebra [27, Theorem A].

For a surjective local map $\varphi\colon R \to S$ with residue field k, we write $\mathrm{D}(S/R;k)$ for the André–Quillen cohomology module of φ with coefficients in k; further details can be found in [21, 25]. Lemma 4.6 can be recast in the theorem below by applying [11, Section 6], a result essentially due to Quillen [30, Theorem 9.5], provided k has charactersitic zero.

Corollary 5.12. *Let $Q \to R \to S_i$ be surjective local ring maps with residue field k for $i = 1,2$. If S_1, S_2 are Tor-independent over R and k has characteristic zero, then there is an isomorphism of graded Lie algebras*

$$\mathrm{D}(S_1 \otimes_R S_2/Q;k) \cong \mathrm{D}(S_1/Q;k) \times_{\mathrm{D}(R/Q;k)} \mathrm{D}(S_2/Q;k)\,.$$

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