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Spherical Schubert varieties and pattern avoidance

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Abstract
A normal variety X is called H-*spherical* for the action of the complex reductive group H if it contains a dense orbit of some Borel subgroup of H. We resolve a conjecture of Hodges–Yong by showing that their *spherical permutations* are characterized by permutation pattern avoidance. Together with results of Gao–Hodges–Yong this implies that the sphericality of a Schubert variety X_w with respect to the largest possible Levi subgroup is characterized by this same pattern avoidance condition.

Keywords Spherical variety · Schubert variety · Permutation pattern avoidance

Mathematics Subject Classification 05E14 · 14M15 · 05A05

1 Introduction

1.1 Spherical varieties

Following [3, 11], a normal variety X is called H-*spherical* for the action of the complex reductive group H if it contains a dense orbit of some Borel subgroup of H. Important examples of spherical varieties include projective and affine toric varieties, complexifications of symmetric spaces, and flag varieties (see Perrin's survey [13]). Producing families of examples of and classifying spherical varieties is of significant interest [10]. In this paper we resolve a conjecture of Hodges–Yong [7], thereby classifying (maximally) spherical Schubert varieties by a permutation pattern avoidance condition.

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1.2 Schubert varieties and pattern avoidance

Let G be a complex reductive algebraic group and B be a Borel subgroup. The Bruhat decomposition decomposes G as

$$G = \coprod_{w \in W} BwB,$$

where W denotes the Weyl group of G. The closures

$$X_w = \overline{BwB/B}$$

of the images of these strata in the flag variety G/B are the *Schubert varieties*, of fundamental importance in algebraic geometry and representation theory.

In the case $G = GL_n(\mathbb{C})$, the Weyl group W is the symmetric group S_n. Beginning with the groundbreaking result of Lakshmibai–Sandhya [9] characterizing smooth Schubert varieties, it has been found that many important geometric and combinatorial properties (see, for example [8, 17]) of X_w are determined by *permutation pattern avoidance* conditions on w. Let $w = w_1 \cdots w_n \in S_n$ be a permutation written in one-line notation, and let $p \in S_k$ be another permutation. Then w is said to have an *occurrence* of the pattern p at positions $1 \leq i_1 < \cdots < i_k \leq n$ if $w_{i_1} \ldots w_{i_k}$ are in the same relative order as $p_1 \ldots p_k$. If w does not contain any occurrences of p, then w is said to *avoid* p.

Under the natural left action of G on G/B, the stabilizer of X_w is the parabolic subgroup $P_{J(w)} \subset G$ corresponding to the left descent set $J(w)$ of w. The parabolic subgroup $P_{J(w)}$ is not reductive, but contains the *Levi subgroup* $L_{J(w)}$ as a maximal reductive subgroup. Following [7], we say X_w is *maximally spherical* if it is $L_{J(w)}$-spherical for the induced action of $L_{J(w)}$. Since all Schubert varieties are known to be normal by the work of DeConcini–Lakshmibai [4] and Ramanan–Ramanathan [14], this is equivalent to the existence of a dense orbit inside X_w of a Borel subgroup of $L_{J(w)}$.

1.3 Hodges and Yong's conjecture

We consider the symmetric group S_n as a Coxeter group with simple generating set $I = \{s_1, \ldots, s_{n-1}\}$, where s_i is the adjacent transposition $(i\; i+1)$, and we write $J(w)$ for the left descent set of $w \in S_n$. See Sect. 2 for background and basic definitions.

Definition 1.1 (Hodges and Yong [7]) A permutation $w \in S_n$ is *spherical* if it has a reduced word $s_{i_1} \cdots s_{i_{\ell(w)}}$ such that:

(S.1) $|\{t \mid s_{i_t} = s_j\}| \leq 1$ for $s_j \in I \setminus J(w)$, and
(S.2) $|\{t \mid s_{i_t} \in C\}| \leq \ell(w_0(C)) + |C|$ for any connected component C of the induced subgraph of the Dynkin diagram on $J(w)$.

Remark Hodges and Yong consider a more general class of spherical elements in finite Coxeter groups. Definition 1.1 is the special case which is relevant to Conjecture 1.3 ($J(w)$-spherical elements in S_n).

Spherical permutations were defined because of Conjecture 1.2, which was recently proven [6] by Gao–Hodges–Yong.

Conjecture 1.2 (Conjectured by Hodges and Yong [7]; proof by Gao–Hodges–Yong [6]) The Schubert variety X_w is maximally spherical if and only if w is spherical.

This geometric property is linked to permutation pattern avoidance by Conjecture 1.3.

Conjecture 1.3 (Hodges and Yong [7]) A permutation w is spherical if and only if it avoids the twenty one patterns in P:

$$\begin{aligned} P = \{&24531, 25314, 25341, 34512, 34521, 35412, 35421, 42531, 45123,\\ &45213, 45231, 45312, 52314, 52341, 53124, 53142, 53412,\\ &53421, 54123, 54213, 54231\}. \end{aligned}$$

Our main result resolves Conjecture 1.3:

Theorem 1.4 *A permutation w is spherical if and only if it avoids the patterns in P.*

Combining this result with Gao–Hodges–Yong's proof of Conjecture 1.2, we thus obtain a characterization of maximally spherical Schubert varieties in terms of pattern avoidance.

Corollary 1.5 *The Schubert variety X_w is maximally spherical if and only if w avoids the patterns from P.*

The following result, an immediate consequence of Theorem 1.4, was conjectured in [7] and proven in [2] using probabilistic methods.

Corollary 1.6

$$\lim_{n\to\infty} |\{\textit{spherical permutations } w \in S_n\}|/n! = 0.$$

Proof The Stanley–Wilf Conjecture, now a theorem of Marcus and Tardos [12], says that the number of permutations in S_n avoiding any fixed set Q of patterns is bounded above by C^n for some constant C. Thus Theorem 1.4 implies that

$$|\{\text{spherical permutations } w \in S_n\}|$$

grows at most exponentially. □

1.4 Outline

Section 2 recalls some basic definitions and facts about Bruhat order as well as a result of Tenner [15] characterizing Boolean intervals in Bruhat order. In Sect. 3 we introduce the notion of *divisible pairs* of permutations and connect these to Boolean permutations and spherical permutations. Divisible pairs, along with a helpful decomposition of the set P of patterns, are applied in Sect. 4 to prove Theorem 1.4.

2 Background

2.1 Bruhat order

For $i = 1, \ldots, n$, let s_i denote the adjacent transposition $(i\ i+1)$ in the symmetric group S_n; the symmetric group is a Coxeter group with respect to the generating set $s_1, \ldots, s_{n-1}$ (see [1] for background on Coxeter groups). For $w \in S_n$, and expression

$$w = s_{i_1} \cdots s_{i_\ell}$$

of minimum length is a *reduced word* for w, and in this case $\ell = \ell(w)$ is the *length* of w.

The *(right) weak order* is the partial order $\leq_R$ on S_n with cover relations $w \lessdot_R ws_i$ whenever $\ell(ws_i) = \ell(w) + 1$. The *Bruhat order* is the partial order $\leq$ on S_n with cover relations $w \lessdot wt$ for t a 2-cycle such that $\ell(wt) = \ell(w) + 1$. Both posets have the identity permutation e as their unique minimal element.

For a permutation $w = w_1 \ldots w_n \in S_n$ and integers $1 \leq a \leq b \leq n$, we write $w[a, b]$ for the set $\{w_a, w_{a+1}, \ldots, w_b\}$. For two k-subsets A, B of $[n] := \{1, \ldots, n\}$ write $A \preceq B$ if $a_1 \leq b_1, \ldots, a_k \leq b_k$, where $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ with $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$. The following well-known property of Bruhat order will be useful:

Proposition 2.1 (Ehresmann [5]) *Let $v, w \in S_n$, then $v \leq w$ if and only if*

$$v[1, i] \preceq w[1, i]$$

for all $i = 1, \ldots, n$.

A generator s_i is a *(left) descent* of w if $\ell(s_i w) < \ell(w)$ (equivalently, if $w^{-1}(i+1) < w^{-1}(i)$). We write $J(w)$ for the set of descents of w. For any $J \subseteq \{s_1, \ldots, s_{n-1}\}$, we write $w_0(J)$ for the unique permutation of maximum length lying in the subgroup of S_n generated by J. Explicitly, the one-line notation for $w_0(J)$ is an increasing sequence of consecutive decreasing runs, where decreasing run consists of $i+d, i+d-1, \cdots, i$ whenever $s_i, s_{i+1}, \ldots, s_{i+d-1} \in J$ while $s_{i-1}, s_{i+d} \notin J$.

2.2 Boolean permutations

Theorem 2.2 (Tenner [15]) *The following are equivalent for a permutation $w \in S_n$:*

(1) *The interval $[e, w]$ in Bruhat order is isomorphic to a Boolean lattice,*
(2) *No simple generator s_i appears more than once in a reduced word for w,*
(3) *w avoids the patterns 321 and 3412.*

We will call a permutation satisfying the equivalent conditions of Theorem 2.2 a *Boolean* permutation. Theorem 2.3 suggests a connection between Boolean permutations and spherical varieties.

Theorem 2.3 (Karuppuchamy [8]) *The Schubert variety X_w is a toric variety if and only if w is a Boolean permutation.*

3 Divisible pairs of permutations

Definition 3.1 Given a pair (v, w) of permutations from S_n, we say that (v, w) is *divisible after position* i if

$$|v[1, i] \cap w[1, i]| \leq i - 2,$$

and *divisible at position* i if $v_i = w_i$ and

$$|v[1, i] \cap w[1, i]| \leq i - 1.$$

We say simply that (v, w) is *divisible* if there exists $1 \leq i \leq n$ such that (v, w) is divisible at or after position i.

The term "divisible" is meant to refer to the fact that if (v, w) is divisible after position i, then if the one-line notation for v is written on top of that for w, a vertical line drawn after position i will divide the occurrences (on the left side of the line) of at least two values in v from the occurrences (on the right side of the line) of these values in w.

Proposition 3.2 *A pair (v, w) of permutations from S_n is divisible if and only if $v^{-1}w$ is not Boolean.*

Proof It is clear from the definition that (v, w) is divisible if and only if (uv, uw) is divisible for all $u \in S_n$, so it suffices to prove the case $v = e$.

Suppose that w is not Boolean, so that w contains a pattern $p \in \{321, 3412\}$ by Theorem 2.2. If $p = 3412$ occurs as $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ then (e, w) is divisible after position i_2, since $\{w_{i_1}, w_{i_2}\} \subseteq w[1, i_2]$ while $e[1, i_2] = \{1, \ldots, i_2\}$ must contain w_{i_3} and w_{i_4} if it contains either w_{i_1} or w_{i_2}. If $p = 321$ occurs as $w_{i_1}w_{i_2}w_{i_3}$, consider three cases: If $w_{i_2} = i_2$ then (e, w) is divisible at i_2, since $w_{i_1} > i_2 \notin e[1, i_2]$; If $w_{i_2} < i_2$, then (e, w) is divisible after $i_2 - 1$, since w_{i_2}, w_{i_3} both lie in $e[1, i_2 - 1]$ but not in $w[1, i_2 - 1]$; Similarly, if $w_{i_2} > i_2$, then (e, w) is divisible after i_2, since w_{i_1}, w_{i_2} both lie in $w[1, i_2]$ but not in $e[1, i_2]$.

Conversely suppose that w is divisible. If w is divisible after position i, then there are two elements $a < b \in w[1, i]$ which are not in $e[1, i] = \{1, \ldots, i\}$ and therefore also two elements $c, d \in w[i+1, n]$ with $c < d \in \{1, \ldots, i\}$. Either $w^{-1}(a) < w^{-1}(b)$ and $w^{-1}(c) < w^{-1}(d)$ in which case w contains 3412 or at least one of these statements fails and w contains 321; in either case w is not Boolean. If w is divisible at position i, then $w_i = e_i = i$ and there is some $a > i$ in $w[1, i-1]$ and some $b < i$ in $w[i+1, n]$; then the values a, i, b form a 321 pattern in w, so w is not Boolean. □

Remark The anonymous referee has helpfully pointed out that Proposition 3.2 is closely related to Theorem 2.2 of [16].

Proposition 3.3 (Gao–Hodges–Yong [6]) *A permutation $w \in S_n$ is spherical if and only if $w_0(J(w))w$ is a Boolean permutation.*

The following characterization of spherical permutations will be convenient for our arguments in Sect. 4.

Corollary 3.4 *A permutation $w \in S_n$ is spherical if and only if $(w_0(J(w)), w)$ is not divisible.*

Proof By Proposition 3.3, w is spherical if and only if $w_0(J(w))w$ is Boolean. Since $w_0(J(w))$ is an involution, Proposition 3.2 implies that this is equivalent to $(w_0(J(w)), w)$ not being divisible. □

4 Proof of Theorem 1.4

The following decomposition of the set P of twenty one patterns will be crucial to the proof of Theorem 1.4: $P = P^{321} \cup P^{3412}$, where

$$\begin{aligned} P^{321} &= \{24531, 25314, 25341, 42531, 45231, 45312, 52314, 52341, \\ &\quad 53124, 53142, 53412\} \\ &= \{p \in P \mid w_0(J(p))p \text{ contains the pattern } 321\}, \end{aligned}$$

and

$$\begin{aligned} P^{3412} &= \{34512, 34521, 35412, 35421, 45123, 45213, 45231, 53412, 53421, \\ &\quad 54123, 54213, 54231\} \\ &= \{p \in P \mid w_0(J(p))p \text{ contains the pattern } 3412\}. \end{aligned}$$

Notice that 45231 and 53412 lie in both P^{321} and P^{3412}. A simple check shows that P^{321} and P^{3412} are also characterized by the following properties:

$$\begin{aligned} P^{321} &= \{p \in S_5 \mid p^{-1}(5) < p^{-1}(3) < p^{-1}(1), p^{-1}(4) \notin [p^{-1}(5), p^{-1}(3)], \\ &\quad p^{-1}(2) \notin [p^{-1}(3), p^{-1}(1)]\}, \qquad (1) \\ P^{3412} &= \{p \in S_5 \mid \max(p^{-1}(4), p^{-1}(5)) < \min(p^{-1}(1), p^{-1}(2)), \\ &\quad p^{-1}(3) \notin [p^{-1}(4), p^{-1}(2)]\}. \qquad (2) \end{aligned}$$

The following proposition is obvious from the definitions, but will be useful to keep in mind throughout the proofs of Lemmas 4.2 and 4.3.

Proposition 4.1 *For $w \in S_n$, let $v = w_0(J(w))$ and $1 \le a < b \le n$. Then $v^{-1}(b) < v^{-1}(a)$ if and only if $w^{-1}(b) < w^{-1}(b-1) < \cdots < w^{-1}(a)$.*

Lemma 4.2 *If $w \in S_n$ avoids the patterns from P then w is spherical.*

Proof We reformulate using Corollary 3.4 and prove the contrapositive: if $(w_0(J(w)), w)$ is divisible, then w contains a pattern from P.

Case 1: Write v for $w_0(J(w))$ and suppose that (v, w) is divisible after i, and furthermore that i is the smallest index for which this is true. Then we have:

$$v[1,i] \setminus w[1,i] = \{a,b\},$$
$$w[1,i] \setminus v[1,i] = \{c,d\},$$

where we may assume without loss of generality that $a < b$ and $c < d$. We have $v \leq_R w$, so in particular $v \leq w$ in Bruhat order; thus by Proposition 2.1 we must have $a < c$ and $b < d$. Suppose that $c \leq b+1$; if $c < b$, then, since c appears after b in v, it must be that b, c lie in the same decreasing run of v, but by Proposition 4.1 this implies that b appears before c in w, a contradiction. If $c = b+1$, then s_c is a left descent of w, so c appears before b in v, again a contradiction. Thus we have $a < b < b+1 < c < d$.

We wish to conclude that w contains a pattern from P. If any value $x \in \{b+1, b+2, \dots, c-1\}$ does not lie between b and c in w, then we are done by (2), since the values $\{a,b,c,d,x\}$ form a pattern from P^{3412} in w. Otherwise, all of these values appear between b and c in w. Suppose that they do not appear in decreasing order, so $w^{-1}(b+j) < w^{-1}(b+j+1)$ for some $j+1 < c-b$. Then the values $\{c, d, b+j, b+j+1, a, b\}$ either contain a pattern from P^{3412}, or appear in w in the order $c, b+j, d, a, b+j+1, b$. In this last case $c, b+j, a, b+j+1, b$ forms an occurrence of the pattern 53142 from P^{321}. Finally, suppose that $\{b+1, b+2, \dots, c-1\}$ appear in decreasing order in w between b and c; then by Proposition 4.1 c appears before b in v, a contradiction. Thus in all cases w contains a pattern from P.

Case 2: Write v for $w_0(J(w))$ and suppose that (v, w) is divisible at i, and furthermore that i is the smallest index at or after which (v, w) is divisible. Then we have $v_i = w_i$ and

$$v[1,i-1] \setminus w[1,i-1] = \{a\},$$
$$w[1,i-1] \setminus v[1,i-1] = \{c\},$$

with $a < c$ by Proposition 2.1. We claim that a is the minimal element in a decreasing run of v. Indeed, otherwise $a-1$ appears immediately after a in v, and thus $a-1$ also appears after a in w. But then $a-1 \in v[1,i] \setminus w[1,i]$, contradicting the minimality of i, thus a is the minimal element in a decreasing run, and is smaller than all values appearing after it in v. Similarly, c is the maximal element in a decreasing run of v and is larger than all values appearing before it in v. Also note that $a < v_i - 1$ and $c > v_i + 1$, for if $a = v_i - 1$ then $w^{-1}(a+1) < w^{-1}(a)$ but $v^{-1}(a+1) > v^{-1}(a)$, contradicting Proposition 4.1, and similarly for c.

We will now see that c, v_i, a participate in an occurrence in w of some pattern $p \in P$. Suppose first that all values $c-1, c-2, \dots, v_i+1$ lie in between c and v_i in w. If these occur in decreasing order, then c and v_i must occur in the same decreasing run of v, but this is not the case since c appears at the beginning of its run, but after v_i in v. Thus there is some $j \leq c - v_i - 2$ such that $w^{-1}(c-j-1) < w^{-1}(c-j)$. In this case $c, c-j-1, c-j, v_i, a$ form an occurrence in w of the pattern $53421 \in P^{3412}$. Similarly, if all values $v_i - 1, v_i - 2, \dots, a+1$ lie in between v_i and a in w, then w contains an occurrence in w of the pattern $54231 \in P$.

In the only remaining case, there is some $x \in \{c-1, c-2, \dots, v_i+1\}$ not lying between c and v_i in w and some $y \in \{v_i - 1, v_i - 2, \dots, a+1\}$ not lying between v_i

and a in w. Then the values $\{c, v_i, a, x, y\}$ form a pattern p from P^{321} in w by (1), with c, v_i, a corresponding to the values 5, 3, 1 in p respectively and x, y corresponding to 4, 2. □

Lemma 4.3 *If $w \in S_n$ is spherical then w avoids the patterns from P.*

Proof We will apply Proposition 3.3 and prove the contrapositive: if w contains a pattern p from P, then vw is not Boolean, where $v = w_0(J(w))$.

Suppose first that w contains a pattern p from P^{321} and that w_i, w_j, w_k with $i < j < k$ correspond to the values 5, 3, 1 in p respectively. Since the 2 and 4 in the pattern p do not lie between w_j, w_k and w_i, w_j respectively, Proposition 4.1 implies that $v(w_i) > v(w_j) > v(w_k)$ since $v = v^{-1}$. Thus vw contains the pattern 321 and is not Boolean by Theorem 2.2.

Suppose now that w contains a pattern p from P^{3412}. Let w_i, w_j, w_k, w_ℓ with $i < j < k < \ell$ correspond to the values $\{1, 2, 4, 5\}$ from p (thus one of w_i, w_j corresponds to 4 and the other to 5, while one of w_k, w_ℓ corresponds to 1 and the other 2). Since the 3 in the pattern p does not lie between the 2 and the 4, Proposition 4.1 implies that $\min(v(w_i), v(w_j)) > \max(v(w_k), v(w_\ell))$. Thus either vw contains the pattern 3412 in these positions or contains a 321 pattern in some subset of them. In either case vw is not Boolean. □

Lemmas 4.2 and 4.3 together yield Theorem 1.4.

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