Locally Differentially Private Decentralized Stochastic Bilevel Optimization with Guaranteed Convergence Accuracy

Anonymous Authors[1]

Abstract

Decentralized bilevel optimization based machine learning techniques are achieving remarkable success in a wide variety of domains. However, the intensive exchange of information (involving nested-loops of consensus or communication iterations) in existing decentralized bilevel-optimization algorithms leads to a great challenge to ensure rigorous differential privacy, which, however, is necessary to bring the benefits of machine learning to domains where involved data are sensitive. By proposing a new decentralized stochastic bilevel-optimization algorithm which avoids nested-loops of information-exchange iterations, we achieve, for the first time, both differential privacy and accurate convergence in decentralized bilevel optimization. This is significant since even for single-level decentralized optimization and learning, existing differential-privacy solutions have to sacrifice convergence accuracy for privacy. Besides characterizing the convergence rate under nonconvex/convex/strongly convex conditions, we also rigorously quantify the price of differential privacy in computational complexities. Experimental results on practical machine learning models confirm the efficacy of our algorithm.

1. Introduction

Bilevel stochastic optimization is evolving as an effective tool for solving many machine learning problems having a nested structure, with typical examples including meta-learning (Bertinetto et al., 2019; Rajeswaran et al., 2019), hyperparameter optimization (Franceschi et al., 2018), imitation learning (Arora et al., 2020), and neural architecture search (Liu et al., 2018). So far, numerous centralized stochastic bilevel-optimization algorithms have been proposed (Ghadimi & Wang, 2018; Khanduri et al., 2021; Ji et al., 2021; Hong et al., 2023). Recently, with the increasingly pressing need to parallelize learning algorithms in order to handle the enormous growth in data and model sizes, the following decentralized stochastic bilevel-optimization (DSBO) problem is gaining increased traction (Lu et al., 2022; Yang et al., 2022; Gao et al., 2023; Chen et al., 2023):

$$\begin{aligned}&\min_{x\in\mathbb{R}^p} F(x), \quad F(x)=\frac{1}{m}\sum_{i=1}^{m} f_i(x,y^*(x)),\\ &\text{s.t. } y^*(x)=\text{argmin}_{y\in\mathbb{R}^q} g(x,y):=\frac{1}{m}\sum_{i=1}^{m} g_i(x,y),\end{aligned} \tag{1}$$

where $x\in\mathbb{R}^p$ and $y\in\mathbb{R}^q$ represent the optimization parameters and m denotes the number of agents. Each agent i only has access to its local upper-level objective function f_i and lower-level objective function g_i, which, in machine learning applications, are usually given by

$$\begin{aligned}f_i(x,y)&=\mathbb{E}_{\varphi_i}[h(x,y;\varphi_i)],\\ g_i(x,y)&=\mathbb{E}_{\xi_i}[l(x,y;\xi_i)].\end{aligned} \tag{2}$$

In (2), φ_i and ξ_i represent random data samples which usually follow unknown and heterogeneous distributions across different agents.

All above DSBO algorithms require participating agents to explicitly share model updates in every iteration, which raises severe privacy concerns when involved data are sensitive. In fact, recent studies (Zhang et al., 2018a; Zhu et al., 2019; Burbano-L et al., 2019; Triastcyn & Faltings, 2020; Wang & Nedić, 2023) have shown that even though raw data are not shared, exploiting information shared in decentralized optimization, external adversaries can still precisely recover the raw data used for training (pixel-wise accurate for images and token-wise matching for texts). As differential privacy is evolving as the de facto standard for privacy preservation due to its rigorous mathematical foundations yet implementation simplicity and post-processing immunity (Dwork et al., 2014), it is of great interest to achieve differential privacy in DSBO. However, given that existing DSBO algorithms all involve nested-loops of com-

[1]Anonymous Institution, Anonymous City, Anonymous Region, Anonymous Country. Correspondence to: Anonymous Author <anon.email@domain.com>.

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munication (consensus) iterations[1], directly incorporating the standard differential-privacy noise injection mechanism in existing DSBO algorithms will inevitably result in an exploding cumulative privacy budget as the iteration proceeds, leading to diminishing privacy protection in the long run. Another challenge is to maintain the accuracy of DSBO under the constraint of differential privacy. In fact, even for the simpler single-level decentralized optimization problem, existing differential-privacy solutions have to trade optimization accuracy for privacy (Bellet et al., 2018; Zhang et al., 2018b; Agarwal et al., 2018; Cyffers et al., 2022), which is undesirable in accuracy-sensitive applications.

1.1. Our Contributions

1. We propose a differentially private DSBO algorithm that can ensure both accurate convergence and rigorous differential privacy, with the cumulative privacy budget bounded even when the number of iterations tends to infinity. To the best of our knowledge, no such results have been reported before. Moreover, by employing the local differential privacy framework, our results can be applied to the fully decentralized setting where no data aggregator or mediator exists to gather data or assist privacy design.

2. A key enabler for our approach to achieving both differential privacy and accurate convergence is a novel algorithm for DSBO. Different from existing DSBO algorithms that all employ nested-loops of communication (consensus) iterations, our new algorithm successfully circumvents nested-loops of communication iterations, which makes it possible to alleviate the growth of the cumulative privacy budget as the number of iterations increases. In fact, given that using intensive (nested-loops of) communication rounds among agents is the only approach in the literature to achieving accurate estimation of hypergraidents when g_i are heterogeneous across the agents, our algorithm is of independent interest in itself even if privacy is not considered.

3. We establish the convergence rate of our algorithm for nonconvex/convex/strongly convex objective functions f_i, which is different from existing DSBO results (Lu et al., 2022; Chen et al., 2022; Gao et al., 2023; Chen et al., 2023) that focus solely on the nonconvex case. Moreover, our convergence analysis relaxes the assumption that g_i is Lipschitz continuous with respect to y, which is widely used in existing DSBO literature (see, e.g., Chen et al. (2022) and Yang et al. (2022)).

4. Despite retaining accurate convergence, our algorithm does pay a price for obtained differential privacy in convergence rate. We systematically quantify the tradeoff between privacy and convergence rate. It is worth noting that by avoiding estimating the full Hessian or Jacobian matrix, our algorithm still achieves improved computational complexity compared with the result for DSBO in Chen et al. (2022), which does not consider privacy protection.

5. We conduct experiment evaluation using several machine learning problems. The results confirm the efficiency of our algorithm on both the synthetic and the real-world datasets.

1.2. Related Work

1.2.1. Bilevel Optimization

Bilevel optimization was first discussed in Bracken & McGill (1973) for solving resource allocation problems. Historically, it was treated by viewing the lower-level optimality condition as constraints to the upper-level problem (Hansen et al., 1992; Shi et al., 2005). More recently, Couellan & Wang (2016) proposed a gradient-based algorithm providing asymptotic convergence and Ghadimi & Wang (2018) developed a nested-loop stochastic approximated algorithm establishing non-asymptotic convergence. Following these developments, various centralized approaches have been introduced, trying to improve the efficiency in solving bilevel-optimization problems (Khanduri et al., 2021; Ji et al., 2021; Hong et al., 2023).

Driven by the need for parallelized learning algorithms to handle the enormous growth in data and model sizes in machine learning, plenty of DSBO algorithms have been proposed recently (Lu et al., 2022; Chen et al., 2022; Yang et al., 2022; Gao et al., 2023; Chen et al., 2023). For example, Lu et al. (2022) and Gao et al. (2023) considered the DSBO problem where the lower-level objective function is fully accessible to every agent. Chen et al. (2022), Yang et al. (2022), and Chen et al. (2023) considered the case where neither the upper-level function nor the lower-level function is fully accessible to every local agent. In addition, the approaches in Chen et al. (2022) and Yang et al. (2022) require computing the full Jacobian and/or Hessian matrix, entailing a computational complexity of the order $\mathcal{O}(pq)$ or $\mathcal{O}(q^2)$ in every iteration. To reduce the computational complexity, Chen et al. (2023) proposed to estimate the Hessian-vector and Jacobian-vector products, which reduces the per-iteration complexity from $\mathcal{O}(pq)$ (or $\mathcal{O}(q^2)$) to $\mathcal{O}(\max\{p, q\})$. However, none of the existing results have addressed differential privacy for DSBO. In fact, as discussed in Section 1, to ensure accurate enough local estimation of the hypergradient, all of these algorithms employ nested-loops of consensus (communication) iterations, which will result in an exploding cumulative privacy budget if we incorporate these algorithms with the standard Laplace-noise mechanism in Dwork et al. (2014) to achieve differential privacy. In Table 1, we summarize the difference between our algorithm and existing results.

[1] Note that the algorithm in Gao et al. (2023) assumes identical data distributions for ξ_i and hence $g_1 = g_2 = \cdots = g_m$ (see equations (2) and (3) in Gao et al. (2023) or Appendix C.2 in Chen et al. (2023)), and thus does not apply to our general setting here.

Table 1. We compare our Algorithm 2 (LDP-DSBO) with existing algorithms, including the centralized bilevel-optimization algorithm BSA (Ghadimi & Wang, 2018), personalized DSBO algorithms SPDB (Lu et al., 2022) and VRDSBO (Gao et al., 2023), and DSBO algorithms DSBO-JHIP (Chen et al., 2022), GBDSBO (Yang et al., 2022), and MA-DSBO (Chen et al., 2023). In the table, we use δ to denote the optimization error. We use "Jacobian" to represent whether the algorithm requires computing the full Hessian or Jacobian matrix. We use "DP" to represent whether the algorithm considers differential privacy. We also use "Privacy Budget" to refer to the cumulative privacy budget of the algorithm when it is combined with the Laplace noise used in our algorithm to enable differential privacy. The detailed cumulative privacy budget calculation is provided in Appendix H.2).

Algorithm	Decentralized?	Computational Complexity	Jacobian	DP	Privacy Budget
BSA	No	$\mathcal{O}(\delta^{-3} + (q^2 \log(\delta^{-1}) + pq)\delta^{-2})$	YES	No	$\mathcal{O}(\delta^{-3})$
SPDB	YES	$\mathcal{O}(\max\{p, q\} \log(\delta^{-1})\delta^{-2})$	No	No	$\mathcal{O}(\delta^{-2})$
VRDSBO	YES	$\mathcal{O}((pq + q^2)\delta^{-\frac{3}{2}})$	YES	No	$\mathcal{O}(\delta^{-\frac{3}{2}})$
DSBO-JHIP	YES	$\mathcal{O}(pq \log(\delta^{-1})\delta^{-3})$	YES	No	$\mathcal{O}(\delta^{-3})$
GBDSBO	YES	$\mathcal{O}((q^2 \log(\delta^{-1}) + pq)\delta^{-2})$	YES	No	$\mathcal{O}(\delta^{-2})$
MA-DSBO	YES	$\mathcal{O}(\max\{p, q\} \log(\delta^{-1})\delta^{-2})$	No	No	$\mathcal{O}(\delta^{-2})$
LDP-DSBO	YES	$\mathcal{O}(\max\{p, q\}\delta^{-2.6})$	No	YES	$\mathcal{O}(1)$

1.2.2. Differential Privacy

Widely regarded as the "gold standard" for privacy protection (Cummings et al., 2021), differential privacy has found numerous applications in distributed computation scenarios, including distributed control systems (Cortés et al., 2016), federated learning (Zhang et al., 2022), and distributed deep learning (Papernot et al., 2018). Note that the commonly used differential-privacy framework assumes the presence of a data aggregator/curator to collect the raw data and inject noise. In the decentralized scenario, to ensure agent-level privacy, we employ the local differential privacy (LDP) framework (Kasiviswanathan et al., 2011), in which random perturbations are performed locally by each agent, thereby protecting individual data against external adversaries and neighboring agents. LDP has been implemented in decentralized optimization and learning algorithms (Bellet et al., 2018; Zhang et al., 2018b; Agarwal et al., 2018; Cyffers et al., 2022); however, these algorithms often face a fundamental tradeoff between optimization accuracy and privacy. It is worth noting that although using the information-theoretic approach, Kasiviswanathan et al. (2011) and Dwork et al. (2014) have proven the possibility to retain accurate convergence in differentially private learning by trading convergence rate for privacy, it is only recently that Wang & Nedić (2023) and Chen & Wang (2023) proposed concrete implementable algorithms that actually achieve this goal in decentralized optimization and learning. Nevertheless, these results are for the conventional single-level decentralized optimization and they cannot be combined with existing bilevel-optimization algorithms to ensure both differential privacy and accurate convergence. In fact, due to the existence of nested-loops of communication (consensus) iterations in existing DSBO algorithms, directly applying the differential-privacy mechanisms in Wang & Nedić (2023) and Chen & Wang (2023) will result in both loss of convergence accuracy and explosion of the cumulative privacy budget.

Notations: We denote $\nabla F(x) \in \mathbb{R}^p$ as the gradient of $F(x)$. We use $\nabla_x g(x, y)$ and $\nabla_y g(x, y)$ to represent the gradients of g with respect to x and y, respectively. We write $\nabla^2_{xy} g(x, y) \in \mathbb{R}^{p\times q}$ for the Jacobian matrix of g and $\nabla^2_{yy} g(x, y) \in \mathbb{R}^{q\times q}$ for the Hessian matrix of g with respect to y. We denote $\|\cdot\|_1$ and $\|\cdot\|$ as the l_1-norm and the l_2-norm of vectors, respectively. We use $\mathbf{1}_p$ to denote the all-ones vector in $\mathbb{R}^p$. We add an overbar to a letter to denote the average of all agents, e.g., $\bar{x}_t = \frac{1}{m}\sum_{i=1}^m x_{i,t}$. We use bold font to represent stacked vectors of all agents, e.g., $\boldsymbol{x}_t = \mathrm{col}(x_{1,t}, \cdots, x_{m,t})$. We write $\mathbb{P}[\mathcal{A}]$ for the probability of an event $\mathcal{A}$. We use $\mathrm{Lap}(\nu)$ to denote the Laplace distribution with a parameter $\nu > 0$, featuring a probability density function $f(x|\nu) = \frac{1}{2\nu} e^{\frac{-|x|}{\nu}}$. $\mathrm{Lap}(\nu)$ has a mean of zero and a variance of $2\nu^2$. We denote the set of m agents as $[m]$ and the neighboring set of agent i as $\mathcal{N}_i$. We denote the coupling weight matrix as $W = \{w_{ij}\} \in \mathbb{R}^{m\times m}$, in which $w_{ij} > 0$ if agent j interacts with agent i, and $w_{ij} = 0$ otherwise.

2. Preliminaries

2.1. Hypergradient Estimation

The major challenge in solving DSBO lies in the absence of explicit knowledge of $y^*(x)$, which makes it impossible for individual agents to evaluate the hypergradient $\nabla F(x, y^*(x))$. By leveraging the results for centralized stochastic bilevel optimization (Ghadimi & Wang, 2018), recently, Chen et al. (2022) proposed to calculate the hypergradient using the following relation:

$$\nabla F(x) = \frac{1}{m}\sum_{i=1}^m \nabla_x f_i(x, y^*(x)) - \nabla^2_{xy} g(x, y^*(x)) \times \left[\nabla^2_{yy} g(x, y^*(x))\right]^{-1} \frac{1}{m}\sum_{i=1}^m \nabla_y f_i(x, y^*(x)). \tag{3}$$

It is evident that computing $\nabla F(x)$ requires global information about g, which is inaccessible to agent i in a decentralized setting. A natural approach is to use ∇g_i as a surrogate; however, due to data heterogeneity across the agents, this approach results in steady-state errors. Therefore, every agent has to maintain local estimates of the global hypergradient. Instead of estimating the entire Hessian/Jacobian matrix, Chen et al. (2023) proposed to estimate the Hessian-inverse-vector product:

$$z^* = \left(\sum_{i=1}^{m} \nabla_{yy} g_i(x, y^*(x))\right)^{-1} \left(\sum_{i=1}^{m} \nabla_y f_i(x, y^*(x))\right). \tag{4}$$

According to (3), the global hypergradient is given by

$$\nabla F(x) = \frac{1}{m}\sum_{i=1}^{m} \left(\nabla_x f_i(x, y^*(x)) - \nabla_{xy}^2 g_i(x, y^*(x)) z^*\right), \tag{5}$$

where $\nabla_{xy}^2 g_i(x, y^*(x))z^*$ will be referred to as the Jacobian-vector product.

From (5), we know that if each agent i can have an accurate enough estimation of $\nabla_x f_i(x, y^*(x))$, z^*, and $\nabla_{xy}^2 g_i(x, y^*(x))z^*$, then every agent can have a good estimate of the global hypergradient. Notably, estimating the vector-valued z^* and $\nabla_{xy}^2 g_i(x, y^*(x))z^*$ circumvents the need for estimating the full Hessian and Jacobian matrices, which substantially reduces the per-iteration computational complexity.

2.2. Assumptions

Assumption 2.1. The weight matrix $W = \{w_{ij}\} \in \mathbb{R}^{m\times m}$ is symmetric and satisfies $\mathbf{1}^T W = \mathbf{0}^T$ and $W\mathbf{1} = \mathbf{0}$. The eigenvalues of $I + W$ (after arranged in an increasing order) satisfy $0 = \delta_1 < \delta_2 \le \cdots \le \delta_m < 1$.

Assumption 2.2. For any $i \in [m]$, functions f_i, ∇f_i, ∇g_i, and $\nabla^2 g_i$ are $L_{f,0}$, $L_{f,1}$, $L_{g,1}$, and $L_{g,2}$ Lipschitz continuous, respectively. Moreover, each function g_i is μ_g-strongly convex in y.

Assumption 2.3. The stochastic oracles $\nabla h(x, y; \varphi)$, $\nabla^2 h(x, y; \varphi)$, $\nabla l(x, y; \xi)$, $\nabla^2 l(x, y; \xi)$, and $\nabla^3 l(x, y; \xi)$ are unbiased with bounded variances, which are represented as $\sigma_{f,1}^2$, $\sigma_{f,2}^2$, $\sigma_{g,1}^2$, $\sigma_{g,2}^2$, and $\sigma_{g,3}^2$, respectively.

Assumptions 2.2 and 2.3 are standard in the DSBO literature (Lu et al., 2022; Chen et al., 2022; Yang et al., 2022; Chen et al., 2023; Gao et al., 2023). They allow f_i and g_i to be heterogeneous across the agents, which are more general than the homogeneous-function assumption in Lu et al. (2022) and Gao et al. (2023). In addition, we relax the assumption that lower-level objective functions g_i are Lipschitz continuous with respect to y, which is used in Chen et al. (2022) and Yang et al. (2022).

2.3. Local Differential Privacy

In this paper, we consider the case where data arrive sequentially in a serial manner, and only one data point is acquired by each agent at each time instant, i.e., at time instant T, the dataset $\mathcal{D}_i$ accessible to agent i is given by $\mathcal{D}_i = \{\xi_{i,1}, \cdots, \xi_{i,T}\}$. Then, we can introduce the following definitions for differential privacy:

Definition 2.4. (Adjacency) Given two local datasets $\mathcal{D}_i = \{\xi_{i,1}, \cdots, \xi_{i,T}\}$ and $\mathcal{D}'_i = \{\xi'_{i,1}, \cdots, \xi'_{i,T}\}$ for any $i \in [m]$ and any time $T \in \mathbb{N}$, $\mathcal{D}_i$ and $\mathcal{D}'_i$ are adjacent if there exists a time instant $k \in \{1, \cdots, T\}$ such that $\xi_{i,k} \neq \xi'_{i,k}$ while $\xi_{i,t} = \xi'_{i,t}$ for all $t \neq k,\ t \in \{1, \cdots, T\}$.

Definition 2.5. (Local Differential Privacy) Denote a DSBO algorithm as a mapping $\mathcal{A}_i(\mathcal{D}_i, x_{-i}) \mapsto \mathcal{O}_i$, where x_{-i} denotes all messages received by agent i and $\mathcal{O}_i$ represents the set of all possible observations on agent i. Then, for any given $\epsilon_i > 0$, we say that $\mathcal{A}_i$ is ϵ_i-locally differentially private if for any adjacent datasets $\mathcal{D}_i$ and $\mathcal{D}'_i$, the following inequality holds:

$$\mathbb{P}[\mathcal{A}_i(\mathcal{D}_i, x_{-i}) \in \mathcal{O}_i] \le e^{\epsilon_i}\mathbb{P}[\mathcal{A}_i(\mathcal{D}'_i, x_{-i}) \in \mathcal{O}_i]. \tag{6}$$

The parameter ϵ_i is referred to as the cumulative privacy budget for iterations $1, 2, \cdots, T$. A smaller ϵ_i indicates closer distributions of observations under adjacent datasets, thereby a higher level of privacy protection. Clearly, if ϵ_i grows to infinity as the number of iterations T tends to infinity, privacy will be lost eventually in the infinite-time horizon.

3. The Proposed Algorithm

In this section, we first introduce an approach for individual agents to locally estimate Hessian-inverse-vector product under the constraint of differential privacy, which is necessary for individual agents to locally estimate the global hypergradient according to (5). Using it as a subroutine, we will then propose our differentially private DSBO algorithm.

Approximating z^* in (4) amounts to letting each agent solve for the following equation:

$$\sum_{i=1}^{m} H_i z^* = \sum_{i=1}^{m} b_i \quad \text{or} \quad z^* \triangleq \left(\sum_{i=1}^{m} H_i\right)^{-1}\left(\sum_{i=1}^{m} b_i\right), \tag{7}$$

where H_i and b_i are given by $H_i = \nabla_{yy}^2 g_i(x, y^*(x))$ and $b_i = \nabla_y f_i(x, y^*(x))$, respectively. Equality (7) is essentially the optimality condition of the following optimization problem:

$$\min_{z\in\mathbb{R}^q} \frac{1}{m}\sum_{i=1}^{m} \phi_i(z), \quad \phi_i(z) = \frac{1}{2} z^T H_i z - b_i^T z. \tag{8}$$

Algorithm 1 Subroutine for Estimating Hessian-Inverse-Vector Product for Agent i, $i \in [m]$

Input: Parameters $x_{i,t}$, $y_{i,t}$, and $z_{i,t}$; Data samples $\{\varphi_{i,k}\}_{k\in[0,t]}$ and $\{\xi_{i,t}\}_{k\in[0,t]}$; Stepsize $\lambda_{z,t} = \frac{\lambda_{z,0}}{(t+1)^{v_z}}$ with $\lambda_{z,0} > 0$ and $v_z \in (0, 1)$; DP-noise $\vartheta_{i,t}$ satisfying Assumption 3.1.
$H_{i,t}z_{i,t} = \nabla^2_{yy} g_{i,t}(x_{i,t}, y_{i,t})z_{i,t}$.
$b_{i,t} = \nabla_y f_{i,t}(x_{i,t}, y_{i,t})$.
$\nabla_z \phi_{i,t}(z_{i,t}) = H_{i,t}z_{i,t} - b_{i,t}$.
$z_{i,t+1} = z_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(z_{j,t} + \vartheta_{j,t} - z_{i,t}) - \lambda_{z,t}\nabla_z \phi_{i,t}(z_{i,t})$.
Output: $z_{i,t+1}$ on agent i.

We present Algorithm 1 that enables all agents to collaboratively find the optimal solution z^* to problem (8).

Since objective functions f_i and g_i in problem (8) are expectations over unknown distributions (see the equations in (2)), they are inaccessible and can only be approximated from sampled data in practical implementations. Therefore, under our setting of serially arriving data, we use $f_{i,t}(x, y) = \frac{1}{t+1}\sum_{k=0}^{t} h(x, y; \varphi_{i,k})$ and $g_{i,t}(x, y) = \frac{1}{t+1}\sum_{k=0}^{t} l(x, y; \xi_{i,k})$.

Building on Algorithm 1, each agent i can estimate the hypergradient $\nabla F(x)$ in (5) locally by using the following equality:

$$u_{i,t} = \nabla_x f_{i,t}(x_{i,t}, y_{i,t}) - \nabla^2_{xy} g_{i,t}(x_{i,t}, y_{i,t})z_{i,t}. \tag{9}$$

With the hypergradient estimation (9), we propose a locally differentially private algorithm to solve the DSBO problem (1) in Algorithm 2. The injected DP noises satisfy the following assumption:

Assumption 3.1. For every $i \in [m]$ and $t \geq 0$, each element of DP-noise vectors $\chi_{i,t}$, $\zeta_{i,t}$, and $\vartheta_{i,t}$ follows Laplace distributions $\mathrm{Lap}\left(\frac{\sigma_{i,x}}{\sqrt{2}(t+1)^{\varsigma_{i,x}}}\right)$, $\mathrm{Lap}\left(\frac{\sigma_{i,y}}{\sqrt{2}(t+1)^{\varsigma_{i,y}}}\right)$, and $\mathrm{Lap}\left(\frac{\sigma_{i,z}}{\sqrt{2}(t+1)^{\varsigma_{i,z}}}\right)$, respectively, where $\sigma_{i,x}$, $\sigma_{i,y}$, and $\sigma_{i,z}$ are positive constants and the rates of DP-noise variances satisfy

$$\max_{i\in[m]}\{\varsigma_{i,x}\} < v_x, \ \max_{i\in[m]}\{\varsigma_{i,y}\} < v_y, \text{ and } \max_{i\in[m]}\{\varsigma_{i,z}\} < v_z,$$

where $v_x, v_y, v_z \in (0, 1)$ are the decaying rates of the stepsizes $\lambda_{x,t}$, $\lambda_{y,t}$, and $\lambda_{z,t}$, respectively, in Algorithm 2.

It is worth noting that different from existing DSBO algorithms in Chen et al. (2022), Yang et al. (2022), and Gao et al. (2023) which estimate the full Hessian matrix or Jacobian matrix, Algorithm 2 only estimates a vector of dimension $\max\{p, q\}$, and hence has reduced computational complexity. In addition, different from existing DSBO algorithms in Chen et al. (2022) and Chen et al. (2023) which use a nested communication (consensus) loop to estimate z^*, Algorithm 2 avoids any nested-loops of consensus operations. The avoidance of nested consensus loops is significant in that under nested-loops of consensus iterations, the cumulative privacy budget will grow quickly as iteration proceeds, making it impossible to ensure a finite cumulative privacy budget in the infinite-time horizon (see detailed explanations in Appendix H.1).

Algorithm 2 LDP Design for DSBO Algorithm for Agent i, $i \in [m]$

Input: Random initialization $x_{i,0} \in \mathbb{R}^p$, $y_{i,0} \in \mathbb{R}^q$, and $z_{i,0} \in \mathbb{R}^q$ for each agent $i \in [m]$. Stepsizes $\lambda_{x,t} = \frac{\lambda_{x,0}}{(t+1)^{v_x}}$ and $\lambda_{y,t} = \frac{\lambda_{y,0}}{(t+1)^{v_y}}$ with $\lambda_{x,0} > 0$, $\lambda_{y,0} > 0$, and $v_x, v_y \in (0, 1)$; DP-noises $\chi_{i,t}$ and $\zeta_{i,t}$ satisfying Assumption 3.1.
for $t = 0, 1, \cdots, T-1$ **do**
  Acquire current data $\varphi_{i,t}$ and $\xi_{i,t}$.
  $y_{i,t+1} = y_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(y_{j,t} + \zeta_{j,t} - y_{i,t}) - \lambda_{y,t}\nabla_y g_{i,t}(x_{i,t}, y_{i,t})$.
  Run Algorithm 1 and obtain the output $z_{i,t+1}$.
  Estimate hypergradient $u_{i,t}$ by using (9).
  $x_{i,t+1} = x_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,t} + \chi_{j,t} - x_{i,t}) - \lambda_{x,t}u_{i,t}$.
end for
Output: $x_{i,T}$ on agent i.

4. Main Results

4.1. Convergence Rate of Algorithm 2

Theorem 4.1. *Denote the lowest decaying rates of DP-noise variances as $\varsigma_x = \min_{i\in[m]}\varsigma_{i,x}$, $\varsigma_y = \min_{i\in[m]}\varsigma_{i,y}$, and $\varsigma_z = \min_{i\in[m]}\varsigma_{i,z}$. Under Assumptions 2.1-2.3, and 3.1, if the stepsize rates satisfy $0 < v_z < v_y < v_x < 1$, then we have the following results for the iterates $\{x_i\}$ generated by Algorithm 2:*

(1) If $F(x)$ is strongly convex and the rates of DP-noise variances satisfy $2\varsigma_x > v_x$, $2\varsigma_x > v_z + v_y$, $2\varsigma_y > v_z + v_y$, and $2\varsigma_z > v_y$, then we have

$$\mathbb{E}\left[\|x_{i,T} - x^*\|^2\right] \leq \mathcal{O}\left(T^{-\beta_1}\right), \tag{10}$$

where the rate β_1 is given by $\beta_1 = \min\{2\varsigma_x - v_x, 2\varsigma_x - 2v_z, 2\varsigma_y - 2v_z, 2\varsigma_z - v_z, 2\varsigma_y - v_y, 2 - 2v_y\}$.

(2) If F is convex and the rates of DP-noise variances satisfy $\varsigma_x > \frac{1}{2}$, $2\varsigma_x > v_z+v_y$, $2\varsigma_x > 2v_z+2-2v_x$, $2\varsigma_y > v_z+v_y$, $2\varsigma_y > 2v_z+2-2v_x$, $2\varsigma_y > v_y+2-2v_x$, $2\varsigma_z > v_z+2-2v_x$, and $2\varsigma_z > v_y$, then we have

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[F(x_{i,t}) - F(x^*)\right] \leq \mathcal{O}\left(T^{-(1-v_x)}\right). \tag{11}$$

(3) If F is nonconvex and the rates of DP-noise variances satisfy $\varsigma_x > \frac{1}{2}$, $2\varsigma_x > v_z + v_y$, $2\varsigma_x > 2v_z + 1 - v_x$, $2\varsigma_y > 2v_z + 1 - v_x$, $2\varsigma_y > v_y + 1 - v_x$, $2\varsigma_y > v_z + v_y$, $2\varsigma_z > v_z + 1 - v_x$, and $2\varsigma_z > v_y$, then we have

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right] \le \mathcal{O}\left(T^{-(1-v_x)}\right). \tag{12}$$

Theorem 4.1 proves that the optimization errors for strongly convex, convex, and nonconvex $F(x)$ decrease with iterations at rates $\mathcal{O}\left(T^{-\beta_1}\right)$, $\mathcal{O}\left(T^{-(1-v_x)}\right)$, and $\mathcal{O}\left(T^{-(1-v_x)}\right)$, respectively.

Moreover, to give a more intuitive description of the computational complexity, we define a δ-solution to problem (1):

Definition 4.2. (Lian et al., 2017) For any $i \in [m]$ and some positive integer T, if $\mathbb{E}\left[\|x_{i,T} - x^*\|^2\right] \le \delta$ holds when F is strongly convex, or $\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[F(x_{i,t}) - F(x^*)\right] \le \delta$ holds when F is convex, or $\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right] \le \delta$ holds when F is nonconvex, then we say that the sequence $\{x_{i,t}\}_{t=0}^{T}$ can reach a δ-solution to problem (1).

Definition 4.2 provides a direct quantitative measure of the optimization error with respect to the optimal solution x^* under strongly convex F. This measure is stronger than the metrics in Ghadimi & Wang (2018) and Yang et al. (2022) that characterize the distance between $F(\bar{x}_T)$ and $F(x^*)$. Moreover, in the nonconvex case, compared with Chen et al. (2023), which uses the minimum hypergradient over all iterations (i.e., $\min_{0<t<T}\mathbb{E}\left[\|\nabla F(\bar{x}_t)\|^2\right] \le \delta$), Definition 4.2 is much more stringent.

Corollary 4.3. *(1) For a strongly convex $F(x)$, if we choose $T = \mathcal{O}(\delta^{-\frac{1}{\beta_1}})$, then the computational complexity of Algorithm 2 is $\mathcal{O}(\max\{p,q\}\delta^{-\frac{1}{\beta_1}})$ in finding a δ-solution. For example, setting $v_x = 0.66$, $v_y = 0.64$, $v_z = 0.43$, $\varsigma_x = 0.65$, $\varsigma_y = 0.63$, and $\varsigma_z = 0.42$ yields ~~a convergence rate of~~ $\beta_1 = 0.4$ and a complexity of $\mathcal{O}(\max\{p,q\}\delta^{-2.5})$.*
(2) For a convex $F(x)$, if we set $T = \mathcal{O}\big(\delta^{-\frac{1}{1-v_x}}\big)$, then the computational complexity of Algorithm 2 is $\mathcal{O}\big(\max\{p,q\}\delta^{-\frac{1}{1-v_x}}\big)$ in finding a δ-solution. For example, with $v_x = 0.77$, $v_y = 0.75$, $v_z = 0.5$, $\varsigma_x = 0.76$, $\varsigma_y = 0.74$, and $\varsigma_z = 0.49$, ~~the convergence rate is~~ $1 - v_x = 0.23$ and ~~the~~ complexity ~~is~~ $\mathcal{O}(\max\{p,q\}\delta^{-4.35})$.
(3) For a nonconvex $F(x)$, if we choose $T = \mathcal{O}\big(\delta^{-\frac{1}{1-v_x}}\big)$, then the computational complexity of Algorithm 2 is $\mathcal{O}\big(\max\{p,q\}\delta^{-\frac{1}{1-v_x}}\big)$ in finding a δ-solution. For example, using $v_x = 0.615$, $v_y = 0.60375$, $v_z = 0.4$, $\varsigma_x = 0.61125$, $\varsigma_y = 0.6$, and $\varsigma_z = 0.398125$ yields ~~a convergence rate of~~ $1 - v_x = 0.385$ and a ~~complexity~~ of $\mathcal{O}(\max\{p,q\}\delta^{-2.6})$.

Corollary 4.3 provides ~~convergence rates and~~ computational complexities under different convexity assumptions. It is more comprehensive than existing DSBO results (Chen et al., 2022; Gao et al., 2023; Chen et al., 2023), which only focus on a nonconvex function F. Moreover, it is worth noting that compared with the computational complexity of $\mathcal{O}(pq\log(\delta^{-1})\delta^{-3})$ in Chen et al. (2022), our Algorithm 2 ensures an improved computational complexity of $\mathcal{O}(\max\{p,q\}\delta^{-2.6})$, even under the additional constraint of differential privacy.

4.2. Differential Privacy Analysis for Algorithm 2

In this subsection, we prove that besides accurate convergence, Algorithm 2 can simultaneously ensure rigorous ϵ_i-LDP for each agent, with a finite cumulative privacy budget even when the number of iterations tends to infinity.

Assumption 4.4. Functions ∇h, ∇l, and $\nabla^2 l$ are $L_{h,1}$, $L_{l,1}$, and $L_{l,2}$ Lipschitz continuous, respectively. Moreover, there exist some positive constants c_{h0} and c_{l0} such that $\|\nabla_y h(x,y;\varphi_i)\|_1 \le c_{h0}$ and $\|\nabla_y l(x,y;\xi_i)\|_1 \le c_{l0}$ hold for all $i \in [m]$.

Assumption 4.4 is commonly used in differential-privacy design for decentralized learning/optimization (Huang et al., 2015; Bellet et al., 2018; Zhang et al., 2018b; Agarwal et al., 2018; Cyffers et al., 2022). Although it is stricter than Assumption 2.2 (which assumes Lipschitz continuity of the gradients of expected functions f_i and g_i), it is not required in our convergence analysis. In fact, existing DSBO results (Chen et al., 2022; Yang et al., 2022; Gao et al., 2023; Chen et al., 2023) often do not clearly differentiate between Assumption 2.2 and Assumption 4.4, and usually assume Lipschitz continuity of loss functions h and l and their first- and second-order moments, similar to Assumption 4.4 (see e.g., Assumptions 3.3 and 3.4 in Yang et al. (2022) and Assumption 2.1 in Chen et al. (2023)).

Theorem 4.5. *Under Assumptions 2.1 and 4.4, if each element of $\chi_{i,t}$, $\zeta_{i,t}$, and $\vartheta_{i,t}$ follows the Laplace distributions given in Assumption 3.1, then $x_{i,t}$ (resp. $F(x_{i,t})$ and $\nabla F(x_{i,t})$ in the general convex case and nonconvex case, respectively) in Algorithm 2 converges in mean square to the optimal solution x^* to problem (1) (resp. in mean to $F(x^*)$ and in mean square to zero, respectively). Furthermore,*

1) For any finite number of iterations T, agent i's implementation of Algorithm 2 is locally differentially private with a cumulative privacy budget bounded by $\epsilon_i = \epsilon_{i,x} + \epsilon_{i,y} + \epsilon_{i,z}$, where $\epsilon_{i,x} \le \sum_{t=1}^{T}\frac{\sqrt{2}\varrho_{t,x}(t+1)^{\varsigma_{i,x}}}{\sigma_{i,x}}$, $\epsilon_{i,y} \le \sum_{t=1}^{T}\frac{\sqrt{2}\varrho_{t,y}(t+1)^{\varsigma_{i,y}}}{\sigma_{i,y}}$, $\epsilon_{i,z} \le \sum_{t=1}^{T}\frac{\sqrt{2}\varrho_{t,z}(t+1)^{\varsigma_{i,z}}}{\sigma_{i,z}}$, $\varrho_{t,x} = 2(c_{h0} + c_z L_{l,1})\sum_{p=1}^{t}(1-\bar{w})^{t-p}\lambda_{x,p-1}$, $\varrho_{t,y} = 2c_{l0}\sum_{p=1}^{t}(1-\bar{w})^{t-p}\lambda_{y,p-1}$, $\varrho_{t,z} = 2(c_z L_{l,1} + c_{h0})\sum_{p=1}^{t}(1-\bar{w})^{t-p}\lambda_{z,p-1}$, $c_z = \max_{t\in[0,T]}\{\|z_{i,t}\|_1\}$, and $\bar{w} = \min_{i\in[m]}\{|w_{ii}|\}$.

2) The cumulative privacy budget ϵ_i is finite even when the number of iterations T tends to infinity.

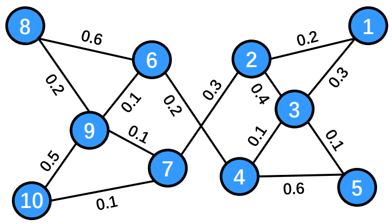


Figure 1. The communication graph of ten agents.

Theorem 4.5 shows that Algorithm 2 can ensure rigorous ϵ_i-LDP and accurate convergence simultaneously. This differs from most existing differential-privacy solutions for decentralized single-level optimization (Bellet et al., 2018; Zhang et al., 2018b; Agarwal et al., 2018; Cyffers et al., 2022), which have to trade convergence accuracy for differential privacy. In fact, our algorithm's accurate convergence comes at the expense of a reduced convergence rate. We use the convergence rate and cumulative privacy budget under a nonconvex $F(x)$ as an example to quantify this tradeoff:

Corollary 4.6. *For any given $\delta \geq 0$, the convergence rate of Algorithm 2 is $\mathcal{O}(T^{v_x-1})$ and the cumulative privacy budget ϵ_i is on the order of $\mathcal{O}(\frac{1}{v_x-0.6})$ with $v_x \in (0.6, 1)$.*

Corollary 4.6 indicates that a higher level of differential privacy, i.e., a smaller cumulative privacy budget ϵ_i, corresponds to a reduced convergence rate $\mathcal{O}(T^{v_x-1})$.

5. Experiments

In this section, we study the application of Algorithm 2 in hyperparameter optimization:

$$\min_{\lambda\in\mathbb{R}^p} \quad \frac{1}{m}\sum_{i=1}^{m} f_i(\lambda,\omega^*(\lambda)),$$

$$\text{s.t.} \quad \omega^*(\lambda) = \text{argmin}_{\omega\in\mathbb{R}^q}\frac{1}{m}\sum_{i=1}^{m} g_i(\lambda,\omega),$$

in which we aim to find an optimal hyperparameter λ under the constraint that $\omega^*(\lambda)$ is the optimal model parameter with a given λ. We conducted experiments on both synthetic and real-world datasets.

In each experiment, we compared Algorithm 2 with state-of-the-art DSBO algorithms, including MA-DSBO (Chen et al., 2023) and GBDSBO (Yang et al., 2022). The interaction pattern associated with the coupling weight matrix W was consistent across all experiments and is depicted in Figure 1.

To evaluate the performance of Algorithm 2 without differential-privacy constraints, we also conducted experiments in the absence of DP noises, with the results given in Appendix A.1. Furthermore, additional comparison results with VRDSBO in Gao et al. (2023) (which only addresses the special case of $g_1 = \cdots = g_m$) were given in Appendix A.2.

5.1. Synthetic Data

Following Chen et al. (2022) and Chen et al. (2023), we define loss functions for each agent i as follows:

$$h(\lambda,\omega;\varphi_i) = \sum_{(x_{i,e},y_{i,e})\in\mathcal{D}^h_{i,t}} L(y_{i,e}x^T_{i,e}\omega),$$

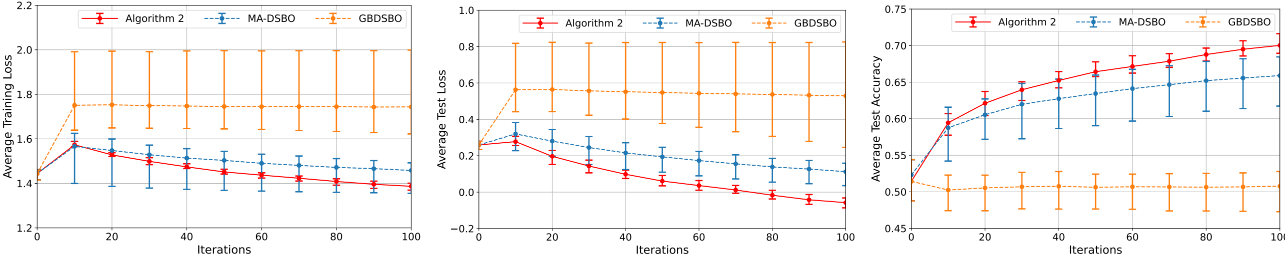
$$l(\lambda,\omega;\xi_i) = \sum_{(x_{i,e},y_{i,e})\in\mathcal{D}^l_{i,t}} L(y_{i,e}x^T_{i,e}\omega) + \frac{1}{2}\sum_{s=1}^{200} e^{\lambda_s}\omega_s^2,$$

where λ_s and ω_s represent the s-th element of $\lambda \in \mathbb{R}^{200}$ and $\omega \in \mathbb{R}^{200}$, respectively. The function $L(\cdot)$ is given by $L(x) = \log(1+e^{-x})$. $\mathcal{D}^l_{i,t}$ and $\mathcal{D}^h_{i,t}$ represent the training dataset and the validation dataset for agent i, at time t, respectively. For each agent i, the data distribution of $x_{i,e}$ was drawn from a normal distribution $\mathcal{N}(0,i^2)$, which is heterogeneous due to the difference in variances. The label y_e was generated by $y_{i,e} = x^T_{i,e}\omega + 0.1\varepsilon$, where $\varepsilon \in \mathbb{R}^{200}$ denotes the noise vector sampled from the standard normal distribution. The algorithm was executed for 100 iterations, with each agent randomly selecting 50 training samples in every iteration. The test dataset contains $20,000$ samples, with $1,000$ samples randomly selected for each iteration. For Algorithm 2, the stepsizes were set to $\lambda_{x,t} = \frac{0.05}{(t+1)^{0.95}}$, $\lambda_{y,t} = \frac{0.05}{(t+1)^{0.87}}$, and $\lambda_{z,t} = \frac{0.02}{(t+1)^{0.75}}$. Each element of DP-noise vectors $\chi_{i,t}$, $\zeta_{i,t}$, and $\vartheta_{i,t}$ for agent i follows Laplace distributions $\text{Lap}(\frac{1}{\sqrt{2}(t+1)^{0.8+0.01i}})$, $\text{Lap}(\frac{1}{\sqrt{2}(t+1)^{0.76+0.01i}})$, and $\text{Lap}(\frac{1}{\sqrt{2}(t+1)^{0.6+0.01i}})$, respectively. In our comparison, near-optimal stepsizes were selected for MA-DSBO and GBDSBO, ensuring that doubling these stepsizes would lead to non-converging behaviors. The number of nested-loops for MA-DSBO and GBDSBO was set to 10. We applied the fastest decaying DP-noise variance $\text{Lap}(\frac{1}{\sqrt{2}(t+1)^{0.8+0.01i}})$ to MA-DSBO and GBDSBO, as using a slower decaying DP noise to make their privacy budget the same as ours results in divergence of both algorithms (this gives them an edge in accuracy comparison).

The resulting training loss, test loss, and test accuracy are shown in Figures 2(a), 2(b), and 2(c), respectively. It is clear that the proposed algorithm has much lower training loss and higher test accuracy under differential-privacy constraints.

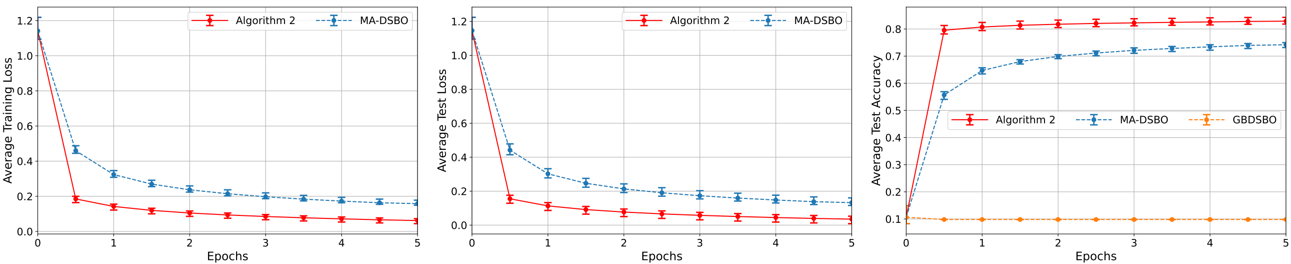
5.2. MNIST

In the second experiment, we evaluated the performance of Algorithm 2 by using the "MNIST" dataset (Grazzi et al.,



(a) Training loss (b) Test loss (c) Test accuracy

Figure 2. Comparison by using the synthetic dataset under differential-privacy constraints.



(a) Training loss (b) Test loss (c) Test accuracy

Figure 3. Comparison by using the "MNIST" dataset under differential-privacy constraints

References

Agarwal, N., Suresh, A. T., Yu, F. X. X., Kumar, S., and McMahan, B. cpSGD: Communication-efficient and differentially-private distributed SGD. *Advances in Neural Information Processing Systems*, 31:7564–7575, 2018.

Arora, S., Du, S., Kakade, S., Luo, Y., and Saunshi, N. Provable representation learning for imitation learning via bi-level optimization. In *International Conference on Machine Learning*, pp. 367–376. PMLR, 2020.

Bellet, A., Guerraoui, R., Taziki, M., and Tommasi, M. Personalized and private peer-to-peer machine learning. In *International Conference on Artificial Intelligence and Statistics*, pp. 473–481. PMLR, 2018.

Bertinetto, L., Henriques, J., Torr, P., and Vedaldi, A. Meta-learning with differentiable closed-form solvers. In *International Conference on Learning Representations*, 2019.

Bracken, J. and McGill, J. T. Mathematical programs with optimization problems in the constraints. *Operations Research*, 21(1):37–44, 1973.

Burbano-L, D. A., George, J., Freeman, R. A., and Lynch, K. M. Inferring private information in wireless sensor networks. In *ICASSP 2019-2019 IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 4310–4314. IEEE, 2019.

Chen, X., Huang, M., and Ma, S. Decentralized bilevel optimization. *arXiv preprint arXiv:2206.05670*, 2022.

Chen, X., Huang, M., Ma, S., and Balasubramanian, K. Decentralized stochastic bilevel optimization with improved per-iteration complexity. In *International Conference on Machine Learning*, pp. 4641–4671. PMLR, 2023.

Chen, Z. and Wang, Y. Locally differentially private gradient tracking for distributed online learning over directed graphs. *arXiv preprint arXiv:2310.16105*, 2023.

Cortés, J., Dullerud, G. E., Han, S., Le Ny, J., Mitra, S., and Pappas, G. J. Differential privacy in control and network systems. In *2016 IEEE 55th Conference on Decision and Control*, pp. 4252–4272. IEEE, 2016.

Couellan, N. and Wang, W. On the convergence of stochastic bi-level gradient methods. *Optimization*, pp. 13833, 2016.

Cummings, R., Kaptchuk, G., and Redmiles, E. M. "I need a better description": an investigation into user expectations for differential privacy. In *Proceedings of the 2021 ACM SIGSAC Conference on Computer and Communications Security*, pp. 3037–3052, 2021.

Cyffers, E., Even, M., Bellet, A., and Massoulié, L. Muffliato: Peer-to-peer privacy amplification for decentralized optimization and averaging. *Advances in Neural Information Processing Systems*, 35:15889–15902, 2022.

Dwork, C., Roth, A., et al. The algorithmic foundations of differential privacy. *Foundations and Trends in Theoretical Computer Science*, 9(3-4):211–407, 2014.

Franceschi, L., Frasconi, P., Salzo, S., Grazzi, R., and Pontil, M. Bilevel programming for hyperparameter optimization and meta-learning. In *International Conference on Machine Learning*, pp. 1568–1577. PMLR, 2018.

Gao, H., Gu, B., and Thai, M. T. On the convergence of distributed stochastic bilevel optimization algorithms over a network. In *International Conference on Artificial Intelligence and Statistics*, pp. 9238–9281. PMLR, 2023.

Ghadimi, S. and Wang, M. Approximation methods for bilevel programming. *arXiv preprint arXiv:1802.02246*, 2018.

Grazzi, R., Franceschi, L., Pontil, M., and Salzo, S. On the iteration complexity of hypergradient computation. In *International Conference on Machine Learning*, pp. 3748–3758. PMLR, 2020.

Hansen, P., Jaumard, B., and Savard, G. New branch-and-bound rules for linear bilevel programming. *SIAM Journal on Scientific and Statistical Computing*, 13(5):1194–1217, 1992.

Hong, M., Wai, H.-T., Wang, Z., and Yang, Z. A two-timescale stochastic algorithm framework for bilevel optimization: Complexity analysis and application to actor-critic. *SIAM Journal on Optimization*, 33(1):147–180, 2023.

Huang, Z., Mitra, S., and Vaidya, N. Differentially private distributed optimization. In *Proceedings of the 16th International Conference on Distributed Computing and Networking*, pp. 1–10, 2015.

Ji, K., Yang, J., and Liang, Y. Bilevel optimization: Convergence analysis and enhanced design. In *International Conference on Machine Learning*, pp. 4882–4892. PMLR, 2021.

Kasiviswanathan, S. P., Lee, H. K., Nissim, K., Raskhodnikova, S., and Smith, A. What can we learn privately? *SIAM Journal on Computing*, 40(3):793–826, 2011.

Khanduri, P., Zeng, S., Hong, M., Wai, H.-T., Wang, Z., and Yang, Z. A near-optimal algorithm for stochastic bilevel optimization via double-momentum. *Advances in Neural Information Processing Systems*, 34:30271–30283, 2021.

Lian, X., Zhang, C., Zhang, H., Hsieh, C.-J., Zhang, W., and Liu, J. Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. *Advances in Neural Information Processing Systems*, 30:5330–5340, 2017.

Liu, H., Simonyan, K., and Yang, Y. DARTS: Differentiable architecture search. In *International Conference on Learning Representations*, 2018.

Lu, S., Cui, X., Squillante, M. S., Kingsbury, B., and Horesh, L. Decentralized bilevel optimization for personalized client learning. In *ICASSP 2022-2022 IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 5543–5547. IEEE, 2022.

Papernot, N., Song, S., Mironov, I., Raghunathan, A., Talwar, K., and Erlingsson, U. Scalable private learning with PATE. In *International Conference on Learning Representations*, 2018.

Rajeswaran, A., Finn, C., Kakade, S. M., and Levine, S. Meta-learning with implicit gradients. *Advances in Neural Information Processing Systems*, 32, 2019.

Shi, C., Lu, J., and Zhang, G. An extended Kuhn–Tucker approach for linear bilevel programming. *Applied Mathematics and Computation*, 162(1):51–63, 2005.

Triastcyn, A. and Faltings, B. Bayesian differential privacy for machine learning. In *International Conference on Machine Learning*, pp. 9583–9592. PMLR, 2020.

Wang, Y. and Nedić, A. Tailoring gradient methods for differentially-private distributed optimization. *IEEE Transactions on Automatic Control (Early Access)*, 2023.

Yang, S., Zhang, X., and Wang, M. Decentralized gossip-based stochastic bilevel optimization over communication networks. *Advances in Neural Information Processing Systems*, 35:238–252, 2022.

Zhang, C., Ahmad, M., and Wang, Y. ADMM based privacy-preserving decentralized optimization. *IEEE Transactions on Information Forensics and Security*, 14(3):565–580, 2018a.

Zhang, X., Khalili, M. M., and Liu, M. Improving the privacy and accuracy of ADMM-based distributed algorithms. In *International Conference on Machine Learning*, pp. 5796–5805. PMLR, 2018b.

Zhang, X., Chen, X., Hong, M., Wu, Z. S., and Yi, J. Understanding clipping for federated learning: Convergence and client-level differential privacy. In *International Conference on Machine Learning*, pp. 26048–26067. PMLR, 2022.

Zhu, L., Liu, Z., and Han, S. Deep leakage from gradients. *Advances in Neural Information Processing Systems*, 32: 14774–14784, 2019.

Outline

The structure of main proofs is given in Figure 4. Given that auxiliary lemmas from Sections B and C are utilized in several lemmas, they are omitted from this figure.

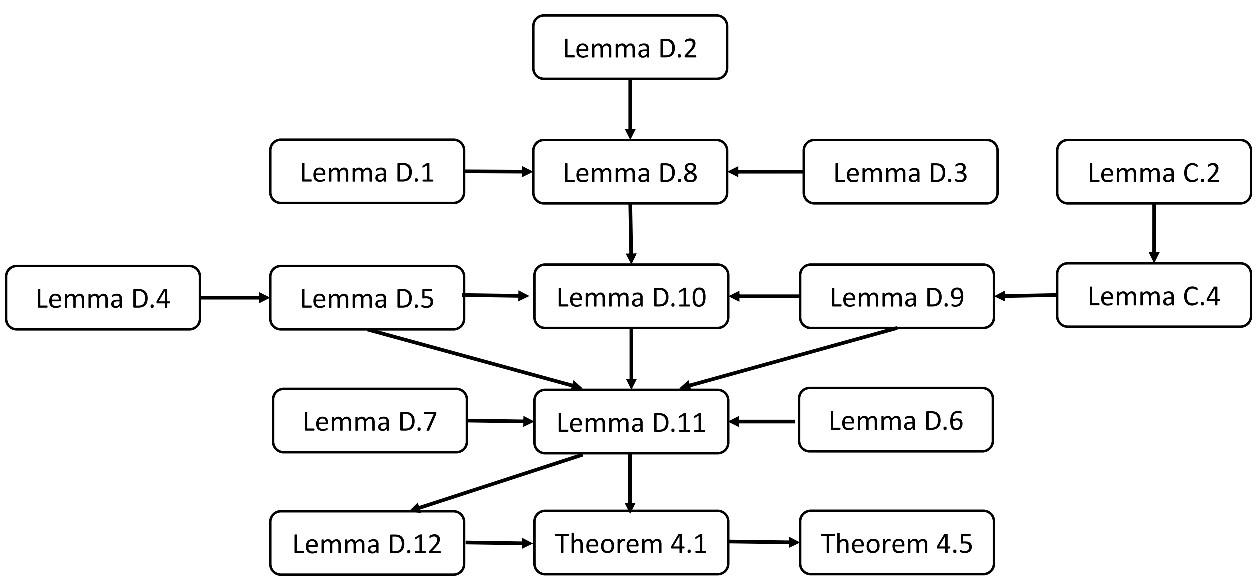
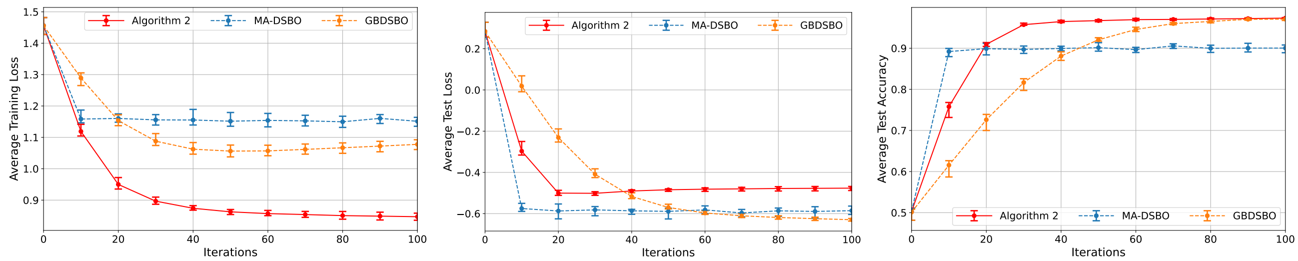
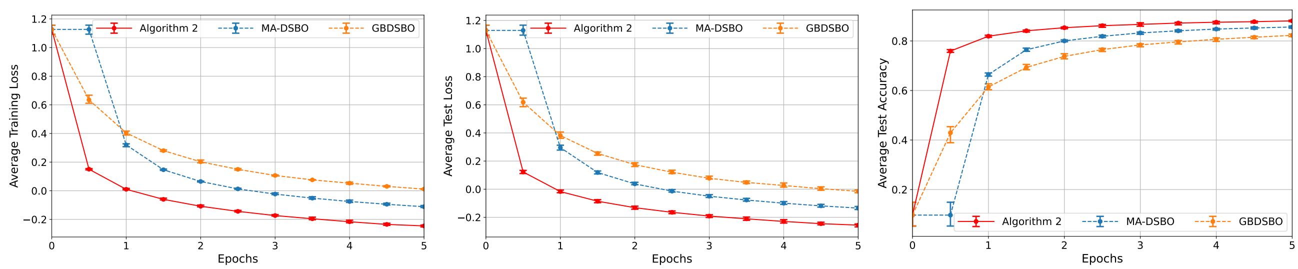


Figure 4. Structure of proofs.



(a) Training loss (b) Test loss (c) Test accuracy

Figure 5. Comparison by using the synthetic dataset in the absence of DP-noises.



(a) Training loss (b) Test loss (c) Test accuracy

Figure 6. Comparison by using the "MNIST" dataset in the absence of DP-noises.

A. Additional Experiments

A.1. Comparison between Algorithm 2 with MA-DSBO and GBDSBO in the absence of DP-Noise

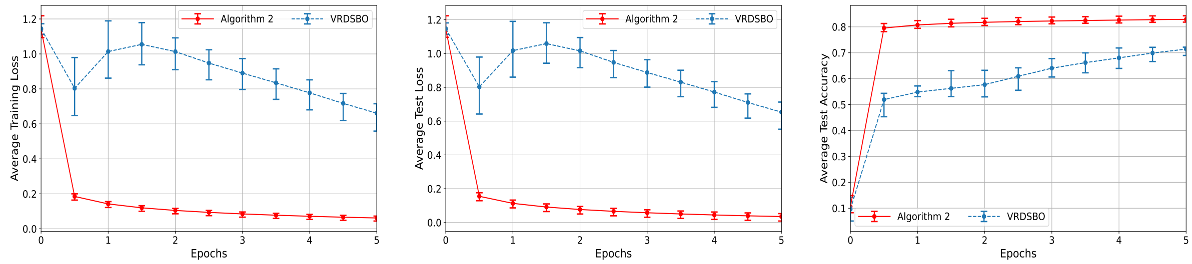
To further assess the performance of our Algorithm 2 in the absence of DP-noise, we conducted additional experiments to compare Algorithm 2 with MA-DSBO and GBDSBO using both synthetic data and real-world data. In the synthetic-data experiment, we chose the stepsizes for our Algorithm 2 as $\lambda_{x,t} = \frac{0.05}{(t+1)^{0.55}}$, $\lambda_{y,t} = \frac{0.05}{(t+1)^{0.5}}$, and $\lambda_{z,t} = \frac{0.02}{(t+1)^{0.45}}$. The stepsizes for MA-DSBO (Chen et al., 2023) were set to $\alpha = \beta = 0.03$ and $\gamma = 0.01$, and the stepsizes for GBDSBO (Yang et al., 2022) were set to $\alpha = \beta = 0.05$ and $\gamma = 0.02$. Those stepsizes were set in accordance with the guidelines provided in these works. In the "MNIST" experiment, the stepsizes for our Algorithm 2 were set to $\lambda_{x,t} = \frac{1.2}{(t+1)^{0.55}}$, $\lambda_{y,t} = \frac{1.2}{(t+1)^{0.5}}$, and $\lambda_{z,t} = \frac{1.2}{(t+1)^{0.45}}$. The stepsizes for MA-DSBO and GBDSBO were all set to 0.1. For all experiments, the number of nested-loops for both MA-DSBO and GBDSBO was set to 10. This setup corresponds to 10 outer iterations, which is equivalent to 100 iterations used in our algorithm, ensuring a fair comparison.

Figure 5 shows that our Algorithm 2 achieves similar test accuracy to GBDSBO and higher test accuracy than MA-DSBO in the synthetic-data experiment. Figure 6 confirms the advantage of our proposed algorithm in both test accuracy and training loss.

A.2. Comparison between Algorithm 2 with VRDSBO

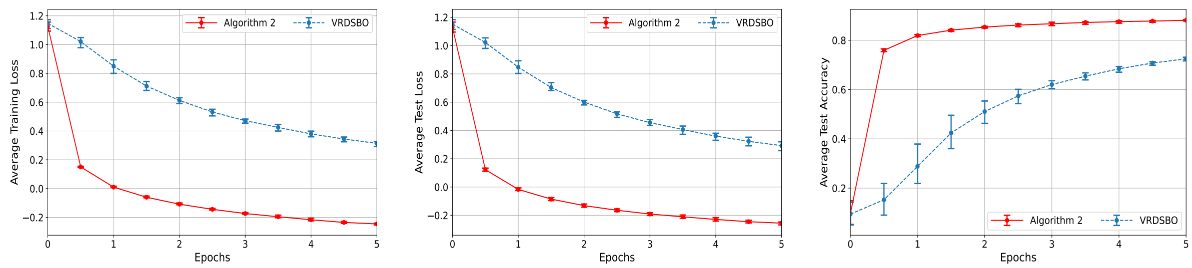
In this subsection, we compared our algorithm with the single-loop algorithm VRDSBO in Gao et al. (2023). While VRDSBO eliminates the need for nested-loops of communication (consensus) iterations, it is not applicable to general DSBO problems because it implicitly assumes homogeneous lower-level functions (a detailed illustration is provided in Appendix C.2 in Chen et al. (2023)). Therefore, we did not include this comparative experiment in the main text.

In the absence of DP-noises, the stepsizes for our Algorithm 2 were set to $\lambda_{x,t} = \frac{1.2}{(t+1)^{0.55}}$, $\lambda_{y,t} = \frac{1.2}{(t+1)^{0.5}}$, and $\lambda_{z,t} = \frac{1.2}{(t+1)^{0.45}}$. For VRDSBO, the stepsizes were set to $\alpha_1 = \alpha_2 = 3$, $\beta_1 = \beta_2 = 1$, and $\eta = 1$. When considering DP-noise, the stepsizes of our Algorithm 2 were set to $\lambda_{x,t} = \frac{1.2}{(t+1)^{0.95}}$, $\lambda_{y,t} = \frac{1.2}{(t+1)^{0.87}}$, and $\lambda_{z,t} = \frac{1.2}{(t+1)^{0.75}}$. The stepsizes for VRDSBO were set to $\alpha_1 = \alpha_2 = 3$, $\beta_1 = \beta_2 = 1$, and $\eta = \frac{1.2}{(t+1)^{0.95}}$ (with η specifically designed to avoid



(a) Training Loss (b) Test Loss (c) Test Accuracy

Figure 7. Comparison Algorithm 2 with VRDSBO by using the "MNIST" dataset under differential-privacy constraints.



(a) Training loss (b) Test loss (c) Test accuracy

Figure 8. Comparison Algorithm 2 with VRDSBO by using the "MNIST" dataset in the absence of DP-noises.

divergent behaviors). The DP-noise variances were the same as those employed in the previous synthetic-data experiment.

Figure 7 and Figure 8 show that under heterogeneous lower-level objective functions, our Algorithm 2 outperforms VRDSBO both in the presence and the absence of differential-privacy constraints.

B. Notations and Auxiliary Lemmas

B.1. Additional Notations

Throughout this paper, we add a bar over a letter to denote the average of all agents and use bold font to represent stacked vectors of m agents. For further notational simplicity, we introduce the following notations:

$$
\begin{array}{lll}
\hat{\boldsymbol{H}}_t = \boldsymbol{H}_t - \mathbf{1}_m \otimes \bar{H}_t, & \hat{\boldsymbol{x}}_t = \boldsymbol{x}_t - \mathbf{1}_m \otimes \bar{x}_t, & \hat{\boldsymbol{y}}_t = \boldsymbol{y}_t - \mathbf{1}_m \otimes \bar{y}_t, \\
\hat{\boldsymbol{z}}_t = \boldsymbol{z}_t - \mathbf{1}_m \otimes \bar{z}_t, & \breve{z}_t = (\nabla^2_{yy} g_t(\bar{x}_t, \bar{y}_t))^{-1} \dfrac{1}{m} \displaystyle\sum_{i=1}^{m} \nabla f_{i,t}(\bar{x}_t, \bar{y}_t), & \hat{\boldsymbol{u}}_t = \boldsymbol{u}_t - \mathbf{1}_m \otimes \bar{u}_t, \\
\chi_{wi,t} = \displaystyle\sum_{j \in \mathcal{N}_i} w_{ij} \chi_{i,t}, & \zeta_{wi,t} = \displaystyle\sum_{j \in \mathcal{N}_i} w_{ij} \zeta_{i,t}, & \vartheta_{wi,t} = \displaystyle\sum_{j \in \mathcal{N}_i} w_{ij} \vartheta_{i,t}, \\
\hat{\boldsymbol{\chi}}_t = \boldsymbol{\chi}_t - \mathbf{1}_m \otimes \bar{\chi}_t, & \hat{\boldsymbol{\zeta}}_t = \boldsymbol{\zeta}_t - \mathbf{1}_m \otimes \bar{\zeta}_t, & \hat{\boldsymbol{\vartheta}}_t = \boldsymbol{\vartheta}_t - \mathbf{1}_m \otimes \bar{\vartheta}_t, \\
\sigma_x^+ = \max\limits_{i \in [m]} \{\sigma_{i,x}\}, & \sigma_y^+ = \max\limits_{i \in [m]} \{\sigma_{i,y}\}, & \sigma_z^+ = \max\limits_{i \in [m]} \{\sigma_{i,z}\}, \\
\varsigma_x = \min\limits_{i \in [m]} \{\varsigma_{i,x}\}, & \varsigma_y = \min\limits_{i \in [m]} \{\varsigma_{i,y}\}, & \varsigma_z = \min\limits_{i \in [m]} \{\varsigma_{i,z}\}, \\
\sigma_{x,t} = \dfrac{\sigma_x^+}{(t+1)^{\varsigma_x}}, & \sigma_{y,t} = \dfrac{\sigma_y^+}{(t+1)^{\varsigma_y}}, & \sigma_{z,t} = \dfrac{\sigma_z^+}{(t+1)^{\varsigma_z}}.
\end{array}
$$

B.2. Auxiliary Lemmas

In this subsection, we introduce some well-known results from the existing literature, along with auxiliary lemmas that will be used in our subsequent convergence analysis.

Lemma B.1. *(Ghadimi & Wang, 2018; Chen et al., 2023) Under Assumption 2.2, $\nabla F(x)$ defined in (1) is L_F-Lipschitz continuous, i.e., for any given $x_1, x_2 \in \mathbb{R}^p$, we have*

$$\|\nabla F(x_1) - \nabla F(x_2)\| \leq L_F \|x_1 - x_2\|, \tag{13}$$

where the Lipschitz constant L_F is given by $L_F = L_{f,1} + \frac{2L_{f,1}L_{g,1}+L_{g,2}L_{f,0}^2}{\mu_g} + \frac{2L_{g,1}L_{f,0}L_{g,2}+L_{g,1}^2L_{f,1}}{\mu_g^2} + \frac{L_{g,2}L_{g,1}^2L_{f,0}}{\mu_g^3}$.

Lemma B.2. *(Wang & Nedić, 2023) Let $\{v_t\}$ be a nonnegative sequence, and $\{a_t\}$ and $\{b_t\}$ be positive sequence satisfying $a_0 < 1$, $\lim_{t\to\infty} a_t = 0$, $\sum_{t=0}^{\infty} a_t = \infty$, and $\lim_{t\to\infty} \frac{b_t}{a_t} = 0$. If $v_{t+1} \leq (1-a_t)v_t + b_t$ holds for all $t > 0$, then we always have $v_t \leq C\frac{b_t}{a_t}$ for all $t > 0$, where C is some positive constant.*

Lemma B.3. *For any given pairs $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, we introduce an auxiliary function $l(x, y; \xi) : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}$ with a random variable ξ. If $\mathbb{E}_\xi [l(x, y; \xi)]$ is L-Lipschitz continuous and $\nabla l(x, y; \xi)$ is unbiased with a bounded variance σ^2, then for any given pairs (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}^p \times \mathbb{R}^q$, the following inequality always holds:*

$$\mathbb{E}_\xi \left[\|l(x_1, y_1; \xi) - l(x_2, y_2; \xi)\|^2\right] \leq 4(L^2 + \sigma^2)(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2). \tag{14}$$

Proof. The mean value theorem implies that there must exist some constant $r \in (0, 1)$ such that for any $x_r = rx_1 + (1-r)x_2$ and $y_r = ry_1 + (1-r)y_2$, the following inequality holds:

$$\begin{aligned}
&\mathbb{E}\left[\|l(x_1, y_1; \xi) - l(x_2, y_2; \xi)\|^2\right] = \mathbb{E}\left[\left(\langle \nabla_x l(x_r, y_r; \xi), x_1 - x_2\rangle + \langle \nabla_y l(x_r, y_r; \xi), y_1 - y_2\rangle\right)^2\right] \\
&\leq 2\mathbb{E}\left[\|\nabla_x l(x_r, y_r; \xi)\|^2\right] \|x_1 - x_2\|^2 + 2\mathbb{E}\left[\|\nabla_y l(x_r, y_r; \xi)\|^2\right] \|y_1 - y_2\|^2.
\end{aligned}$$

Since both terms $\mathbb{E}[\|\nabla_x l(x_r, y_r; \xi)\|^2]$ and $\mathbb{E}[\|\nabla_y l(x_r, y_r; \xi)\|^2]$ are no larger than $\mathbb{E}[\|\nabla l(x_r, y_r; \xi)\|^2]$, we can arrive at (14) based on the relationship $\mathbb{E}[\|\nabla l(x_r, y_r; \xi)\|^2] \leq 2L^2 + 2\sigma^2$. □

C. Empirical Risk Minimization Problems and Useful Properties of Empirical Functions

C.1. Empirical Risk Minimization Problem with respect to Problem (1)

We introduce the following ERM problem to approximate problem (1) under sequentially arriving data:

$$\begin{aligned}
&\min_{x\in\mathbb{R}^p} F_t(x), \quad F_t(x) = \frac{1}{m}\sum_{i=1}^{m} f_{i,t}(x, y_t^*(x)), \\
&\text{s.t.} \quad y_t^*(x) = \text{argmin}_{y\in\mathbb{R}^q} g_t(x, y) := \frac{1}{m}\sum_{i=1}^{m} g_{i,t}(x, y),
\end{aligned} \tag{15}$$

for any $t \geq 0$, where empirical functions $f_{i,t}$ and $g_{i,t}$ are given by $f_{i,t}(x, y) = \frac{1}{t+1}\sum_{k=0}^{t} h(x, y; \varphi_{i,k})$ and $g_{i,t}(x, y) = \frac{1}{t+1}\sum_{k=0}^{t} l(x, y; \xi_{i,k})$, respectively.

In the following lemmas, we present the useful properties of empirical functions $F_t(x)$ and $g_t(x, y)$.

Lemma C.1 proves the boundedness properties of $F_t(x)$ and $g_t(x, y)$.

Lemma C.1. *Under Assumptions 2.2 and 2.3, for any given pair $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, the following inequalities hold:*

$$\begin{aligned}
&\mathbb{E}\left[\|\nabla_y F_t(x)\|^2\right] \leq 2\sigma_{f,1}^2 + 2L_{f,0}^2, \quad \mathbb{E}[\|\nabla_{yy}^2 g_t(x, y)\|^2] \leq 2\sigma_{g,2}^2 + 2L_{g,1}^2, \\
&\mathbb{E}[\|\nabla_{xy}^2 g_t(x, y)\|^2] \leq 2\sigma_{g,2}^2 + 2L_{g,1}^2, \quad \mathbb{E}[\|\nabla_{yy}^2 g_t(x, y)\|^2] \geq \mu_g^2.
\end{aligned} \tag{16}$$

Proof. By using the definition of F_t, Assumption 2.2, and Assumption 2.3, we have

$$\begin{aligned}\mathbb{E}\left[\|\nabla_y F_t(x)\|^2\right] &\le \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[\left\|\frac{1}{t+1}\sum_{k=0}^{t}\nabla_y h(x,y;\varphi_{i,k}) - \nabla_y f_i(x,y) + \nabla_y f_i(x,y)\right\|^2\right] \\ &\le \frac{2\sigma_{f,1}^2}{t+1} + \frac{2}{m}\sum_{i=1}^{m}\|\nabla_y f_i(x,y)\|^2 \le \frac{2\sigma_{f,1}^2}{t+1} + 2L_{f,0}^2 \le 2\sigma_{f,1}^2 + 2L_{f,0}^2.\end{aligned}$$

Similarly, based on the definition of g_t, Assumption 2.2, and Assumption 2.3, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\nabla_{yy}^2 g_t(x,y)\|^2\right] &\le \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[\left\|\frac{1}{t+1}\sum_{k=0}^{t}\nabla_{yy}^2 l(x,y;\xi_{i,k}) - \nabla_{yy}^2 g_i(x,y) + \nabla_{yy}^2 g_i(x,y)\right\|^2\right] \\ &\le \frac{2\sigma_{g,2}^2}{t+1} + \frac{2}{m}\sum_{i=1}^{m}\|\nabla_{yy}^2 g_i(x,y)\|^2 \le \frac{2\sigma_{g,2}^2}{t+1} + 2L_{g,1}^2 \le 2\sigma_{g,2}^2 + 2L_{g,1}^2,\end{aligned}$$

and the following inequality:

$$\mathbb{E}\left[\|\nabla_{xy}^2 g_t(x,y)\|^2\right] \le \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[\left\|\frac{1}{t+1}\sum_{k=0}^{t}\nabla_{xy}^2 l(x,y;\xi_{i,k}) - \nabla_{xy}^2 g_i(x,y) + \nabla_{xy}^2 g_i(x,y)\right\|^2\right] \le 2\sigma_{g,2}^2 + 2L_{g,1}^2.$$

The μ_g-strongly convexity of lower-level functions g_i in Assumption 2.2 implies

$$\mathbb{E}\left[\nabla_{yy}^2 g_t(x,y)\right] = \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[\frac{1}{t+1}\sum_{k=0}^{t}\nabla_{yy}^2 l(x,y;\xi_{i,k}) - \nabla_{yy}^2 g_i(x,y) + \nabla_{yy}^2 g_i(x,y)\right] = \nabla_{yy}^2 g(x,y) \ge \mu_g I_q,$$

which implies the last inequality in (16). □

By using Lemma B.3, we establish Lemma C.2 for Lipschitz continuity of functions $F_t(x)$ and $g_t(x,y)$.

Lemma C.2. *Under Assumptions 2.2 and 2.3, we have the following statements:*

(i) For any given pairs $(x_1,y_1) \in \mathbb{R}^p \times \mathbb{R}^q$ and $(x_2,y_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and any $t > 0$, we have

$$\mathbb{E}\left[\|\nabla_y F_t(x_2) - \nabla_y F_t(x_1)\|^2\right] \le 4(L_{f,1}^2 + \sigma_{f,2}^2)\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right). \tag{17}$$

(ii) For any given pairs $(x_1,y_1) \in \mathbb{R}^p \times \mathbb{R}^q$ and $(x_2,y_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and any $t > 0$, we obtain

$$\mathbb{E}\left[\left\|\left(\nabla_{yy}^2 g_t(x_2,y_2)\right)^{-1} - \left(\nabla_{yy}^2 g_t(x_1,y_1)\right)^{-1}\right\|^2\right] \le \frac{4(L_{g,2}^2 + \sigma_{g,3}^2)}{\mu_g^4}\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right), \tag{18}$$

$$\mathbb{E}\left[\left\|\nabla_y^2 g_t(x_2,y_2) - \nabla_y^2 g_t(x_1,y_1)\right\|^2\right] \le 4(L_{g,1}^2 + \sigma_{g,2}^2)\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right). \tag{19}$$

Proof. (i) By using the definition of F_t and Lemma B.3, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\nabla_y F_t(x_2) - \nabla_y F_t(x_1)\|^2\right] &\le \frac{1}{m}\sum_{i=1}^{m}\frac{1}{t+1}\sum_{k=0}^{t}\mathbb{E}\left[\|\nabla_y h(x_2,y_2;\varphi_{i,k}) - \nabla_y h(x_1,y_1;\varphi_{i,k})\|^2\right] \\ &\le 4(L_{f,1}^2 + \sigma_{f,2}^2)\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right),\end{aligned}$$

where we have used $\nabla_y f_i(x,y) = \mathbb{E}\left[\nabla_y h(x,y;\varphi_{i,k})\right]$, $L_{f,1}$-Lipschitz continuity of $\nabla_y f_i(x,y)$, and the bounded variance $\sigma_{f,2}^2$ of $\nabla^2 h(x,y;\varphi_{i,k})$ in the last inequality.

(ii) According to the definition of g_t, we use Lemma C.1 and Lemma B.3 to obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\left(\nabla^2_{yy}g_t(x_2,y_2)\right)^{-1}-\left(\nabla^2_{yy}g_t(x_1,y_1)\right)^{-1}\right\|^2\right]&\leq\frac{\mathbb{E}\left[\left\|\nabla^2_{yy}g_t(x_2,y_2)-\nabla^2_{yy}g_t(x_1,y_1)\right\|^2\right]}{\mu_g^4}\\&\leq\frac{4(L_{g,2}^2+\sigma_{g,3}^2)}{\mu_g^4}\left(\|x_2-x_1\|^2+\|y_2-y_1\|^2\right),\end{aligned}$$

where in the derivation we have used the following inequality from the proof of Lemma 2.2 in Ghadimi & Wang (2018) for any symmetrical matrices $A_1\in\mathbb{R}^{q\times q}$ and $A_2\in\mathbb{R}^{q\times q}$ satisfying $A_1\geq\mu_g I$ and $A_2\geq\mu_g I$:

$$\|A_1^{-1}-A_2^{-1}\|=\|A_1^{-1}(A_2-A_1)A_2^{-1}\|\leq\|A_1^{-1}\|\|A_2^{-1}\|\|A_2-A_1\|\leq\frac{\|A_2-A_1\|}{\mu_g^2}. \tag{20}$$

Additionally, using an argument similar to the derivation of (17), we arrive at (19). □

Lemma C.3 establishes the variations of functions $\nabla_y F_{t+1}(x)$ and $\nabla_{yy}g_t(x,y)$ over iterations.

Lemma C.3. *Under Assumptions 2.2 and 2.3, for any given pairs (x,y) and any $t>0$, the following inequalities hold:*

$$\mathbb{E}\left[\|\nabla_y F_{t+1}(x)-\nabla_y F_t(x)\|^2\right]\leq\frac{8(\sigma_{f,1}^2+L_{f,0}^2)}{(t+2)^2}\quad\textit{and}\quad\mathbb{E}\left[\|\nabla_{yy}g_{t+1}(x,y)-\nabla_{yy}g_t(x,y)\|^2\right]\leq\frac{8(\sigma_{g,2}^2+L_{g,1}^2)}{(t+2)^2}. \tag{21}$$

Proof. We estimate an upper bound on $\mathbb{E}\left[\|\nabla_y F_{t+1}(x)-\nabla_y F_t(x)\|^2\right]$ by using the definition of F_t:

$$\begin{aligned}&\mathbb{E}\left[\|\nabla_y F_{t+1}(x)-\nabla_y F_t(x)\|^2\right]\\&\leq\frac{1}{m}\sum_{i=1}^m\mathbb{E}\left[\left\|\frac{1}{t+2}\nabla_y h(x,y;\varphi_{i,t+1})+\frac{1}{t+2}\sum_{k=0}^t\nabla_y h(x,y;\varphi_{i,k})-\frac{1}{t+1}\sum_{k=0}^t\nabla_y h(x,y;\varphi_{i,k})\right\|^2\right]\\&\leq\frac{2}{m(t+2)^2}\sum_{i=1}^m\mathbb{E}\left[\|\nabla_y h(x,y;\varphi_{i,t+1})\|^2\right]+\frac{2}{m}\sum_{i=1}^m\left(\frac{1}{(t+2)(t+1)}\right)^2\mathbb{E}\left[\left\|\sum_{k=0}^t\nabla_y h(x,y;\varphi_{i,k})\right\|^2\right].\end{aligned} \tag{22}$$

The first term on the right hand side of (22) satisfies

$$\mathbb{E}\left[\|\nabla_y h(x,y;\varphi_{i,t+1})\|^2\right]\leq\mathbb{E}\left[2\left\|\nabla_y h(x,y;\varphi_{i,t+1})-\nabla_y f_i(x,y)\right\|^2+2\left\|\nabla_y f_i(x,y)\right\|^2\right]\leq 2\sigma_{f,1}^2+2L_{f,0}^2. \tag{23}$$

The second term on the right hand side of (22) satisfies

$$\mathbb{E}\left[\left\|\sum_{k=0}^t\nabla_y h(x,y;\varphi_{i,k})\right\|^2\right]\leq(t+1)\sum_{k=0}^t\mathbb{E}\left[\|\nabla_y h(x,y;\varphi_{i,k})\|^2\right]\leq 2(t+1)^2(\sigma_{f,1}^2+L_{f,0}^2), \tag{24}$$

where we have used $(a_1+\cdots+a_n)^2\leq n(a_1^2+\cdots+a_n^2)$ in the first inequality and (23) in the last inequality.

After substituting (23) and (24) into (22), we arrive at the first term in (21). Furthermore, by employing an argument similar to the derivation of the first term in (21), we can obtain the second term in (21). □

Lemma C.4 quantifies the distance between the optimal solution $y_t^*(x)$ to the lower-level ERM problem in (15) and the true optimal solution $y^*(x)$ to the lower-level optimization problem in (1):

Lemma C.4. *Under Assumptions 2.2 and 2.3, for any given $x\in\mathbb{R}^p$ and any $t>0$, we have*

$$\mathbb{E}\left[\|y_t^*(x)-y^*(x)\|^2\right]\leq\frac{4\sigma_{g,1}^2}{\mu_g^2(t+1)}. \tag{25}$$

Proof. We introduce the auxiliary functions $\bar{g}_{x,t}(y) = g_t(x,y)$ and $\bar{g}_x(y) = g(x,y)$, each with its optimal solution denoted as $y_t^* = \operatorname{argmin}_{y\in\mathbb{R}^q}\bar{g}_{x,t}(y)$ and $y^* = \operatorname{argmin}_{y\in\mathbb{R}^q}\bar{g}_x(y)$, respectively. For any given $x \in \mathbb{R}^p$, at time t, it follows that $y_t^* = y_t^*(x)$ and $y^* = y^*(x)$.

Given the definition of y_t^*, we obtain $\bar{g}_{x,t}(y_t^*) \le \bar{g}_{x,t}(y^*)$, which further implies

$$\bar{g}_x(y_t^*) - \bar{g}_x(y^*) \le (\bar{g}_x(y_t^*) - \bar{g}_{x,t}(y_t^*)) - (\bar{g}_x(y^*) - \bar{g}_{x,t}(y^*)). \tag{26}$$

By applying the mean value theorem to (26), we have

$$\bar{g}_x(y_t^*) - \bar{g}_x(y^*) \le \langle \nabla_y\bar{g}_x(\theta) - \nabla_y\bar{g}_{x,t}(\theta), y_t^* - y^*\rangle \le \|\nabla_y\bar{g}_x(\theta) - \nabla_y\bar{g}_{x,t}(\theta)\|\|y_t^* - y^*\|, \tag{27}$$

where the variable θ is given by $\theta = ry_t^* + (1-r)y^*$ with some constant $r \in (0,1)$.

The definition $\nabla_y\bar{g}_x(\theta) = \frac{1}{m}\sum_{i=1}^m \mathbb{E}[\nabla_y l(x,\theta;\xi_i)]$ implies

$$\begin{aligned}\mathbb{E}\left[\|\nabla_y\bar{g}_{x,t}(\theta) - \nabla_y\bar{g}_x(\theta)\|\right] &= \mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m \nabla_y g_{i,t}(x,\theta) - \nabla_y g(x,\theta)\right\|\right] \\ &\le \frac{1}{m}\sum_{i=1}^m \frac{1}{t+1}\sum_{k=0}^t \mathbb{E}\left[\|\nabla_y l(x,\theta;\xi_{i,k}) - \mathbb{E}\left[\nabla_y l(x,\theta;\xi_{i,k})\right]\|\right].\end{aligned} \tag{28}$$

Considering that the data points $\xi_{i,k}$ are independently and identically distributed across iterations, we use Assumption 2.3 and the Lyapunov inequality $E[\|X\|] \le (E[\|X\|^p])^{\frac{1}{p}}$, $\forall p \ge 1$ to obtain

$$\begin{aligned}\sum_{k=0}^t \mathbb{E}\left[\|\nabla_y l(x,\theta;\xi_{i,k}) - \mathbb{E}\left[\nabla_y l(x,\theta;\xi_{i,k})\right]\|\right] &\le \sqrt{\mathbb{E}\left[\left(\sum_{k=0}^t \|\nabla_y l(x,\theta;\xi_{i,k}) - \mathbb{E}\left[\nabla_y l(x,\theta;\xi_{i,k})\right]\|\right)^2\right]} \\ &\le \sqrt{\mathbb{E}\left[\sum_{k=0}^t \|\nabla_y l(x,\theta;\xi_{i,k}) - \nabla_y g_i(x,\theta)\|^2\right]} \le \sigma_{g,1}\sqrt{t+1}.\end{aligned} \tag{29}$$

Substituting (29) into (28) yields $\mathbb{E}\left[\|\nabla_y\bar{g}_{x,t}(\theta) - \nabla_y\bar{g}_x(\theta)\|\right] \le \frac{\sigma_{g,1}}{\sqrt{t+1}}$. Further combing this relation with (27) leads to

$$\mathbb{E}\left[\|\bar{g}_x(y_t^*) - \bar{g}_x(y^*)\|\right] \le \frac{\sigma_{g,1}}{\sqrt{t+1}}\mathbb{E}\left[\|y_t^* - y^*\|\right]. \tag{30}$$

The μ_g-strongly convex of g_i implies $\frac{\mu_g}{2}\|y_t^* - y^*\|^2 \le \bar{g}_x(y_t^*) - \bar{g}_x(y^*)$. By combing this relation with (30), we have

$$\frac{\mu_g}{2}\mathbb{E}\left[\|y_t^* - y^*\|^2\right] \le \frac{\sigma_{g,1}}{\sqrt{t+1}}\mathbb{E}\left[\|y_t^* - y^*\|\right], \tag{31}$$

which implies $\mathbb{E}[\|y_t^* - y^*\|] \le \frac{2\sigma_{g,1}}{\mu_g\sqrt{t+1}}$. Substituting this inequality into (31), we obtain $\mathbb{E}[\|y_t^* - y^*\|^2] \le \frac{4\sigma_{g,1}^2}{\mu_g^2(t+1)}$. Recalling relationships $y_t^* = y_t^*(x)$ and $y^* = y^*(x)$ for any given $x \in \mathbb{R}^p$, at time t, we arrive at (25). $\square$

Remark C.5. Since $\nabla_y g(x, y^*(x)) = 0$ is valid for any given $x \in \mathbb{R}^p$, it follows from Lemma C.4 that

$$\mathbb{E}\left[\|\nabla_y g(x,y_t^*(x))\|^2\right] = \mathbb{E}\left[\|\nabla_y g(x,y_t^*(x)) - \nabla_y g(x,y^*(x))\|^2\right] \le L_{g,1}^2\mathbb{E}\left[\|y_t^*(x) - y^*(x)\|^2\right] \le \frac{4L_{g,1}^2\sigma_{g,1}^2}{\mu_g^2(t+1)}. \tag{32}$$

We would like to point out that the relation (32) is a key to circumventing the assumption of Lipschitz continuity of the lower-level objective function $g(x,y)$ with respect to y, which are used in existing DSBO results (see Assumption 2.1 in Chen et al. (2022) and Assumption 3.4(iv) in Yang et al. (2022).)

Furthermore, we define $y_i^*(x) = \operatorname{argmin}_{y\in\mathbb{R}^q} g_i(x,y)$ for any given $x \in \mathbb{R}^p$. By using an argument similar to the derivation of (25), we can obtain

$$\mathbb{E}\left[\|\nabla_y g_i(x,y_t^*(x))\|^2\right] = \mathbb{E}\left[\|\nabla_y g_i(x,y_t^*(x)) - \nabla_y g_i(x,y_i^*(x))\|^2\right] \le L_{g,1}^2\mathbb{E}\left[\|y_t^*(x) - y_i^*(x)\|^2\right] \le \frac{4L_{g,1}^2\sigma_{g,1}^2}{\mu_g^2(t+1)}. \tag{33}$$

In Lemma C.6, we quantify the variation of $y_t^*(x)$ over iteration t.

Lemma C.6. *Under Assumptions 2.2 and 2.3, for any given $x \in \mathbb{R}^p$, the following inequality always holds:*

$$\mathbb{E}\left[\|y_{t+1}^*(x) - y_t^*(x)\|^2\right] \leq \frac{2\sigma_{g,1}^2(\mu_g^2 + 4L_{g,1}^2)}{\mu_g^4(t+1)^2}. \tag{34}$$

Proof. For any given $x \in \mathbb{R}^p$, the definition of $y_t^*(x)$ implies $\nabla_y g_t(x, y_t^*(x)) = 0$, which further implies

$$\nabla_{yx}^2 g_t(x, y_t^*(x)) + \nabla_{yy}^2 g_t(x, y_t^*(x)) \nabla_x y_t^*(x) = 0 \quad \text{or} \quad \nabla_x y_t^*(x) = -\left(\nabla_{yy}^2 g_t(x, y_t^*(x))\right)^{-1} \nabla_{yx}^2 g_t(x, y_t^*(x)). \tag{35}$$

Taking the squared norm and expectation on both sides of (35), we obtain the following inequality based on Lemma C.1:

$$\mathbb{E}\left[\|\nabla_x y_t^*(x)\|^2\right] \leq \frac{2\sigma_{g,2}^2 + 2L_{g,1}^2}{\mu_g^2}. \tag{36}$$

The differential mean value theorem implies Lipschitz continuity of $y_t^*(x)$:

$$\mathbb{E}\left[\|y_t^*(x_2) - y_t^*(x_1)\|^2\right] \leq \frac{2\sigma_{g,2}^2 + 2L_{g,1}^2}{\mu_g^2}\|x_2 - x_1\|^2. \tag{37}$$

We proceed to estimate an upper bound on $\mathbb{E}\left[\|y_{t+1}^*(x) - y_t^*(x)\|\right]$.

For any given $x \in \mathbb{R}^p$, we define an auxiliary function $g_{x,t}(y) \triangleq \frac{1}{m}\sum_{i=1}^m l(x, y; \xi_{i,t})$. Considering the definition of $g_t(x, y)$, we obtain the relation $g_t(x, y) = \frac{1}{t+1}\sum_{k=0}^t g_{x,k}(y)$, which further implies the following two inequalities based on $y_t^*(x) = \text{argmin}_{y \in \mathbb{R}^q} g_t(x, y)$:

$$\sum_{k=0}^{t} \nabla_y g_{x,k}(y_t^*(x)) = 0 \quad \text{and} \quad \sum_{k=0}^{t+1} \nabla_y g_{x,k}(y_{t+1}^*(x)) = 0. \tag{38}$$

Given $\sum_{k=0}^{t+1} \nabla_y g_{x,k}(y_{t+1}^*(x)) = \sum_{k=0}^{t} \nabla_y g_{x,k}(y_{t+1}^*(x)) + \nabla_y g_{x,t+1}(y_{t+1}^*(x))$, we use (38) to obtain

$$\begin{aligned}
&\sum_{k=0}^{t} \left\langle y_{t+1}^*(x) - y_t^*(x), \nabla_y g_{x,k}(y_{t+1}^*(x)) - \nabla_y g_{x,k}(y_t^*(x)) \right\rangle \\
&= \left\langle y_{t+1}^*(x) - y_t^*(x), \sum_{k=0}^{t+1} \nabla_y g_{x,k}(y_{t+1}^*(x)) - \nabla_y g_{x,t+1}(y_{t+1}^*(x)) - \sum_{k=0}^{t} \nabla_y g_{x,k}(y_t^*(x)) \right\rangle \\
&= -\left\langle y_{t+1}^*(x) - y_t^*(x), \nabla_y g_{x,t+1}(y_{t+1}^*(x)) \right\rangle.
\end{aligned} \tag{39}$$

Recalling the definition $g_t(x, y) = \frac{1}{t+1}\sum_{k=0}^t g_{x,k}(y)$, Assumptions 2.2, and 2.3, for any given $x \in \mathbb{R}^p$, $y_1 \in \mathbb{R}^q$, and $y_2 \in \mathbb{R}^q$, the following inequality always holds:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{t} \langle y_1 - y_2, \nabla_y g_{x,k}(y_1) - \nabla_y g_{x,k}(y_2) \rangle\right] = (t+1)\mathbb{E}\left[\langle y_1 - y_2, \nabla_y g_t(x, y_1) - \nabla_y g_t(x, y_2) \rangle\right] \\
&= (t+1)\langle y_1 - y_2, \nabla_y g(x, y_1) - \nabla_y g(x, y_2) \rangle \geq \mu_g (t+1)\|y_1 - y_2\|^2,
\end{aligned}$$

which further implies

$$\mathbb{E}\left[\sum_{k=0}^{t} \left\langle y_{t+1}^*(x) - y_t^*(x), \nabla_y g_{x,k}(y_{t+1}^*(x)) - \nabla_y g_{x,k}(y_t^*(x)) \right\rangle\right] \geq \mu_g (t+1)\mathbb{E}\left[\|y_{t+1}^*(x) - y_t^*(x)\|^2\right]. \tag{40}$$

Combing (39) and (40) leads to

$$-\mathbb{E}\left[\left\langle y_{t+1}^*(x) - y_t^*(x), \nabla_y g_{x,t+1}(y_{t+1}^*(x)) \right\rangle\right] \geq (t+1)\mu_g \mathbb{E}\left[\|y_{t+1}^*(x) - y_t^*(x)\|^2\right]. \tag{41}$$

By using Assumption 2.2, Assumption 2.3, and Lemma C.4, we have

$$\mathbb{E}\left[\|\nabla_y g_{x,t+1}(y^*_{t+1}(x))\|^2\right] = \mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^{m}\nabla_y l(x, y^*_{t+1}(x); \xi_{i,t+1}) - \nabla_y g(x, y^*_{t+1}(x)) + \nabla_y g(x, y^*_{t+1}(x))\right\|^2\right]$$

$$\leq 2\sigma_{g,1}^2 + 2\mathbb{E}\left[\left\|\nabla_y g(x, y^*_{t+1}(x)) - \nabla_y g(x, y^*(x))\right\|^2\right] \leq 2\sigma_{g,1}^2 + \frac{8L_{g,1}^2\sigma_{g,1}^2}{\mu_g^2(t+1)},$$

which implies $\mathbb{E}\left[\|\nabla_y g_{x,t+1}(y^*_{t+1}(x))\|\right] \leq \sigma_{g,1}\sqrt{2 + \frac{8L_{g,1}^2}{\mu_g^2}}$. Further combing this inequality and (41), we arrive at

$$\sigma_{g,1}\sqrt{2 + \frac{8L_{g,1}^2}{\mu_g^2}}\mathbb{E}\left[\|y^*_{t+1}(x) - y^*_t(x)\|\right] \geq (t+1)\mu_g\mathbb{E}\left[\|y^*_{t+1}(x) - y^*_t(x)\|^2\right], \tag{42}$$

which implies (34) in Lemma C.6. $\square$

C.2. Empirical Risk Minimization Problem with respect to Problem (8)

We introduce the following ERM problem to approximate problem (8) under sequentially arriving data:

$$\min_{z\in\mathbb{R}^q}\frac{1}{m}\sum_{i=1}^{m}\phi_{i,t}(z), \quad \phi_{i,t}(z) = \frac{1}{2}z^T H_{i,t}z - b_{i,t}^T z, \tag{43}$$

where $H_{i,t}$ and $b_{i,t}$ are given by $H_{i,t} = \nabla^2_{yy}g_{i,t}(x_{i,t}, y_{i,t})$ and $b_{i,t} = \nabla_y f_{i,t}(x_{i,t}, y_{i,t})$.

Considering the optimality conditions of (8) and (43), for any given $x \in \mathbb{R}^p$ and any $t > 0$, the optimal solution z^* to problem (8) and the optimal solution z^*_t to problem (43) satisfy the following relationships, respectively:

$$z^* = \left(\nabla^2_{yy}g(x, y^*(x))\right)^{-1}\nabla_y F(x, y^*(x)) \quad \text{and} \quad z^*_t = \left(\nabla^2_{yy}g_t(x, y^*(x))\right)^{-1}\nabla_y F_t(x, y^*(x)). \tag{44}$$

In the following lemma, we quantify the distance between z_t and z^*:

Lemma C.7. *For any given $x \in \mathbb{R}^p$, we denote z^*_t as the optimal solution to problem (43) at time t and z^* as the optimal solution to the original problem (8). Under Assumptions 2.2 and 2.3, we have*

$$\mathbb{E}\left[\|z^*_t - z^*\|^2\right] \leq \left(\frac{2\sigma_{f,1}^2}{\mu_g^2} + \frac{2L_{f,0}^2\sigma_{g,2}^2}{\mu_g^4}\right)\frac{1}{t+1}. \tag{45}$$

Proof. By using (44), (20), Assumption 2.2, Assumption 2.3, and Lemma C.1, we arrive at

$$\mathbb{E}\left[\|z^*_t - z^*\|^2\right] \leq 2\mathbb{E}\left[\left\|\left(\nabla^2_{yy}g_t(x, y^*(x))\right)^{-1}\right\|^2\|\nabla_y F_t(x, y^*(x)) - \nabla_y F(x, y^*(x))\|^2\right]$$

$$+ 2\mathbb{E}\left[\left\|\left(\nabla^2_{yy}g_t(x, y^*(x))\right)^{-1} - \left(\nabla^2_{yy}g(x, y^*(x))\right)^{-1}\right\|^2\|\nabla_y F(x, y^*(x))\|^2\right]$$

$$\leq 2\mathbb{E}\left[\left\|\left(\nabla^2_{yy}g_t(x, y^*(x))\right)^{-1}\right\|^2\left\|\frac{1}{m}\sum_{i=1}^{m}\frac{1}{t+1}\sum_{k=0}^{t}\nabla_y h(x, y^*(x); \varphi_{i,k}) - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[\nabla_y h(x, y^*(x); \varphi_i)\right]\right\|^2\right]$$

$$+ 2\mathbb{E}\left[\frac{\left\|\nabla^2_{yy}g_t(x, y^*(x)) - \nabla^2_{yy}g(x, y^*(x))\right\|^2}{\mu_g^4}\|\nabla_y F(x, y^*(x))\|^2\right]$$

$$\leq \left(\frac{2\sigma_{f,1}^2}{\mu_g^2} + \frac{2L_{f,0}^2\sigma_{g,2}^2}{\mu_g^4}\right)\frac{1}{t+1},$$

where we have used the definition $g_t(x, y^*(x)) = \frac{1}{t+1}\sum_{k=0}^{t}l(x, y^*(x); \xi_{i,k})$ in the last inequality. $\square$

Lemma C.7 demonstrates that the optimal solution z^*_t to the ERM problem (43) converges in mean square to the true optimal solution z^* to problem (8).

D. Results of Algorithm 2

This section is devoted to analyzing the consensus error of the iterative variables generated by Algorithm 2. To this end, several technical lemmas are presented in Subsections D.1-D.10, with their interrelationships depicted in Figure 4.

D.1. Estimation of $\mathbb{E}\left[\|\bar{x}_{t+1}-\bar{x}_t\|^2\right]$ in Lemma D.1 and Its Proof

Recalling Algorithm 2 Step 7: $x_{i,t+1}=x_{i,t}+\sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,t}+\chi_{j,t}-x_{i,t})-\lambda_{x,t}u_{i,t}$, we express the update rule of $\bar{x}_{t+1}$ as follows:

$$\bar{x}_{t+1}=\bar{x}_t+\bar{\chi}_t-\lambda_{x,t}\bar{u}_t \quad \text{with} \quad \bar{u}_t=\frac{1}{m}\sum_{i=1}^{m}\left(\nabla_x f_{i,t}(x_{i,t},y_{i,t})-\nabla^2_{xy}g_{i,t}(x_{i,t},y_{i,t})z_{i,t}\right). \tag{46}$$

Lemma D.1. *Under Assumptions 2.1-2.3 and 3.1, for any $t>0$, we have*

$$\begin{aligned}\mathbb{E}\left[\|\bar{x}_{t+1}-\bar{x}_t\|^2\right]\le{}&\sigma^2_{x,t}+c_{\bar{x}1}\lambda^2_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+c_{\bar{x}2}\lambda^2_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+c_{\bar{x}3}\lambda^2_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]+c_{\bar{x}4}\lambda^2_{x,t}\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]\\&+c_{\bar{x}5}\lambda^2_{x,t}\mathbb{E}\left[\|\bar{y}_t-y^*_t(\bar{x}_t)\|^2\right]+c_{\bar{x}6}\lambda^2_{x,t},\end{aligned} \tag{47}$$

where the constants $c_{\bar{x}1}$ to $c_{\bar{x}6}$ are given by $c_{\bar{x}1}=\frac{36L^2_{f,0}(\sigma^2_{g,2}+L^2_{g,1})}{m\mu_g^2}$, $c_{\bar{x}2}=\frac{18L^2_{f,0}}{m}$, $c_{\bar{x}3}=\frac{12(\sigma^2_{g,2}+L^2_{g,1})}{m}$, $c_{\bar{x}4}=c_{\bar{x}3}m$, $c_{\bar{x}5}=c_{\bar{x}2}m$, and $c_{\bar{x}6}=6(\sigma^2_{f,1}+L^2_{f,0})+\frac{c_{\bar{x}4}L^2_{f,0}}{\mu_g^2}$.

Proof. Considering the definition of $\bar{u}_t$ in (46), we have

$$\begin{aligned}&\mathbb{E}\left[\|\bar{u}_t\|^2\right]=\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[\|\nabla_x f_{i,t}(x_{i,t},y_{i,t})-\nabla^2_{xy}g_{i,t}(x_{i,t},y_{i,t})z_{i,t}\|^2\right]\\&\le\frac{2}{m}\sum_{i=1}^{m}\mathbb{E}\left[\|\nabla_x f_{i,t}(x_{i,t},y_{i,t})\|^2\right]+\frac{2}{m}\sum_{i=1}^{m}\mathbb{E}\left[\|\nabla^2_{xy}g_{i,t}(x_{i,t},y_{i,t})\|^2\|\bar{z}_t\|^2\right],\\&\le\frac{2}{m}\sum_{i=1}^{m}\mathbb{E}\left[\|\nabla_x f_{i,t}(x_{i,t},y_{i,t})-\nabla_x f_i(x_{i,t},y_{i,t})+\nabla_x f_i(x_{i,t},y_{i,t})-\nabla_x f_i(x_{i,t},y^*_t(x_{i,t}))+\nabla_x f_i(x_{i,t},y^*_t(x_{i,t}))\|^2\right]\\&\quad+\frac{2}{m}\sum_{i=1}^{m}\mathbb{E}\left[\|\nabla^2_{xy}g_{i,t}(x_{i,t},y_{i,t})\|^2\|\bar{z}_t\|^2\right]\\&\le\frac{6\sigma^2_{f,1}}{t+1}+\frac{6}{m}\sum_{i=1}^{m}\mathbb{E}\left[\|\nabla_x f_i(x_{i,t},y_{i,t})-\nabla_x f_i(x_{i,t},y^*_t(x_{i,t}))\|^2\right]+6L^2_{f,0}+\frac{4}{m}\left(\sigma^2_{g,2}+L^2_{g,1}\right)\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right]\\&\le\frac{6\sigma^2_{f,1}}{t+1}+\frac{6L^2_{f,0}}{m}\mathbb{E}\left[\|\boldsymbol{y}_t-\boldsymbol{y}^*_t(\boldsymbol{x})\|^2\right]+6L^2_{f,0}+\frac{4}{m}\left(\sigma^2_{g,2}+L^2_{g,1}\right)\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right],\end{aligned} \tag{48}$$

where $\boldsymbol{y}_t$ and $\boldsymbol{y}^*_t(\boldsymbol{x})$ are given by $\boldsymbol{y}_t=\mathrm{col}(y_{1,t},\cdots,y_{m,t})$ and $\boldsymbol{y}^*_t(\boldsymbol{x})=\mathrm{col}(y^*_t(x_{1,t}),\cdots,y^*_t(x_{m,t}))$.

To further analyze the term $\mathbb{E}\left[\|\boldsymbol{y}_t-\boldsymbol{y}^*_t(\boldsymbol{x})\|^2\right]$ in (48), we use the following decomposition:

$$\begin{aligned}\mathbb{E}\left[\|\boldsymbol{y}_t-\boldsymbol{y}^*_t(\boldsymbol{x})\|^2\right]&\le\mathbb{E}\left[\|\boldsymbol{y}_t-\mathbf{1}_m\otimes\bar{y}_t+\mathbf{1}_m\otimes\bar{y}_t-\mathbf{1}_m\otimes y^*_t(\bar{x}_t)+\mathbf{1}_m\otimes y^*_t(\bar{x}_t)-\boldsymbol{y}^*_t(\boldsymbol{x})\|^2\right]\\&\le3\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+3m\mathbb{E}\left[\|\bar{y}_t-y^*_t(\bar{x}_t)\|^2\right]+3\sum_{i=1}^{m}\mathbb{E}\left[\|y^*_t(\bar{x}_t)-y^*_t(x_{i,t})\|^2\right]\\&\le3\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+3m\mathbb{E}\left[\|\bar{y}_t-y^*_t(\bar{x}_t)\|^2\right]+\frac{6(\sigma^2_{g,2}+L^2_{g,1})}{\mu_g^2}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right],\end{aligned} \tag{49}$$

with $\hat{\boldsymbol{y}}_t=\boldsymbol{y}_t-\mathbf{1}_m\otimes\bar{y}_t$ and $\hat{\boldsymbol{x}}_t=\boldsymbol{x}_t-\mathbf{1}_m\otimes\bar{x}_t$. In the last inequality, we have used (37).

We now focus on characterizing the term $\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right]$ in (48). Considering that both the first term in (16) from Lemma C.1 and Assumption 2.2 lead to $\mathbb{E}\left[\|\breve{z}_t\|^2\right]=\mathbb{E}\left[\|(\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t,\bar{y}_t)\|^2\right]\le\frac{L^2_{f,0}}{\mu_g^2}$, where $\nabla_y F_t(\bar{x}_t,\bar{y}_t)\triangleq$

$\frac{1}{m}\sum_{i=1}^{m} f_{i,t}(\bar{x}_t, \bar{y}_t)$. We subsequently obtain

$$\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right] = \mathbb{E}\left[\|\hat{\boldsymbol{z}}_t + \mathbf{1}_m \otimes (\bar{z}_t - \check{z}_t) + \mathbf{1}_m \otimes \check{z}_t\|^2\right] \leq 3\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 3m\mathbb{E}\left[\|\bar{z}_t - \check{z}_t\|^2\right] + \frac{3mL_{f,0}^2}{\mu_g^2}, \tag{50}$$

where $\hat{\boldsymbol{z}}_t$ is defined as $\hat{\boldsymbol{z}}_t = \boldsymbol{z}_t - \mathbf{1}_m \otimes \bar{z}_t$.

Substituting (49) and (50) into (48), we arrive at

$$\begin{aligned}\mathbb{E}\left[\|\bar{u}_t\|^2\right] \leq{}& \frac{6\sigma_{f,1}^2}{t+1} + \frac{36L_{f,0}^2(\sigma_{g,2}^2 + L_{g,1}^2)}{m\mu_g^2}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \frac{18L_{f,0}^2}{m}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + \frac{12\left(\sigma_{g,2}^2 + L_{g,1}^2\right)}{m}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \\ &+ 18L_{f,0}^2\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + 12\left(\sigma_{g,2}^2 + L_{g,1}^2\right)\mathbb{E}\left[\|\bar{z}_t - \check{z}_t\|^2\right] + \frac{12\left(\sigma_{g,2}^2 + L_{g,1}^2\right)L_{f,0}^2}{\mu_g^2} + 6L_{f,0}^2.\end{aligned} \tag{51}$$

Taking the squared norm and expectation on both sides of (46) and then substituting (51) into (46), we arrive at (47). $\square$

D.2. Estimation of $\mathbb{E}\left[\|\bar{y}_{t+1} - \bar{y}_t\|^2\right]$ in Lemma D.2 and Its Proof

Recalling Algorithm 2 Step 4: $y_{i,t+1} = y_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(y_{j,t} + \zeta_{j,t} - y_{i,t}) - \lambda_{y,t}\nabla_y g_{i,t}(x_{i,t}, y_{i,t})$, we express the update rule of $\bar{y}_{t+1}$ as follows:

$$\bar{y}_{t+1} = \bar{y}_t + \bar{\zeta}_t - \lambda_{y,t}\frac{1}{m}\sum_{i=1}^{m}\nabla_y g_{i,t}(x_{i,t}, y_{i,t}). \tag{52}$$

Lemma D.2. *Under Assumptions 2.1-2.3 and 3.1, for any $t > 0$, we have*

$$\mathbb{E}\left[\|\bar{y}_{t+1} - \bar{y}_t\|^2\right] \leq \sigma_{y,t}^2 + c_{\bar{y}1}\lambda_{y,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{\bar{y}2}\lambda_{y,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + c_{\bar{y}3}\lambda_{y,t}^2\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + c_{\bar{y}4}\frac{\lambda_{y,t}^2}{t+1}, \tag{53}$$

with $c_{\bar{y}1} = \frac{24(L_{g,1})^2(\sigma_{g,2}^2 + L_{g,1}^2)}{m\mu_g^2}$, $c_{\bar{y}2} = \frac{12L_{g,1}^2}{m}$, $c_{\bar{y}3} = c_{\bar{y}2}m$, *and* $c_{\bar{y}4} = 2\sigma_{g,1}^2\left(1 + \frac{8L_{g,1}^2}{\mu_g^2}\right)$.

Proof. By taking the squared norm and expectation on both sides of (52), we have

$$\begin{aligned}&\mathbb{E}\left[\|\bar{y}_{t+1} - \bar{y}_t\|^2\right] \leq \mathbb{E}\left[\|\bar{\zeta}_t\|^2\right] + \lambda_{y,t}^2\mathbb{E}\left[\frac{2}{m}\sum_{i=1}^{m}\|\nabla_y g_{i,t}(x_{i,t}, y_{i,t}) - \nabla_y g_i(x_{i,t}, y_{i,t})\|^2 + 2\left\|\frac{1}{m}\sum_{i=1}^{m}\nabla_y g_i(x_{i,t}, y_{i,t})\right\|^2\right] \\ &\leq \sigma_{y,t}^2 + \frac{2\sigma_{g,1}^2\lambda_{y,t}^2}{t+1} + 2\lambda_{y,t}^2\mathbb{E}\left[2\frac{1}{m}\sum_{i=1}^{m}\|\nabla_y g_i(x_{i,t}, y_{i,t}) - \nabla_y g_i(x_{i,t}, y_t^*(x_{i,t}))\|^2 + 2\left\|\frac{1}{m}\sum_{i=1}^{m}\nabla_y g_i(x_{i,t}, y_t^*(x_{i,t}))\right\|^2\right] \\ &\leq \sigma_{y,t}^2 + \frac{2\sigma_{g,1}^2\lambda_{y,t}^2}{t+1} + \frac{4L_{g,1}^2}{m}\lambda_{y,t}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + 4\lambda_{y,t}^2\mathbb{E}\left[\|\nabla_y g(x_{i,t}, y_t^*(x_{i,t})) - \nabla_y g(x_{i,t}, y^*(x_{i,t}))\|^2\right] \\ &\leq \sigma_{y,t}^2 + \frac{4L_{g,1}^2}{m}\lambda_{y,t}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + 2\sigma_{g,1}^2\left(1 + \frac{8L_{g,1}^2}{\mu_g^2}\right)\frac{\lambda_{y,t}^2}{t+1},\end{aligned} \tag{54}$$

where we have used (32) in the last inequality. Further substituting (49) into (54) yields (53). $\square$

D.3. Estimation of $\mathbb{E}\left[\|\check{z}_{t+1} - \check{z}_t\|^2\right]$ in Lemma D.3 and Its Proof

Recalling the definition $\check{z}_t = (\nabla_{yy}^2 g_t(\bar{x}_t, \bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t, \bar{y}_t)$ with $\nabla_y F_t(\bar{x}_t, \bar{y}_t) \triangleq \frac{1}{m}\sum_{i=1}^{m}\nabla f_{i,t}(\bar{x}_t, \bar{y}_t)$, we express $\check{z}_{t+1} - \check{z}_t$ as follows:

$$\check{z}_{t+1} - \check{z}_t = (\nabla_{yy}^2 g_{t+1}(\bar{x}_{t+1}, \bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1}, \bar{y}_{t+1}) - (\nabla_{yy}^2 g_t(\bar{x}_t, \bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t, \bar{y}_t). \tag{55}$$

Lemma D.3. *Under Assumptions 2.2 and 2.3, for any $t > 0$, we have*

$$\mathbb{E}\left[\|\check{z}_{t+1} - \check{z}_t\|^2\right] < c_{\check{z}1}\mathbb{E}\left[\|\bar{x}_{t+1} - \bar{x}_t\|^2\right] + c_{\check{z}1}\mathbb{E}\left[\|\bar{y}_{t+1} - \bar{y}_t\|^2\right] + \frac{c_{\check{z}2}}{(t+2)^2}, \tag{56}$$

with $c_{\check{z}1} = \frac{16(L_{f,1}^2 + \sigma_{f,2}^2)}{\mu_g^2} + \frac{32(L_{g,2}^2 + \sigma_{g,3}^2)(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^4}$ *and* $c_{\check{z}2} = \frac{32(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^2}\left(1 + \frac{2(\sigma_{g,2}^2 + L_{g,1}^2)}{\mu_g^2}\right)$.

Proof. By taking the squared norm and expectation on both sides of (55), we have

$$\begin{aligned}
&\mathbb{E}\left[\|\breve{z}_{t+1}-\breve{z}_t\|^2\right] = \mathbb{E}\left[\left\|(\nabla^2_{yy}g_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t,\bar{y}_t)\right\|^2\right]\\
&\le 4\mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t,\bar{y}_t)\right\|^2\right]\\
&\quad + 4\mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\right\|^2\right]\\
&\quad + 4\mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\right\|^2\right]\\
&\quad + 4\mathbb{E}\left[\left\|(\nabla^2_{yy}g_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1})\right\|^2\right].
\end{aligned} \tag{57}$$

Using both (16) in Lemma C.1 and (17) in Lemma C.2, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t,\bar{y}_t)\right\|^2\right]\\
&\le \mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\right\|^2\right]\mathbb{E}\left[\|\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1}) - \nabla_y F_t(\bar{x}_t,\bar{y}_t)\|^2\right]\\
&\le \frac{4(L_{f,1}^2+\sigma_{f,2}^2)}{\mu_g^2}\left(\mathbb{E}\left[\|\bar{x}_{t+1}-\bar{x}_t\|^2\right]+\mathbb{E}\left[\|\bar{y}_{t+1}-\bar{y}_t\|^2\right]\right).
\end{aligned} \tag{58}$$

Similarly, using (16) in Lemma C.1 and (18) in Lemma C.2, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\right\|^2\right]\\
&\le \mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1} - (\nabla^2_{yy}g_t(\bar{x}_t,\bar{y}_t))^{-1}\right\|^2\right]\mathbb{E}\left[\|\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\|^2\right]\\
&\le \frac{8(L_{g,2}^2+\sigma_{g,3}^2)(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^4}\left(\mathbb{E}\left[\|\bar{x}_{t+1}-\bar{x}_t\|^2\right]+\mathbb{E}\left[\|\bar{y}_{t+1}-\bar{y}_t\|^2\right]\right).
\end{aligned} \tag{59}$$

Using (16) in Lemma C.1 and the first term in (21) of Lemma C.3, one yields

$$\begin{aligned}
&\mathbb{E}\left[\left\|(\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\right\|^2\right]\\
&\le \frac{1}{\mu_g^2}\mathbb{E}\left[\|\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - \nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\|^2\right] \le \frac{8(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2(t+2)^2}.
\end{aligned} \tag{60}$$

Utilizing (20), the results in (16) from Lemma C.1 and the second term in (21) of Lemma C.3, we arrive at

$$\begin{aligned}
&\mathbb{E}\left[\left\|(\nabla^2_{yy}g_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - (\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1})\right\|^2\right]\\
&\le \mathbb{E}\left[\left\|(\nabla^2_{yy}g_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1} - (\nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1}))^{-1}\right\|^2\right]\mathbb{E}\left[\|\nabla_y F_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1})\|^2\right]\\
&\le \frac{1}{\mu_g^4}\mathbb{E}\left[\|\nabla^2_{yy}g_{t+1}(\bar{x}_{t+1},\bar{y}_{t+1}) - \nabla^2_{yy}g_t(\bar{x}_{t+1},\bar{y}_{t+1})\|^2\right]\mathbb{E}\left[\|\nabla_y F_t(\bar{x}_{t+1},\bar{y}_{t+1})\|\right]\\
&\le \frac{16(\sigma_{g,2}^2+L_{g,1}^2)(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^4(t+2)^2}.
\end{aligned} \tag{61}$$

Substituting (58) to (61) into (57), we arrive at (56). □

In the following Subsections D.4-D.7, we quantify the distance between the iterative variables generated by Algorithm 2 and their corresponding average values.

D.4. Estimation of $\mathbb{E}\left[\|\hat{\boldsymbol{u}}_t\|^2\right]$ in Lemma D.4 and Its Proof

Here, we use the definitions $\hat{\boldsymbol{u}}_t = \boldsymbol{u}_t - \mathbf{1}_m \otimes \bar{u}_t$, $\boldsymbol{u}_t = \mathrm{col}(u_{1,t},\cdots,u_{m,t})$, and $\bar{u}_t = \frac{1}{m}\sum_{i=1}^m u_{i,t}$ with $u_{i,t}$ given by

$$u_{i,t} = \nabla_x f_{i,t}(x_{i,t},y_{i,t}) - \nabla^2_{xy}g_{i,t}(x_{i,t},y_{i,t})z_{i,t}. \tag{62}$$

Lemma D.4. *Under Assumptions 2.2 and 2.3, for any $t > 0$, the following inequality always holds:*

$$\mathbb{E}\left[\|\hat{\boldsymbol{u}}_t\|^2\right] \leq c_{\hat{u}1}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{\hat{u}2}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + c_{\hat{u}3}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + c_{\hat{u}4}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + c_{\hat{u}5}\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + c_{\hat{u}6}, \quad (63)$$

where the constants $c_{\hat{u}1}$ to $c_{\hat{u}6}$ are given by $c_{\hat{u}1} = \frac{144L_{f,0}^2(\sigma_{g,2}^2 + L_{g,1}^2)}{\mu_g^2}$, $c_{\hat{u}2} = 72L_{f,0}^2$, $c_{\hat{u}3} = 48(\sigma_{g,2}^2 + L_{g,1}^2)$, $c_{\hat{u}4} = c_{\hat{u}3}m$, $c_{\hat{u}5} = c_{\hat{u}2}m$, and $c_{\hat{u}6} = 24m\sigma_{f,1}^2 + 24mL_{f,0}^2 + \frac{c_{\hat{u}4}L_{f,0}^2}{\mu_g^2}$.

Proof. We first determine an upper bound on $\mathbb{E}\left[\|\boldsymbol{u}_t\|^2\right]$. Based on (62) and Lemma C.1, we have

$$\begin{aligned}
&\mathbb{E}\left[\|\boldsymbol{u}_t\|^2\right] \leq 2\sum_{i=1}^m \mathbb{E}\left[\|\nabla_x f_{i,t}(x_{i,t}, y_{i,t})\|^2 + \|\nabla_{xy}^2 g_{i,t}(x_{i,t}, y_{i,t})\|^2\|z_{i,t}\|^2\right] \\
&\leq 2\sum_{i=1}^m \mathbb{E}\left[\|\nabla_x f_{i,t}(x_{i,t}, y_{i,t}) - \nabla_x f_i(x_{i,t}, y_{i,t}) + \nabla_x f_i(x_{i,t}, y_{i,t})\|^2\right] + 2\sum_{i=1}^m \mathbb{E}\left[\|\nabla_{xy}^2 g_{i,t}(x_{i,t}, y_{i,t})\|^2\|z_{i,t}\|^2\right] \\
&\leq \frac{6m\sigma_{f,1}^2}{t+1} + 6\sum_{i=1}^m \mathbb{E}\left[\|\nabla_x f_i(x_{i,t}, y_{i,t}) - \nabla_x f_i(x_{i,t}, y_t^*(x_{i,t}))\|^2\right] + 6mL_{f,0}^2 + 4\left(\frac{\sigma_{g,2}^2}{t+1} + L_{g,1}^2\right)\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right] \\
&\leq \frac{6m\sigma_{f,1}^2}{t+1} + 6L_{f,0}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + 6mL_{f,0}^2 + 4\left(\sigma_{g,2}^2 + L_{g,1}^2\right)\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right].
\end{aligned} \quad (64)$$

Then, we characterize the term $\mathbb{E}\left[\|\mathbf{1}_m \otimes \bar{u}_t\|^2\right]$. By using (48), we have

$$\mathbb{E}\left[\|\mathbf{1}_m \otimes \bar{u}_t\|^2\right] \leq \frac{6m\sigma_{f,1}^2}{t+1} + 6L_{f,0}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + 6mL_{f,0}^2 + 4\left(\sigma_{g,2}^2 + L_{g,1}^2\right)\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right]. \quad (65)$$

Based on the relation $\|\hat{\boldsymbol{u}}_t\|^2 = 2\|\boldsymbol{u}_t\|^2 + 2\|\mathbf{1}_m \otimes \bar{u}_t\|^2$, by summing up the corresponding sides of (64) and (65), we obtain

$$\mathbb{E}\left[\|\hat{\boldsymbol{u}}_t\|^2\right] \leq \frac{24m\sigma_{f,1}^2}{t+1} + 24L_{f,0}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + 24mL_{f,0}^2 + 16\left(\sigma_{g,2}^2 + L_{g,1}^2\right)\mathbb{E}\left[\|\boldsymbol{z}_t\|^2\right]. \quad (66)$$

Substituting (49) and (50) into (66), we can arrive at (63). □

D.5. Estimation of $\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]$ in Lemma D.5 and Its Proof

Recalling the definitions $\hat{\boldsymbol{x}}_t = \boldsymbol{x}_t - \mathbf{1}_m \otimes \bar{x}_t$, $\boldsymbol{x}_t = \text{col}(x_{1,t}, \cdots, x_{m,t})$, and $\bar{x}_t = \frac{1}{m}\sum_{i=1}^m x_{i,t}$ with $x_{i,t+1} = x_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,t} + \chi_{j,t} - x_{i,t}) - \lambda_{x,t}u_{i,t}$ in Algorithm 2 Step 8, we have

$$\hat{\boldsymbol{x}}_{t+1} = (I + W \otimes I_q)\hat{\boldsymbol{x}}_t + \hat{\boldsymbol{\chi}}_t - \lambda_{x,t}\hat{\boldsymbol{u}}_t. \quad (67)$$

Lemma D.5. *Under Assumptions 2.1-2.3 and 3.1, for any $t > 0$, we have*

$$\begin{aligned}
\mathbb{E}\left[\|\hat{\boldsymbol{x}}_{t+1}\|^2\right] \leq &\left(1 - \frac{\delta_2}{2} + c_{\hat{x}1}\lambda_{x,t}^2\right)\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + 4m\sigma_{x,t}^2 + c_{\hat{x}2}\lambda_{x,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + c_{\hat{x}3}\lambda_{x,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \\
&+ c_{\hat{x}4}\lambda_{x,t}^2\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + c_{\hat{x}5}\lambda_{x,t}^2\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + c_{\hat{x}6}\lambda_{x,t}^2,
\end{aligned} \quad (68)$$

where $c_{\hat{x}1}$ to $c_{\hat{x}6}$ are given by $c_{\hat{x}i} = \left(1 + \frac{2}{\delta_2}\right)c_{\hat{u}i},\ i = \{1, \cdots, 6\}$ with $c_{\hat{u}i}$ given in the statement of Lemma D.4.

Proof. By taking the squared norm and expectation on both sides of (67), we obtain

$$\begin{aligned}
\mathbb{E}\left[\|\hat{\boldsymbol{x}}_{t+1}\|^2\right] &= \|I + W \otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + 4m\sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{u}}_t\|^2\right] - 2\mathbb{E}\left[\langle (I + W \otimes I_q)\hat{\boldsymbol{x}}_t, \lambda_{x,t}\hat{\boldsymbol{u}}_t\rangle\right] \\
&\leq \left(1 - \frac{\delta_2}{2}\right)\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + 4m\sigma_{x,t}^2 + \left(1 + \frac{2}{\delta_2}\right)\lambda_{x,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{u}}_t\|^2\right],
\end{aligned} \quad (69)$$

where in the derivation we have used Assumptions 2.1, Assumption 3.1, and the following inequality:

$$-2\mathbb{E}\left[\langle (I + W \otimes I_q)\hat{\boldsymbol{x}}_t, \lambda_{x,t}\hat{\boldsymbol{u}}_t\rangle\right] \leq \frac{\delta_2}{2}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \frac{2}{\delta_2}\mathbb{E}\left[\|\hat{\boldsymbol{u}}_t\|^2\right].$$

Substituting (63) from Lemma D.4 into (69), we arrive at (68). □

D.6. Estimation of $\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]$ in Lemma D.6 and Its Proof

Recalling the definitions $\hat{\boldsymbol{y}}_t = \boldsymbol{y}_t - \mathbf{1}_m \otimes \bar{y}_t$, $\boldsymbol{y}_t = \mathrm{col}(y_{1,t}, \cdots, y_{m,t})$, and $\bar{y}_t = \frac{1}{m}\sum_{i=1}^m y_{i,t}$ with $y_{i,t+1} = y_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,t} + \zeta_{j,t} - y_{i,t}) - \lambda_{y,t}\nabla_y g_{i,t}(x_{i,t}, y_{i,t})$ given in Algorithm 2 Step 5, we have

$$\hat{\boldsymbol{y}}_{t+1} = (I + W \otimes I_q)\hat{\boldsymbol{y}}_t + \hat{\boldsymbol{\zeta}}_t - \lambda_{y,t}\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t), \tag{70}$$

with $\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t) = \mathrm{col}(\nabla_y \hat{g}_{1,t}, \cdots, \hat{g}_{m,t})$ and $\nabla_y \hat{g}_{i,t} = \nabla_y g_{i,t}(x_{i,t}, y_{i,t}) - \frac{1}{m}\sum_{i=1}^m \nabla_y g_{i,t}(x_{i,t}, y_{i,t})$.

Lemma D.6. *Under Assumptions 2.1- 2.3 and 3.1, for any $t > 0$, the following inequality always holds:*

$$\mathbb{E}\left[\|\hat{\boldsymbol{y}}_{t+1}\|^2\right] \le \left(1 - \frac{\delta_2}{2} + c_{\hat{y}1}\lambda_{y,t}^2\right)\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + 4m\sigma_{y,t}^2 + c_{\hat{y}2}\lambda_{y,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{\hat{y}3}\lambda_{y,t}^2\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + c_{\hat{y}4}\frac{\lambda_{y,t}^2}{t+1}, \tag{71}$$

where the constants $c_{\hat{y}1}$ to $c_{\hat{y}4}$ are given by $c_{\hat{y}1} = 48L_{g,1}^2\left(1 + \frac{2}{\delta_2}\right)$, $c_{\hat{y}2} = \left(1 + \frac{2}{\delta_2}\right)\frac{96(\sigma_{g,2}^2 + L_{g,1}^2)L_{g,1}^2}{\mu_g^2}$, $c_{\hat{y}3} = c_{\hat{y}1}m$, and $c_{\hat{y}4} = 8\sigma_{g,1}^2 m\left(1 + \frac{2}{\delta_2}\right)\left(1 + \frac{8L_{g,1}^2}{\mu_g^2}\right)$.

Proof. By taking the squared norm and expectation on both sides of (70), we obtain

$$\begin{aligned}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_{t+1}\|^2\right] &= \|I + W \otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + 4m\sigma_{y,t}^2 + \lambda_{y,t}^2\mathbb{E}\left[\|\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t)\|^2\right] - 2\mathbb{E}\left[\langle (I + W \otimes I_q)\hat{\boldsymbol{y}}_t, \lambda_{y,t}\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t)\rangle\right] \\ &\le \left(1 - \frac{\delta_2}{2}\right)\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + 4m\sigma_{y,t}^2 + \left(1 + \frac{2}{\delta_2}\right)\lambda_{y,t}^2\mathbb{E}\left[\|\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t)\|^2\right].\end{aligned} \tag{72}$$

We proceed to characterize the term $\mathbb{E}\left[\|\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t)\|^2\right]$ in (72). Considering the definition of $\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t)$, we have

$$\mathbb{E}\left[\|\nabla_y \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t, \boldsymbol{y}_t)\|^2\right] \le 2\sum_{i=1}^m \mathbb{E}\left[\|\nabla_y g_{i,t}(x_{i,t}, y_{i,t})\|^2\right] + 2\sum_{i=1}^m \mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m \nabla_y g_{i,t}(x_{i,t}, y_{i,t})\right\|^2\right]. \tag{73}$$

We first analyze the first term on the right hand side of (73):

$$\begin{aligned}&\sum_{i=1}^m \mathbb{E}\left[\|\nabla_y g_{i,t}(x_{i,t}, y_{i,t})\|^2\right] \le \frac{2m\sigma_{g,1}^2}{t+1} + 2\sum_{i=1}^m \mathbb{E}\left[\|\nabla_y g_i(x_{i,t}, y_{i,t})\|^2\right] \\ &\le \frac{2m\sigma_{g,1}^2}{t+1} + 2\sum_{i=1}^m \mathbb{E}\left[2\|\nabla_y g_i(x_{i,t}, y_{i,t}) - \nabla_y g_i(x_{i,t}, y_t^*(x_{i,t}))\|^2 + 2\|\nabla_y g_i(x_{i,t}, y_t^*(x_{i,t}))\|^2\right] \\ &\le 4L_{g,1}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + \frac{2\sigma_{g,1}^2 m}{t+1}\left(1 + \frac{8L_{g,1}^2}{\mu_g^2}\right),\end{aligned} \tag{74}$$

where we have used (33) in the last inequality. Similarly, the second term on the right hand side of (73) satisfies

$$\sum_{i=1}^m \mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m \nabla_y g_{i,t}(x_{i,t}, y_{i,t})\right\|^2\right] \le 4L_{g,1}^2\mathbb{E}\left[\|\boldsymbol{y}_t - \boldsymbol{y}_t^*(\boldsymbol{x})\|^2\right] + \frac{2\sigma_{g,1}^2 m}{t+1}\left(1 + \frac{8L_{g,1}^2}{\mu_g^2}\right). \tag{75}$$

Substituting (74) and (75) into (73) and subsequently substituting (73) and (49) into (72), we arrive at (71). □

D.7. Estimation of $\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]$ in Lemma D.7 and Its Proof

Using $\bar{z}_t = \frac{1}{m}\sum_{i=1}^m z_{i,t}$, $z_{i,t+1} = z_{i,t} + \sum_{j\in\mathcal{N}_i} w_{ij}(x_{j,t} + \vartheta_{j,t} - z_{i,t}) - \lambda_{z,t}\nabla_z \varphi_{i,t}(z_{i,t})$ from Algorithm 1 Step 5, and $\nabla_z \phi_{i,t}(z_{i,t}) = H_{i,t}z_{i,t} - b_{i,t}$ from Algorithm 1 Step 4, we have

$$\bar{z}_{t+1} = \bar{z}_t + \bar{\vartheta}_t - \lambda_{z,t}\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t} + \lambda_{z,t}\bar{b}_t, \tag{76}$$

with $\bar{b}_t = \frac{1}{m}\sum_{i=1}^m b_{i,t}$ and $b_{i,t} = \nabla_y f_{i,t}(x_{i,t}, y_{i,t})$.

Recalling definitions $\hat{z}_{i,t} = z_{i,t} - \bar{z}_t$, $H_{i,t} = \nabla^2_{yy} g_{i,t}(x_{i,t}, y_{i,t})$, and $\bar{H}_t = \frac{1}{m}\sum_{i=1}^m H_{i,t}$, we obtain

$$\begin{aligned} H_{i,t}z_{i,t} - \frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t} &= H_{i,t}z_{i,t} - \frac{1}{m}\sum_{i=1}^m H_{i,t}(\hat{z}_{i,t} + \bar{z}_t) = H_{i,t}z_{i,t} - \frac{1}{m}\sum_{i=1}^m H_{i,t}\hat{z}_{i,t} - \bar{H}_t\bar{z}_t \\ &= H_{i,t}\hat{z}_{i,t} - \frac{1}{m}\sum_{i=1}^m H_{i,t}\hat{z}_{i,t} + \left(H_{i,t} - \bar{H}_t\right)\bar{z}_t. \end{aligned} \tag{77}$$

We define auxiliary variables $\tilde{\boldsymbol{H}}_t = \breve{\boldsymbol{H}}_t - \frac{1}{m}(\mathbf{1}_m \otimes I_q)(\boldsymbol{H}_t)^T \in \mathbb{R}^{mq\times mq}$ with $\breve{\boldsymbol{H}}_t = \mathrm{diag}(H_{1,t}, \cdots, H_{m,t}) \in \mathbb{R}^{mq\times mq}$ and $\boldsymbol{H}_t = \mathrm{col}(H_{1,t}, \cdots, H_{m,t})$. Further using the definitions $\hat{\boldsymbol{z}}_t = \boldsymbol{z}_t - \mathbf{1}_m \otimes \bar{z}_t \in \mathbb{R}^{mq}$, $\hat{\boldsymbol{b}}_t = \boldsymbol{b}_t - \mathbf{1}_m \otimes \bar{b}_t \in \mathbb{R}^{mq}$, and $\hat{\boldsymbol{H}}_t = \boldsymbol{H}_t - \mathbf{1}_m \otimes \bar{H}_t \in \mathbb{R}^{mq\times q}$, and then combining (76) and (77), we obtain the following equality:

$$\hat{\boldsymbol{z}}_{t+1} = (I + W \otimes I_q)\,\hat{\boldsymbol{z}}_t + \hat{\boldsymbol{\vartheta}}_t - \lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t - \lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t + \lambda_{z,t}\hat{\boldsymbol{b}}_t. \tag{78}$$

Lemma D.7. *Under Assumptions 2.1-2.3 and 3.1, for any $t > 0$, the following inequality always holds:*

$$\mathbb{E}\left[\|\hat{\boldsymbol{z}}_{t+1}\|^2\right] \leq \left(1 - \frac{\delta_2}{2}\right)\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 4m\sigma_{z,t}^2 + c_{\hat{z}1}\lambda_{z,t}^2\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + c_{\hat{z}2}\lambda_{z,t}^2, \tag{79}$$

where $c_{\hat{z}1}$ and $c_{\hat{z}2}$ are given by $c_{\hat{z}1} = 8mL_{g,1}^2\left(3 + \frac{8(1-\delta_2)^2}{\delta_2}\right)$ and $c_{\hat{z}2} = \frac{c_{\hat{z}1}}{2L_{g,1}^2}\left(\frac{4L_{g,1}^2(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^2} + L_{f,0}^2\right)$.

Proof. By taking the squared norm and expectation on both sides of (78), and then using inequality $(a+b+c+d)^2 \leq a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$, we have

$$\begin{aligned} \mathbb{E}\left[\|\hat{\boldsymbol{z}}_{t+1}\|^2\right] &= \mathbb{E}\left[\|\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{\vartheta}}_t\|^2\right] + \lambda_{z,t}^2\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|^2\|\hat{\boldsymbol{z}}_t\|^2\right] + \lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{H}}_t\|^2\|\bar{z}_t\|^2\right] + \lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{b}}_t\|^2\right] \\ &\quad - 2\mathbb{E}\left[\left\langle\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t, \lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t\right\rangle\right] - 2\mathbb{E}\left[\left\langle\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t\right\rangle\right] + 2\mathbb{E}\left[\left\langle\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{b}}_t\right\rangle\right] \\ &\quad + 2\mathbb{E}\left[\left\langle\lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t\right\rangle\right] - 2\mathbb{E}\left[\left\langle\lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{b}}_t\right\rangle\right] - 2\mathbb{E}\left[\left\langle\lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t, \lambda_{z,t}\hat{\boldsymbol{b}}_t\right\rangle\right], \end{aligned} \tag{80}$$

where in the derivation we have used Assumption 3.1, which implies $\mathbb{E}[\langle\cdot, \hat{\boldsymbol{\vartheta}}_t\rangle] = 0$.

By using the relationships $2ab \leq a^2 + b^2$ and $2\langle a, \lambda_{z,t}b\rangle \leq \kappa_1 a^2 + \frac{1}{\kappa_1}\lambda_{z,t}^2 b^2$ holding for all $\kappa_1 > 0$, we can obtain

$$\begin{cases} -2\mathbb{E}\left[\left\langle\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t, \lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t\right\rangle\right] \leq 2\lambda_{z,t}\mathbb{E}\left[\|I + W\otimes I_q\|\,\|\tilde{\boldsymbol{H}}_t\|\,\|\hat{\boldsymbol{z}}_t\|^2\right], \\ -2\mathbb{E}\left[\left\langle\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t\right\rangle\right] \leq \kappa_1\|I + W\otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + \dfrac{\lambda_{z,t}^2}{\kappa_1}\mathbb{E}\left[\|\hat{\boldsymbol{H}}_t\|^2\|\bar{z}_t\|^2\right], \\ 2\mathbb{E}\left[\left\langle\left(I + W\otimes I_q\right)\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{b}}_t\right\rangle\right] \leq \kappa_1\|I + W\otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + \dfrac{\lambda_{z,t}^2}{\kappa_1}\mathbb{E}\left[\|\hat{\boldsymbol{b}}_t\|^2\right], \\ 2\mathbb{E}\left[\left\langle\lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t\right\rangle\right] \leq \lambda_{z,t}^2\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|^2\|\hat{\boldsymbol{z}}_t\|^2\right] + \lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{H}}_t\|^2\|\bar{z}_t\|^2\right], \\ 2\mathbb{E}\left[\left\langle\lambda_{z,t}\tilde{\boldsymbol{H}}_t\hat{\boldsymbol{z}}_t, \lambda_{z,t}\hat{\boldsymbol{b}}_t\right\rangle\right] \leq \lambda_{z,t}^2\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|^2\|\hat{\boldsymbol{z}}_t\|^2\right] + \lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{b}}_t\|^2\right], \\ -2\mathbb{E}\left[\left\langle\lambda_{z,t}\hat{\boldsymbol{H}}_t\bar{z}_t, \lambda_{z,t}\hat{\boldsymbol{b}}_t\right\rangle\right] \leq \lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{H}}_t\|^2\|\bar{z}_t\|^2\right] + \lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{b}}_t\|^2\right]. \end{cases} \tag{81}$$

Substituting (81) into (80), we arrive at

$$\begin{aligned} \mathbb{E}\left[\|\hat{\boldsymbol{z}}_{t+1}\|^2\right] &= \|I + W\otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{\vartheta}}_t\|^2\right] + \left(3\lambda_{z,t}^2 + \frac{\lambda_{z,t}^2}{\kappa_1}\right)\mathbb{E}\left[\|\hat{\boldsymbol{H}}_t\|^2\|\bar{z}_t\|^2\right] + (3\lambda_{z,t}^2 + \frac{\lambda_{z,t}^2}{\kappa_1})\mathbb{E}\left[\|\hat{\boldsymbol{b}}_t\|^2\right] \\ &\quad + 3\lambda_{z,t}^2\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|^2\right]\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 2\kappa_1\|I + W\otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 2\lambda_{z,t}\|I + W\otimes I_q\|\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|\right]\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]. \end{aligned} \tag{82}$$

By using the definition of $\tilde{\boldsymbol{H}}_t$ and Assumption 2.2, we obtain

$$\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|^2\right] \leq 2\mathbb{E}\left[\|\breve{\boldsymbol{H}}_t\|^2\right] + 2\mathbb{E}\left[\left\|\frac{1}{m}(\mathbf{1}_m \otimes I_q)(\boldsymbol{H}_t)^T\right\|^2\right] \leq 4mL_{g,1}^2. \tag{83}$$

We choose $\kappa_1 \leq \frac{\delta_2}{8(1-\delta_2)^2}$, leading to $2\kappa_1(1-\delta_2)^2 \leq \frac{\delta_2}{4}$. Additionally, since the stepsize $\lambda_{z,t}$ decays with time, the inequality $12mL_{g,1}^2\lambda_{z,t}^2 + 4\sqrt{m}L_{g,1}\lambda_{z,t}(1-\delta_2) \leq \frac{\delta_2}{4}$ always holds for a sufficiently large iteration T. Without loss of generality, we can set $\lambda_{z,0}$ as a small constant, ensuring that above inequality is satisfied. This strategy is commonly used in the DSBO result, such as Yang et al. (2022). Then, the summation of the last three terms on the right hand side of (82) can be simplified as follows:

$$3\lambda_{z,t}^2\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|^2\right]\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 2\lambda_{z,t}\|I + W \otimes I_q\|\mathbb{E}\left[\|\tilde{\boldsymbol{H}}_t\|\right]\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 2\kappa_1\|I + W \otimes I_q\|^2\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \leq \frac{\delta_2}{2}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right], \tag{84}$$

where in the derivation we have used (83) and $\|I + W \otimes I_q\| \leq 1 - \delta_2$ from Assumption 2.1.

Substituting (84) into (82) and using $(1-\delta_2)^2 < 1 - \delta_2$ based on $\delta_2 < 1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\|\hat{\boldsymbol{z}}_{t+1}\|^2\right] \leq \left(1 - \frac{\delta_2}{2}\right)\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + \left(3 + \frac{1}{\kappa_1}\right)\lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{H}}_t\|^2\|\bar{z}_t\|^2\right] + \left(3 + \frac{1}{\kappa_1}\right)\lambda_{z,t}^2\mathbb{E}\left[\|\hat{\boldsymbol{b}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{\vartheta}}_t\|^2\right] \\
&\leq \left(1 - \frac{\delta_2}{2}\right)\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 4mL_{g,1}^2\left(3 + \frac{1}{\kappa_1}\right)\lambda_{z,t}^2\left(2\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + 2\mathbb{E}\left[\|\breve{z}_t\|^2\right]\right) + 4mL_{f,0}^2\left(3 + \frac{1}{\kappa_1}\right)\lambda_{z,t}^2 + 4m\sigma_{z,t}^2 \\
&\leq \left(1 - \frac{\delta_2}{2}\right)\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 4m\sigma_{z,t}^2 + c_{\hat{z}1}\lambda_{z,t}^2\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + c_{\hat{z}2}\lambda_{z,t}^2,
\end{aligned}$$

where we have used $\mathbb{E}[\|\hat{\boldsymbol{H}}_t\|^2] \leq 4mL_{g,1}^2$ and $\mathbb{E}[\|\hat{\boldsymbol{b}}_t\|^2] \leq 4mL_{f,0}^2$ from Assumption 2.2, as well as $\mathbb{E}[\|\hat{\boldsymbol{\vartheta}}_t\|^2] \leq 4m\sigma_{z,t}^2$ from Assumption 2.3 in the second inequality. Moreover, we have utilized $\mathbb{E}[\|\breve{z}_t\|^2] \leq \frac{2\sigma_{f,1}^2 + 2L_{f,0}^2}{\mu_g^2}$ from Lemma C.1 in the last inequality. □

D.8. Estimation of $\mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_{t+1}\|^2\right]$ in Lemma D.8 and Its Proof

Here, we use definitions $\bar{z}_t = \frac{1}{m}\sum_{i=1}^m z_{i,t}$ and $\breve{z}_t = (\nabla_{yy}^2 g_t(\bar{x}_t, \bar{y}_t))^{-1}\nabla_y F_t(\bar{x}_t, \bar{y}_t)$. The update of $\bar{z}_{t+1}$ satisfies

$$\bar{z}_{t+1} = \bar{z}_t + \bar{\vartheta}_t - \lambda_{z,t}\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t} + \lambda_{z,t}\bar{b}_t. \tag{85}$$

Lemma D.8. *Under Assumptions 2.1-2.3 and 3.1, for any $t > 0$, we have*

$$\begin{aligned}
\mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_{t+1}\|^2\right] \leq &\left(1 - \frac{\lambda_{z,t}\mu_g}{4} + c_{z1}\lambda_{z,t}^2 + c_{z2}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] \\
&+ \left(c_{z3}\lambda_{z,t} + c_{z4}\kappa_2 + c_{z5}\frac{\lambda_{x,t}^2}{\lambda_{z,t}} + c_{z6}\frac{\lambda_{y,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \left(c_{z3}\lambda_{z,t} + c_{z4}\kappa_2 + c_{z7}\frac{\lambda_{x,t}^2}{\lambda_{z,t}} + c_{z8}\frac{\lambda_{y,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] \\
&+ \left(c_{z9}\lambda_{z,t} + c_{z10}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + \left(c_{z11}\frac{\lambda_{x,t}^2}{\lambda_{z,t}} + c_{z12}\frac{\lambda_{y,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] \\
&+ c_{z13}\sigma_{z,t}^2 + c_{z14}\frac{\sigma_{x,t}^2}{\lambda_{z,t}} + c_{z14}\frac{\sigma_{y,t}^2}{\lambda_{z,t}} + c_{z15}(\lambda_{z,t})^2 + c_{z16}\frac{\lambda_{x,t}^2}{\lambda_{z,t}} + c_{z17}\frac{\lambda_{y,t}^2}{\lambda_{z,t}(t+1)} + c_{z18}\frac{1}{\lambda_{z,t}(t+2)^2},
\end{aligned} \tag{86}$$

where the constants c_{z1} to c_{z18} are given by $c_{z1} = c_{\bar{z}1}\left(1 + \frac{\lambda_{z,0}\mu_g}{4}\right)$, $c_{z2} = c_{\breve{z}1}c_{\bar{x}4}\left(\lambda_{z,0} + \frac{4}{\mu_g}\right)$, $c_{z3} = \frac{c_{z1}c_{\bar{z}2}}{c_{\bar{z}1}}$, $c_{z4} = \frac{c_{z1}c_{\breve{z}3}}{c_{\bar{z}1}}$, $c_{z5} = \frac{c_{z2}c_{\bar{x}1}}{c_{\bar{x}4}}$, $c_{z6} = \frac{c_{z2}c_{\bar{y}1}}{c_{\bar{x}4}}$, $c_{z7} = \frac{c_{z2}c_{\bar{x}2}}{c_{\bar{x}4}}$, $c_{z8} = \frac{c_{z2}c_{\bar{y}2}}{c_{\bar{x}4}}$, $c_{z9} = \frac{c_{z1}c_{\bar{z}3}}{c_{\bar{z}1}}$, $c_{z10} = \frac{c_{z2}c_{\bar{x}3}}{c_{\bar{x}4}}$, $c_{z11} = \frac{c_{z2}c_{\bar{x}5}}{c_{\bar{x}4}}$, $c_{z12} = \frac{c_{z2}c_{\bar{y}3}}{c_{\bar{x}4}}$, $c_{z13} = \frac{c_{z1}}{c_{\bar{z}1}}$, $c_{z14} = \frac{c_{z2}}{c_{\bar{x}4}}$, $c_{z15} = \frac{c_{z1}c_{\bar{z}4}}{c_{\bar{z}1}}$, $c_{z15} = c_{z12}c_{\bar{z}4}$, $c_{z16} = \frac{c_{z2}c_{\bar{x}6}}{c_{\bar{x}4}}$, $c_{z17} = \frac{c_{z2}c_{\bar{y}4}}{c_{\bar{x}4}}$, *and* $c_{z18} = \frac{c_{z2}c_{\breve{z}2}}{c_{\bar{x}4}}$.

Proof. According to the update of $\bar{z}_{t+1}$ in (85) and the definition of $\breve{z}_t$, we have

$$\begin{aligned}\mathbb{E}\left[\|\bar{z}_{t+1}-\breve{z}_t\|^2\right]=\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]+\mathbb{E}\left[\|\bar{\vartheta}_t\|^2\right]+\lambda_{z,t}^2\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t}-\bar{b}_t\right\|^2\right]\\ -2\mathbb{E}\left[\left\langle\bar{z}_t-\breve{z}_t,\lambda_{z,t}\left(\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t}-\bar{b}_t\right)\right\rangle\right].\end{aligned}\tag{87}$$

The definition of $\hat{z}_{i,t}$ implies $z_{i,t}=\hat{z}_{i,t}+\bar{z}_t$, which further implies

$$\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t}-\bar{H}_t\bar{z}_t\right\|^2\right]=\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m H_{i,t}\hat{z}_{i,t}\right\|^2\right]\le\frac{2(\sigma_{g,2}^2+L_{g,1}^2)}{m}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right],\tag{88}$$

where in the derivation we have used $\mathbb{E}[\|H_{i,t}\|^2]=\mathbb{E}[\|\nabla_{yy}^2 g_{i,t}(x_{i,t},y_{i,t})\|^2]\le 2(\sigma_{g,2}^2+L_{g,1}^2)$ from Lemma C.1. Substituting (88) into the third term on the right hand side of (87) yields

$$\begin{aligned}&\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t}-\bar{b}_t\right\|^2\right]\le 2\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m H_{i,t}z_{i,t}-\bar{H}_t\bar{z}_t\right\|^2\right]+2\mathbb{E}\left[\left\|\bar{H}_t\bar{z}_t-\bar{b}_t\right\|^2\right]\\ &\le\frac{4(\sigma_{g,2}^2+L_{g,1}^2)}{m}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]+16\left(\sigma_{g,2}^2+L_{g,1}^2\right)\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]+32\left(\sigma_{g,2}^2+L_{g,1}^2\right)\frac{\sigma_{f,1}^2+L_{f,0}^2}{\mu_g^2}+8\left(\sigma_{f,1}^2+L_{f,0}^2\right),\end{aligned}\tag{89}$$

where we have used the following inequality in the last inequality:

$$\begin{aligned}\mathbb{E}\left[\left\|\bar{H}_t\bar{z}_t-\bar{b}_t\right\|^2\right]&\le\mathbb{E}\left[2\left\|\bar{H}_t\right\|^2\left(2\left\|\bar{z}_t-\breve{z}_t\right\|^2+2\|\breve{z}_t\|^2\right)+2\left\|\bar{b}_t\right\|^2\right]\\ &\le 8\left(\sigma_{g,2}^2+L_{g,1}^2\right)\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]+\frac{16\left(\sigma_{g,2}^2+L_{g,1}^2\right)\left(\sigma_{f,1}^2+L_{f,0}^2\right)}{\mu_g^2}+4\left(\sigma_{f,1}^2+L_{f,0}^2\right),\end{aligned}\tag{90}$$

and relations $\mathbb{E}[\|\bar{H}_t\|^2]\le 2\sigma_{g,2}^2+2L_{g,1}^2$, $\mathbb{E}[\|\breve{z}_t\|^2]\le\frac{2\sigma_{f,1}^2+2L_{f,0}^2}{\mu_g^2}$ and $\mathbb{E}[\|\bar{b}_t\|^2]\le 2\sigma_{f,1}^2+2L_{f,0}^2$ from Lemma C.1.

To characterize the last term on the right hand side of (87), we define an auxiliary variable $\breve{z}_t'$ as follows:

$$\breve{z}_t'=(\bar{H}_t)^{-1}\bar{b}_t=\left(\frac{1}{m}\sum_{i=1}^m\nabla_{yy}^2 g_{i,t}(x_{i,t},y_{i,t})\right)^{-1}\left(\frac{1}{m}\sum_{i=1}^m\nabla_y f_{i,t}(x_{i,t},y_{i,t})\right)=\left(\nabla_{yy}^2 g_t(x_{i,t},y_{i,t})\right)^{-1}\nabla_y F_t(x_{i,t},y_{i,t}).$$

Then, we can obtain the following relationship:

$$\lambda_{z,t}\mathbb{E}\left[\left\langle\bar{z}_t-\breve{z}_t',(\bar{H}_t\bar{z}_t-\bar{b}_t)\right\rangle\right]=\lambda_{z,t}\mathbb{E}\left[\left\langle\bar{z}_t-\breve{z}_t',(\bar{H}_t\bar{z}_t-\bar{H}_t\breve{z}_t')\right\rangle\right]\ge\lambda_{z,t}\mu_g\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t'\|^2\right],\tag{91}$$

where we have used $\mathbb{E}_\xi\left[\nabla_{yy}g_t(x,y)\right]=\nabla_{yy}g(x,y)$ for any given (x,y) and Assumption 2.2 in the last inequality.

By using (91) and $2\langle a,\lambda_{z,t}b\rangle\le\kappa_2 a^2+\frac{1}{\kappa_2}\lambda_{z,t}^2 b^2$ holding for all $\kappa_2>0$, we obtain the following inequality:

$$\begin{aligned}&2\lambda_{z,t}\mathbb{E}\left[\left\langle\bar{z}_t-\breve{z}_t,\bar{H}_t\bar{z}_t-\bar{b}_t\right\rangle\right]=2\lambda_{z,t}\mathbb{E}\left[\left\langle\bar{z}_t-\breve{z}_t',\bar{H}_t\bar{z}_t-\bar{b}_t\right\rangle\right]+2\lambda_{z,t}\mathbb{E}\left[\left\langle\breve{z}_t'-\breve{z}_t,\bar{H}_t\bar{z}_t-\bar{b}_t\right\rangle\right]\\ &\ge 2\lambda_{z,t}\mu_g\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t'\|^2\right]+2\lambda_{z,t}\mathbb{E}\left[\left\langle\breve{z}_t'-\breve{z}_t,(\bar{H}_t\bar{z}_t-\bar{b}_t)\right\rangle\right]\\ &\ge\lambda_{z,t}\mu_g\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]-2\lambda_{z,t}\mu_g\mathbb{E}\left[\|\breve{z}_t'-\breve{z}_t\|^2\right]-\left(\kappa_2\mathbb{E}\left[\|\breve{z}_t'-\breve{z}_t\|^2\right]+\frac{\lambda_{z,t}^2}{\kappa_2}\mathbb{E}\left[\|\bar{H}_t\bar{z}_t-\bar{b}_t\|^2\right]\right),\end{aligned}\tag{92}$$

where in the last inequality we have used the inequality $\|b\|^2\le 2\|a\|^2+2\|b-a\|^2$ resulting in $\|a\|^2\ge\frac{\|b\|^2}{2}-\|b-a\|^2$ for any $a,b,c\in\mathbb{R}^q$.

According to definitions $\breve{z}_t = (\nabla^2_{yy} g_t(\bar{x}_t, \bar{y}_t))^{-1} \nabla_y F_t(\bar{x}_t, \bar{y}_t)$ and $\breve{z}'_t = (\nabla^2_{yy} g_t(x_{i,t}, y_{i,t}))^{-1} \nabla_y F_t(x_{i,t}, y_{i,t})$, we estimate an upper bound on $\mathbb{E}\left[\|\breve{z}'_t - \breve{z}_t\|^2\right]$ as follows:

$$
\begin{aligned}
&\mathbb{E}\left[\|\breve{z}'_t - \breve{z}_t\|^2\right] = \mathbb{E}\left[\|(\nabla^2_{yy} g_t(x_{i,t}, y_{i,t}))^{-1} \nabla_y F_t(x_{i,t}, y_{i,t}) - (\nabla^2_{yy} g_t(\bar{x}_t, \bar{y}_t))^{-1} \nabla_y F_t(\bar{x}_t, \bar{y}_t)\|^2\right] \\
&\le 2\mathbb{E}\left[\|(\nabla^2_{yy} g_t(x_{i,t}, y_{i,t}))^{-1} - (\nabla^2_{yy} g_t(\bar{x}_t, \bar{y}_t))^{-1}\|^2\right] \mathbb{E}\left[\|\nabla_y F_t(x_{i,t}, y_{i,t})\|^2\right] \\
&\quad + 2\mathbb{E}\left[\|(\nabla^2_{yy} g_t(\bar{x}_t, \bar{y}_t))^{-1}\|^2\right] \mathbb{E}\left[\|\nabla_y F_t(\bar{x}_t, \bar{y}_t) - \nabla_y F_t(x_{i,t}, y_{i,t})\|^2\right] \\
&\le c_{\breve{z}3} \mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{\breve{z}3} \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right],
\end{aligned}
\tag{93}
$$

where we have used Lemma C.1, as well as (17) and (18) from Lemma C.2 in the second inequality. The constants $c_{\breve{z}3}$ is given by $c_{\breve{z}3} = \frac{c_{\bar{z}1}}{2m}$ with $c_{\bar{z}1}$ given in the statement of Lemma D.3.

By using inequalities (88), (90), (92), and (93), the last term on the right hand side of (87) satisfies

$$
\begin{aligned}
&-2\mathbb{E}\left[\left\langle \bar{z}_t - \breve{z}_t, \lambda_{z,t}\left(\frac{1}{m}\sum_{i=1}^m H_{i,t} z_{i,t} - \bar{b}_t\right)\right\rangle\right] \\
&= -2\lambda_{z,t}\mathbb{E}\left[\left\langle \bar{z}_t - \breve{z}_t, \bar{H}_t \bar{z}_t - \bar{b}_t\right\rangle\right] + 2\lambda_{z,t}\mathbb{E}\left[\left\langle \bar{z}_t - \breve{z}_t, \bar{H}_t \bar{z}_t - \frac{1}{m}\sum_{i=1}^m H_{i,t} z_{i,t}\right\rangle\right] \\
&\le -\lambda_{z,t}\mu_g \mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + (2\lambda_{z,t}\mu_g + \kappa_2)\mathbb{E}\left[\|\breve{z}'_t - \breve{z}_t\|^2\right] + \frac{\lambda_{z,t}^2}{\kappa_2}\mathbb{E}\left[\|\bar{H}_t \bar{z}_t - \bar{b}_t\|^2\right] \\
&\quad + \left(\frac{\lambda_{z,t}\mu_g}{2}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + \frac{2\lambda_{z,t}}{\mu_g}\mathbb{E}\left[\left\|\bar{H}_t \bar{z}_t - \frac{1}{m}\sum_{i=1}^m H_{i,t} z_{i,t}\right\|^2\right]\right) \\
&\le -\frac{\lambda_{z,t}\mu_g}{2}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + (2\lambda_{z,t}\mu_g + \kappa_2)c_{\breve{z}3}\left(\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\right) + \frac{4(\sigma_{g,2}^2 + L_{g,1}^2)}{m\mu_g}\lambda_{z,t}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \\
&\quad + \frac{8\left(\sigma_{g,2}^2 + L_{g,1}^2\right)}{\kappa_2}\lambda_{z,t}^2\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + \frac{4\left(\sigma_{f,1}^2 + L_{f,0}^2\right)}{\kappa_2}\left(\frac{4\left(\sigma_{g,2}^2 + L_{g,1}^2\right)}{\mu_g^2} + 1\right)\lambda_{z,t}^2.
\end{aligned}
\tag{94}
$$

Substituting (89) and (94) into (87), we arrive at

$$
\begin{aligned}
\mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_t\|^2\right] &\le \left(1 - \frac{\lambda_{z,t}\mu_g}{2} + c_{\bar{z}1}\lambda_{z,t}^2\right)\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + \sigma_{z,t}^2 \\
&\quad + (c_{\bar{z}2}\lambda_{z,t} + \kappa_2 c_{\breve{z}3})\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + (c_{\bar{z}2}\lambda_{z,t} + \kappa_2 c_{\breve{z}3})\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + c_{\bar{z}3}\lambda_{z,t}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + c_{\bar{z}4}\lambda_{z,t}^2,
\end{aligned}
\tag{95}
$$

where the constants $c_{\bar{z}1}$ to $c_{\bar{z}4}$ are given by $c_{\bar{z}1} = 8\left(\sigma_{g,2}^2 + L_{g,1}^2\right)\left(2 + \frac{1}{\kappa_2}\right)$, $c_{\bar{z}2} = 2\mu_g c_{\breve{z}3}$, $c_{\bar{z}3} = \frac{4(\sigma_{g,2}^2 + L_{g,1}^2)}{m}(\frac{1}{\mu_g} + \lambda_{z,0})$, and $c_{\bar{z}4} = \left(\frac{16(\sigma_{g,2}^2 + L_{g,1}^2)(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^2} + 2(\sigma_{f,1}^2 + L_{f,0}^2)\right)(2 + \frac{1}{\kappa_2})$.

We proceed to use the following decomposition:

$$
\|\bar{z}_{t+1} - \breve{z}_{t+1}\|^2 \le \left(1 + \frac{\lambda_{z,t}\mu_g}{4}\right)\|\bar{z}_{t+1} - \breve{z}_t\|^2 + \left(1 + \frac{4}{\lambda_{z,t}\mu_g}\right)\|\breve{z}_{t+1} - \breve{z}_t\|^2. \tag{96}
$$

Substituting (56) in Lemma D.3 into (96) yields

$$
\begin{aligned}
\mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_{t+1}\|^2\right] &\le \left(1 + \frac{\lambda_{z,t}\mu_g}{4}\right)\mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_t\|^2\right] \\
&\quad + \left(1 + \frac{4}{\lambda_{z,t}\mu_g}\right)\left(c_{\breve{z}1}\mathbb{E}\left[\|\bar{x}_{t+1} - \bar{x}_t\|^2\right] + c_{\breve{z}1}\mathbb{E}\left[\|\bar{y}_{t+1} - \bar{y}_t\|^2\right] + \frac{c_{\breve{z}2}}{(t+2)^2}\right).
\end{aligned}
\tag{97}
$$

Further substituting (47) in Lemma D.1, (53) in Lemma D.2, and (95) into (97), we arrive at (86). □

D.9. Estimation of $\mathbb{E}\left[\|\bar{y}_{t+1}-y_{t+1}^*(\bar{x}_{t+1})\|^2\right]$ in Lemma D.9 and Its Proof

Here, we use definitions $\bar{y}_t=\frac{1}{m}\sum_{i=1}^m y_{i,t}$ and $y_t^*(\bar{x}_t):=\operatorname{argmin}_{y\in\mathbb{R}^q} g_t(\bar{x}_t,y)$ with $\bar{x}_t=\frac{1}{m}\sum_{i=1}^m x_{i,t}$. We express the update rule of $\bar{y}_{t+1}$ as follows:

$$\bar{y}_{t+1}=\bar{y}_t+\bar{\zeta}_t-\lambda_{y,t}\frac{1}{m}\sum_{i=1}^m \nabla_y g_{i,t}(x_{i,t},y_{i,t}). \tag{98}$$

Lemma D.9. *Under Assumptions 2.1-2.3 and 3.1, for any $t>0$, we have*

$$\begin{aligned}\mathbb{E}\left[\|\bar{y}_{t+1}-y_{t+1}^*(\bar{x}_{t+1})\|^2\right]\le\left(1-\frac{\lambda_{y,t}\mu_g}{4}+c_{y1}\lambda_{y,t}^2\right)\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+c_{y2}\sigma_{y,t}^2+c_{y3}\frac{\lambda_{y,t}^2}{t+1}\\+c_{y4}\lambda_{y,t}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+c_{y5}\lambda_{y,t}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+\frac{c_{y6}}{\lambda_{y,t}(t+1)^2},\end{aligned}\tag{99}$$

where the constants c_{y1} to c_{y6} are given by $c_{y1}=\left(1+\frac{\lambda_{y,0}\mu_g}{4}\right)c_{\bar{y}3}$, $c_{y2}=\frac{c_{y1}}{c_{\bar{y}3}}$, $c_{y3}=c_{y2}c_{\bar{y}4}$, $c_{y4}=c_{y2}\left(\frac{8(L_{g,1}^2+\sigma_{g,2}^2)}{m\mu_g}+c_{\bar{y}1}\lambda_{y,0}\right)$, $c_{y5}=c_{y2}\left(\frac{8(L_{g,1}^2+\sigma_{g,2}^2)}{m\mu_g}+c_{\bar{y}2}\lambda_{y,0}\right)$, and $c_{y6}=\left(\lambda_{y,0}+\frac{4}{\mu_g}\right)\frac{2\sigma_{g,1}^2(\mu_g^2+4L_{g,1}^2)}{\mu_g^4}$.

Proof. Taking the squared norm and expectation on both sides of (98), we obtain

$$\begin{aligned}\mathbb{E}\left[\|\bar{y}_{t+1}-y_t^*(\bar{x}_t)\|^2\right]\le\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\sigma_{y,t}^2+\lambda_{y,t}^2\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m\nabla_y g_{i,t}(x_{i,t},y_{i,t})\right\|^2\right]\\-2\lambda_{y,t}\mathbb{E}\left[\left\langle\bar{y}_t-y_t^*(\bar{x}_t),\frac{1}{m}\sum_{i=1}^m\nabla_y g_{i,t}(x_{i,t},y_{i,t})\right\rangle\right].\end{aligned}\tag{100}$$

By using an argument similar to the derivation of (53), we have

$$\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m\nabla_y g_{i,t}(x_{i,t},y_{i,t})\right\|^2\right]\le c_{\bar{y}1}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+c_{\bar{y}2}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+c_{\bar{y}3}\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\frac{c_{\bar{y}4}}{t+1}.\tag{101}$$

By using (19) in Lemma C.2, we obtain

$$\mathbb{E}\left[\left\|\frac{1}{m}\sum_{i=1}^m\nabla_y g_{i,t}(x_{i,t},y_{i,t})-\nabla_y g_t(\bar{x}_t,\bar{y}_t)\right\|^2\right]\le\frac{4(L_{g,1}^2+\sigma_{g,2}^2)}{m}\left(\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\right),\tag{102}$$

which further implies

$$\begin{aligned}&-2\lambda_{y,t}\mathbb{E}\left[\left\langle\bar{y}_t-y_t^*(\bar{x}_t),\frac{1}{m}\sum_{i=1}^m\nabla_y g_{i,t}(x_{i,t},y_{i,t})\right\rangle\right]=-2\lambda_{y,t}\mathbb{E}\left[\langle\bar{y}_t-y_t^*(x),\nabla_y g_t(\bar{x}_t,\bar{y}_t)\rangle\right]\\&+2\lambda_{y,t}\mathbb{E}\left[\left\langle\bar{y}_t-y_t^*(\bar{x}_t),\nabla_y g_t(\bar{x}_t,\bar{y}_t)-\frac{1}{m}\sum_{i=1}^m\nabla_y g_{i,t}(x_{i,t},y_{i,t})\right\rangle\right]\\&\le-\lambda_{y,t}\mu_g\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\frac{\lambda_{y,t}\mu_g}{2}\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\frac{8(L_{g,1}^2+\sigma_{g,2}^2)\lambda_{y,t}}{m\mu_g}\left(\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\right),\end{aligned}\tag{103}$$

where we have used Assumption 2.2 and (102) in the last inequality.

Substituting (101) and (103) into (100), we obtain

$$\begin{aligned}\mathbb{E}\left[\|\bar{y}_{t+1}-y_t^*(\bar{x}_t)\|^2\right]\le\left(1-\frac{\lambda_{y,t}\mu_g}{2}+c_{\bar{y}3}\lambda_{y,t}^2\right)\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\sigma_{y,t}^2+c_{\bar{y}4}\frac{\lambda_{y,t}^2}{t+1}\\+\left(\frac{8(L_{g,1}^2+\sigma_{g,2}^2)}{m\mu_g}+c_{\bar{y}1}\lambda_{y,0}\right)\lambda_{y,t}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+\left(\frac{8(L_{g,1}^2+\sigma_{g,2}^2)}{m\mu_g}+c_{\bar{y}2}\lambda_{y,0}\right)\lambda_{y,t}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right].\end{aligned}\tag{104}$$

We proceed to use the following decomposition:

$$\mathbb{E}\left[\|\bar{y}_{t+1}-y^*_{t+1}(\bar{x}_{t+1})\|^2\right]=\left(1+\frac{\lambda_{y,t}\mu_g}{4}\right)\mathbb{E}\left[\|\bar{y}_{t+1}-y^*_t(\bar{x}_t)\|^2\right]+\left(1+\frac{4}{\lambda_{y,t}\mu_g}\right)\mathbb{E}\left[\|y^*_{t+1}(\bar{x}_{t+1})-y^*_t(\bar{x}_t)\|^2\right]. \tag{105}$$

By substituting (34) and (104) into (105), we arrive at (99). □

D.10. Consensus Errors of Algorithm 2

In this subsection, we summarize the consensus errors of the iterative variables generated by Algorithm 2. The analysis is based on the definitions: $\hat{\boldsymbol{x}}_t=\boldsymbol{x}_t-\mathbf{1}_m\otimes\bar{x}_t$, $\hat{\boldsymbol{y}}_t=\boldsymbol{y}_t-\mathbf{1}_m\otimes\bar{y}_t$, and $\hat{\boldsymbol{z}}_t=\boldsymbol{z}_t-\mathbf{1}_m\otimes\bar{z}_t$.

Lemma D.10. *Under Assumptions 2.1-2.3 and 3.1, if the stepsize rates satisfy $1>v_x>v_y>v_z>0$ and the rates of DP-noise variances satisfy $2\varsigma_x>v_z+v_y$, $2\varsigma_y>v_z+v_y$ and $2\varsigma_z>v_y$, then the following inequality always holds:*

$$\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]+\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]+\mathbb{E}\left[\|\bar{y}_t-y^*_t(\bar{x}_t)\|^2\right]\le\frac{C_0}{(t+1)^{\beta_0}}, \tag{106}$$

where the rate β_0 is given by $\beta_0=\min\{2\varsigma_x-v_z-v_y,2\varsigma_y-v_z-v_y,2\varsigma_z-v_y,2-2v_y\}$ and $C_0>0$ is some constant.

Proof. We sum up both sides of (68), (71), (79), (86), and (99) to obtain

$$\begin{aligned}
&\mathbb{E}\left[\|\hat{\boldsymbol{x}}_{t+1}\|^2\right]+\mathbb{E}\left[\|\hat{\boldsymbol{y}}_{t+1}\|^2\right]+\mathbb{E}\left[\|\hat{\boldsymbol{z}}_{t+1}\|^2\right]+\mathbb{E}\left[\|\bar{z}_{t+1}-\breve{z}_{t+1}\|^2\right]+\mathbb{E}\left[\|\bar{y}_{t+1}-y^*_t(\bar{x}_{t+1})\|^2\right]\\
&\le\left(1-\frac{\delta_2}{2}+c_{\hat{x}1}\lambda_{x,t}^2+c_{\hat{y}2}\lambda_{y,t}^2+c_{z3}\lambda_{z,t}+c_{z4}\kappa_2+c_{z5}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}+c_{z6}\frac{\lambda_{y,t}^2}{\lambda_{z,t}}+c_{y4}\lambda_{y,t}\right)\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]\\
&\quad+\left(1-\frac{\delta_2}{2}+c_{\hat{y}1}\lambda_{y,t}^2+c_{\hat{x}2}\lambda_{x,t}^2+c_{z3}\lambda_{z,t}+c_{z4}\kappa_2+c_{z7}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}+c_{z8}\frac{\lambda_{y,t}^2}{\lambda_{z,t}}+c_{y5}\lambda_{y,t}\right)\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\\
&\quad+\left(1-\frac{\delta_2}{2}+c_{\hat{x}3}\lambda_{x,t}^2+c_{z9}\lambda_{z,t}+c_{z10}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]\\
&\quad+\left(1-\frac{\lambda_{z,t}\mu_g}{4}+c_{z1}\lambda_{z,t}^2+c_{z2}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}+c_{\hat{x}4}\lambda_{x,t}^2+c_{\hat{z}1}\lambda_{z,t}^2\right)\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]\\
&\quad+\left(1-\frac{\lambda_{y,t}\mu_g}{4}+c_{y1}\lambda_{y,t}^2+c_{\hat{x}5}\lambda_{x,t}^2+c_{\hat{y}3}\lambda_{y,t}^2+c_{z11}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}+c_{z12}\frac{\lambda_{y,t}^2}{\lambda_{z,t}}\right)\mathbb{E}\left[\|\bar{y}_t-y^*_t(\bar{x}_t)\|^2\right]\\
&\quad+4m\sigma_{x,t}^2+(4m+c_{y2})\sigma_{y,t}^2+(4m+c_{z13})\sigma_{z,t}^2+c_{z14}\frac{\sigma_{x,t}^2}{\lambda_{z,t}}+c_{z14}\frac{\sigma_{y,t}^2}{\lambda_{z,t}}+c_{\hat{x}6}\lambda_{x,t}^2+(c_{\hat{y}4}+c_{y3})\frac{\lambda_{y,t}^2}{t+1}\\
&\quad+(c_{\hat{z}2}+c_{z15})(\lambda_{z,t}^2)+c_{z16}\frac{\lambda_{x,t}^2}{\lambda_{z,t}}+c_{z17}\frac{\lambda_{y,t}^2}{\lambda_{z,t}(t+1)}+\frac{c_{y6}}{\lambda_{y,t}(t+1)^2}+\frac{c_{z18}}{\lambda_{z,t}(t+1)^2}.
\end{aligned} \tag{107}$$

To guarantee $c_{z4}\kappa_2\le\frac{\delta_2}{4}$, we select $\kappa_2\le\frac{\delta_2}{4c_{z4}}$. Furthermore, considering decaying stepsizes satisfying $\lambda_{x,t}\le\lambda_{x,0}$, $\lambda_{y,t}\le\lambda_{y,0}$, and $\lambda_{z,t}\le\lambda_{z,0}$, we can choose the initial stepsizes $\lambda_{x,0}$, $\lambda_{y,0}$, and $\lambda_{z,0}$ to satisfy the following inequalities:

$$\begin{cases}
\frac{\delta_2}{4}\ge\frac{\lambda_{y,0}\mu_g}{8}+c_{\hat{x}1}\lambda_{x,0}^2+c_{\hat{y}2}\lambda_{y,0}^2+c_{z3}\lambda_{z,0}+c_{z5}\frac{\lambda_{x,0}^2}{\lambda_{z,0}}+c_{z6}\frac{\lambda_{y,0}^2}{\lambda_{z,0}}+c_{y4}\lambda_{y,0},\\
\frac{\delta_2}{4}\ge\frac{\lambda_{y,0}\mu_g}{8}+c_{\hat{y}1}\lambda_{y,0}^2+c_{\hat{x}2}\lambda_{x,0}^2+c_{z3}\lambda_{z,0}+c_{z7}\frac{\lambda_{x,0}^2}{\lambda_{z,0}}+c_{z8}\frac{\lambda_{y,0}^2}{\lambda_{z,t}}+c_{y5}\lambda_{y,0},\\
\frac{\delta_2}{2}\ge\frac{\lambda_{y,0}\mu_g}{8}+c_{\hat{x}3}\lambda_{x,0}^2+c_{z9}\lambda_{z,0}+c_{z10}\frac{\lambda_{x,0}^2}{\lambda_{z,0}},\\
\frac{\mu_g}{8}\ge c_{z1}\lambda_{z,0}+c_{z2}\frac{\lambda_{x,0}^2}{\lambda_{z,0}^2}+c_{\hat{x}4}\frac{\lambda_{x,0}^2}{\lambda_{z,0}}+c_{\hat{z}1}\lambda_{z,0},\\
\frac{\mu_g}{8}\ge c_{y1}\lambda_{y,0}+c_{\hat{x}5}\frac{\lambda_{x,0}^2}{\lambda_{y,0}}+c_{\hat{y}3}\lambda_{y,0}+c_{z11}\frac{\lambda_{x,0}^2}{\lambda_{z,0}\lambda_{y,0}}+c_{z12}\frac{\lambda_{y,0}}{\lambda_{z,0}}.
\end{cases} \tag{108}$$

It should be noted that in practical applications, the initial stepsizes $\lambda_{x,0}$, $\lambda_{y,0}$, and $\lambda_{z,0}$ can be chosen as any positive constants, without strictly following (108). This flexibility is due to the decaying property of the terms on the right hand side of (108), which guarantees that there will be a time instant $T_0 > 0$ such that (108) is valid for all $t > T_0$.

Considering the relations in (108), inequality (107) can be rewritten as

$$\begin{aligned}&\mathbb{E}\left[\|\hat{\boldsymbol{x}}_{t+1}\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{y}}_{t+1}\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{z}}_{t+1}\|^2\right] + \mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_{t+1}\|^2\right] + \mathbb{E}\left[\|\bar{y}_{t+1} - y_t^*(\bar{x}_{t+1})\|^2\right] \\ &\leq \left(1 - \frac{\lambda_{y,0}\mu_g}{8(t+1)^{v_y}}\right)\left(\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + \mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + \mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right]\right) + \Phi_t,\end{aligned} \tag{109}$$

where Φ_t is given by

$$\begin{aligned}\Phi_t = &\frac{4m(\sigma_x^+)^2}{(t+1)^{2\varsigma_x}} + \frac{(4m + c_{y2})(\sigma_y^+)^2}{(t+1)^{2\varsigma_y}} + \frac{(4m + c_{z13})(\sigma_z^+)^2}{(t+1)^{2\varsigma_z}} + \frac{c_{z14}(\sigma_x^+)^2}{\lambda_{z,0}^2(t+1)^{2\varsigma_x - v_z}} + \frac{c_{z14}(\sigma_y^+)^2}{\lambda_{z,0}^2(t+1)^{2\varsigma_y - v_z}} \\ &+ \frac{c_{\hat{x}6}\lambda_{x,0}^2}{(t+1)^{2v_x}} + \frac{(c_{\hat{y}4} + c_{y3})\lambda_{y,0}^2}{(t+1)^{2v_y+1}} + \frac{(c_{\hat{z}2} + c_{z15})\lambda_{z,0}^2}{(t+1)^{2v_z}} + \frac{c_{z16}\lambda_{x,0}^2}{\lambda_{z,0}(t+1)^{2v_x - v_z}} + \frac{c_{z17}\lambda_{y,0}^2}{\lambda_{z,0}(t+1)^{2v_y+1-v_z}} \\ &+ \frac{c_{y6}}{\lambda_{y,0}(t+1)^{2-v_y}} + \frac{c_{z18}}{\lambda_{z,0}(t+1)^{2-v_z}} \leq \frac{c_1}{(t+1)^s},\end{aligned} \tag{110}$$

with $c_1 = 4m(\sigma_x^+)^2 + (4m + c_{y2})(\sigma_y^+)^2 + (4m + c_{z13})(\sigma_z^+)^2 + c_{z14}(\sigma_x^+)^2 + c_{z14}(\sigma_y^+)^2 + c_{\hat{x}6}\lambda_{x,0}^2 + (c_{\hat{y}4} + c_{y3})\lambda_{y,0}^2 + (c_{\hat{z}2} + c_{z15})(\lambda_{z,0})^2 + c_{z16}\lambda_{x,0}^2 + c_{z17}\lambda_{y,0}^2 + \frac{c_{y6}}{\lambda_{y,0}} + \frac{c_{z18}}{\lambda_{z,0}}$, and $s = \min\{2\varsigma_x - v_z, 2\varsigma_y - v_z, 2\varsigma_z, 2 - v_y\}$.

Recalling the conditions $1 > v_x > v_y > v_z > 0$, $2\varsigma_x > v_z + v_y$, $2\varsigma_y > v_z + v_y$, and $2\varsigma_z > v_y$ given in the lemma statement, we know that $s > v_y$ always holds. Hence, using Lemma B.2 leads to (106). □

To accurately characterize the consensus error of iterative variables generated by Algorithm 2, we present the following lemma, which is derived from Lemma D.10.

Lemma D.11. *Under the same assumptions given in Lemma D.10, we have*

$$\begin{aligned}&\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] \leq \frac{\hat{c}_x}{(t+1)^{2\varsigma_x}}, \qquad \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] \leq \frac{\hat{c}_y}{(t+1)^{2\varsigma_y}}, \qquad \mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] \leq \frac{\bar{c}_y}{(t+1)^{\min\{2\varsigma_y - v_y, 2-2v_y\}}}, \\ &\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \leq \frac{\hat{c}_z}{(t+1)^{2\varsigma_z}}, \quad \mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] \leq \frac{\bar{c}_z}{(t+1)^{\min\{2\varsigma_x - 2v_z, 2\varsigma_y - 2v_z, 2\varsigma_z - v_z\}}},\end{aligned} \tag{111}$$

where the constants $\hat{c}_x$, $\hat{c}_y$, $\hat{c}_z$, $\bar{c}_y$, and $\bar{c}_z$ are given in (113), (114), (115), (117), and (119), respectively.

Proof. Combing (106) in Lemma D.10 with (68) in Lemma D.5 yields

$$\begin{aligned}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_{t+1}\|^2\right] &\leq \left(1 - \frac{\delta_2}{2} + c_{\hat{x}1}\lambda_{x,t}^2\right)\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \frac{4m(\sigma_x^+)^2}{(t+1)^{2\varsigma_x}} + \frac{\sum_{i=2}^5 c_{\hat{x}i}C_0\lambda_{x,0}^2}{(t+1)^{2v_x+\beta_0}} + \frac{c_{\hat{x}6}\lambda_{x,0}^2}{(t+1)^{2v_x}} \\ &\leq \left(1 - \frac{\delta_2}{4}\right)\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \frac{c_x}{(t+1)^{2\varsigma_x}},\end{aligned} \tag{112}$$

where the constant c_x is given by $c_x = 4m(\sigma_x^+)^2 + \sum_{i=2}^5 c_{\hat{x}i}C_0\lambda_{x,0}^2 + c_{\hat{x}6}\lambda_{x,0}^2$.

By using Lemma 11 from Chen & Wang (2023), we can obtain the following inequality:

$$\mathbb{E}\left[\|\hat{x}_t\|^2\right] \leq \mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] \leq \frac{\hat{c}_x}{(t+1)^{2\varsigma_x}} \quad \text{with} \quad \hat{c}_x = c_x\left(\frac{8\varsigma_x}{e\ln(\frac{8}{8-\delta_2})}\right)^{2\varsigma_x}\left(\frac{\mathbb{E}\left[\|\hat{\boldsymbol{x}}_0\|^2\right](4-\delta_2)}{4c_x} + \frac{8}{\delta_2}\right). \tag{113}$$

By combining (106) in Lemma D.10 with (71) in Lemma D.6 and (79) in Lemma D.7, we use again Lemma 11 from Chen & Wang (2023) to obtain

$$\mathbb{E}\left[\|\hat{y}_t\|^2\right] \leq \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] \leq \frac{\hat{c}_y}{(t+1)^{2\varsigma_y}} \quad \text{with} \quad \hat{c}_y = c_y\left(\frac{8\varsigma_y}{e\ln(\frac{8}{8-\delta_2})}\right)^{2\varsigma_y}\left(\frac{\mathbb{E}\left[\|\hat{\boldsymbol{y}}_0\|^2\right](4-\delta_2)}{4c_y} + \frac{8}{\delta_2}\right), \tag{114}$$

$$\mathbb{E}\left[\|\hat{z}_t\|^2\right] \leq \mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \leq \frac{\hat{c}_z}{(t+1)^{2\varsigma_z}} \quad \text{with} \quad \hat{c}_z = c_z\left(\frac{8\varsigma_z}{e\ln(\frac{8}{8-\delta_2})}\right)^{2\varsigma_z}\left(\frac{\mathbb{E}\left[\|\hat{\boldsymbol{z}}_0\|^2\right](4-\delta_2)}{4c_z} + \frac{8}{\delta_2}\right), \tag{115}$$

where c_y and c_z are given by $c_y = 4m(\sigma_y^+)^2 + (c_{\hat{y}2} + c_{\hat{y}3})C_0\lambda_{y,0}^2 + c_{\hat{y}4}\lambda_{y,0}^2$ and $c_z = 4m(\sigma_z^+)^2 + c_{\hat{z}1}C_0\lambda_{z,0}^2 + c_{\hat{z}2}\lambda_{z,0}^2$.

Utilizing (106) in Lemma D.10, (108), (113), (114), and (99) in Lemma D.9, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\bar{y}_{t+1} - y_{t+1}^*(\bar{x}_{t+1})\|^2\right] &\le \left(1 - \frac{\lambda_{y,0}\mu_g}{8(t+1)^{v_y}}\right)\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + \frac{c_{y2}(\sigma_y^+)^2}{(t+1)^{2\varsigma_y}} + \frac{c_{y3}\lambda_{y,0}^2}{(t+1)^{2v_y+1}} + \frac{c_{y4}\lambda_{y,0}\hat{c}_x}{(t+1)^{2\varsigma_x+v_y}}\\ &+ \frac{c_{y5}\lambda_{y,0}\hat{c}_y}{(t+1)^{2\varsigma_y+v_y}} + \frac{c_{y6}}{\lambda_{y,0}(t+1)^{2-v_y}} \le \left(1 - \frac{\lambda_{y,0}\mu_g}{8(t+1)^{v_y}}\right)\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + \frac{c_{\bar{y}*}}{(t+1)^{\min\{2\varsigma_y, 2-v_y\}}},\end{aligned} \tag{116}$$

where the constant $c_{\bar{y}*}$ is given by $c_{\bar{y}*} = c_{y2}(\sigma_y^+)^2 + c_{y3}\lambda_{y,0}^2 + c_{y4}\lambda_{y,0}\hat{c}_x + c_{y5}\lambda_{y,0}\hat{c}_y + c_{y6}$.

Applying Lemma B.2 to (116), we have

$$\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] \le \frac{\bar{c}_y}{(t+1)^{\beta_{\bar{y}}}}, \tag{117}$$

where the rate $\beta_{\bar{y}}$ is given by $\beta_{\bar{y}} = \min\{2\varsigma_y - v_y, 2 - 2v_y\}$ and $\bar{c}_y$ is some positive constant.

Furthermore, we use (106) in Lemma D.10, (108), (113), (114), (115), and (86) in Lemma D.8 to obtain

$$\begin{aligned}\mathbb{E}\left[\|\bar{z}_{t+1} - \breve{z}_{t+1}\|^2\right] &\le \left(1 - \frac{\lambda_{z,t}\mu_g}{8}\right)\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + \left(c_{z3}\lambda_{z,0} + c_{z4}\kappa_2 + c_{z5}\frac{\lambda_{x,0}^2}{\lambda_{z,0}} + c_{z6}\frac{\lambda_{y,0}^2}{\lambda_{z,0}}\right)\frac{\hat{c}_x}{(t+1)^{2\varsigma_x}}\\ &+ \left(c_{z3}\lambda_{z,0} + c_{z4}\kappa_2 + c_{z7}\frac{\lambda_{x,0}^2}{\lambda_{z,0}} + c_{z8}\frac{\lambda_{y,0}^2}{\lambda_{z,0}}\right)\frac{\hat{c}_x}{(t+1)^{2\varsigma_y}} + \left(c_{z9} + c_{z10}\frac{\lambda_{x,0}^2}{\lambda_{z,0}^2}\right)\frac{\lambda_{z,0}\hat{c}_z}{(t+1)^{v_z+2\varsigma_z}}\\ &+ \left(c_{z11}\frac{\lambda_{x,0}^2}{\lambda_{y,0}^2} + c_{z12}\right)\frac{\lambda_{y,0}^2\bar{c}_y}{\lambda_{z,0}(t+1)^{2v_y-v_z+\beta_{\bar{y}}}} + \frac{c_{z13}(\sigma_z^+)^2}{(t+1)^{2\varsigma_z}} + \frac{c_{z14}(\sigma_x^+)^2}{\lambda_{z,0}(t+1)^{2\varsigma_x-v_z}} + \frac{c_{z14}(\sigma_y^+)^2}{\lambda_{z,0}(t+1)^{2\varsigma_y-v_z}}\\ &+ \frac{c_{z15}(\lambda_{z,0})^2}{(t+1)^{2v_z}} + \frac{c_{z16}(\lambda_{x,0})^2}{\lambda_{z,0}(t+1)^{2v_x-v_z}} + \frac{c_{z17}(\lambda_{y,0})^2}{\lambda_{z,0}(t+1)^{2v_y+1-v_z}} + \frac{c_{z18}}{\lambda_{z,0}(t+1)^{2-v_z}}\\ &\le \left(1 - \frac{\lambda_{z,0}\mu_g}{8(t+1)^{v_z}}\right)\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + \frac{c_{\bar{z}\breve{z}}}{(t+1)^{\min\{2\varsigma_x-v_z, 2\varsigma_y-v_z, 2\varsigma_z\}}},\end{aligned} \tag{118}$$

where the constant $c_{c_{\bar{z}\breve{z}}}$ is given by $c_{c_{\bar{z}\breve{z}}} = 2c_{z4}\kappa_2\hat{c}_x + (2c_{z3}\hat{c}_x + c_{z9}\hat{c}_z)\lambda_{z,0} + ((c_{z5} + c_{z7})\hat{c}_x + c_{z10}\hat{c}_z + c_{z11}\bar{c}_y + c_{z16})\frac{\lambda_{x,0}^2}{\lambda_{z,0}} + c_{z13}(\sigma_z^+)^2 + ((c_{z6} + c_{z8})\hat{c}_x + c_{z12}\bar{c}_y + c_{z17})\frac{\lambda_{y,0}^2}{\lambda_{z,0}} + (c_{z14}((\sigma_x^+)^2 + (\sigma_y^+)^2) + c_{z18})\frac{1}{\lambda_{z,0}} + c_{z15}\lambda_{z,0}^2$.

By applying Lemma B.2 to (118), we arrive at

$$\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] \le \frac{\bar{c}_z}{(t+1)^{\beta_{\bar{z}}}}, \tag{119}$$

where the rate $\beta_{\bar{z}}$ is given by $\beta_{\bar{z}} = \min\{2\varsigma_x - 2v_z, 2\varsigma_y - 2v_z, 2\varsigma_z - v_z\}$ and $\bar{c}_z$ is some positive constant. □

D.11. Estimation of $\mathbb{E}\left[\|\bar{u}_t - u_t^*\|^2\right]$ in Lemma D.12 and Its Proof

Here, we use the definitions $\bar{u}_t = \frac{1}{m}\sum_{i=1}^m u_{i,t}$ and $\breve{u}_t = \nabla_x F_t(\bar{x}_t, \bar{y}_t) + \nabla_{xy}^2 g_t(\bar{x}_t, \bar{y}_t)\breve{z}_t$. Moreover, we define the following auxiliary variables:

$$\begin{aligned}\bar{z}_t^* &= \left(\nabla_{yy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1}\nabla_y F_t(\bar{x}_t, y^*(\bar{x}_t)), \quad \bar{u}_t^* = \nabla_x F_t(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_{xy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\bar{z}_t^*,\\ z_t^* &= \left(\nabla_{yy}^2 g(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1}\nabla_y F(\bar{x}_t, y^*(\bar{x}_t)), \quad u_t^* = \nabla_x F(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_{xy}^2 g(\bar{x}_t, y^*(\bar{x}_t))z_t^*.\end{aligned} \tag{120}$$

Lemma D.12. *Under Assumptions 2.1- 2.3 and 3.1, for any $t > 0$, the following inequality always holds:*

$$\begin{aligned}\mathbb{E}\left[\|\bar{u}_t - u_t^*\|^2\right] &\le \frac{3(c_{\bar{u}_1^*} + c_{\bar{u}_2^*} + c_{\bar{u}_4^*})}{t+1} + 3c_{\bar{u}_3^*}\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + 3c_{\bar{u}_5^*}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + 3c_{\bar{u}_5^*}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\\ &+ 3c_{\bar{u}_6^*}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + 3c_{\bar{u}_7^*}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right],\end{aligned} \tag{121}$$

where the constants $c_{\bar{u}_1^*}$ *to* $c_{\bar{u}_7^*}$ *are given by* $c_{\bar{u}_1^*} = 2\sigma_{f,1}^2 + \frac{8\sigma_{g,2}^2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2} + 4L_{g,1}^2 c_{\bar{z}^*}$ *with* $c_{\bar{z}^*} = \frac{2\sigma_{f,1}^2}{\mu_g^2} + \frac{2L_{f,0}^2\sigma_{g,2}^2}{\mu_g^4}$, $c_{\bar{u}_2^*} = 12\sigma_{f,1}^2\left(1+\frac{2(\sigma_{g,2}^2+L_{g,1}^2)}{\mu_g^2}\right) + \frac{48(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}\left(\sigma_{g,2}^2 + \frac{\sigma_{g,2}^2(\sigma_{g,2}^2+L_{g,1}^2)}{\mu_g^2}\right)$, $c_{\bar{u}_3^*} = 12L_{f,1}^2\left(1+\frac{2(\sigma_{g,2}^2+L_{g,1}^2)}{\mu_g^2}\right) + \frac{48(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}\left(L_{g,2}^2 + \frac{L_{g,2}^2(\sigma_{g,2}^2+L_{g,1}^2)}{\mu_g^2}\right)$, $c_{\bar{u}_4^*} = 12\sigma_{f,1}^2 + \frac{48\sigma_{g,2}^2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}$, $c_{\bar{u}_5^*} = \frac{12L_{f,1}^2}{m} + \frac{48L_{g,2}^2(\sigma_{f,1}^2+L_{f,0}^2)}{m\mu_g^2}$, $c_{\bar{u}_6^*} = \frac{16(\sigma_{g,2}^2+L_{g,1}^2)}{m}$, *and* $c_{\bar{u}_7^*} = 16(\sigma_{g,2}^2+L_{g,1}^2)$.

Proof. We use the following decomposition:

$$\mathbb{E}\left[\|u_t^* - \bar{u}_t\|^2\right] \le 3\mathbb{E}\left[\|u_t^* - \bar{u}_t^*\|^2\right] + 3\mathbb{E}\left[\|\bar{u}_t^* - \breve{u}_t\|^2\right] + 3\mathbb{E}\left[\|\breve{u}_t - \bar{u}_t\|^2\right]. \tag{122}$$

By using Assumption 3.1, the definitions of $\bar{z}_t^*$ and z_t^*, and Lemma C.1, we have

$$\begin{aligned}\mathbb{E}\left[\|\bar{z}_t^* - z_t^*\|^2\right] &\le 2\mathbb{E}\left[\left\|\left(\nabla_{yy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1}\right\|^2 \|\nabla_y F_t(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_y F(\bar{x}_t, y^*(\bar{x}_t))\|^2\right] \\ &\quad + 2\mathbb{E}\left[\left\|\left(\nabla_{yy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1} - \left(\nabla_{yy}^2 g(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1}\right\|^2 \|\nabla_y F(\bar{x}_t, y^*(\bar{x}_t))\|^2\right] \le \frac{c_{\bar{z}^*}}{t+1},\end{aligned} \tag{123}$$

where $c_{\bar{z}^*}$ is given by $c_{\bar{z}^*} = \frac{2\sigma_{f,1}^2}{\mu_g^2} + \frac{2L_{f,0}^2\sigma_{g,2}^2}{\mu_g^4}$. Using the definitions of $\bar{u}_t^*$ and u_t^* and inequality (123), we further obtain

$$\begin{aligned}&\mathbb{E}\left[\|\bar{u}_t^* - u_t^*\|^2\right] \le 2\mathbb{E}\left[\|\nabla_x F_t(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_x F(\bar{x}_t, y^*(\bar{x}_t))\|^2\right] + 2\mathbb{E}\left[\|\nabla_{xy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\bar{z}_t^* - \nabla_{xy}^2 g(\bar{x}_t, y^*(\bar{x}_t)) z_t^*\|^2\right] \\ &\le \frac{2\sigma_{f,1}^2}{t+1} + 4\mathbb{E}\left[\|\nabla_{xy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_{xy}^2 g(\bar{x}_t, y^*(\bar{x}_t))\|^2\|\bar{z}_t^*\|^2\right] + 4\mathbb{E}\left[\|\nabla_{xy}^2 g(\bar{x}_t, y^*(\bar{x}_t))\|^2\|\bar{z}_t^* - z_t^*\|^2\right] \\ &\le \frac{2\sigma_{f,1}^2}{t+1} + \frac{8\sigma_{g,2}^2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2(t+1)} + \frac{4L_{g,1}^2 c_{\bar{z}^*}}{t+1} = \frac{c_{\bar{u}_1^*}}{t+1},\end{aligned} \tag{124}$$

where we have used the relationship $\mathbb{E}\left[\|\bar{z}_t^*\|^2\right] \le \frac{2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}$ from Lemma C.1 in the last inequality.

We proceed to estimate an upper bound on $\mathbb{E}\left[\|\bar{u}_t^* - \breve{u}_t\|^2\right]$ in (122) based on the definitions of $\bar{u}_t^*$ and $\breve{u}_t$:

$$\begin{aligned}&\mathbb{E}\left[\|\bar{u}_t^* - \breve{u}_t\|^2\right] \le 2\mathbb{E}\left[\|\nabla_x F_t(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_x F_t(\bar{x}_t, \bar{y}_t)\|^2\right] + 2\mathbb{E}\left[\|\nabla_{xy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\bar{z}_t^* - \nabla_{xy}^2 g_t(\bar{x}_t, \bar{y}_t)\breve{z}_t\|^2\right] \\ &\le 2\left(\frac{6\sigma_{f,1}^2}{t+1} + 6L_{f,1}^2\mathbb{E}\left[\|y_t^*(\bar{x}_t) - \bar{y}_t\|^2\right]\right) \\ &\quad + 4\mathbb{E}\left[\|\nabla_{xy}^2 g_t(\bar{x}_t, y^*(\bar{x}_t))\bar{z}_t^* - \nabla_{xy}^2 g_t(\bar{x}_t, \bar{y}_t)\bar{z}_t^*\|^2\right] + 4\mathbb{E}\left[\|\nabla_{xy}^2 g_t(\bar{x}_t, \bar{y}_t)\bar{z}_t^* - \nabla_{xy}^2 g_t(\bar{x}_t, \bar{y}_t)\breve{z}_t\|^2\right] \\ &\le \left(12\sigma_{f,1}^2 + \frac{48\sigma_{g,2}^2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}\right)\frac{1}{t+1} + \left(12L_{f,1}^2 + \frac{48L_{g,2}^2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}\right)\mathbb{E}\left[\|y_t^*(\bar{x}_t) - \bar{y}_t\|^2\right] \\ &\quad + 8(\sigma_{g,2}^2+L_{g,1}^2)\mathbb{E}\left[\|\bar{z}_t^* - \breve{z}_t\|^2\right],\end{aligned} \tag{125}$$

where in the derivation we have used the following inequalities:

$$\begin{aligned}\mathbb{E}\left[\left\|\nabla_x F_t(x_2, y_2) - \nabla_x F_t(x_1, y_1)\right\|^2\right] &\le \frac{6\sigma_{f,1}^2}{t+1} + 6L_{f,1}^2\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right), \\ \mathbb{E}\left[\left\|\nabla_{xy}^2 g_t(x_2, y_2) - \nabla_{xy}^2 g_t(x_1, y_1)\right\|^2\right] &\le \frac{6\sigma_{g,2}^2}{t+1} + 6L_{g,2}^2\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right),\end{aligned} \tag{126}$$

for any given pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and any $t > 0$. Moreover, we have utilized $\mathbb{E}\left[\|\bar{z}_t^*\|^2\right] \le \frac{2(\sigma_{f,1}^2+L_{f,0}^2)}{\mu_g^2}$ and $\mathbb{E}\left[\|\nabla_{xy}^2 g_t(\bar{x}_t, \bar{y}_t)\|^2\right] \le 2(\sigma_{g,2}^2+L_{g,1}^2)$ in the last inequality.

Next, we characterize the term $\mathbb{E}\left[\|\bar{z}_t^* - \breve{z}_t\|^2\right]$ in (125) as follows:

$$\begin{aligned}
&\mathbb{E}\left[\|\bar{z}_t^* - \breve{z}_t\|^2\right] \le 2\mathbb{E}\left[\| \left(\nabla^2_{yy} g_t(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1} \|^2 \|\nabla_y F_t(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_y F_t(\bar{x}_t, \bar{y}_t)\|^2\right] \\
&\quad + 2\mathbb{E}\left[\| \left(\nabla^2_{yy} g_t(\bar{x}_t, y^*(\bar{x}_t))\right)^{-1} - \left(\nabla^2_{yy} g_t(\bar{x}_t, \bar{y}_t)\right)^{-1} \|^2 \|\nabla_y F_t(\bar{x}_t, \bar{y}_t)\|^2\right] \\
&\le \left(\frac{12\sigma_{f,1}^2}{\mu_g^2} + \frac{24\sigma_{g,2}^2(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^4}\right)\frac{1}{t+1} + \left(\frac{12L_{f,1}^2}{\mu_g^2} + \frac{24L_{g,2}^2(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^4}\right)\mathbb{E}\left[\|y_t^*(\bar{x}_t) - \bar{y}_t\|^2\right],
\end{aligned} \tag{127}$$

where we have used the following relationship in the last inequality:

$$\mathbb{E}\left[\left\|\left(\nabla^2_{yy} g_t(x_2, y_2)\right)^{-1} - \left(\nabla^2_{yy} g_t(x_1, y_1)\right)^{-1}\right\|^2\right] \le \frac{6\sigma_{g,2}^2}{\mu_g^4(t+1)} + \frac{6L_{g,2}^2}{\mu_g^4}\left(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2\right), \tag{128}$$

for any given pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and any $t > 0$.

Substituting (127) into (125), we arrive at

$$\mathbb{E}\left[\|\bar{u}_t^* - \breve{u}_t\|^2\right] \le \frac{c_{\bar{u}_2^*}}{t+1} + c_{\bar{u}_3^*}\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right]. \tag{129}$$

Now we estimate an upper bound on $\mathbb{E}\left[\|\breve{u}_t - \bar{u}_t\|^2\right]$ in (122):

$$\mathbb{E}\left[\|\bar{u}_t - \breve{u}_t\|^2\right] \| \le \frac{2}{m}\sum_{i=1}^m \left(\mathbb{E}\left[\|\nabla_x f_{i,t}(x_{i,t}, y_{i,t}) - \nabla_x f_{i,t}(\bar{x}_t, \bar{y}_t)\|^2\right] + \mathbb{E}\left[\|\nabla^2_{xy} g_{i,t}(x_{i,t}, y_{i,t}) z_{i,t} - \nabla^2_{xy} g_{i,t}(\bar{x}_t, \bar{y}_t)\breve{z}_t\|^2\right]\right). \tag{130}$$

The last term on the right hand side of (130) satisfies

$$\begin{aligned}
&\frac{2}{m}\sum_{i=1}^m \mathbb{E}\left[\|\nabla^2_{xy} g_{i,t}(x_{i,t}, y_{i,t}) z_{i,t} - \nabla^2_{xy} g_{i,t}(\bar{x}_t, \bar{y}_t)\breve{z}_t\|^2\right] \\
&\le \frac{4}{m}\sum_{i=1}^m \mathbb{E}\left[\|\nabla^2_{xy} g_{i,t}(x_{i,t}, y_{i,t}) z_{i,t} - \nabla^2_{xy} g_{i,t}(x_{i,t}, y_{i,t})\breve{z}_t\|^2\right] + \frac{4}{m}\sum_{i=1}^m \mathbb{E}\left[\|\nabla^2_{xy} g_{i,t}(x_{i,t}, y_{i,t})\breve{z}_t - \nabla^2_{xy} g_{i,t}(\bar{x}_t, \bar{y}_t)\breve{z}_t\|^2\right] \\
&\le \frac{8(\sigma_{g,2}^2 + L_{g,1}^2)}{m}\mathbb{E}\left[\|\boldsymbol{z}_t - \mathbf{1}_m \otimes \breve{z}_t\|^2\right] + \left(\frac{24\sigma_{g,2}^2}{t+1} + \frac{24L_{g,2}^2}{m}\left(\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\right)\right)\frac{2(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^2}, \\
&\le \frac{48\sigma_{g,2}^2(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^2(t+1)} + \frac{48L_{g,2}^2(\sigma_{f,1}^2 + L_{f,0}^2)}{m\mu_g^2}\left(\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + \mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\right) + \frac{16(\sigma_{g,2}^2 + L_{g,1}^2)}{m}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] \\
&\quad + 16(\sigma_{g,2}^2 + L_{g,1}^2)\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right],
\end{aligned} \tag{131}$$

where we have used the relationship $\mathbb{E}\left[\|\breve{z}_t\|^2\right] \le \frac{2(\sigma_{f,1}^2 + L_{f,0}^2)}{\mu_g^2}$ in the third inequality.

By using (126) and substituting (131) into (130), we obtain

$$\mathbb{E}\left[\|\bar{u}_t - \breve{u}_t\|^2\right] \| \le \frac{c_{\bar{u}_4^*}}{t+1} + c_{\bar{u}_5^*}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{\bar{u}_5^*}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + c_{\bar{u}_6^*}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + c_{\bar{u}_7^*}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right], \tag{132}$$

where the constants $c_{\bar{u}_5^*}$ to $c_{\bar{u}_7^*}$ are given in the lemma statement.

Substituting (124), (129), and (132) into (122), we arrive at (121). □

E. Proof of Theorem 4.1

In this section, we establish convergence rates of Algorithm 2 under different convexity assumptions on the upper-level objective function F. Specifically, the convergence rate for a strongly convex F is given in Theorem E.1, for a convex F is given in Theorem E.2, and for a nonconvex F is given in Theorem E.3.

E.1. Convergence Rate for a Strongly Convex Upper-Level Objective Function

Theorem E.1. *Under Assumptions 2.1-2.3 and 3.1, if the upper-level objective function $F(x)$ is μ_f-strongly convex, the stepsize rates satisfy $0 < v_z < v_y < v_x < 1$, and the rates of DP-noise variances satisfy $2\varsigma_x > v_z + v_y$, $2\varsigma_y > v_z + v_y$, $2\varsigma_z > v_y$ and $2\varsigma_x > v_x$, then the following inequality always holds:*

$$\mathbb{E}\left[\|x_{i,T} - x^*\|^2\right] \le \mathcal{O}\left(T^{-\beta_1}\right), \tag{133}$$

for all $T > 0$ and any $i \in [m]$, where β_1 is given by $\beta_1 = \min\{2\varsigma_x - v_x, 2\varsigma_x - 2v_z, 2\varsigma_y - 2v_z, 2\varsigma_z - v_z, 2\varsigma_y - v_y, 2 - 2v_y\}$.

Proof. We first characterize the distance between the average sequence $\bar{x}_{t+1}$ and the optimal solution x^* to problem (1). Recalling the update of $x_{i,t}$ in Algorithm 2 Step 7, we have $\bar{x}_{t+1} = \bar{x}_t + \bar{\chi}_t - \lambda_{x,t}\bar{u}_t$, which further implies

$$\begin{aligned}
&\mathbb{E}\left[\|\bar{x}_{t+1} - x^*\|^2\right] \le \mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right] - 2\lambda_{x,t}\mathbb{E}\left[\langle \bar{x}_t - x^*, \bar{u}_t\rangle\right]\\
&\le \mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right] - 2\lambda_{x,t}\mathbb{E}\left[\langle \bar{x}_t - x^*, u_t^*\rangle\right] + 2\lambda_{x,t}\mathbb{E}\left[\langle \bar{x}_t - x^*, u_t^* - \bar{u}_t\rangle\right]\\
&\le \mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right] - \lambda_{x,t}\mu_f\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \frac{\lambda_{x,t}\mu_f}{2}\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \frac{2\lambda_{x,t}}{\mu_f}\mathbb{E}\left[\|u_t^* - \bar{u}_t\|^2\right]\\
&\le \left(1 - \frac{\lambda_{x,t}\mu_f}{2}\right)\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right] + \frac{2\lambda_{x,t}}{\mu_f}\mathbb{E}\left[\|u_t^* - \bar{u}_t\|^2\right],
\end{aligned} \tag{134}$$

where we have used the μ_f-strong convexity of $F(x)$, i.e., $2\lambda_{x,t}\langle \bar{x}_t - x^*, u_t^*\rangle \ge \lambda_{x,t}\mu_f\|\bar{x}_t - x^*\|^2$.

By substituting (49) and (50) into (48), we can obtain an upper bound on $\mathbb{E}\left[\|\bar{u}_t\|^2\right]$:

$$\mathbb{E}\left[\|\bar{u}_t\|^2\right] \le c_{\bar{x}1}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{\bar{x}2}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right] + c_{\bar{x}3}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + c_{\bar{x}4}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + c_{\bar{x}5}\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + c_{\bar{x}6}. \tag{135}$$

By further substituting (135) and (121) in Lemma D.12 into (134), inequality (134) can be rewritten as

$$\begin{aligned}
\mathbb{E}\left[\|\bar{x}_{t+1} - x^*\|^2\right] \le &\left(1 - \frac{\lambda_{x,t}\mu_f}{2}\right)\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \sigma_{x,t}^2 + c_{x1}\lambda_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right] + c_{x2}\lambda_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\\
&+ c_{x3}\lambda_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right] + c_{x4}\lambda_{x,t}\mathbb{E}\left[\|\bar{z}_t - \breve{z}_t\|^2\right] + c_{x5}\lambda_{x,t}\mathbb{E}\left[\|\bar{y}_t - y_t^*(\bar{x}_t)\|^2\right] + c_{x6}\lambda_{x,t}^2 + c_{x7}\frac{\lambda_{x,t}}{t+1},
\end{aligned} \tag{136}$$

where the constants c_{x1} to c_{x7} are given by $c_{x1} = c_{\bar{x}1}\lambda_{x,0} + \frac{6c_{\bar{u}_5^*}}{\mu_f}$, $c_{x2} = c_{\bar{x}2}\lambda_{x,0} + \frac{6c_{\bar{u}_5^*}}{\mu_f}$, $c_{x3} = c_{\bar{x}3}\lambda_{x,0} + \frac{6c_{\bar{u}_6^*}}{\mu_f}$, $c_{x4} = c_{\bar{x}4}\lambda_{x,0} + \frac{6c_{\bar{u}_7^*}}{\mu_f}$, $c_{x5} = c_{\bar{x}5}\lambda_{x,0} + \frac{6c_{\bar{u}_3^*}}{\mu_f}$, $c_{x6} = c_{\bar{x}6}$, and $c_{x7} = \frac{6(c_{\bar{u}_1^*} + c_{\bar{u}_2^*} + c_{\bar{u}_4^*})}{\mu_f}$.

Using the results in Lemma D.11, we rewrite inequality (136) as follows:

$$\begin{aligned}
\mathbb{E}\left[\|\bar{x}_{t+1} - x^*\|^2\right] \le &\left(1 - \frac{\lambda_{x,0}\mu_f}{2(t+1)^{v_x}}\right)\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \frac{(\sigma_x^+)^2}{(t+1)^{2\varsigma_x}} + \frac{c_{x1}\lambda_{x,0}\hat{c}_x}{(t+1)^{2\varsigma_x + v_x}} + \frac{c_{x2}\lambda_{x,0}\hat{c}_y}{(t+1)^{2\varsigma_y + v_x}} + \frac{c_{x3}\lambda_{x,0}\hat{c}_z}{(t+1)^{2\varsigma_z + v_x}}\\
&+ \frac{c_{x4}\lambda_{x,0}\bar{c}_z}{(t+1)^{\min\{2\varsigma_x - 2v_z + v_x, 2\varsigma_y - 2v_z + v_x, 2\varsigma_z - v_z + v_x\}}} + \frac{c_{x5}\lambda_{x,0}\bar{c}_y}{(t+1)^{\min\{2\varsigma_y - v_y + v_x, 2 - 2v_y + v_x\}}} + \frac{c_{x6}\lambda_{x,0}^2}{(t+1)^{2v_x}} + \frac{c_{x7}\lambda_{x,0}}{(t+1)^{1+v_x}}\\
\le &\left(1 - \frac{\lambda_{x,0}\mu_f}{2(t+1)^{v_x}}\right)\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \frac{c_2}{(t+1)^{s_1}},
\end{aligned} \tag{137}$$

with $c_2 = (\sigma_x^+)^2 + (c_{x1}\hat{c}_x + c_{x2}\hat{c}_y + c_{x3}\hat{c}_z + c_{x4}\bar{c}_z + c_{x5}\bar{c}_y)\lambda_{x,0} + c_{x7}\lambda_{x,0}$ and $s_1 = \min\{2\varsigma_x, 2\varsigma_x - 2v_z + v_x, 2\varsigma_y - 2v_z + v_x, 2\varsigma_z - v_z + v_x, 2\varsigma_y - v_y + v_x, 2 - 2v_y + v_x\}$.

According to the conditions given in the theorem statement (or given in the statement of Theorem 4.1-(1)), we know that $s_1 > v_x$ always holds. Therefore, by using Lemma B.2, we arrive at

$$\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] \le \frac{c_3}{(t+1)^{\beta_1}}. \tag{138}$$

where the rate β_1 is given by $\beta_1 = \min\{2\varsigma_x - v_x, 2\varsigma_x - 2v_z, 2\varsigma_y - 2v_z, 2\varsigma_z - v_z, 2\varsigma_y - v_y, 2 - 2v_y\}$ and c_3 is some positive constant.

By using the definition $\hat{\boldsymbol{x}}_t = \boldsymbol{x}_t - \mathbf{1}_m \otimes \bar{x}_t$ and the first term of inequality (111) in Lemma D.11, we obtain

$$\mathbb{E}\left[\|x_{i,t} - x^*\|^2\right] \leq 2\mathbb{E}\left[\|x_{i,t} - \bar{x}_t\|^2\right] + 2\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] \leq C_1(t+1)^{-\beta_1}, \tag{139}$$

where the constant C_1 is given by $C_1 = 2(\hat{c}_x + c_3)$ and the rate β_1 satisfies $\beta_1 = \min\{2\varsigma_x - v_x, 2\varsigma_x - 2v_z, 2\varsigma_y - 2v_z, 2\varsigma_z - v_z, 2\varsigma_y - v_y, 2 - 2v_y\}$. Inequality (139) directly implies (133) in Theorem E.1 and (10) in Theorem 4.1-(1). □

E.2. Convergence Rate for a Convex Upper-Level Objective Function

Theorem E.2. *Under Assumptions 2.1-2.3 and 3.1, if the upper-level objective function $F(x)$ is convex, the stepsize rates satisfy $0 < v_z < v_y < v_x < 1$, and the rates of DP-noise variances satisfy $\varsigma_x > \frac{1}{2}$, $2\varsigma_x > v_z + v_y$, $2\varsigma_x > 2v_z + 2 - 2v_x$, $2\varsigma_y > 2v_z + 2 - 2v_x$, $2\varsigma_y > v_y + 2 - 2v_x$, $2\varsigma_y > v_z + v_y$, $2\varsigma_z > v_z + 2 - 2v_x$, and $2\varsigma_z > v_y$, then the following inequalities always hold:*

$$\begin{aligned}\mathbb{E}\left[\|\boldsymbol{x}_T - \mathbf{1}_m \otimes \bar{x}_T\|^2\right] &\leq \mathcal{O}\left(T^{-2\varsigma_x}\right),\\ \frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[F(\bar{x}_t) - F(x^*)\right] &\leq \mathcal{O}\left(T^{v_x-1}\right),\\ \frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[F(x_{i,t}) - F(x^*)\right] &\leq \mathcal{O}\left(T^{v_x-1}\right),\end{aligned} \tag{140}$$

for all $T > 0$ and any $i \in [m]$, where v_x is the rate of stepsize $\lambda_{x,t}$ given in Algorithm 2 satisfying $v_x - 1 < 0$.

Proof. (i) Based on the definition $\hat{\boldsymbol{x}}_t = \boldsymbol{x}_t - \mathbf{1}_m \otimes \bar{x}_t$, the first inequality in (140) follows naturally from (111) in Lemma D.11.

(ii) We now proceed to prove the second inequality in (140). Taking the squared norm and expectation on both sides of equality $\bar{x}_{t+1} = \bar{x}_t + \bar{\chi}_t - \lambda_{x,t}\bar{u}_t$ yields

$$\mathbb{E}\left[\|\bar{x}_{t+1} - x^*\|^2\right] \leq \mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right] - 2\mathbb{E}\left[\langle \bar{x}_t - x^*, \lambda_{x,t}\bar{u}_t\rangle\right]. \tag{141}$$

According to the definition $u_t^* = \nabla_x F(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_{xy}^2 g(\bar{x}_t, y^*(\bar{x}_t))z_t^*$, we have $u_t^* = \nabla F(\bar{x}_t)$. Using this relation and the convexity of F, the last term on the right hand side of (141) satisfies

$$\begin{aligned}&-2\mathbb{E}\left[\langle \bar{x}_t - x^*, \lambda_{x,t}\bar{u}_t\rangle\right] = 2\mathbb{E}\left[\langle x^* - \bar{x}_t, \lambda_{x,t}u_t^*\rangle\right] - 2\mathbb{E}\left[\langle \bar{x}_t - x^*, \lambda_{x,t}(\bar{u}_t - u_t^*)\rangle\right]\\ &\leq -2\lambda_{x,t}\mathbb{E}\left[F(\bar{x}_t) - F(x^*)\right] + a_t\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\bar{u}_t - u_t^*\|^2\right],\end{aligned} \tag{142}$$

where a_t is an auxiliary decaying sequence satisfying $a_t = \frac{1}{(t+1)^r}$ with $1 < r < 2v_x$.

Substituting (142) into (141) leads to

$$\mathbb{E}\left[\|\bar{x}_{t+1} - x^*\|^2\right] \leq -2\lambda_{x,t}\mathbb{E}\left[F(\bar{x}_t) - F(x^*)\right] + (1 + a_t)\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \Phi_t, \tag{143}$$

where the term Φ_t is given by

$$\Phi_t = \frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\bar{u}_t - u_t^*\|^2\right] + \sigma_{x,t}^2 + \lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right]. \tag{144}$$

Since the relation $F(\bar{x}_t) \geq F(x^*)$ always holds, we drop the negative term $-2\lambda_{x,t}\mathbb{E}\left[F(\bar{x}_t) - F(x^*)\right]$ in (143) to obtain

$$\mathbb{E}\left[\|\bar{x}_{t+1} - x^*\|^2\right] \leq (1 + a_t)\mathbb{E}\left[\|\bar{x}_t - x^*\|^2\right] + \Phi_t \leq \left(\prod_{t=0}^{T}(1 + a_t)\right)\left(\mathbb{E}\left[\|\bar{x}_0 - x^*\|^2\right] + \sum_{t=0}^{T}\Phi_t\right). \tag{145}$$

By using the relation $\ln(1+u) \leq u$ holding for any $u > 0$ and the definition $a_t = \frac{1}{(t+1)^r}$ with $1 < r < 2v_x$, we have

$$\ln\left(\prod_{t=0}^{T}(1 + a_t)\right) = \sum_{t=0}^{T}\ln(1 + a_t) \leq \sum_{t=0}^{T}a_t \leq a_0 + \sum_{t=1}^{T}\frac{1}{(t+1)^r} \leq a_0 + \int_1^{\infty}\frac{1}{x^r}dx \leq \frac{a_0(r-1)}{r-1}, \tag{146}$$

which implies $\prod_{t=0}^{T}(1+a_t) \le e^{\frac{a_0(r-1)}{r-1}}$. Then, inequality (145) can be rewritten as follows:

$$\mathbb{E}\left[\|\bar{x}_{t+1}-x^*\|^2\right] \le e^{\frac{a_0(r-1)}{r-1}}\left(\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]+\sum_{t=0}^{T}\Phi_t\right). \tag{147}$$

Next, we estimate an upper bound on $\sum_{t=0}^{T}\Phi_t$, where Φ_t is defined in (144).

Substituting (121) and (135) into (144) and subsequently using (111) and the relation $a_t \le 1$, we obtain

$$\begin{aligned}
\sum_{t=0}^{T}\Phi_t \le& \sum_{t=0}^{T}\Bigg(\frac{3(c_{\bar{u}_1^*}+c_{\bar{u}_2^*}+c_{\bar{u}_4^*})\lambda_{x,t}^2}{a_t(t+1)}+\left(3c_{\bar{u}_3^*}+c_{\bar{x}5}\right)\frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\left(3c_{\bar{u}_5^*}+c_{\bar{x}1}\right)\frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]\\
&+\left(3c_{\bar{u}_5^*}+c_{\bar{x}2}\right)\frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]+\left(3c_{\bar{u}_6^*}+c_{\bar{x}3}\right)\frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]+\left(3c_{\bar{u}_1^*}+c_{\bar{x}4}\right)\frac{\lambda_{x,t}^2}{a_t}\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]+\sigma_{x,t}^2+c_{\bar{x}6}\lambda_{x,t}^2\Bigg)\\
\le& \sum_{t=0}^{T}\frac{3\lambda_{x,0}^2(c_{\bar{u}_1^*}+c_{\bar{u}_2^*}+c_{\bar{u}_4^*})}{(t+1)^{2v_x-r+1}}+\sum_{t=0}^{T}\frac{\left(3c_{\bar{u}_3^*}+c_{\bar{x}5}\right)\bar{c}_y\lambda_{x,0}^2}{(t+1)^{\min\{2v_x-r+2\varsigma_y-v_y,2v_x-r+2-2v_y\}}}+\sum_{t=0}^{T}\frac{\left(3c_{\bar{u}_5^*}+c_{\bar{x}1}\right)\hat{c}_x\lambda_{x,0}^2}{(t+1)^{2v_x-r+2\varsigma_x}}\\
&+\sum_{t=0}^{T}\frac{\left(3c_{\bar{u}_5^*}+c_{\bar{x}2}\right)\hat{c}_y\lambda_{x,0}^2}{(t+1)^{2v_x-r+2\varsigma_y}}+\sum_{t=0}^{T}\frac{\left(3c_{\bar{u}_6^*}+c_{\bar{x}3}\right)\hat{c}_z\lambda_{x,0}^2}{(t+1)^{2v_x-r+2\varsigma_z}}+\sum_{t=0}^{T}\frac{\left(3c_{\bar{u}_1^*}+c_{\bar{x}4}\right)\bar{c}_z\lambda_{x,0}^2}{(t+1)^{\min\{2v_x-r+2\varsigma_x-2v_z,2v_x-r+2\varsigma_y-2v_z,2v_x-r+2\varsigma_z-v_z\}}}\\
&+\sum_{t=0}^{T}\frac{(\sigma_x^+)^2}{(t+1)^{2\varsigma_x}}+\sum_{t=0}^{T}\frac{c_{\bar{x}6}\lambda_{x,0}^2}{(t+1)^{2v_x}}.
\end{aligned} \tag{148}$$

By using the following inequality:

$$\sum_{t=0}^{T}\frac{1}{(t+1)^r}=1+\sum_{t=2}^{T+1}\frac{1}{t^s}\le 1+\int_1^\infty\frac{1}{x^r}dx\le\frac{r}{r-1}, \tag{149}$$

and the constant r satisfying $1<r<2v_x$, we can rewrite inequality (148) as follows:

$$\begin{aligned}
\sum_{t=0}^{T}\Phi_t \le& \frac{3\lambda_{x,0}^2(c_{\bar{u}_1^*}+c_{\bar{u}_2^*}+c_{\bar{u}_4^*})(2v_x-r+1)}{2v_x-r}+\left(3c_{\bar{u}_3^*}+c_{\bar{x}5}\right)\bar{c}_y\lambda_{x,0}^2\max\left\{\frac{2v_x-r+2\varsigma_y-v_y}{2v_x-r+2\varsigma_y-v_y-1},\frac{2v_x-r+2-2v_y}{2v_x-r+1-2v_y}\right\}\\
&+\frac{\left(3c_{\bar{u}_5^*}+c_{\bar{x}1}\right)\hat{c}_x\lambda_{x,0}^2(2v_x-r+2\varsigma_x)}{2v_x-r+2\varsigma_x-1}+\frac{\left(3c_{\bar{u}_5^*}+c_{\bar{x}2}\right)\hat{c}_y\lambda_{x,0}^2(2v_x-r+2\varsigma_y)}{2v_x-r+2\varsigma_y-1}+\frac{\left(3c_{\bar{u}_6^*}+c_{\bar{x}3}\right)\hat{c}_z\lambda_{x,0}^2(2v_x-r+2\varsigma_z)}{2v_x-r+2\varsigma_z-1}\\
&+\left(3c_{\bar{u}_1^*}+c_{\bar{x}4}\right)\bar{c}_z\lambda_{x,0}^2\max\left\{\frac{2v_x-r+2\varsigma_x-2v_z}{2v_x-r+2\varsigma_x-2v_z-1},\frac{2v_x-r+2\varsigma_y-2v_z}{2v_x-r+2\varsigma_y-2v_z-1},\frac{2v_x-r+2\varsigma_z-v_z}{2v_x-r+2\varsigma_z-v_z-1}\right\}\\
&+\frac{2(\sigma_x^+)^2\varsigma_x}{2\varsigma_x-1}+\frac{2c_{\bar{x}6}\lambda_{x,0}^2v_x}{2v_x-1}\triangleq c_4,
\end{aligned} \tag{150}$$

Substituting (150) into (147), we can arrive at

$$\mathbb{E}\left[\|\bar{x}_{t+1}-x^*\|^2\right]\le e^{\frac{a_0(r-1)}{r-1}}\left(\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]+c_4\right). \tag{151}$$

We proceed to sum up both sides of (143) from 0 to T (T can be any positive integer):

$$\sum_{t=0}^{T}2\lambda_{x,t}\mathbb{E}\left[F(\bar{x}_t)-F(x^*)\right]\le-\sum_{t=0}^{T}\mathbb{E}\left[\|\bar{x}_{t+1}-x^*\|^2\right]+\sum_{t=0}^{T}(1+a_t)\mathbb{E}\left[\|\bar{x}_t-x^*\|^2\right]+\sum_{t=0}^{T}\Phi_t. \tag{152}$$

The first and second terms on the right hand side of (152) can be simplified as follows:

$$\begin{aligned}
&\sum_{t=0}^{T}(1+a_t)\mathbb{E}\left[\|\bar{x}_t-x^*\|^2\right]-\sum_{t=0}^{T}\mathbb{E}\left[\|\bar{x}_{t+1}-x^*\|^2\right]\\
&\le a_0\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]+\sum_{t=1}^{T}a_t\mathbb{E}\left[\|\bar{x}_t-x^*\|^2\right]+\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]-\mathbb{E}\left[\|\bar{x}_{t+1}-x^*\|^2\right]\\
&\le \sum_{t=1}^{T}\frac{1}{(t+1)^r}\left(e^{\frac{a_0(r-1)}{r-1}}\left(\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]+c_4\right)\right)+(1+a_0)\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]\\
&\le \left(\frac{re^{\frac{a_0(r-1)}{r-1}}}{r-1}+(1+a_0)\right)\mathbb{E}\left[\|\bar{x}_0-x^*\|^2\right]+\frac{c_4 r}{r-1}\triangleq c_5,
\end{aligned}\tag{153}$$

where we have used (151) in the second inequality and (149) in the last inequality.

Substituting (150) and (153) into (152) and using $\lambda_{x,T}\le\lambda_{x,t}$ for any $t\in[0,T]$ yield $\sum_{t=0}^{T}2\lambda_{x,t}\mathbb{E}\left[F(\bar{x}_t)-F(x^*)\right]\le c_4+c_5$, which further implies

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[F(\bar{x}_t)-F(x^*)\right]\le\frac{c_4+c_5}{2\lambda_{x,0}(T+1)^{1-v_x}}=\frac{C_2'}{(T+1)^{1-v_x}},\tag{154}$$

with $C_2'=\frac{c_4+c_5}{2\lambda_{x,0}}$. Inequality (154) directly implies the second inequality in (140).

(iii) We now prove the third inequality in (140).

Assumption 2.2 implies $\mathbb{E}\left[F(x_{i,t})-F(\bar{x}_t)\right]\le L_{f,0}(\mathbb{E}\left[\|\hat{x}_t\|\right]+\mathbb{E}\left[\|\hat{y}_t\|\right])$. By using Lemma D.11, we have

$$\mathbb{E}\left[F(x_{i,t})-F(\bar{x}_t)\right]\le L_{f,0}\left(\frac{\sqrt{\hat{c}_x}}{(t+1)^{\varsigma_x}}+\frac{\sqrt{\hat{c}_y}}{(t+1)^{\varsigma_y}}\right).\tag{155}$$

Since $\sum_{t=0}^{T}\frac{1}{(t+1)^p}\le\int_{x=0}^{T+1}\frac{1}{x^p}dx\le\frac{(T+1)^{1-p}}{1-p}$ always holds for any $p\in(0,1)$, we arrive at

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[F(x_{i,t})-F(\bar{x}_t)\right]\le L_{f,0}\left(\frac{\sqrt{\hat{c}_x}}{(T+1)^{\varsigma_x}}+\frac{\sqrt{\hat{c}_y}}{(T+1)^{\varsigma_y}}\right)=\frac{C_2}{(T+1)^{\min\{\varsigma_x,\varsigma_y\}}},\tag{156}$$

where the constant C_2 is given by $C_2=L_{f,0}(\sqrt{\hat{c}_x}+\sqrt{\hat{c}_y})$.

According to the conditions $2\varsigma_x>v_z+v_y+2-2v_x$ and $2\varsigma_y>v_z+v_y+2-2v_x$ given in the theorem statement (or given in the statement of Theorem 4.1-(2)), we have $1-v_x<\varsigma_x$ and $1-v_x<\varsigma_y$. Hence, by using (154), we arrive at the third inequality in (140) and (11) in Theorem 4.1-(2). □

E.3. Convergence Rate for a Nonconvex Upper-Level Objective Function

Theorem E.3. *Under Assumptions 2.1-2.3 and 3.1, if the upper-level objective function $F(x)$ is nonconvex, the stepsize rates satisfy $0<v_z<v_y<v_x<1$, and the rates of DP-noise variances satisfy $\varsigma_x>\frac{1}{2}$, $2\varsigma_x>v_z+v_y$, $2\varsigma_x>2v_z+1-v_x$, $2\varsigma_y>2v_z+1-v_x$, $2\varsigma_y>v_y+1-v_x$, $2\varsigma_y>v_z+v_y$, $2\varsigma_z>v_z+1-v_x$, and $2\varsigma_z>v_y$, then the following inequalities always hold:*

$$\begin{aligned}
&\mathbb{E}\left[\|\boldsymbol{x}_T-\mathbf{1}_m\otimes\bar{x}_T\|^2\right]\le\mathcal{O}\left(T^{-2\varsigma_x}\right),\\
&\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right]\le\mathcal{O}\left(T^{v_x-1}\right),
\end{aligned}\tag{157}$$

for all $T>0$ and any $i\in[m]$, where v_x is the rate of stepsize $\lambda_{x,t}$ given in Algorithm 2 satisfying $v_x-1<0$.

Proof. The first inequality in (157) follows naturally from (111) in Lemma D.11.

We proceed to prove the second inequality in (157).

Assumption 2.2 implies

$$F(\bar{x}_{t+1}) \le F(\bar{x}_t) + \langle \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L_{f,1}}{2} \|\bar{x}_{t+1} - \bar{x}_t\|. \tag{158}$$

Taking expectation on both sides of (158) yields

$$\mathbb{E}\left[F(\bar{x}_{t+1}) - F(\bar{x}_t)\right] \le \mathbb{E}\left[\langle \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle\right] + \frac{L_{f,1}}{2} \mathbb{E}\left[\|\bar{x}_{t+1} - \bar{x}_t\|^2\right]. \tag{159}$$

Substituting the relation $\bar{x}_{t+1} - \bar{x}_t = \bar{\chi}_t - \lambda_{x,t}\bar{u}_t$ into the terms on the right hand side of (159) yields

$$\begin{aligned}
&\mathbb{E}\left[\langle \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle\right] + \frac{L_{f,1}}{2} \mathbb{E}\left[\|\bar{x}_{t+1} - \bar{x}_t\|^2\right] \\
&= -\mathbb{E}\left[\langle \nabla F(\bar{x}_t), \lambda_{x,t}\bar{u}_t \rangle\right] + \frac{L_{f,1}}{2} \mathbb{E}\left[\|\bar{\chi}_t - \lambda_{x,t}\bar{u}_t\|^2\right] \\
&\le -\mathbb{E}\left[\langle \nabla F(\bar{x}_t), \lambda_{x,t}\bar{u}_t \rangle\right] + \frac{L_{f,1}}{2} \left(\sigma_{x,t}^2 + \lambda_{x,t}^2 \mathbb{E}\left[\|\bar{u}_t\|^2\right]\right).
\end{aligned} \tag{160}$$

The definition of u_t^* implies $u_t^* = \nabla F(\bar{x}_t)$. Hence, the first term on the right hand side of (160) satisfies

$$\begin{aligned}
-\mathbb{E}\left[\langle \nabla F(\bar{x}_t), \lambda_{x,t}\bar{u}_t \rangle\right] &= -\lambda_{x,t}\mathbb{E}\left[\langle \nabla F(\bar{x}_t), u_t^* \rangle\right] - \lambda_{x,t}\mathbb{E}\left[\langle \nabla F(\bar{x}_t), \bar{u}_t - u_t^* \rangle\right] \\
&\le -\lambda_{x,t}\mathbb{E}\left[\|\nabla F(\bar{x}_t)\|^2\right] + \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\|\nabla F(\bar{x}_t)\|^2\right] + \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\bar{u}_t - u_t^*\|^2\right] \\
&\le -\frac{\lambda_{x,t}}{2}\mathbb{E}\left[\|\nabla F(\bar{x}_t)\|^2\right] + \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\bar{u}_t - u_t^*\|^2\right].
\end{aligned} \tag{161}$$

By substituting (160) and (161) into (159), we have

$$\mathbb{E}\left[F(\bar{x}_{t+1}) - F(\bar{x}_t)\right] \le -\frac{\lambda_{x,t}}{2}\mathbb{E}\left[\|\nabla F(\bar{x}_t)\|^2\right] + \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\bar{u}_t - u_t^*\|^2\right] + \frac{L_{f,1}}{2}\sigma_{x,t}^2 + \frac{L_{f,1}}{2}\lambda_{x,t}^2\mathbb{E}\left[\|\bar{u}_t\|^2\right]. \tag{162}$$

Summing up both sides of (162) from 0 to T and using the relationship $F(x^*) \le F(\bar{x}_{t+1})$, we obtain

$$\begin{aligned}
&\sum_{t=0}^{T} \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\|\nabla F(\bar{x}_t)\|^2\right] \\
&\le \mathbb{E}\left[F(\bar{x}_0) - F(x^*)\right] + \sum_{t=0}^{T} \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\bar{u}_t - u_t^*\|^2\right] + \sum_{t=0}^{T} \frac{L_{f,1}(\sigma_x^+)^2}{2(t+1)^{2\varsigma_x}} + \sum_{t=0}^{T} \frac{L_{f,1}\lambda_{x,t}^2}{2}\mathbb{E}\left[\|\bar{u}_t\|^2\right].
\end{aligned} \tag{163}$$

Combining (163) and the relation $\lambda_{x,t}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right] \le \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\|\nabla F(x_{i,t}) - F(\bar{x}_t)\|^2\right] + \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right]$ yields

$$\sum_{t=0}^{T} \lambda_{x,t}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right] \le \mathbb{E}\left[F(\bar{x}_0) - F(x^*)\right] + \sum_{t=0}^{t} \Phi_t. \tag{164}$$

where the term Φ_t is given by

$$\Phi_t = \lambda_{x,t}\mathbb{E}\left[\|\nabla F(\bar{x}_t) - \nabla F(x_{i,t})\|^2\right] + \frac{\lambda_{x,t}}{2}\mathbb{E}\left[\bar{u}_t - u_t^*\|^2\right] + \frac{L_{f,1}(\sigma_x^+)^2}{2(t+1)^{2\varsigma_x}} + \frac{L_{f,1}\lambda_{x,t}^2}{2}\mathbb{E}\left[\|\bar{u}_t\|^2\right]. \tag{165}$$

We proceed to estimate an upper bound on $\sum_{t=0}^{T} \Phi_t$.

Substituting (121) and (135) into (165), and then using Lemma B.1 and Lemma D.11, we have

$$
\begin{aligned}
\sum_{t=0}^{T}\Phi_t \leq& \sum_{t=0}^{T}\Bigg[\left(\frac{L_F}{m}+\frac{3c_{\bar{u}_5^*}+L_{f,1}c_{\bar{x}1}\lambda_{x,0}}{2}\right)\lambda_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{x}}_t\|^2\right]+\left(\frac{3c_{\bar{u}_5^*}+L_{f,1}c_{\bar{x}2}\lambda_{x,0}}{2}\right)\lambda_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{y}}_t\|^2\right]\\
&+\left(\frac{3c_{\bar{u}_6^*}+L_{f,1}c_{\bar{x}3}\lambda_{x,0}}{2}\right)\lambda_{x,t}\mathbb{E}\left[\|\hat{\boldsymbol{z}}_t\|^2\right]+\left(\frac{3c_{\bar{u}_7^*}+L_{f,1}c_{\bar{x}4}\lambda_{x,0}}{2}\right)\lambda_{x,t}\mathbb{E}\left[\|\bar{z}_t-\breve{z}_t\|^2\right]\\
&+\left(\frac{3c_{\bar{u}_3^*}+L_{f,1}c_{\bar{x}5}\lambda_{x,0}}{2}\right)\lambda_{x,t}\mathbb{E}\left[\|\bar{y}_t-y_t^*(\bar{x}_t)\|^2\right]+\frac{3(c_{\bar{u}_1^*}+c_{\bar{u}_2^*}+c_{\bar{u}_4^*})}{2}\frac{\lambda_{x,t}}{t+1}+\frac{L_{f,1}(\sigma_x^+)^2}{2(t+1)^{2\varsigma_x}}+\frac{L_{f,1}c_{\bar{x}6}}{2}\lambda_{x,t}^2\Bigg]\\
\leq& \sum_{t=0}^{T}\frac{3\lambda_{x,0}(c_{\bar{u}_1^*}+c_{\bar{u}_2^*}+c_{\bar{u}_4^*})}{2(t+1)^{1+v_x}}+\sum_{t=0}^{T}\frac{L_{f,1}(\sigma_x^+)^2}{2(t+1)^{2\varsigma_x}}+\sum_{t=0}^{T}\frac{L_{f,1}c_{\bar{x}6}(\lambda_{x,0})^2}{2(t+1)^{2v_x}}\\
&+\sum_{t=0}^{T}\left(\frac{L_F}{m}+\frac{3c_{\bar{u}_5^*}+L_{f,1}c_{\bar{x}1}\lambda_{x,0}}{2}\right)\frac{\hat{c}_x\lambda_{x,0}}{(t+1)^{2\varsigma_x+v_x}}+\sum_{t=0}^{T}\left(\frac{3c_{\bar{u}_5^*}+L_{f,1}c_{\bar{x}2}\lambda_{x,0}}{2}\right)\frac{\hat{c}_y\lambda_{x,0}}{(t+1)^{2\varsigma_y+v_x}}\\
&+\sum_{t=0}^{T}\left(\frac{3c_{\bar{u}_6^*}+L_{f,1}c_{\bar{x}3}\lambda_{x,0}}{2}\right)\frac{\hat{c}_z\lambda_{x,0}}{(t+1)^{2\varsigma_z+v_x}}\\
&+\sum_{t=0}^{T}\left(\frac{3c_{\bar{u}_7^*}+L_{f,1}c_{\bar{x}4}\lambda_{x,0}}{2}\right)\frac{\bar{c}_z\lambda_{x,0}}{(t+1)^{\min\{2\varsigma_x-2v_z+v_x,2\varsigma_y-2v_z+v_x,2\varsigma_z-v_z+v_x\}}}\\
&+\sum_{t=0}^{T}\left(\frac{3c_{\bar{u}_3^*}+L_{f,1}c_{\bar{x}5}\lambda_{x,0}}{2}\right)\frac{\bar{c}_y\lambda_{x,0}}{(t+1)^{\min\{2\varsigma_y-v_y+v_x,2-2v_y+v_x\}}}.
\end{aligned}
\tag{166}
$$

Using inequality (149) yields

$$
\begin{aligned}
\sum_{t=0}^{T}\Phi_t \leq& \frac{3\lambda_{x,0}(c_{\bar{u}_1^*}+c_{\bar{u}_2^*}+c_{\bar{u}_4^*})(1+v_x)}{2v_x}+\frac{L_{f,1}(\sigma_x^+)^2\varsigma_x}{2\varsigma_x-1}+\frac{L_{f,1}c_{\bar{x}6}(\lambda_{x,0})^2v_x}{2v_x-1}\\
&+\left(\frac{L_F}{m}+\frac{3c_{\bar{u}_5^*}+L_{f,1}c_{\bar{x}1}\lambda_{x,0}}{2}\right)\frac{\hat{c}_x\lambda_{x,0}(2\varsigma_x+v_x)}{2\varsigma_x+v_x-1}+\left(\frac{3c_{\bar{u}_5^*}+L_{f,1}c_{\bar{x}2}\lambda_{x,0}}{2}\right)\frac{\hat{c}_y\lambda_{x,0}(2\varsigma_y+v_x)}{2\varsigma_y+v_x-1}\\
&+\left(\frac{3c_{\bar{u}_6^*}+L_{f,1}c_{\bar{x}3}\lambda_{x,0}}{2}\right)\frac{\hat{c}_z\lambda_{x,0}(2\varsigma_z+v_x)}{2\varsigma_z+v_x-1}\\
&+\left(\frac{3c_{\bar{u}_7^*}+L_{f,1}c_{\bar{x}4}\lambda_{x,0}}{2}\right)\bar{c}_z\lambda_{x,0}\max\left\{\frac{2\varsigma_x-2v_z+v_x}{2\varsigma_x-2v_z+v_x-1},\frac{2\varsigma_y-2v_z+v_x}{2\varsigma_y-2v_z+v_x-1},\frac{2\varsigma_z-v_z+v_x}{2\varsigma_z-v_z+v_x-1}\right\}\\
&+\left(\frac{3c_{\bar{u}_3^*}+L_{f,1}c_{\bar{x}5}\lambda_{x,0}}{2}\right)\bar{c}_y\lambda_{x,0}\max\left\{\frac{2\varsigma_y-v_y+v_x}{2\varsigma_y-v_y+v_x-1},\frac{2-2v_y+v_x}{1-2v_y+v_x}\right\}\triangleq c_6.
\end{aligned}
\tag{167}
$$

Substituting (167) into (164) and defining $c_7\triangleq\mathbb{E}\left[F(\bar{x}_0)-F(x^*)\right]$, we can obtain $\sum_{t=0}^{T}\lambda_{x,t}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right]\leq c_6+c_7$, which implies

$$
\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right]\leq\frac{c_6+c_7}{\lambda_{x,0}(T+1)^{1-v_x}}=\frac{C_3}{(T+1)^{1-v_x}},
\tag{168}
$$

with $C_3=\frac{c_6+c_7}{2\lambda_{x,0}}$. Inequality (168) directly implies the second inequality in (157) and (12) in Theorem 4.1-(3). □

F. Proof of Theorem 4.5

In this section, we prove that in addition to accurate convergence, Algorithm 2 can also simultaneously ensure rigorous ϵ_i-LDP for each agent, even when the number of iterations T tends to infinity. To this end, we first provide a definition for the sensitivity of agent i's implementation $\mathcal{A}_i$:

Definition F.1. (Sensitivity) The sensitivity of agent i's implementation $\mathcal{A}_i$ is

$$
\Delta_{i,t}=\max_{\mathrm{Adj}(\mathcal{D}_i,\mathcal{D}_i')}\left\|\mathcal{A}_i(\mathcal{D}_i,\theta_{-i,t})-\mathcal{A}_i(\mathcal{D}_i',\theta_{-i,t})\right\|_1,
\tag{169}
$$

where $\mathrm{Adj}(\mathcal{D}_i, \mathcal{D}_i')$ represents the adjacent relationship between agent i's adjacent datasets $\mathcal{D}_i$ and $\mathcal{D}_i'$, and $\theta_{-i,t}$ represents all information agent i receives from its neighbors at time t.

According to Definition F.1, under Algorithm 2, each agent i's implementation involves three sensitivities: $\Delta_{i,t,x}$, $\Delta_{i,t,y}$, and $\Delta_{i,t,z}$, which correspond to $x_{i,t}$, $y_{i,t}$, and $z_{i,t}$, respectively. With this understanding, we have the following lemma:

Lemma F.2. *(Huang et al., 2015) At each time $t \geq 0$, if agent i injects into each of its shared variables $x_{i,t}$, $y_{i,t}$, and $z_{i,t}$ noise vectors $\chi_{i,t}$, $\zeta_{i,t}$, and $\vartheta_{i,t}$ consisting of p, q, and q independent Laplace noises with parameters $\nu_{i,t,x}$, $\nu_{i,t,y}$, and $\nu_{i,t,z}$, respectively, such that $\sum_{t=1}^{T}\left(\frac{\Delta_{i,t,x}}{\nu_{i,t,x}} + \frac{\Delta_{i,t,y}}{\nu_{i,t,y}} + \frac{\Delta_{i,t,z}}{\nu_{i,t,z}}\right) \leq \epsilon_i$, then agent i's implementation $\mathcal{A}_i$ of Algorithm 2 is ϵ_i-LDP for time $t = 0$ to $t = T$.*

For the convenience of privacy analysis, we represent the different data points between upper-level adjacent datasets $\mathcal{D}_{f_i}$ and $\mathcal{D}_{f_i}'$ (as well as between lower-level adjacent datasets $\mathcal{D}_{g_i}$ and $\mathcal{D}_{g_i}'$) as the k-th one, i.e., $\varphi_{i,k}$ in $\mathcal{D}_{f_i}$ and $\varphi_{i,k}'$ in $\mathcal{D}_{f_i}'$ ($\xi_{i,k}$ in $\mathcal{D}_{g_i}$ and $\xi_{i,k}'$ in $\mathcal{D}_{g_i}'$), without loss of generality. We further denote $x_{i,t}$, $y_{i,t}$, and $z_{i,t}$ as the parameters generated by Algorithm 2 based on $\mathcal{D}_{f_i}$ and $\mathcal{D}_{g_i}$. We also use $x_{i,t}'$, $y_{i,t}'$, and $z_{i,t}'$ to represent the parameters generated by Algorithm 2 based on $\mathcal{D}_{f_i}'$ and $\mathcal{D}_{g_i}'$.

Now, we are in position to prove Theorem 4.5.

Proof. The convergence results follow naturally from Theorem 4.1.

(1) To prove the statement on privacy, we first analyze the sensitivities of agent i's implementation under Algorithm 2.

According to the definition of sensitivity, we have $z_{j,t}+\vartheta_{j,t} = z_{j,t}'+\vartheta_{j,t}'$, $y_{j,t}+\zeta_{j,t} = y_{j,t}'+\zeta_{j,t}'$, and $x_{j,t}+\chi_{j,t} = x_{j,t}'+\chi_{j,t}'$ for all $t \geq 0$ and $j \in \mathcal{N}_i$. Since we assume that only the k-th data point is different between $\mathcal{D}_{f_i}$ and $\mathcal{D}_{f_i}'$, as well as between $\mathcal{D}_{g_i}$ and $\mathcal{D}_{g_i}'$, when $t < k$, we have $z_{i,t} = z_{i,t}'$, $y_{i,t} = y_{i,t}'$, and $x_{i,t} = x_{i,t}'$. However, when $t \geq k$, since the difference in loss functions kicks in at iteration k, i.e., $h(x,y;\varphi_{i,k}) \neq h(x,y;\varphi_{i,k}')$ and $l(x,y;\xi_{i,k}) \neq l(x,y;\xi_{i,k}')$, we have $z_{i,t} \neq z_{i,t}'$, $y_{i,t} \neq y_{i,t}'$, and $x_{i,t} \neq x_{i,t}'$. Hence, for agent i's implementation of Algorithm 2, we have

$$\|z_{i,t+1} - z_{i,t+1}'\|_1 = \|(1+w_{ii})(z_{i,t} - z_{i,t}') - \lambda_{z,t}(H_{i,t}z_{i,t} - H_{i,t}'z_{i,t}') + \lambda_{z,t}(b_{i,t} - b_{i,t}')\|_1, \tag{170}$$

for all $t \geq 0$. Let $\bar{w} = \min\{|w_{ii}|\},\ i \in [m]$, the sensitivity $\Delta_{i,t,z}$ satisfies

$$\begin{aligned}
\Delta_{i,t+1,z} \leq\, & (1-\bar{w})\Delta_{i,t,z} + \frac{\lambda_{z,t}}{t+1}\sum_{p=k}^{t}\|\nabla_{yy}^2 l(x_{i,t},y_{i,t};\xi_{i,p})z_{i,t} - \nabla_{yy}^2 l(x_{i,t}',y_{i,t}';\xi_{i,p}')z_{i,t}'\|_1 \\
& + \frac{\lambda_{z,t}}{t+1}\sum_{p=k}^{t}\|\nabla_y h(x_{i,t},y_{i,t};\varphi_{i,p}) - \nabla_y h(x_{i,t}',y_{i,t}';\varphi_{i,p}')\|_1 \\
\leq\, & (1-\bar{w})\Delta_{i,t,z} + \frac{c_z\lambda_{z,t}}{t+1}\sum_{p=0}^{t}\|\nabla_{yy}^2 l(x_{i,t},y_{i,t};\xi_{i,p}) - \nabla_{yy}^2 l(x_{i,t}',y_{i,t}';\xi_{i,p}')\|_1 \\
& + \frac{\lambda_{z,t}}{t+1}\sum_{p=0}^{t}\|\nabla_y h(x_{i,t},y_{i,t};\varphi_{i,p}) - \nabla_y h(x_{i,t}',y_{i,t}';\varphi_{i,p}')\|_1,
\end{aligned} \tag{171}$$

where we used $\|z_{i,t}\|_1 \leq c_z$ from the convergence of Algorithm 2. Given that the difference in loss functions kicks in at iteration k, we have $\sum_{p=0}^{k-1}\nabla_{yy}^2 l(x_{i,t},y_{i,t};\xi_{i,p})z_{i,t} = \sum_{p=0}^{k-1}\nabla_{yy}^2 l(x_{i,t}',y_{i,t}';\xi_{i,p}')z_{i,t}'$, and $\sum_{p=0}^{k-1}\nabla_y h(x_{i,t},y_{i,t};\varphi_{i,p}) = \sum_{p=0}^{k-1}\nabla_y h(x_{i,t}',y_{i,t}';\varphi_{i,p}')$, which are used in the last inequality.

By using Assumption 4.4 and the relation $\Delta_{i,0,z} = 0$, we iterate (171) from $t = 1$ to $t = T$ to obtain

$$\Delta_{i,t,z} \leq 2(c_z L_{l,1} + c_{h0})\sum_{p=1}^{t}(1-\bar{w})^{t-p}\lambda_{z,p-1}. \tag{172}$$

Similarly, by using the update of $y_{i,t}$ in Algorithm 2 Step 4, we have

$$\|y_{i,t+1} - y_{i,t+1}'\|_1 = \|(1+w_{ii})(y_{i,t} - y_{i,t}') - \lambda_{y,t}(\nabla_y g_{i,t}(x_{i,t},y_{i,t}) - \nabla_y g_{i,t}'(x_{i,t}',y_{i,t}'))\|_1, \tag{173}$$

for all $t \geq 0$. Then, the sensitivity $\Delta_{i,t,y}$ satisfies

$$\begin{aligned}\Delta_{i,t+1,y} &\leq (1-\bar{w})\Delta_{i,t,y} + \frac{\lambda_{y,t}}{t+1}\sum_{p=k}^{t}\|\nabla_y l(x_{i,t}, y_{i,t}; \xi_{i,p}) - \nabla_y l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})\|_1 \\ &\leq (1-\bar{w})\Delta_{i,t,y} + \frac{\lambda_{y,t}}{t+1}\sum_{p=0}^{t}\|\nabla_y l(x_{i,t}, y_{i,t}; \xi_{i,p}) - \nabla_y l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})\|_1.\end{aligned} \tag{174}$$

By using Assumption 4.4 and the relation $\Delta_{i,0,y} = 0$, we iterate (174) from $t = 1$ to $t = T$ to obtain

$$\Delta_{i,t,y} \leq 2c_{l0}\sum_{p=1}^{t}(1-\bar{w})^{t-p}\lambda_{z,p-1}. \tag{175}$$

Furthermore, by using the update of $x_{i,t}$ in Algorithm 2 Step 7, we have

$$\|x_{i,t+1} - x'_{i,t+1}\|_1 = \|(1+w_{ii})(x_{i,t} - x'_{i,t}) - \lambda_t(u_{i,t} - u'_{i,t})\|_1, \tag{176}$$

for all $t \geq 0$. Then, the sensitivity $\Delta_{i,t,x}$ satisfies

$$\begin{aligned}\Delta_{i,t+1,x} \leq{}& (1-\bar{w})\Delta_{i,t,x} + \frac{\lambda_{x,t}}{t+1}\sum_{p=k}^{t}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,p}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi'_{i,p})\|_1 \\ &+ \frac{\lambda_{x,t}}{t+1}\sum_{p=k}^{t}\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,p}) z_{i,t} - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p}) z'_{i,t}\|_1 \\ \leq{}& (1-\bar{w})\Delta_{i,t,x} + \frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,p}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi'_{i,p})\|_1 \\ &+ \frac{c_z\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,p}) - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})\|_1.\end{aligned} \tag{177}$$

By using Assumption 4.4 and the relation $\Delta_{i,0,y} = 0$, we iterate (177) from $t = 1$ to $t = T$ to obtain

$$\Delta_{i,t,x} \leq 2(c_{h0} + c_z L_{l,1})\sum_{p=1}^{t}(1-\bar{w})^{t-p}\lambda_{x,p-1}. \tag{178}$$

Inequalities (172), (175), and (178) imply that for agent i, the cumulative privacy budgets in T iterations $\epsilon_{i,z}$, $\epsilon_{i,y}$, and $\epsilon_{i,x}$ are bounded by $\sum_{t=1}^{T}\frac{\sqrt{2}\varrho_{t,z}(t+1)^{\varsigma_{i,z}}}{\sigma_{i,z}}$, $\sum_{t=1}^{T}\frac{\sqrt{2}\varrho_{t,y}(t+1)^{\varsigma_{i,y}}}{\sigma_{i,y}}$, and $\sum_{t=1}^{T}\frac{\sqrt{2}\varrho_{t,x}(t+1)^{\varsigma_{i,x}}}{\sigma_{i,x}}$, where $\varrho_{t,z}$, $\varrho_{t,y}$, and $\varrho_{t,x}$ are given in the theorem statement.

(2) By leveraging inequality (174) and the relation $\xi_{i,p} = \xi'_{i,p}$ for $p \neq k$, we have

$$\begin{aligned}\Delta_{i,t+1,y} \leq{}& (1-\bar{w})\Delta_{i,t,y} + \frac{\lambda_{y,t}}{t+1}\sum_{p=0,p\neq k}^{t}\|\nabla_y l(x_{i,t}, y_{i,t}; \xi_{i,p}) - \nabla_y l(x'_{i,t}, y'_{i,t}; \xi_{i,p})\|_1 \\ &+ \frac{\lambda_{y,t}}{t+1}\|\nabla_y l(x_{i,t}, y_{i,t}; \xi_{i,k}) - \nabla_y l(x'_{i,t}, y'_{i,t}; \xi'_{i,k})\|_1.\end{aligned} \tag{179}$$

Assumption 4.4 implies that for the same data $\xi_{i,p}$, we can rewrite (179) as follows:

$$\Delta_{i,t+1,y} \leq \left(1 - \bar{w} + \frac{L_{l,1}\lambda_{y,t}t}{t+1}\right)\Delta_{i,t,y} + \frac{L_{l,1}\lambda_{y,t}t}{t+1}\Delta_{i,t,x} + \frac{2c_{l0}\lambda_{y,t}}{t+1}. \tag{180}$$

By using inequality (177), we obtain

$$\begin{aligned}\Delta_{i,t+1,x} \leq{}& (1-\bar{w})\Delta_{i,t,x} + \frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,p}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi'_{i,p})\|_1 \\ &+ \frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,p}) z_{i,t} - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p}) z'_{i,t}\|_1.\end{aligned} \tag{181}$$

By using the relation $\varphi_{i,p} = \varphi'_{i,p}$ for all $p \neq k$, the second term on the right hand side of (181) satisfies

$$\frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,p}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi'_{i,p})\|_1 \le \frac{\lambda_{x,t}}{t+1}\sum_{p=0,p\neq k}^{t}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,p}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi_{i,p})\|_1$$
$$+ \frac{\lambda_{x,t}}{t+1}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,k}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi'_{i,k})\|_1 \le \frac{L_{h,1}\lambda_{x,t}t}{t+1}(\Delta_{i,t,x} + \Delta_{i,t,y}) + \frac{2c_{h0}\lambda_{x,t}}{t+1}. \tag{182}$$

Using an argument similar to the derivation of (182), the third term on the right hand side of (181) satisfies

$$\frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,p})z_{i,t} - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})z'_{i,t}\|_1$$
$$\le \frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\left(\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,p})z_{i,t} - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})z_{i,t}\|_1 + \|\nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})z_{i,t} - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})z'_{i,t}\|_1\right)$$
$$\le \frac{\lambda_{x,t}}{t+1}\sum_{p=0,p\neq k}^{t}\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,p}) - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi_{i,p})\|_1\|z_{i,t}\|_1$$
$$+ \frac{\lambda_{x,t}}{t+1}\|\nabla^2_{xy} l(x_{i,t}, y_{i,t}; \xi_{i,k}) - \nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,k})\|_1\|z_{i,t}\|_1 + \frac{\lambda_{x,t}}{t+1}\sum_{p=0}^{t}\|\nabla^2_{xy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})\|_1\|z_{i,t} - z'_{i,t}\|_1$$
$$\le \frac{c_z L_{l,2}\lambda_{x,t}t}{t+1}(\Delta_{i,t,x} + \Delta_{i,t,y}) + \frac{2c_z L_{l,1}\lambda_{x,t}}{t+1} + L_{l,1}\lambda_{x,t}\Delta_{i,t,z}. \tag{183}$$

Substituting (182) and (183) into (181) yields

$$\Delta_{i,t+1,x} \le \left(1 - \bar{w} + \frac{(L_{h,1} + c_z L_{l,2})\lambda_{x,t}t}{t+1}\right)\Delta_{i,t,x} + \frac{(L_{h,1} + c_z L_{l,2})\lambda_{x,t}t}{t+1}\Delta_{i,t,y} + \frac{2(c_{h0} + c_z L_{l,1})\lambda_{x,t}}{t+1} + L_{l,1}\lambda_{x,t}\Delta_{i,t,z}. \tag{184}$$

Furthermore, by leveraging (171) and using an argument similar to the derivation of (184), we have

$$\Delta_{i,t+1,z} \le (1 - \bar{w})\Delta_{i,t,z} + \frac{\lambda_{z,t}}{t+1}\sum_{p=0}^{t}\|\nabla_y h(x_{i,t}, y_{i,t}; \varphi_{i,p}) - \nabla_y h(x'_{i,t}, y'_{i,t}; \varphi'_{i,p})\|_1$$
$$+ \frac{\lambda_{z,t}}{t+1}\sum_{p=0}^{t}\|\nabla^2_{yy} l(x_{i,t}, y_{i,t}; \xi_{i,p})z_{i,t} - \nabla^2_{yy} l(x'_{i,t}, y'_{i,t}; \xi'_{i,p})z'_{i,t}\|_1 \tag{185}$$
$$\le (1 - \bar{w} + c_{l1}\lambda_{z,t})\Delta_{i,t,z} + (L_{h,1} + c_z L_{l,2})\frac{\lambda_{z,t}t}{t+1}(\Delta_{i,t,x} + \Delta_{i,t,y}) + \frac{2(c_{h0} + c_z L_{l,1})\lambda_{z,t}}{t+1}.$$

Summing up both sides of (180), (184), and (185), we obtain

$$\Delta_{i,t+1,x} + \Delta_{i,t+1,y} + \Delta_{i,t+1,z} \le \left(1 - \bar{w} + \frac{L_{l,1}\lambda_{y,t}t}{t+1} + (L_{h,1} + c_z L_{l,2})\frac{\lambda_{x,t}t}{t+1} + (L_{h,1} + c_z L_{l,2})\frac{\lambda_{z,t}t}{t+1}\right)\Delta_{i,t,x}$$
$$+ \left(1 - \bar{w} + \frac{L_{l,1}\lambda_{y,t}t}{t+1} + (L_{h,1} + c_z L_{l,2})\frac{\lambda_{x,t}t}{t+1} + (L_{h,1} + c_z L_{l,2})\frac{\lambda_{z,t}t}{t+1}\right)\Delta_{i,t,y} \tag{186}$$
$$+ (1 - \bar{w} + L_{l,1}\lambda_{x,t} + c_{l1}\lambda_{z,t})\Delta_{i,t,z} + \frac{2c_{l0}\lambda_{y,t}}{t+1} + \frac{2(c_{h0} + c_z L_{l,1})\lambda_{x,t}}{t+1} + \frac{2(c_{h0} + c_z L_{l,1})\lambda_{z,t}}{t+1}.$$

Since stepsizes $\lambda_{x,t}$, $\lambda_{y,t}$, and $\lambda_{z,t}$ are decaying sequences, we can choose proper initial stepsizes such that the following inequality always holds:

$$\Delta_{i,t+1,x} + \Delta_{i,t+1,y} + \Delta_{i,t+1,z}$$
$$\le \left(1 - \frac{\bar{w}}{2}\right)(\Delta_{i,t,x} + \Delta_{i,t,y} + \Delta_{i,t,z}) + \frac{2c_{l0}\lambda_{y,t}}{t+1} + \frac{2(c_{h0} + c_z L_{l,1})\lambda_{x,t}}{t+1} + \frac{2(c_{h0} + c_z L_{l,1})\lambda_{z,t}}{t+1} \tag{187}$$
$$\le \left(1 - \frac{\bar{w}}{2}\right)(\Delta_{i,t,x} + \Delta_{i,t,y} + \Delta_{i,t,z}) + \frac{M_1}{(t+1)^{\beta_\epsilon}},$$

with $M_1 = 2c_{l0}\lambda_{y,0} + 2(c_{h0} + c_z L_{l,1})\lambda_{x,0} + 2(c_{h0} + c_z L_{l,1})\lambda_{z,0}$ and $\beta_\epsilon = \min\{1 + v_x, 1 + v_y, 1 + v_z\}$.

According to Lemma 11 in Chen & Wang (2023), the following inequality holds:

$$\Delta_{i,t,x} + \Delta_{i,t,y} + \Delta_{i,t,z} \le M_2 t^{-\beta_\epsilon}, \tag{188}$$

where the constant M_2 is given by $M_2 = \left(\frac{4\beta_\epsilon}{e\ln(\frac{4}{2-2\bar{w}})}\right)^{\beta_\epsilon}\left(\frac{(\Delta_{i,0,x}+\Delta_{i,0,y}+\Delta_{i,0,z})(1-\frac{\bar{w}}{2})}{M_1} + \frac{4}{\bar{w}}\right)$.

According to (188), we have $\Delta_{i,t,x} \le M_2$, $\Delta_{i,t,y} \le M_2$, and $\Delta_{i,t,z} \le M_2$ for all $t > 0$. Substituting $\Delta_{i,t,x} \le M_2$ into (180) and using again Lemma 11 in Chen & Wang (2023), we have

$$\Delta_{i,t,y} \le \frac{C_{\epsilon y}}{(t+1)^{1+v_y}}, \text{ with } C_{\epsilon y} = \left(\frac{4(1+v_y)}{e\ln(\frac{4}{2-2\bar{w}})}\right)^{1+v_y}\left(\frac{\Delta_{i,0,y}(1-\frac{\bar{w}}{2})}{L_{l,1}\lambda_{y,0}M_2 + 2c_{l0}\lambda_{y,0}} + \frac{4}{\bar{w}}\right). \tag{189}$$

Similarly, substituting $\Delta_{i,t,x} \le M_2$ and $\Delta_{i,t,y} \le M_2$ into (185), we have

$$\Delta_{i,t,z} \le \frac{C_{\epsilon z}}{(t+1)^{1+v_z}}, \text{ with } C_{\epsilon z} = \left(\frac{4(1+v_z)}{e\ln(\frac{4}{2-2\bar{w}})}\right)^{1+v_z}\left(\frac{\Delta_{i,0,z}(1-\frac{\bar{w}}{2})}{(2M_2(L_{h,1}+c_zL_{l,2}) + 2(c_{h0}+c_zL_{l,1}))\,\lambda_{z,0}} + \frac{4}{\bar{w}}\right). \tag{190}$$

Furthermore, substituting $\Delta_{i,t,y} \le M_2$ and $\Delta_{i,t,z} \le \frac{C_{\epsilon z}}{(t+1)^{1+v_z}}$ into (184) yields

$$\Delta_{i,t,x} \le \frac{C_{\epsilon z}}{(t+1)^{1+v_x}} \text{ with } C_{\epsilon x} = \left(\frac{4(1+v_x)}{e\ln(\frac{4}{2-2\bar{w}})}\right)^{1+v_x}\left(\frac{\Delta_{i,0,x}(1-\frac{\bar{w}}{2})}{((L_{h,1}+c_zL_{l,2})M_2 + 2(c_{h0}+c_zL_{l,1}) + L_{l,1}C_{\epsilon z})\,\lambda_{x,0}} + \frac{4}{\bar{w}}\right). \tag{191}$$

By using (189)-(191) and Lemma F.2, we arrive at

$$\epsilon_{i,x} \le \sum_{t=1}^{T}\frac{\sqrt{2}C_{\epsilon x}}{\sigma_{i,x}(t+1)^{1+v_x-\varsigma_x}}, \quad \epsilon_{i,y} \le \sum_{t=1}^{T}\frac{\sqrt{2}C_{\epsilon y}}{\sigma_{i,y}(t+1)^{1+v_y-\varsigma_y}}, \quad \epsilon_{i,z} \le \sum_{t=1}^{T}\frac{\sqrt{2}C_{\epsilon z}}{\sigma_{i,z}(t+1)^{1+v_z-\varsigma_z}}, \tag{192}$$

implying that $\epsilon_i = \epsilon_{i,z} + \epsilon_{i,y} + \epsilon_{i,x}$ is finite even when T tends to infinity since $v_x > \varsigma_x$, $v_y > \varsigma_y$, and $v_z > \varsigma_z$. □

G. Proofs of Corollaries 4.3 and 4.6

G.1. Proof of Corollary 4.3

Proof. (1) For a strongly convex $F(x)$, the convergence rate of Algorithm 2 is $\mathcal{O}(T^{-\beta_1})$ based on (10). Therefore, setting $T^{-\beta_1} = \delta$ yields that the iteration complexity of Algorithm 2 is $\mathcal{O}(\delta^{-\frac{1}{\beta_1}})$ in finding a δ-solution. Furthermore, since the per-iteration complexity of Algorithm 2 is $\max\{p, q\}$, the computational complexity of Algorithm 2 is $\mathcal{O}(\max\{p, q\}\delta^{-\frac{1}{\beta_1}})$ in finding a δ-solution.

According to the conditions $0 < v_z < v_y < v_z < 1$, $2\varsigma_x > v_x$, $2\varsigma_x > v_z + v_y$, $2\varsigma_y > v_z + v_y$, and $2\varsigma_z > v_y$ given in Theorem 4.1-(1), we can choose $v_x = 0.66$, $v_y = 0.64$, $v_z = 0.43$, $\varsigma_x = 0.65$, $\varsigma_y = 0.63$, and $\varsigma_z = 0.42$. Under these parameters, the convergence rate is $\beta_1 = \min\{0.64, 0.44, 0.4, 0.43, 0.62, 0.72\} = 0.4$ and the computational complexity is $\mathcal{O}(\max\{p, q\}\delta^{-2.5})$.

(2) Similarly, for a convex $F(x)$, the convergence rate of Algorithm 2 is $\mathcal{O}(T^{-(1-v_x)})$ based on (11). Therefore, the computational complexity of Algorithm 2 is $\mathcal{O}(\max\{p, q\}\delta^{-\frac{1}{1-v_x}})$ in finding a δ-solution. Recalling the conditions $0 < v_z < v_y < v_z < 1$, $\varsigma_x > \frac{1}{2}$, $2\varsigma_x > v_z + v_y$, $2\varsigma_x > 2v_z + 2 - 2v_x$, $2\varsigma_y > v_z + v_y$, $2\varsigma_y > 2v_z + 2 - 2v_x$, $2\varsigma_y > v_y + 2 - 2v_x$, $2\varsigma_z > v_z + 2 - 2v_x$, and $2\varsigma_z > v_y$ given in Theorem 4.1-(2), we can select $v_x = 0.77$, $v_y = 0.75$, $v_z = 0.5$, $\varsigma_x = 0.76$, $\varsigma_y = 0.74$, and $\varsigma_z = 0.49$ yielding a convergence rate of $1 - v_x = 0.23$ and a computational complexity of $\mathcal{O}(\max\{p, q\}\delta^{-4.35})$.

(3) For a nonconvex $F(x)$, the convergence rate of Algorithm 2 is $\mathcal{O}(T^{-(1-v_x)})$ based on (12). Therefore, the computational complexity of Algorithm 2 is $\mathcal{O}(\max\{p, q\}\delta^{-\frac{1}{1-v_x}})$ in finding a δ-solution. We use $v_x = 0.615$, $v_y = 0.60375$, $v_z = 0.4$,

$\varsigma_x = 0.61125$, $\varsigma_y = 0.6$, and $\varsigma_z = 0.398125$ to satisfy the conditions $0 < v_z < v_y < v_x < 1$, $\varsigma_x > \frac{1}{2}$, $2\varsigma_x > v_z + v_y$, $2\varsigma_x > 2v_z + 1 - v_x$, $2\varsigma_y > 2v_z + 1 - v_x$, $2\varsigma_y > v_y + 1 - v_x$, $2\varsigma_y > v_z + v_y$, $2\varsigma_z > v_z + 1 - v_x$, and $2\varsigma_z > v_y$ given in Theorem 4.1-(3). Under these parameters, the convergence rate is $1 - v_x = 0.385$ and the complexity is $\mathcal{O}(\max\{p, q\}\delta^{-2.6})$. □

G.2. Proof of Corollary 4.6

Proof. The convergence rate $\mathcal{O}\left(T^{v_x-1}\right)$ follows naturally from Theorem 4.1-(3).

Next, we characterize the cumulative privacy budget. We select $v_x = \frac{3}{5} + \kappa$, $v_y = \frac{3}{5} + \frac{\kappa}{4}$, $v_z = \frac{2}{5}$, $\varsigma_x = v_x - \frac{\kappa}{4}$, $\varsigma_y = v_y - \frac{\kappa}{4}$, and $\varsigma_z = v_z - \frac{\kappa}{8}$ with $\kappa \in (0, \frac{2}{5})$ that satisfy the conditions given in Theorem 4.1-(3). In this case, based on the convergence rate in (12) from Theorem 4.1-(3), we have

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\left[\|\nabla F(x_{i,t})\|^2\right] \leq \mathcal{O}\left(T^{\kappa-\frac{2}{5}}\right), \tag{193}$$

which implies that the iteration complexity of Algorithm 2 is no more than $\mathcal{O}\big(\delta^{-\frac{5}{2-5\kappa}}\big)$. Moreover, it is evident that a smaller κ corresponds to a faster convergence rate and less iteration complexity.

We proceed to characterize the cumulative privacy budget for agent i's implementation under Algorithm 2. Based on (182), we can obtain

$$\begin{aligned}\epsilon_i = \epsilon_{i,x} + \epsilon_{i,y} + \epsilon_{i,y} \leq & \frac{\sqrt{2}C_{\epsilon x}}{\sigma_{i,x}(v_x - \varsigma_x)}\left(1 - (T+1)^{-(v_x-\varsigma_x)}\right) + \frac{\sqrt{2}C_{\epsilon y}}{\sigma_{i,y}(v_y - \varsigma_y)}\left(1 - (T+1)^{-(v_y-\varsigma_y)}\right) \\ & + \frac{\sqrt{2}C_{\epsilon z}}{\sigma_{i,z}(v_z - \varsigma_z)}\left(1 - (T+1)^{-(v_z-\varsigma_z)}\right),\end{aligned} \tag{194}$$

where in the derivation we have used the following inequality:

$$\sum_{t=1}^{T}\frac{1}{(t+1)^r} \leq \int_0^T \frac{1}{(x+1)^r}dx = \frac{1}{1-r}\left((T+1)^{1-r} - 1\right). \tag{195}$$

Substituting the given parameters $v_x = \frac{3}{5} + \kappa$, $v_y = \frac{3}{5} + \frac{\kappa}{4}$, $v_z = \frac{2}{5}$, $\varsigma_x = v_x - \frac{\kappa}{4}$, $\varsigma_y = v_y - \frac{\kappa}{4}$, and $\varsigma_z = v_z - \frac{\kappa}{8}$ with $\kappa \in (0, \frac{2}{5})$ into (194), we arrive at

$$\epsilon_i = \epsilon_{i,x} + \epsilon_{i,y} + \epsilon_{i,y} \leq O\left(\frac{1}{\kappa} - \frac{1}{\kappa(T+1)^{\frac{\kappa}{8}}}\right) = O\left(\frac{1-(T+1)^{-\frac{\kappa}{8}}}{\kappa}\right). \tag{196}$$

By substituting the obtained relations $T + 1 = \mathcal{O}\big(\delta^{-\frac{5}{2-5\kappa}}\big)$ into (196), we have that the cumulative privacy budget for each agent i's implementation is in the order $\mathcal{O}\big(\frac{1}{\kappa} - \frac{\delta^{-\frac{5\kappa}{16-40\kappa}}}{\kappa}\big)$ when Algorithm 2 achieves a δ-solution.

It is evident that for a given $\delta > 0$, the cumulative privacy budget is no more than $\mathcal{O}\big(\frac{1}{\kappa}\big)$. Since the constant κ was set to $\kappa = v_x - \frac{3}{5}$, we can obtain the cumulative privacy budget scaled as $\mathcal{O}\big(\frac{1}{v_x-0.6}\big)$ with $v_x \in (0.6, 1)$. □

H. The Reason why Existing DSBO Algorithms cannot Ensure a Finite Cumulative Privacy Budget ϵ_i

H.1. The Limitation of Existing DSBO Algorithms under Differential-Privacy Constraints

In this section, we explain the limitation of existing DSBO algorithms in Chen et al. (2022), Yang et al. (2022), and Chen et al. (2023) under LDP constraints. Specifically, to obtain good approximations of the hypergradient and/or the optimal solution y^* to the lower-level optimization problem in (1), these algorithms incorporate inner-loop iterations into the outer algorithmic iteration, which leads to a cumulative privacy budget that grows to infinity as the number of outer iterations tends to infinity.

We use the DSBO-HIGP algorithm in Chen et al. (2023) as an example to illustrate this idea. To ensure privacy, persistent DP-noises have to be added to messages transmitted in each iteration of the DSBO-HIGP algorithm. Then, the modified

Algorithm 3 LDP design for DSBO-HIGP

1: **Input:** Stepsizes α_t, β_t, and γ; Iterations $T > 0$, $K > 0$, and $N = \log(T)$; Initialization $y_{i,k}^0 = 0$, $x_{i,0} = r_{i,0} = 0$, $d_{i,t}^0 = -b_{i,t}^0$, $s_{i,t}^0 = -b_{i,t}^0$, and $z_{i,t}^0 = 0$; DP-noises $\vartheta_{i,t}^k$, $\zeta_{i,t}^k$, and $\chi_{i,t}^k$ satisfying Assumption 3.1.
2: **for** $t = 0, 1, \cdots, T-1$ **do**
3: $\quad y_{i,t}^0 = y_{i,t-1}^K$.
4: $\quad$ **for** $k = 0, 1, \cdots, K-1$ **do**
5: $\quad\quad$ **for** $i = 0, 1, \cdots, m-1$ **do**
6: $\quad\quad\quad y_{i,t}^{k+1} = y_{i,t}^k + \sum_{j \in \mathcal{N}_i} w_{ij}(y_{j,t}^k + \zeta_{j,t}^k - y_{i,t}^k) - \beta_t v_{i,t}^k$ with $v_{i,t}^k = \nabla_y g_i(x_{i,t}, y_{i,t}^k; \xi_{i,t}^k)$.
7: $\quad\quad$ **end for**
8: $\quad$ **end for**
9: $\quad$ **for** $k = 0, 1, \cdots, N-1$ **do**
10: $\quad\quad$ **for** $i = 0, 1, \cdots, m-1$ **do**
11: $\quad\quad\quad z_{i,t}^{k+1} = z_{i,t}^k + \sum_{j \in \mathcal{N}_i} w_{ij}(z_{j,t}^k + \vartheta_{j,t}^k - z_{i,t}^k) - \gamma d_{i,t}^k$,
12: $\quad\quad\quad s_{i,t}^{k+1} = H_{i,t}^{k+1} z_{i,t}^{k+1} - b_{i,t}^{k+1}$,
13: $\quad\quad\quad d_{i,t}^{k+1} = d_{i,t}^k + \sum_{j \in \mathcal{N}_i} w_{ij}(d_{j,t}^k + \vartheta_{j,t}^k - d_{i,t}^k) + s_{i,t}^{k+1} - s_{i,t}^k$.
14: $\quad\quad$ **end for**
15: $\quad$ **end for**
16: $\quad u_{i,t} = \nabla_x f_i(x_{i,t}, y_{i,t}^K; \varphi_{i,0}) - \nabla_{xy}^2 g_i(x_{i,t}, y_{i,t}^K; \xi_{i,0}) z_{i,t}^N$.
17: $\quad$ **for** $i = 0, 1, \cdots, m-1$ **do**
18: $\quad\quad x_{i,t+1} = x_{i,t} + \sum_{j \in \mathcal{N}_i} w_{ij}(x_{j,t} + \chi_{j,t} - x_{i,t}) - \alpha_t r_{i,t}$,
19: $\quad\quad r_{i,t+1} = (1-\alpha_t) r_{i,t} + \alpha_t u_{i,t}$.
20: $\quad$ **end for**
21: **end for**
22: **Output:** $\bar{x}_T = \frac{1}{m}\sum_{i=1}^m x_{i,T}$.

DSBO-HIGP algorithm with injected DP-noises is described in the following Algorithm 3. It can be seen that Algorithm 3 has double inner-loops: a K-step inner-loop (lines 4-8) for achieving a good approximation of y^* (the optimal solution to the lower-level optimization problem in (1)) and an N-step inner-loop (lines 9-15) for a good estimation of the hypergradient $\nabla F(x)$. DP-noises have been injected into all communication steps to enable privacy. According to Theorem 3.3 in Chen et al. (2023), the convergence of the original DSBO-HIGP can be guaranteed only when $K = \log(T)$, $N \geq 1$, $\alpha_t = \mathcal{O}(\frac{1}{\sqrt{T}})$, $\forall T > 0$, $\beta_t = \mathcal{O}(\frac{1}{\sqrt{T}})$, $\forall T > 0$, and $\gamma \in (c_1, c_2)$ with $0 < c_1 < c_2$. It is worth noting that when T tends to infinity, the number of iterations K also tends to infinity.

With this understanding, we first analyze the cumulative privacy budget $\epsilon_{i,y}$ associated with $y_{i,t}$ in Algorithm 3. By leveraging (192), the cumulative privacy budget $\epsilon_{i,y}$ of Algorithm 3 satisfies

$$\epsilon_{i,y} \leq \sum_{t=1}^{T}\sum_{k=1}^{K} \mathcal{O}\left(\frac{\beta_t}{\sigma_{i,y,t}^k (t+1)}\right), \tag{197}$$

where $\sigma_{i,y,t}^k$ represents the variance of the DP-noise $\zeta_{i,t}^k$.

When the DP-noise variance decays over the outer-loop iteration t (in this case, a fixed DP-noise is injected into the consensus operation at Algorithm 3 Step 6 during each inner-loop iteration, which degrades the estimation performance of the global y^*), the convergence of Algorithm 3 is significantly affected. Therefore, we consider the following two designs for $\sigma_{i,y,t}^k$:

(1) The DP-noise variance decays over both inner-loop iterations k and outer-loop iterations t, i.e., $\sigma_{i,y,t}^k = \mathcal{O}\left(\frac{1}{(t+1)^{\varsigma_y}(k+1)^{\varsigma_y}}\right)$,

(2) The DP-noise variance decays over inner-loop iterations k, i.e.,$\sigma_{i,y,t}^k = \mathcal{O}\left(\frac{1}{(k+1)^{\varsigma_y}}\right)$.

By using the decaying stepsize $\beta_t = \mathcal{O}(\frac{1}{(t+1)^{v_y}})$ with $v_y \in (0,1)$, the cumulative privacy budget $\epsilon_{i,y}$ for the aforementioned two scenarios satisfy

$$(1)\quad \epsilon_{i,y} \leq \sum_{t=1}^{T} \mathcal{O}\left(\frac{1}{(t+1)^{1+v_y-\varsigma_y}}\right) \sum_{k=1}^{K} \mathcal{O}\left((k+1)^{\varsigma_y}\right), \quad (2)\quad \epsilon_{i,y} \leq \sum_{t=1}^{T} \mathcal{O}\left(\frac{1}{(t+1)^{1+v_y}}\right) \sum_{k=1}^{K} \mathcal{O}\left((k+1)^{\varsigma_y}\right),$$

which imply that the cumulative privacy budget $\epsilon_{i,y}$ in both scenarios will grow to infinity when the number of outer iterations T tends to infinity, thus violating rigorous ϵ_i-LDP privacy constraints. Of course, employing a constant stepsize γ in the N-step inner-loop (lines 9-15) of Algorithm 3 exacerbates this issue, leading to a significant increase in the cumulative privacy budget $\epsilon_{i,z}$ (see the following Section H.2 for details).

The above mentioned issue also exists in other inner-loop-based DSBO algorithms (Chen et al., 2022; Yang et al., 2022).

H.2. The Calculations of the Cumulative Privacy Budget for the Algorithms Listed in Table 1

First, we compute the computational complexity and the cumulative privacy budget of our Algorithm 2, i.e., LDP-DSBO. We select $v_x = \frac{3}{5} + \kappa$, $v_y = \frac{3}{5} + \frac{\kappa}{4}$, $v_z = \frac{2}{5}$, $\varsigma_x = v_x - \frac{\kappa}{4}$, $\varsigma_y = v_y - \frac{\kappa}{4}$, and $\varsigma_z = v_z - \frac{\kappa}{8}$ with $\kappa \in (0, \frac{2}{5})$ that satisfy the conditions given in Theorem 4.1-(3) (Since all results in Table 1 are obtained for a nonconvex F). Under these settings, the iteration complexity of Algorithm 2 is $\mathcal{O}\big(\delta^{-\frac{5}{2-5\kappa}}\big)$ and the cumulative privacy budget is $\mathcal{O}\big(\frac{1}{\kappa}\big)$ (Detailed computations of the iteration complexity and the cumulative privacy budget have been given in the proof of Corollary 4.6 in Appendix G.2). In this case, we can choose $\kappa \approx 0.015$ such that the iteration complexity of Algorithm 2 is no more than $\mathcal{O}\big(\delta^{-2.6}\big)$ and the cumulative privacy budget is 66.67, which is a constant and hence has an order of $\mathcal{O}(1)$.

Then, we compute the cumulative privacy budget of the remaining algorithms (except LDP-DSBO) listed in Table 1. For these algorithms, we employ the same Laplace noise used in our algorithm.

Given that all remaining algorithms in Table 1 use a constant stepsize, we estimate their cumulative privacy budgets ϵ_i under a stepsize $\gamma > 0$ and the DP-noise variance $\mathcal{O}\left(\frac{1}{(t+1)^{\varsigma}}\right)$ for some $\varsigma \in (0,1)$. Additionally, we do not include inner-loops in this estimation. As explained in Subsection H.1, inner-loops cannot ensure a finite cumulative privacy budget in the infinite-time horizon, and thus a relaxed condition is considered for these algorithms, which makes the results better than the actual case. Based on (192), we obtain

$$\epsilon_i \leq \sum_{t=1}^{T} \mathcal{O}\left(\frac{\gamma}{\sigma_t(t+1)}\right) \leq \sum_{t=1}^{T} \mathcal{O}\left(\frac{1}{(t+1)^{1-\varsigma}}\right) \leq \mathcal{O}\left((T+1)^{\varsigma}\right), \tag{198}$$

where σ_t is the DP-noise variance and we have used the following relation for the last inequality:

$$\sum_{t=1}^{T} \frac{1}{(t+1)^{1-\varsigma}} \leq \int_{1}^{T+1} x^{\varsigma-1} dx \leq \frac{1}{\varsigma}(T+1)^{\varsigma} - \frac{1}{\varsigma} \leq \frac{1}{\varsigma}(T+1)^{\varsigma}. \tag{199}$$

By substituting the respective complexities of the algorithms listed in Table 1 into (199), we can obtain the results given in the last column of Table 1.