A PARTIAL CONVERSE GHOST LEMMA FOR THE DERIVED CATEGORY OF A COMMUTATIVE NOETHERIAN RING

JIAN LIU AND JOSH POLLITZ

Abstract. In this article a condition is given to detect the containment among thick subcategories of the bounded derived category of a commutative noetherian ring. More precisely, for a commutative noetherian ring R and complexes of R-modules with finitely generated homology M and N, we show N is in the thick subcategory generated by M if and only if the ghost index of $N_\mathfrak{p}$ with respect to $M_\mathfrak{p}$ is finite for each prime $\mathfrak{p}$ of R. To do so, we establish a "converse coghost lemma" for the bounded derived category of a non-negatively graded DG algebra with noetherian homology.

Introduction

This article is concerned with certain numerical invariants and thick subcategories in the bounded derived category of a commutative noetherian ring. Let R be a commutative noetherian ring, $\mathsf{D}(R)$ will denote its derived category, and $\mathsf{D}^f_b(R)$ will be the full subcategory of $\mathsf{D}(R)$ consisting of objects with finitely generated homology.

An object N of $\mathsf{D}(R)$ is in the thick subcategory generated by M, denoted $\mathsf{thick}_{\mathsf{D}(R)}(M)$, provided that N can be inductively built from M using the triangulated structure of $\mathsf{D}(R)$ (see 2.1 for more details). There are cases where a notion of support reports on whether N is in $\mathsf{thick}_{\mathsf{D}(R)}(M)$. For example, there is the celebrated theorem of Hopkins [13, Theorem 11] and Neeman [19, Theorem 1.5] that applies when M and N are perfect complexes. Another instance is when R is locally complete intersection by using support varieties; this was proved by Stevenson for thick subcategories containing R when R is a quotient of a regular ring [22, Corollary 10.5], and in general in [18, Theorem 3.1]. However, in general, detecting containment among thick subcategories can be an intractable task.

In this article, we give a new criterion to determine the containment among thick subcategories of $\mathsf{D}^f_b(R)$ based on certain numerical invariants being locally finite. We quickly define these below; see 2.1, 2.3, and 2.10 for precise definitions.

For a triangulated category T, fix objects G and X. The level of X with respect to G counts the minimal number of cones needed to generate X, up to suspensions and direct summand, starting from G. We denote this by $\mathsf{level}^G_\mathsf{T}(X)$ and note that this is finite exactly when X is in $\mathsf{thick}_\mathsf{T}(G)$. The coghost index of X with respect to G, denoted $\mathsf{cogin}^G_\mathsf{T}(X)$, is the minimal number n satisfying that any composition

$$X^n \xrightarrow{f^n} X^{n-1} \to \dots \xrightarrow{f^1} X^0 = X,$$

2010 *Mathematics Subject Classification.* 18E30 (primary); 13B30, 16E45 (secondary).

Key words and phrases. derived category, (co)ghost index, (co)ghost lemma, level, DG algebra, Koszul complex.

where each $\mathrm{Hom}_{\mathsf{T}}(f^i, \Sigma^j G) = 0$, must be zero in T. Switching the variance in the definition above determines the ghost index of X with respect to G, denoted $\mathsf{gin}^G_{\mathsf{T}}(X)$.

These invariants are of independent interest and have been studied in [2, 3, 5, 7, 8, 10, 16, 17, 21]. In general, they are related by the following well-known *(co)ghost lemma*:

$$\max\{\mathsf{cogin}^G_{\mathsf{T}}(X), \mathsf{gin}^G_{\mathsf{T}}(X)\} \le \mathsf{level}^G_{\mathsf{T}}(X).$$

Oppermann and Šťovíček proved a so-called *converse coghost lemma:* Namely, $\mathsf{cogin}^M_{\mathsf{D}^f_b(R)}(N)$ and $\mathsf{level}^M_{\mathsf{D}^f_b(R)}(N)$ agree whenever M and N are objects of $\mathsf{D}^f_b(R)$, see [20, Theorem 24]. Letz extracted a notable consequence from the converse co-ghost lemma in [17, Theorem 3.6]: for M and N in $\mathsf{D}^f_b(R)$,

$$\mathsf{level}^M_{\mathsf{D}^f_b(R)}(N) < \infty \iff \mathsf{level}^{M_{\mathfrak{p}}}_{\mathsf{D}^f_b(R_{\mathfrak{p}})}(N_{\mathfrak{p}}) < \infty \text{ for all prime ideals } \mathfrak{p} \text{ of } R.$$

In this article we ask whether finiteness of certain ghost indices can determine finiteness of level, and hence containment among thick subcategories. The main result in this direction is the following which is contained in Theorem 3.1.

Theorem A. *Let R be a commutative noetherian ring. For M and N in $\mathsf{D}^f_b(R)$, the following are equivalent:*

(1) $\mathsf{level}^M_{\mathsf{D}^f_b(R)}(N) < \infty$*;*

(2) $\mathsf{gin}^{M_{\mathfrak{p}}}_{\mathsf{D}^f_b(R_{\mathfrak{p}})}(N_{\mathfrak{p}}) < \infty$ *for all prime ideals $\mathfrak{p}$ of R.*

One of the main steps in the proof of Theorem A is establishing a converse coghost lemma for graded-commutative, bounded below DG algebras A with $\mathrm{H}(A)$ a noetherian $\mathrm{H}_0(A)$-module (cf. Theorem 2.6). We follow the proof of [20, Theorem 24] closely, however, extra care is needed when working with such DG algebras. Namely, we make use of certain ascending semifree filtrations, see 1.6, as the truncations used by Oppermann and Šťovíček are no longer available in this setting. It is also worth mentioning this recovers [20, Theorem 2] for noetherian rings, and more generally for the noetherian DG algebras discussed above (see Corollary 2.7).

Acknowledgements. We thank Srikanth Iyengar and Janina Letz for stimulating discussions related to this work. We also thank an anonymous referee for a careful reading of the manuscript, and their helpful comments and suggestions. The second author was supported by NSF grant numbers 1840190 and 2002173.

1. Derived Category of a DG Algebra and Semifree DG Modules

Much of this section is devoted to setting notation and reviewing the necessary background regarding the topics from the title of the section. Proposition 1.8 is the main technical result of the section and will be put to use in the next section.

Throughout this article objects will be graded homologically. By a DG algebra we will implicitly assume A is non-negatively graded and graded-commutative. For the rest of the section fix a DG algebra A.

1.1. Let $\mathsf{D}(A)$ denote the derived category of (left) DG A-modules (see, for example, [3, Sections 2 & 3] or [15, Section 4]). We use Σ to denote the suspension functor on the triangulated category $\mathsf{D}(A)$ where Σ is the autoequivalence defined by

$$(\Sigma M)_i := M_{i-1},\ a \cdot (\Sigma m) := (-1)^{|a|} am,\ \text{and } \partial^{\Sigma M} := -\partial^M.$$

For a DG A-module M, its homology $\mathrm{H}(M) = \{\mathrm{H}_i(M)\}_{i\in\mathbb{Z}}$ is naturally a graded $\mathrm{H}(A)$-module. Also, define the *infimum of* M to be $\inf(M) \coloneqq \inf\{n \mid \mathrm{H}_n(M) \neq 0\}$; its *supremum* is $\sup(M) \coloneqq \sup\{n \mid \mathrm{H}_n(M) \neq 0\}$.

1.2. The following triangulated subcategories of $\mathsf{D}(A)$ will be of interest in the sequel. First, let $\mathsf{D}^f(A)$ denote the full subcategory of $\mathsf{D}(A)$ consisting of DG A-modules M such that each $\mathrm{H}_i(M)$ is a noetherian $\mathrm{H}_0(A)$-module. We let $\mathsf{D}^f_+(A)$ be the full subcategory of objects M of $\mathsf{D}^f(A)$ such that $\inf(M) > -\infty$. Finally, $\mathsf{D}^f_b(A)$ consists of those objects M of $\mathsf{D}^f(A)$ satisfying $\mathrm{H}_i(M) = 0$ for all $|i| \gg 0$. When $\mathrm{H}(A)$ is noetherian as a module over $\mathrm{H}_0(A)$ and $\mathrm{H}_0(A)$ is noetherian, $\mathsf{D}^f_b(A)$ is exactly the full subcategory of $\mathsf{D}(A)$ whose objects M are those with $\mathrm{H}(M)$ being finitely generated as a graded $\mathrm{H}(A)$-module.

1.3. A DG A-module F is *semifree* if it admits a filtration of DG A-submodules

$$\ldots \subseteq F(-1) \subseteq F(0) \subseteq F(1) \subseteq \ldots$$

where $F(i) = 0$ for $i \ll 0$, $F = \cup F(i)$ and each $F(i)/F(i-1)$ is a direct sum of shifts of A. The filtration above is called a *semifree filtration* of F.[1] By [15, Section 3], F is homotopy colimit of the $F(i)$ and so there is the following exact triangle in $\mathsf{D}(A)$

$$\coprod_{i\in\mathbb{Z}} F(i) \xrightarrow{1-s} \coprod_{i\in\mathbb{Z}} F(i) \to F \to \Sigma \coprod_{i\in\mathbb{Z}} F(i), \tag{1}$$

where s is induced by the canonical inclusions $F(j) \hookrightarrow F(j+1) \hookrightarrow \coprod F(i)$.

1.4. For the following background on semifree resolutions see [11, Chapter 6] (or [1, Section 1.3]). Let M be a DG A-module. There exists a surjective quasi-isomorphism of DG A-modules $\epsilon\colon F \xrightarrow{\simeq} M$ where F is a semifree DG A-module, see [11, Proposition 6.6]; the map ϵ is called a *semifree resolution* of M over A. Semifree resolutions of M are unique up to homotopy equivalence.

1.5. Fix a DG A-module M with semifree resolution $\epsilon\colon F \xrightarrow{\simeq} M$. For any DG A-module N, it is clear that

$$\mathrm{Hom}_{\mathsf{D}(A)}(M, N) = \mathrm{Hom}_{\mathsf{D}(A)}(F, N)$$

and the right-hand side is computed as the degree zero homology of the DG A-module $\mathrm{Hom}_A(F, N)$. That is,

$$\mathrm{Hom}_{\mathsf{D}(A)}(M, -) = \mathrm{H}_0(\mathrm{Hom}_A(F, -)). \tag{2}$$

In particular, $\mathrm{Hom}_{\mathsf{D}(A)}(M, N)$ naturally inherits an $\mathrm{H}_0(A)$-module structure and since A is non-negatively graded, $\mathrm{Hom}_{\mathsf{D}(A)}(M, N)$ inherits an A_0-module structure. As semifree resolutions are homotopy equivalent, this $\mathrm{H}_0(A)$-module is independent of choice of semifree resolution.

1.6. Assume each $\mathrm{H}_i(A)$ is finitely generated over $\mathrm{H}_0(A)$ and $\mathrm{H}_0(A)$ is itself noetherian. Let M be an object of $\mathsf{D}^f_+(A)$. By [4, Appendix B.2], there exists a semifree resolution $F \xrightarrow{\simeq} M$ with $F_i = 0$ for all $i < \inf(M)$ and F admits a semifree filtration $\{F(i)\}_{i\in\mathbb{Z}}$ equipped with exact sequences of DG A-modules

$$0 \to F(i-1) \to F(i) \to \Sigma^i A^{\beta_i} \to 0$$

[1] The choice to allow arbitrary indices for the start of the filtration is a non-standard one, but this simplifies notation in the proof of Theorem 2.6.

for some non-negative integer $\beta_i \geq 0$.

Lemma 1.7. *Assume* $\mathrm{H}_0(A)$ *is noetherian and that each* $\mathrm{H}_i(A)$ *is finitely generated over it. Let* N *be an object of* $\mathsf{D}(A)$ *such that* $\sup(N) < \infty$. *For an object* M *in* $\mathsf{D}(A)$ *with* $\inf(M) > \sup(N)$, $\mathrm{Hom}_{\mathsf{D}(A)}(M, N) = 0$.

Proof. Fix a semifree resolution $F \xrightarrow{\simeq} M$ as in 1.6. By (2) in 1.5,

$$\mathrm{Hom}_{\mathsf{D}(A)}(\Sigma^i A^{\beta_i}, N) \cong \mathrm{H}_i(N)^{\beta_i} = 0$$

for each $i \geq \inf(M)$. Combining these isomorphisms with the exact sequences

$$0 \to F(i-1) \to F(i) \to \Sigma^i A^{\beta_i} \to 0$$

yields by induction that $\mathrm{Hom}_{\mathsf{D}(A)}(F(i), N) = 0$ for all $i \geq \inf(M)$. Finally, (1) in 1.3 implies $\mathrm{Hom}_{\mathsf{D}(A)}(F, N) = 0$, and hence $\mathrm{Hom}_{\mathsf{D}(A)}(M, N) = 0$ (cf. 1.5). □

Proposition 1.8. *Assume* $\mathrm{H}_0(A)$ *is noetherian and each* $\mathrm{H}_i(A)$ *is a finitely generated* $\mathrm{H}_0(A)$*-module. Let* M *be in* $\mathsf{D}^f_+(A)$ *and* N *be an object in* $\mathsf{D}(A)$ *such that* $\sup(N) < \infty$. *Suppose* $F \xrightarrow{\simeq} M$ *is a semifree resolution of* M *as in 1.6, then for all* $i > \sup(N)$ *the natural map below is an isomorphism*

$$\mathrm{Hom}_{\mathsf{D}(A)}(M, N) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{D}(A)}(F(i), N).$$

Proof. For each $i \geq \inf(M)$, there is an exact sequence of DG A-modules

$$0 \to F(i) \to F \to F' \to 0 \qquad (3)$$

where by choice of F we have that $\inf(F') > i$. Applying $\mathrm{Hom}_{\mathsf{D}(A)}(-, N)$ to (3) and appealing to Lemma 1.7 yields the desired isomorphisms whenever $i > \sup(N)$. □

2. Levels and Coghost Index

We begin by briefly recalling the notion of level. For more details, see [3, Section 2], [8, Section 2] or [21, Section 3].

2.1. Let T be a triangulated category and C be a full subcategory of T. We say C is *thick* if it is closed under suspensions, retracts and cones. The smallest thick subcategory of T containing an object X is denoted $\mathsf{thick}_{\mathsf{T}}(X)$; this consists of all objects Y such that one can obtain Y from X using finitely many suspensions, retracts and cones.

We set $\mathsf{level}^X_{\mathsf{T}}(Y)$ to be smallest non-negative integer n such that Y can be built starting from X using finitely many suspensions, finitely many retracts and exactly $n-1$ cones in T. If no such n exists, we set $\mathsf{level}^X_{\mathsf{T}}(Y) = \infty$. Note Y is in $\mathsf{thick}_{\mathsf{T}}(X)$ if and only if $\mathsf{level}^X_{\mathsf{T}}(Y) < \infty$. Also, if C is a thick subcategory of T containing X and Y, then $\mathsf{level}^X_{\mathsf{T}}(Y) = \mathsf{level}^X_{\mathsf{C}}(Y)$.

Example 2.2. Let A be a DG algebra. A DG A-module M is *perfect* if M is an object of $\mathsf{thick}_{\mathsf{D}(A)}(A)$. In this case, M is a retract of a semifree DG A-module F with finite semifree filtration

$$0 \subseteq F(0) \subseteq F(1) \subseteq \ldots \subseteq F(n) = F.$$

If n is the minimal such value, then $\mathsf{level}^A_{\mathsf{D}(A)}(M) = n+1$, see [3, Theorem 4.2].

2.3. Let T be a triangulated category with suspension functor Σ. A morphism $f\colon X \to Y$ in T is called *G-coghost* if

$$\mathrm{Hom}_{\mathsf{T}}(f, \Sigma^i G)\colon\ \mathrm{Hom}_{\mathsf{T}}(Y, \Sigma^i G) \to \mathrm{Hom}_{\mathsf{T}}(X, \Sigma^i G)$$

is zero for all $i \in \mathbb{Z}$. Following [17, Defnition 2.4], we define the *coghost index of X with respect to G in* T, denoted $\mathsf{cogin}_T^G(X)$, to be the smallest non-negative integer n such that any composition of G-coghost maps

$$X^n \xrightarrow{f^n} X^{n-1} \xrightarrow{f^{n-1}} \ldots \to X^1 \xrightarrow{f^1} X^0 = X$$

is zero in T.

2.4. Let T be a triangulated category with objects G and X. In this generality, level bounds cogin from above. That is,

$$\mathsf{cogin}_{\mathsf{T}}^G(X) \leq \mathsf{level}_{\mathsf{T}}^G(X),$$

see [5, Lemma 2.2(1)] (see also [21, Lemma 4.11]). However, there are known instances when equality holds. For example, $\mathsf{level}_{\mathsf{T}}^G(-)$ and $\mathsf{cogin}_{\mathsf{T}}^G(-)$ agree provided every object in T has an appropriate left approximation by G, see [5, Lemma 2.2(2)] for more details. Another instance is when R is a commutative noetherian ring (or more generally, a noether algebra)

$$\mathsf{cogin}_{\mathsf{D}_b^f(R)}^G(X) = \mathsf{level}_{\mathsf{D}_b^f(R)}^G(X)$$

for each G and X in $\mathsf{D}_b^f(R)$; this has been coined the *converse coghost lemma* (see [20, Theorem 24]).

Example 2.5. Let A be the ring $\mathbb{Z}/(4)$ and consider the complex

$$X = 0 \to A \xrightarrow{2\cdot} A \xrightarrow{2\cdot} A \to 0\,.$$

It is straightforward to see that $2 \cdot \mathrm{id}^X$ is A-coghost yet it is nonzero in $\mathsf{D}_b^f(A)$. In fact, it is also A-ghost; cf. 2.10.

We now get to the main result of the section which generalizes a particular case of [20, Theorem 24] mentioned above. It is worth noting that [20, Theorem 24] was proved for derived categories satisfying certain finiteness conditions; however, it does not apply directly to the case considered in the theorem below. The proof of [20, Theorem 24] is suitably adapted to the setting under consideration with the main observation being that truncations need to be replaced with the ascending filtrations discussed in 1.6. We have indicated the necessary changes below, while attempting to not recast the parts of the proof of [20, Theorem 24] that carry over with only minor changes.

Theorem 2.6. *Let A be a DG algebra with $\mathrm{H}(A)$ noetherian as an $\mathrm{H}_0(A)$-module. If M and N are in $\mathsf{D}_b^f(A)$, then*

$$\mathsf{cogin}_{\mathsf{D}_b^f(A)}^M(N) = \mathsf{level}_{\mathsf{D}_b^f(A)}^M(N).$$

Recall that a triangulated category T is called *strongly finitely generated* if there exists G in T and $d \in \mathbb{N}$ such that $\mathsf{level}_{\mathsf{T}}^G(X) \leq d$ for all X in T, see [21, 3.1.1]. For example, let A be an artinian ring, then $\mathsf{D}_b^f(A)$ is strongly finitely generated by $G = A/J$ where J is the Jacobson radical of A; see [21, Proposition 7.38]. Using Theorem 2.6 the same argument in [20, Theorem 7] yields a DG version of Oppermann and Šťovíček's result recorded below.

Corollary 2.7. *Let A be as in Theorem 2.6 and T be a thick subcategory of $\mathsf{D}^f_b(A)$ containing $\mathsf{thick}_{\mathsf{D}^f_b(A)}(A)$. If T is strongly finitely generated, then $\mathsf{T} = \mathsf{D}^f_b(A)$.*

Before beginning the proof of Theorem 2.6, we record the following remark and lemma; both are easy but important pieces in establishing Theorem 2.6.

Remark 2.8. For M and N in $\mathsf{D}^f_b(R)$,

$$\mathsf{cogin}^M_{\mathsf{D}^f_+(A)}(N) = \mathsf{level}^M_{\mathsf{D}^f_+(A)}(N).$$

Indeed, one can directly apply the argument from [20, Theorem 24] once it is noted that, by restricting scalars along the map of commutative rings $A_0 \to \mathrm{H}_0(R)$, $\mathrm{Hom}_{\mathsf{D}^f_+(A)}(X, \Sigma^i N)$ is finitely generated over A_0 for X in $\mathsf{D}^f_+(A)$ and $i \in \mathbb{Z}$.

To see the latter holds, such an X admits a resolution with a semifree filtration whose subquotients are perfect DG A-modules. Also, since N is in $\mathsf{D}^f_b(A)$ we can apply Proposition 1.8 to get

$$\mathrm{Hom}_{\mathsf{D}^f_+(A)}(X, \Sigma^i N) \cong \mathrm{Hom}_{\mathsf{D}^f_+(A)}(P, \Sigma^i N)$$

where P a perfect DG A-module with a finite semifree filtration as in Example 2.2. Therefore, induction on the length of this filtration finishes the proof of the claim, where we are again using that N is in $\mathsf{D}^f_b(A)$.

Lemma 2.9. *Let A be a DG algebra. Assume $\alpha\colon F^1 \to F^2$ is a morphism of bounded below semifree DG A-modules with $F^j_i = 0$ for $i < \inf(F^j)$ and semifree filtrations $\{F^j(i)\}_{i\in\mathbb{Z}}$ for $j = 1, 2$ satisfying*

$$0 \to F^j(i-1) \to F^j(i) \to \coprod_{\ell \le i} \Sigma^\ell A^{\beta^j_\ell(i)} \to 0$$

for non-negative integers $\beta^j_\ell(i)$ and $j = 1, 2$. For each $i \in \mathbb{Z}$, α restricts to a morphism of DG A-modules $\alpha(i)\colon F^1(i) \to F^2(i)$.

Proof. Indeed, $F^1(i) = 0$ for all $i < \inf(F^1)$ and so there is nothing to show for such values of i. Now for $i \ge \inf(F^1)$, the DG A-module $F^1(i)$ is generated in degrees at most i and since α is degree preserving $\alpha(F^1(i))$ is generated in degrees at most i. However, the assumption on the filtration $\{F^2(j)\}$ also implies

$$\alpha(F^1(i)) \subseteq F^2(i).$$

Hence, setting $\alpha(i) \coloneqq \alpha\,|_{F^1(i)}$ proves the claim by induction. □

Proof of Theorem 2.6. First, by 2.1 and Remark 2.8

$$\mathsf{level}^M_{\mathsf{D}(A)}(N) = \mathsf{level}^M_{\mathsf{D}^f_+(A)}(N) = \mathsf{cogin}^M_{\mathsf{D}^f_+(A)}(N),$$

while the inequality

$$\mathsf{cogin}^M_{\mathsf{D}^f_+(A)}(N) \ge \mathsf{cogin}^M_{\mathsf{D}^f_b(A)}(N) \tag{4}$$

is standard. So it suffices to prove the reverse inequality of (4) holds.

Set $n = \mathsf{cogin}^M_{\mathsf{D}^f_b(A)}(N)$ and consider a composition

$$N^n \xrightarrow{f^n} N^{n-1} \xrightarrow{f^{n-1}} \dots \xrightarrow{f^2} N^1 \xrightarrow{f^1} N^0 = N,$$

where each f^i is a M-coghost map in $\mathsf{D}^f_+(A)$. Using the assumptions on $\mathrm{H}(A)$ and that each N^i is in $\mathsf{D}^f_+(A)$, there exist semifree resolution $F^i \xrightarrow{\simeq} N^i$ with corresponding semifree filtrations $\{F^i(j)\}_{j\in\mathbb{Z}}$ as in 1.6. Moreover, by 1.5(2), each f^i determines a morphism of DG A-modules $\alpha^i\colon F^i \to F^{i-1}$ such that the following diagram commutes in $\mathsf{D}(A)$

$$\begin{array}{ccc} F^i & \xrightarrow{\alpha^i} & F^{i-1} \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ N^i & \xrightarrow{f^i} & N^{i-1}. \end{array} \tag{5}$$

Furthermore, by Lemma 2.9 there are the following commutative diagrams of DG A-modules

$$\begin{array}{ccccc} F^i(j) & \xrightarrow{\alpha^i(j)} & F^{i-1}(j) & \hookrightarrow & F^{i-1}(j') \\ \downarrow & & & \searrow & \downarrow \\ F^i & & \xrightarrow{\alpha^i} & & F^{i-1} \end{array} \tag{6}$$

whenever $j' \geq j$. Moreover, since each $F^i(j)$ is a perfect DG A-module and M is in $\mathsf{D}^f_b(A)$, the commutativity of the diagrams in (5) and the assumption that each f^i is M-coghost imply the compositions along the top of (6) are M-coghost for all $j' \geq j \gg 0$; the same argument as in proof of [20, Theorem 24] works in this setting.

Combining this with Proposition 1.8 there exists integers i_j such that

$$\begin{array}{ccccccc} F^n(i_n) & \xrightarrow{\beta^n} & F^{n-1}(i_{n-1}) & \xrightarrow{\beta^{n-1}} & \dots & \xrightarrow{\beta^1} & F^0(i_0) \\ \downarrow & & \downarrow & & & & \downarrow \\ F^n & \xrightarrow{\alpha^n} & F^{n-1} & \xrightarrow{\alpha^{n-1}} & \dots & \xrightarrow{\alpha^1} & F^0 \\ \downarrow{\simeq} & & \downarrow{\simeq} & & & & \downarrow{\simeq} \\ N^n & \xrightarrow{f^n} & N^{n-1} & \xrightarrow{f^{n-1}} & \dots & \xrightarrow{f^1} & N^0 \end{array}$$

commutes in $\mathsf{D}(A)$, the natural map

$$\operatorname{Hom}_{\mathsf{D}^f_+(A)}(F^n, N) \xrightarrow{\cong} \operatorname{Hom}_{\mathsf{D}^f_+(A)}(F^n(i_n), N) \tag{7}$$

is an isomorphism and each β^i is M-coghost. Now since each β^i is an M-coghost map between perfect DG A-modules then by choice of n the composition along the top and then down to N, denoted β, must be zero. It is worth noting that the previous step needs the assumption that $\mathrm{H}(A)$ is finitely generated over $\mathrm{H}_0(A)$ since, in this case, each map in the composition defining β must be in $\mathsf{D}^f_b(A)$.

Finally, the isomorphism in (7) identifies β with $f\sigma$ where $f = f^1 f^2 \dots f^n$ and σ is the quasi-isomorphism $F^n \xrightarrow{\simeq} N^n$ defined in (5). Hence, $f = 0$ and so

$$\mathsf{cogin}^M_{\mathsf{D}^f_+(A)}(N) \leq n = \mathsf{cogin}^M_{\mathsf{D}^f_b(A)}(N),$$

as needed. □

2.10. Let T be a triangulated category and fix G and X in T. The *ghost index of* X *with respect to* G *in* T, denoted $\mathsf{gin}^G_T(X)$, to be the least non-negative integer n such that any composition of G-ghost maps

$$X = X^n \xrightarrow{f^n} X^{n-1} \xrightarrow{f^{n-1}} \dots \to X^1 \xrightarrow{f^1} X^0$$

is zero in T, where a map g is G-*ghost* provided $\operatorname{Hom}_{\mathsf{T}}(\Sigma^i G, g) = 0$ for all $i \in \mathbb{Z}$. That is, $\mathsf{gin}^G_{\mathsf{T}}(X) = \mathsf{cogin}^G_{\mathsf{T}^{op}}(X)$. In general, $\mathsf{gin}^G_{\mathsf{T}}(X) \leq \mathsf{level}^G_{\mathsf{T}}(X)$ and it is unknown whether equality holds when R is a commutative noetherian ring and $\mathsf{T} = \mathsf{D}^f_b(R)$. The point of the next section is to provide a partial "converse."

3. A Partial Converse Ghost Lemma

In this section R is a commutative noetherian ring. As localization defines an exact functor $\mathsf{D}(R) \to \mathsf{D}(R_{\mathfrak{p}})$, level cannot increase upon localization. Hence, for M and N in $\mathsf{D}^f_b(R)$, if N is in $\mathsf{thick}_{\mathsf{D}(R)}(M)$, then

$$\mathsf{gin}^{M_{\mathfrak{p}}}_{\mathsf{D}^f_b(R_{\mathfrak{p}})}(N_{\mathfrak{p}}) < \infty \text{ for all } \mathfrak{p} \in \operatorname{Spec} R.$$

The converse and an evident corollary are established below.

Theorem 3.1. *Let R be a commutative noetherian ring and fix M and N in $\mathsf{D}^f_b(R)$. If $\mathsf{gin}^{M_{\mathfrak{p}}}_{\mathsf{D}^f_b(R_{\mathfrak{p}})}(N_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R$, then N is an object of $\mathsf{thick}_{\mathsf{D}(R)}(M)$.*

Corollary 3.2. *If $\mathsf{gin}^{M_{\mathfrak{p}}}_{\mathsf{D}^f_b(R_{\mathfrak{p}})}(N_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathsf{gin}^{M}_{\mathsf{D}^f_b(R)}(N) < \infty$.*

To prove Theorem 3.1, there are essentially two steps. We first go to derived categories of certain Koszul complexes where it is shown that cogin, gin and level all agree using Theorem 2.6. Second, we apply a local-to-global principle to conclude the desired result. We explain this below and give the proof of the theorem at the end of the section.

3.3. Assume R is local with maximal ideal $\mathfrak{m}$, we let K^R be the Koszul complex on a minimal generating set for $\mathfrak{m}$. It is regarded as a DG algebra in the usual way and is well-defined up to an isomorphism of DG R-algebras, see [9, Section 1.6]. For any $\mathfrak{p} \in \operatorname{Spec} R$, let M be an object of $\mathsf{D}(R)$. We set

$$M(\mathfrak{p}) \coloneqq M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} K^{R_{\mathfrak{p}}}$$

which is a DG $K^{R_{\mathfrak{p}}}$-module. Restricting scalars along the morphism of DG algebras $R_{\mathfrak{p}} \to K^{R_{\mathfrak{p}}}$ we may regard $M(\mathfrak{p})$ as an object of $\mathsf{D}(R_{\mathfrak{p}})$. In [6, Theorem 5.10], Benson, Iyengar and Krause proved the following local-to-global principle: For objects M and N in $\mathsf{D}^f_b(R)$, N is in $\mathsf{thick}_{\mathsf{D}(R)}(M)$ if and only if $N(\mathfrak{p})$ is in $\mathsf{thick}_{\mathsf{D}(R_{\mathfrak{p}})}(M(\mathfrak{p}))$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Lemma 3.4. *Let R be a commutative noetherian local ring. For M and N in $\mathsf{D}^f_b(K^R)$,*

$$\mathsf{level}^M_{\mathsf{D}(K^R)}(N) = \mathsf{cogin}^M_{\mathsf{D}^f_b(K^R)}(N) = \mathsf{gin}^M_{\mathsf{D}^f_b(K^R)}(N).$$

Proof. The natural map $K^R \to K^{\widehat{R}}$ is a quasi-isomorphism of DG algebras and so it induces an exact equivalence

$$\mathsf{D}^f_b(K^R) \xrightarrow{\equiv} \mathsf{D}^f_b(K^{\widehat{R}}).$$

Since cogin, gin and level are invariant under exact equivalences we can assume R is complete and set $K = K^R$.

As R is complete, it is well known that R admits a dualizing DG module ω; see, for example, [14, Corollay 1.4]. Now applying [12, Theorem 2.1], $\operatorname{Hom}_R(K, \omega)$ is a dualizing DG K-module. In particular, setting $(-)^{\dagger} \coloneqq \operatorname{Hom}_K(-, \operatorname{Hom}_R(K, \omega))$

then for any M in $\mathsf{D}^f_b(K)$, $M^\dagger$ is in $\mathsf{D}^f_b(K)$ and the natural biduality map $M \xrightarrow{\simeq} M^{\dagger\dagger}$ is an isomoprhism in $\mathsf{D}^f_b(K)$. Hence, $(-)^\dagger$ restricts to an exact auto-equivalence of $\mathsf{D}^f_b(K)$.

Finally, as $(-)^\dagger$ is an exact auto-equivalence of $\mathsf{D}^f_b(K)$ interchanging coghost and ghost maps, from Theorem 2.6 the desired equality follows. □

Remark 3.5. The lemma holds for any DG algebra A satisfying the hypotheses of Theorem 2.6 which admits a dualizing DG module as defined in [12, 1.8]. Another example, generalizing the Koszul complex above, would be the DG fiber of any local ring map of finite flat dimension whose target ring admits a dualizing complex [12, Theorem VI].

Lemma 3.6. *Let R be a commutative noetherian local ring and let $\mathsf{t}\colon \mathsf{D}(R) \to \mathsf{D}(K^R)$ denote $- \otimes_R K^R$. If M and N are objects of $\mathsf{D}^f_b(R)$, then*

$$\mathsf{gin}^{\mathsf{t}M}_{\mathsf{D}(K^R)}(\mathsf{t}N) \leq \mathsf{gin}^{M}_{\mathsf{D}(R)}(N).$$

Proof. We set $K = K^R$. For X in $\mathsf{D}(R)$ and Y in $\mathsf{D}(K)$, there is an adjunction isomorphism

$$\operatorname{Hom}_{\mathsf{D}(K)}(\mathsf{t}X, Y) \cong \operatorname{Hom}_{\mathsf{D}(R)}(X, Y), \tag{8}$$

which is induced by the natural map $\eta_X\colon X \to \mathsf{t}X$ given by $x \mapsto x \otimes 1$. Moreover, when $f\colon Y \to Z$ is a $\mathsf{t}M$-ghost map in $\mathsf{D}^f_b(K)$, then (8) implies that f is a M-ghost map in $\mathsf{D}^f_b(R)$.

Assume $n \coloneqq \mathsf{gin}^M_{\mathsf{D}^f_b(R)}(N) < \infty$ and suppose $g\colon \mathsf{t}N \to Y$ in $\mathsf{D}^f_b(K)$ factors as the composition of n maps in $\mathsf{D}^f_b(K)$ which are $\mathsf{t}M$-ghost. By the adjunction above any $\mathsf{t}M$-ghost is M-ghost. Hence, g is the composition of n maps in $\mathsf{D}^f_b(R)$ which are M-ghost, and thus so is $g \circ \eta_N$. Therefore, by assumption $g \circ \eta_N = 0$ and so from (8) we conclude that $g = 0$ in $\mathsf{D}^f_b(K)$, completing the proof. □

Proof of Theorem 3.1. Let $\mathfrak{p} \in \operatorname{Spec} R$. Hence, by assumption $\mathsf{gin}^{M_\mathfrak{p}}_{\mathsf{D}^f_b(R_\mathfrak{p})}(N_\mathfrak{p}) < \infty$. Also,

$$\begin{aligned}\mathsf{gin}^{M_\mathfrak{p}}_{\mathsf{D}^f_b(R_\mathfrak{p})}(N_\mathfrak{p}) &\geq \mathsf{gin}^{M(\mathfrak{p})}_{\mathsf{D}^f_b(K^{R_\mathfrak{p}})}(N(\mathfrak{p})) \\ &= \mathsf{cogin}^{M(\mathfrak{p})}_{\mathsf{D}^f_b(K^{R_\mathfrak{p}})}(N(\mathfrak{p})) \\ &= \mathsf{level}^{M(\mathfrak{p})}_{\mathsf{D}(K^{R_\mathfrak{p}})}(N(\mathfrak{p}))\end{aligned}$$

where the inequality is from Lemma 3.6 and the equalities are from Lemma 3.4. Thus $\mathsf{level}^{M(\mathfrak{p})}_{\mathsf{D}(K^{R_\mathfrak{p}})}(N(\mathfrak{p})) < \infty$, and so $N(\mathfrak{p})$ is in $\mathsf{thick}_{\mathsf{D}(K^{R_\mathfrak{p}})}(M(\mathfrak{p}))$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Now by restricting scalars along $R_\mathfrak{p} \to K^{R_\mathfrak{p}}$ we conclude that $N(\mathfrak{p})$ is in $\mathsf{thick}_{\mathsf{D}(R_\mathfrak{p})}(M(\mathfrak{p}))$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Finally, we apply 3.3 to obtain the desired result. □

References

1. Luchezar L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118. MR 1648664
2. Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Srikanth Iyengar, *Class and rank of differential modules*, Invent. Math. **169** (2007), no. 1, 1–35. MR 2308849

3. Luchezar L. Avramov, Ragnar-Olaf Buchweitz, Srikanth B. Iyengar, and Claudia Miller, *Homology of perfect complexes*, Adv. Math. **223** (2010), no. 5, 1731–1781. MR 2592508
4. Luchezar L. Avramov, Srikanth B. Iyengar, Saeed Nasseh, and Sean Sather-Wagstaff, *Homology over trivial extensions of commutative DG algebras*, Comm. Algebra **47** (2019), no. 6, 2341–2356. MR 3957101
5. Apostolos Beligiannis, *Some ghost lemmas, lecture notes for the conference "the representation dimension of artin algebras"*, (Bielefeld 2008).
6. Dave Benson, Srikanth B. Iyengar, and Henning Krause, *A local-global principle for small triangulated categories*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 3, 451–476. MR 3335421
7. Petter Andreas Bergh, *Representation dimension and finitely generated cohomology*, Adv. Math. **219** (2008), no. 1, 389–400. MR 2435428
8. A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258. MR 1996800
9. Winfried Bruns and Jürgen Herzog, *Cohen-macaulay rings*, 2 ed., Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1998.
10. J. Daniel Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta*, Adv. Math. **136** (1998), no. 2, 284–339. MR 1626856
11. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR 1802847
12. Anders Frankild, Srikanth Iyengar, and Peter Jørgensen, *Dualizing differential graded modules and Gorenstein differential graded algebras*, J. London Math. Soc. (2) **68** (2003), no. 2, 288–306. MR 1994683
13. Michael J. Hopkins, *Global methods in homotopy theory*, Homotopy theory (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 73–96. MR 932260
14. Takesi Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149. MR 1859029
15. Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102. MR 1258406
16. G. Maxwell Kelly, *Chain maps inducing zero homology maps*, Proc. Cambridge Philos. Soc. **61** (1965), 847–854. MR 188273
17. Janina C. Letz, *Local to global principles for generation time over commutative noetherian rings*, Homology Homotopy Appl. **23** (2021), no. 2, 165–182. MR 4259573
18. Jian Liu and Josh Pollitz, *Duality and symmetry of complexity over complete intersections via exterior homology*, Proc. Amer. Math. Soc. **149** (2021), no. 2, 619–631. MR 4198070
19. Amnon Neeman, *The chromatic tower for $D(R)$*, Topology **31** (1992), no. 3, 519–532, With an appendix by Marcel Bökstedt. MR 1174255
20. Steffen Oppermann and Jan Šťovíček, *Generating the bounded derived category and perfect ghosts*, Bull. Lond. Math. Soc. **44** (2012), no. 2, 285–298. MR 2914607
21. Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256. MR 2434186
22. Greg Stevenson, *Subcategories of singularity categories via tensor actions*, Compos. Math. **150** (2014), no. 2, 229–272. MR 3177268

School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China.
Email address: liuj231@sjtu.edu.cn

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, U.S.A.
Email address: pollitz@math.utah.edu