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The quasi-periodic Cauchy problem for the generalized Benjamin-Bona-Mahony equation on the real line

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This paper studies the existence and uniqueness problem for the generalized Benjamin-Bona-Mahony (gBBM) equation with quasi-periodic initial data on the real line. We obtain an existence and uniqueness result in the classical sense with arbitrary time horizon under the assumption of polynomially decaying initial Fourier data using the combinatorial analysis method developed in earlier papers by Christ [6], Damanik-Goldstein [11], and the present authors [12]. Our result is valid for exponentially decaying initial Fourier data and hence can be viewed as a Cauchy-Kovalevskaya theorem in the space

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Generalized Benjamin-Bona-Mahony equation

variable for the gBBM equation with quasi-periodic initial data.

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1. Introduction

The equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{1.1}$$

was introduced by Benjamin, Bona, and Mahony in [3] as an improvement of the Korteweg–de Vries equation

$$u_t + u_{xxx} + uu_x = 0 \tag{1.2}$$

for modeling unidirectional propagation of long waves of small amplitude on $\mathbb{R}$. We will for simplicity refer to (1.1) as BBM and to (1.2) as KdV.

Olver showed in [28] that BBM possesses exactly three independent and non-trivial conservation laws, whereas KdV is known to possess infinitely many [24]. Both equations admit solitary wave solutions. KdV can be described via a Lax pair [22], whereas BBM cannot.

Establishing the existence of solutions to the Cauchy problem associated with BBM is simpler for decaying or periodic initial data. For some foundational results in these two special cases we refer the reader to [3] and [25], respectively. In the present paper we are interested in studying spatially quasi-periodic solutions, which is a more challenging task.

For KdV with quasi-periodic initial data, the existence and uniqueness of solutions was studied by Tsugawa [30] and Damanik-Goldstein [11]. More recently, the analogous problem for the generalized Korteweg–de Vries (gKdV) equation

$$u_t + u_{xxx} + u^{p-1} u_x = 0 \tag{1.3}$$

was studied by the three of us in [12]. Via these works it is known that sufficiently small quasi-periodic initial data with exponentially decaying Fourier coefficients admit a local in time solution that remains quasi-periodicity in the spatial variable with exponentially decaying Fourier coefficients. Indeed, within this class of functions, the solution is unique. For KdV one can go further and show that, for Diophantine frequency vector, the local result can be iterated in constant time steps. In this way, one obtains global existence and uniqueness [11]. We mention in passing that the dependence on time is in this setting known to be almost periodic [4], which is a result in line with (and providing evidence for) the Deift conjecture [8,9], which states that the KdV equation with almost periodic initial data admits a global solution that is almost periodic both in space and time.[5]

This passage from a local to a global result in [11] rests on a rather involved spectral analysis of quasi-periodic Schrödinger operators [10]. As an input of this kind is not available for gKdV, it is at present unclear how to leverage the local result from [12] to a global result.

In this paper we want to discuss the existence and uniqueness of spatially quasi-periodic solutions of BBM, and in fact more generally of the generalized Benjamin-Bona-Mahony (gBBM) equation

$$u_t - u_{xxt} + u_x + u^{p-1} u_x = 0, \tag{1.4}$$

on the real line $\mathbb{R}$, where $2 \leq p \in \mathbb{N}$.

As initial data we consider quasi-periodic functions of the form

$$u(0,x) = \sum_{n \in \mathbb{Z}^\nu} \hat{u}(n) e^{\mathrm{i}\langle n \rangle x}, \tag{1.5}$$

where $x \in \mathbb{R}$, $2 \leq \nu \in \mathbb{N}$, $\omega = (\omega_1, \cdots, \omega_\nu) \in \mathbb{R}^\nu$ (a given frequency vector), $n = (n_1, \cdots, n_\nu) \in \mathbb{Z}^\nu$ (the lattice vector), and $\langle n \rangle \triangleq \langle n, \omega \rangle$ is the standard inner product defined by letting $\langle n, \omega \rangle := \sum_{j=1}^\nu n_j \omega_j$. As usual we assume that the frequency vector is non-resonant or rationally independent, that is, $\langle n \rangle = 0$ implies that $n = 0 \in \mathbb{Z}^\nu$.

What we are interested in is the existence and uniqueness of spatially quasi-periodic solutions defined by the Fourier series

$$u(t,x) = \sum_{n \in \mathbb{Z}^\nu} \hat{u}(t,n) e^{\mathrm{i}\langle n \rangle x} \tag{1.6}$$

to the quasi-periodic Cauchy problem (1.4)–(1.5) in the classical sense.

[5] While our paper was under consideration, the Deift conjecture as formulated in [8,9] has been disproved by Chapouto, Killip, and Visan in [7]. It remains an interesting open problem to establish a positive result in the spirit of the Deift conjecture, that is, a proof of an accordingly modified conjecture.

Throughout this paper, we will use $|n|$ to denote the $\ell^1(\mathbb{Z}^\nu)$-norm of $n = (n_1, \cdot, n_\nu) \in \mathbb{Z}^\nu$, that is, $|n| := \sum_{j=1}^{\nu} |n_j|$.

Our main results are the following Theorem A (exponential decay) and Theorem B (polynomial decay) below.

Theorem A. *Suppose the Fourier coefficients* $\hat{u}(n)$ *of the initial data satisfy the following exponential decay condition,*

$$|\hat{u}(n)| \le \mathcal{A}^{\frac{1}{p-1}} e^{-\rho|n|}, \quad \forall n \in \mathbb{Z}^\nu, \tag{1.7}$$

where $\mathcal{A} > 0$ *and* $0 < \rho \le 1$.

Then, on the time interval $[0, \mathcal{L}_p]$, *where*

$$\mathcal{L}_p \triangleq 2\left(1 - \frac{1}{p}\right)^{p-1} \frac{\rho^{(p-1)\nu}}{\mathcal{A}6^{(p-1)\nu}}, \tag{1.8}$$

the quasi-periodic Cauchy problem (1.4)–(1.5) *has a spatially quasi-periodic solution in the classical sense of the form*

$$u(t, x) = \sum_{n \in \mathbb{Z}^\nu} \hat{u}(t, n) e^{\mathrm{i}\langle n \rangle x}$$

with

$$|\hat{u}(t, n)| \le \mathcal{B}_p e^{-\frac{\rho}{2}|n|},$$

where $\mathcal{B}_p \triangleq \frac{p}{p-1} \mathcal{A}^{\frac{1}{p-1}} (6\rho^{-1})^\nu$.

Moreover, this solution is unique among all solutions subject to this exponential decay estimate for the Fourier coefficients.

Remark 1.1. Theorem A is a local existence and uniqueness result with arbitrary time horizon. That is, given any $T > 0$, we provide an explicit class of quasi-periodic initial data with exponential Fourier decay (namely those obeying (1.7) with parameters $\mathcal{A}, \rho, \nu$ subject to the condition

$$2\left(1 - \frac{1}{p}\right)^{p-1} \frac{\rho^{(p-1)\nu}}{\mathcal{A}6^{(p-1)\nu}} \ge T$$

for the prescribed value of T) for which we establish the existence of a unique spatially quasi-periodic solution to (1.4)–(1.5) up to the time horizon T.

Furthermore, we may replace the exponential decay condition (1.7) by the polynomial decay condition (1.9) below and obtain the same conclusions for gBBM (1.4) as in

Theorem A. To express the time horizon as a function of the parameters of the decay parameters, it is convenient to introduce

$$\mathfrak{b}(\mathtt{s};\nu) \triangleq 1 + \sum_{j=1}^{\nu} \binom{\nu}{j} 2^j j^{-\mathtt{s}} \left\{ \zeta\left(\frac{\mathtt{s}}{j}\right) \right\}^j,$$

where ζ is the Riemann zeta function,

$$\zeta(\mathtt{s}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathtt{s}}}.$$

Theorem B. *Suppose the Fourier coefficients $\hat{u}(n)$ of the initial data satisfy the following polynomial decay condition,*

$$|\hat{u}(n)| \le \mathtt{A}^{\frac{1}{p-1}} (1+|n|)^{-\mathtt{r}}, \quad \forall n \in \mathbb{Z}^{\nu}, \tag{1.9}$$

where $\mathtt{A} > 0$ and $2 \le \nu < \frac{\mathtt{r}}{4} - 2$.

Then, on the time interval $[0, \mathcal{L}_p']$, where

$$\mathcal{L}_p' \triangleq 2\left(1 - \frac{1}{p}\right)^{p-1} \mathtt{A}^{-1} \mathfrak{b}\left(\frac{\mathtt{r}}{2};\nu\right)^{-(p-1)}. \tag{1.10}$$

the quasi-periodic Cauchy problem (1.4)–(1.5) has a spatially quasi-periodic solution in the classical sense of the form

$$u(t,x) = \sum_{n\in\mathbb{Z}^{\nu}} \hat{u}(t,n) e^{\mathrm{i}\langle n \rangle x}$$

with

$$|\hat{u}(t,n)| \le \mathtt{B}_p (1+|n|)^{-\frac{\mathtt{r}}{2}},$$

where $\mathtt{B}_p = \frac{p}{p-1} \mathtt{A}^{\frac{1}{p-1}} \mathfrak{b}\left(\frac{\mathtt{r}}{2};\nu\right)$.

Moreover, this solution is unique among all solutions subject to this polynomial decay estimate for the Fourier coefficients.

Remark 1.2. (a) What was pointed out in Remark 1.1 applies equally well here. Theorem B is a local existence and uniqueness result with arbitrary time horizon. That is, given any $T > 0$, we provide an explicit class of quasi-periodic initial data with polynomial Fourier decay (namely those obeying (1.9) with parameters $\mathtt{A}, \mathtt{r}, \nu$ subject to the condition

$$2\left(1 - \frac{1}{p}\right)^{p-1} \mathtt{A}^{-1} \mathfrak{b}\left(\frac{\mathtt{r}}{2};\nu\right)^{-(p-1)} \ge T$$

for the prescribed value of T) for which we establish the existence of a unique spatially quasi-periodic solution to (1.4)–(1.5) up to the time horizon T.

(b) It follows from the exponential (resp. polynomial) decay that the solution we construct is in the classical sense. In addition, the exponential decay property implies that our result can be viewed as a Cauchy-Kovalevskaya theorem in the space variable for the gBBM equation with quasi-periodic initial data, based on a basic fact: a quasi-periodic Fourier series with exponentially decaying Fourier coefficients is analytic.

(c) The extension of the existence result for BBM with smooth and decaying initial data from [3] to the case of gBBM was discussed by Albert in [1], see also [2]. Moreover, solutions for gBBM with $p = 5$ that are periodic in space and quasi-periodic in time were discussed by Shi and Yan in [29]. Let us also mention that Wang discussed in [32] solutions to nonlinear PDEs that are periodic in space and quasi-periodic in time from a more general perspective.

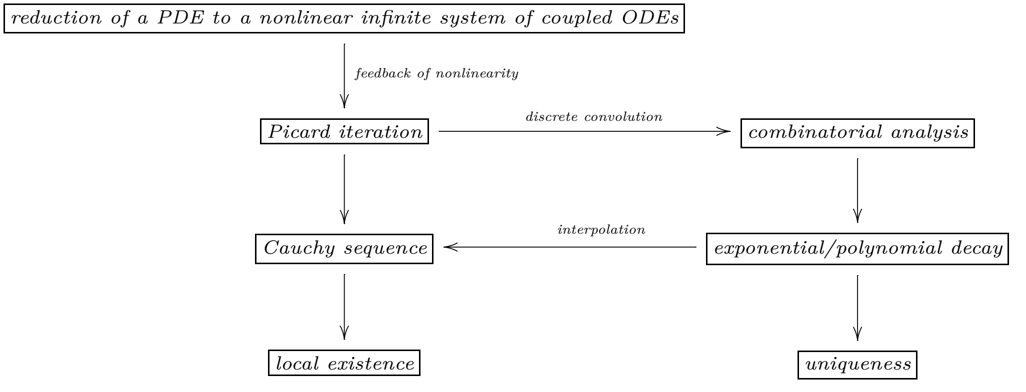
(d) The dependence on the spatial variable in our setting is neither decaying nor periodic. There are only a few existing results for initial data lacking these two properties. In addition to the works already mentioned, Oh discussed the nonlinear Schrödinger equation in one dimension with almost periodic initial data [26,27] and Wang presented spatially quasi-periodic standing wave solutions to the nonlinear Schrödinger equation in arbitrary dimension [33]. We also refer the reader to [13,14,19] for a broader discussion and to [17,18,20,23], which are primarily based on inverse spectral theory and the preservation of reflectionlessness by equations in the KdV hierarchy (see also [5,15,16,31] for related work).

(e) The absence of decay and periodicity makes the problem at hand significantly more difficult. As in the works [11] and [12] we have to deal with the higher dimensional discrete convolution operation

$$\hat{u}^{*p}(\text{fixed total distance}) = \sum_{\substack{q_1,\cdots,q_p\in\mathbb{Z}^\nu \\ q_1+\cdots+q_p=\text{ fixed total distance}}} \prod_{j=1}^{p} \hat{u}(q_j)$$

appearing in the Picard iteration, during which the number of terms will increase exponentially. More precisely, let $\mathbf{N}_k$ be the number of terms for the Picard sequence. It is easy to see that $\mathbf{N}_1 = 2$ and $\mathbf{N}_k = 1 + \mathbf{N}_{k-1}^p$ for all $k \geq 2$. The key point to overcoming this difficulty is an explicit combinatorial analysis in order to obtain the exponential (resp. polynomial) decay of the Picard sequence; see [6], [11] and [12] for an implementation of this strategy for NLS, KdV and gKdV, respectively.

(f) The structure of the proofs of Theorem A and Theorem B is given by the following diagram:



2. The special case $p = 2$: BBM

For the sake of convenience and readability, we first study the quasi-periodic Cauchy problem (1.4)–(1.5) for $p = 2$. Whenever we refer to (1.4) in this section we tacitly assume that $p = 2$.

We will denote the Fourier coefficients at time 0 and time t by $c(n) \triangleq \hat{u}(n)$ and $c(t, n) \triangleq \hat{u}(t, n)$, respectively.

2.1. Reduction

The first step in our proof is a reduction of the PDE in question to a nonlinear infinite system of coupled ODEs. For the latter we then consider a suitable Picard sequence.

Formally, by the Cauchy product for infinite series (i.e., the discrete convolution operation), we have

$$(u^2)(t,x) = \sum_{n \in \mathbb{Z}^\nu} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^\nu \\ n_1 + n_2 = n}} \prod_{j=1}^{2} c(t, n_j) e^{\mathrm{i}\langle n \rangle x}. \tag{2.1}$$

Assuming that ∂ and $\sum$ can be interchanged, we have

$$u_t - u_{xxt} = \sum_{n \in \mathbb{Z}^\nu} (1 + \langle n \rangle^2)(\partial_t c)(t, n) e^{\mathrm{i}\langle n \rangle x}, \tag{2.2a}$$

$$u_x = \sum_{n \in \mathbb{Z}^\nu} \mathrm{i}\langle n \rangle c(t, n) e^{\mathrm{i}\langle n \rangle x}, \tag{2.2b}$$

$$u u_x = \partial_x \left(\frac{u^2}{2} \right) = \sum_{n \in \mathbb{Z}^\nu} \frac{\mathrm{i}\langle n \rangle}{2} \sum_{\substack{n_1, n_2 \in \mathbb{Z}^\nu \\ n_1 + n_2 = n}} \prod_{j=1}^{2} c(t, n_j) e^{\mathrm{i}\langle n \rangle x}. \tag{2.2c}$$

Substituting (2.2a)-(2.2c) into (1.4) yields

$$\sum_{n\in\mathbb{Z}^\nu}\left\{(1+\langle n\rangle^2)(\partial_t c)(t,n)+\mathrm{i}\langle n\rangle c(t,n)+\frac{\mathrm{i}\langle n\rangle}{2}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}c(t,n_j)\right\}e^{\mathrm{i}\langle n\rangle x}=0.$$

By the orthogonality of $\{e^{\mathrm{i}\langle n\rangle x}: x\in\mathbb{R}\}$ relative to

$$< u,v >_{L^2(\mathbb{R})}:=\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{+L}u(x)\bar{v}(x)\mathrm{d}x,$$

we see that (1.4) is equivalent to the nonlinear infinite system of coupled ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t,n)-\lambda(n)c(t,n)=\frac{\lambda(n)}{2}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}c(t,n_j),\tag{2.3}$$

where

$$\lambda(n)\triangleq\frac{-\mathrm{i}\langle n\rangle}{1+\langle n\rangle^2}\quad\text{(a purely imaginary number)}\tag{2.4}$$

obeys the uniform bound $|\lambda(n)|\leq\frac{1}{2}$ for all $n\in\mathbb{Z}^\nu$. Here we use "$\frac{\mathrm{d}}{\mathrm{d}t}$" rather than "$\partial_t$" to emphasize that (2.3) is an ODE for any given $n\in\mathbb{Z}^\nu$.

Motivated by an idea from [21], we observe that $c(t,n)$ is determined by the following integral equation,

$$c(t,n)=e^{\lambda(n)t}c(n)+\frac{\lambda(n)}{2}\int_0^t e^{\lambda(n)(t-\tau)}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}c(\tau,n_j)\,\mathrm{d}\tau.\tag{2.5}$$

To determine $c(t,n)$, we construct a Picard sequence $\{c_k(t,n)\}_{k\geq0}$ to approximate it. We choose $e^{\lambda(n)t}c(n)$ as the initial guess $c_0(t,n)$ and obtain $\{c_k(t,n)\}_{k\geq1}$ via the following iteration,

$$c_k(t,n):=c_0(t,n)+\frac{\lambda(n)}{2}\int_0^t e^{\lambda(n)(t-\tau)}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}c_{k-1}(\tau,n_j)\,\mathrm{d}\tau,\quad\forall k\geq1.\tag{2.6}$$

2.2. Combinatorial tree for the Picard sequence

Our goal is to show that the Picard sequence converges. To this end, it is convenient to express it via a combinatorial tree. This is the aim of the present subsection.

Set

$$\spadesuit^{(k)} := \begin{cases} \{0,1\}, & k=1; \\ \{0\} \cup (\spadesuit^{(k-1)})^2, & k \geq 2. \end{cases}$$

For $\gamma^{(k)} = 0 \in \spadesuit^{(k)}, k \geq 1$, $\mathfrak{N}^{(k,0)} := \mathbb{Z}^\nu$; for $\gamma^{(1)} = 1 \in \spadesuit^{(1)}$, $\mathfrak{N}^{(1,1)} := (\mathbb{Z}^\nu)^2$; for $\gamma^{(k)} = (\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) \in (\spadesuit^{(k-1)})^2, k \geq 2$, $\mathfrak{N}^{(k,\gamma^{(k)})} := \prod_{j=1}^2 \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})}$.

Define a function $\mu : \cup_{k=1}^\infty (\mathbb{Z}^\nu)^k \to \mathbb{Z}^\nu$ by letting $\mu(\clubsuit) = \sum_{j=1}^k \clubsuit_j$ if $\clubsuit = (\clubsuit_j \in \mathbb{Z}^\nu)_{1 \leq j \leq k} \in (\mathbb{Z}^\nu)^k$.

For $k \geq 1, \gamma^{(k)} = 0 \in \spadesuit^{(k)}, n = n^{(k)} \in \mathfrak{N}^{(k,0)}$,

$$\mathfrak{C}^{(k,0)}(n^{(k)}) := c(n),$$
$$\mathfrak{I}^{(k,0)}(t, n^{(k)}) := e^{\lambda(n)t},$$
$$\mathfrak{F}^{(k,0)}(n^{(k)}) := 1;$$

for $k = 1, \gamma^{(1)} = 1 \in \spadesuit^{(1)}$, $(n_1, n_2) = n^{(1)} \in \mathfrak{N}^{(1,1)}$,

$$\mathfrak{C}^{(1,1)}(n^{(1)}) := \prod_{j=1}^2 c(n_j),$$
$$\mathfrak{I}^{(1,1)}(t, n^{(1)}) := \int_0^t e^{\lambda(\mu(n^{(1)}))(t-\tau)} \prod_{j=1}^2 e^{\lambda(n_j)\tau} \mathrm{d}\tau,$$
$$\mathfrak{F}^{(1,1)}(n^{(1)}) := \frac{\lambda(\mu(n^{(1)}))}{2};$$

for $k \geq 2, \gamma^{(k)} = (\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) \in (\spadesuit^{(k-1)})^2$, $(n_1^{(k-1)}, n_2^{(k-1)}) = n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})}$,

$$\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) := \prod_{j=1}^2 \mathfrak{C}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}),$$
$$\mathfrak{I}^{(k,\gamma^{(k)})}(t, n^{(k)}) := \int_0^t e^{\lambda(\mu(n^{(k)}))(t-\tau)} \prod_{j=1}^2 \mathfrak{I}^{(k-1,\gamma_j^{(k-1)})}(\tau, n_j^{(k-1)}) \mathrm{d}\tau,$$
$$\mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)}) := \frac{\lambda(\mu(n^{(k)}))}{2} \prod_{j=1}^2 \mathfrak{F}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}).$$

Remark 2.1. For the convenience of understanding these abstract notations, we give an interpretation as follows; see also [12].

These symbols are produced and defined in the Picard iteration process. We use $\spadesuit^{(k)}$ to label the set of trees at iteration k. The coefficient $\mathfrak{C}$ is associated with the initial Fourier data c and it can be viewed as the multi-linear accumulative product of c, the coefficient $\mathfrak{I}$ stands for the time integration and the coefficient $\mathfrak{F}$ is the rest. This separation and independence form is exactly the power of the combinatorial analysis method in dealing with an infinite product of Fourier series.

Lemma 2.2. *The Picard sequence $\{c_k(t,n)\}$ can be reformulated as the following combinatorial tree,*

$$c_k(t,n) = \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \sum_{\substack{n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})} \\ \mu(n^{(k)})=n}} \mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) \mathfrak{I}^{(k,\gamma^{(k)})}(t,n^{(k)}) \mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)}), \quad \forall k \geq 1. \tag{2.7}$$

Proof. We first notice that

$$c_0(t,n) = \sum_{\gamma^{(k)}=0 \in \spadesuit^{(k)}} \sum_{\substack{n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})} \\ \mu(n^{(k)})=n}} \mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) \mathfrak{I}^{(k,\gamma^{(k)})}(t,n^{(k)}) \mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)}), \quad \forall k \geq 1.$$

For $k = 1$, we have

$$\begin{aligned} c_1(t,n) - c_0(t,n) &= \frac{\lambda(n)}{2} \int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^{2} c_0(\tau, n_j)\, \mathrm{d}\tau \\ &= \sum_{\substack{n_1,n_2 \in \mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^{2} c(n_j) \cdot \frac{\lambda(n)}{2} \cdot \int_0^t e^{\lambda(n)(t-\tau)} \prod_{j=1}^{2} e^{\lambda(n_j)\tau}\, \mathrm{d}\tau \\ &= \sum_{\gamma^{(1)}=1 \in \spadesuit^{(1)}} \sum_{\substack{n^{(1)} \in \mathfrak{N}^{(1,\gamma^{(1)})} \\ \mu(n^{(1)})=n}} \mathfrak{C}^{(1,\gamma^{(1)})}(n^{(1)}) \mathfrak{I}^{(1,\gamma^{(1)})}(t,n^{(1)}) \mathfrak{F}^{(1,\gamma^{(1)})}(n^{(1)}). \end{aligned}$$

Hence we have

$$\begin{aligned} &c_1(t,n) \\ &= \left(\sum_{\gamma^{(1)}=0 \in \spadesuit^{(1)}} + \sum_{\gamma^{(1)}=1 \in \spadesuit^{(1)}} \right) \sum_{\substack{n^{(1)} \in \mathfrak{N}^{(1,\gamma^{(1)})} \\ \mu(n^{(1)})=n}} \mathfrak{C}^{(1,\gamma^{(1)})}(n^{(1)}) \mathfrak{I}^{(1,\gamma^{(1)})}(t,n^{(1)}) \mathfrak{F}^{(1,\gamma^{(1)})}(n^{(1)}) \\ &= \sum_{\gamma^{(1)} \in \spadesuit^{(1)}} \sum_{\substack{n^{(1)} \in \mathfrak{N}^{(1,\gamma^{(1)})} \\ \mu(n^{(1)})=n}} \mathfrak{C}^{(1,\gamma^{(1)})}(n^{(1)}) \mathfrak{I}^{(1,\gamma^{(1)})}(t,n^{(1)}) \mathfrak{F}^{(1,\gamma^{(1)})}(n^{(1)}). \end{aligned}$$

This shows that (2.7) holds for $k = 1$.

Let $k \geq 2$ and assume that (2.7) is true for $1, \cdots, k-1$. For k, we have

$$
\begin{aligned}
& c_k(t,n) - c_0(t,n) \\
&= \frac{\lambda(n)}{2} \int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^2 c_{k-1}(\tau, n_j)\, \mathrm{d}\tau \\
&= \frac{\lambda(n)}{2} \int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^2 \sum_{\gamma_j^{(k-1)} \in \spadesuit^{(k-1)}} \sum_{\substack{n_j^{(k-1)} \in \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})} \\ \mu(n_j^{(k-1)})=n_j}} \\
&\qquad \mathfrak{C}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}) \mathfrak{I}^{(k-1,\gamma_j^{(k-1)})}(\tau, n_j^{(k-1)}) \mathfrak{F}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)})\, \mathrm{d}\tau \\
&= \sum_{\substack{\gamma_j^{(k-1)} \in \spadesuit^{(k-1)} \\ j=1,2}} \sum_{\substack{n_1,n_2 \in \mathbb{Z}^\nu \\ n_1+n_2=n}} \sum_{\substack{n_j^{(k-1)} \in \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})} \\ \mu(n_j^{(k-1)})=n_j \\ j=1,2}} \prod_{j=1}^2 \mathfrak{C}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}) \\
&\qquad \times \frac{\lambda(n)}{2} \prod_{j=1}^2 \mathfrak{F}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}) \\
&\qquad \times \int_0^t e^{\lambda(n)(t-\tau)} \prod_{j=1}^2 \mathfrak{I}^{(k-1,\gamma_j^{(k-1)})}(\tau, n_j^{(k-1)})\, \mathrm{d}\tau \\
&= \sum_{\gamma^{(k)} \in (\spadesuit^{(k-1)})^2} \sum_{\substack{n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})} \\ \mu(n^{(k)})=n}} \mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) \mathfrak{I}^{(k,\gamma^{(k)})}(t, n^{(k)}) \mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)}).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
c_k(t,n) &= \left(\sum_{\gamma^{(k)}=0 \in \spadesuit^{(k)}} + \sum_{\gamma^{(k)} \in (\spadesuit^{(k-1)})^2} \right) \\
&\qquad \sum_{\substack{n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})} \\ \mu(n^{(k)})=n}} \mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) \mathfrak{I}^{(k,\gamma^{(k)})}(t, n^{(k)}) \mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)}) \\
&= \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \sum_{\substack{n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})} \\ \mu(n^{(k)})=n}} \mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) \mathfrak{I}^{(k,\gamma^{(k)})}(t, n^{(k)}) \mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)}).
\end{aligned}
$$

This shows that (2.7) holds for k. By induction, it follows that (2.7) is true for all $k \geq 1$. This completes the proof of Lemma 2.2. □

Remark 2.3. We refer to (2.7) as a combinatorial tree to emphasize both the tree structure of the summation formula and the importance of understanding the combinatorics of the expansion terms for the given PDE problem. Indeed, while the general philosophy underlying the approach worked out in present paper is the same as in our previous study of the generalized KdV equation, [12], the details of dealing with the combinatorial tree are dependent on the given PDE and pose model-specific challenges. Motivated by a question posed to us by the late Thomas Kappeler, there is forthcoming work that applies this approach to the standard nonlinear Schrödinger equation, as well as the derivative nonlinear Schrödinger equation. The key will again be to find a way to deal with the combinatorial tree arising from the NLSE in question.

2.3. Uniform exponential decay of the Picard sequence

In this subsection we show with the help of notations established in the previous subsection that the Picard sequence for the Fourier coefficients obeys a uniform exponential decay estimate.

Indeed, we have the following result:

Lemma 2.4. *Assume that the initial Fourier coefficients c satisfy the exponential decay property* (1.7)*. With the constants* $\mathcal{A}$ *and* ρ *from* (1.7) *and the dimension* ν*, set*

$$\mathcal{B}_2 \triangleq 2\mathcal{A}(6\rho^{-1})^{\nu} \tag{2.8}$$

and

$$\mathcal{L}_2 \triangleq \frac{\rho^{\nu}}{\mathcal{A}6^{\nu}}. \tag{2.9}$$

Then, we have

$$\sup_{\substack{t\in[0,\mathcal{L}_2]\\ k\geq 0}} |c_k(t,n)| \leq \mathcal{B}_2 e^{-\frac{\rho}{2}|n|} \tag{2.10}$$

for every $n \in \mathbb{Z}^{\nu}$*.*

To prove Lemma 2.4, we need the following lemmas.

Lemma 2.5. *For all* $k \geq 1$ *we have*

$$|\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)})| \leq \mathcal{A}^{\sigma(\gamma^{(k)})} e^{-\rho|n^{(k)}|}, \tag{2.11}$$

$$|\mathfrak{I}^{(k,\gamma^{(k)})}(t,n^{(k)})| \leq \frac{t^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})}, \tag{2.12}$$

$$|\mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)})| \leq \frac{1}{4^{\ell(\gamma^{(k)})}} \leq 1, \tag{2.13}$$

where $\sigma(0) = 1, \ell(0) = 0, \mathfrak{D}(0) = 1$*;* $\sigma(1) = 2, \ell(1) = 1, \mathfrak{D}(1) = 1$*; for* $k \geq 2, \gamma^{(k)} = (\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) \in (\spadesuit^{(k-1)})^2$*,*

$$\sigma(\gamma^{(k)}) = \sum_{j=1}^{2} \sigma(\gamma_j^{(k-1)}),$$

$$\ell(\gamma^{(k)}) = 1 + \sum_{j=1}^{2} \ell(\gamma_j^{(k-1)}),$$

$$\mathfrak{D}(\gamma^{(k)}) = \ell(\gamma^{(k)}) \prod_{j=1}^{2} \mathfrak{D}^{(k-1,\gamma_j^{(k-1)})},$$

and $|n^{(k)}| = \sum_{j=1}^{2} |n_j^{(k-1)}|$ *if* $n^{(k)} = (n_j^{(k-1)})_{1 \leq j \leq 2}$*.*

Remark 2.6. Regarding these notations, there is an intuitive understanding: σ means the degree or the multiplicity (of nonlinearity), ℓ means the number of time integrations, and $\mathfrak{D}$ quantifies the decay after time integration.

Proof. Recall that $\lambda(n)$ is a purely imaginary number; see (2.4).

For $k \geq 1, 0 = \gamma^{(k)} \in \spadesuit^{(k)}, n = n^{(k)} \in \mathfrak{N}^{(k,0)}$,

$$|\mathfrak{C}^{(k,0)}(n^{(k)})| = |c(n)| \leq \mathcal{A} e^{-\rho|n|} = \mathcal{A}^{\sigma(0)} e^{-\rho|n^{(k)}|};$$

$$|\mathfrak{I}^{(k,0)}(t, n^{(k)})| = |e^{\lambda(n)t}| \leq 1 = \frac{t^{\ell(0)}}{\mathfrak{D}(0)};$$

$$|\mathfrak{F}^{(k,0)}(n^{(k)})| = 1 = \frac{1}{4^{\ell(0)}} \leq 1.$$

For $k = 1, 1 = \gamma^{(1)} \in \spadesuit^{(1)}, (n_1, n_2) = n^{(1)} \in \mathfrak{N}^{(1,1)}$,

$$|\mathfrak{C}^{(1,1)}(n_1, n_2)| = \prod_{j=1}^{2} |c(n_j)| \leq \prod_{j=1}^{2} \mathcal{A} e^{-\rho|n_j|} = \mathcal{A}^2 e^{-\rho(|n_1|+|n_2|)} = \mathcal{A}^{\sigma(1)} e^{-\rho|n^{(1)}|};$$

$$|\mathfrak{I}^{(1,1)}(t, n^{(1)})| \leq \int_0^t |e^{\lambda(\mu(n^{(1)}))(t-\tau)}| \prod_{j=1}^{2} |e^{\lambda(n_j)\tau}| \mathrm{d}\tau = t = \frac{t^{\ell(1)}}{\mathfrak{D}(1)};$$

$$|\mathfrak{F}^{(1,1)}(n^{(1)})| = \frac{|\lambda(\mu(n^{(1)}))|}{2} \leq \frac{1}{4} = \frac{1}{4^{\ell(1)}} < 1.$$

Hence (2.11)–(2.13) hold for $k = 1$.

Let $k \geq 2$ and assume that they are true for $1, \cdots, k-1$. For $k, (\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) = \gamma^{(k)} \in (\spadesuit^{(k-1)})^2$ and $(n_1^{(k-1)}, n_2^{(k-1)}) = n^{(k)} \in \prod_{j=1}^{2} \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})}$, one can derive that

$$
\begin{aligned}
|\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)})| &= \prod_{j=1}^{2} |\mathfrak{C}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)})| \\
&\le \prod_{j=1}^{2} \mathcal{A}^{\sigma(\gamma_j^{(k-1)})} e^{-\rho|n_j^{(k-1)}|} \\
&= \mathcal{A}^{\sum_{j=1}^{2} \sigma(\gamma_j^{(k-1)})} e^{-\rho \sum_{j=1}^{2} |n_j^{(k-1)}|} \\
&= \mathcal{A}^{\sigma(\gamma^{(k)})} e^{-\rho|n^{(k)}|}; \\
|\mathfrak{I}^{(k,\gamma^{(k)})}(t,n^{(k)})| &\le \int_0^t |e^{\lambda(\mu(n^{(k)}))(t-\tau)}| \prod_{j=1}^{2} |\mathfrak{I}^{(k-1,\gamma_j^{(k-1)})}(\tau, n_j^{(k-1)})| \mathrm{d}\tau \\
&\le \int_0^t \prod_{j=1}^{2} \frac{\tau^{\ell(\gamma_j^{(k-1)})}}{\mathfrak{D}(\gamma_j^{(k-1)})} \mathrm{d}\tau \\
&= \frac{t^{1+\sum_{j=1}^{2} \ell(\gamma_j^{(k-1)})}}{(1+\sum_{j=1}^{2} \ell(\gamma_j^{(k-1)})) \prod_{j=1}^{2} \mathfrak{D}(\gamma_j^{(k-1)})} \\
&= \frac{t^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})}; \\
|\mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)})| &\le \frac{|\lambda(\mu(n^{(k)}))|}{2} \prod_{j=1}^{2} |\mathfrak{F}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)})| \\
&\le \frac{1}{4} \prod_{j=1}^{2} \frac{1}{4^{\ell(\gamma_j^{(k-1)})}} \\
&= \frac{1}{4^{1+\sum_{j=1}^{2} \ell(\gamma_j^{(k-1)})}} \\
&= \frac{1}{4^{\ell(\gamma^{(k)})}} \\
&< 1.
\end{aligned}
$$

These imply that (2.11)–(2.13) are true for k. By induction, they hold for all $k \ge 1$. This completes the proof of Lemma 2.5. □

Lemma 2.7.

(1) *For all* $k \ge 1$,

$$
\sigma(\gamma^{(k)}) = \ell(\gamma^{(k)}) + 1. \tag{2.14}
$$

(2) *If* $0 < \rho \leq 1$, *then*

$$\sum_{n\in\mathbb{Z}} e^{-\rho|n|} \leq 3\rho^{-1}. \tag{2.15}$$

(3) *Let* $\dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(k,\gamma^{(k)})}$ *be the integer number of components per* $\mathbb{Z}^\nu$, *that is,* $\mathfrak{N}^{(k,\gamma^{(k)})} = (\mathbb{Z}^\nu)^{\dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(k,\gamma^{(k)})}}$. *Then*

$$\dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(k,\gamma^{(k)})} = \sigma(\gamma^{(k)}). \tag{2.16}$$

(4) *If* $0 < \flat \leq \frac{1}{4}$, *then*

$$\diamondsuit_k \triangleq \sum_{\gamma^{(k)}\in\spadesuit^{(k)}} \frac{\flat^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \leq 2. \tag{2.17}$$

Proof. (1) Since $\sigma(0) = 1, \ell(0) = 0; \sigma(1) = 2, \ell(1) = 1$, then $\sigma(0) = \ell(0) + 1$ and $\sigma(1) = \ell(1) + 1$. Hence (2.14) holds for $k = 1$. Let $k \geq 2$. Assume that is true for $1, \cdots, k-1$. For k and $(\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) = \gamma^{(k)} \in (\spadesuit^{(k-1)})^2$, one has

$$\sigma(\gamma^{(k)}) = \sum_{j=1}^{2} \sigma(\gamma_j^{(k-1)}) = \sum_{j=1}^{2} (\ell(\gamma_j^{(k-1)}) + 1) = 1 + \ell(\gamma^{(k)}).$$

Hence (2.14) holds for all $k \geq 1$ by induction.

(2) Let $z(y) := (3-y)e^y - (3+y), 0 < y \leq 1$. After a simple calculation, we find

$$z'(y) = (2-y)e^y - 1, \quad z''(y) = (1-y)e^y.$$

Since $0 < y \leq 1$, then $z''(y) \geq 0$. Hence z' is monotonically increasing and $z'(y) \geq z'(0) = 1 > 0$. Similarly one can see that $z(y) \geq z(0) = 0$, that is $\frac{e^y+1}{e^y-1} \leq 3y^{-1}$ provided that $0 < y \leq 1$. By the symmetry of $\mathbb{Z}$ and $0 < \rho \leq 1$,

$$\sum_{n\in\mathbb{Z}} e^{-\rho|n|} = 2\sum_{n=0}^{\infty} e^{-\rho n} - 1 \overset{(\rho\geq 0)}{=} \frac{2}{1-e^{-\rho}} - 1 = \frac{e^\rho + 1}{e^\rho - 1} \overset{(0<\rho\leq 1)}{\leq} 3\rho^{-1}. \tag{2.18}$$

(3) For $k \geq 1, \mathfrak{N}^{(k,0)} = \mathbb{Z}^\nu$, we know that $\dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(k,0)} = 1 = \sigma(0)$. Also $\mathfrak{N}^{(1,1)} = (\mathbb{Z}^\nu)^2$, hence $\dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(1,1)} = 2 = \sigma(1)$. This implies that (2.16) holds for $k = 1$. Let $k \geq 2$. Assume that it is true for $1, \cdots, k-1$. For k, $\gamma^{(k)} = (\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) \in (\spadesuit^{(k-1)})^2$, and $\mathfrak{N}^{(k,\gamma^{(k)})} = \prod_{j=1}^2 \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})}$, we have

$$\dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(k,\gamma^{(k)})} = \sum_{j=1}^{2} \dim_{\mathbb{Z}^\nu} \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})} = \sum_{j=1}^{2} \sigma(\gamma_j^{(k-1)}) = \sigma(\gamma^{(k)}).$$

By induction, (2.16) holds for all $k \geq 1$.

(4) For $k=1$, by the definition of $\spadesuit^{(1)}, \ell$ and $\mathfrak{D}$, we have

$$\diamondsuit_1 = \frac{\flat^{\ell(0)}}{\mathfrak{D}(0)} + \frac{\flat^{\ell(1)}}{\mathfrak{D}(1)} = 1+\flat \leq \frac{5}{4} \leq 2.$$

Let $k \geq 2$. Suppose that (2.17) holds for $1, \cdots, k-1$. For k, we first have

$$\begin{aligned}
&\sum_{\gamma^{(k)}=(\gamma_1^{(k-1)},\gamma_2^{(k-1)})\in(\spadesuit^{(k-1)})^2} \frac{\flat^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \\
&= \sum_{\substack{\gamma_j^{(k-1)}\in\spadesuit^{(k-1)}\\ j=1,2}} \frac{\flat^{1+\sum_{j=1}^2 \ell(\gamma_j^{(k-1)})}}{\left(1+\sum_{j=1}^2 \ell(\gamma_j^{(k-1)})\right)\prod_{j=1}^2 \mathfrak{D}(\gamma_j^{(k-1)})} \\
&\leq \flat \sum_{\substack{\gamma_j^{(k-1)}\in\spadesuit^{(k-1)}\\ j=1,2}} \prod_{j=1}^2 \frac{\flat^{\ell(\gamma_j^{(k-1)})}}{\mathfrak{D}(\gamma_j^{(k-1)})} \\
&= \flat \prod_{j=1}^2 \sum_{\gamma_j^{(k-1)}\in\spadesuit^{(k-1)}} \frac{\flat^{\ell(\gamma_j^{(k-1)})}}{\mathfrak{D}(\gamma_j^{(k-1)})} \\
&\leq 2^2\flat.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\diamondsuit_k &= \frac{\flat^{\ell(0)}}{\mathfrak{D}(0)} + \sum_{\gamma^{(k)}=(\gamma_1^{(k-1)},\gamma_2^{(k-1)})\in(\spadesuit^{(k-1)})^2} \frac{\flat^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \\
&\leq 1+2^2\flat \\
&\leq 2.
\end{aligned}$$

This shows that (2.17) is true for k. By induction, (2.17) holds for all $k \geq 1$. □

Proof of Lemma 2.4. If $0 \leq t \leq \mathcal{L}_2$, then

$$\begin{aligned}
|c_k(t,n)| &\overset{(2.7)}{\leq} \sum_{\gamma^{(k)}\in\spadesuit^{(k)}} \sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}} |\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)})||\mathfrak{I}^{(k,\gamma^{(k)})}(t,n^{(k)})||\mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)})| \\
&\overset{(2.11)-(2.14)}{\leq} \mathcal{A} \sum_{\gamma^{(k)}\in\spadesuit^{(k)}} \frac{(4^{-1}\mathcal{A}t)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}} e^{-\rho|n^{(k)}|}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \mathcal{A} \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \frac{(4^{-1}\mathcal{A}t)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \sum_{n^{(k)} \in \mathfrak{N}^{(k,\gamma^{(k)})}} e^{-\frac{\rho}{2}|n^{(k)}|} \cdot e^{-\frac{\rho}{2}|n|} \\
&\overset{(2.16)}{\leq} \quad \mathcal{A} \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \frac{(4^{-1}\mathcal{A}t)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \sum_{n^{(k)} \in (\mathbb{Z}^\nu)^{\sigma(\gamma^{(k)})}} e^{-\frac{\rho}{2}|n^{(k)}|} \cdot e^{-\frac{\rho}{2}|n|} \\
&= \quad \mathcal{A} \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \frac{(4^{-1}\mathcal{A}t)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \prod_{j=1}^{\sigma(\gamma^{(k)})} \prod_{j'=1}^{\nu} \sum_{n_{jj'} \in \mathbb{Z}} e^{-\frac{\rho}{2}|n_{jj'}|} \cdot e^{-\frac{\rho}{2}|n|} \\
&\overset{(2.15)}{\leq} \quad \mathcal{A} \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \frac{(4^{-1}\mathcal{A}t)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} (6\rho^{-1})^{\sigma(\gamma^{(k)})\nu} \cdot e^{-\frac{\rho}{2}|n|} \\
&\overset{(2.14)}{=} \quad \mathcal{A}(6\rho^{-1})^\nu \sum_{\gamma^{(k)} \in \spadesuit^{(k)}} \frac{\left(4^{-1}\mathcal{A}(6\rho^{-1})^\nu t\right)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \cdot e^{-\frac{\rho}{2}|n|} \\
&\overset{(2.17)}{\leq} \quad \mathcal{B}_2 e^{-\frac{\rho}{2}|n|},
\end{aligned}
$$

provided that $0 \leq t \leq \mathcal{L}_2$, where $\mathcal{B}_2$ is given by (2.8), i.e., $\mathcal{B}_2 = 2\mathcal{A}(6\rho^{-1})^\nu$. This completes the proof of Lemma 2.4. □

2.4. Convergence of the Picard sequence

We are now in a position to show that the Picard sequence for the Fourier coefficients converges. The following lemma establishes a bound for the magnitude of $c_k(t,n) - c_{k-1}(t,n)$, from which the desired conclusion follows.

Lemma 2.8. *For $k \geq 1$, $0 \leq t \leq \mathcal{L}_2$ with $\mathcal{L}_2$ from (2.9), and $n \in \mathbb{Z}^\nu$, we have*

$$
|c_k(t,n) - c_{k-1}(t,n)| \leq \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1} \in \mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|} \tag{2.19}
$$

$$
\leq \frac{\mathcal{B}_2(12\rho^{-1})^\nu}{2} \cdot \frac{\left(2^{-1}\mathcal{B}_2(12\rho^{-1})^\nu t\right)^k}{k!} \cdot e^{-\frac{\rho}{4}|n|}. \tag{2.20}
$$

Hence, $\{c_k(t,n)\}$ is a Cauchy sequence.

Proof. It follows from Lemma 2.4 and induction that we can complete this proof.

In fact, for $k = 1$ we have

$$
|c_1(t,n) - c_0(t,n)| \leq \frac{|\lambda(n)|}{2} \int_0^t |e^{\lambda(n)(t-\tau)}| \sum_{\substack{n_1,n_2 \in \mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^{2} |c_0(\tau,n_j)| \, \mathrm{d}\tau
$$

$$\le \frac{\mathcal{B}_2^2 t}{4} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^{2} e^{-\frac{\rho}{2}|n_j|}.$$

This shows that (2.19) holds for $k=1$.

Let $k \ge 2$ and assume that (2.19) is true for $1,\cdots,k-1$. For k, we first have

$$\begin{aligned}
&|c_k(t,n) - c_{k-1}(t,n)| \\
&\quad\le \frac{|\lambda(n)|}{2} \int_0^t |e^{\lambda(n)(t-\tau)}| \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \Big| \prod_{j=1}^{2} c_{k-1}(\tau,n_j) - \prod_{j=1}^{2} c_{k-2}(\tau,n_j) \Big| \,\mathrm{d}\tau \\
&\quad\le \frac{1}{4} \int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} |c_{k-1}(\tau,n_1)||c_{k-1}(\tau,n_2) - c_{k-2}(\tau,n_2)| \,\mathrm{d}\tau \triangleq (I) \\
&\qquad + \frac{1}{4} \int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} |c_{k-1}(\tau,n_1) - c_{k-2}(\tau,n_1)||c_{k-2}(\tau,n_2)| \,\mathrm{d}\tau \triangleq (II).
\end{aligned}$$

For the first component, it follows from the induction hypothesis and Lemma 2.4 that

$$\begin{aligned}
(I) &\le \frac{1}{4} \int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \mathcal{B}_2 e^{-\frac{\rho}{2}|n_1|} \cdot \frac{2^{k-2}\mathcal{B}_2^k \tau^{k-1}}{4^{k-1}\cdot(k-1)!} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}^\nu \\ m_1+\cdots+m_k=n_2}} \prod_{j=1}^{k} e^{-\frac{\rho}{2}|m_j|} \,\mathrm{d}\tau \\
&= \frac{2^{k-2}\mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}^\nu \\ m_1+\cdots+m_k=n_2}} e^{-\frac{\rho}{2}|n_1|} \prod_{j=1}^{k} e^{-\frac{\rho}{2}|m_j|} \\
&= \frac{2^{k-2}\mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}.
\end{aligned}$$

Analogously, for the second component we have

$$(II) \le \frac{2^{k-2}\mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}.$$

Thus we find that

$$\begin{aligned}
|c_k(t,n) - c_{k-1}(t,n)| &\le (I) + (II) \\
&\le 2 \times \frac{2^{k-2}\mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}
\end{aligned}$$

$$\leq \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}.$$

By induction, we see that (2.19) holds for all $k \geq 1$. Furthermore,

$$\begin{aligned}
|c_k(t,n) - c_{k-1}(t,n)| &\leq \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|} \\
&\leq \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu} \prod_{j=1}^{k+1} e^{-\frac{\rho}{4}|n_j|} \cdot e^{-\frac{\rho}{4}|n|} \\
&= \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \prod_{j=1}^{k+1}\prod_{j'=1}^{\nu}\sum_{n_{jj'}\in\mathbb{Z}} e^{-\frac{\rho}{4}|n_{jj'}|} \cdot e^{-\frac{\rho}{4}|n|} \\
&\overset{(2.15)}{\leq} \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} (12\rho^{-1})^{(k+1)\nu} \cdot e^{-\frac{\rho}{4}|n|} \\
&= \frac{\mathcal{B}_2(12\rho^{-1})^\nu}{2} \cdot \frac{\left(2^{-1}\mathcal{B}_2(12\rho^{-1})^\nu t\right)^k}{k!} \cdot e^{-\frac{\rho}{4}|n|}.
\end{aligned}$$

Hence $\{c_k(t,n)\}$ is a Cauchy sequence on $[0,\mathcal{L}_2] \times \mathbb{Z}^\nu$, completing the proof of Lemma 2.8. □

2.5. Proof of Theorem A: existence

In this subsection we prove the existence part of Theorem A.

Proof. By Lemma 2.8 we know that $\{c_k(t,n)\}$ is a Cauchy sequence and there exists a limit function, denoted by $c^\dagger(t,n)$, where $0 \leq t \leq \mathcal{L}_2$ and $n \in \mathbb{Z}^\nu$. By the triangle inequality we have

$$|c^\dagger(t,n)| \leq |c^\dagger(t,n) - c_k(t,n)| + |c_k(t,n)|, \quad \forall k \geq 1.$$

Since $k \geq 1$ is arbitrary in this estimate and the $c_k(t,n)$ obey the uniform upper bound (2.10), it follows that the same upper bound holds for $c^\dagger(t,n)$, that is,

$$|c^\dagger(t,n)| \leq \mathcal{B}_2 e^{-\frac{\rho}{2}|n|}. \tag{2.21}$$

Naturally we regard the coefficients $c^\dagger(t,n)$ as the Fourier coefficients of a candidate solution $u^\dagger(t,x)$ of the Cauchy problem in question, and hence set

$$u^\dagger(t,x) := \sum_{n\in\mathbb{Z}^\nu} c^\dagger(t,n) e^{\mathrm{i}\langle n\rangle x};$$

$$(\partial_x^{\#} u^\dagger)(t,x) := \sum_{n\in\mathbb{Z}^\nu} (\mathrm{i}\langle \mathrm{n}\rangle)^{\#} c^\dagger(t,n) e^{\mathrm{i}\langle n\rangle x}, \quad \# = 1,2;$$

$$(u^\dagger u_x^\dagger)(t,x) := \sum_{n\in\mathbb{Z}^\nu} \frac{\mathrm{i}\langle n\rangle}{2} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} c^\dagger(t,n_j) e^{\mathrm{i}\langle n\rangle x};$$

$$(u_t^\dagger)(t,x) := \sum_{n\in\mathbb{Z}^\nu} \left\{ \lambda(n) c^\dagger(t,n) + \frac{\lambda(n)}{2} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} c^\dagger(t,n_j) \right\} e^{\mathrm{i}\langle n\rangle x};$$

$$(u_{xxt}^\dagger)(t,x) := \sum_{n\in\mathbb{Z}^\nu} (\mathrm{i}\langle n\rangle)^2 \left\{ \lambda(n) c^\dagger(t,n) + \frac{\lambda(n)}{2} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} c^\dagger(t,n_j) \right\} e^{\mathrm{i}\langle n\rangle x}.$$

We claim that $u^\dagger$ is a classical spatially quasi-periodic solution to quasi-periodic Cauchy problem (1.4)–(1.5), that is, $u_t^\dagger, u_{xxt}^\dagger, u_x^\dagger, u^\dagger u_x^\dagger$ satisfy BBM (1.4) in the classical sense and $u^\dagger$ has initial data (1.5). On the one hand, by the exponential decay of $c^\dagger(t,n)$, one can see that $u^\dagger, u_t^\dagger, u_{xxt}^\dagger, u_x^\dagger, u^\dagger u_x^\dagger$ are uniformly and absolutely convergent. It is sufficient to verify that

$$\sum_{n\in\mathbb{Z}^\nu} (1+|n|+|n|^2) \left\{ c^\dagger(t,n) + \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} |c^\dagger(t,n_j)| \right\} < \infty.$$

In fact, for $\# = 0,1,2$, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^\nu} |n|^{\#} |c^\dagger(t,n)| &\overset{(2.21)}{\lesssim} \sum_{n\in\mathbb{Z}^\nu} |n|^{\#} e^{-\frac{\rho}{2}|n|} \\
&= \sum_{n\in\mathbb{Z}^\nu} \underbrace{|n|^{\#} e^{-\frac{\rho}{4}|n|}}_{\text{bounded}} e^{-\frac{\rho}{4}|n|} \\
&\lesssim \sum_{n\in\mathbb{Z}^\nu} e^{-\frac{\rho}{4}|n|} \\
&\overset{(2.15)}{\leq} (12\rho^{-1})^\nu \\
&< \infty
\end{aligned}$$

and

$$\sum_{n\in\mathbb{Z}^\nu} |n|^{\#} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} |c^\dagger(t,n_j)| \overset{(2.21)}{\lesssim} \sum_{n\in\mathbb{Z}^\nu} |n|^{\#} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} e^{-\frac{\rho}{2}|n_j|}$$

$$
\begin{aligned}
&\le \sum_{n\in\mathbb{Z}^\nu} \underbrace{|n|^{\#} e^{-\frac{\rho}{8}|n|}}_{\text{bounded}} \underbrace{\sum_{n_1,n_2\in\mathbb{Z}^\nu} \prod_{j=1}^{2} e^{-\frac{\rho}{4}|n_j|}\, e^{-\frac{\rho}{8}|n|}}_{\text{bounded by (2.15)}}\\
&\lesssim \sum_{n\in\mathbb{Z}^\nu} e^{-\frac{\rho}{8}|n|}\\
&\overset{(2.15)}{\le} (24\rho^{-1})^\nu\\
&< \infty.
\end{aligned}
$$

On the another hand, since $c^\dagger$ is the limit function of the Picard sequence $\{c_k\}$ defined by (2.6), it satisfies (2.5). Hence $c^\dagger$ is a solution to (2.3) and satisfies the initial condition $c^\dagger(0,n) = c(n)$. This implies that $u^\dagger$ is a classical spatially quasi-periodic solution to the Cauchy problem (1.4)–(1.5). Hence the existence part of Theorem A is proved. □

2.6. Proof of Theorem A: uniqueness

In this subsection we prove the uniqueness part of Theorem A.

Proof. Let

$$
v(t,x) = \sum_{n\in\mathbb{Z}^\nu} \hat{v}(t,n) e^{\mathrm{i}\langle n\rangle x} \quad \text{and} \quad w(t,x) = \sum_{n\in\mathbb{Z}^\nu} \hat{w}(t,n) e^{\mathrm{i}\langle n\rangle x}
$$

be two quasi-periodic solutions to (1.4)–(1.5), where $\hat{v}$ and $\hat{w}$ satisfy the following conditions:

- (same initial data)

$$
\hat{v}(0,n) = \hat{w}(0,n), \quad \forall n \in \mathbb{Z}^\nu;
$$

- (integral equation)

$$
\hat{v}(t,n) = e^{\lambda(n)t}\hat{v}(0,n) + \frac{\lambda(n)}{2}\int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} \hat{v}(\tau,n_j)\mathrm{d}\tau,
$$

$$
\hat{w}(t,n) = e^{\lambda(n)t}\hat{w}(0,n) + \frac{\lambda(n)}{2}\int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} \hat{w}(\tau,n_j)\mathrm{d}\tau;
$$

- (exponential decay)

$$
|\hat{v}(t,n)| \le \mathcal{B}_2 e^{-\frac{\rho}{2}|n|} \quad \text{and} \quad |\hat{w}(t,n)| \le \mathcal{B}_2 e^{-\frac{\rho}{2}|n|}, \qquad 0 \le t \le \mathcal{L}_2, n \in \mathbb{Z}^\nu.
$$

For all $k \geq 1$, one can derive that

$$|\hat{v}(t,n) - \hat{w}(t,n)| \leq \frac{2^k \mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}. \tag{2.22}$$

In fact, we first have

$$\begin{aligned} |\hat{v}(t,n) - \hat{w}(t,n)| &\leq \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} |\prod_{j=1}^{2} \hat{v}(\tau,n_j) - \prod_{j=1}^{2} \hat{w}(\tau,n_j)|\, \mathrm{d}\tau \\ &\leq \frac{\mathcal{B}_2^2 t}{2} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \prod_{j=1}^{2} e^{-\frac{\rho}{2}|n_j|}. \end{aligned}$$

Hence (2.22) holds for $k = 1$. Let $k \geq 2$ and assume that it holds for $1, \cdots, k-1$. For k, we have

$$\begin{aligned} |\hat{v}(t,n) - \hat{w}(t,n)| &\leq \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \Big|\prod_{j=1}^{2} \hat{v}(\tau,n_j) - \prod_{j=1}^{2} \hat{w}(\tau,n_j)\Big|\, \mathrm{d}\tau \\ &\leq \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} |\hat{v}(\tau,n_1) - \hat{w}(\tau,n_1)||\hat{v}(\tau,n_2)|\, \mathrm{d}\tau \triangleq (I') \\ &\quad + \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} |\hat{w}(\tau,n_1)||\hat{v}(\tau,n_2) - \hat{w}(\tau,n_2)|\, \mathrm{d}\tau \triangleq (II'). \end{aligned}$$

For the first component we have

$$\begin{aligned} (I') &\leq \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \frac{2^{k-1}\mathcal{B}_2^k \tau^{k-1}}{4^{k-1}\cdot(k-1)!} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}^\nu \\ m_1+\cdots+m_k=n_1}} \prod_{j=1}^{k} e^{-\frac{\rho}{2}|m_j|} \cdot \mathcal{B}_2 e^{-\frac{\rho}{2}|n_2|}\mathrm{d}\tau \\ &= \frac{2^{k-1}\mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu \\ n_1+n_2=n}} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}^\nu \\ m_1+\cdots+m_k=n_1}} e^{-\frac{\rho}{2}|n_2|} \prod_{j=1}^{k} e^{-\frac{\rho}{2}|m_j|} \\ &= \frac{2^{k-1}\mathcal{B}_2^{k+1} t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}. \end{aligned}$$

After a similar argument we find

$$(II') \leq \frac{2^{k-1}\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}.$$

Hence we have

$$\begin{aligned}|\hat{v}(t,n)-\hat{w}(t,n)| &\leq (I')+(II') \\ &\leq \frac{2^k\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|}.\end{aligned}$$

By induction, (2.22) holds for all $k \geq 1$. By (2.15) one can derive that

$$\begin{aligned}|\hat{v}(t,n)-\hat{w}(t,n)| &\leq \frac{2^k\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \prod_{j=1}^{k+1} e^{-\frac{\rho}{2}|n_j|} \\ &\leq \frac{2^k\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu} \prod_{j=1}^{k+1} e^{-\frac{\rho}{4}|n_j|} \cdot e^{-\frac{\rho}{4}|n|} \\ &= \frac{2^k\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!} \prod_{j=1}^{k+1}\prod_{j'=1}^{\nu} \sum_{n_{jj'}\in\mathbb{Z}^\nu} e^{-\frac{\rho}{4}|n_{jj'}|} \cdot e^{-\frac{\rho}{4}|n|} \\ &\leq \frac{2^k\mathcal{B}_2^{k+1}t^k}{4^k \cdot k!}(12\rho^{-1})^{(k+1)\nu} \cdot e^{-\frac{\rho}{4}|n|} \\ &\leq \mathcal{B}_2(12\rho^{-1})^\nu \cdot \frac{(2^{-1}\mathcal{B}_2(12\rho^{-1})^\nu t)^k}{k!} \cdot e^{-\frac{\rho}{4}|n|}.\end{aligned}$$

As these estimates hold for arbitrary $k \geq 1$, we can send $k \to \infty$ in the upper bound for $|\hat{v}(t,n)-\hat{w}(t,n)|$, and since the limit vanishes, we have

$$\hat{v}(t,n) \equiv \hat{w}(t,n), \quad 0 \leq t \leq \mathcal{L}_2, n \in \mathbb{Z}^\nu.$$

This implies that the spatially quasi-periodic solution to the quasi-periodic Cauchy problem (1.4)–(1.5) is unique. Hence the uniqueness component of Theorem A is proved. □

3. The general case: gBBM

In this section we study the general case, that is, $p \geq 2$. In particular we address the remaining cases $p \geq 3$. The key point is to propose new indices for the term $u^{p-1}u_x$ and their relations (compared with [11]). Proofs that are similar to the ones in Section 2 are omitted; we only give those proofs that need significant new arguments.

To avoid confusion of symbols, set $\hat{u}(n) = \mathfrak{c}(n)$ and $\hat{u}(t,n) = \mathfrak{c}(t,n)$. The counterparts of $\mathfrak{N}$, $\mathfrak{C}$, $\mathfrak{I}$, $\mathfrak{F}$, σ, ℓ and $\mathfrak{D}$ will be denoted by $\mathscr{N}$, $\mathscr{C}$, $\mathscr{I}$, $\mathscr{F}$, α, β and $\mathscr{D}$, respectively. Their definitions will be introduced below.

In the Fourier space, the quasi-periodic Cauchy problem (1.4)–(1.5) is again reduced to a nonlinear infinite system of coupled ODEs,

$$(\partial_t \mathfrak{c})(t,n) - \lambda(n)\mathfrak{c}(t,n) = \frac{\lambda(n)}{p} \sum_{\substack{n_1,\cdots,n_p \in \mathbb{Z}^\nu \\ n_1+\cdots+n_p=n}} \prod_{j=1}^{p} \mathfrak{c}(t,n_j), \quad \forall n \in \mathbb{Z}^\nu \tag{3.1}$$

with initial data

$$\mathfrak{c}(0,n) = \mathfrak{c}(n), \quad \forall n \in \mathbb{Z}^\nu. \tag{3.2}$$

Obviously $\mathfrak{c}(t,0) \equiv \mathfrak{c}(0)$. According to the variation of constants formula, the Cauchy problem (3.1)–(3.2) is equivalent to the following integral equation,

$$\mathfrak{c}(t,n) = e^{\lambda(n)t}\mathfrak{c}(n) + \frac{\lambda(n)}{p} \int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,\cdots,n_p \in \mathbb{Z}^\nu \\ n_1+\cdots+n_p=n}} \prod_{j=1}^{p} \mathfrak{c}(\tau,n_j)\,\mathrm{d}\tau, \quad \forall n \in \mathbb{Z}^\nu \backslash \{0\}.$$

Define the Picard sequence $\{\mathfrak{c}_k(t,n)\}$ to approximate $\mathfrak{c}(t,n)$ by letting

$$\mathfrak{c}_k(t,n) := \begin{cases} e^{\lambda(n)t}\mathfrak{c}(n), & k=0; \\ \mathfrak{c}_0(t,n) + \frac{\lambda(n)}{p} \int_0^t e^{\lambda(n)(t-\tau)} \sum_{\substack{n_1,\cdots,n_p \in \mathbb{Z}^\nu \\ n_1+\cdots+n_p=n}} \prod_{j=1}^{p} \mathfrak{c}_{k-1}(\tau,n_j)\mathrm{d}\tau, & k \geq 1. \end{cases}$$

Next we will use the combinatorial tree form of the Picard sequence to prove its exponential decay property and then to prove that it is a Cauchy sequence.

Recall the definition of μ; see subsection 2.2. Set

$$¶^{(k)} := \begin{cases} \{0,1\}, & k=1; \\ \{0\} \cup (¶^{(k-1)})^p, & k \geq 2. \end{cases}$$

$$\mathscr{N}^{(k,\gamma^{(k)})} := \begin{cases} \mathbb{Z}^\nu, & 0 = \gamma^{(k)} \in ¶^{(k)}, k \geq 1; \\ (\mathbb{Z}^\nu)^p, & 1 = \gamma^{(1)}; \\ \prod_{j=1}^{p} \mathscr{N}^{(k-1,\gamma_j^{(k-1)})}, & (\gamma_j^{(k-1)})_{1\leq j\leq p} = \gamma^{(k)} \in (¶^{(k-1)})^p, k \geq 2. \end{cases}$$

$\mathscr{C}^{(k,\gamma^{(k)})}(n^{(k)}) :=$

$$\begin{cases} \mathfrak{c}(n), & 0=\gamma^{(k)}\in\P^{(k)}, n=n^{(k)}\in\mathscr{N}^{(k,0)}, k\geq 1;\\ \prod_{j=1}^{p}\mathfrak{c}(n_j), & 1=\gamma^{(1)}\in\P^{(1)}, (n_j)_{1\leq j\leq p}=n^{(1)}\in\mathscr{N}^{(1,1)};\\ \prod_{j=1}^{p}\mathscr{C}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}), & (\gamma_j^{(k-1)})_{1\leq j\leq p}=\gamma^{(k)}\in(\P^{(k-1)})^p,\\ & (n_j^{(k-1)})_{1\leq j\leq p}=n^{(k)}\in\prod_{j=1}^{p}\mathscr{N}^{(k-1,\gamma_j^{(k-1)})},\\ & k\geq 2. \end{cases}$$

$$\mathscr{I}^{(k,\gamma^{(k)})}(t,n^{(k)}):=$$

$$\begin{cases} e^{\lambda(n)t}, & 0=\gamma^{(k)}\in\P^{(k)},\\ & n=n^{(k)}\in\mathscr{N}^{(k,0)},\\ & k\geq 1;\\ \int_0^t e^{\lambda(\mu(n^{(1)})(t-\tau))}\prod_{j=1}^{p}e^{\lambda(n_j)\tau}\mathrm{d}\tau, & 1=\gamma^{(1)}\in\P^{(1)},\\ & (n_j)_{1\leq j\leq p}=n^{(1)}\\ & \in\mathscr{N}^{(1,1)};\\ \int_0^t e^{\lambda(\mu(n^{(k)}))(t-\tau)}\prod_{j=1}^{p}\mathscr{I}^{(k-1,\gamma_j^{(k-1)})}(\tau,n_j^{(k-1)})\mathrm{d}\tau, & (\gamma_j^{(k-1)})_{1\leq j\leq p}=\gamma^{(k)}\\ & \in(\P^{(k-1)})^p,\\ & (n_j^{(k-1)})_{1\leq j\leq p}=n^{(k)}\\ & \in\prod_{j=1}^{p}\mathscr{N}^{(k-1,\gamma_j^{(k-1)})},\\ & k\geq 2. \end{cases}$$

$$\mathscr{F}^{(k,\gamma^{(k)})}(n^{(k)}):=$$

$$\begin{cases} 1, & 0=\gamma^{(k)}\in\P^{(k)},\\ & n=n^{(k)}\in\mathscr{N}^{(k,0)},\\ & k\geq 1;\\ \frac{\lambda(\mu(n^{(1)}))}{p}, & 1=\gamma^{(1)}\in\P^{(1)},\\ & (n_j)_{1\leq j\leq p}=n^{(1)}\in\mathscr{N}^{(1,1)};\\ \frac{\lambda(\mu(n^{(k)}))}{p}\prod_{j=1}^{p}\mathscr{F}^{(k-1,\gamma_j^{(k-1)})}(n_j^{(k-1)}), & (\gamma_j^{(k-1)})_{1\leq j\leq p}=\gamma^{(k)}\in(\P^{(k-1)})^p,\\ & (n_j^{(k-1)})_{1\leq j\leq p}=n^{(k)}\\ & \in\prod_{j=1}^{p}\mathscr{N}^{(k-1,\gamma_j^{(k-1)})}, k\geq 2. \end{cases}$$

With the help of these abstract symbols, we have the following:

Lemma 3.1. *The Picard sequence* $\{\mathfrak{c}_k(t,n)\}$ *can be rewritten as a combinatorial tree, that is,*

$$\mathfrak{c}_k(t,n)=\sum_{\gamma^{(k)}\in\P^{(k)}}\sum_{\substack{n^{(k)}\in\mathscr{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}}\mathscr{C}^{(k,\gamma^{(k)})}(n^{(k)})\mathscr{I}^{(k,\gamma^{(k)})}(t,n^{(k)})\mathscr{F}^{(k,\gamma^{(k)})}(n^{(k)}),\quad \forall k\geq 1.$$

By induction, we can prove the following estimates for $\mathscr{C}, \mathscr{I}$ and $\mathscr{F}$.

Lemma 3.2. *For all* $k \geq 1$,

$$|\mathscr{C}^{(k,\gamma^{(k)})}(n^{(k)})| \leq \mathcal{A}^{\alpha(\gamma^{(k)})} e^{-\rho|n^{(k)}|},$$

$$|\mathscr{I}^{(k,\gamma^{(k)})}(t, n^{(k)})| \leq \frac{t^{\beta(\gamma^{(k)})}}{\mathscr{D}(\gamma^{(k)})},$$

$$|\mathscr{F}^{(k,\gamma^{(k)})}(n^{(k)})| \leq \frac{1}{(2p)^{\beta(\gamma^{(k)})}} < 1,$$

where

$$\alpha(\gamma^{(k)}) := \begin{cases} \frac{1}{p-1}, & 0 = \gamma^{(k)} \in \P^{(k)}, k \geq 1; \\ \frac{p}{p-1}, & 1 = \gamma^{(1)} \in \P^{(1)}; \\ \sum_{j=1}^{p} \alpha(\gamma_j^{(k-1)}), & (\gamma_j^{(k-1)})_{1 \leq j \leq p} = \gamma^{(k)} \in (\P^{(k-1)})^p, \\ & k \geq 2, \end{cases}$$

$$\beta(\gamma^{(k)}) := \begin{cases} 0, & 0 = \gamma^{(k)} \in \P^{(k)}, k \geq 1; \\ 1, & 1 = \gamma^{(1)} \in \P^{(1)}; \\ 1 + \sum_{j=1}^{p} \beta(\gamma_j^{(k-1)}), & (\gamma_j^{(k-1)})_{1 \leq j \leq p} = \gamma^{(k)} \in (\P^{(k-1)})^p, \\ & k \geq 2, \end{cases}$$

$$\mathscr{D}(\gamma^{(k)}) := \begin{cases} 1, & 0 = \gamma^{(k)} \in \P^{(k)}, k \geq 1; \\ 1, & 1 = \gamma^{(1)} \in \P^{(1)}; \\ \ell(\gamma^{(k)}) \prod_{j=1}^{p} \mathscr{D}(\gamma_j^{(k-1)}), & (\gamma_j^{(k-1)})_{1 \leq j \leq p} = \gamma^{(k)} \in (\P^{(k-1)})^p, \\ & k \geq 2. \end{cases}$$

The following lemma contains some observations we will need below:

Lemma 3.3. *For all* $k \geq 1$,

(1) $\dim_{\mathbb{Z}^\nu} \mathscr{N}^{(k,\gamma^{(k)})} = (p-1)\alpha(\gamma^{(k)})$*;*
(2) $\alpha(\gamma^{(k)}) = \beta(\gamma^{(k)}) + \frac{1}{p-1}$*;*
(3) *If* $0 \leq \natural \leq \frac{(p-1)^{p-1}}{p^p}$, *then*

$$\blacklozenge_k \triangleq \sum_{\gamma^{(k)} \in \P^{(k)}} \frac{\natural^{\beta(\gamma^{(k)})}}{\mathscr{D}(\gamma^{(k)})} \leq \frac{p}{p-1}, \quad \forall k \geq 1.$$

Proof. We first prove the first two identities. It is easy to see that they are true for $0 = \gamma^{(k)} \in \P^{(k)}$, $k \geq 1$, and $1 = \gamma^{(1)}$. Assume that they hold for $1, \cdots, k-1$, where

$k \geq 2$. For $(\gamma_j^{(k-1)})_{1\leq j\leq p} = \gamma^{(k)} \in \P^{(k)}$, by the definition of α, β and $\mathscr{N}^{(k,\gamma^{(k)})}$, one can derive that

$$\begin{aligned}\dim_{\mathbb{Z}^\nu} \mathscr{N}^{(k,\gamma^{(k)})} &= \dim_{\mathbb{Z}^\nu} \prod_{j=1}^{p} \mathscr{N}^{(k-1,\gamma_j^{(k-1)})} \\ &= \sum_{j=1}^{p} \dim_{\mathbb{Z}^\nu} \mathscr{N}^{(k-1,\gamma_j^{(k-1)})} \\ &= (p-1)\sum_{j=1}^{p} \alpha(\gamma_j^{(k-1)}) \\ &= (p-1)\alpha(\gamma^{(k)})\end{aligned}$$

and

$$\begin{aligned}\alpha(\gamma^{(k)}) &= \sum_{j=1}^{p} \alpha(\gamma_j^{(k-1)}) \\ &= \sum_{j=1}^{p} \left(\beta(\gamma_j^{(k-1)}) + \frac{1}{p-1}\right) \\ &= 1 + \sum_{j=1}^{p} \beta(\gamma_j^{(k-1)}) + \frac{1}{p-1} \\ &= \beta(\gamma^{(k)}) + \frac{1}{p-1}.\end{aligned}$$

By induction, it follows that the first two identities hold for all $k \geq 1$.

Next we will prove the last inequality. For $k = 1$, we have

$$\blacklozenge_1 = 1 + \natural \leq 1 + \frac{(p-1)^{p-1}}{p^p} \leq \frac{p}{p-1}.$$

This shows that it holds for $k = 1$. Let $k \geq 2$ and assume that it is true for $1, \cdots, k-1$. For k, we have

$$\blacklozenge_k \leq 1 + \natural \prod_{j=1}^{p} \sum_{\gamma_j^{(k-1)} \in \P^{(k-1)}} \frac{\natural^{\beta(\gamma_j^{(k-1)})}}{\mathscr{D}(\gamma_j^{(k-1)})} \leq 1 + \frac{(p-1)^{(p-1)}}{p^p} \cdot \left(\frac{p}{p-1}\right)^p = \frac{p}{p-1}.$$

Hence the last inequality is true for all $k \geq 1$. This completes the proof of Lemma 3.3. □

In a similar way as in the proof of Lemma 2.4, we can obtain uniform exponential decay for the Picard sequence $\{\mathfrak{c}_k(t,n)\}$:

Lemma 3.4. *Assume that the initial Fourier coefficients $\mathfrak{c}(n)$ obey (1.7). With the constants $\mathcal{A}$ and ρ from (1.7) and the dimension ν set*

$$\mathcal{B}_p \triangleq \frac{p}{p-1}\mathcal{A}^{\frac{1}{p-1}}(6\rho^{-1})^{\nu} \tag{3.3}$$

and

$$\mathcal{L}_p \triangleq \frac{2(p-1)^{p-1}\rho^{(p-1)\nu}}{p^{p-1}\mathcal{A}6^{(p-1)\nu}}. \tag{3.4}$$

Then, we have

$$\sup_{\substack{t\in[0,\mathcal{L}_p]\\ k\geq 0}} |\mathfrak{c}_k(t,n)| \leq \mathcal{B}_p e^{-\frac{\rho}{2}|n|}$$

for every $n \in \mathbb{Z}^{\nu}$.

Furthermore using the following pattern decomposition,

$$|\prod_{j=1}^{\star}\blacktriangle_j - \prod_{j=1}^{\star}\blacksquare_j| \leq \sum_{j'=1}^{\star}\prod_{j=1}^{j'-1}|\blacksquare_j|\cdot|\blacktriangle_{j'} -, \blacksquare_{j'}|\cdot\prod_{j=j'+1}^{\star}|\blacktriangle_j|,$$

where

$$\prod_{j=1}^{0}|\blacksquare_j| := 1 \quad \text{and} \quad \prod_{j=\star+1}^{\star}|\blacktriangle_j| := 1,$$

we can prove the following by induction:

Lemma 3.5. *For all $k \geq 1$,*

$$\begin{aligned}|\mathfrak{c}_k(t,n) - \mathfrak{c}_{k-1}(t,n)| &\leq \frac{p^{k-1}\mathcal{B}_p^{(p-1)k+1}t^k}{(2p)^k\cdot k!}\sum_{\substack{n_1,\cdots,n_{(p-1)k+1}\in\mathbb{Z}^{\nu}\\ n_1+\cdots+n_{(p-1)k+1}=n}}\prod_{j=1}^{(p-1)k+1}e^{-\frac{\rho}{2}|n_j|}\\ &\leq \frac{\mathcal{B}_p(12\rho^{-1})^{\nu}}{p}\cdot\frac{\left(2^{-1}\mathcal{B}_p^{p-1}(12\rho^{-1})^{(p-1)\nu}t\right)^k}{k!}\cdot e^{-\frac{\rho}{4}|n|}.\end{aligned}$$

Hence $\{\mathfrak{c}_k(t,n)\}$ is a Cauchy sequence on $[0,\mathcal{L}_p]\times\mathbb{Z}^{\nu}$.

With these estimates in hand, Theorem A for the quasi-periodic Cauchy problem (1.4)–(1.5), i.e. gBBM, follows in the same way as it did in Section 2 for BBM. This completes the proof of Theorem A.

4. Proof of Theorem B

In this section we generalize the decay condition from exponential to polynomial for the Fourier coefficients of the quasi-periodic initial data. For the sake of convenience and readability, we take the case of $p = 2$ (BBM) as an illustration. This generalization works for the general case (we will give a proof for the general case in the forthcoming paper on the nonlinear Schrödinger equation with quasi-periodic initial data mentioned in Remark 2.3).

Specifically, the exponential decay condition (1.7) is replaced by the polynomial decay condition (1.9), where $2 \le \nu < \frac{\mathtt{r}}{4} - 2$.

From the proof above, we need to re-estimate only $\mathfrak{C}$.

Lemma 4.1. *If the initial Fourier coefficients satisfy the polynomial decay estimate* (1.9)*, then*

$$|\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)})| \le \mathtt{A}^{\sigma(\gamma^{(k)})} \prod_{j=1}^{\sigma(\gamma^{(k)})} \left(1 + |(n^{(k)})_j|\right)^{-\mathtt{r}}, \quad \forall k \ge 1. \tag{4.1}$$

Proof. We first prove the following equality: for all $k \ge 1$ and $n^{(k)} = (n_j)_{1 \le j \le \sigma(\gamma^{(k)})} \in \mathfrak{N}^{(k,\gamma^{(k)})}$,

$$\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) = \prod_{j=1}^{\sigma(\gamma^{(k)})} c(n_j). \tag{4.2}$$

It is not difficult to see that (4.2) holds for $0 = \gamma^{(k)} \in \spadesuit^{(k)}, k \ge 1$, and $1 = \gamma^{(1)} \in \spadesuit^{(1)}$.

Let $k \ge 2$. Assume that it holds for $1, \cdots, k-1$. For $k, (\gamma_1^{(k-1)}, \gamma_2^{(k-1)}) = \gamma^{(k)} \in (\spadesuit^{(k-1)})^2$ and $(n_1^{(k-1)}, n_2^{(k-1)}) = n^{(k)} \in \prod_{j=1}^{2} \mathfrak{N}^{(k-1,\gamma_j^{(k-1)})}$, where $n_1^{(k-1)} = (n_j)_{1 \le j \le \sigma(\gamma_1^{(k-1)})}$ and $n_2^{(k-1)} = (n_{\sigma(\gamma_1^{(k-1)})+j})_{1 \le j \le \sigma(\gamma_1^{(k-1)})}$, by the definition of $\mathfrak{C}$, one can derive that

$$\begin{aligned}
\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)}) &= \mathfrak{C}^{(k-1,\gamma_1^{(k-1)})}(n_1^{(k-1)}) \cdot \mathfrak{C}^{(k-1,\gamma_2^{(k-1)})}(n_2^{(k-1)}) \\
&= \prod_{j=1}^{\sigma(\gamma_1^{(k-1)})} c(n_j) \cdot \prod_{j=1}^{\sigma(\gamma_2^{(k-1)})} c(n_{\sigma(\gamma_1^{(k-1)})+j}) \\
&= \prod_{j=1}^{\sigma(\gamma^{(k)})} c(n_j).
\end{aligned}$$

By induction, (4.2) holds for all $k \ge 1$. It follows from polynomial decay (1.9) for c that

$$
\begin{aligned}
|\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)})| &= \prod_{j=1}^{\sigma(\gamma^{(k)})} |c((n^{(k)})_j)| \\
&\leq \prod_{j=1}^{\sigma(\gamma^{(k)})} \mathtt{A}(1+|(n^{(k)})_j|)^{-\mathtt{r}} \\
&= \mathtt{A}^{\sigma(\gamma^{(k)})} \prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)^{-\mathtt{r}}.
\end{aligned}
$$

This completes the proof of Lemma 4.1. □

In addition, we need the following basic statements:

Lemma 4.2.

(1) *(Mean value inequality)*

$$
\prod_{j=1}^{n} \mathtt{a}_j \leq \left(\frac{1}{n}\sum_{j=1}^{n}\mathtt{a}_j\right)^{n}, \quad \mathtt{a}_j > 0, j = 1, \cdots, n \in \mathbb{N}. \tag{4.3}
$$

(2) *(Bound for the Riemann zeta function on $\mathbb{R}$)*

$$
\sum_{n=1}^{\infty} \frac{1}{n^{\mathtt{s}}} =: \zeta(\mathtt{s}) \leq 1 + \frac{1}{\mathtt{s}-1}, \quad \mathtt{s} > 1. \tag{4.4}
$$

(3) *Set*

$$
\sum_{n\in\mathbb{Z}^{\nu}} \frac{1}{(1+|n|)^{\mathtt{s}}} := \mathscr{H}(\mathtt{s};\nu). \tag{4.5}
$$

If $2 \leq \nu < \mathtt{s}$, then

$$
\mathscr{H}(\mathtt{s};\nu) \leq \mathfrak{b}(\mathtt{s};\nu) \triangleq 1 + \sum_{j_0=1}^{\nu} \binom{\nu}{j_0} 2^{j_0} j_0^{-\mathtt{s}} \left\{\zeta\left(\frac{\mathtt{s}}{j_0}\right)\right\}^{j_0}. \tag{4.6}
$$

(4) *(Generalized Bernoulli inequality)*

$$
\prod_{j=1}^{m}(1+x_j) \geq 1 + \sum_{j=1}^{m} x_j, \tag{4.7}
$$

where $2 \leq m \in \mathbb{N}$, $x_1, \cdots, x_r$ are real numbers, all greater than -1, and all with the same sign.

Proof. (1) This is just the Arithmetic Mean-Geometric Mean inequality, so the proof is omitted.

(2) Let $g(x) = x^{-\mathtt{s}}, \mathtt{s} > 1$. Notice that g is monotonically decreasing on $[1, \infty)$. Thus, we have

$$g(n+1) \leq \int_n^{n+1} g(\tau) \mathrm{d}\tau, \quad \forall n \geq 1. \tag{4.8}$$

For all $N \geq 2$, the summation of (4.8) over $n = 1, \ldots, N-1$ yields

$$\sum_{n=1}^{N} g(n) \leq g(1) + \int_1^N g(\tau)\, \mathrm{d}\tau,$$

that is,

$$\sum_{n=1}^{N} \frac{1}{n^{\mathtt{s}}} \leq 1 + \frac{1}{\mathtt{s}-1}, \quad \text{uniformly for} \ \ N, \quad \text{provided that } \mathtt{s} > 1.$$

Hence (4.4) holds for all $\mathtt{s} > 1$.

(3) Set $\wp := \{0, \cdots, \nu-1, \nu\}$. For every $j_0 \in \wp$, define

$$\mathcal{S}_{j_0} := \{n = (n_1, \cdots, n_\nu) \in \mathbb{Z}^\nu : \text{exactly } \nu - j_0 \text{ components are equal to zero}\}.$$

Hence we have the following decomposition,

$$\mathscr{H}(\mathtt{s}; \nu) = 1 + \sum_{n \in \bigcup_{j_0 \in \wp \setminus \{0\}} \mathcal{S}_{j_0}} (1+|n|)^{-\mathtt{s}} \triangleq 1 + (IV)$$

On the one hand, for all $j_0 \in \wp \setminus \{0\}$, we have

$$\begin{aligned}
\sum_{n \in \mathcal{S}_{j_0}} (1+|n|)^{-\mathtt{s}} &= \binom{\nu}{j_0} \sum_{\substack{n=(n_1,\cdots,n_{j_0},0,\cdots,0) \in \mathbb{Z}^\nu \\ n_1,\cdots,n_{j_0} \in \mathbb{Z}\setminus\{0\}}} (1+|n|)^{-\mathtt{s}} \\
&= \binom{\nu}{j_0} \sum_{n_1,\cdots,n_{j_0} \in \mathbb{Z}\setminus\{0\}} \left(1 + \sum_{j=1}^{j_0} |n_j|\right)^{-\mathtt{s}} \\
&\leq \binom{\nu}{j_0} \sum_{n_1,\cdots,n_{j_0} \in \mathbb{Z}\setminus\{0\}} \left(\sum_{j=1}^{j_0} |n_j|\right)^{-\mathtt{s}} \\
&\overset{(4.3)}{\leq} \binom{\nu}{j_0} j_0^{-\mathtt{s}} \sum_{n_1,\cdots,n_{j_0} \in \mathbb{Z}\setminus\{0\}} \prod_{j=1}^{j_0} |n_j|^{-\frac{\mathtt{s}}{j_0}}
\end{aligned}$$

$$= \binom{\nu}{j_0} j_0^{-\mathtt{s}} \prod_{j=1}^{j_0} \sum_{n_j \in \mathbb{Z}\setminus\{0\}} |n_j|^{-\frac{\mathtt{s}}{j_0}}$$

$$= \binom{\nu}{j_0} 2^{j_0} j_0^{-\mathtt{s}} \prod_{j=1}^{j_0} \sum_{n_j=1}^{\infty} n_j^{-\frac{\mathtt{s}}{j_0}}$$

$$= \binom{\nu}{j_0} 2^{j_0} j_0^{-\mathtt{s}} \prod_{j=1}^{j_0} \zeta\left(\frac{\mathtt{s}}{j_0}\right)$$

$$= \binom{\nu}{j_0} 2^{j_0} j_0^{-\mathtt{s}} \left\{\zeta\left(\frac{\mathtt{s}}{j_0}\right)\right\}^{j_0}.$$

Hence we have

$$\begin{aligned}(IV) &= \sum_{n\in\bigcup_{j_0\in\wp\setminus\{0\}} \mathcal{S}_{j_0}} (1+|n|)^{-\mathtt{s}} \\ &= \sum_{j_0\in\wp\setminus\{0\}} \sum_{n\in\mathcal{S}_{j_0}} (1+|n|)^{-\mathtt{s}} \\ &= \sum_{j_0=1}^{\nu} \binom{\nu}{j_0} 2^{j_0} j_0^{-\mathtt{s}} \left\{\zeta\left(\frac{\mathtt{s}}{j_0}\right)\right\}^{j_0}.\end{aligned}$$

Combining these estimates, we arrive at the following inequality,

$$\mathscr{H}(\mathtt{s};\nu) \le 1 + \sum_{j_0=1}^{\nu} \binom{\nu}{j_0} 2^{j_0} j_0^{-\mathtt{s}} \left\{\zeta\left(\frac{\mathtt{s}}{j_0}\right)\right\}^{j_0} \triangleq \mathfrak{b}(\mathtt{s};\nu).$$

It follows from (4.4) that $\mathscr{H}(\mathtt{s};\nu)$ is a bounded positive number for any fixed $\mathtt{s}$ and ν for which $2 \le \nu < \mathtt{s}$.

(4) Clearly, for $m = 2$, it follows from the same sign condition of x_1 and x_2 that

$$(1+x_1)(1+x_2) = 1 + x_1 + x_2 + x_1x_2 \ge 1 + x_1 + x_2.$$

Let $m \ge 3$. Assume that (4.7) holds for all $2 < m' < m$. For m, by the induction hypothesis, $x_m > -1$ and the same sign condition, we have

$$\begin{aligned}\prod_{j=1}^{m}(1+x_j) &= \prod_{j=1}^{m-1}(1+x_j) \times (1+x_m) \\ &\ge (1+x_1+\cdots+x_{m-1})(1+x_m) \\ &= 1+x_1+\cdots+x_{m-1}+x_m+\sum_{j=1}^{m-1} x_jx_m \\ &\ge 1+x_1+\cdots+x_{m-1}+x_m.\end{aligned}$$

By induction, (4.7) holds for all $2 \le m \in \mathbb{N}$. □

In what follows we will prove that the Picard sequence satisfies a uniform polynomial decay estimate (Lemma 4.3) and is fundamental (i.e., a Cauchy sequence; Lemma 4.4).

Lemma 4.3. *If* $0 \le t \le \frac{1}{\mathtt{A}\mathfrak{b}(\frac{\mathtt{r}}{2};\nu)} \triangleq \mathcal{L}_2'$ *and* $2 \le \nu < \frac{\mathtt{r}}{2}$, *then*

$$|c_k(t,n)| \le \mathtt{B}_2(1+|n|)^{-\frac{\mathtt{r}}{2}}, \quad \textit{for all } \mathbb{N} \ni k \ge 1, \quad \textit{where } \mathtt{B}_2 \triangleq 2\mathtt{A}\mathfrak{b}\left(\frac{\mathtt{r}}{2};\nu\right). \tag{4.9}$$

Proof. We first have

$$\begin{aligned} |c_k(t,n)| &\overset{(2.7)}{\le} \sum_{\gamma^{(k)}\in\spadesuit^{(k)}} \sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}} |\mathfrak{C}^{(k,\gamma^{(k)})}(n^{(k)})||\mathfrak{I}^{(k,\gamma^{(k)})}(t,n^{(k)})||\mathfrak{F}^{(k,\gamma^{(k)})}(n^{(k)})| \\ &\overset{(4.1),(2.12)-(2.14)}{\le} \mathtt{A} \sum_{\gamma^{(k)}\in\spadesuit^{(k)}} \frac{(4^{-1}\mathtt{A}t)^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})} \sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}} \prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)^{-\mathtt{r}}. \end{aligned} \tag{4.10}$$

The main difference, compared to the proof in the exponential decay case, is to deal with the term

$$\begin{aligned} &\sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}} \prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)^{-\mathtt{r}} \\ &= \sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}} \prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)^{-\frac{\mathtt{r}}{2}} \cdot \underbrace{\prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)^{-\frac{\mathtt{r}}{2}}}_{\boxdot}. \end{aligned} \tag{4.11}$$

It follows from the generalized Bernoulli inequality (4.7) and $\mu(n^{(k)}) = n$ that

$$\begin{aligned} \boxdot &= \prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)^{-\frac{\mathtt{r}}{2}} \\ &= \left(\prod_{j=1}^{\sigma(\gamma^{(k)})} (1+|(n^{(k)})_j|)\right)^{-\frac{\mathtt{r}}{2}} \\ &\overset{(4.7)}{\le} \left(1+\sum_{j=1}^{\sigma(\gamma^{(k)})} |(n^{(k)})_j|\right)^{-\frac{\mathtt{r}}{2}} \end{aligned}$$

$$= (1+|n^{(k)}|)^{-\frac{\mathfrak{r}}{2}}$$
$$\leq (1+|n|)^{-\frac{\mathfrak{r}}{2}}. \tag{4.12}$$

Inserting (4.12) into (4.11) yields

$$\begin{aligned}\sum_{\substack{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}\\ \mu(n^{(k)})=n}}\prod_{j=1}^{\sigma(\gamma^{(k)})}(1+|(n^{(k)})_j|)^{-\mathfrak{r}} &\leq \sum_{n^{(k)}\in\mathfrak{N}^{(k,\gamma^{(k)})}}\prod_{j=1}^{\sigma(\gamma^{(k)})}(1+|(n^{(k)})_j|)^{-\frac{\mathfrak{r}}{2}}\cdot(1+|n|)^{-\frac{\mathfrak{r}}{2}}\\ &\leq \sum_{n_1,\cdots,n_{\sigma(\gamma^{(k)})}\in\mathbb{Z}^\nu}\prod_{j=1}^{\sigma(\gamma^{(k)})}(1+|n_j|)^{-\frac{\mathfrak{r}}{2}}\cdot(1+|n|)^{-\frac{\mathfrak{r}}{2}}\\ &= \prod_{j=1}^{\sigma(\gamma^{(k)})}\underbrace{\sum_{n_j\in\mathbb{Z}^\nu}(1+|n_j|)^{-\frac{\mathfrak{r}}{2}}}_{=\mathscr{H}(\frac{\mathfrak{r}}{2};\nu)\leq\mathfrak{b}(\frac{\mathfrak{r}}{2};\nu)\text{ by (4.6)}}\cdot(1+|n|)^{-\frac{\mathfrak{r}}{2}}\\ &\leq \left\{\mathfrak{b}\left(\frac{\mathfrak{r}}{2};\nu\right)\right\}^{\sigma(\gamma^{(k)})}(1+|n|)^{-\frac{\mathfrak{r}}{2}}. \end{aligned}\tag{4.13}$$

Inserting (4.13) into (4.10), by (2.14), we obtain

$$|c_k(t,n)| \leq \mathtt{A}\mathfrak{b}\left(\frac{\mathfrak{r}}{2};\nu\right)\sum_{\gamma^{(k)}\in\spadesuit^{(k)}}\frac{\left\{4^{-1}\mathtt{A}\mathfrak{b}\left(\frac{\mathfrak{r}}{2};\nu\right)t\right\}^{\ell(\gamma^{(k)})}}{\mathfrak{D}(\gamma^{(k)})}\cdot(1+|n|)^{-\frac{\mathfrak{r}}{2}}.$$

It follows from (2.17) that

$$|c_k(t,n)| \leq \mathtt{B}_2(1+|n|)^{-\frac{\mathfrak{r}}{2}}, \quad \text{where } \mathtt{B}_2 \triangleq 2\mathtt{A}\mathfrak{b}\left(\frac{\mathfrak{r}}{2};\nu\right),$$

provided that

$$0 \leq t \leq \mathcal{L}_2' = \frac{1}{\mathtt{A}\mathfrak{b}\left(\frac{\mathfrak{r}}{2};\nu\right)}.$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *If $2 \leq \nu < \frac{\mathfrak{r}}{4}$, then for all $k \geq 1$,*

$$|c_k(t,n) - c_{k-1}(t,n)| \leq \frac{2^{k-1}\mathtt{B}_2^{k+1}t^k}{4^k\cdot k!}\sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu\\ n_1+\cdots+n_{k+1}=n}}\left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{2}} \tag{4.14}$$

$$\leq \frac{\mathtt{B}_2\mathfrak{b}\left(\frac{\mathfrak{r}}{4};\nu\right)}{2}\cdot\frac{\left\{2^{-1}\mathtt{B}_2\mathfrak{b}\left(\frac{\mathfrak{r}}{4};\nu\right)t\right\}^k}{k!}\cdot\{1+|n|\}^{-\frac{\mathfrak{r}}{4}}. \tag{4.15}$$

This implies that $\{c_k(t,n)\}$ is a Cauchy sequence on $(t,n) \in [0,\mathcal{L}_2'] \times \mathbb{Z}^\nu$.

Proof. For $k=1$ one has

$$
\begin{aligned}
|c_1(t,n)-c_0(t,n)| &\le \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} |c_0(s,n_j)|\,\mathrm{d}\tau \\
&= \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \prod_{j=1}^{2} |c(n_j)|\,\mathrm{d}\tau \\
&\le \frac{\mathtt{B}_2^2 t}{4} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \left\{\prod_{j=1}^{2}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{2}}.
\end{aligned}
$$

This shows that (4.14) is true for $k=1$. Let $k \ge 2$ and suppose that it holds for $1,\cdots,k-1$. For k, one can derive that

$$
\begin{aligned}
|c_k(t,n)-c_{k-1}(t,n)| \le{}& \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} |c_{k-1}(\tau,n_1)||c_{k-1}(\tau,n_2)-c_{k-2}(\tau,n_2)|\,\mathrm{d}\tau \triangleq (I') \\
&+ \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} |c_{k-1}(\tau,n_1)-c_{k-2}(\tau,n_1)||c_{k-2}(\tau,n_2)|\,\mathrm{d}\tau \triangleq (II'),
\end{aligned}
$$

where

$$
\begin{aligned}
(I') &\le \frac{1}{4}\int_0^t \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \mathtt{B}_2(1+|n_1|)^{-\frac{\mathfrak{r}}{2}} \cdot \frac{2^{k-2}\mathtt{B}_2^k \tau^{k-1}}{4^{k-1}\cdot(k-1)!} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}^\nu\\ m_1+\cdots+m_k=n_2}} \left\{\prod_{j=1}^{k}(1+|m_j|)\right\}^{-\frac{\mathfrak{r}}{2}} \mathrm{d}\tau \\
&= \frac{2^{k-2}\mathtt{B}_2^{k+1}t^k}{4^k\cdot k!} \sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}} \sum_{\substack{m_1,\cdots,m_k\in\mathbb{Z}^\nu\\ m_1+\cdots+m_k=n_2}} (1+|n_1|)^{-\frac{\mathfrak{r}}{2}} \left\{\prod_{j=1}^{k}(1+|m_j|)\right\}^{-\frac{\mathfrak{r}}{2}} \\
&= \frac{2^{k-2}\mathtt{B}_2^{k+1}t^k}{4^k\cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu\\ n_1+\cdots+n_{k+1}=n}} \left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{2}},
\end{aligned}
$$

and analogously

$$(II') \leq \frac{2^{k-2}\mathtt{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{2}}.$$

Hence, according to (4.7), we have

$$\begin{aligned}
&|c_k(t,n) - c_{k-1}(t,n)| \\
&\quad \leq (I') + (II') \\
&\quad \leq \frac{2^{k-1}\mathtt{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{2}} \\
&\quad \leq \frac{2^{k-1}\mathtt{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{\substack{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu \\ n_1+\cdots+n_{k+1}=n}} \left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{4}} \cdot \left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{4}} \\
&\quad \leq \frac{2^{k-1}\mathtt{B}_2^{k+1}t^k}{4^k \cdot k!} \sum_{n_1,\cdots,n_{k+1}\in\mathbb{Z}^\nu} \left\{\prod_{j=1}^{k+1}(1+|n_j|)\right\}^{-\frac{\mathfrak{r}}{4}} \cdot \left\{1+\sum_{j=1}^{k+1}|n_j|\right\}^{-\frac{\mathfrak{r}}{4}} \\
&\quad \leq \frac{2^{k-1}\mathtt{B}_2^{k+1}t^k}{4^k \cdot k!} \prod_{j=1}^{k+1} \underbrace{\sum_{n_j\in\mathbb{Z}^\nu} \{(1+|n_j|)\}^{-\frac{\mathfrak{r}}{4}}}_{=\mathscr{H}\left(\frac{\mathfrak{r}}{4};\nu\right)\leq\mathfrak{b}\left(\frac{\mathfrak{r}}{4};\nu\right) \quad \text{by} \quad (4.6)} \cdot \{1+|n|\}^{-\frac{\mathfrak{r}}{4}} \\
&\quad = \frac{\mathtt{B}_2\mathfrak{b}\left(\frac{\mathfrak{r}}{4};\nu\right)}{2} \cdot \frac{\left\{2^{-1}\mathtt{B}_2\mathfrak{b}\left(\frac{\mathfrak{r}}{4};\nu\right)t\right\}^k}{k!} \cdot \{1+|n|\}^{-\frac{\mathfrak{r}}{4}}.
\end{aligned}$$

This completes the proof of Lemma 4.4. □

We are now in a position to prove our second main result, Theorem B.

Proof of Theorem B. The existence proof is similar to the case of exponential decay. The uniqueness proof is analogous to proving that the Picard sequence is a Cauchy sequence. We mainly give a convergence analysis to show that the solution we construct is in the classical sense. In fact, for $\# = 0, 1, 2$, one can derive that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^\nu} |n|^{\#}|c(t,n)| &\lesssim \sum_{n\in\mathbb{Z}^\nu} |n|^{\#}(1+|n|)^{-\frac{\mathfrak{r}}{2}} \\
&\leq \sum_{n\in\mathbb{Z}^\nu} (1+|n|)^{\#-\frac{\mathfrak{r}}{2}} \\
&= \mathscr{H}\left(\frac{\mathfrak{r}}{2} - \#;\nu\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^\nu}|n|^{\#}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}|c(t,n_j)| \\
&\lesssim \sum_{n\in\mathbb{Z}^\nu}|n|^{\#}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}(1+|n_j|)^{-\frac{\mathtt{r}}{2}} \\
&\overset{(4.7)}{\leq} \sum_{n\in\mathbb{Z}^\nu}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}(|n_1|+|n_2|)^{\#}\prod_{j=1}^{2}(1+|n_j|)^{-\frac{\mathtt{r}}{4}}\cdot(1+|n|)^{-\frac{\mathtt{r}}{4}} \\
&\leq \sum_{n\in\mathbb{Z}^\nu}(1+|n|)^{-\frac{\mathtt{r}}{4}}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}((1+|n_1|)(1+|n_2|))^{\#} \\
&\quad \prod_{j=1}^{2}(1+|n_j|)^{-\frac{\mathtt{r}}{4}} \\
&= \sum_{n\in\mathbb{Z}^\nu}(1+|n|)^{-\frac{\mathtt{r}}{4}}\sum_{\substack{n_1,n_2\in\mathbb{Z}^\nu\\ n_1+n_2=n}}\prod_{j=1}^{2}(1+|n_j|)^{\#-\frac{\mathtt{r}}{4}} \\
&\leq \sum_{n\in\mathbb{Z}^\nu}(1+|n|)^{-\frac{\mathtt{r}}{4}}\sum_{n_1,n_2\in\mathbb{Z}^\nu}\prod_{j=1}^{2}(1+|n_j|)^{\#-\frac{\mathtt{r}}{4}} \\
&\leq \sum_{n\in\mathbb{Z}^\nu}(1+|n|)^{-\frac{\mathtt{r}}{4}}\prod_{j=1}^{2}\sum_{n_j\in\mathbb{Z}^\nu}(1+|n_j|)^{\#-\frac{\mathtt{r}}{4}} \\
&= \mathscr{H}\left(\frac{\mathtt{r}}{4};\nu\right)\left\{\mathscr{H}\left(\frac{\mathtt{r}}{4}-\#;\nu\right)\right\}^2.
\end{aligned}$$

It follows from (4.5) that the convergence needed can be guaranteed if

$$2\leq\nu<\frac{\mathtt{r}}{4}-2.$$

This completes the proof of Theorem B. □

Data availability

No data was used for the research described in the article.

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