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PRINCIPAL SPECTRAL THEORY OF TIME-PERIODIC NONLOCAL DISPERSAL COOPERATIVE SYSTEMS AND APPLICATIONS*

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Abstract. This paper is concerned with the principal spectral theory of time-periodic cooperative systems with nonlocal dispersal and Neumann boundary condition. First we present a sufficient condition for the existence of principal eigenvalues by using the theory of resolvent positive operators with their perturbations. Then we establish the monotonicity of principal eigenvalues with respect to the frequency and investigate the limiting properties of principal eigenvalues as the frequency tends to zero or infinity. We also study the effects of dispersal rates and dispersal ranges on the principal eigenvalues, and the difficulty is that principal eigenvalues of time-periodic cooperative systems with Neumann boundary conditions are not monotone with respect to the domain. Finally, we apply our theory to a man-environment-man epidemic model and consider the impacts of dispersal rates, frequency, and dispersal ranges on the basic reproduction number and positive time-periodic solutions.

Key words. principal eigenvalue, nonlocal dispersal, cooperative system, resolvent positive operator, basic reproduction number

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1. Introduction.

It is known that the principal eigenvalue of a linearized system can be regarded as a threshold in determining the dynamics of the corresponding nonlinear system. In this paper we study the principal spectral theory of the following time-periodic cooperative system with nonlocal dispersal and Neumann boundary condition:

$$\begin{cases} \omega\partial_t\varphi(x,t) = d\mathcal{K}[\varphi](x,t) - d\mathcal{J}[\varphi](x,t) + A(x,t)\varphi(x,t) - \lambda\varphi(x,t), & (x,t)\in\bar{\Omega}\times\mathbb{R}, \\ \varphi(x,t+1) = \varphi(x,t), & (x,t)\in\bar{\Omega}\times\mathbb{R}, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth bounded domain; $\omega>0$ represents the frequency; $\varphi = (\varphi_1,\dots,\varphi_m)^T$; $d=\mathrm{diag}(d_1,\dots,d_m)$ with each d_i being a positive constant; $\mathcal{K} = \mathrm{diag}(\mathcal{K}_1,\dots,\mathcal{K}_m)$ with $\mathcal{K}_i$ being defined by

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$$\mathcal{K}_i[\varphi](x,t) = \int_\Omega J_i(x-y)\varphi_i(y,t)\mathrm{d}y,\ 1 \le i \le m;$$

$\mathcal{J} = \mathrm{diag}(\mathcal{J}_1, \ldots, \mathcal{J}_m)$ with $\mathcal{J}_i$ being defined by

$$\mathcal{J}_i[\varphi](x,t) = \int_\Omega J_i(x-y)\mathrm{d}y\varphi_i(x,t),\ 1 \le i \le m;$$

$A(x,t) = (a_{ij}(x,t))_{m\times m}$ with $a_{ij}(x,t+1) = a_{ij}(x,t)$ for $1 \le i,j \le m$ and $(x,t) \in \bar{\Omega}\times\mathbb{R}$. Recall that a square matrix is said to be cooperative if its off-diagonal elements are nonnegative. Throughout the paper, we make the following assumptions:

(J) $J_i \in C(\mathbb{R}^n), J_i(0) > 0, J_i(x) \ge 0, \int_{\mathbb{R}^n} J_i(x)\mathrm{d}x = 1$ for all $1 \le i \le m$;

(A1) $A(x,t)$ is cooperative for any $(x,t) \in \bar{\Omega}\times\mathbb{R}$.

It is well-known that principal eigenvalues of nonlocal dispersal operators may not exist or more conditions may be required to ensure the existence (see, e.g., [10, 34]). To overcome this difficulty, some studies focus on principal spectrum points or generalized principal eigenvalues instead of principal eigenvalues (see, e.g., [6, 10, 20, 36]). On the other hand, it is natural to find some suitable conditions for the existence of principal eigenvalues of nonlocal dispersal equations. By using the generalized Krein–Rutman theorem (see, e.g., [13]), Coville [10] gave a sufficient condition for the existence of the principal eigenvalue of nonlocal elliptic equations; Liang, Zhang, and Zhao [22, 23] studied the eigenvalue problem associated with a linear time-periodic nonlocal dispersal cooperative system with and without time delay, respectively. By employing the results about perturbations of positive semigroups in Bürger [7], Rawal and Shen [30] gave a necessary and sufficient condition for the existence of principal eigenvalues of time-periodic nonlocal dispersal equations; Bao and Shen [5] extended some existing results about principal eigenvalues of time-periodic nonlocal dispersal equations to time-periodic cooperative and irreducible systems with nonlocal dispersal. However, the condition in [5] requires irreducibility and the condition in [23] is not easy to verify. In this paper, we use the theory of resolvent positive operators (see, e.g., [40, 41]) to study this problem. It should be pointed out that Kang and Ruan [16] investigated the existence of principal eigenvalues of age-structured operators with nonlocal dispersal by means of this method.

In addition to the existence of the principal eigenvalue of system (1.1), another central question of interest is to investigate the dependence of principal eigenvalues (principal spectrum points or generalized principal eigenvalues) on parameters such as frequency, dispersal rate, and dispersal range. For time-periodic nonlocal dispersal equations, Shen and Vo [32] investigated the effects of dispersal rates and dispersal ranges on principal spectrum points; Su et al. [36] considered the monotonicity of generalized principal eigenvalues with respect to the frequency and the asymptotic behavior as frequency tends to zero or infinity; Vo [43] overcame the difficulty that principal eigenvalues of operators with Neumann boundary conditions are not monotone with respect to the domain and obtained the limiting properties of generalized principal eigenvalues as the dispersal range tends to zero. To the best of our knowledge, the dependence of principal eigenvalues of time-periodic nonlocal dispersal systems on parameters such as frequency, dispersal rate, and dispersal range has not been considered in the literature.

It is worth mentioning that there have been quite a few results about principal eigenvalues for cooperative systems with local (random) dispersal. As for cooperative elliptic systems, Sweers [38] established the existence of a unique first eigenfunction;

Dancer [11] and Lam and Lou [18] considered the asymptotic behavior of the principal eigenvalue with small dispersal rates. As for time-periodic cooperative parabolic systems, Antón and López-Gómez [2] showed the existence and uniqueness of the principal eigenvalue; Bai and He [3] and Zhang and Zhao [46] analyzed the asymptotic behavior of principal eigenvalues with small dispersal rates and large dispersal rates, respectively. There are also some results about principal eigenvalues of time-periodic patch models. For example, Liu, Lou, and Song [26] examined the monotonicity of principal eigenvalues with respect to frequency; Zhang and Zhao [47] investigated the asymptotic behavior of principal eigenvalues as the dispersal rate tends to zero and infinity, respectively. Recently, there also have been a number of studies on elliptic-type nonlocal systems. For instance, Nguyen and Vo [28] and Ninh and Vo [29] derived the existence, simplicity, and qualitative properties of the principal eigenvalue for the cooperative system; Su and his collaborators [35, 37] studied the principal spectral theory and variational characterizations for cooperative systems with matrix-type nonlocal operators which come from stem cell regeneration models.

The purpose of this paper is to study the existence and qualitative property of the principal eigenvalue of (1.1). Motivated by Kang and Ruan [16], we establish a sufficient condition for the existence of the principal eigenvalue of (1.1) by using the theory of resolvent positive operators. Inspired by Liu and Lou [24], Liu et al. [25], and Liu, Lou, and Song [26], we find a new type of monotonicity of the principal eigenvalue of (1.1) with respect to the frequency. It should be pointed out that the condition for this monotonicity is different from that of random dispersal operators (see the details in Remark 3.3). We believe that this difference reveals an essential difference between random dispersal operators and nonlocal dispersal operators in new insights. Moreover, we investigate the limiting properties as the frequency approaches zero or infinity. When considering the effects of the dispersal rates and dispersal ranges on principal eigenvalues, we also need to overcome the difficulty that principal eigenvalues of operators with Neumann boundary conditions are not monotone with respect to the domain.

As far as we know, the basic reproduction number $\mathcal{R}_0$ is a significant threshold in population dynamics. Recently, Zhang and Zhao [46] studied the asymptotic behavior of the basic reproduction number for periodic reaction-diffusion systems in the case of small and large dispersal coefficients. Based on the theory of resolvent positive operators, they reduced the problem on the asymptotic behavior of $\mathcal{R}_0$ into that of the principal eigenvalue associated with linear periodic systems. Motivated by this idea, we consider the impacts of frequency, dispersal rate, and dispersal range on the basic reproduction number and positive periodic solutions of a man-environment-man epidemic model.

The rest of the paper is organized as follows. In section 2, we establish a sufficient condition for the existence of principal eigenvalues of (1.1). In section 3, we show the monotonicity of principal eigenvalues with respect to the frequency and study the effects of frequency on principal eigenvalues. In section 4, we investigate the limiting properties of principal eigenvalues as the dispersal rate or dispersal range tends to zero or infinity. In section 5, we apply our theory to an epidemic model and consider the properties of the basic reproduction number and positive periodic solutions.

2. Principal spectral theory. In this section, we establish a sufficient condition for the existence of the principal eigenvalue of (1.1) by using the theory of resolvent positive operators with their perturbations. For any $\alpha = (\alpha_1, \dots, \alpha_m)^T \in \mathbb{R}^m$, we put $|\alpha| = \sqrt{\sum_{i=1}^m \alpha_i^2}$. Let

$$(\mathbb{R}^m)^+ = \left\{\alpha = (\alpha_1, \dots, \alpha_m)^T \mid \alpha_i \in \mathbb{R}, \alpha_i \geq 0, i = 1, 2, \dots, m\right\}$$

and

$$(\mathbb{R}^m)^{++} = \left\{\alpha = (\alpha_1, \dots, \alpha_m)^T \mid \alpha_i \in \mathbb{R}, \alpha_i > 0, i = 1, 2, \dots, m\right\}.$$

Set $\mathcal{X} = \{\mathbf{u} \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R}^m) |\ \mathbf{u}(x, t+1) = \mathbf{u}(x, t), (x, t) \in \bar{\Omega} \times \mathbb{R}\}$ with norm $\|\mathbf{u}\|_{\mathcal{X}} = \sup_{(x,t)\in\bar{\Omega}\times\mathbb{R}} |\mathbf{u}(x, t)|$, and

$$\mathcal{X}^+ = \{\mathbf{u} \in \mathcal{X} |\ \mathbf{u}(x, t) \in (\mathbb{R}^m)^+, (x, t) \in \bar{\Omega} \times \mathbb{R}\},$$
$$\mathcal{X}^{++} = \{\mathbf{u} \in \mathcal{X} |\ \mathbf{u}(x, t) \in (\mathbb{R}^m)^{++}, (x, t) \in \bar{\Omega} \times \mathbb{R}\}.$$

Let $X = C(\bar{\Omega}, \mathbb{R}^m)$ with norm $\|\mathbf{u}\|_X = \sup_{x\in\bar{\Omega}} |\mathbf{u}(x)|$, and

$$X^+ = \left\{\mathbf{u} \in X |\ \mathbf{u}(x) \in (\mathbb{R}^m)^+, x \in \bar{\Omega}\right\},$$
$$X^{++} = \left\{\mathbf{u} \in X |\ \mathbf{u}(x) \in (\mathbb{R}^m)^{++}, x \in \bar{\Omega}\right\}.$$

For $\mathbf{u}, \mathbf{v} \in \mathscr{X}$, we write

$$\mathbf{u} \geq \mathbf{v} \quad \text{if} \quad \mathbf{u} - \mathbf{v} \in \mathscr{X}^+,$$
$$\mathbf{u} > \mathbf{v} \quad \text{if} \quad \mathbf{u} - \mathbf{v} \in \mathscr{X}^+ \setminus \{\mathbf{0}\},$$
$$\mathbf{u} \gg \mathbf{v} \quad \text{if} \quad \mathbf{u} - \mathbf{v} \in \mathscr{X}^{++},$$

where $\mathscr{X} = \mathbb{R}^m, X, \mathcal{X}$.

2.1. Resolvent positive operators. First we recall some results about resolvent positive operators. For more details, we refer to Thieme [39, 40, 41]. Let Z denote a Banach space and Z^+ be a closed convex cone that is normal and generating. Denote the interior of Z^+ by Z^{++}. A bounded linear operator L on Z is said to be *positive* if $L : Z^+ \to Z^+$ and *strongly positive* if $L : Z^+ \setminus \{0\} \to Z^{++}$.

DEFINITION 2.1. *A closed operator A in Z is said to be* resolvent positive *if the resolvent set of A, $\rho(A)$, contains a ray (ϱ, ∞) and the resolvent $(\lambda I - A)^{-1}$ is a positive operator for all $\lambda > \varrho$.*

DEFINITION 2.2. *Define the spectral bound of a closed operator A by*

$$s(A) = \sup\{\operatorname{Re} \lambda \in \mathbb{R} \mid \lambda \in \sigma(A)\},$$

the real spectral bound *of A by*

$$s_{\mathbb{R}}(A) = \sup\{\lambda \in \mathbb{R} \mid \lambda \in \sigma(A)\},$$

and the spectral radius *of A by*

$$r(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\}.$$

If $s(A)$ is an isolated eigenvalue of A with a positive eigenfunction φ (i.e., $\varphi \in Z^+ \setminus \{0\}$), then $s(A)$ is called the principal eigenvalue *of A.*

THEOREM 2.3 (Thieme [40, Theorem 3.5]). *Let A be a resolvent positive operator in Z. Then $s(A) = s_{\mathbb{R}}(A) < \infty$ and $s(A) \in \sigma(A)$ whenever $s(A) > -\infty$. Moreover, there is a constant $c > 0$ such that*

$$\left\|(\lambda I - A)^{-1}\right\| \leq c \left\|(\operatorname{Re} \lambda I - A)^{-1}\right\| \ \textit{whenever}\ \operatorname{Re} \lambda > s(A).$$

Define

$$F_\lambda = C(\lambda I - B)^{-1}, \quad \lambda > s(B).$$

Definition 2.4. *The operator* $C: D(B) \to Z$ *is called* a compact perturbator *of* B *and* $A = B + C$ a compact perturbation *of* B *if*

$$(\lambda I - B)^{-1} F_\lambda : \overline{D(B)} \to \overline{D(B)} \text{ is compact for some } \lambda > s(B)$$

and

$$(\lambda I - B)^{-1} (F_\lambda)^2 : Z \to Z \text{ is compact for some } \lambda > s(B).$$

C *is called* an essentially compact perturbator *of* B *and* $A = B + C$ an essentially compact perturbation *of* B *if there is some* $n \in \mathbb{N}$ *such that* $(\lambda I - B)^{-1} (F_\lambda)^n$ *is compact for all* $\lambda > s(B)$.

Theorem 2.5 (Thieme [41, Theorem 3.6]). *Let* $A = B + C$ *be a positive perturbation of* B. *Then* $r(F_\lambda)$ *is a decreasing convex function of* $\lambda > s(B)$ *and exactly one of the following three cases holds:*

(i) *If* $r(F_\lambda) \geq 1$ *for all* $\lambda > s(B)$, *then* A *is not resolvent positive.*
(ii) *If* $r(F_\lambda) < 1$ *for all* $\lambda > s(B)$, *then* A *is resolvent positive and* $s(A) = s(B)$.
(iii) *If there exists* $v > \lambda > s(B)$ *such that* $r(F_v) < 1 \leq r(F_\lambda)$, *then* A *is resolvent positive and* $s(B) < s(A) < \infty$; *further* $s = s(A)$ *is characterized by* $r(F_s) = 1$.

Theorem 2.6 (Thieme [40, Theorem 4.7]). *Assume that* C *is an essentially compact perturbator of* B. *Moreover assume that there exist* $\lambda_2 > \lambda_1 > s(B)$ *such that* $r(F_{\lambda_1}) \geq 1 > r(F_{\lambda_2})$. *Then* $s(B) < s(A)$ *and the following statements hold:*

(i) $s(A)$ *is an eigenvalue of* A *associated with positive eigenfunctions of* A *and* A^*, *has finite algebraic multiplicity, and is a pole of the resolvent of* A. *If* C *is a compact perturbator of* B, *then all spectral values* λ *of* A *with* $\operatorname{Re}\lambda \in (s(B), s(A)]$ *are poles of the resolvent of* A *and are eigenvalues of* A *with finite algebraic multiplicity.*
(ii) 1 *is an eigenvalue of* $F_{s(A)}$ *and is associated with an eigenfunction* $w \in Z$ *of* $F_{s(A)}$ *such that* $(\lambda I - B)^{-1} w \in Z^+$. *Actually* $s(A)$ *is the largest* $\lambda \in \mathbb{R}$ *for which* 1 *is an eigenvalue of* F_λ.

2.2. Existence of principal eigenvalues. Define

$$\begin{aligned}
&\mathcal{A}[\mathbf{u}](x,t) = -\omega \partial_t \mathbf{u}(x,t) + d\mathcal{K}[\mathbf{u}](x,t) - d\mathcal{J}[\mathbf{u}](x,t) + A(x,t)\mathbf{u}(x,t),\\
&\mathcal{B}[\mathbf{u}](x,t) = -\omega \partial_t \mathbf{u}(x,t) - d\mathcal{J}[\mathbf{u}](x,t) + A(x,t)\mathbf{u}(x,t),\\
&\mathcal{C}[\mathbf{u}](x,t) = d\mathcal{K}[\mathbf{u}](x,t).
\end{aligned}$$

Obviously, $\mathcal{A} = \mathcal{B} + \mathcal{C}$. Note that if $\eta \in \mathbb{C}$ such that $(\eta I - \mathcal{B})^{-1}$ exists, then

$$(\mathcal{B} + \mathcal{C})\mathbf{u} = \eta \mathbf{u}$$

has nontrivial solutions in $\mathcal{X} \oplus i\mathcal{X}$ is equivalent to

$$\mathcal{C}(\eta I - \mathcal{B})^{-1} \mathbf{v} = \mathbf{v}$$

has nontrivial solutions in $\mathcal{X} \oplus i\mathcal{X}$, where

$$\mathcal{X} \oplus i\mathcal{X} = \{\mathbf{u} + i\mathbf{v} \mid \mathbf{u}, \mathbf{v} \in \mathcal{X}\}.$$

Without loss of generality, we assume that $a_{ii}(x,t) > 0$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$ and $1 \leq i \leq m$. Otherwise, choose a sufficiently large constant $C > 0$ such that $a_{ii}(x,t) + C > 0$

for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$ and $1 \le i \le m$. Let $\{\mathcal{U}_\lambda(t,\tau) \mid t \ge \tau\}$ and $\{\mathcal{F}_\lambda(t,\tau) \mid t \ge \tau\}$ respectively be the evolution family on X determined by

$$\omega \partial_t \mathbf{u}(x,t) = -d\mathcal{J}[\mathbf{u}](x,t) + A(x,t)\mathbf{u}(x,t) - \lambda \mathbf{u}(x,t), \quad x \in \bar{\Omega}, \quad t \in \mathbb{R},$$

and

$$\omega \partial_t \mathbf{u}(x,t) = d\mathcal{K}[\mathbf{u}](x,t) - d\mathcal{J}[\mathbf{u}](x,t) + A(x,t)\mathbf{u}(x,t) - \lambda \mathbf{u}(x,t), \quad x \in \bar{\Omega}, \quad t \in \mathbb{R}.$$

Recall that a square matrix is said to be irreducible if it is not similar, via a permutation, to a block upper triangular matrix. To guarantee that the principal eigenfunction is strongly positive, we need the following additional assumption:

(A2) There exists some $x_0 \in \Omega$ such that $A(x_0,t)$ is irreducible for any $t \in \mathbb{R}$.

By [23, Lemma B.3], we have the following result.

Lemma 2.7. *$\mathcal{U}_\lambda(t,\tau)$ and $\mathcal{F}_\lambda(t,\tau)$ are positive on X for any $t \ge \tau$ and $\lambda \in \mathbb{R}$. If, in addition,* (A2) *holds, then $\mathcal{F}_\lambda(t,\tau)$ is strongly positive on X for any $t > \tau$ and $\lambda \in \mathbb{R}$.*

In view of [3, Theorem 1.4], we have the following lemma.

Lemma 2.8. *For any given $x \in \bar{\Omega}$, the eigenvalue problem*

$$\begin{cases} -\omega \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + A(x,t)\phi(t) = \lambda \phi(t), & t \in \mathbb{R}, \\ \phi(t+1) = \phi(t), & t \in \mathbb{R}, \end{cases} \tag{2.1}$$

has a principal eigenvalue $\lambda(x)$ with a positive eigenfunction $\phi(x,t)$.

By [3, Lemma 3.6], we know that $\lambda(x)$ and $\phi(x,t)$ are as smooth in x as $A(x,t)$ in x, and when $A(x,t) \equiv A(x)$, $\lambda(x)$ is the largest real part of the eigenvalues of the matrix $A(x)$. Let $\alpha(x)$ denote the principal eigenvalue determined in Lemma 2.8 with $A(x,t)$ replaced by

$$B(x,t) := A(x,t) + \mathrm{diag}\left(-d_1 \int_\Omega J_1(x-y)\mathrm{d}y, \dots, -d_m \int_\Omega J_m(x-y)\mathrm{d}y\right).$$

Proposition 2.9. *The resolvent operator $(\eta I - \mathcal{B})^{-1}$ exists when $\mathrm{Re}\,\eta > \alpha^* =: \max_{x\in\bar{\Omega}} \alpha(x)$. Moreover, $\mathcal{B}$ is a resolvent positive operator and $s(\mathcal{B}) = \alpha^*$.*

Proof. Similar to [5, Proposition 3.2], we know that $(\eta I - \mathcal{B})^{-1}$ exists when $\mathrm{Re}\,\eta > \alpha^* =: \max_{x\in\bar{\Omega}} \alpha(x)$, which implies that $s(\mathcal{B}) \le \alpha^*$. It follows from Lemma 2.7 that $\mathcal{B}$ generates a positive semigroup. This together with [41, Theorem 3.12] gives that $\mathcal{B}$ is a resolvent positive operator. By using the same argument as in the proof of [5, Proposition 3.1], we know that $s(\mathcal{B}) \ge \alpha^*$. Hence, $s(\mathcal{B}) = \alpha^*$. The proof is completed. ∎

Recall that $\{\mathcal{F}_0(t,\tau) \mid t \ge \tau\}$ is the evolution family on X determined by

$$\omega \partial_t \mathbf{u}(x,t) = d\mathcal{K}[\mathbf{u}](x,t) - d\mathcal{J}[\mathbf{u}](x,t) + A(x,t)\mathbf{u}(x,t), \quad x \in \bar{\Omega}, \quad t \in \mathbb{R}.$$

Define an operator $\mathcal{Q}_\lambda \in \mathcal{L}(X)$ by

$$\mathcal{Q}_\lambda \psi = e^{-\frac{\lambda}{\omega}} \mathcal{F}_0(1,0)\psi \quad \text{for} \quad \psi \in X.$$

Proposition 2.10. *There exists $\lambda_0 \in \mathbb{R}$ such that $r(\mathcal{Q}_{\lambda_0}) = r(e^{-\frac{\lambda_0}{\omega}} \mathcal{F}_0(1,0)) = 1$. In addition, the operator $\mathcal{A}$ is resolvent positive and $s(\mathcal{A}) = \lambda_0 = \omega \ln r(\mathcal{F}_0(1,0))$.*

Proof. Consider the resolvent equation

$$\varphi = (\lambda I - \mathcal{A})^{-1}\psi, \quad \psi \in \mathcal{X}, \lambda \in \rho(\mathcal{A}).$$

It follows from the variation of constants formula that

$$\varphi(x,t) = e^{-\frac{\lambda}{\omega}t}\mathcal{F}_0(t,0)\varphi(x,0) + \int_0^t e^{-\frac{\lambda}{\omega}(t-\tau)}\mathcal{F}_0(t,\tau)\frac{\psi(x,\tau)}{\omega}\mathrm{d}\tau. \tag{2.2}$$

Since $\varphi(x,t) = \varphi(x,t+1)$, we derive from (2.2) that

$$(I - \mathcal{Q}_\lambda)\varphi(x,0) = \int_0^1 e^{-\frac{\lambda}{\omega}(1-\tau)}\mathcal{F}_0(1,\tau)\frac{\psi(x,\tau)}{\omega}\mathrm{d}\tau.$$

Thus, if $1 \in \rho(\mathcal{Q}_\lambda)$, then

$$\begin{aligned}[(\lambda I - \mathcal{A})^{-1}\psi](x,t) = e^{-\frac{\lambda}{\omega}t}\mathcal{F}_0(t,0)(I - \mathcal{Q}_\lambda)^{-1}\left[\int_0^1 \mathcal{F}_0(1,s)e^{-\frac{\lambda}{\omega}(1-s)}\frac{\psi(x,s)}{\omega}\mathrm{d}s\right]\\ + \int_0^t \mathcal{F}_0(t,s)e^{-\frac{\lambda}{\omega}(t-s)}\frac{\psi(x,s)}{\omega}\mathrm{d}s.\end{aligned} \tag{2.3}$$

Moreover, $\lambda \in \rho(\mathcal{A})$ if and only if $1 \in \rho(\mathcal{Q}_\lambda)$.

Set

$$m_{ij} = \min_{(x,t)\in\bar{\Omega}\times[0,1]} a_{ij}(x,t) \quad \text{and} \quad M = (m_{ij})_{m\times m}.$$

By Lemma 2.7, we derive $\mathcal{F}_0(t,\tau) \geq \mathcal{W}(t-\tau)$ in the sense of positive operators, where $\mathcal{W}(t)$ is the semigroup generated by the operator $\frac{1}{\omega}[d\mathcal{K} - d\mathcal{J} + M]$. By [18, Theorem 1.3], M admits a real eigenvalue λ_M corresponding to a positive eigenvector φ_M. It is easily seen that $\frac{\lambda_M}{\omega}$ is an eigenvalue of the following eigenvalue problem:

$$\frac{1}{\omega}[d\mathcal{K}[\varphi](x) - d\mathcal{J}[\varphi](x) + M\varphi(x)] = \lambda\varphi(x).$$

By virtue of the spectral mapping theorem (Thieme [41, Lemma 5.8]),

$$e^{\sigma\left(\frac{d\mathcal{K}-d\mathcal{J}+M}{\omega}\right)t} = \sigma(\mathcal{W}(t)) \setminus \{0\} \quad \text{for all} \quad t > 0. \tag{2.4}$$

We derive from Lemma 2.7 that $\mathcal{W}(t)$ is a positive operator. Then by [27, Proposition 4.1.1], $r(\mathcal{W}(t)) \in \sigma(\mathcal{W}(t))$ for any $t > 0$. By (2.4),

$$e^{s\left(\frac{d\mathcal{K}-d\mathcal{J}+M}{\omega}\right)t} = r(\mathcal{W}(t)) \quad \text{for all} \quad t > 0.$$

Then we have

$$e^{\frac{\lambda_M}{\omega}t} \leq e^{s\left(\frac{d\mathcal{K}-d\mathcal{J}+M}{\omega}\right)t} = r(\mathcal{W}(t)) \quad \text{for all} \quad t > 0.$$

As a result,

$$r(\mathcal{Q}_{\lambda_M}) = r\left(e^{-\frac{\lambda_M}{\omega}}\mathcal{F}_0(1,0)\right) \geq r\left(e^{-\frac{\lambda_M}{\omega}}\mathcal{W}(1)\right) \geq 1.$$

On the other hand, $r(\mathcal{Q}_\lambda) \to 0$ as $\lambda \to +\infty$.

Thus, we have that $r(\mathcal{Q}_\lambda)$ is strictly decreasing with respect to $\lambda \in \mathbb{R}$ and there is a unique λ_0 such that $r(\mathcal{Q}_{\lambda_0}) = 1$. Then for any $\lambda \in \mathbb{R}$, we have $r(\mathcal{Q}_\lambda) < r(\mathcal{Q}_{\lambda_0}) = 1$

if $\lambda > \lambda_0$, implying that $(I-\mathcal{Q}_\lambda)^{-1}$ exists and $\rho(\mathcal{A})$ contains a ray $(\lambda_0,+\infty)$. In addition, $(\lambda I-\mathcal{A})^{-1}$ is positive by (2.3) for all $\lambda>\lambda_0$. As a result, $\mathcal{A}$ is a resolvent positive operator.

Since $\mathcal{Q}_\lambda$ is positive, $1=r(\mathcal{Q}_{\lambda_0})\in\sigma(\mathcal{Q}_{\lambda_0})$; that is, $\sigma(\mathcal{Q}_{\lambda_0})$ is nonempty, which implies that $\lambda_0\in\sigma(\mathcal{A})$ and $\sigma(\mathcal{A})$ is nonempty. In view of the fact that λ_0 is larger than any other real spectral value in $\sigma(\mathcal{A})$, we derive $\lambda_0=s_{\mathbb{R}}(\mathcal{A})$. Note that $\mathcal{X}$ is a Banach space with a normal and generating cone $\mathcal{X}^+$ and $s(\mathcal{A})\geq\lambda_0>-\infty$ due to $\lambda_0\in\sigma(\mathcal{A})$. It follows from Theorem 2.3 that $s(\mathcal{A})=s_{\mathbb{R}}(\mathcal{A})=\lambda_0$. The proof is completed. ∎

Proposition 2.11. *For any* $\mathrm{Re}\,\eta>\alpha^*$, $\mathcal{C}(\eta I-\mathcal{B})^{-1}$ *is a compact operator in* $\mathcal{X}\oplus i\mathcal{X}$.

Proof. For any bounded sequence $\{\mathbf{u}_n\}\in\mathcal{X}\oplus i\mathcal{X}$, set

$$\mathbf{v}_n=(\eta I-\mathcal{B})^{-1}\mathbf{u}_n.$$

By virtue of the boundedness of $\mathcal{B}+\omega\partial_t$, both $\{\mathbf{v}_n\}$ and $\{\partial_t\mathbf{v}_n\}$ are bounded sequences in $\mathcal{X}\oplus i\mathcal{X}$. Then $\{\mathcal{C}\mathbf{v}_n\}$ is uniformly bounded and equicontinuous due to the assumption (J). It follows from the Arzelá–Ascoli theorem that $\{\mathcal{C}\mathbf{v}_n\}$ is relatively compact in $\mathcal{X}$. Thus, $\mathcal{C}(\eta I-\mathcal{B})^{-1}$ is a compact operator in $\mathcal{X}$. The proof is completed. ∎

Corollary 2.12. *The operator* $\mathcal{C}$ *is a compact perturbator and also an essentially compact perturbator of* $\mathcal{B}$. *Thus the operator* $\mathcal{A}=\mathcal{B}+\mathcal{C}$ *is a compact perturbation and also an essentially compact perturbation of* $\mathcal{B}$.

Proof. $(\eta I-\mathcal{B})^{-1}\mathcal{C}(\eta I-\mathcal{B})^{-1}$ is compact for any $\eta>s(\mathcal{B})$ since $\mathcal{C}(\eta I-\mathcal{B})^{-1}$ is compact by Proposition 2.11. ∎

Now we present the main result of this section.

Theorem 2.13. *Suppose that* $s(\mathcal{A})>s(\mathcal{B})$; *then* $s(\mathcal{A})$ *is the principal eigenvalue of* $\mathcal{A}$ *with an eigenfunction* $\varphi\in\mathcal{X}^+\setminus\{\mathbf{0}\}$. *Moreover, if* (A2) *holds, then* $s(\mathcal{A})$ *is an algebraically simple eigenvalue of* $\mathcal{A}$ *with an eigenfunction* $\varphi\in\mathcal{X}^{++}$. *Conversely, if* λ *is an eigenvalue of* $\mathcal{A}$ *with an eigenfunction* $\varphi\in\mathcal{X}^{++}$, *then* $\lambda=s(\mathcal{A})>s(\mathcal{B})$.

Proof. Set

$$\mathcal{G}_\lambda=\mathcal{C}(\lambda I-\mathcal{B})^{-1},\quad \lambda>s(\mathcal{B}).$$

Since $\mathcal{A}$ is resolvent positive by Proposition 2.10, case (i) in Theorem 2.5 is impossible. We derive from the assumption $s(\mathcal{A})>s(\mathcal{B})$ that case (iii) in Theorem 2.5 will happen. Thus, there exist $\lambda_2>\lambda_1>s(\mathcal{B})$ such that $r(\mathcal{G}_{\lambda_1})\geq 1>r(\mathcal{G}_{\lambda_2})$. Now applying Theorem 2.6 yields that $s(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with an eigenfunction $\varphi\in\mathcal{X}^+\setminus\{\mathbf{0}\}$ and has finite algebraic multiplicity and is a pole of the resolvent of $\mathcal{A}$, which implies that $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. If (A2) holds, then we derive from Lemma 2.7 that the corresponding principal eigenfunction $\varphi\in\mathcal{X}^{++}$. Similar to the proof of [5, Theorem 2.3], we have $s(\mathcal{A})$ is an algebraically simple eigenvalue.

Now assume that $\lambda\in\mathbb{R}$ is an eigenvalue of $\mathcal{A}$ with an eigenfunction $\varphi\in\mathcal{X}^{++}$. It is readily verified that $\mathcal{F}_0(t,0)\varphi(x,0)=e^{\frac{\lambda}{\omega}t}\varphi(x,t)$. Since $\varphi(x,t)>0$ for all $(x,t)\in\bar{\Omega}\times\mathbb{R}$, for any $\mathbf{u}_0\in X^+$ with

$$\mathbf{u}_0\leq M_0\varphi(x,0),\quad x\in\bar{\Omega},$$

where $M_0=\frac{\|\mathbf{u}_0\|}{\min_{1\leq i\leq m}\min_{x\in\bar{\Omega}}\varphi_i(x,0)}$, we derive from Lemma 2.7 that

$$\mathcal{F}_0(t,0)\mathbf{u}_0\leq M_0\mathcal{F}_0(t,0)\varphi(\cdot,0)=M_0e^{\frac{\lambda}{\omega}t}\varphi(\cdot,t)\quad\text{for all}\quad t>0,$$

from which we have

$$\frac{s(\mathcal{A})}{\omega} \le \omega(\mathcal{F}_0(t,0)) := \lim_{t\to+\infty} \frac{\ln \|\mathcal{F}_0(t,0)\|}{t} \le \frac{\lambda}{\omega},$$

where $\omega(\mathcal{F}_0(t,0))$ is the growth bound. By the definition of $s(\mathcal{A})$, we have $s(\mathcal{A}) \ge \lambda$. Hence, $s(\mathcal{A}) = \lambda$. Finally following the argument similar to the proof of [5, Theorem 2.1], we prove $\lambda = s(\mathcal{A}) > s(\mathcal{B})$. The proof is completed. ∎

Remark 2.14. Theorem 2.13 also holds for the operator with Dirichlet or periodic boundary condition. In this case $\alpha(x)$ is the principal eigenvalue determined in Lemma 2.8 with $A(x,t)$ replaced by $A(x,t) + \mathrm{diag}(-d_1, \dots, -d_m)$.

COROLLARY 2.15. *$s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ if one of the following assumptions holds:*

(i) *$A(x,t) \equiv A(t)$, that is, $a_{ij}(x,t) \equiv a_{ij}(t)$ for all $1 \le i,j \le m$.*

(ii) *(A2) holds and $\frac{1}{\alpha^* - \alpha(\cdot)} \notin L^1(\Omega_0)$ for some bounded domain $\Omega_0 \subset \Omega$.*

(iii) *$\min_{1\le i\le m} d_i$ is sufficiently large.*

Proof. (i) By Lemma 2.8, there exists $(\lambda, \phi(t))$ satisfying

$$-\omega \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + A(t)\phi(t) = \lambda\phi(t).$$

Then $-\mathcal{A}[\phi](t) + \lambda\phi(t) = 0$ and $s(\mathcal{A}) \ge \lambda > s(\mathcal{B})$. We derive from Theorem 2.13 that $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.

(ii) Note that $s(\mathcal{A}) \ge s(\mathcal{B})$. If $s(\mathcal{A}) > s(\mathcal{B})$, there is nothing to prove due to Theorem 2.13. Suppose $s(\mathcal{A}) = s(\mathcal{B}) = \alpha^*$. Similar to the proof of [5, Proposition 3.4], we have $r(\mathcal{C}(\eta I - \mathcal{B})^{-1}) > 1$ for $\eta > s(\mathcal{B}) = \alpha^*$ and $\eta - \alpha^* \ll 1$, which contradicts Theorem 2.5. Thus, $s(\mathcal{A}) > s(\mathcal{B})$ and $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.

(iii) The matrix M admits a real eigenvalue λ_M corresponding to a positive eigenvector with $M = (m_{ij})_{m\times m}$ and $m_{ij} = \min_{(x,t)\in\bar{\Omega}\times[0,1]} a_{ij}(x,t)$. Obviously, $s(\mathcal{A}) \ge \lambda_M$. Set $j_i = \min_{x\in\bar{\Omega}} \int_\Omega J_i(x-y)\mathrm{d}y$, $\tilde{m}_{ij} = \max_{(x,t)\in\bar{\Omega}\times[0,1]} a_{ij}(x,t)$, $\tilde{M} = (\tilde{m}_{ij})_{m\times m}$, and $Q = \mathrm{diag}(-d_1 j_1, \dots, -d_m j_m) + \tilde{M}$. Then we have $s(\mathcal{B}) \le \lambda(Q)$, where $\lambda(Q)$ is the principal eigenvalue of Q. Note that $\lambda(Q) \to -\infty$ as $\min_{1\le i\le m} d_i \to +\infty$. Hence, there exists $d_* > 0$ such that $s(\mathcal{A}) > s(\mathcal{B})$ for all $\min_{1\le i\le m} d_i \ge d_*$. It follows from Theorem 2.13 that $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ for all $\min_{1\le i\le m} d_i \ge d_*$. The proof is completed. ∎

3. Monotonicity with respect to the frequency. In this section, we always suppose that assumption (A2) holds and investigate the monotonicity of principal eigenvalues with respect to the frequency ω and the limiting properties as ω tends to zero or infinity. For any 1-periodic function $u(x,t) \in C(\bar{\Omega}\times\mathbb{R})$, set

$$\hat{u}(x) = \int_0^1 u(x,t)\mathrm{d}t, \quad \bar{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\mathrm{d}x,$$

$$\hat{A}(x) = (\hat{a}_{ij}(x))_{m\times m}, \quad A^* = \left(\frac{1}{|\Omega|}\int_\Omega \hat{a}_{ij}(x)\mathrm{d}x\right)_{m\times m}.$$

Let

$$\mathcal{X}_1 = \{\mathbf{u} \in C^{0,1}(\bar{\Omega}\times\mathbb{R}, \mathbb{R}^m) | \mathbf{u}(x,t+1) = \mathbf{u}(x,t), (x,t) \in \bar{\Omega}\times\mathbb{R}\}.$$

For $\mathbf{f}, \mathbf{g} \in L^2(\Omega\times[0,1], \mathbb{R}^m)$, we set

$$(\mathbf{f},\mathbf{g})_0 = \sum_{i=1}^m \int_0^1 \int_\Omega f_i(x,t) g_i(x,t)\mathrm{d}x\mathrm{d}t.$$

In order to prove the monotonicity of principal eigenvalues with respect to the frequency ω, we need the following result.

LEMMA 3.1. *Assume that $J_i(-x) = J_i(x)$, $d_i = d_i(\omega) \in C^1((0,\infty))$ for $1 \le i \le m$, $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, and $a_{ij}(x,t) = a_{ji}(x,t)$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$ and $1 \le i,j \le m$. Let φ and ψ be the principal eigenfunctions of* (1.1) *and the adjoint problem of* (1.1) *corresponding to $s(\mathcal{A})$, respectively. Then the following statements hold:*

(i) *Given $\mathbf{f} \in \mathcal{X}$, there exists $\mathbf{u} \in \mathcal{X}_1$ such that $\mathcal{A}\mathbf{u} - s(\mathcal{A})\mathbf{u} = \mathbf{f}$ if and only if $(\mathbf{f},\psi)_0 = 0$. Also there exists $\mathbf{v} \in \mathcal{X}_1$ such that $\mathcal{A}^*\mathbf{v} - s(\mathcal{A})\mathbf{v} = \mathbf{f}$ if and only if $(\mathbf{f},\varphi)_0 = 0$, where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ defined by*

$$\mathcal{A}^*[\mathbf{v}](x,t) = \omega\partial_t\mathbf{v}(x,t) + d\mathcal{K}[\mathbf{v}](x,t) - d\mathcal{J}[\mathbf{v}](x,t) + A(x,t)\mathbf{v}(x,t).$$

(ii) *$(s(\mathcal{A}),\varphi,\psi)$ is continuously differentiable with respect to ω.*

Proof. (i) If there exists $\mathbf{u} \in \mathcal{X}_1$ such that $\mathcal{A}\mathbf{u} - s(\mathcal{A})\mathbf{u} = \mathbf{f}$, then $(\mathbf{f},\psi)_0 = (\mathcal{A}\mathbf{u} - s(\mathcal{A})\mathbf{u},\psi)_0 = (\mathbf{u},\mathcal{A}^*\psi - s(\mathcal{A})\psi)_0 = 0$.

Set $\tilde{\mathcal{A}} = \mathcal{A} - s(\mathcal{A})\mathcal{I}$. By virtue of the proof of Theorem 2.13, $s(\tilde{\mathcal{A}}) = 0$ is an algebraically simple eigenvalue of $\tilde{\mathcal{A}}$ and is isolated in the spectrum $\sigma(\tilde{\mathcal{A}})$. Let $\sigma_1 = \{0\}$ and $\sigma_2 = \sigma(\tilde{\mathcal{A}}) \setminus \sigma_1$. It follows from [17, Theorem 6.17, p. 178] that there exists a decomposition of $\tilde{\mathcal{A}}$ according to a decomposition $\mathcal{X} = \mathcal{M}_1 \oplus \mathcal{M}_2$ of the space in such a way that the spectra of the parts $\tilde{\mathcal{A}}_{\mathcal{M}_1}$ and $\tilde{\mathcal{A}}_{\mathcal{M}_2}$ coincide with σ_1 and σ_2, respectively, where $\tilde{\mathcal{A}}_{\mathcal{M}_i}$ is an operator in the space $\mathcal{M}_i$ with $D(\tilde{\mathcal{A}}_{\mathcal{M}_i}) = D(\tilde{\mathcal{A}}) \cap \mathcal{M}_i$ such that $\tilde{\mathcal{A}}_{\mathcal{M}_i}\mathbf{u} = \tilde{\mathcal{A}}\mathbf{u} \in \mathcal{M}_i$, $i = 1,2$. Then, $s(\tilde{\mathcal{A}}_{\mathcal{M}_2}) < 0$. Clearly, we have $\mathcal{M}_1 = \{c\varphi \mid c \in \mathbb{C}\}$. If $(\mathbf{f},\psi)_0 = 0$, then $\mathbf{f} \in \mathcal{M}_2$. Since 0 is in the resolvent set of $\tilde{\mathcal{A}}_{\mathcal{M}_2}$, there exists $\mathbf{u} \in \mathcal{X}_1$ such that $\mathcal{A}\mathbf{u} - s(\mathcal{A})\mathbf{u} = \mathbf{f}$.

Let

$$\check{\mathcal{A}}[\mathbf{u}](x,t) = -\omega\partial_t\mathbf{u}(x,t) + d\mathcal{K}[\mathbf{u}](x,t) - d\mathcal{J}[\mathbf{u}](x,t) + A(x,-t)\mathbf{u}(x,t),$$

$\check{\psi}(x,t) = \psi(x,-t)$, $\check{\varphi}(x,t) = \varphi(x,-t)$, and $\check{\mathbf{f}}(x,t) = \mathbf{f}(x,-t)$ for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$. Then $\check{\psi}$ satisfies $\check{\mathcal{A}}\check{\psi} = s(\mathcal{A})\check{\psi}$. We derive from Theorem 2.13 that $s(\check{\mathcal{A}}) = s(\mathcal{A})$ is an algebraically simple eigenvalue of $\check{\mathcal{A}}$ and is isolated in the spectrum $\sigma(\check{\mathcal{A}})$. By the above arguments, there exists $\check{\mathbf{v}} \in \mathcal{X}_1$ such that $\check{\mathcal{A}}\check{\mathbf{v}} - s(\mathcal{A})\check{\mathbf{v}} = \check{\mathbf{f}}$ if and only if $(\check{\mathbf{f}},\check{\varphi})_0 = 0$. Thus, there exists $\mathbf{v} \in \mathcal{X}_1$ such that $\mathcal{A}^*\mathbf{v} - s(\mathcal{A})\mathbf{v} = \mathbf{f}$ if and only if $(\mathbf{f},\varphi)_0 = 0$.

(ii) Inspired by [19, Proposition 1.3.15 and Theorem 4.3.4], we prove the continuous differentiability of $s(\mathcal{A})$ and φ with respect to ω via the implicit function theorem. Normalize φ and ψ such that $\frac{1}{2}(\varphi,\varphi)_0 = (\varphi,\psi)_0 = 1$ for any $\omega > 0$. Define a mapping $\mathcal{F} : \mathcal{X}_1 \times \mathbb{R} \times (0,\infty) \to \mathcal{X} \times \mathbb{R}$ by

$$\mathcal{F}(\mathbf{u},\mu,\omega) := \left(\mathcal{A}\mathbf{u} - \mu\mathbf{u}, \frac{1}{2}(\mathbf{u},\mathbf{u})_0 - 1\right).$$

Clearly, $\mathcal{F}(\varphi, s(\mathcal{A}),\omega) = (\mathbf{0},0)$. In order to prove the continuous differentiability of $s(\mathcal{A})$ and φ with respect to ω, it suffices to show that for each fixed $\omega > 0$, the linear mapping

$$D_{(\mathbf{u},\mu)}\mathcal{F}(\varphi, s(\mathcal{A}),\omega) : \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X} \times \mathbb{R}$$

is invertible. To this end, given $(\mathbf{f},c) \in \mathcal{X} \times \mathbb{R}$, we need to prove the existence and uniqueness of $(\mathbf{g},h) \in \mathcal{X}_1 \times \mathbb{R}$ such that

$$\begin{cases} \mathcal{A}[\mathbf{g}](x,t) - s(\mathcal{A})\mathbf{g}(x,t) - h\varphi(x,t) = \mathbf{f}(x,t), & (x,t) \in \bar{\Omega} \times \mathbb{R}, \\ (\varphi, \mathbf{g})_0 = c. \end{cases} \tag{3.1}$$

First we show the existence. To this end, we choose $h = -(\mathbf{f}, \psi)_0$, so that $(h\varphi + \mathbf{f}, \psi)_0 = 0$. By (i), there exists $\mathbf{u} \in \mathcal{X}_1$ such that $\mathcal{A}\mathbf{u} - s(\mathcal{A})\mathbf{u} = h\varphi + \mathbf{f}$. Set $\mathbf{g} = \mathbf{u} + \frac{c - (\mathbf{u}, \varphi)_0}{2}\varphi$; then $(\mathbf{g}, h)$ satisfies (3.1).

To show uniqueness, we set $(\mathbf{f}, c) = (\mathbf{0}, 0)$ in (3.1) and proceed to show $(\mathbf{g}, h) = (\mathbf{0}, 0)$. In view of $\mathbf{f} = \mathbf{0}$, we have

$$0 = (\psi, \mathbf{f})_0 = (\psi, \mathcal{A}\mathbf{g} - s(\mathcal{A})\mathbf{g} - h\varphi)_0 = -h.$$

Thus $h = 0$ and (3.1) becomes

$$\begin{cases} \mathcal{A}[\mathbf{g}](x,t) - s(\mathcal{A})\mathbf{g}(x,t) = \mathbf{0}, & (x,t) \in \bar{\Omega} \times \mathbb{R}, \\ (\varphi, \mathbf{g})_0 = 0. \end{cases}$$

Since $s(\mathcal{A})$ is an algebraically simple eigenvalue, we have $\mathbf{g} = k\varphi$ for some $k \in \mathbb{C}$. Then, $(\varphi, \mathbf{g})_0 = 2k = 0$. As a result, $\mathbf{g} = \mathbf{0}$.

For the continuous differentiability of ψ with respect to ω, we define a mapping $\mathcal{G} : \mathcal{X}_1 \times \mathbb{R} \times (0, \infty) \to \mathcal{X} \times \mathbb{R}$ by

$$\mathcal{G}(\mathbf{u}, \mu, \omega) := \left(\mathcal{A}^*\mathbf{u} - \mu\mathbf{u}, (\varphi, \mathbf{u})_0 - 1\right).$$

Clearly, $\mathcal{G}(\psi, s(\mathcal{A}), \omega) = (\mathbf{0}, 0)$. It remains to show that for each fixed $\omega > 0$, the linear mapping

$$D_{(\mathbf{u},\mu)}\mathcal{G}(\psi, s(\mathcal{A}), \omega) : \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X} \times \mathbb{R}$$

is invertible. Given $(\mathbf{w}, b) \in \mathcal{X} \times \mathbb{R}$, by the same arguments as above, we can prove the existence and uniqueness of $(\mathbf{p}, q) \in \mathcal{X}_1 \times \mathbb{R}$ such that

$$\begin{cases} \mathcal{A}^*[\mathbf{p}](x,t) - s(\mathcal{A})\mathbf{p}(x,t) - q\psi(x,t) = \mathbf{w}(x,t), & (x,t) \in \bar{\Omega} \times \mathbb{R}, \\ (\varphi, \mathbf{p})_0 = b. \end{cases}$$

The proof is completed. □

In the following, for convenience, the $'$ notation denotes differentiation with respect to ω.

THEOREM 3.2. *Suppose the assumptions of Lemma* 3.1 *hold. If* $d_i'(\omega) \geq 0$ *and* $(\frac{d_i(\omega)}{\omega})' \leq 0$ *for* $1 \leq i \leq m$*, then* $s(\mathcal{A})$ *is nonincreasing with respect to* ω*. In addition,*

(i) *if* $d_i'(\omega) > 0$ *for* $1 \leq i \leq m$*, then* $s'(\mathcal{A}) = 0$ *if and only if there exists some* 1*-periodic function* $\zeta(t) \in C(\mathbb{R})$ *with* $\int_0^1 \zeta(t)\mathrm{d}t = 0$ *satisfying*

$$\left[\zeta(t)I - (A(x,t) - A^*)\right]\Phi = 0 \quad \textit{for any} \quad x \in \bar{\Omega} \quad \textit{and} \quad t \in [0,1],$$

where Φ *is the principal eigenvector corresponding to the principal eigenvalue* $\tilde{\lambda}$ *of* A^**;*

(ii) *if* $d_i'(\omega) = 0$, $(\frac{d_i(\omega)}{\omega})' < 0$ *for* $1 \leq i \leq m$ *and the eigenvalue problem*

$$d\mathcal{K}[\varphi](x) - d\mathcal{J}[\varphi](x) + \hat{A}(x)\varphi(x) = \lambda\varphi(x) \tag{3.2}$$

admits a principal eigenvalue, then $s'(\mathcal{A})=0$ *if and only if there exists some* 1*-periodic function* $r(t)\in C(\mathbb{R})$ *with* $\int_0^1 r(t)\mathrm{d}t=0$ *satisfying*

$$\left[r(t)I-(A(x,t)-\hat{A}(x))\right]\phi(x)=0 \quad \text{for any} \quad x\in\bar{\Omega} \quad \text{and} \quad t\in[0,1],$$

where $\phi(x)$ *is the principal eigenfunction of* (3.2)*.*

Remark 3.3. A typical example for Theorem 3.2 is $d_i(\omega)=a_i\omega+b_i$ for some nonnegative constants a_i and b_i satisfying $(a_i,b_i)\neq(0,0)$. Our results are new even for scalar time periodic nonlocal equations when $d'(\omega)>0$ and $(\frac{d(\omega)}{\omega})'\leq 0$. However, the condition for random dispersal equations is $(\frac{d(\omega)}{\omega^2})'\leq 0$ (see [24, Theorem 1.1]). This observation may reflect an essential difference between nonlocal dispersal operators and random dispersal operators.

Proof of Theorem 3.2. By the definition of the principal eigenvalue of $\mathcal{A}$, there exists $\varphi\in\mathcal{X}^{++}$ such that

$$\omega\partial_t\varphi(x,t)=d\mathcal{K}[\varphi](x,t)-d\mathcal{J}[\varphi](x,t)+A(x,t)\varphi(x,t)-s(\mathcal{A})\varphi(x,t). \tag{3.3}$$

Let $\psi\in\mathcal{X}^{++}$ be the eigenfunction of the adjoint problem to (1.1) given by

$$-\omega\partial_t\psi(x,t)=d\mathcal{K}[\psi](x,t)-d\mathcal{J}[\psi](x,t)+A(x,t)\psi(x,t)-s(\mathcal{A})\psi(x,t). \tag{3.4}$$

Set

$$\alpha_i=\sqrt{\varphi_i\psi_i} \quad \text{and} \quad \beta_i=\frac{1}{2}\ln\left(\frac{\varphi_i}{\psi_i}\right) \quad \text{for} \quad 1\leq i\leq m.$$

Then some computation yields that

$$\begin{aligned}&-d_i\int_\Omega J_i(x-y)(\alpha_i(y,t)-\alpha_i(x,t))\mathrm{d}y-\sum_{j=1}^m a_{ij}(x,t)\alpha_j(x,t)-c_i(x,t)\alpha_i(x,t)\\&\quad=-s(\mathcal{A})\alpha_i(x,t),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}c_i(x,t)=&\frac{d_i}{2}\int_\Omega J_i(x-y)\left(\sqrt{\frac{\varphi_i(y,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_i(y,t)}{\psi_i(x,t)}}\right)^2\mathrm{d}y\\&+\frac{1}{2}\sum_{j=1}^m a_{ij}(x,t)\left(\sqrt{\frac{\varphi_j(x,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_j(x,t)}{\psi_i(x,t)}}\right)^2-\omega\partial_t\beta_i(x,t).\end{aligned}$$

In view of Lemma 3.1, we can differentiate both sides of (3.5) with respect to ω to find

(3.6)

$$\begin{aligned}&-d_i'(\omega)\int_\Omega J_i(x-y)(\alpha_i(y,t)-\alpha_i(x,t))\mathrm{d}y-d_i\int_\Omega J_i(x-y)(\alpha_i'(y,t)-\alpha_i'(x,t))\mathrm{d}y\\&\quad-\sum_{j=1}^m a_{ij}(x,t)\alpha_j'(x,t)-c_i'(x,t)\alpha_i(x,t)-c_i(x,t)\alpha_i'(x,t)\\&\quad=-s'(\mathcal{A})\alpha_i(x,t)-s(\mathcal{A})\alpha_i'(x,t).\end{aligned}$$

Multiplying (3.5) by α_i', multiplying (3.6) by α_i, subtracting the resulting equations, and integrating over $\Omega\times(0,1)$ yield

$$s'(\mathcal{A})\int_0^1\int_\Omega \alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t = -\frac{d_i'(\omega)}{2}\int_0^1\int_\Omega\int_\Omega J_i(x-y)(\alpha_i(y,t)-\alpha_i(x,t))^2\mathrm{d}y\mathrm{d}x\mathrm{d}t + \int_0^1\int_\Omega c_i'(x,t)\alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t.$$

Adding the above equations from $i=1$ to m yields

$$\begin{aligned} s'(\mathcal{A})\sum_{i=1}^m\int_0^1\int_\Omega \alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t &= -\sum_{i=1}^m\frac{d_i'(\omega)}{2}\int_0^1\int_\Omega\int_\Omega J_i(x-y)(\alpha_i(y,t)-\alpha_i(x,t))^2\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &\quad + \sum_{i=1}^m\int_0^1\int_\Omega c_i'(x,t)\alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.7}$$

A simple computation gives that

$$\begin{aligned} c_i'(x,t) =& \frac{d_i'(\omega)}{2}\int_\Omega J_i(x-y)\left(\sqrt{\frac{\varphi_i(y,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_i(y,t)}{\psi_i(x,t)}}\right)^2\mathrm{d}y \\ &+ \frac{d_i}{2}\int_\Omega J_i(x-y)\left[\left(\sqrt{\frac{\varphi_i(y,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_i(y,t)}{\psi_i(x,t)}}\right)^2\right]'\mathrm{d}y \\ &+ \frac{1}{2}\sum_{j=1}^m a_{ij}(x,t)\left[\left(\sqrt{\frac{\varphi_j(x,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_j(x,t)}{\psi_i(x,t)}}\right)^2\right]' - \partial_t\beta_i(x,t) - \omega\partial_t\beta_i'(x,t). \end{aligned}$$

And a further computation yields

$$\begin{aligned} &\frac{d_i}{2}\int_0^1\int_\Omega\int_\Omega J_i(x-y)\left[\left(\sqrt{\frac{\varphi_i(y,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_i(y,t)}{\psi_i(x,t)}}\right)^2\right]'\alpha_i^2(x,t)\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &= \frac{d_i}{2}\int_0^1\int_\Omega\int_\Omega J_i(x-y)\Big[\varphi_i'(y,t)\psi_i(x,t) - \frac{\varphi_i(y,t)\psi_i(x,t)}{\varphi_i(x,t)}\varphi_i'(x,t) \\ &\quad +\psi_i'(y,t)\varphi_i(x,t) - \frac{\varphi_i(x,t)\psi_i(y,t)}{\psi_i(x,t)}\psi_i'(x,t)\Big]\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &= -\frac{d_i}{2}\int_0^1\int_\Omega\int_\Omega J_i(x-y)\frac{\psi_i(x,t)}{\varphi_i(x,t)}\left(\frac{\varphi_i(x,t)}{\psi_i(x,t)}\right)'(\varphi_i(y,t)\psi_i(x,t) \\ &\quad - \psi_i(y,t)\varphi_i(x,t))\mathrm{d}y\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\sum_{i=1}^m\frac{1}{2}\int_0^1\int_\Omega\sum_{j=1}^m a_{ij}(x,t)\left[\left(\sqrt{\frac{\varphi_j(x,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_j(x,t)}{\psi_i(x,t)}}\right)^2\right]'\alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t \\ &= \sum_{i=1}^m\frac{1}{2}\int_0^1\int_\Omega\sum_{j=1}^m a_{ij}(x,t)\Big[\varphi_j'(x,t)\psi_i(x,t) - \frac{\varphi_j(x,t)\psi_i(x,t)}{\varphi_i(x,t)}\varphi_i'(x,t) \\ &\quad +\psi_j'(x,t)\varphi_i(x,t) - \frac{\varphi_i(x,t)\psi_j(x,t)}{\psi_i(x,t)}\psi_i'(x,t)\Big]\mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{2}\int_0^1\int_\Omega\sum_{i=1}^m\sum_{j=1}^m a_{ij}(x,t)\frac{\psi_i(x,t)}{\varphi_i(x,t)}\left(\frac{\varphi_i(x,t)}{\psi_i(x,t)}\right)'(\varphi_j(x,t)\psi_i(x,t) \\ &\quad - \psi_j(x,t)\varphi_i(x,t))\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.9}$$

and

(3.10)
$$\begin{aligned}
&\sum_{i=1}^{m}\int_0^1\int_\Omega \omega\partial_t\beta_i'(x,t)\alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t\\
&\quad=-\sum_{i=1}^{m}\int_0^1\int_\Omega \omega\beta_i'(x,t)[\partial_t\varphi_i(x,t)\psi_i(x,t)+\varphi_i(x,t)\partial_t\psi_i(x,t)]\mathrm{d}x\mathrm{d}t\\
&\quad=-\int_0^1\int_\Omega\sum_{i=1}^{m}\frac{1}{2}\frac{\psi_i(x,t)}{\varphi_i(x,t)}\left(\frac{\varphi_i(x,t)}{\psi_i(x,t)}\right)'\Bigg[d_i\int_\Omega J_i(x-y)(\varphi_i(y,t)\psi_i(x,t)\\
&\qquad-\psi_i(y,t)\varphi_i(x,t))\mathrm{d}y+\sum_{j=1}^{m}a_{ij}(x,t)(\varphi_j(x,t)\psi_i(x,t)-\psi_j(x,t)\varphi_i(x,t))\Bigg]\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now in view of (3.7)–(3.10), we have

(3.11)
$$\begin{aligned}
&s'(\mathcal{A})\sum_{i=1}^{m}\int_0^1\int_\Omega \alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t\\
&\quad=-\sum_{i=1}^{m}\frac{d_i'(\omega)}{2}\int_0^1\int_\Omega\int_\Omega J_i(x-y)(\alpha_i(y,t)-\alpha_i(x,t))^2\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\qquad+\sum_{i=1}^{m}\int_0^1\int_\Omega\frac{d_i'(\omega)}{2}\int_\Omega J_i(x-y)\left(\sqrt{\frac{\varphi_i(y,t)}{\varphi_i(x,t)}}-\sqrt{\frac{\psi_i(y,t)}{\psi_i(x,t)}}\right)^2\alpha_i^2(x,t)\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\qquad-\sum_{i=1}^{m}\int_0^1\int_\Omega\partial_t\beta_i(x,t)\alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Moreover, we have

(3.12)
$$\begin{aligned}
&\sum_{i=1}^{m}\int_0^1\int_\Omega \partial_t\beta_i(x,t)\alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t\\
&\quad=-\sum_{i=1}^{m}\int_0^1\int_\Omega \beta_i(x,t)[\partial_t\varphi_i(x,t)\psi_i(x,t)+\varphi_i(x,t)\partial_t\psi_i(x,t)]\mathrm{d}x\mathrm{d}t\\
&\quad=-\sum_{i=1}^{m}\int_0^1\int_\Omega\frac{1}{2}\ln\left(\frac{\varphi_i(x,t)}{\psi_i(x,t)}\right)\Bigg[\frac{d_i}{\omega}\int_\Omega J_i(x-y)(\varphi_i(y,t)\psi_i(x,t)\\
&\qquad-\varphi_i(x,t)\psi_i(y,t))\mathrm{d}y+\sum_{j=1}^{m}\frac{1}{\omega}a_{ij}(x,t)(\varphi_j(x,t)\psi_i(x,t)-\varphi_i(x,t)\psi_j(x,t))\Bigg]\mathrm{d}x\mathrm{d}t\\
&\quad=-\int_0^1\int_\Omega\sum_{i=1}^{m}d_i'(\omega)\int_\Omega J_i(x-y)\varphi_i(y,t)\psi_i(x,t)\ln\sqrt{\frac{\varphi_i(x,t)\psi_i(y,t)}{\varphi_i(y,t)\psi_i(x,t)}}\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^1\int_\Omega\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{1}{\omega}a_{ij}(x,t)\varphi_j(x,t)\psi_i(x,t)\ln\sqrt{\frac{\varphi_i(x,t)\psi_j(x,t)}{\varphi_j(x,t)\psi_i(x,t)}}\mathrm{d}x\mathrm{d}t+\sum_{i=1}^{m}\varsigma_i,
\end{aligned}$$

where

$$\varsigma_i = \int_0^1 \int_\Omega \frac{\omega}{2} \left(\frac{d_i(\omega)}{\omega}\right)' \ln\left(\frac{\varphi_i(x,t)}{\psi_i(x,t)}\right) \int_\Omega J_i(x-y)(\varphi_i(y,t)\psi_i(x,t) - \varphi_i(x,t)\psi_i(y,t))\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

And

$$\begin{aligned}\sum_{i=1}^m \int_0^1 \int_\Omega \frac{d_i'(\omega)}{2} \int_\Omega J_i(x-y) \left(\sqrt{\frac{\varphi_i(y,t)}{\varphi_i(x,t)}} - \sqrt{\frac{\psi_i(y,t)}{\psi_i(x,t)}}\right)^2 \alpha_i^2(x,t)\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ = \sum_{i=1}^m \int_0^1 \int_\Omega \frac{d_i'(\omega)}{2} \int_\Omega J_i(x-y)(\varphi_i(y,t)\psi_i(x,t) + \psi_i(y,t)\varphi_i(x,t))\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ - \sum_{i=1}^m \int_0^1 \int_\Omega d_i'(\omega) \int_\Omega J_i(x-y)\sqrt{\varphi_i(y,t)\varphi_i(x,t)\psi_i(y,t)\psi_i(x,t)}\mathrm{d}y\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.13}$$

Define $g(z) := z - 1 - \ln z$, $z > 0$, and $h(z_1, z_2) := (z_1 - z_2)(\ln z_1 - \ln z_2)$, $z_1, z_2 > 0$. By virtue of (3.11)–(3.13), we obtain

$$\begin{aligned} s'(\mathcal{A}) \sum_{i=1}^m \int_0^1 \int_\Omega \alpha_i^2(x,t)\mathrm{d}x\mathrm{d}t \\ = -\sum_{i=1}^m \frac{d_i'(\omega)}{2} \int_0^1 \int_\Omega \int_\Omega J_i(x-y)(\alpha_i(y,t) - \alpha_i(x,t))^2 \mathrm{d}y\mathrm{d}x\mathrm{d}t \\ - \frac{1}{4\omega} \int_0^1 \int_\Omega \sum_{\substack{i,j=1 \\ i\neq j}}^m a_{ij}(x,t) h(\varphi_j(x,t)\psi_i(x,t), \psi_j(x,t)\varphi_i(x,t))\mathrm{d}x\mathrm{d}t \\ + \frac{1}{4} \sum_{i=1}^m \omega \left(\frac{d_i(\omega)}{\omega}\right)' \rho_i - \sum_{i=1}^m d_i'(\omega)\theta_i, \end{aligned} \tag{3.14}$$

where

$$\rho_i = \int_0^1 \int_\Omega \int_\Omega J_i(x-y) h(\varphi_i(y,t)\psi_i(x,t), \psi_i(y,t)\varphi_i(x,t))\mathrm{d}y\mathrm{d}x\mathrm{d}t$$

and

$$\theta_i = \int_0^1 \int_\Omega \int_\Omega J_i(x-y)\varphi_i(y,t)\psi_i(x,t) g\left(\sqrt{\frac{\varphi_i(x,t)\psi_i(y,t)}{\varphi_i(y,t)\psi_i(x,t)}}\right)\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

Since $g(z) \geq 0$ for all $z > 0$, $a_{ij}(x,t) \geq 0$ for $i \neq j$ and $(x,t) \in \bar{\Omega} \times \mathbb{R}$, and $h(z_1, z_2) \geq 0$ for all $z_1, z_2 > 0$, we derive from (3.14) that $s'(\mathcal{A}) \leq 0$ for all $\omega > 0$.

(i) If $s'(\mathcal{A}) = 0$ for some $\omega > 0$, then (3.14) gives that

$$\int_0^1 \int_\Omega \int_\Omega J_i(x-y)(\alpha_i(y,t) - \alpha_i(x,t))^2 \mathrm{d}y\mathrm{d}x\mathrm{d}t = 0 \quad \text{for all} \quad 1 \leq i \leq m, \tag{3.15}$$

$$\int_0^1 \int_\Omega \int_\Omega J_i(x-y)\varphi_i(y,t)\psi_i(x,t) g\left(\sqrt{\frac{\varphi_i(x,t)\psi_i(y,t)}{\varphi_i(y,t)\psi_i(x,t)}}\right)\mathrm{d}y\mathrm{d}x\mathrm{d}t = 0, \tag{3.16}$$

and

$$\varphi_j(x,t)\psi_i(x,t)=\psi_j(x,t)\varphi_i(x,t) \quad \text{for each} \quad (x,t)\in\bar{\Omega}\times\mathbb{R} \quad \text{and} \quad 1\le i,j\le m, \tag{3.17}$$

which implies that $\alpha_i=\alpha_i(t)$ is independent of $x\in\bar{\Omega}$ and $\frac{\varphi_i(x,t)\psi_i(y,t)}{\varphi_i(y,t)\psi_i(x,t)}\equiv 1$ for each $x,y\in\bar{\Omega}$ and $t\in[0,1]$. Thus, $\varphi_i=\kappa_i(t)\psi_i$ for some 1-periodic function $\kappa_i(t)$. By virtue of (3.17), we have $\kappa_i(t)=\kappa_j(t)=\kappa(t)$ with $\kappa(t)$ being some 1-periodic function for $t\in[0,1]$ and $1\le i,j\le m$. A direct computation yields that

$$\partial_t(\alpha_i^2(x,t))=\frac{d_i}{\omega}\int_\Omega J_i(x-y)(\varphi_i(y,t)\psi_i(x,t)-\psi_i(y,t)\varphi_i(x,t))\mathrm{d}y=0,$$

implying that α_i is independent of t and is a constant. By (3.5), we have

$$\sum_{j=1}^m a_{ij}(x,t)\alpha_j-\omega\frac{1}{2}\frac{\mathrm{d}\ln\kappa(t)}{\mathrm{d}t}\alpha_i=s(\mathcal{A})\alpha_i.$$

Integrating the above equality over $\Omega\times(0,1)$ gives $s(\mathcal{A})=\tilde{\lambda}$. Thus, the conclusion holds with $\zeta(t)=\omega\frac{1}{2}\frac{\mathrm{d}\ln\kappa(t)}{\mathrm{d}t}$.

If there exists some 1-periodic function $\zeta(t)\in C(\mathbb{R})$ with $\int_0^1\zeta(t)\mathrm{d}t=0$ satisfying

$$[\zeta(t)I-(A(x,t)-A^*)]\,\Phi=0 \quad \text{for any} \quad x\in\bar{\Omega} \quad \text{and} \quad t\in[0,1],$$

set $\varphi(t)=\exp[\frac{1}{\omega}\int_0^t\zeta(s)\mathrm{d}s]\Phi$. Then,

$$\omega\varphi'(t)=A(x,t)\varphi(t)-\tilde{\lambda}\varphi(t),$$

implying that $s(\mathcal{A})=\tilde{\lambda}$ for all $\omega>0$. As a result, $s'(\mathcal{A})\equiv 0$.

(ii) If there exists some 1-periodic function $r(t)\in C(\mathbb{R})$ with $\int_0^1 r(t)\mathrm{d}t=0$ satisfying

$$\left[r(t)I-(A(x,t)-\hat{A}(x))\right]\phi(x)=0 \quad \text{for any} \quad x\in\bar{\Omega} \quad \text{and} \quad t\in[0,1],$$

where $\phi(x)$ is the principal eigenfunction of (3.2), set $\varphi(x,t)=\exp[\frac{1}{\omega}\int_0^t r(s)\mathrm{d}s]\phi(x)$. Then,

$$\begin{aligned}
&-\omega\partial_t\varphi_i(x,t)+d_i\int_\Omega J_i(x-y)(\varphi_i(y,t)-\varphi_i(x,t))\mathrm{d}y+\sum_{j=1}^m a_{ij}(x,t)\varphi_j(x,t)\\
&\quad-\hat{\lambda}\varphi_i(x,t)\\
&\quad=-r(t)\varphi_i(x,t)+\sum_{j=1}^m(a_{ij}(x,t)-\hat{a}_{i,j}(x))\varphi_j(x,t)\\
&\quad=0,
\end{aligned}$$

where $\hat{\lambda}$ is the principal eigenvalue of (3.2). Thus, $s(\mathcal{A})=\hat{\lambda}$ and $s'(\mathcal{A})=0$ for all $\omega>0$.

If $s'(\mathcal{A})=0$ for some $\omega>0$, then (3.14) gives that

$$\varphi_j(x,t)\psi_i(x,t)=\psi_j(x,t)\varphi_i(x,t) \quad \text{and} \quad \varphi_i(y,t)\psi_i(x,t)=\psi_i(y,t)\varphi_i(x,t)$$

for each $x, y \in \Omega$, $t \in [0,1]$, and $1 \le i, j \le m$. This implies that $\varphi_i = \varrho_i(t)\psi_i$ for some 1-periodic function $\varrho_i(t)$ and $\varrho_i(t) = \varrho_j(t) = \varrho(t)$ for some 1-periodic function $\varrho(t)$. Thus, $\varphi = \varrho(t)\psi$ and $\beta_i = \frac{1}{2}\ln \varrho(t)$. A direct computation yields that

$$\partial_t(\alpha_i^2(x,t)) = \frac{d_i}{\omega}\int_\Omega J_i(x-y)(\varphi_i(y,t)\psi_i(x,t) - \psi_i(y,t)\varphi_i(x,t))\mathrm{d}y = 0,$$

implying that $\alpha_i = \alpha_i(x)$ is independent of t. By (3.5), we have

$$d_i\int_\Omega J_i(x-y)(\alpha_i(y)-\alpha_i(x))\mathrm{d}y + \sum_{j=1}^m a_{ij}(x,t)\alpha_j(x) - \omega\frac{\mathrm{d}\beta_i(t)}{\mathrm{d}t}\alpha_i(x) = s(\mathcal{A})\alpha_i(x). \tag{3.18}$$

Integrating the above equality over $(0,1)$ gives

$$d_i\int_\Omega J_i(x-y)(\alpha_i(y)-\alpha_i(x))\mathrm{d}y + \sum_{j=1}^m \hat{a}_{ij}(x)\alpha_j(x) = s(\mathcal{A})\alpha_i(x). \tag{3.19}$$

Now we derive from (3.18) and (3.19) that

$$\sum_{j=1}^m (a_{ij}(x,t) - \hat{a}_{ij}(x))\alpha_j(x) - \frac{1}{2}\omega\frac{\mathrm{d}\ln\varrho(t)}{\mathrm{d}t}\alpha_i(x) = 0.$$

Hence, the conclusion holds with $r(t) = \frac{1}{2}\omega\frac{\mathrm{d}\ln\varrho(t)}{\mathrm{d}t}$. The proof is completed. ∎

Next we give the limiting properties of principal eigenvalues as ω tends to zero or infinity. We obtain these results with the help of *generalized principal eigenvalues* defined by

$$\lambda_p(\mathcal{A}) := \sup\left\{\lambda \in \mathbb{R} \mid \exists \varphi \in \mathcal{X}^{++} \text{ s.t. } (-\mathcal{A}+\lambda)[\varphi] \le 0 \text{ in } \bar{\Omega}\times\mathbb{R}\right\}$$

and

$$\lambda_p'(\mathcal{A}) := \inf\left\{\lambda \in \mathbb{R} \mid \exists \varphi \in \mathcal{X}^{++} \text{ s.t. } (-\mathcal{A}+\lambda)[\varphi] \ge 0 \text{ in } \bar{\Omega}\times\mathbb{R}\right\}.$$

PROPOSITION 3.4. *Assume that $(r_{ij}(x,t))_{m\times m}$ satisfies the same conditions as $(a_{ij}(x,t))_{m\times m}$. If $a_{ij}(x,t) \le r_{ij}(x,t)$ for all $1 \le i,j \le m$ and $(x,t) \in \bar{\Omega}\times\mathbb{R}$, then*

$$\lambda_p(\mathcal{A}) + l \le \lambda_p(\mathcal{R}),$$

where

$$l = \min_{1\le i\le m}\left\{\min_{\bar{\Omega}\times[0,1]}[r_{ii}(x,t) - a_{ii}(x,t)]\right\}$$

and $\mathcal{R}$ is defined by replacing $(a_{ij}(x,t))_{m\times m}$ by $(r_{ij}(x,t))_{m\times m}$ in the definition of $\mathcal{A}$.

Proof. For any $\lambda < \lambda_p(\mathcal{A})$, there exists $\varphi \in \mathcal{X}^{++}$ such that

$$-\mathcal{A}[\varphi](x,t) + \lambda\varphi(x,t) \le 0 \text{ in } \bar{\Omega}\times\mathbb{R}.$$

Then

$$\begin{aligned}&\omega\partial_t\varphi_i(x,t)-d_i\int_\Omega J_i(x-y)(\varphi_i(y,t)-\varphi_i(x,t))\mathrm{d}y-\sum_{j=1}^m r_{ij}(x,t)\varphi_j(x,t)\\&+(\lambda+l)\varphi_i(x,t)\\&\quad\le\omega\partial_t\varphi_i(x,t)-d_i\int_\Omega J_i(x-y)(\varphi_i(y,t)-\varphi_i(x,t))\mathrm{d}y\\&\qquad-\sum_{\substack{j=1\\j\neq i}}^m a_{ij}(x,t)\varphi_j(x,t)+(-r_{ii}(x,t)+\lambda+l)\varphi_i(x,t)\\&\quad\le 0,\end{aligned}$$

which implies that $\lambda+l\le\lambda_p(\mathcal{R})$. Thus, $\lambda_p(\mathcal{A})+l\le\lambda_p(\mathcal{R})$. The proof is completed. ∎

PROPOSITION 3.5. *If $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then*

$$s(\mathcal{A})=\lambda_p(\mathcal{A})=\lambda_p'(\mathcal{A}).$$

Proof. First we prove $s(\mathcal{A})=\lambda_p(\mathcal{A})$. If $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then there exists $\phi\in\mathcal{X}^{++}$ such that

$$\mathcal{A}\phi-s(\mathcal{A})\phi=0\quad\text{in}\quad\bar\Omega\times\mathbb{R}.$$

By the definition of $\lambda_p(\mathcal{A})$, we have $s(\mathcal{A})\le\lambda_p(\mathcal{A})$. Suppose to the contrary that $s(\mathcal{A})<\lambda_p(\mathcal{A})$. By virtue of the definition of $\lambda_p(\mathcal{A})$, there exist $\lambda\in(s(\mathcal{A}),\lambda_p(\mathcal{A}))$ and $\varphi\in\mathcal{X}^{++}$ such that

$$-\mathcal{A}\varphi+\lambda\varphi\le 0\quad\text{in}\quad\bar\Omega\times\mathbb{R}.$$

Then $\varphi(\cdot,t)\le e^{-\frac{\lambda}{\omega}t}\mathcal{F}_0(t,0)\varphi(\cdot,0)$. Together with $\varphi(\cdot,0)=\varphi(\cdot,1)$, we have

$$\varphi(\cdot,0)\le e^{-\frac{\lambda}{\omega}}\mathcal{F}_0(1,0)\varphi(\cdot,0).$$

It follows from [31, Proposition 3] that $r(e^{-\frac{\lambda}{\omega}}\mathcal{F}_0(1,0))\ge 1$. We derive from Proposition 2.10 that $r(e^{-\frac{s(\mathcal{A})}{\omega}}\mathcal{F}_0(1,0))=1$. We obtain from the proof of Proposition 2.10 that $r(e^{-\frac{\lambda}{\omega}}\mathcal{F}_0(1,0))$ is strictly decreasing with respect to $\lambda\in\mathbb{R}$. As a result, $s(\mathcal{A})\ge\lambda$. A contradiction occurs implying $s(\mathcal{A})=\lambda_p(\mathcal{A})$.

Next we prove $s(\mathcal{A})=\lambda_p'(\mathcal{A})$. If $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then it is easy to see that $\lambda_p'(\mathcal{A})\le s(\mathcal{A})$. The eigenfunction ϕ corresponding to $s(\mathcal{A})$ satisfies

(3.20)
$$\omega\partial_t\phi_i(x,t)-d_i\int_\Omega J_i(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y-\sum_{j=1}^m a_{ij}(x,t)\phi_j(x,t)+s(\mathcal{A})\phi_i(x,t)=0.$$

Assume that $\lambda_p'(\mathcal{A})<s(\mathcal{A})$. By the definition of $\lambda_p'(\mathcal{A})$, there exist $\lambda_*\in(\lambda_p'(\mathcal{A}),s(\mathcal{A}))$ and $\psi\in\mathcal{X}^{++}$ such that

(3.21)
$$\omega\partial_t\psi_i(x,t)-d_i\int_\Omega J_i(x-y)(\psi_i(y,t)-\psi_i(x,t))\mathrm{d}y-\sum_{j=1}^m a_{ij}(x,t)\psi_j(x,t)+\lambda_*\psi_i(x,t)\ge 0.$$

Let

$$\delta^* := \sup\left\{\delta \mid \psi > \delta\phi \text{ for all } (x,t)\in\bar{\Omega}\times[0,1]\right\}.$$

Due to ϕ, $\psi \in \mathcal{X}^{++}$, we have $\delta^* \in (0,+\infty)$. By the definition of δ^*, we have $V = \psi - \delta^*\phi \geq 0$. We derive from (3.20) and (3.21) that V satisfies

$$\begin{aligned} &\omega\partial_t V_i(x,t) - d_i\int_\Omega J_i(x-y)(V_i(y,t)-V_i(x,t))\mathrm{d}y + KV_i(x,t)\\ &\quad\geq \sum_{j=1}^m a_{ij}(x,t)\left(\psi_j(x,t)-\delta^*\phi_j(x,t)\right) + (s(\mathcal{A})-\lambda_*)\psi_i(x,t) + (K-s(\mathcal{A}))V_i(x,t)\\ &\quad> 0, \end{aligned} \tag{3.22}$$

provided that K is sufficiently large. And $V_i(x,0) = V_i(x,1) \geq 0$. By the strong maximum principle and the periodicity of ϕ and ψ, we get $V_i > 0$ for all $1 \leq i \leq m$. Hence, there exists $\theta^* > 0$ small enough such that $\psi > (\delta^* + \theta^*)\phi$, contradicting the definition of δ^*. As a result, $s(\mathcal{A}) = \lambda_p'(\mathcal{A})$. The proof is completed. ∎

THEOREM 3.6. *Suppose that $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Then the following conclusions hold:*

(i) *Assume that $a_{ij} \in C^{0,1}(\bar{\Omega}\times\mathbb{R})$ and $d_i(\omega) \to b_i$ as $\omega \to 0$ for some nonnegative constant b_i. If $s(\mathcal{P}(t))$ is the principal eigenvalue of $\mathcal{P}(t)$, then*

$$\lim_{\omega\to 0} s(\mathcal{A}) = \int_0^1 s(\mathcal{P}(t))\mathrm{d}t,$$

where $\mathcal{P}(t)$ is defined by

$$\mathcal{P}(t)[\mathbf{u}](x) := b_i\mathcal{K}[\mathbf{u}](x) - b_i\mathcal{J}[\mathbf{u}](x) + A(x,t)\mathbf{u}(x), \quad \mathbf{u}\in C(\bar{\Omega},\mathbb{R}^m).$$

(ii) *Assume that the conditions of Theorem 3.2 hold. Then the following assertions hold:*

(a) *If $d_i(\omega) \to p_i$ as $\omega \to +\infty$ for some positive constant p_i, then*

$$\lim_{\omega\to+\infty} s(\mathcal{A}) = s_\infty = \sup_{\|\mathbf{u}\|=1}\langle\mathcal{L}[\mathbf{u}],\mathbf{u}\rangle,$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}[\mathbf{u}](x) := p_i\mathcal{K}[\mathbf{u}](x) - p_i\mathcal{J}[\mathbf{u}](x) + \hat{A}(x)\mathbf{u}(x), \quad \mathbf{u}\in C(\bar{\Omega},\mathbb{R}^m),$$

and s_∞ is the principal eigenvalue of $\mathcal{L}$.

(b) *If $d_i(\omega) \to +\infty$ as $\omega \to +\infty$ and $\frac{d_i(\omega)}{\omega} \to q_i$ for some nonnegative constant q_i, then*

$$\lim_{\omega\to+\infty} s(\mathcal{A}) = \lambda(A^*),$$

where $\lambda(A^*)$ is the principal eigenvalue of A^*.

Proof. (i) For fixed $t\in[0,1]$, since $s(\mathcal{P}(t))$ is the principal eigenvalue of $\mathcal{P}(t)$, there exists $\mathbf{v}(\cdot,t)\in C(\bar\Omega,\mathbb{R}^m)$ with $\mathbf{v}(\cdot,t)\gg\mathbf{0}$ in $\bar\Omega$ such that $\mathcal{P}(t)\mathbf{v}=s(\mathcal{P}(t))\mathbf{v}$. It follows from the classical perturbation theory (see Kato [17]) that $\mathbf{v}\in C^1([0,1],C(\bar\Omega,\mathbb{R}^m))$ and $\mathbf{v}(x,t+1)=\mathbf{v}(x,t)$.

Set $\varphi(x,t)=b(t)\mathbf{v}(x,t)$ with

$$b(t)=\exp\left\{-\frac{1}{\omega}\left[t\int_0^1 s(\mathcal{P}(s))\mathrm{d}s-\int_0^t s(\mathcal{P}(s))\mathrm{d}s\right]\right\}.$$

Given an arbitrary $\varepsilon>0$, there exists a sufficiently small $\omega_0>0$ such that $\omega|\partial_t v_i|\le\frac{\varepsilon}{2}v_i$ and

$$\left|(d_i-b_i)\int_\Omega J_i(x-y)(v_i(y,t)-v_i(x,t))\mathrm{d}y\right|\le\frac{\varepsilon}{2}v_i\quad\text{for all}\quad\omega\le\omega_0.$$

In addition,

$$\begin{aligned}&-\mathcal{A}[\varphi]+\left(\int_0^1 s(\mathcal{P}(s))\mathrm{d}s-\varepsilon\right)\varphi\\ &\quad\le\omega b(t)\partial_t\mathbf{v}+\left[-\int_0^1 s(\mathcal{P}(s))\mathrm{d}s+s(\mathcal{P}(t))\right]\varphi-\mathcal{P}(t)[\varphi]+\frac{\varepsilon}{2}\varphi\\ &\qquad+\left(\int_0^1 s(\mathcal{P}(s))\mathrm{d}s-\varepsilon\right)\varphi\\ &\quad\le 0.\end{aligned}$$

By the definition of $\lambda_p(\mathcal{A})$, we get

$$\int_0^1 s(\mathcal{P}(s))\mathrm{d}s-\varepsilon\le\lambda_p(\mathcal{A})=s(\mathcal{A})\quad\text{for all}\quad\omega\le\omega_0.$$

Similarly, we have

$$-\mathcal{A}[\varphi]+\left(\int_0^1 s(\mathcal{P}(s))\mathrm{d}s+\varepsilon\right)\varphi\ge 0.$$

It follows from the definition of $\lambda_p'(\mathcal{A})$ that

$$s(\mathcal{A})=\lambda_p'(\mathcal{A})\le\int_0^1 s(\mathcal{P}(s))\mathrm{d}s+\varepsilon\quad\text{for all}\quad\omega\le\omega_0.$$

As a result, $\lim_{\omega\to0}s(\mathcal{A})=\int_0^1 s(\mathcal{P}(t))\mathrm{d}t$.

(ii) Choose a sequence $\{\omega_n\}$ with $\omega_n\to+\infty$ as $n\to+\infty$ and denote $(s_n(\mathcal{A}),\varphi_n)$ the corresponding eigenpairs; that is, $(s_n(\mathcal{A}),\varphi_n)$ with $\varphi_n\in\mathcal{X}^{++}$ normalized by $\|\varphi_n\|_{\mathcal{X}}=1$ satisfies

(3.23)
$$\omega_n\partial_t\varphi_{n_i}=d_i\int_\Omega J_i(x-y)(\varphi_{n_i}(y,t)-\varphi_{n_i}(x,t))\mathrm{d}y+\sum_{j=1}^m a_{ij}(x,t)\varphi_{n_j}-s_n(\mathcal{A})\varphi_{n_i}.$$

Dividing (3.23) by φ_{n_i} and integrating it over $\Omega \times (0,1)$ yield

$$\begin{aligned} s_n(\mathcal{A}) =& \frac{d_i}{2|\Omega|} \int_0^1 \int_\Omega \int_\Omega J_i(x-y) \left(\sqrt{\frac{\varphi_{n_i}(y,t)}{\varphi_{n_i}(x,t)}} - \sqrt{\frac{\varphi_{n_i}(x,t)}{\varphi_{n_i}(y,t)}} \right)^2 \mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{|\Omega|} \int_0^1 \int_\Omega \sum_{j=1}^m a_{ij}(x,t) \frac{\varphi_{n_j}(x,t)}{\varphi_{n_i}(x,t)} \mathrm{d}x\mathrm{d}t \\ \geq& \frac{1}{|\Omega|} \int_0^1 \int_\Omega a_{ii}(x,t)\mathrm{d}x\mathrm{d}t, \end{aligned}$$

from which we obtain

$$s_n(\mathcal{A}) \geq \frac{1}{m|\Omega|} \sum_{i=1}^m \int_0^1 \int_\Omega a_{ii}(x,t)\mathrm{d}x\mathrm{d}t. \tag{3.24}$$

We derive from Theorem 3.2 that $s_n(\mathcal{A})$ is nonincreasing and thus $\lim_{n\to+\infty} s_n(\mathcal{A}) = s_\infty$ exists. Moreover, $\{s_n(\mathcal{A})\}$ is a bounded sequence.

(a) If $d_i(\omega) \to p_i$ as $\omega \to +\infty$ for some positive constant p_i, dividing (3.23) by ω_n and letting $n \to +\infty$ yields $\partial_t \varphi_{n_i} \to 0$ uniformly on $\bar{\Omega} \times [0,1]$ as $n \to +\infty$ due to the boundedness of $s_n(\mathcal{A})$ and φ_n. Note that $\|\varphi_n\|_{\mathcal{X}} = 1$. There exists some subsequence of $\{\varphi_n\}$, still denoted by itself, such that $\varphi_n \to \varphi$ weakly in $L^2(\Omega \times (0,1), \mathbb{R}^m)$ and $\partial_t \varphi_n \to \partial_t \varphi = 0$ for some function φ as $n \to +\infty$. This implies that φ is independent of t. Integrating (3.23) over $(0,1)$ and letting $n \to +\infty$ yield

$$p_i \int_\Omega J_i(x-y)(\varphi_i(y) - \varphi_i(x))\mathrm{d}y + \sum_{j=1}^m \hat{a}_{ij}(x)\varphi_j(x) - s_\infty \varphi_i(x) = 0. \tag{3.25}$$

Applying the dominated convergence theorem, we conclude that $\|\varphi_n\|_{L^2(\Omega\times(0,1),\mathbb{R}^m)} \to \|\varphi\|_{L^2(\Omega,\mathbb{R}^m)}$ as $n \to +\infty$. Hence, $\varphi_n \to \varphi$ in $L^2(\Omega \times (0,1), \mathbb{R}^m)$ as $n \to +\infty$. Obviously, φ is not identically $\mathbf{0}$. In view of (3.25), $\varphi \in X^{++}$ and s_∞ is the principal eigenvalue of $\mathcal{L}$. This together with the symmetry of $A(x,t)$ gives $s_\infty = \sup_{\|\mathbf{u}\|=1} \langle \mathcal{L}[\mathbf{u}], \mathbf{u} \rangle$.

(b) If $d_i(\omega) \to +\infty$ and $\frac{d_i(\omega)}{\omega} \to q_i$ for some nonnegative constant q_i as $\omega \to +\infty$, dividing (3.23) by ω_n implies that $\partial_t \varphi_{n_i}$ is uniformly bounded due to $\|\varphi_n\|_{\mathcal{X}} = 1$ and the boundedness of $\frac{d_i(\omega_n)}{\omega_n}$ and $s_n(\mathcal{A})$. Multiplying (3.23) by $\partial_t \varphi_{n_i}$, integrating over $\Omega \times (0,1)$, and adding the resulting equations from $i = 1$ to m yield

$$\begin{aligned} &\omega_n \sum_{i=1}^m \int_0^1 \int_\Omega |\partial_t \varphi_{n_i}(x,t)|^2 \mathrm{d}x\mathrm{d}t \\ &= \sum_{i=1}^m \int_0^1 \int_\Omega d_i \int_\Omega J_i(x-y)(\varphi_{n_i}(y,t) - \varphi_{n_i}(x,t))\mathrm{d}y \partial_t \varphi_{n_i}(x,t)\mathrm{d}x\mathrm{d}t \\ &\quad + \sum_{i=1}^m \int_0^1 \int_\Omega \sum_{j=1}^m a_{ij}(x,t)\varphi_{n_j}(x,t)) \partial_t \varphi_{n_i}(x,t)\mathrm{d}x\mathrm{d}t \\ &\quad - s_n(\mathcal{A}) \sum_{i=1}^m \int_0^1 \int_\Omega \varphi_{n_i}(x,t) \partial_t \varphi_{n_i}(x,t)\mathrm{d}x\mathrm{d}t \\ &= \sum_{i,j=1}^m \int_0^1 \int_\Omega a_{ij}(x,t)\varphi_{n_j}(x,t)) \partial_t \varphi_{n_i}(x,t)\mathrm{d}x\mathrm{d}t \\ &\leq M, \end{aligned}$$

where $M>0$ is a constant. Hence, $\|\partial_t \varphi_{n_i}\|_{L^2(\Omega\times(0,1))} \to 0$ as $n\to+\infty$. By the above result and the boundedness of $\|\varphi_{n_i}\|_{L^2(\Omega\times(0,1))}$, up to an extraction, there exists $\psi_i \in W^{1,2}((0,1),L^2(\Omega))$ such that $\varphi_{n_i}\to\psi_i$ and $\partial_t\varphi_{n_i}\to\partial_t\psi_i$ weakly in $L^2(\Omega\times(0,1))$ as $n\to+\infty$. In addition, we have $\|\partial_t\psi_i\| \le \liminf_{n\to+\infty}\|\partial_t\varphi_{n_i}\|_{L^2(\Omega\times(0,1))} = 0$ and thus ψ_i is independent of t.

If $q_i>0$, dividing (3.23) by ω_n, integrating over $(0,1)$ and letting $n\to+\infty$ yield

$$q_i\int_\Omega J_i(x-y)(\psi_i(y)-\psi_i(x))\mathrm{d}y = 0,$$

which implies that ψ_i is a constant and ψ_i is positive due to the normalization of φ_n. Integrating (3.23) over $\Omega\times(0,1)$ and sending $n\to+\infty$ yield

$$\sum_{j=1}^m \frac{1}{|\Omega|}\int_0^1\int_\Omega a_{ij}(x,t)\mathrm{d}x\mathrm{d}t\psi_j = s_\infty\psi_i,$$

which implies that s_∞ is the principal eigenvalue of A^*.

If $q_i=0$, dividing (3.23) by $d_i(\omega_n)$, integrating over $(0,1)$, and letting $n\to+\infty$ yield

$$\int_\Omega J_i(x-y)(\psi_i(y)-\psi_i(x))\mathrm{d}y = 0.$$

The remaining proof is the same as above. The proof is completed. ∎

4. Effects of dispersal rates and dispersal ranges. In this section, we investigate the effects of small and large dispersal rates or dispersal ranges on principal eigenvalues. Set $\bar{A}(t) = (\bar{a}_{ij}(t))_{m\times m}$. Let $\{\Psi(t,s)\mid t\ge s\}$ be the evolution family on $\mathbb{R}^m$ of $\omega\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \bar{A}(t)\mathbf{u}, t\ge s$.

THEOREM 4.1. *Suppose that $J_i(-x)=J_i(x)$ for $1\le i\le m$; then the following statements hold:*

(i) *$s(\mathcal{A})\to\lambda^*$ as $\min_{1\le i\le m} d_i\to+\infty$, where λ^* is the principal eigenvalue of the eigenvalue problem*

$$\begin{cases}\omega\frac{d\mathbf{u}(t)}{dt} = \bar{A}(t)\mathbf{u}(t)-\lambda\mathbf{u}(t), & t\in\mathbb{R},\\ \mathbf{u}(t+1)=\mathbf{u}(t), & t\in\mathbb{R}.\end{cases} \tag{4.1}$$

(ii) *Assume $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Then $s(\mathcal{A})\to\max_{x\in\bar\Omega}\lambda(x)$ as $\max_{1\le i\le m}\{d_i\}\to 0$, where $\lambda(x)$ is defined in Lemma 2.8.*

Proof. (i) It follows from Corollary 2.15(iii) that there exists $d_*>0$ such that $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ for all $\min_{1\le i\le m} d_i\ge d_*$. Then there exists $\varphi\in\mathcal{X}^+\setminus\{\mathbf{0}\}$ such that for $(x,t)\in\Omega\times\mathbb{R}$,

$$\omega\partial_t\varphi_i(x,t) = d_i\int_\Omega J_i(x-y)[\varphi_i(y,t)-\varphi_i(x,t)]\mathrm{d}y + \sum_{j=1}^m a_{ij}(x,t)\varphi_j(x,t) - s(\mathcal{A})\varphi_i(x,t). \tag{4.2}$$

Integrating (4.2) over $\Omega\times(0,1)$ and adding the resulting equations from $i=1$ to m give

$$s(\mathcal{A})\sum_{i=1}^m\int_0^1\int_\Omega\varphi_i(x,t)\mathrm{d}x\mathrm{d}t = \sum_{i=1}^m\sum_{j=1}^m\int_0^1\int_\Omega a_{ij}(x,t)\varphi_j(x,t)\mathrm{d}x\mathrm{d}t,$$

which implies that $s(\mathcal{A}) \geq 0$. Normalize φ by

$$\sum_{i=1}^{m} \int_0^1 \int_\Omega \varphi_i^2(x,t)\mathrm{d}x\mathrm{d}t = 1. \tag{4.3}$$

Multiplying (4.2) by φ_i, integrating over $\Omega \times (0,1)$ and adding the resulting equations from $i = 1$ to m yield

$$\begin{aligned} 0 \leq s(\mathcal{A}) =& -\sum_{i=1}^{m} \frac{d_i}{2} \int_0^1 \int_\Omega \int_\Omega J_i(x-y)(\varphi_i(y,t) - \varphi_i(x,t))^2 \mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &+ \sum_{i=1}^{m} \int_0^1 \int_\Omega \sum_{j=1}^{m} a_{ij}(x,t)\varphi_j(x,t)\varphi_i(x,t)\mathrm{d}x\mathrm{d}t \\ \leq& \frac{1}{2} \sum_{i,j=1}^{m} \max_{(x,t)\in\bar{\Omega}\times[0,1]} |a_{ij}(x,t)| \left(\int_0^1 \int_\Omega \varphi_j^2(x,t)\mathrm{d}x\mathrm{d}t + \int_0^1 \int_\Omega \varphi_i^2(x,t)\mathrm{d}x\mathrm{d}t \right) \\ \leq& \sum_{i,j=1}^{m} \max_{(x,t)\in\bar{\Omega}\times[0,1]} |a_{ij}(x,t)|. \end{aligned} \tag{4.4}$$

Then we have

$$\begin{aligned} &\int_0^1 \int_\Omega \int_\Omega J_i(x-y)(\varphi_i(y,t) - \varphi_i(x,t))^2 \mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &\quad \leq \frac{2}{d_i} \left[-s(\mathcal{A}) + \sum_{i=1}^{m} \int_0^1 \int_\Omega \sum_{j=1}^{m} a_{ij}(x,t)\varphi_j(x,t)\varphi_i(x,t)\mathrm{d}x\mathrm{d}t \right] \\ &\quad \leq \frac{2}{d_i} \left(\sum_{i,j=1}^{m} \max_{(x,t)\in\bar{\Omega}\times[0,1]} |a_{ij}(x,t)| \right). \end{aligned} \tag{4.5}$$

Set $\phi(x,t) = \varphi(x,t) - \bar{\varphi}(t)$. Then $\int_\Omega \phi(x,t)\mathrm{d}x = \mathbf{0}$. Note that

$$\int_\Omega \int_\Omega J_i(x-y)(\varphi_i(y,t) - \varphi_i(x,t))^2 \mathrm{d}y\mathrm{d}x = \int_\Omega \int_\Omega J_i(x-y)(\phi_i(y,t) - \phi_i(x,t))^2 \mathrm{d}y\mathrm{d}x. \tag{4.6}$$

By [33, Formula (5.6), p. 1688], there exists $C_i > 0$ such that

$$\int_\Omega \phi_i^2(x,t)\mathrm{d}x \leq \frac{1}{2C_i} \int_\Omega \int_\Omega J_i(x-y)(\phi_i(y,t) - \phi_i(x,t))^2 \mathrm{d}y\mathrm{d}x \quad \text{for all} \quad d_i \gg 1. \tag{4.7}$$

In view of (4.5), (4.6), and (4.7), we have

$$\int_0^1 \int_\Omega \phi_i^2(x,t)\mathrm{d}x\mathrm{d}t \leq \frac{1}{C_i d_i} \left(\sum_{i,j=1}^{m} \max_{(x,t)\in\bar{\Omega}\times[0,1]} |a_{ij}(x,t)| \right). \tag{4.8}$$

Integrating (4.2) over Ω yields

$$\omega \frac{\mathrm{d}\bar{\varphi}_i(t)}{\mathrm{d}t} = \sum_{j=1}^{m} \frac{1}{|\Omega|} \int_\Omega a_{ij}(x,t)\mathrm{d}x \bar{\varphi}_j(t) - s(\mathcal{A})\bar{\varphi}_i(t) + \sum_{j=1}^{m} \frac{1}{|\Omega|} \int_\Omega a_{ij}(x,t)\phi_j(x,t)\mathrm{d}x.$$

In view of (4.8), we have

$$\sum_{j=1}^{m}\frac{1}{|\Omega|}\int_0^1\int_\Omega a_{ij}(x,t)\phi_j(x,t)\mathrm{d}x\mathrm{d}t = O\left(\sum_{i=1}^{m} d_i^{-\frac{1}{2}}\right) \text{ for all } \min_{1\le i\le m} d_i \gg 1.$$

The variation of constants formula gives rise to

$$\bar{\varphi}(t) = e^{-\frac{s(\mathcal{A})}{\omega}t}\Psi(t,0)\bar{\varphi}(0) + O\left(\sum_{i=1}^{m} d_i^{-\frac{1}{2}}\right) \text{ for all } \min_{1\le i\le m} d_i \gg 1. \tag{4.9}$$

Note that $\bar{\varphi}(1)=\bar{\varphi}(0)$. We derive

$$\bar{\varphi}(0) = e^{-\frac{s(\mathcal{A})}{\omega}}\Psi(1,0)\bar{\varphi}(0) + O\left(\sum_{i=1}^{m} d_i^{-\frac{1}{2}}\right) \text{ for all } \min_{1\le i\le m} d_i \gg 1. \tag{4.10}$$

If $\liminf_{\min_{1\le i\le m} d_i\to+\infty}\sum_{j=1}^m \bar{\varphi}_j(0)=0$, we derive from (4.9) that

$$\liminf_{\min_{1\le i\le m} d_i\to+\infty}\sum_{j=1}^{m}\bar{\varphi}_j(t) = 0 \text{ uniformly for } t\in[0,1],$$

which implies that $\liminf_{\min_{1\le i\le m} d_i\to+\infty}\sum_{j=1}^m\int_0^1\int_\Omega \bar{\varphi}_j^2(t)\mathrm{d}x\mathrm{d}t=0$. Since

$$\int_0^1\int_\Omega \varphi_j^2(x,t)\mathrm{d}x\mathrm{d}t \le 2\left(\int_0^1\int_\Omega \phi_j^2(x,t)\mathrm{d}x\mathrm{d}t + \int_0^1\int_\Omega \bar{\varphi}_j^2(t)\mathrm{d}x\mathrm{d}t\right),$$

combining with (4.8), we have $\liminf_{\min_{1\le i\le m} d_i\to+\infty}\sum_{j=1}^m\int_0^1\int_\Omega \varphi_j^2(x,t)\mathrm{d}x\mathrm{d}t = 0$. This contradicts (4.3). Thus,

$$\liminf_{\min_{1\le i\le m} d_i\to+\infty}\sum_{j=1}^{m}\bar{\varphi}_j(0) > 0. \tag{4.11}$$

If $\limsup_{\min_{1\le i\le m} d_i\to+\infty}\max_{1\le j\le m}\bar{\varphi}_j(0)=+\infty$, we derive from (4.9) that

$$\limsup_{\min_{1\le i\le m} d_i\to+\infty}\max_{1\le j\le m}\bar{\varphi}_j(t) = +\infty \text{ uniformly for } t\in[0,1].$$

Note that

$$\int_0^1\bar{\varphi}_j^2(t)\mathrm{d}t = \int_0^1\left(\frac{1}{|\Omega|}\int_\Omega\varphi_j(x,t)\mathrm{d}x\right)^2\mathrm{d}t \le \frac{1}{|\Omega|}\int_0^1\int_\Omega\varphi_j^2(x,t)\mathrm{d}x\mathrm{d}t.$$

We obtain

$$\limsup_{\min_{1\le i\le m} d_i\to+\infty}\max_{1\le j\le m}\int_0^1\int_\Omega\varphi_j^2(x,t)\mathrm{d}x\mathrm{d}t = +\infty,$$

contradicting (4.3). Thus,

$$\limsup_{\min_{1\le i\le m} d_i\to+\infty}\max_{1\le j\le m}\bar{\varphi}_j(0) < +\infty. \tag{4.12}$$

Case 1. $\Psi(1,0)$ is irreducible. For any sequence $\{d_k\}$ with $d_k=(d_{k_1},\dots,d_{k_m})^T$, there exists a subsequence $\{d_{k_l}\}$ such that $\bar{\varphi}_{k_l}(0)\to\varphi^*$ and $s(\mathcal{A})\to s^*$ as $\min_{1\le i\le m} d_{k_{l_i}}\to+\infty$ for some $\varphi^*\in(\mathbb{R}^m)^+$ and $s^*\ge 0$. It follows from (4.10) that $\varphi^*=e^{-\frac{s^*}{\omega}}\Psi(1,0)\varphi^*$. Since $\Psi(1,0)$ is irreducible, we derive from the Perron–Frobenius theorem that $s^*=\omega\ln r(\Psi(1,0))$. By the arbitrariness of $\{d_k\}$, we get the desired conclusion.

Case 2. $\Psi(1,0)$ is reducible. Although the remaining proof is similar to the proof of [46, Theorem 3.3], we still present the proof here for the sake of completeness. We first prove that $s_\infty:=\lim_{\min_{1\le i\le m} d_i\to+\infty} s(\mathcal{A})$ exists. By (4.4), we know $s_-:=\liminf_{\min_{1\le i\le m} d_i\to+\infty} s(\mathcal{A})$ and $s_+:=\limsup_{\min_{1\le i\le m} d_i\to+\infty} s(\mathcal{A})$ exist. It suffices to show $s_-=s_+$. If $s_-<s_+$, for any $\tilde{s}\in[s_-,s_+]$, we derive from (4.11), (4.12), and [46, Lemma 3.13] that there exists a sequence $\{d_l\}$ with $d_l=(d_{l_1},\dots,d_{l_m})^T$ such that $\bar{\varphi}_l(0)\to\tilde{\varphi}$ and $s(\mathcal{A})\to\tilde{s}$ as $\min_{1\le i\le m} d_{l_i}\to+\infty$ for some $\tilde{\varphi}\in(\mathbb{R}^m)^+$ with $\tilde{\varphi}\ne\mathbf{0}$. It follows from (4.10) that $\tilde{\varphi}=e^{-\frac{\tilde{s}}{\omega}}\Psi(1,0)\tilde{\varphi}$, which implies that $e^{\frac{\tilde{s}}{\omega}}$ is an eigenvalue of $\Psi(1,0)$ for any $\tilde{s}\in[s_-,s_+]$. This is a contradiction. Thus, $s_-=s_+$.

Next we prove $s_\infty=\omega\ln r(\Psi(1,0))$. For any given $\varepsilon>0$, let $a^\varepsilon_{ij}(x,t)=a_{ij}(x,t)+\varepsilon$ and $A^\varepsilon(x,t)=(a^\varepsilon_{ij}(x,t))_{m\times m}$ for all $(x,t)\in\bar{\Omega}\times\mathbb{R}$. Let $\{\Psi^\varepsilon(t,s)\mid t\ge s\}$ be the evolution family on $\mathbb{R}^m$ of $\omega\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\bar{A}^\varepsilon(t)\mathbf{u}$, $t\ge s$ with $\bar{A}^\varepsilon(t)=(\frac{\int_\Omega a^\varepsilon_{ij}(x,t)\mathrm{d}x}{|\Omega|})_{m\times m}$. In view of Lemma 2.7 and Proposition 2.10, $s(\mathcal{A}^\varepsilon)\ge s(\mathcal{A})$, where $\mathcal{A}^\varepsilon$ is defined by replacing $A(x,t)$ by $A^\varepsilon(x,t)$ in the definition of $\mathcal{A}$. Since $s(\mathcal{A}^\varepsilon)\to\omega\ln r(\Psi^\varepsilon(1,0))$ as $\min_{1\le i\le m} d_i\to+\infty$ and $\ln r(\Psi^\varepsilon(1,0))\to\ln r(\Psi(1,0))$ as $\varepsilon\to 0$, we have $s_\infty\le\omega\ln r(\Psi(1,0))$. Via a suitable permutation, we may rewrite $\Psi(1,0)$ into the following form:

$$\begin{pmatrix} M_{11} & 0 & \cdots & 0\\ M_{21} & M_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ M_{\tilde{m}1} & M_{\tilde{m}2} & \cdots & M_{\tilde{m}\tilde{m}}\end{pmatrix},$$

where M_{kk} is an $i_k\times i_k$ irreducible matrix and $\sum_{k=1}^{\tilde{m}} i_k=m$. Without loss of generality, we assume that $\Psi(1,0)$ is already in the above form. We know $r(\Psi(1,0))=\max_{1\le k\le\tilde{m}} r(M_{kk})$. We split $d,\mathcal{K},\mathcal{J}$, $A(x,t)$, and $\bar{A}(t)$ into $d=\mathrm{diag}(\mathscr{D}_1,\dots,\mathscr{D}_{\tilde{m}})$, $\mathcal{K}=\mathrm{diag}(\mathscr{K}_1,\dots,\mathscr{K}_{\tilde{m}})$, $\mathcal{J}=\mathrm{diag}(\mathscr{J}_1,\dots,\mathscr{J}_{\tilde{m}})$,

$$A(x,t)=\begin{pmatrix} A_{11}(x,t) & A_{12}(x,t) & \cdots & A_{1\tilde{m}}(x,t)\\ A_{21}(x,t) & A_{22}(x,t) & \cdots & A_{2\tilde{m}}(x,t)\\ \vdots & \vdots & \ddots & \vdots\\ A_{\tilde{m}1}(x,t) & A_{\tilde{m}2}(x,t) & \cdots & A_{\tilde{m}\tilde{m}}(x,t)\end{pmatrix},$$

and

$$\bar{A}(t)=\begin{pmatrix} \bar{A}_{11}(t) & \bar{A}_{12}(t) & \cdots & \bar{A}_{1\tilde{m}}(t)\\ \bar{A}_{21}(t) & \bar{A}_{22}(t) & \cdots & \bar{A}_{2\tilde{m}}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \bar{A}_{\tilde{m}1}(t) & \bar{A}_{\tilde{m}2}(t) & \cdots & \bar{A}_{\tilde{m}\tilde{m}}(t)\end{pmatrix}.$$

For each $k=1,2,\dots,\tilde{m}$, let $\{\Psi_k(t,s)\mid t\ge s\}$ be the evolution family of $\omega\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\bar{A}_{kk}(t)\mathbf{u}$, $t\ge s$. It follows from [46, Lemma 3.11] that for each $k=1,2,\dots,\tilde{m}$ and $l>k$, $\bar{A}_{kl}(t)$, and hence $A_{kl}(x,t)$ is a zero matrix for any $t\in\mathbb{R}$ and $(x,t)\in\bar{\Omega}\times\mathbb{R}$, respectively. Define $\mathscr{A}_k$ by

$$\mathscr{A}_k[\mathbf{u}_k](x,t):=-\omega\partial_t\mathbf{u}_k(x,t)+\mathscr{D}\mathscr{K}_k[\mathbf{u}_k](x,t)-\mathscr{D}\mathscr{J}_k[\mathbf{u}_k](x,t)+A_{kk}(x,t)\mathbf{u}_k(x,t).$$

By virtue of Lemma 2.7 and Proposition 2.10, $s(\mathcal{A})\geq s(\mathscr{A}_k)$. By the results obtained in Case 1, we have $s(\mathscr{A}_k)\to\omega\ln r(\Psi_k(1,0))$ as $\min_{1\leq i\leq m} d_i\to+\infty$. As a result, $s_\infty\geq \max_{1\leq k\leq\tilde{m}}\omega\ln r(\Psi_k(1,0))$. Now we have $s_\infty=\omega\ln r(\Psi(1,0))$. We know $\omega\ln r(\Psi(1,0))$ is the principal eigenvalue of (4.1). The proof of (i) is completed.

(ii) We first assume that

$$a_{ij}(x,t)>0 \quad \text{for all} \quad (x,t)\in\bar{\Omega}\times\mathbb{R} \quad\text{and}\quad i\neq j. \tag{4.13}$$

It follows from [3, Theorem 1.4] that there exists $\phi(x,t)\in\mathcal{X}^{++}$ satisfying

$$-\omega\partial_t\phi(x,t)+A(x,t)\phi(x,t)=\lambda(x)\phi(x,t),\quad (x,t)\in\bar{\Omega}\times[0,1].$$

Then for any $\varepsilon>0$, there is $d_0>0$ such that for all $\max_{1\leq i\leq m}\{d_i\}\leq d_0$,

$$\begin{aligned}
&-\mathcal{A}[\phi](x,t)+\left(\max_{x\in\bar{\Omega}}\lambda(x)+\varepsilon\right)\phi(x,t)\\
&=-d\mathcal{K}[\phi](x,t)+d\mathcal{J}[\phi](x,t)-\lambda(x)\phi(x,t)+\left(\max_{x\in\bar{\Omega}}\lambda(x)+\varepsilon\right)\phi(x,t)\\
&\geq 0,
\end{aligned}$$

which implies that $s(\mathcal{A})\leq\max_{x\in\bar{\Omega}}\lambda(x)+\varepsilon$ for all $\max_{1\leq i\leq m}\{d_i\}\leq d_0$ by the definition of $\lambda_p'(\mathcal{A})$. Hence, $\limsup_{\max_{1\leq i\leq m}\{d_i\}\to 0} s(\mathcal{A})\leq\max_{x\in\bar{\Omega}}\lambda(x)$.

Now we remove the extra assumption (4.13). Let $\delta>0$ be any small constant and define $\mathcal{A}_\delta$ by replacing $A(x,t)$ by $A_\delta(x,t)=(a_{ij}(x,t)+\delta)_{m\times m}$ in the definition of $\mathcal{A}$. The above arguments give that $\limsup_{\max_{1\leq i\leq m}\{d_i\}\to 0} s(\mathcal{A}_\delta)\leq\max_{x\in\bar{\Omega}}\lambda_\delta(x)$, where $\lambda_\delta(x)$ is defined in Lemma 2.8 by replacing $A(x,t)$ by $A_\delta(x,t)$. By virtue of Lemma 2.7 and Proposition 2.10, $s(\mathcal{A}_\delta)\geq s(\mathcal{A})$. We derive from the proof of [3, Theorem 1.5] that $\lambda_\delta(x)\to\lambda(x)$ uniformly on $\bar{\Omega}$ as $\delta\to 0$. Hence, $\limsup_{\max_{1\leq i\leq m}\{d_i\}\to 0} s(\mathcal{A})\leq \max_{x\in\bar{\Omega}}\lambda(x)$.

By Propositions 2.9 and 2.10 we know that both $\mathcal{A}$ and $\mathcal{B}$ are resolvent positive. By [22, Lemma 2.2], we obtain $s(\mathcal{A})\geq s(\mathcal{B})=\alpha^*$. Hence, $s(\mathcal{A})\to\max_{x\in\bar{\Omega}}\lambda(x)$ as $\max_{1\leq i\leq m}\{d_i\}\to 0$. The proof is completed. ∎

Next we investigate the effect of dispersal ranges. Set $J_{\sigma_i}(x)=\frac{1}{\sigma_i^n}J_i(\frac{x}{\sigma_i})$ and denote $\mathcal{A}_\sigma$ the corresponding operator replacing J_i by J_{σ_i} and d_i by $\frac{d_i}{\sigma_i^{m_i}}$ for all $1\leq i\leq m$ in the definition of $\mathcal{A}$, where $\sigma_i>0$ is the dispersal range and $m_i>0$ is the cost parameter (see Hutson et al. [15]). Motivated by Shen and Vo [32] and Vo [43], we have the following results.

THEOREM 4.2. *Suppose that J_i is compactly supported and $s(\mathcal{A}_\sigma)$ is the principal eigenvalue of $\mathcal{A}_\sigma$. Then the following statements hold:*

(i) $s(\mathcal{A}_\sigma)\to\max_{x\in\bar{\Omega}}\lambda(x)$ *as* $\min_{1\leq i\leq m}\{\sigma_i\}\to+\infty$ *for* $m_i\geq 0$ *and* $1\leq i\leq m$.

(ii) *If* (A2) *holds,* J_i *is symmetric with respect to each component, and* $a_{ij}\in C^{4,0}(\bar{\Omega}\times[0,1])$ *for all* $1\leq i,j\leq m$, *then* $s(\mathcal{A}_\sigma)\to\max_{x\in\bar{\Omega}}\lambda(x)$ *as* $\max_{1\leq i\leq m}\{\sigma_i\}\to 0$ *for* $m_i\in[0,2)$ *and* $1\leq i\leq m$.

Proof. (i) We derive from [22, Lemma 2.2] that $s(\mathcal{A}_\sigma)\geq\max_{\bar{\Omega}}\alpha_\sigma(x)$, where $\alpha_\sigma(x)$ is the principal eigenvalue determined in Lemma 2.8 with $A(x,t)$ replaced by

$$A(x,t) + \mathrm{diag}\left(-\frac{d_1}{\sigma_1^{m_1}}\int_\Omega J_{\sigma_1}(x-y)\mathrm{d}y, \dots, -\frac{d_m}{\sigma_m^{m_m}}\int_\Omega J_{\sigma_m}(x-y)\mathrm{d}y\right),$$

which implies that

$$\liminf_{\min\limits_{1\le i\le m}\{\sigma_i\}\to+\infty} s(\mathcal{A}_\sigma) \ge \max_{x\in\bar\Omega}\lambda(x).$$

It suffices to show that

$$\limsup_{\min\limits_{1\le i\le m}\{\sigma_i\}\to+\infty} s(\mathcal{A}_\sigma) \le \max_{x\in\bar\Omega}\lambda(x).$$

We first assume that

$$a_{ij}(x,t) > 0 \quad \text{for all} \quad (x,t)\in\bar\Omega\times\mathbb{R} \quad \text{and} \quad i\ne j. \tag{4.14}$$

It follows from [3, Theorem 1.4] that there exists $\phi\in\mathcal{X}^{++}$ satisfying

$$-\omega\partial_t\phi(x,t) + A(x,t)\phi(x,t) = \lambda(x)\phi(x,t), \quad (x,t)\in\bar\Omega\times[0,1].$$

Since $\phi(x,t)\in\mathcal{X}^{++}$ and

$$\left\|\frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y\right\|_{L^\infty(\Omega\times(0,1))} \to 0 \quad \text{as} \quad \min_{1\le i\le m}\{\sigma_i\}\to+\infty,$$

for any $\varepsilon>0$, there exists $\sigma_0>0$ such that for all $\min_{1\le i\le m}\{\sigma_i\}\ge\sigma_0$,

$$\begin{aligned}
&\omega\partial_t\phi_i(x,t) - \frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y \\
&\quad -\sum_{j=1}^m a_{ij}(x,t)\phi_j(x,t) + \left(\max_{x\in\bar\Omega}\lambda(x)+\varepsilon\right)\phi_i(x,t) \\
&\quad\quad = -\frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y - \lambda(x)\phi(x,t) \\
&\quad\quad\quad + \left(\max_{x\in\bar\Omega}\lambda(x)+\varepsilon\right)\phi_i(x,t) \\
&\quad\quad \ge -\frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y + \varepsilon\phi_i(x,t) \\
&\quad\quad \ge 0.
\end{aligned}$$

By the definition of $\lambda_p'(\mathcal{A}_\sigma)$, we have $s(\mathcal{A}_\sigma)=\lambda_p'(\mathcal{A}_\sigma)\le(\max_{x\in\bar\Omega}\lambda(x)+\varepsilon)$. Hence,

$$\limsup_{\min\limits_{1\le i\le m}\{\sigma_i\}\to+\infty} s(\mathcal{A}_\sigma) \le \max_{x\in\bar\Omega}\lambda(x).$$

By using the same argument as the proof of Theorem 4.1(ii), we can remove the extra assumption (4.14).

(ii) Since $a_{ij}\in C^{4,0}(\bar\Omega\times[0,1])$, we have $\lambda(x)\in C^4(\bar\Omega)$ and the corresponding eigenfunction $\phi(x,t)\in C^{4,0}(\bar\Omega\times[0,1],\mathbb{R}^m)$. Normalize ϕ by $\|\phi\|_{\mathcal{X}}=1$. Then for any $\varepsilon>0$,

$$\begin{aligned}
&\omega\partial_t\phi_i(x,t) - \frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y \\
&\quad - \sum_{j=1}^m a_{ij}(x,t)\phi_j(x,t) + \left(\max_{x\in\bar\Omega}\lambda(x)+\varepsilon\right)\phi_i(x,t) \\
&\quad = -\frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y - \lambda(x)\phi_i(x,t) \\
&\qquad + \left(\max_{x\in\bar\Omega}\lambda(x)+\varepsilon\right)\phi_i(x,t) \\
&\quad \geq -\frac{d_i}{\sigma_i^{m_i}}\int_\Omega J_{\sigma_i}(x-y)(\phi_i(y,t)-\phi_i(x,t))\mathrm{d}y + \varepsilon\phi_i(x,t) \\
&\quad = -\frac{d_i}{\sigma_i^{m_i}}\int_{\frac{\Omega-x}{\sigma_i}} J_i(z)(\phi_i(x+\sigma_i z,t)-\phi_i(x,t))\mathrm{d}z + \varepsilon\phi_i(x,t). \qquad (4.15)
\end{aligned}$$

Since J_i is compactly supported, there exists $\sigma^* > 0$ such that $\operatorname{supp} J_i \subset \frac{\Omega-x}{\sigma_i}$ for all $\sigma_i \leq \sigma^*$ and $1 \leq i \leq m$. Thus, by Taylor's expansion and the symmetry of J_i, there holds

$$\begin{aligned}
&\frac{d_i}{\sigma_i^{m_i}}\int_{\frac{\Omega-x}{\sigma_i}} J_i(z)(\phi_i(x+\sigma_i z,t)-\phi_i(x,t))\mathrm{d}z \\
&\quad = \frac{d_i}{\sigma_i^{m_i}}\int_{\mathbb{R}^n} J_i(z)(\phi_i(x+\sigma_i z,t)-\phi_i(x,t))\mathrm{d}z \\
&\quad = \frac{d_i}{\sigma_i^{m_i}}\int_{\mathbb{R}^n} J_i(z)\left[D\phi_i(x,t)(\sigma_i z) + \frac{1}{2}(\sigma_i z)^T D^2\phi_i(x,t)(\sigma_i z) + o\left(\sigma_i^2\right)\right]\mathrm{d}z \\
&\quad = \frac{d_i\sigma_i^{2-m_i}}{2}\int_{\mathbb{R}^n} J_i(z) z^T D^2\phi_i(x,t) z\,\mathrm{d}z + o\left(\sigma_i^{2-m_i}\right). \qquad (4.16)
\end{aligned}$$

Combining (4.15) with (4.16) yields

$$-\mathcal{A}_\sigma[\phi](x,t) + \left(\max_{x\in\bar\Omega}\lambda(x)+\varepsilon\right)\phi(x,t) \geq 0 \quad \text{for all} \quad \max_{1\leq i\leq m}\sigma_i \leq \sigma^*.$$

By the definition of $\lambda_p'(\mathcal{A}_\sigma)$, we have

$$s(\mathcal{A}_\sigma) = \lambda_p'(\mathcal{A}_\sigma) \leq \max_{x\in\bar\Omega}\lambda(x) + \varepsilon.$$

Then it implies that

$$\limsup_{\max_{1\leq i\leq m}\sigma_i\to 0} s(\mathcal{A}_\sigma) \leq \max_{x\in\bar\Omega}\lambda(x).$$

It suffices to show that

$$\liminf_{\max_{1\leq i\leq m}\sigma_i\to 0} s(\mathcal{A}_\sigma) \geq \max_{x\in\bar\Omega}\lambda(x).$$

For any $\epsilon > 0$, there exists an open ball $B_\epsilon \subset \Omega$ of radius ϵ such that $\lambda(x) + \epsilon > \max_{\bar\Omega}\lambda(x)$ in B_ϵ. Let $\tilde\phi_\epsilon \in C^{4,1}(\mathbb{R}^n\times\mathbb{R},\mathbb{R}^m)$ be nonnegative and 1-periodic in t and satisfy

$$\tilde\phi_\epsilon = \phi \text{ in } \bar B_\epsilon\times\mathbb{R}, \quad \tilde\phi_\epsilon = 0 \text{ in } (\mathbb{R}^n\backslash B_{2\epsilon})\times\mathbb{R}, \quad \text{and} \quad \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}|\tilde\phi_\epsilon(x,t)| \leq \|\phi\|_{\mathcal{X}} = 1.$$

Let $\tilde{\lambda}_p(\mathcal{A}_\sigma^{\mathcal{D}})$ be the generalized principal eigenvalue of $\mathcal{A}_\sigma^{\mathcal{D}}$, where $\mathcal{A}_\sigma^{\mathcal{D}}$ is defined by

$$\mathcal{A}_\sigma^{\mathcal{D}}[\mathbf{u}](x,t) := -\omega\partial_t \mathbf{u}(x,t) + \mathcal{O}\mathcal{K}_{\mathcal{D}}[\mathbf{u}](x,t) - \mathcal{O}\mathbf{u}(x,t) + A(x,t)\mathbf{u}(x,t)$$

with $\mathcal{O} = \operatorname{diag}(\frac{d_1}{\sigma_1^{m_1}}, \dots, \frac{d_m}{\sigma_m^{m_m}})$ and

$$\mathcal{K}_{\mathcal{D}}[\mathbf{u}](x,t) = \left(\int_{\mathcal{D}} J_{\sigma_1}(x-y)\mathbf{u}_1(y,t)\mathrm{d}y, \dots, \int_{\mathcal{D}} J_{\sigma_m}(x-y)\mathbf{u}_m(y,t)\mathrm{d}y\right).$$

Then we have for $(x,t) \in \bar{B}_\epsilon \times \mathbb{R}$ that

$$\begin{aligned}
&\omega\partial_t\phi_i(x,t) - \frac{d_i}{\sigma_i^{m_i}}\left[\int_{B_\epsilon} J_{\sigma_i}(x-y)\phi_i(y,t)\mathrm{d}y - \phi_i(x,t)\right] \\
&\quad - \sum_{j=1}^m a_{ij}(x,t)\phi_j(x,t) + \left(\max_{x\in\bar{\Omega}}\lambda(x) - \epsilon - \frac{1}{|\ln\epsilon|}\right)\phi_i(x,t) \\
&\quad = -\frac{d_i}{\sigma_i^{m_i}}\left[\int_{B_\epsilon} J_{\sigma_i}(x-y)\phi_i(y,t)\mathrm{d}y - \phi_i(x,t)\right] \\
&\qquad + \left(-\lambda(x) + \max_{x\in\bar{\Omega}}\lambda(x) - \epsilon - \frac{1}{|\ln\epsilon|}\right)\phi_i(x,t) \\
&\quad \le -\frac{d_i}{\sigma_i^{m_i}}\left[\int_{B_\epsilon} J_{\sigma_i}(x-y)\phi_i(y,t)\mathrm{d}y - \phi_i(x,t)\right] - \frac{1}{|\ln\epsilon|}\phi_i(x,t) \\
&\quad = -\frac{d_i}{\sigma_i^{m_i}}\left[\int_{\mathbb{R}^n} J_{\sigma_i}(x-y)\tilde{\phi}_{\epsilon_i}(y,t)\mathrm{d}y - \tilde{\phi}_{\epsilon_i}(x,t) - \int_{B_{2\epsilon}\setminus B_\epsilon} J_{\sigma_i}(x-y)\tilde{\phi}_{\epsilon_i}(y,t)\mathrm{d}y\right] \\
&\qquad - \frac{1}{|\ln\epsilon|}\phi_i(x,t).
\end{aligned}$$

Set $\sigma = \max_{1\le i\le m}\sigma_i$ and $m^* = \max_{1\le i\le m} m_i$. Now following a similar argument as in Shen and Vo [32, Theorem D] and choosing $\epsilon = \sigma^k$ and $k = \frac{m^*+2n}{n}$, we have for $0 < \sigma \ll 1$ and $(x,t) \in \bar{B}_{\sigma^k} \times \mathbb{R}$ that

$$\begin{aligned}
&\omega\partial_t\phi_i(x,t) - \frac{d_i}{\sigma_i^{m_i}}\left[\int_{B_{\sigma^k}} J_{\sigma_i}(x-y)\phi_i(y,t)\mathrm{d}y - \phi_i(x,t)\right] \\
&\quad - \sum_{j=1}^m a_{ij}(x,t)\phi_j(x,t) + \left(\max_{x\in\bar{\Omega}}\lambda(x) - \sigma^k - \frac{1}{|\ln(\sigma^k)|}\right)\phi_i(x,t) \le 0.
\end{aligned}$$

Then by the definition of $\tilde{\lambda}_p(\mathcal{A}_\sigma^{B_{\sigma^k}})$, we have

$$\tilde{\lambda}_p\left(\mathcal{A}_\sigma^{B_{\sigma^k}}\right) \ge \max_{x\in\bar{\Omega}}\lambda(x) - \sigma^k - \frac{1}{|\ln(\sigma^k)|}.$$

It is easy to check that $\tilde{\lambda}_p\left(\mathcal{A}_\sigma^{\Omega}\right) \ge \tilde{\lambda}_p(\mathcal{A}_\sigma^{B_{\sigma^k}})$ and thus

$$\tilde{\lambda}_p\left(\mathcal{A}_\sigma^{\Omega}\right) \ge \max_{x\in\bar{\Omega}}\lambda(x) - \sigma^k - \frac{1}{|\ln(\sigma^k)|}. \tag{4.17}$$

Set

$$\tilde{a}_{ii}(x,t) = a_{ii}(x,t) + \frac{d_i}{\sigma_i^{m_i}} - \frac{d_i}{\sigma_i^{m_i}} \int_{\frac{\Omega - x}{\sigma_i}} J_i(z)\mathrm{d}z.$$

It is obvious to see that $\int_{\frac{\Omega-x}{\sigma_i}} J_i(z)\mathrm{d}z = 1$ for a sufficiently small σ_i, which implies that $\lim_{\sigma_i \to 0} \|\tilde{a}_{ii} - a_{ii}\|_{C(\bar{\Omega}\times[0,1])} = 0$. We derive from Proposition 3.4 that

$$\tilde{\lambda}_p\left(\tilde{\mathcal{A}}_\sigma^\Omega\right) \geq \tilde{\lambda}_p\left(\mathcal{A}_\sigma^\Omega\right) + \min_{1\leq i\leq m}\left\{\min_{\bar{\Omega}\times[0,1]}[\tilde{a}_{ii}(x,t) - a_{ii}(x,t)]\right\}, \tag{4.18}$$

where $\tilde{\mathcal{A}}_\sigma^\Omega$ is defined by replacing $a_{ii}(x,t)$ by $\tilde{a}_{ii}(x,t)$ in the definition of $\mathcal{A}_\sigma^\Omega$. By virtue of (4.17) and (4.18), we obtain

$$\liminf_{\max_{1\leq i\leq m} \sigma_i \to 0} s(\mathcal{A}_\sigma) = \liminf_{\max_{1\leq i\leq m} \sigma_i \to 0} \tilde{\lambda}_p\left(\tilde{\mathcal{A}}_\sigma^\Omega\right) \geq \max_{x\in\bar{\Omega}} \lambda(x).$$

The proof is completed. ∎

5. An application. In this section, we apply our theory to study the following epidemic model [8, 9, 45, 14, 21]:

(5.1)
$$\begin{cases}\omega\partial_t u = d_1 \int_\Omega J_1(x-y)(u(y,t)-u(x,t))\mathrm{d}y - a(x,t)u + H(x,t,v), & (x,t)\in\bar{\Omega}\times\mathbb{R},\\ \omega\partial_t v = d_2 \int_\Omega J_2(x-y)(v(y,t)-v(x,t))\mathrm{d}y - b(x,t)v + G(x,t,u), & (x,t)\in\bar{\Omega}\times\mathbb{R},\end{cases}$$

where $u(x,t)$ and $v(x,t)$ denote the spatial density of the bacterial population and the infective human population at location x in the habit region and time t, respectively; the positive constant ω is the frequency; positive constants d_1 and d_2 are the dispersal coefficients; $a(x,t)$ and $b(x,t)$ denote the unit natural death rates of bacteria and infective human population, respectively; the nonlinearity $H(x,t,v(x,t))$ means the growth rate of bacteria caused by infective humans; and $G(x,t,u(x,t))$ stands for the infection rate of the human population under the assumption that the total susceptible human population is constant during the evolution of epidemic. Assume that the kernel functions J_i satisfy

(K) $J_i \in C(\mathbb{R}^n), J_i(0) > 0, J_i(x) \geq 0, J_i(-x) = J_i(x), \int_{\mathbb{R}^n} J_i(x)\mathrm{d}x = 1$ for $i = 1,2$.

Functions $a(x,t)$, $b(x,t)$, $H(x,t,z)$, and $G(x,t,z)$ satisfy

(F) $a(x,t)$ and $b(x,t)$ are positive continuous functions on $\bar{\Omega}\times\mathbb{R}$ and $a(x,t+1) = a(x,t)$, $b(x,t+1) = b(x,t)$ for all $\bar{\Omega}\times\mathbb{R}$. $H, G \in C^{0,0,2}(\bar{\Omega}\times\mathbb{R}\times[0,+\infty))$, $H(x,t,0) = G(x,t,0) = 0$ for all $(x,t)\in\bar{\Omega}\times\mathbb{R}$ and $\frac{\partial G(x,t,z)}{\partial z}$, $\frac{\partial H(x,t,z)}{\partial z} > 0$, $H(x,t+1,z) = H(x,t,z)$, $G(x,t+1,z) = G(x,t,z)$ for all $(x,t,z)\in\bar{\Omega}\times\mathbb{R}\times[0,+\infty)$. There exists a constant $M > 0$ such that $\mathbf{M} = (M,M)$ is an upper solution of (5.1); $\frac{\partial^2 G(x,t,z)}{\partial z^2}$, $\frac{\partial^2 H(x,t,z)}{\partial z^2} < 0$ for all $(x,t,z)\in\bar{\Omega}\times\mathbb{R}\times[0,+\infty)$.

We aim to investigate the impacts of dispersal rates, frequency, and dispersal ranges on the basic reproduction number and positive periodic solutions. Now we choose $m = 2$. It then follows that $X = C(\bar{\Omega},\mathbb{R}^2)$,

$$X^+ = \left\{\mathbf{u}\in C(\bar{\Omega},\mathbb{R}^2) \mid \mathbf{u}(x)\in(\mathbb{R}^2)^+, x\in\bar{\Omega}\right\},$$
$$\mathcal{X} = \left\{\mathbf{u}\in C(\bar{\Omega}\times\mathbb{R},\mathbb{R}^2) \mid \mathbf{u}(x,t+1) = \mathbf{u}(x,t), (x,t)\in\bar{\Omega}\times\mathbb{R}\right\}.$$

Define

$$
\begin{aligned}
&C(x,t) := \begin{pmatrix} -a(x,t) & \frac{\partial H}{\partial z}(x,t,0) \\ \frac{\partial G}{\partial z}(x,t,0) & -b(x,t) \end{pmatrix},\\
&\hat{C}(x) := \begin{pmatrix} -\int_0^1 a(x,t)\mathrm{d}t & \int_0^1 \frac{\partial H}{\partial z}(x,t,0)\mathrm{d}t \\ \int_0^1 \frac{\partial G}{\partial z}(x,t,0)\mathrm{d}t & -\int_0^1 b(x,t)\mathrm{d}t \end{pmatrix},\\
&\bar{C}(t) := \begin{pmatrix} -\frac{1}{|\Omega|}\int_\Omega a(x,t)\mathrm{d}x & \frac{1}{|\Omega|}\int_\Omega \frac{\partial H}{\partial z}(x,t,0)\mathrm{d}x \\ \frac{1}{|\Omega|}\int_\Omega \frac{\partial G}{\partial z}(x,t,0)\mathrm{d}x & -\frac{1}{|\Omega|}\int_\Omega b(x,t)\mathrm{d}x \end{pmatrix},\\
&F(x,t) := \begin{pmatrix} 0 & 0 \\ \frac{\partial G}{\partial z}(x,t,0) & 0 \end{pmatrix}, \quad \hat{F}(x) := \begin{pmatrix} 0 & 0 \\ \int_0^1 \frac{\partial G}{\partial z}(x,t,0)\mathrm{d}t & 0 \end{pmatrix},\\
&\bar{F}(t) := \begin{pmatrix} 0 & 0 \\ \frac{1}{|\Omega|}\int_\Omega \frac{\partial G}{\partial z}(x,t,0)\mathrm{d}x & 0 \end{pmatrix},\\
&\mathscr{A}[\mathbf{u}](x,t) := -\omega\partial_t \mathbf{u}(x,t) + d\mathcal{K}[\mathbf{u}](x,t) - d\mathcal{J}[\mathbf{u}](x,t) + C(x,t)\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X},\\
&\mathscr{B}[\mathbf{u}](x,t) := -\omega\partial_t \mathbf{u}(x,t) + d\mathcal{K}[\mathbf{u}](x,t) - d\mathcal{J}[\mathbf{u}](x,t)\\
&\qquad\qquad\qquad + (C(x,t)-F(x,t))\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X},\\
&\mathscr{L}[\mathbf{u}](x) := d\mathcal{K}[\mathbf{u}](x) - d\mathcal{J}[\mathbf{u}](x) + (\hat{C}(x)-\hat{F}(x))\mathbf{u}(x), \quad \mathbf{u}\in X,\\
&\mathscr{P}(t)[\mathbf{u}](x) := d\mathcal{K}[\mathbf{u}](x) - d\mathcal{J}[\mathbf{u}](x) + (C(x,t)-F(x,t))\mathbf{u}(x), \quad \mathbf{u}\in X,\\
&\mathscr{C}[\mathbf{u}](x,t) := -\omega\partial_t \mathbf{u}(x,t) - d\mathcal{J}[\mathbf{u}](x,t) + (C(x,t)-F(x,t))\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X},\\
&\mathscr{D}[\mathbf{u}](x,t) := -\omega\partial_t \mathbf{u}(x,t) + (C(x,t)-F(x,t))\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X},\\
&\bar{\mathscr{D}}[\mathbf{u}](x,t) := -\omega\partial_t \mathbf{u}(x,t) + (\bar{C}(t)-\bar{F}(t))\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X},\\
&\mathscr{F}[\mathbf{u}](x,t) := F(x,t)\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X},\\
&\hat{\mathscr{F}}[\mathbf{u}](x) := \hat{F}(x)\mathbf{u}(x), \quad \mathbf{u}\in X,\\
&\bar{\mathscr{F}}[\mathbf{u}](x,t) := \bar{F}(t)\mathbf{u}(x,t), \quad \mathbf{u}\in\mathcal{X}.
\end{aligned}
$$

Let $\{\mathscr{T}(t,s)\,|\,t\geq s\}$ be the evolution family on X associated with $\mathscr{B}$.

PROPOSITION 5.1. *$\mathscr{B}$ is a resolvent positive operator on $\mathcal{X}$ and $s(\mathscr{B})<0$.*

Proof. We know that $\mathscr{T}(t,s)$ is a positive operator in the sense that $\mathscr{T}(t,s)X^+\subset X^+$ for all $t\geq s$. It follows from [41, Theorem 3.12] that $\mathscr{B}$ is resolvent positive. It is clear that $s(\mathscr{C})<0$. Suppose for the contrary that $s(\mathscr{B})\geq 0$. Then by Theorem 2.13, $s(\mathscr{B})>s(\mathscr{C})$ implies that $s(\mathscr{B})$ is the principal eigenvalue of $\mathscr{B}$ with the corresponding eigenfunction φ; that is, $(s(\mathscr{B}),\varphi)$ satisfies

$$
(5.2)\quad \begin{cases} -\omega\partial_t\varphi_1 + d_1\int_\Omega J_1(x-y)(\varphi_1(y,t)-\varphi_1(x,t))\mathrm{d}y - a(x,t)\varphi_1 + \frac{\partial H}{\partial z}(x,t,0)\varphi_2 \\ \quad = s(\mathscr{B})\varphi_1, \\ -\omega\partial_t\varphi_2 + d_2\int_\Omega J_2(x-y)(\varphi_2(y,t)-\varphi_2(x,t))\mathrm{d}y - b(x,t)\varphi_2 = s(\mathscr{B})\varphi_2. \end{cases}
$$

Integrating (5.2) over $\Omega\times(0,1)$ yields

$$
\begin{cases} -\int_0^1\int_\Omega a(x,t)\varphi_1(x,t)\mathrm{d}x\mathrm{d}t + \int_0^1\int_\Omega \frac{\partial H}{\partial z}(x,t,0)\varphi_2(x,t)\mathrm{d}x\mathrm{d}t \\ \quad = s(\mathscr{B})\int_0^1\int_\Omega\varphi_1(x,t)\mathrm{d}x\mathrm{d}t, \\ -\int_0^1\int_\Omega b(x,t)\varphi_2(x,t)\mathrm{d}x\mathrm{d}t = s(\mathscr{B})\int_0^1\int_\Omega\varphi_2(x,t)\mathrm{d}x\mathrm{d}t, \end{cases}
$$

which implies that $s(\mathscr{B})<0$ due to $\varphi\not\equiv\mathbf{0}$. A contradiction occurs. Thus, $s(\mathscr{B})<0$. The proof is completed. ∎

Define

$$\mathscr{N}[\phi](s) := \int_0^{+\infty} F(\cdot,s)\mathscr{T}(s,s-t)\phi(\cdot,s-t)\mathrm{d}t, \quad \phi \in \mathcal{X}.$$

Suppose that $\phi(x,t)$ is the density distribution of infected individuals at the spatial location $x \in \Omega$ and time t. Then $F(x,s)\mathscr{T}(s,s-t)\phi(x,s-t)$ is the distribution of individuals newly infected at time s by those infected individuals who were introduced at time $s-t$. Inspired by the ideas of next generation operators (see [12, 42, 44]), define the spectral radius of $\mathscr{N}$

$$\mathscr{R}_0 = r(\mathscr{N})$$

as the basic reproduction number of system (5.1).

In order to investigate the impact of dispersal range, we set $J_{\sigma_i}(x) = \frac{1}{\sigma_i^n} J_i(\frac{x}{\sigma_i})$ and denote $\mathscr{B}_\sigma$ and $\mathscr{C}_\sigma$ the corresponding operator replacing J_i by J_{σ_i} and d_i by $\frac{d_i}{\sigma_i^{m_i}}$ for $i=1,2$ in the definition of $\mathscr{B}$ and $\mathscr{C}$, respectively, where $\sigma_i > 0$ is the dispersal range and $m_i > 0$ is the cost parameter. Similar to the definition of $\mathscr{R}_0$, we define

$$\mathscr{R}_0^\sigma = r(\mathscr{N}_\sigma),$$

where

$$\mathscr{N}_\sigma[\phi](s) := \int_0^{+\infty} F(\cdot,s)\mathscr{T}_\sigma(s,s-t)\phi(\cdot,s-t)\mathrm{d}t, \quad \phi \in \mathcal{X},$$

with $\mathscr{T}_\sigma(t,s)$ being the evolution operator associated with $\mathscr{B}_\sigma$.

THEOREM 5.2. *The following statements hold:*
(i) $\mathscr{R}_0 - 1$ *has the same sign as* $s(\mathscr{A}) = s(\mathscr{B}+\mathscr{F})$.
(ii) *If* $\mathscr{R}_0 > 0$, *then* $\mu = \mathscr{R}_0$ *is the unique solution of* $s(\mathscr{B} + \frac{1}{\mu}\mathscr{F}) = 0$.

Proof. We derive from Proposition 5.1 that $\mathscr{B}$ is resolvent positive and $s(\mathscr{B}) < 0$. Then [41, Theorem 3.12] gives

$$(5.3) \quad (\lambda I - \mathscr{B})^{-1}\phi = \int_0^{+\infty} e^{-\lambda t}\mathscr{T}(s,s-t)\phi(\cdot,s-t)\mathrm{d}t \ \text{ for any } \ \lambda > s(\mathscr{B}), \phi \in \mathcal{X}.$$

Choosing $\lambda = 0$ in (5.3) yields

$$-\mathscr{B}^{-1}\phi = \int_0^{+\infty} \mathscr{T}(s,s-t)\phi(\cdot,s-t)\mathrm{d}t \ \text{ for all } \ \phi \in \mathcal{X}.$$

Then we have $\mathscr{N} = -\mathscr{F}\mathscr{B}^{-1}\phi$. By virtue of Proposition 2.10, we know that $\mathscr{B} + \frac{1}{\mu}\mathscr{F}$ is resolvent positive for any $\mu > 0$. It follows from [41, Theorem 3.5] that $s(\mathscr{A})$ has the same sign as $r(-\mathscr{F}\mathscr{B}^{-1}) - 1 = \mathscr{R}_0 - 1$. If $r(-\mathscr{F}\mathscr{B}^{-1}) = \mathscr{R}_0 > 0$, we derive from [46, Lemma 2.5(ii)] that $\mu = \mathscr{R}_0$ is the unique solution of $s(\mathscr{B} + \frac{1}{\mu}\mathscr{F}) = 0$. The proof is completed. ∎

Remark 5.3. Proposition 5.1 also holds for $\mathscr{B}_\sigma$ and Theorem 5.2 also holds for $\mathscr{R}_0^\sigma$.

Next we study the effects of small and large dispersal rates, frequency, and dispersal ranges. Set

$$\mathscr{P} := \{\mathbf{u} \in C(\mathbb{R},\mathbb{R}^2) \mid \mathbf{u}(t) = \mathbf{u}(t+1), t \in \mathbb{R}\}.$$

For each $x\in\bar{\Omega}$, let $\{\Gamma_x(t,s)\,|\,t\geq s\}$ be the evolution family on $\mathbb{R}^2$ of

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=-a(x,t)u+\frac{\partial H}{\partial z}(x,t,0)v, & t\geq s,\\ \frac{\mathrm{d}v}{\mathrm{d}t}=-b(x,t)v, & t\geq s.\end{cases}\tag{5.4}$$

We use $\{\bar{\Gamma}(t,s)\,|\,t\geq s\}$ to denote the evolution family on $\mathbb{R}^2$ of

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=-\bar{a}(t)u+\frac{1}{|\Omega|}\int_\Omega\frac{\partial H}{\partial z}(x,t,0)\mathrm{d}xv, & t\geq s,\\ \frac{\mathrm{d}v}{\mathrm{d}t}=-\bar{b}(t)v, & t\geq s.\end{cases}\tag{5.5}$$

Let $\{\Phi(t,s)\,|\,t\geq s\}$ and $\{\bar{\Phi}(t,s)\,|\,t\geq s\}$ be the evolution families on X of

$$\begin{cases}\frac{\partial u}{\partial t}=-a(x,t)u+\frac{\partial H}{\partial z}(x,t,0)v, & t\geq s,\\ \frac{\partial v}{\partial t}=-b(x,t)v, & t\geq s,\end{cases}$$

and

$$\begin{cases}\frac{\partial u}{\partial t}=-\bar{a}(t)u+\frac{1}{|\Omega|}\int_\Omega\frac{\partial H}{\partial z}(x,t,0)\mathrm{d}xv, & t\geq s,\\ \frac{\partial v}{\partial t}=-\bar{b}(t)v, & t\geq s,\end{cases}$$

respectively.

Let $\{\Phi_d(t)\}_{t\geq 0}$ be the semigroup generated by $\mathscr{L}$. We define a series of bounded linear operators $\mathcal{Q}:\mathcal{X}\to\mathcal{X}$, $\bar{\mathcal{Q}}:\mathcal{X}\to\mathcal{X}$, $\mathscr{Q}_x:\mathscr{P}\to\mathscr{P}$, $\bar{\mathscr{Q}}:\mathscr{P}\to\mathscr{P}$, and $\hat{\mathcal{Q}}:X\to X$ by

$$\begin{aligned}\mathcal{Q}[\mathbf{u}](s)&:=\int_0^{+\infty}F(x,s)\Phi(s,s-t)\mathbf{u}(\cdot,s-t)\mathrm{d}t, \quad s\in\mathbb{R},\mathbf{u}\in\mathcal{X},\\ \bar{\mathcal{Q}}[\mathbf{u}](s)&:=\int_0^{+\infty}\bar{F}(s)\bar{\Phi}(s,s-t)\mathbf{u}(\cdot,s-t)\mathrm{d}t, \quad s\in\mathbb{R},\mathbf{u}\in\mathcal{X},\\ \mathscr{Q}_x[\mathbf{u}](s)&:=\int_0^{+\infty}F(x,s)\Gamma_x(s,s-t)\mathbf{u}(s-t)\mathrm{d}t, \quad s\in\mathbb{R},\mathbf{u}\in\mathscr{P},\\ \bar{\mathscr{Q}}[\mathbf{u}](s)&:=\int_0^{+\infty}\bar{F}(s)\bar{\Gamma}(s,s-t)\mathbf{u}(s-t)\mathrm{d}t, \quad s\in\mathbb{R},\mathbf{u}\in\mathscr{P},\\ \hat{\mathcal{Q}}[\mathbf{u}](x)&:=\int_0^{+\infty}\hat{F}(x)\Phi_d(t)\mathbf{u}\mathrm{d}t, \quad \mathbf{u}\in X.\end{aligned}$$

Let us define $R_0:=r(\mathcal{Q})$, $\bar{R}_0:=r(\bar{\mathcal{Q}})$, $\mathcal{R}_0(x):=r(\mathscr{Q}_x)$ for any $x\in\bar{\Omega}$, $\bar{\mathcal{R}}_0:=r(\bar{\mathscr{Q}})$, and $\hat{\mathcal{R}}_0:=r(\hat{\mathcal{Q}})$. In the following, we always suppose that $s(\mathscr{B}_\sigma+\frac{1}{\mu}\mathscr{F})>s(\mathscr{C}_\sigma+\frac{1}{\mu}\mathscr{F})$ for any $\mu>0$. Then $s(\mathscr{B}_\sigma+\frac{1}{\mu}\mathscr{F})$ is the principal eigenvalue of $\mathscr{B}_\sigma+\frac{1}{\mu}\mathscr{F}$.

THEOREM 5.4. *The following statements hold:*

(i) $\mathscr{R}_0\to\max_{x\in\bar{\Omega}}\mathcal{R}_0(x)$ *as* $\max\{d_1,d_2\}\to 0$.

(ii) $\mathscr{R}_0\to\bar{\mathcal{R}}_0$ *as* $\min\{d_1,d_2\}\to+\infty$.

(iii) *If* $\frac{\partial H}{\partial z}(x,t,0)$, $\frac{\partial G}{\partial z}(x,t,0)\in C^{0,1}(\bar{\Omega}\times\mathbb{R})$ *and* $s(\mathcal{P}_\mu(t))$ *is the principal eigenvalue of* $\mathcal{P}_\mu(t)$ *for* $t\in[0,1]$, *then* $\mathscr{R}_0\to\mathcal{R}_0^*$ *as* $\omega\to 0$, *where* $\mathcal{R}_0^*$ *satisfies* $\int_0^1 s\left(\mathcal{P}_{\mathcal{R}_0^*}(t)\right)\mathrm{d}t=0$ *and* $\mathcal{P}_\mu(t)$ *is defined by*

$$\mathcal{P}_\mu(t)[\mathbf{u}](x):=d\mathcal{K}[\mathbf{u}](x)-d\mathcal{J}[\mathbf{u}](x)+\left(C(x,t)-F(x,t)+\frac{1}{\mu}F(x,t)\right)\mathbf{u}(x).$$

(iv) *If* $\frac{\partial H}{\partial z}(x,t,0)=\frac{\partial G}{\partial z}(x,t,0)$ *for all* $(x,t)\in\bar{\Omega}\times\mathbb{R}$, *then* $\mathscr{R}_0\to\hat{\mathcal{R}}_0$ *as* $\omega\to+\infty$.

(v) *If J_i is compactly supported and symmetric with respect to each component, $\frac{\partial H}{\partial z}(x,t,0)$, $\frac{\partial G}{\partial z}(x,t,0) \in C^{4,0}(\bar{\Omega}\times[0,1])$, and $m_i \in [0,2)$ for $i=1,2$, then $\mathscr{R}_0^\sigma \to \max_{x\in\bar{\Omega}} \mathcal{R}_0(x)$ as $\max\{\sigma_1,\sigma_2\}\to 0$.*

(vi) *If J_i is compactly supported and $m_i \in [0,+\infty)$ for $i=1,2$, then $\mathscr{R}_0^\sigma \to \max_{x\in\bar{\Omega}} \mathcal{R}_0(x)$ as $\min\{\sigma_1,\sigma_2\}\to+\infty$.*

Proof. Since $s(\mathscr{B}+\frac{1}{\mu}\mathscr{F}) > s(\mathscr{C}+\frac{1}{\mu}\mathscr{F})$ for any $\mu>0$, $s(\mathscr{B}+\frac{1}{\mu}\mathscr{F})$ is the principal eigenvalue of $\mathscr{B}+\frac{1}{\mu}\mathscr{F}$ due to Theorem 2.13. It follows from Theorem 4.1 that $s(\mathscr{B}+\frac{1}{\mu}\mathscr{F}) \to s(\bar{\mathscr{D}}+\frac{1}{\mu}\bar{\mathscr{F}})$ as $\min\{d_1,d_2\}\to+\infty$ and $s(\mathscr{B}+\frac{1}{\mu}\mathscr{F}) \to \max_{x\in\bar{\Omega}}\lambda^*(x)$ as $\max\{d_1,d_2\}\to 0$, where $\lambda^*(x)$ is the principal eigenvalue of the following eigenvalue problem:

$$
\begin{cases} -\omega\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + \left(C(x,t) - F(x,t) + \frac{1}{\mu}F(x,t)\right)\phi(t) = \lambda\phi(t), \quad \phi(t)\in\mathbb{R}^2, \\ \phi(t+1) = \phi(t). \end{cases} \tag{5.6}
$$

We derive from Theorem 3.6 that $s(\mathscr{B}+\frac{1}{\mu}\mathscr{F}) \to s(\mathscr{L}+\frac{1}{\mu}\hat{\mathscr{F}})$ as $\omega\to+\infty$ for any $\mu>0$. We conclude from Proposition 2.9 that $s(\mathscr{D}+\frac{1}{\mu}\mathscr{F}) = \max_{x\in\bar{\Omega}}\lambda^*(x)$ for any $\mu>0$. Note that $\mathscr{B}+\frac{1}{\mu}\mathscr{F}$ is resolvent positive for any $\mu>0$. It follows from [46, Theorem 2.6] that

$$
\lim_{\min\{d_1,d_2\}\to+\infty} \mathscr{R}_0 = \lim_{\min\{d_1,d_2\}\to+\infty} r(-\mathscr{F}\mathscr{B}^{-1}) = r(-\bar{\mathscr{F}}\bar{\mathscr{D}}^{-1}),
$$

$$
\lim_{\max\{d_1,d_2\}\to 0} \mathscr{R}_0 = \lim_{\max\{d_1,d_2\}\to 0} r(-\mathscr{F}\mathscr{B}^{-1}) = r(-\mathscr{F}\mathscr{D}^{-1}),
$$

and

$$
\lim_{\omega\to+\infty} \mathscr{R}_0 = \lim_{\omega\to+\infty} r(-\mathscr{F}\mathscr{B}^{-1}) = r(-\hat{\mathscr{F}}\mathscr{L}^{-1}),
$$

where we use $\mathscr{R}_0 = r(-\mathscr{F}\mathscr{B}^{-1})$ derived in the proof of Theorem 5.2. By the same arguments as the proof of Theorem 5.2, we have $\bar{R}_0 = r(-\bar{\mathscr{F}}\bar{\mathscr{D}}^{-1})$, $R_0 = r(-\mathscr{F}\mathscr{D}^{-1})$, and $\hat{\mathcal{R}}_0 = r(-\hat{\mathscr{F}}\mathscr{L}^{-1})$. We derive from [46, Lemma 4.2] that $\bar{R}_0 = \bar{\mathcal{R}}_0$ and $R_0 = \max_{x\in\bar{\Omega}}\mathcal{R}_0(x)$. By Theorem 3.6(i), $s(\mathscr{B}+\frac{1}{\mu}\mathscr{F}) \to \int_0^1 \mathcal{P}_\mu(t)\mathrm{d}t$ as $\omega\to 0$. Together with Theorem 5.2, we can derive (iii). (v) and (vi) can be proved by the same arguments as above. The proof is completed. ∎

Now we focus on positive periodic solutions of (5.1).

THEOREM 5.5. *Suppose that $\mathscr{R}_0>1$. Then* (5.1) *admits a unique bounded positive periodic solution $\mathbf{u}^*$. Moreover, for any $\mathbf{u}_0 \in X^+\setminus\{\mathbf{0}\}$ with $\mathbf{u}_0 \le \mathbf{M} = (M,M)$, the solution $\mathbf{u}(\cdot,t;\mathbf{u}_0)$ of* (5.1) *with initial data $\mathbf{u}_0$ satisfies*

$$
\|\mathbf{u}(\cdot,t;\mathbf{u}_0) - \mathbf{u}^*(\cdot,t)\|_X \to 0 \quad \text{as} \quad t\to+\infty.
$$

Proof. Since $\mathscr{R}_0>1$, we derive from Theorem 5.2(i) that $s(\mathscr{A})>0$. Since $s(\mathscr{A})$ is the principal eigenvalue of $\mathscr{A}$, there exists $\varphi\in\mathcal{X}^{++}$ which satisfies

$$
\begin{cases} -\omega\partial_t\varphi_1 + d_1\int_\Omega J_1(x-y)(\varphi_1(y,t)-\varphi_1(x,t))\mathrm{d}y - a(x,t)\varphi_1 + \frac{\partial H}{\partial z}(x,t,0)\varphi_2 \\ \quad = s(\mathscr{A})\varphi_1, \\ -\omega\partial_t\varphi_2 + d_2\int_\Omega J_2(x-y)(\varphi_2(y,t)-\varphi_2(x,t))\mathrm{d}y + \frac{\partial G}{\partial z}(x,t,0)\varphi_1 - b(x,t)\varphi_2 \\ \quad = s(\mathscr{A})\varphi_2. \end{cases} \tag{5.7}
$$

Set $\underline{\mathbf{u}} = (\underline{u}_1, \underline{u}_2) = (\delta\varphi_1, \delta\varphi_2)$ for some positive constant δ. Then we have

$$\begin{aligned}\omega\partial_t\underline{u}_1 - d_1\int_\Omega J_1(x-y)(\underline{u}_1(y,t) - \underline{u}_1(x,t))\mathrm{d}y + a(x,t)\underline{u}_1(x,t) - H(x,t,\underline{u}_2(x,t))\\ = \frac{\partial H}{\partial z}(x,t,0)\underline{u}_2(x,t) - s(\mathscr{A})\underline{u}_1(x,t) - H(x,t,\underline{u}_2(x,t))\\ \le -\delta s(\mathscr{A})\varphi_1(x,t) + \left|\frac{\partial^2 H}{\partial z^2}(x,t,\hat{u}_2)\right|(\delta\varphi_2(x,t))^2\\ \le 0,\end{aligned}$$

provided $\delta > 0$ small enough and here $\hat{u}_2$ is between 0 and $\underline{u}_2(x,t)$. Similarly, if $\delta > 0$ small enough, we get

$$\omega\partial_t\underline{u}_2 - d_2\int_\Omega J_2(x-y)(\underline{u}_2(y,t) - \underline{u}_2(x,t))\mathrm{d}y + b(x,t)\underline{u}_2(x,t) - G(x,t,\underline{u}_1(x,t)) \le 0.$$

As a result, $\underline{\mathbf{u}}$ is a lower solution of (5.1). By assumption (F), $\mathbf{M} = (M,M)$ is an upper solution of (5.1). The desired conclusion follows from the standard argument and one can see the proof of [4, Theorem 1.1]. ◻

For each $x \in \bar{\Omega}$, let $\rho(x)$ be the principal eigenvalue of

$$\begin{cases}\omega\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} = C(x,t)\mathbf{u}(t) - \lambda\mathbf{u}(t), & t\in\mathbb{R},\\ \mathbf{u}(t+1) = \mathbf{u}(t), & t\in\mathbb{R}.\end{cases}$$

Consider

$$\begin{cases}\omega\frac{\mathrm{d}u}{\mathrm{d}t} = -a(x,t)u(t) + H(x,t,v(t)),\\ \omega\frac{\mathrm{d}v}{\mathrm{d}t} = -b(x,t)v(t) + G(x,t,u(t)).\end{cases} \tag{5.8}$$

By [48, Theorem 3.1.2], we have the following lemma.

LEMMA 5.6. *Assume that* $\min_{x\in\bar{\Omega}}\rho(x) > 0$. *Then system* (5.8) *admits a unique positive* 1*-periodic solution, denoted by* $\mathbf{w}(x,t)$. *Moreover,* $\mathbf{w}(x,t)$ *is continuous on* $\bar{\Omega}\times\mathbb{R}$.

LEMMA 5.7. *If* $\bar{\mathcal{R}}_0 > 1$, *then there exists* $d_0 > 0$ *such that* (5.1) *admits a unique positive* 1*-periodic solution* (u,v) *for all* $\min\{d_1,d_2\} > d_0$. *In addition,*

$$\lim_{\min\{d_1,d_2\}\to+\infty}(\bar{u}(t),\bar{v}(t)) = (w_1^*(t), w_2^*(t)) \quad \textit{uniformly on} \quad \mathbb{R},$$

where $(w_1^*(t), w_2^*(t))$ *is the unique positive* 1*-periodic solution of*

$$\begin{cases}\omega\frac{du}{dt} = -\frac{1}{|\Omega|}\int_\Omega a(x,t)\mathrm{d}x u(t) + \frac{1}{|\Omega|}\int_\Omega H(x,t,v(t))\mathrm{d}x,\\ \omega\frac{dv}{dt} = -\frac{1}{|\Omega|}\int_\Omega b(x,t)\mathrm{d}x v(t) + \frac{1}{|\Omega|}\int_\Omega G(x,t,u(t))\mathrm{d}x.\end{cases} \tag{5.9}$$

Proof. Since $\bar{\mathcal{R}}_0 > 1$, we derive from Theorem 5.4(ii) and Theorem 5.5 that there exists $d_0 > 0$ such that (5.1) admits a unique positive 1-periodic solution (u,v) for all $\min\{d_1,d_2\} > d_0$. It follows from [48, Theorem 3.1.2] that (5.9) admits a unique positive 1-periodic solution $(w_1^*(t), w_2^*(t))$. Integrating (5.1) over Ω yields

$$\begin{cases}\omega\frac{\mathrm{d}\bar{u}(t)}{\mathrm{d}t} = -\frac{1}{|\Omega|}\int_\Omega a(x,t)u(x,t)\mathrm{d}x + \frac{1}{|\Omega|}\int_\Omega H(x,t,v(x,t))\mathrm{d}x, & t\in\mathbb{R},\\ \omega\frac{\mathrm{d}\bar{v}(t)}{\mathrm{d}t} = -\frac{1}{|\Omega|}\int_\Omega b(x,t)v(x,t)\mathrm{d}x + \frac{1}{|\Omega|}\int_\Omega G(x,t,u(x,t))\mathrm{d}x, & t\in\mathbb{R}.\end{cases} \tag{5.10}$$

In view of assumption (F), we have

$$u(x,t), v(x,t) \leq M \quad \text{for all} \quad (x,t) \in \bar{\Omega} \times \mathbb{R}, \tag{5.11}$$

and there exists $C_0 > 0$ independent of d_1 and d_2 such that

$$H(x,t,z), G(x,t,z) \leq C_0 \quad \text{for all} \quad (x,t,z) \in \bar{\Omega} \times \mathbb{R} \times [0,M]. \tag{5.12}$$

Thus,

(5.13)
$$\left|\frac{\mathrm{d}\bar{u}(t)}{\mathrm{d}t}\right|, \left|\frac{\mathrm{d}\bar{v}(t)}{\mathrm{d}t}\right| \leq \frac{C_0 + \left(\max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} a(x,t) + \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} b(x,t)\right) M}{\omega} \quad \text{for all} \quad t \in \mathbb{R}.$$

Integrating (5.10) from 0 to t gives

(5.14)
$$\begin{cases} \omega\bar{u}(t) - \omega\bar{u}(0) = -\int_0^t \bar{a}(s)\bar{u}(s)\mathrm{d}s + \int_0^t \frac{1}{|\Omega|}\int_\Omega H(x,s,\bar{v}(s))\mathrm{d}x\mathrm{d}s + h_1(t), & t \in \mathbb{R}, \\ \omega\bar{v}(t) - \omega\bar{v}(0) = -\int_0^t \bar{b}(s)\bar{v}(s)\mathrm{d}s + \int_0^t \frac{1}{|\Omega|}\int_\Omega G(x,s,\bar{u}(s))\mathrm{d}x\mathrm{d}s + h_2(t), & t \in \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} h_1(t) = &-\frac{1}{|\Omega|}\int_0^t \int_\Omega a(x,s)\left[u(x,s) - \bar{u}(s)\right]\mathrm{d}x\mathrm{d}s \\ &+ \frac{1}{|\Omega|}\int_0^t \int_\Omega \left[H(x,s,v(x,s)) - H(x,s,\bar{v}(s))\right]\mathrm{d}x\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} h_2(t) = &-\frac{1}{|\Omega|}\int_0^t \int_\Omega b(x,s)\left[v(x,s) - \bar{v}(s)\right]\mathrm{d}x\mathrm{d}s \\ &+ \frac{1}{|\Omega|}\int_0^t \int_\Omega \left[G(x,s,u(x,s)) - G(x,s,\bar{u}(s))\right]\mathrm{d}x\mathrm{d}s. \end{aligned}$$

Next we aim to prove that $h_i(t) \to 0$ uniformly on $[0,1]$ as $\min\{d_1,d_2\} \to +\infty$ for $i = 1,2$. Multiplying the first equation of (5.1) by u and the second equation by v and then integrating over $\Omega \times (0,1)$ give

$$\begin{aligned} &\frac{d_1}{2}\int_0^1 \int_\Omega \int_\Omega J_1(x-y)(u(y,t) - u(x,t))^2 \mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &= -d_1 \int_0^1 \int_\Omega \int_\Omega J_1(x-y)(u(y,t) - u(x,t))u(x,t)\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &= -\int_0^1 \int_\Omega a(x,t)u^2(x,t)\mathrm{d}x\mathrm{d}t + \int_0^1 \int_\Omega H(x,t,v(x,t))u(x,t)\mathrm{d}x\mathrm{d}t \\ &\leq |\Omega| M \max_{(x,t,z)\in\bar{\Omega}\times\mathbb{R}\times[0,M]} H(x,t,z) := C_1 \end{aligned}$$

and

$$\begin{aligned}&\frac{d_2}{2}\int_0^1\int_\Omega\int_\Omega J_2(x-y)(v(y,t)-v(x,t))^2\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&=-d_2\int_0^1\int_\Omega\int_\Omega J_2(x-y)(v(y,t)-v(x,t))v(x,t)\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&=-\int_0^1\int_\Omega b(x,t)v^2(x,t)\mathrm{d}x\mathrm{d}t+\int_0^1\int_\Omega G(x,t,u(x,t))v(x,t)\mathrm{d}x\mathrm{d}t\\&\le|\Omega|M\max_{(x,t,z)\in\bar\Omega\times\mathbb{R}\times[0,M]}G(x,t,z):=C_2.\end{aligned}$$

Set $\tilde u(x,t)=u(x,t)-\bar u(t)$ and $\tilde v(x,t)=v(x,t)-\bar v(t)$. Then we have $\int_\Omega\tilde u(x,t)\mathrm{d}x=\int_\Omega\tilde v(x,t)\mathrm{d}x=0$ for all $t\in\mathbb{R}$. By [33, Formula (5.6), p. 1688], there exists $C_3>0$ such that

$$\int_\Omega\tilde u^2(x,t)\mathrm{d}x\le\frac{1}{2C_3}\int_\Omega\int_\Omega J_1(x-y)(\tilde u(y,t)-\tilde u(x,t))^2\mathrm{d}y\mathrm{d}x\quad\text{for all}\quad d_1\gg1$$

and

$$\int_\Omega\tilde v^2(x,t)\mathrm{d}x\le\frac{1}{2C_3}\int_\Omega\int_\Omega J_2(x-y)(\tilde v(y,t)-\tilde v(x,t))^2\mathrm{d}y\mathrm{d}x\quad\text{for all}\quad d_2\gg1.$$

Note that

$$\int_\Omega\int_\Omega J_1(x-y)(u(y,t)-u(x,t))^2\mathrm{d}y\mathrm{d}x=\int_\Omega\int_\Omega J_1(x-y)(\tilde u(y,t)-\tilde u(x,t))^2\mathrm{d}y\mathrm{d}x$$

and

$$\int_\Omega\int_\Omega J_2(x-y)(v(y,t)-v(x,t))^2\mathrm{d}y\mathrm{d}x=\int_\Omega\int_\Omega J_2(x-y)(\tilde v(y,t)-\tilde v(x,t))^2\mathrm{d}y\mathrm{d}x.$$

Then we have

$$\int_0^1\int_\Omega\tilde u^2(x,t)\mathrm{d}x\mathrm{d}t\le\frac{C_1}{d_1C_3}\quad\text{for all}\quad d_1\gg1$$

and

$$\int_0^1\int_\Omega\tilde v^2(x,t)\mathrm{d}x\mathrm{d}t\le\frac{C_2}{d_2C_3}\quad\text{for all}\quad d_2\gg1.$$

By the Hölder inequality, there exists $C_4>0$ such that

$$\int_0^1\int_\Omega|\tilde u(x,t)|\,\mathrm{d}x\mathrm{d}t\le\frac{C_4}{\sqrt{d_1}}\quad\text{for all}\quad d_1\gg1$$

and

$$\int_0^1\int_\Omega|\tilde v(x,t)|\,\mathrm{d}x\mathrm{d}t\le\frac{C_4}{\sqrt{d_2}}\quad\text{for all}\quad d_2\gg1.$$

Since $H,G\in C^{0,0,2}(\bar\Omega\times\mathbb{R}\times[0,+\infty))$, there exists $L>0$ independent of x, t, d_1, and d_2 such that

$$|H(x,t,v(x,t))-H(x,t,\bar v(t))|\le L\,|v(x,t)-\bar v(t)|\quad\text{for all}\quad(x,t)\in\bar\Omega\times\mathbb{R}\tag{5.15}$$

and

$$|G(x,t,u(x,t)) - G(x,t,\bar{u}(t))| \le L\,|u(x,t) - \bar{u}(t)| \quad \text{for all} \quad (x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{5.16}$$

A simple calculation gives for all $t \in [0,1]$ that

$$\begin{aligned}|h_1(t)| \le & \frac{1}{|\Omega|} \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} a(x,t) \int_0^t \int_\Omega |u(x,s) - \bar{u}(s)|\,\mathrm{d}x\mathrm{d}s \\ & + \frac{1}{|\Omega|} L \int_0^t \int_\Omega |v(x,s) - \bar{v}(s)|\,\mathrm{d}x\mathrm{d}s \\ \le & \frac{1}{|\Omega|} \frac{C_4}{\sqrt{d_1}} \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} a(x,t) + \frac{1}{|\Omega|} L \frac{C_4}{\sqrt{d_2}}\end{aligned}$$

and

$$\begin{aligned}|h_2(t)| \le & \frac{1}{|\Omega|} \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} b(x,t) \int_0^t \int_\Omega |v(x,s) - \bar{v}(s)|\,\mathrm{d}x\mathrm{d}s \\ & + \frac{1}{|\Omega|} L \int_0^t \int_\Omega |u(x,s) - \bar{u}(s)|\,\mathrm{d}x\mathrm{d}s \\ \le & \frac{1}{|\Omega|} \frac{C_4}{\sqrt{d_2}} \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} b(x,t) + \frac{1}{|\Omega|} L \frac{C_4}{\sqrt{d_1}},\end{aligned}$$

which imply that $h_i(t) \to 0$ uniformly on $[0,1]$ as $\min\{d_1, d_2\} \to +\infty$ for $i = 1, 2$.

Suppose to the contrary that there exists a sequence $\{d_l = (d_{l_1}, d_{l_2})^T\}$ with $\min\{d_{l_1}, d_{l_2}\} \to +\infty$ as $l \to +\infty$ such that

$$\|(\bar{u}_l, \bar{v}_l) - (w_1^*, w_2^*)\|_{C([0,1],\mathbb{R}^2)} \ge \epsilon_0 \quad \text{for some} \quad \epsilon_0 > 0,$$

where (u_l, v_l) is the corresponding positive 1-periodic solution of (5.1) and

$$(\bar{u}_l(t), \bar{v}_l(t)) = \left(\frac{1}{|\Omega|} \int_\Omega u_l(x,t)\mathrm{d}x, \frac{1}{|\Omega|} \int_\Omega v_l(x,t)\mathrm{d}x \right).$$

By virtue of (5.13), we derive from the Ascoli–Arzelà theorem that there is a subsequence $\{d_{l_k}\}$ such that

$$(\bar{u}_{l_k}, \bar{v}_{l_k}) \to (U, V) \quad \text{in} \quad C([0,1], \mathbb{R}^2) \quad \text{as} \quad \min\{d_{l_{k_1}}, d_{l_{k_2}}\} \to +\infty$$

and $(U(0), V(0)) = (U(1), V(1))$. By (5.14), (U, V) satisfies

$$\begin{cases} \omega U(t) - \omega U(0) = -\int_0^t \bar{a}(s) U(s)\mathrm{d}s + \int_0^t \frac{1}{|\Omega|} \int_\Omega H(x,s,V(s))\mathrm{d}x\mathrm{d}s, & t \in \mathbb{R}, \\ \omega V(t) - \omega V(0) = -\int_0^t \bar{b}(s) V(s)\mathrm{d}s + \int_0^t \frac{1}{|\Omega|} \int_\Omega G(x,s,U(s))\mathrm{d}x\mathrm{d}s, & t \in \mathbb{R}, \end{cases}$$

which imply that $(U, V) = (w_1^*, w_2^*)$. This is a contradiction. The proof is completed. ∎

THEOREM 5.8. *The following statements hold:*

(i) *If* $\max_{x\in\bar{\Omega}} \mathcal{R}_0(x) > 1$, *then there exists* $d_0^* > 0$ *such that* (5.1) *admits a unique positive* 1*-periodic solution* $\mathbf{u}$ *for all* $\max\{d_1, d_2\} < d_0^*$. *Furthermore, if* $\min_{x\in\bar{\Omega}} \rho(x) > 0$, *then*

$$\lim_{\max\{d_1,d_2\}\to 0} \mathbf{u}(x,t) = \mathbf{w}(x,t) \quad \text{uniformly in} \quad (x,t) \in \bar{\Omega} \times \mathbb{R},$$

where $\mathbf{w}$ *is the unique positive* 1*-periodic solution of* (5.8).

(ii) *If $\bar{\mathcal{R}}_0 > 1$, then there exists $d_0 > 0$ such that* (5.1) *admits a unique positive* 1*-periodic solution* $\mathbf{u}$ *for all* $\min\{d_1, d_2\} > d_0$. *In addition,*

$$\lim_{\min\{d_1,d_2\}\to+\infty} \mathbf{u}(x,t) = \mathbf{w}^*(t) \quad \textit{uniformly in} \quad (x,t) \in \bar{\Omega} \times \mathbb{R},$$

where $\mathbf{w}^*(t)$ *is the unique positive* 1*-periodic solution of* (5.9).

Proof. (i) Since $\max_{x\in\bar{\Omega}} \mathcal{R}_0(x) > 1$, we derive from Theorems 5.4(i) and 5.5 that there exists $d_0^* > 0$ such that (5.1) admits a unique positive 1-periodic solution $\mathbf{u}$ for all $\max\{d_1, d_2\} < d_0^*$. Lemma 5.6 gives the existence of the unique positive 1-periodic solution $\mathbf{w}$ of (5.8). Similar to [32, Theorem C], we can prove

$$\lim_{\max\{d_1,d_2\}\to 0} \mathbf{u}(x,t) = \mathbf{w}(x,t) \quad \text{uniformly in} \quad (x,t) \in \bar{\Omega} \times \mathbb{R}.$$

(ii) Our argument is motivated by [46]. Set $\tilde{u}(x,t) = u(x,t) - \bar{u}(t)$ and $\tilde{v}(x,t) = v(x,t) - \bar{v}(t)$. In view of Lemma 5.7, it suffices to prove $(\tilde{u}(x,t), \tilde{v}(x,t)) \to (0,0)$ uniformly on $\bar{\Omega} \times \mathbb{R}$ as $\min\{d_1, d_2\} \to +\infty$. By (5.1) and (5.10), $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ satisfy

$$\begin{aligned}\omega\partial_t\tilde{u}(x,t) =& d_1 \int_\Omega J_1(x-y)(\tilde{u}(y,t) - \tilde{u}(x,t))\mathrm{d}y - a(x,t)u(x,t) + H(x,t,v(x,t)) \\ &+ \frac{1}{|\Omega|}\int_\Omega a(x,t)u(x,t)\mathrm{d}x - \frac{1}{|\Omega|}\int_\Omega H(x,t,v(x,t))\mathrm{d}x \\ =& d_1 \int_\Omega J_1(x-y)(\tilde{u}(y,t) - \tilde{u}(x,t))\mathrm{d}y + g_1(x,t) + g_2(x,t) + g_3(x,t)\end{aligned}$$

and

$$\begin{aligned}\omega\partial_t\tilde{v}(x,t) =& d_2 \int_\Omega J_2(x-y)(\tilde{v}(y,t) - \tilde{v}(x,t))\mathrm{d}y - b(x,t)v(x,t) + G(x,t,u(x,t)) \\ &+ \frac{1}{|\Omega|}\int_\Omega b(x,t)v(x,t)\mathrm{d}x - \frac{1}{|\Omega|}\int_\Omega G(x,t,u(x,t))\mathrm{d}x \\ =& d_2 \int_\Omega J_2(x-y)(\tilde{v}(y,t) - \tilde{v}(x,t))\mathrm{d}y + f_1(x,t) + f_2(x,t) + f_3(x,t),\end{aligned}$$

where

$$\begin{aligned}g_1(x,t) &= -a(x,t)u(x,t) + a(x,t)\bar{u}(t) + H(x,t,v(x,t)) - H(x,t,\bar{v}(t)), \\ g_2(x,t) &= -a(x,t)\bar{u}(t) + \frac{1}{|\Omega|}\int_\Omega a(x,t)\mathrm{d}x\bar{u}(t) + H(x,t,\bar{v}(t)) - \frac{1}{|\Omega|}\int_\Omega H(x,t,\bar{v}(t))\mathrm{d}x, \\ g_3(x,t) &= \frac{1}{|\Omega|}\int_\Omega a(x,t)[u(x,t) - \bar{u}(t)]\mathrm{d}x + \frac{1}{|\Omega|}\int_\Omega [H(x,t,\bar{v}(t)) - H(x,t,v(x,t))]\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}f_1(x,t) &= -b(x,t)v(x,t) + b(x,t)\bar{v}(t) + G(x,t,u(x,t)) - G(x,t,\bar{u}(t)), \\ f_2(x,t) &= -b(x,t)\bar{v}(t) + \frac{1}{|\Omega|}\int_\Omega b(x,t)\mathrm{d}x\bar{v}(t) + G(x,t,\bar{u}(t)) - \frac{1}{|\Omega|}\int_\Omega G(x,t,\bar{u}(t))\mathrm{d}x, \\ f_3(x,t) &= \frac{1}{|\Omega|}\int_\Omega b(x,t)[v(x,t) - \bar{v}(t)]\mathrm{d}x + \frac{1}{|\Omega|}\int_\Omega [G(x,t,\bar{u}(t)) - G(x,t,u(x,t))]\mathrm{d}x.\end{aligned}$$

By (5.15) and (5.16), we have

$$|g_1(x,t)| \le |\tilde{u}(x,t)| \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} a(x,t) + L|\tilde{v}(x,t)|,$$
$$|f_1(x,t)| \le |\tilde{v}(x,t)| \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} b(x,t) + L|\tilde{u}(x,t)|,$$
$$|g_3(x,t)| \le |\Omega|^{-\frac{1}{2}} \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} a(x,t)\|\tilde{u}(\cdot,t)\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}} L\|\tilde{v}(\cdot,t)\|_{L^2(\Omega)},$$
$$|f_3(x,t)| \le |\Omega|^{-\frac{1}{2}} \max_{(x,t)\in\bar{\Omega}\times\mathbb{R}} b(x,t)\|\tilde{v}(\cdot,t)\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}} L\|\tilde{u}(\cdot,t)\|_{L^2(\Omega)}.$$

In addition, there exists some $C>0$ such that

$$|g_2(x,t)|, |f_2(x,t)| \le C \quad \text{for all } (x,t)\in\bar{\Omega}\times\mathbb{R}.$$

Set

$$\mathscr{J}_i[\phi](x) = \int_\Omega J_i(x-y)(\phi(y)-\phi(x))\mathrm{d}y, \quad \phi\in C(\bar{\Omega},\mathbb{R}), i=1,2.$$

Let $\{\mathscr{U}_i(t)\}_{t\ge 0}$ be the semigroup generated by $\frac{d_i}{\omega}\mathscr{J}_i$ for $i=1,2$. It follows from the variation of constants formula that

$$\tilde{u}(\cdot,t) = \mathscr{U}_1(t)\tilde{u}(\cdot,0) + \frac{1}{\omega}\int_0^t \mathscr{U}_1(t-s)(g_1+g_2+g_3)(\cdot,s)\mathrm{d}s$$

and

$$\tilde{v}(\cdot,t) = \mathscr{U}_2(t)\tilde{v}(\cdot,0) + \frac{1}{\omega}\int_0^t \mathscr{U}_2(t-s)(f_1+f_2+f_3)(\cdot,s)\mathrm{d}s.$$

Define

$$\beta_i := \inf_{u\in L^2(\Omega), \int_\Omega u\,dx=0, u\not\equiv 0} \frac{\frac{1}{2}\int_\Omega\int_\Omega J_i(x-y)(u(y)-u(x))^2\mathrm{d}y\mathrm{d}x}{\int_\Omega u^2(x)\mathrm{d}x} \quad \text{for} \quad i=1,2.$$

Note that for all $t\in\mathbb{R}$,

$$\int_\Omega \tilde{u}(x,t)\mathrm{d}x = 0, \quad \int_\Omega [g_1(x,t)+g_3(x,t)]\mathrm{d}x = 0, \quad \int_\Omega g_2(x,t)\mathrm{d}x = 0,$$
$$\int_\Omega \tilde{v}(x,t)\mathrm{d}x = 0, \quad \int_\Omega [f_1(x,t)+f_3(x,t)]\mathrm{d}x = 0, \quad \int_\Omega f_2(x,t)\mathrm{d}x = 0.$$

It follows from [1, Theorem 3.6] that

$$\begin{aligned}\|\tilde{u}(\cdot,t)\|_{L^2(\Omega)} \le{}& e^{-\frac{d_1}{\omega}\beta_1 t}\|\tilde{u}(\cdot,0)\|_{L^2(\Omega)} + \frac{C}{\omega}\int_0^t e^{-\frac{d_1}{\omega}\beta_1(t-s)}\mathrm{d}s\\ &+ \frac{1}{\omega}\int_0^t e^{-\frac{d_1}{\omega}\beta_1(t-s)}\|(g_1+g_3)(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\|\tilde{v}(\cdot,t)\|_{L^2(\Omega)} \le{}& e^{-\frac{d_2}{\omega}\beta_2 t}\|\tilde{v}(\cdot,0)\|_{L^2(\Omega)} + \frac{C}{\omega}\int_0^t e^{-\frac{d_2}{\omega}\beta_2(t-s)}\mathrm{d}s\\ &+ \frac{1}{\omega}\int_0^t e^{-\frac{d_2}{\omega}\beta_2(t-s)}\|(f_1+f_3)(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s.\end{aligned}$$

Set $\beta=\min\{\beta_1,\beta_2\}$, $d^*=\min\{d_1,d_2\}$ and $m(t)=\max\{\|\tilde{u}(\cdot,t)\|_{L^2(\Omega)},\|\tilde{v}(\cdot,t)\|_{L^2(\Omega)}\}$. Then

$$m(t)\le e^{-\frac{d^*}{\omega}\beta t}m(0)+\frac{C}{\beta d^*}+\frac{\tilde{C}}{\omega}\int_0^t e^{-\frac{d^*}{\omega}\beta(t-s)}m(s)\mathrm{d}s,$$

where

$$\tilde{C}=\left(1+|\Omega|^{-\frac{1}{2}}\right)\max\left\{\max_{(x,t)\in\bar{\Omega}\times\mathbb{R}}a(x,t),L,\max_{(x,t)\in\bar{\Omega}\times\mathbb{R}}b(x,t)\right\}.$$

Choose $\nu\in(0,\beta)$ and define $\chi(t):=e^{\nu\frac{d^*}{\omega}t}m(t),\tilde{\chi}(t):=\sup\{\chi(s)\mid 0\le s\le t\}$. Then we get

$$\chi(t)\le e^{-\frac{d^*}{\omega}(\beta-\nu)t}m(0)+\frac{C}{\beta d^*}e^{\frac{d^*}{\omega}\nu t}+\frac{\tilde{C}}{\beta d^*}\tilde{\chi}(t),$$

and hence,

$$\tilde{\chi}(t)\le m(0)+\frac{C}{\beta d^*}e^{\frac{d^*}{\omega}\nu t}+\frac{\tilde{C}}{\beta d^*}\tilde{\chi}(t). \tag{5.17}$$

Since $d^*\to+\infty$, without loss of generality, we assume $\frac{\tilde{C}}{\beta d^*}<\frac{1}{2}$. We derive from (5.17) that

$$m(t)\le e^{-\nu\frac{d^*}{\omega}t}\tilde{\chi}(t)\le 2\left(m(0)e^{-\nu\frac{d^*}{\omega}t}+\frac{C}{\beta d^*}\right),$$

which implies that $\limsup_{t\to+\infty}m(t)\le 2\frac{C}{\beta d^*}$. Since $m(t)$ is periodic in $t\in\mathbb{R}$, we have $\sup_{t\in\mathbb{R}}m(t)\le 2\frac{C}{\beta d^*}$. As a result, $\sup_{t\in\mathbb{R}}m(t)\to 0$ as $d^*\to+\infty$, implying that $m(t)\to 0$ uniformly on $\mathbb{R}$ as $d^*\to+\infty$. Set $j_i(x)=\int_\Omega J_i(x-y)\mathrm{d}y$ for $i=1,2$. By [1, Lemma 3.5], $\beta_i\le\min_{x\in\bar{\Omega}}j_i(x)$. The variation of constants formula gives

$$\begin{aligned}\tilde{u}(\cdot,t)&=\tilde{u}(\cdot,0)e^{-\frac{d_1}{\omega}j_1(x)t}\\&\quad+\frac{1}{\omega}\int_0^t e^{-\frac{d_1}{\omega}j_1(x)(t-s)}\left[d_1\int_\Omega J_1(x-y)\tilde{u}(y,s)\mathrm{d}y+(g_1+g_2+g_3)(\cdot,s)\right]\mathrm{d}s.\end{aligned}$$

By the Hölder inequality, we have

$$\int_\Omega J_1(x-y)\tilde{u}(y,s)\mathrm{d}y\le C^*\|\tilde{u}(\cdot,s)\|_{L^2(\Omega)}\ \text{ for some constant }\ C^*>0.$$

As a result,

$$|\tilde{u}(\cdot,t)|\le|\tilde{u}(\cdot,0)|e^{-\frac{d_1}{\omega}\beta_1 t}+\frac{C^*}{\beta_1}\sup_{t\in\mathbb{R}}m(t)+\bar{C}\frac{1}{d_1\beta_1},$$

where constant $\bar{C}>0$ is the upper bound of g_1, g_2, and g_3. Hence, $\tilde{u}(x,t)\to 0$ uniformly on $\bar{\Omega}\times\mathbb{R}$ as $d^*\to+\infty$. Similarly, we can prove that $\tilde{v}(x,t)\to 0$ uniformly on $\bar{\Omega}\times\mathbb{R}$ as $d^*\to+\infty$. The proof is completed. ∎

Now we study the effects of dispersal range and replace J_i by J_{σ_i} and d_i by $\frac{d_i}{\sigma_i^{m_i}}$ for $i=1,2$ in (5.1). By the same proof of Theorem 5.8(i), we have the following result.

THEOREM 5.9. *The following statements hold:*

(i) *Suppose that the conditions of Theorem* 5.4(v) *hold. If* $\max_{x\in\bar{\Omega}}\mathcal{R}_0(x)>1$, *then there exists* $\sigma_0>0$ *such that* (5.1) *admits a unique positive* 1*-periodic solution* $\mathbf{u}$ *for all* $\max\{\sigma_1,\sigma_2\}<\sigma_0$. *In addition, if* $\min_{x\in\bar{\Omega}}\rho(x)>0$, *then*

$$\lim_{\max\{\sigma_1,\sigma_2\}\to 0}\mathbf{u}(x,t)=\mathbf{w}(x,t)\quad \textit{uniformly in}\quad (x,t)\in\bar{\Omega}\times\mathbb{R},$$

where $\mathbf{w}$ *is the unique positive* 1*-periodic solution of* (5.8).

(ii) *Suppose that the conditions of Theorem* 5.4(vi) *hold. If* $\max_{x\in\bar{\Omega}}\mathcal{R}_0(x)>1$, *then there exists* $\sigma_1>0$ *such that* (5.1) *admits a unique positive* 1*-periodic solution* $\mathbf{u}$ *for all* $\min\{\sigma_1,\sigma_2\}>\sigma_1$. *In addition, if* $\min_{x\in\bar{\Omega}}\rho(x)>0$, *then*

$$\lim_{\min\{\sigma_1,\sigma_2\}\to +\infty}\mathbf{u}(x,t)=\mathbf{w}(x,t)\quad \textit{uniformly in}\quad (x,t)\in\bar{\Omega}\times\mathbb{R},$$

where $\mathbf{w}$ *is the unique positive* 1*-periodic solution of* (5.8).

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