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Short Communication: Is a Sophisticated Agent Always a Wise One?*

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Abstract. For time-inconsistent optimal control problems, a quite popular approach is the equilibrium approach, taken by sophisticated agents. In this short note, we construct a deterministic continuous-time example where the unique equilibrium is dominated by another control. Therefore, in this situation, it may not be wise to take the equilibrium strategy.

Key words. time inconsistency, Pareto optimal, sophisticated agents, equilibrium strategy, naive strategy, precommitted strategy

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1. Introduction.

For time-inconsistent optimal control problems, different types of agents would choose different strategies, for example, the precommitted strategy, the naive strategy, and the equilibrium strategy; see, e.g., the survey paper by Strotz [14]. Among them, the equilibrium approach by sophisticated agents has received very strong attention. In particular, in the past decade, this approach has been extended to continuous-time models by many authors; see, e.g., Björk, Khapko, and Murgoci [1], Björk, Murgoci, and Zhou [2], Ekeland and Lazrak [4], and Yong [15], to mention a few. We also refer to the recent paper by Hernández and Possamaï [8] for a nice literature review. In this approach, the sophisticated agent will play a game with (infinitely many) future selves, and the goal is to find an equilibrium which is suboptimal and time consistent in a certain sense.

In this short note, we construct an example in the deterministic continuous-time framework such that the unique equilibrium is not Pareto optimal. To be precise, our optimal control problem has a unique equilibrium α^*, but we can construct another control $\hat{\alpha}$ such that

$$J(t,\hat{\alpha}) < J(t,\alpha^*) \quad \text{for all } t < T, \quad \text{and} \quad J(T,\hat{\alpha}) = J(T,\alpha^*), \tag{1.1}$$

where $J(t,\alpha)$ is the dynamic cost function with control α. This raises the serious question on the rationale of using the equilibrium α^*.

In a noncooperative game, it is not surprising that an equilibrium may not be Pareto optimal, for example, in the well-known prisoner's dilemma; cf. Nash [13, Example 2] and Lacey [12]. In that case, since the players do not play cooperatively, typically due to lack of mutual trust, they may still choose the equilibrium. For the sophisticated agent in our time-inconsistent problem, however, the agent is "playing" the game with future selves, and

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there is no reason that the agent would "play" in a noncooperative way. In particular, in our example, it is not rational or, say, not wise for the agent to choose α^* instead of $\hat{\alpha}$. We shall point out, though, that we are not claiming any optimality of the constructed $\hat{\alpha}$.

It will be very interesting to explore possible alternative notions of equilibrium, or of "good" strategies, which we shall leave to future research. In our opinion, a good strategy in a dynamic approach should satisfy at least two basic properties: (i) time consistency and (ii) Pareto optimality. Time consistency has been a natural consideration for time-inconsistent problems, as [14, p. 173] points out that when there is intertemporal conflict (namely, time inconsistency), the player's problem "*is then to find the best plan among those that he will actually follow*." Pareto optimality is a basic requirement in cooperative game theory; in fact, it is exactly the core of the notion; cf. Gillies [6]. In this short note, we want to bring to attention Pareto optimality, which seems less addressed for time-inconsistent problems. We note that the precommitted strategy is by definition Pareto optimal but time inconsistent (unless the original problem is time consistent) and that the equilibrium strategy is time consistent but may not be Pareto optimal, as shown in this note. We also note that the naive strategy, while less interesting and less popular in the literature, is also time consistent; see, e.g., the recent paper by Chen and Zhou [3]. However, in the same spirit of Remark 2.4, we can easily construct an example such that the naive strategy is not Pareto optimal.

We remark that the time consistency of strategies relies on the criterion we take; for example, the equilibrium strategy and the naive strategy satisfy time consistency in different senses. When exploring good alternative strategies, it will be a crucial and intrinsic component to specify this criterion, which in practice relies on the agent's preference. We would also like to mention the dynamic utility function in [11] and the moving scalarization in Feinstein and Rudloff [5] (see also the recent paper İşeri and Zhang [10]), where the precommitted strategy satisfies both time consistency and Pareto optimality. In this approach, the utility function for the subproblem over $[t,T]$ is modified and thus leads to a different function $J(t,\alpha)$ when $t>0$. Similarly, whether to use the dynamic utility function or the original utility function relies on the agent's preference.

The rest of the paper is organized as follows. In section 2, we construct the example and prove (1.1). In section 3, we verify that our example is indeed time inconsistent. Finally, in section 4, we construct another example where the naive strategy is not Pareto optimal.

2. An example.

Set the time horizon $[0,T]$ with $T=1$, and denote

$$t_n := 1-2^{-n} \quad \text{and} \quad s_n := \frac{1}{4}[t_n+3t_{n+1}], \qquad n\geq 0. \tag{2.1}$$

The admissible control set $\mathcal{A}$ consists of Borel-measurable functions $\alpha:[0,1]\to[-1,1]$. Consider the following deterministic two-dimensional backward controlled system:

$$\begin{gathered} Y_t^{1,\alpha} := \int_t^T \alpha_s ds, \quad Y_t^{2,\alpha} := \int_t^T c(s)\alpha_s Y_s^{1,\alpha} ds, \quad t\in[0,T], \\ \text{where} \quad c(t) := \sum_{n=0}^{\infty} [\mathbf{1}_{[t_n,s_n)}(t) + 6\mathbf{1}_{[s_n,t_{n+1})}(t)]. \end{gathered} \tag{2.2}$$

Our time-inconsistent optimization problem is

$$V_t := \inf_{\alpha\in\mathcal{A}} J(t,\alpha), \quad \text{where} \quad J(t,\alpha) := Y_t^{2,\alpha}. \tag{2.3}$$

We remark that besides the hyperbolic discounting, this type of multidimensional optimization problem is also typically time inconsistent, which includes the well-known mean variance optimization problem; see Karnam, Ma, and Zhang [11] for more discussion. We shall prove rigorously the time inconsistency for this example in the next section, and here we focus on the equilibrium approach and Pareto optimality.

We first recall the notion of equilibrium.

Definition 2.1. *We say $\alpha^*\in\mathcal{A}$ is an equilibrium if, for any $t\in[0,T)$ and $\alpha\in\mathcal{A}$,*

$$\varliminf_{\delta\downarrow 0}\frac{1}{\delta}[J(t,\alpha\oplus_{t+\delta}\alpha^*) - J(t,\alpha^*)] \ge 0, \quad \textit{where} \quad \alpha\oplus_{t+\delta}\alpha^* := \alpha\mathbf{1}_{[0,t+\delta)} + \alpha^*\mathbf{1}_{[t+\delta,T]}. \tag{2.4}$$

Proposition 2.2. *$\alpha^*\equiv 0$ is the unique equilibrium.*

Proof. (i) We first show that $\alpha^*\equiv 0$ is an equilibrium. Indeed, for any $t\in[0,T)$, by our construction of c, there exists $\delta_t>0$ such that $c(s)\equiv c(t)>0$ for all $s\in[t,t+\delta_t]$. Then, for any $0<\delta\le\delta_t$ and any $\alpha\in\mathcal{A}$, denoting $\alpha^\delta := \alpha\otimes_{t+\delta}\alpha^*$,

$$Y_s^{1,\alpha^\delta} = \mathbf{1}_{[0,t+\delta]}(s)\int_s^{t+\delta}\alpha_r dr, \tag{2.5}$$

$$J(t,\alpha^\delta) = Y_t^{2,\alpha^\delta} = \int_t^{t+\delta} c(s)\alpha_s Y_s^{1,\alpha^\delta} ds = c(t)\int_t^{t+\delta}\alpha_s\int_s^{t+\delta}\alpha_r dr ds = \frac{c(t)}{2}\left(\int_t^{t+\delta}\alpha_r dr\right)^2 \ge 0.$$

It is obvious that $J(t,\alpha^*) = 0 \le J(t,\alpha^\delta)$. Then (2.4) holds true. In fact, α^* is an equilibrium in a stronger sense, as in He and Jiang [7] and Huang and Zhou [9].

(ii) We next show the uniqueness. Let $\alpha^*\in\mathcal{A}$ be an arbitrary equilibrium. For any $t\in[0,T)$ and $\alpha\in\mathcal{A}$, let $\delta_t>0$ and α^δ be as in (i). Then, for any $0<\delta\le\delta_t$, noting that $|\alpha|, |\alpha^*|\le 1$,

$$J(t,\alpha^\delta) - J(t,\alpha^*) = c(t)\int_t^{t+\delta}[\alpha_s Y_s^{1,\alpha^\delta} - \alpha_s^* Y_s^{1,\alpha^*}]ds = c(t)Y_t^{1,\alpha^*}\int_t^{t+\delta}[\alpha_s-\alpha_s^*]ds + O(\delta^2). \tag{2.6}$$

Thus, by setting $\alpha\equiv -1$ when $Y_t^{1,\alpha^*}>0$ and $\alpha\equiv 1$ when $Y_t^{1,\alpha^*}<0$, it follows from (2.4) that

$$\begin{aligned} &\varliminf_{\delta\downarrow 0}\frac{1}{\delta}\int_t^{t+\delta}[-1-\alpha_s^*]ds \ge 0 \quad \text{whenever } Y_t^{1,\alpha^*}>0,\\ &\varlimsup_{\delta\downarrow 0}\frac{1}{\delta}\int_t^{t+\delta}[1-\alpha_s^*]ds \le 0 \quad \text{whenever } Y_t^{1,\alpha^*}<0. \end{aligned} \tag{2.7}$$

We recall again that $|\alpha^*|\le 1$ and that Y_t^{1,α^*} is continuous in t. Then clearly

$$\begin{aligned} \alpha_t^* = -1 &\quad \text{for Leb-a.e. } t\in\{s\in[0,T]: Y_s^{1,\alpha^*}>0\},\\ \alpha_t^* = 1 &\quad \text{for Leb-a.e. } t\in\{s\in[0,T]: Y_s^{1,\alpha^*}<0\}. \end{aligned} \tag{2.8}$$

Note further that

$$\int_0^T \alpha_t^* Y_t^{1,\alpha^*} dt = \frac{1}{2}\left(\int_0^T \alpha_t^* dt\right)^2 \geq 0. \tag{2.9}$$

This, together with (2.8), implies that $Y_t^{1,\alpha^*} \equiv 0$ for all $t \in [0,1]$. Therefore, $\alpha^* \equiv 0$. ■

Proposition 2.3. *The following $\hat{\alpha}$ dominates $\alpha^* \equiv 0$ in the sense of* (1.1):

$$\hat{\alpha}_t := \sum_{n=0}^{\infty} \Big[\mathbf{1}_{[t_n,s_n)}(t) - \mathbf{1}_{[s_n,t_{n+1})}(t)\Big]. \tag{2.10}$$

Proof. First, we note that

$$s_n - t_n = \frac{3}{2^{n+3}}, \quad t_{n+1} - s_n = \frac{1}{2^{n+3}}.$$

For any n and $t \in [s_n, t_{n+1})$, we have

$$\begin{aligned} Y_t^{1,\hat{\alpha}} &= -(t_{n+1}-t) + \sum_{m=n+1}^{\infty} [(s_m - t_m) - (t_{m+1} - s_m)] = -(t_{n+1}-t) + \sum_{m=n+1}^{\infty} \frac{1}{2^{m+2}} \\ &= \frac{1}{2^{n+2}} - (t_{n+1}-t) \geq \frac{1}{2^{n+2}} - (t_{n+1}-s_n) = \frac{1}{2^{n+3}} > 0, \end{aligned} \tag{2.11}$$

and for $t \in [t_n, s_n)$,

$$Y_t^{1,\hat{\alpha}} = Y_{s_n}^{1,\hat{\alpha}} + (s_n - t) = \frac{1}{2^{n+3}} + (s_n - t) > 0. \tag{2.12}$$

Then, for any n, recalling the $c(t)$ in (2.2) and $\hat{\alpha}$ in (2.10),

$$\begin{aligned} Y_{t_n}^{2,\hat{\alpha}} &= \sum_{m=n}^{\infty}\left[\int_{t_m}^{s_m} Y_t^{1,\hat{\alpha}} dt - 6\int_{s_m}^{t_{m+1}} Y_t^{1,\hat{\alpha}} dt\right] \\ &= \sum_{m=n}^{\infty}\left[\int_{t_m}^{s_m}\left[\frac{1}{2^{m+3}} + (s_m - t)\right] dt - 6\int_{s_m}^{t_{m+1}}\left[\frac{1}{2^{m+2}} - (t_{m+1}-t)\right] dt\right] \\ &= \sum_{m=n}^{\infty}\left[\left[\frac{1}{2^{m+3}} \times \frac{3}{2^{m+3}} + \frac{1}{2}\left(\frac{3}{2^{m+3}}\right)^2\right] - 6\left[\frac{1}{2^{m+2}} \times \frac{1}{2^{m+3}} - \frac{1}{2}\left(\frac{1}{2^{m+3}}\right)^2\right]\right] \\ &= -\sum_{m=n}^{\infty} \frac{3}{2^{2m+7}} = -\frac{1}{2^{2n+5}} < 0. \end{aligned} \tag{2.13}$$

Recall by (2.11), (2.12) that $Y_t^{1,\alpha} > 0$ for all $t < T$. Then, for each n,

$$\begin{aligned} t \in [t_n, s_n): &\quad Y_t^{2,\hat{\alpha}} = Y_{t_n}^{2,\hat{\alpha}} - \int_{t_n}^{t} Y_s^{1,\hat{\alpha}} ds \leq Y_{t_n}^{2,\hat{\alpha}} < 0, \\ t \in [s_n, t_{n+1}): &\quad Y_t^{2,\hat{\alpha}} = Y_{t_{n+1}}^{2,\hat{\alpha}} - 6\int_t^{t_{n+1}} Y_s^{1,\hat{\alpha}} ds \leq Y_{t_{n+1}}^{2,\hat{\alpha}} < 0. \end{aligned} \tag{2.14}$$

Since $J(t, \alpha^*) = 0$, this proves (1.1) immediately. ■

Remark 2.4. In a discrete-time setting, $0 = t_0 < \cdots < t_n = T$. By the definition of equilibrium, we must have $J(t_{n-1}, \alpha^*) \le J(t_{n-1}, \alpha)$, so there is no $\hat\alpha$ satisfying (1.1) at t_{n-1}. However, following the same spirit, one can easily construct examples such that $J(t_i, \hat\alpha) < J(t_i, \alpha^*)$ for all $i = 0, \cdots, n-2$. Here in the continuous-time model, the "last" step vanishes, and thus the strict inequality holds for all $t < T$.

3. Time inconsistency.

We now show that the dynamic optimization problem (2.3) is time inconsistent. We first note that (2.3) admits an optimal control for any fixed t.

Proposition 3.1. *For any $t \in [0,T)$, the V_t in (2.3) has an optimal control $\bar\alpha^* \in \mathcal{A}$.*

Before we prove this result, we use it to show the time inconsistency.

Proposition 3.2. *The dynamic problem (2.3) is time inconsistent.*

Proof. Assume by contradiction that (2.3) is time consistent. Then there exists $\bar\alpha^* \in \mathcal{A}$, which is optimal for all $t \in [0,T)$. By Definition 2.1, this $\bar\alpha^*$ is an equilibrium, and thus, by Proposition 2.2, we must have $\bar\alpha^* \equiv 0$. However, by Proposition 2.3, we see that $\alpha^* \equiv 0$ is not optimal for all $t < T$, which is the desired contradiction. ■

Proof of Proposition 3.1. Without loss of generality, we prove the result only at $t = 0$. Let $\mathcal{X}$ denote the set of functions $X : [0,T] \to \mathbb{R}$ such that

$$|X_t - X_s| \le |t-s| \quad \text{and} \quad X_T = 0, \tag{3.1}$$

and we equip $\mathcal{X}$ with the uniform norm. Then the set $\mathcal{X}$ is compact, and $\alpha \in \mathcal{A}$ has one-to-one correspondence with $X \in \mathcal{X}$ in the sense that $X_t = \int_t^T \alpha_s ds$ and $\alpha_t = -X'_t$. Therefore,

$$\begin{aligned} V_0 &= \inf_{X\in\mathcal{X}} \left[-\int_0^T c(t) X_t X'_t dt \right] = -\sup_{X\in\mathcal{X}} F_\infty(X), \quad \text{where} \\ F_n(x) &:= \int_0^{t_n} c(t) X_t X'_t dt, \quad F_\infty(X) := \int_0^T c(t) X_t X'_t dt. \end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned} F_n(X) &= \sum_{m=0}^{n-1} \left[c(t_m) \int_{t_m}^{s_m} X_t X'_t dt + c(s_m) \int_{s_m}^{t_{m+1}} X_t X'_t dt \right] \\ &= \sum_{m=0}^{n-1} \left[\frac{c(t_m)}{2} (X_{s_m}^2 - X_{t_m}^2) + \frac{c(s_m)}{2} (X_{t_{m+1}}^2 - X_{s_m}^2) \right]. \end{aligned} \tag{3.3}$$

It is obvious that F_n is continuous in X. Moreover,

$$\begin{aligned} \sup_{X\in\mathcal{X}} |F_\infty(X) - F_n(X)| &\le \int_{t_n}^T c(t)|X'_t| \int_t^T |X'_s| ds dt \\ &\le \int_{t_n}^T c(t)(T-t) dt \le \frac{C}{2^{2n}} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.4}$$

Then F_∞ is also continuous, and thus, by the compactness of $\mathcal{X}$, there exists $X^* \in \mathcal{X}$ such that $F_\infty(X^*) = \sup_{X\in\mathcal{X}} F_\infty(X)$. Therefore, $\bar\alpha^*_t := -\frac{d}{dt} X^*_t$ is an optimal control for V_0. ■

4. Nonoptimality of the naive strategy. In this section, we investigate briefly the naive strategy, which is much less popular in the literature. It is well understood that a naive strategy may not be optimal, except in the last step in a discrete-time model, as in Remark 2.4. We now provide an example in a continuous-time framework such that the naive strategy is Pareto dominated by another strategy.

Let $\mathcal{A}$ consist of Borel-measurable functions $\alpha : [0,T] \to \mathbb{R}$. Consider

$$V_t := \inf_{\alpha\in\mathcal{A}} J(t,\alpha), \quad \text{where} \quad J(t,\alpha) := \int_t^T |\alpha_s - K(t,s)|ds, \quad K(t,s) := 2(s-t). \tag{4.1}$$

It is obvious that, for each $t \in [0,T]$, the optimization problem (4.1) on $[t,T]$ has a unique optimal control $\alpha^t_s := K(t,s)$, $s \in [t,T]$. In particular, for $t_1 < t_2$,

$$\alpha^{t_1}_s = K(t_1,s) \neq K(t_2,s) = \alpha^{t_2}_s, \quad s \geq t_2. \tag{4.2}$$

That is, the dynamic problem (4.1) is time inconsistent. The naive strategy is defined as

$$\alpha^*_t := \alpha^t_t = K(t,t) = 0, \quad 0 \leq t \leq T, \tag{4.3}$$

and thus

$$J(t,\alpha^*) = \int_t^T |\alpha^*_s - K(t,s)|ds = \int_t^T 2(s-t)ds = (T-t)^2. \tag{4.4}$$

We now set

$$\hat{\alpha}_t := T - t, \quad 0 \leq t \leq T. \tag{4.5}$$

Then, for any $0 \leq t \leq T$,

$$J(t,\hat{\alpha}) = \int_t^T |\hat{\alpha}_s - K(t,s)|ds = \int_t^T |T + 2t - 3s|ds = \int_0^{T-t} |T - t - 3s|ds = \frac{5}{6}(T-t)^2. \tag{4.6}$$

Thus, $\hat{\alpha}$ Pareto dominates the naive strategy α^* in the sense of (1.1).

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