Balanced Convective Circulations in a Stratified Atmosphere. Part I: A Framework for Assessing Radiation, the Coriolis Force, and Drag

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ABSTRACT: The so-called traditional approximation, wherein the component of the Coriolis force proportional to the cosine of latitude is ignored, is frequently made in order to simplify the equations of atmospheric circulation. For velocity fields whose vertical component is comparable to their horizontal component (such as convective circulations), and in the tropics where the sine of latitude vanishes, the traditional approximation is not justified. We introduce a framework for studying the effect of diabatic heating on circulations in the presence of both traditional and non-traditional terms in the Coriolis force. The framework is intended to describe steady convective circulations on an f-plane in the presence of radiation and momentum damping. We derive a single elliptic equation for the horizontal velocity potential, which is a generalization of the weak temperature Gradient (WTG) approximation. The elliptic operator depends on latitude, radiative damping, and momentum damping coefficients. We show how all other dynamical fields can be diagnosed from this velocity potential; the horizontal velocity induced by the Coriolis force has a particularly simple expression in terms of the velocity potential. Limiting examples occur at the equator, where only the non-traditional terms are present, at the poles, where only the traditional terms appear, and in the absence of radiative damping where the WTG approximation is recovered. We discuss how the framework will be used to construct dynamical, nonlinear convective models, in order to diagnose their consequent upscale momentum and temperature fluxes.

1. Introduction

The full Coriolis force contains terms proportional to the sine and cosine of latitude. The former are referred to as the traditional Coriolis terms, and couple the zonal and meridional momentum equations. The latter, referred to as the non-traditional Coriolis terms (NCTs), couple the zonal and vertical momentum equations. Scaling arguments have often been used to justify the neglect of the NCTs. For instance, in midlatitude, synoptic scale meteorology, it can be shown that the non-traditional Coriolis term in the zonal momentum equation is relatively small, and in the vertical momentum equation, it is negligible compared to vertical accelerations, gravity, and the vertical pressure gradient. Under these circumstances, the “traditional approximation" is made, whereby the NCT are neglected but the traditional Coriolis terms (TCT) are retained. However, near the equator, the cosine and sine of latitude approach unity and zero, respectively, and it becomes more difficult to justify the outright neglect of the non-traditional terms for circulations which are not in hydrostatic balance.

The effect of the non-traditional Coriolis terms have been studied in different contexts. They have been considered in convection (Igel and Biello 2020), tropical waves (Ong and Roundy 2020; Ong and Yang 2022), convective momentum transport (LeMone 1983), oceanic dynamics (Marshall and Schott 1999), and idealized studies of the planetary boundary layer (Dubos et al. 2008). The work of Igel and Biello (2020) shows how the NCT and the pressure field induced by convective circulations create a purely horizontal force which acts on the circulation. In the framework described below, this horizontal force will manifest as a secondary horizontal circulation added to the primary convective circulation. The non-traditional Coriolis terms have also shown to be important in shallow water approximations (Stewart and Dellar 2013, 2012, 2010). In addition, a set of equations that retain the non-traditional Coriolis terms, and possess conservation principles for mass, energy, and potential vorticity were derived in Tort and Dubos (2014). However, it is largely case that the influence of the NCTs on atmospheric flows remains incompletely understood and poorly appreciated. Studies of the non-traditional terms tend to conclude that, when considered diligently, the NCTs should not be ignored in low-latitude meteorological situations with the potential for or the occurrence of sustained vertical motion.

Our original intention for this work was to study the NCTs only in a broad way. We wanted to introduce a mathematical framework for understanding tropical dynamics under the influence of the

NCTs that would be applicable from the synoptic scales to the mesoscales and would not necessarily invoke wave dynamics, the latter having been the focus of most previous work on the NCTs. To do so, we introduced a scaling of the incompressible Euler equations on an equatorial beta-plane that would allow us to study the NCTs' effect on the corresponding steady state equations. However, we realized that our analysis could easily be extended to the Euler equations at an arbitrary latitude, and the case where only the non-traditional terms are present could be obtained by evaluating the theory at zero latitude.

To yield a general, albeit linear, framework, we consider the impacts of radiation and dissipation of momentum on the dynamics. The latter allows the possibility of steady state solutions. Consideration of the former is motivated by mesoscale studies of tropical systems which tend to emphasize the important role of radiation, especially in horizontal gradients of radiative heating (Wing et al. 2017), and by its fundamental role in the energy balance of the tropical atmosphere (Manabe and Strickler 1964). As a consequence of our choice of time and length scales, and in the absence of radiation, there is a simplification of our equations that yields one of the fundamental features of the weak temperature gradient (WTG) approximation: the direct diagnosis of vertical velocity from the heating. The WTG approximation has been applied on mesoscales and synoptic scales in the tropics to understand, among other things, tropical cyclone formation (Raymond et al. 2007; Adames et al. 2021), the Madden-Julian Oscillation (Chikira 2014), and the Walker Cell (Bretherton and Sobel 2002). At first glance, it may be counterintuitive that convection can be described by a diagnostic equation for the vertical velocity, since it is understood to be achieved on meso and synoptic scales in the tropics. However, balance of the form of WTG requires that the waves travel across the region of interest more quickly than the circulation transports the fluid. In this framework, the gravity wave travel time across an isolated convective element is much faster than a convective turnover time, which are the timescales under consideration. This time scale separation means that gravity waves quickly re-stratify the potential temperature (or buoyancy) in the vicinity of the convection, so that the time derivative of the buoyancy equation can be neglected in favor of its balanced state (a radiation modified version of WTG). A WTG balance on convective scales was first developed by Klein and collaborators and was summarized nicely by (Klein 2010). More recently, a diagnostic equation for the vertical velocity in deep convection was also derived by (Hittmeir and Klein 2018) using the method of asymptotic scale analysis.

The derivation of our framework will begin with a nondimensionalization and scale analysis, but will set aside a systematic asymptotic analysis for the future. We split our work into two parts. Here in Part I, we derive sets of diagnostic equations for velocity, pressure, and buoyancy perturbation. We consider three distinct cases to elucidate the effect of the Coriolis force on convective flows; when the full Coriolis force, only the non-traditional terms, or only the traditional terms are retained. The last two cases occur at the equator and pole, respectively. Since the equatorial, non-traditional Coriolis case is of the most interest to us, it is presented fully in Part II (Marsico et al. In Preparation).

This paper is organized as follows. In section 2, we discuss the velocity, and time scales for which the incompressible Euler equations yield solutions corresponding to equilibrated circulations on atmospheric convective length scales, as would be used for sub-grid convective parameterizations in large scale computations. Since this is a preliminary framework, we focus on flow strengths that can be described by linear theory because they are weak enough. The effects of turbulent dissipation on sub-grid scales are often approximated by drag damping, or enhanced, turbulent diffusivity. In our model, we will use linear dissipation on convective scales to account for the enhanced diffusivity associated with sub-grid turbulence. We also focus on time scales where the zonal and meridional components of the full Coriolis force balance the pressure gradients and damping, while the vertical component balances the vertical pressure gradient, damping, and buoyancy.

In order to solve the resulting steady linear equations, it is necessary to introduce damping, and we consider two forms: first, constant drag damping in the momentum, and Newtonian cooling in the buoyancy equations; second, diffusive damping in the momentum equations and Newtonian cooling in the buoyancy equation. In section 3, we use the Helmholtz decomposition to separate the velocity field into two components. The poloidal component of the velocity field is horizontally convergent and directly responds to the heating; we thus describe it as the primary circulation (Zhang and Schubert 1997). A purely horizontal velocity field is generated from the poloidal circulation, the Coriolis force, and the momentum damping; we describe it as the secondary circulation.

There are two significant physical predictions of our framework regarding the effect of NCT and radiation. The first is expressed by equation (18), which arises as a balance between the "net

Coriolis force" (Igel and Biello 2020) and momentum damping. It provides a simple relationship between the vertical derivative of the stream function of the secondary circulation and the derivative of the potential function of the primary (poloidal) circulation along the axis of rotation of the Earth. The second is expressed in equation (19), where the potential of the primary, poloidal circulation is related to the latent heating through an elliptic operator. In the absence of radiation, this expression reduces to the weak temperature gradient approximation; that is to say, the vertical velocity is proportional to the latent heating. Radiation allows the effect of latent heating to be felt away from its source, thereby providing a mechanism for descent or ascent away from the center of convection. In section 4, we contrast solutions to these equations at the equator (purely NCT) versus the poles (purely TCT). In section 5, our results are summarized.

2. Length and time scales of the Primitive Equations appropriate to convective circulations

Our framework describes steady, convective circulations under the influence of buoyancy, NCT, TCT, and damping. In this and our companion manuscript the framework will be linear. Our reasoning is that nonlinearity will primarily create turbulent dissipation (modelled as a linear damping), and can be mostly accounted for by eddy diffusivity. Future work will extend these results to circulations where advective nonlinearities cannot be neglected, yet the weak temperature gradient will be maintained. It is the versatility of the WTG simplification that allows for simple solutions in both linear and nonlinear steady circulations. Furthermore, in the linear regime, the various properties of the circulation and buoyancy response to diabatic heating can be straightforwardly associated with their sources and sinks, making this framework a natural starting point for a dynamical convective parameterization.

In the following paragraphs, we non-dimensionalize the equations of motion and describe the relevant spatial, temporal, velocity and buoyancy scales. Although we will ultimately work with a linear and dimensional model, the discussion of non-dimensionalization is important to ensure our framework remains consistent with flows we seek to describe. Furthermore, we envision this framework as the first step toward a multi-scale analysis of the nonlinear effects of convection on meso- and synoptic scale circulations in keeping with (Klein 2010), (Hittmeir and Klein 2018), and (Hirt et al. 2023). A careful multi-scale analysis must begin with a clear non-dimensionalization of the equations of motion in order to identify the relevant small parameters used in the asymptotic method. Therefore, with an eye to future applications, we proceed with the scale analysis.

We begin with the incompressible, stratified, damped Euler equations on an f-plane at a latitude λ,

$$\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u - 2\Omega v \sin(\lambda) + 2\Omega\cos(\lambda) w = -\frac{\partial \phi}{\partial x} - d_1 u, \tag{1a}$$

$$\frac{\partial v}{\partial t} + \mathbf{u}\cdot\nabla v + 2\Omega u \sin(\lambda) = -\frac{\partial \phi}{\partial y} - d_1 v, \tag{1b}$$

$$\frac{\partial w}{\partial t} + \mathbf{u}\cdot\nabla w - 2\Omega u \cos(\lambda) = -\frac{\partial \phi}{\partial z} + b - d_1 w, \tag{1c}$$

$$\frac{Db}{Dt} + N^2 w = S \tag{1d}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1e}$$

where $b = g\theta/\theta_0$ is the buoyancy perturbation, θ is the potential temperature perturbation, θ_0 is a reference potential temperature, d_1 is the damping coefficient due to the sub-cloud scale turbulent dissipation (or damping operator, if e.g. a drag parameterization is used), $N^2 = (g/\theta_0)(d\tilde{\theta}/dz)$ is the squared buoyancy frequency of the unperturbed atmosphere, $\tilde{\theta}(z)$ is the background potential temperature stratification, and $\phi = p/\rho_0 + gz$ is the Montgomery potential for a constant density fluid, ρ_0. The buoyancy source is related to the diabatic heating through $S = (g/\theta_0)S_\theta$. Since we consider an idealized theoretical framework, we use the incompressible equation (1e), instead of the anelastic continuity equation.

To non-dimensionalize the equations, we introduce the length, time, velocity, buoyancy, pressure and latent heating scales, $(L, T, U, b_0, \phi_0, S_0)$, as follows: $(x, y, z) = L(x', y', z')$, $t = Tt'$, $(u, v, w) = U(u', v', w')$, $b = b_0 b'$, $\phi = \phi_0\phi'$, and $S = S_0 S'$. Since the scaling is isotropic in the vertical and horizontal directions, the resulting vertical momentum equation will not express hydrostatic balance. Instead we allow for the possibility that all of the linear forces participate in the dominant balance at lowest order. Rewriting equations (1a)-(1e) in terms of the non-dimensional variables (and dropping primes for readability) we find

$$\frac{\partial u}{\partial t}+\frac{UT}{L}\mathbf{u}\cdot\nabla u-2\Omega T\sin(\lambda)v+2\Omega T\cos(\lambda)w=-\frac{\phi_0 T}{LU}\frac{\partial\phi}{\partial x}-d_1Tu, \tag{2a}$$

$$\frac{\partial v}{\partial t}+\frac{UT}{L}\mathbf{u}\cdot\nabla v+2\Omega T\sin(\lambda)u=-\frac{\phi_0 T}{LU}\frac{\partial\phi}{\partial y}-d_1Tv, \tag{2b}$$

$$\frac{\partial w}{\partial t}+\frac{UT}{L}\mathbf{u}\cdot\nabla w-2\Omega T\cos(\lambda)u=\frac{\phi_0 T}{LU}\left(-\frac{\partial\phi}{\partial z}+\frac{b_0L}{\phi_0}b\right)-d_1Tw, \tag{2c}$$

$$\frac{b_0}{N^2UT}\left[\frac{\partial b}{\partial t}+\frac{UT}{L}\mathbf{u}\cdot\nabla b\right]+w=\frac{S_0}{N^2U}S \tag{2d}$$

$$\nabla\cdot\mathbf{u}=0. \tag{2e}$$

As with all asymptotically inspired methods, one attains a simplified model by seeking a dominant balance between different terms in the primitive equations. However, the vertical and horizontal length scales under consideration are fixed by the troposphere height. Choosing $L = 7\,\text{km}$ allows for deep convective circulations (order $2L$) as well as developing convection (order $L/2$).

The Coriolis force participates in the dominant balance when $2\Omega T \geq 1$, which means that we consider time scales of $T = (2\Omega)^{-1} \approx 2$ hours or larger. Notwithstanding that on a 2 hour time scale the time derivatives in the momentum equation may not necessarily be negligible, the balanced circulations we consider herein can be thought of as either the equilibration of a convective circulation under Coriolis and damping, or a quasi-stationary, slowly evolving circulation pattern due to latent heating.

The relative strength of the nonlinear terms to the linear terms is measured by the Rossby number

$$\frac{UT}{L}=\frac{U}{2\Omega L}\equiv \text{Ro}.$$

A linear regime is applicable if the Rossby number of the flow is less than one. So $\text{Ro} < 1$ implies the velocity U is less than the scale $2\Omega L \approx 1\,\text{m/s}$. From the perspective of small scale turbulent motions in atmospheric convection, this is indeed a small velocity. However, we expect that this velocity scale is appropriate to the large scale envelope of convection, and that the smaller scale, faster motions contribute to the sub-cloud enhanced turbulent diffusion.

Buoyancy driven circulations of low Mach number (the ratio of the characteristic speed to the speed of sound) result in incompressible (or anelastic) velocity fields to a high degree of approximation, and this is maintained by the pressure gradient. Therefore we expect that the buoyancy and pressure perturbation are the same order of magnitude, $b_0 = \phi_0/L$. Indeed this is often observed in simulated active convection (Jeevanjee and Romps 2016; Peters 2016). We also expect that the pressure gradient and buoyancy be of the same order of magnitude as the Coriolis force, thereby $\phi_0 = (UL)/T = \mathrm{Ro}(L^2)/T^2$, which yields the buoyancy scale

$$b_0 = \mathrm{Ro}\,\frac{L}{T^2}.$$

Using $UT = \mathrm{Ro}\,L$ we can estimate the coefficient multiplying the temperature transport term on the left side of equation (2d) to be

$$\frac{b_0}{N^2UT} = \frac{\mathrm{Ro}\,L}{T^2}\,\frac{1}{N^2\mathrm{Ro}\,L} = (NT)^{-2} \equiv \epsilon^2,$$

where the last equality is the definition of ϵ. Since the Brunt Vaisala frequency in the troposphere is approximately $N = .02\,\mathrm{s}^{-1}$, and using a Coriolis time scale $T \approx 7200\,\mathrm{s}$ we find

$$\epsilon \approx \frac{1}{144},$$

so that the temperature advection term on the left hand side of (2d) is extremely small compared to the vertical transport of the background stratification (the w term on the left hand side of equation (2d)). Effectively this means that gravity waves are extremely fast compared with advection. Therefore, the weak temperature Gradient approximation, where the vertical velocity balances the diabatic heating in a diagnostic equation, is an excellent approximation even on convective scales. We also expect that the momentum damping will balance the Coriolis force, so that the damping rate, d_1^{-1}, is of the order $T \approx 2$ hrs.

These scale arguments establish the time, length, and diabatic heating scales for which the linear, steady approximation provides an excellent description of the circulation. Convective circulations do not necessarily satisfy these constraints throughout their development, but the linear steady theory can still provide insights to the induced circulation, even if nonlinear advection would tend to slowly evolve such a circulation.

a. Radiative Damping

We are ultimately interested in the effect that radiative cooling has on steady state circulations and can model its effect by introducing a Newtonian cooling term of the form $-d_2 b$ to the right hand side of equation (1d). This term would then be non-dimensionalized as $-(d_2 b_0)/(N^2 U)b$ on the right hand side of equation (2d). Inclusion of this radiative term in no way changes any of the previous scaling arguments. Now, if the diabatic heating source and radiative sink on the right hand side of the temperature equation are to be in balance with the vertical velocity, then $S_0 \approx d_2 b_0 \approx UN^2 = \mathrm{Ro} \times 1\,\mathrm{m/s} \times (.02s^{-1})^2 = 1.44\,\mathrm{m/s^2\,hr^{-1}}$. At the small buoyancy perturbations considered here, this balance requires a somewhat large Newtonian cooling parameter, d_2.

3. Linear Convective WTG with full Coriolis force

In this section we derive the framework of the Linear Convective WTG with the full Coriolis force. As we discussed above, we consider the linear, steady versions of equations (1a) - (1e) with a heating source and linear cooling in the temperature equation, and damping in the momentum equations. For the momentum equations, we will discuss both linear drag and enhanced turbulent diffusion.

In order to elucidate the physics of the problem, as well as simplify the mathematics, we will begin by using the Helmholtz decomposition to separate the horizontally convergent flow which directly responds to diabatic heating from the horizontally non-convergent flow which arises as a balance between the Coriolis force, and the momentum damping. The theory will consist of an elliptic (Poisson-like) equation for the horizontal velocity potential with a source term given by the diabatic heating. We will show how the other variables, the horizontal stream function, pressure, buoyancy, and the three components of the velocity, can all be diagnosed from this velocity potential. The linear convective WTG equations with Coriolis force are

$$-2\Omega\sin(\lambda)v + 2\Omega\cos(\lambda)w = -\frac{\partial\phi}{\partial x} - d_1 u \tag{3a}$$

$$2\Omega\sin(\lambda)u = -\frac{\partial\phi}{\partial y} - d_1 v \tag{3b}$$

$$-2\Omega\cos(\lambda)u = -\frac{\partial\phi}{\partial z} + b - d_1 w \tag{3c}$$

$$N^2 w = S - d_2 b, \tag{3d}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3e}$$

where d_1 is the momentum damping coefficient, and d_2 is the radiative damping coefficient. We first consider the case when d_1 and d_2 are due to Newtonian drag and radiative damping and then show how the theory can be easily extended to account for turbulent diffusion.

To describe the analytic solution of these equations, we follow the Helmholtz decomposition (Helmholtz 1867; Lebovitz 1989), introducing the stream function, ψ, and velocity potential, Φ, and write the velocity field as

$$\mathbf{u} = (-\Phi_x - \psi_y)\mathbf{i} + (-\Phi_y + \psi_x)\mathbf{j} + w\mathbf{k} \tag{4}$$

The horizontally irrotational component, described by Φ, can converge in the horizontal direction (it was described as horizontally confluent in (Igel and Biello 2020)) and constitutes a poloidal vector field which is directly tied to the vertical velocity through a kinematic expression. The horizontally rotational component is described by a stream function, ψ, and therefore has no convergence in the horizontal plane. Its relationship to the velocity potential is a consequence of physics, as we will describe below. Setting the divergence of equation (4) to zero yields the well known Poisson equation for the velocity potential in terms of the vertical velocity

$$\nabla_h^2\Phi = w_z, \tag{5}$$

where ∇_h^2 is the Laplacian operator in the horizontal (x, y) direction alone. Taking the vertical component of the curl of the velocity field, yields the (also) well known expression of the stream function in terms of the vertical component of vorticity

$$\nabla_h^2 \psi = v_x - u_y. \tag{6}$$

We now derive an equation for Φ in terms of S by eliminating the pressure and buoyancy from the momentum equations. We will then use equations (4)-(6) to write this equation in terms of Φ. Differentiating equation (3a) with respect to z, and equation (3c) with respect to x and eliminating ϕ yields

$$d_1 u_z - 2\Omega \sin(\lambda) v_z + 2\Omega \cos(\lambda) w_z = -2\Omega \cos(\lambda) u_x - b_x + d_1 w_x. \tag{7}$$

Differentiating equation (3b) with respect to z, equation (3c) with respect to y and eliminating ϕ yields

$$2\Omega \sin(\lambda) u_z + d_1 v_z = -2\Omega \cos(\lambda) u_y - b_y + d_1 w_y. \tag{8}$$

Now we use equation (3d) to eliminate b from (7)

$$d_1 u_z - 2\Omega \sin(\lambda) v_z + 2\Omega \cos(\lambda) w_z = -2\Omega \cos(\lambda) u_x + \frac{N^2 w_x - S_x}{d_2} + d_1 w_x \tag{9}$$

and from equation (8)

$$2\Omega \sin(\lambda) u_z + d_1 v_z = -2\Omega \cos(\lambda) u_y + \frac{N^2 w_y - S_y}{d_2} + d_1 w_y. \tag{10}$$

Notice d_2 appears in the denominator in both equations (9) and (10), and this term would be singular if d_2 were zero. In this limit, the WTG approximation is recovered for w and therefore Φ, i.e. $N^2 w = S$. Upon differentiating equation (9) with respect to x, equation (10) with respect to y, adding the results, taking the z-derivative, and making some rearrangements we obtain

$$d_1 (u_{xz} + v_{yz}) + 2\Omega \left\{ \sin(\lambda)(u_{yz} - v_{xz}) + \cos(\lambda) \left[w_{xz} + \nabla_h^2 u \right] \right\} = \frac{\left(N^2 + d_1 d_2\right) \nabla_h^2 w - \nabla_h^2 S}{d_2}. \tag{11}$$

Using the incompressibility constraint, (11) simplifies to

$$-d_1 d_2 w_{zz} - 2\Omega d_2 \left\{ \sin(\lambda)(v_{xz} - u_{yz}) + \cos(\lambda) \left[v_{xy} - u_{yy} \right] \right\} = \left(N^2 + d_1 d_2\right) \nabla_h^2 w - \nabla_h^2 S. \tag{12}$$

One can recognize the vertical component of vorticity, $v_x - u_y$, in both Coriolis terms on the left hand side of (12), which we will eliminate in favor of the Laplacian of the stream function. The vorticity is operated on by the derivative

$$\frac{\partial}{\partial n} \equiv \cos(\lambda)\frac{\partial}{\partial y} + \sin(\lambda)\frac{\partial}{\partial z}. \tag{13}$$

This is the directional derivative parallel to the direction of the North polar axis (thus our choice of 'n') as viewed from the tangent plane at latitude λ. A way to visualize this derivative is that, at an latitude λ, the derivative is taken in a direction which points toward the North Star. Taking the derivative of (12) with respect to z, we can use (5) to replace w in favor of Φ, and upon rearranging the expression we find

$$\left(N^2 + d_1 d_2\right) \nabla_h^4 \Phi + d_1 d_2 \nabla_h^2 \Phi_{zz} = \nabla_h^2 S_z - 2\Omega\, d_2 \nabla_h^2 \psi_{nz}. \tag{14}$$

Next, we invert one instance of the horizontal Laplacian throughout the expression (14). The resulting expression would have an arbitrary harmonic function, which is the kernel of the Laplacian, added to the right hand side. However, all harmonic functions either grow at infinity (corresponding to solutions growing away from the source) or are singular at a point in the domain (corresponding to solutions which blow up at a point). Therefore we can set the harmonic function to zero, and we arrive at one expression which relates the horizontal stream function, the horizontal convergence (potential Φ), and the diabatic heating

$$\left(N^2 + d_1 d_2\right) \nabla_h^2 \Phi + d_1 d_2 \Phi_{zz} = S_z - 2\Omega\, d_2 \psi_{nz}. \tag{15}$$

We have chosen to work with the potential for the horizontal convergence, Φ, in order to attain an expression which does not contain any horizontal derivatives of S. In the companion paper, we consider diabatic heating profiles with horizontal discontinuities, such as would be expected during cloud formation, and wish to avoid second derivatives of discontinuous functions. By setting $d_2 = 0$, we recover the WTG approximation from (15).

In order to construct a single elliptic PDE for Φ, we need another expression relating the stream function to the potential. Note that the derivatives we have used to arrive at (15) construct the horizontal components of the vorticity equation. Subtracting the y-derivative of (3a) from the x-derivative of (3b) eliminates the horizontal pressure gradient and describes vertical component of the vorticity equation, which is not directly affected by buoyancy

$$2\Omega\left\{\sin(\lambda)\left[u_x+v_y\right]-\cos(\lambda)w_y\right\}=-d_1\left[v_x-u_y\right]. \tag{16}$$

Taking the z-derivative of (16) and replacing the components of the velocity with the stream function and potential

$$-2\Omega\left\{\sin(\lambda)\nabla_h^2\Phi_z+\cos(\lambda)\nabla_h^2\Phi_y\right\}=-d_1\nabla_h^2\psi_z. \tag{17}$$

Again, using the expression for the directional derivative along the north polar axis, inverting an instance of the horizontal Laplacian on each term, and swapping the sides of the equality yields the extremely simple relationship relating the stream function to the velocity potential

$$d_1\frac{\partial\psi}{\partial z}=2\Omega\frac{\partial\Phi}{\partial n}. \tag{18}$$

Expression (18) is elegant, deceptively simple, and merits some elucidation. Although the right hand side is measured in units of acceleration, it arose from the vertical *torque* due to the Coriolis force acting on a poloidal velocity field described by Φ (Igel and Biello 2020). From the Helmholtz decomposition, the poloidal component of a velocity field is uniquely determined from its vertical component, yielding the convergence in the horizontal plane which compensates for the vertical circulation; that is to say it is the solution of equation (5) substituted into (4). This is a significant relationship between the convective, primary circulation described by Φ, and the horizontal, secondary circulation described by the stream function, ψ. Its derivation was motivated by the computation in (Igel and Biello 2020), of the divergence free portion of the Coriolis force induced by a convective velocity field. When this divergence free component of the Coriolis force is balanced by momentum drag (or dissipation), equation (18) results.

The left hand side of equation (18) arises from the damping of the vertical component of the vorticity. That vertical component of vorticity is, itself, due to the secondary circulation, described by ψ, in the horizontal plane (again refer to equation (4)). Therefore, equation (18) is the statement that the vertical torque due to the Coriolis force acting on the convective circulation must be in balance with the torque associated with vorticity damping (later dissipation); in the absence of this damping ($d_1 = 0$) there is no balanced circulation. Since we have chosen to model damping linearly, then the response, ψ, corresponds to a secondary horizontal circulation which is linearly related to the primary poloidal (convective) circulation. That the secondary circulation is singular in the damping coefficient, d_1, is notable, but not surprising given that equilibrium flow must be in, or nearly in, force balance. Ultimately, in any convective model, it will be the upscale fluxes of momentum, and thermodynamic quantities that are of interest to convective parameterizations, and we will discuss these fluxes in a subsequent manuscript.

We can now eliminate ψ from equation (15) using equation (18) to arrive at an elliptic equation for the velocity potential in terms of the diabatic heating

$$\nabla_h^2\Phi + \frac{d_1 d_2}{N^2 + d_1 d_2}\left[\Phi_{zz} + \left(\frac{2\Omega}{d_1}\right)^2 \Phi_{nn}\right] = \frac{S_z}{\left(N^2 + d_1 d_2\right)}. \tag{19}$$

From equations (19) and (18), along with the relations (4) and (5), we can construct all three components of the velocity field from a diabatic heating source. There only involves one elliptic inversion to compute Φ from equation (19), a vertical integration of equation (18) to compute ψ

$$\psi = -\frac{2\Omega}{d_1}\int_z^{\infty} \frac{\partial\Phi}{\partial n}\, dz' \tag{20}$$

where the constant of integration is chosen so that the horizontal velocity vanishes at infinite height, and a vertical integration of equation (5),

$$w = \int_0^z \nabla_h^2\Phi\, dz', \tag{21}$$

where the constant of integration is chosen so the vertical velocity vanishes at $z = 0$. Taking the necessary partial derivatives of Φ and ψ in (4), we have then computed horizontal components of the velocity field.

From the buoyancy equation (3d), we could easily compute b as the deviation of the vertical velocity from WTG, but this expression would be singular in the radiative damping parameter, d_2, and not illuminating in the WTG limit. Instead, by subtracting the z derivative of the meridional acceleration equation (3c) from the y derivative of the vertical acceleration equation (3d) eliminating ψ using equation (18), and performing some antiderivatives, we arrive at the expression for the buoyancy in terms of the velocity potential

$$b = -d_1 \int_z^\infty \left[\nabla_{\mathrm{H}}^2 \Phi + \Phi_{zz} + \left(\frac{2\Omega}{d_1}\right)^2 \Phi_{nn} \right] dz', \tag{22}$$

where we have chosen the constant of integration so that the buoyancy vanishes at infinite heights. This equation (22) makes the effect of rotation on buoyancy explicit through the presence of the last term in the integral, and it will be useful when constructing upscale fluxes for convective parameterizations. To determine the pressure perturbation, ϕ, we vertically integrate equation (3c) using the condition that ϕ vanishes at infinite height

$$\phi = \int_z^\infty \left[d_1 w - 2\Omega \cos(\lambda)\, u - b \right]\, dz'. \tag{23}$$

The exact expression for ϕ in terms of Φ or S is not particularly illuminating, so we leave equation (23) as it is. We note, however, that in the absence of buoyancy and damping, equation (23) expresses the vertical geostrophic balance discussed by Igel and Biello (2020).

a. Diffusive Momentum Damping

Now we briefly examine the equations when the damping in the momentum equations takes the form of enhanced turbulent diffusivity. Effectively, this corresponds to replacing the momentum drag coefficient with the diffusion operator; $d_1 \to -\mu\nabla^2$ and every instance of d_1 in the denominator should be interpreted as the inversion of the Laplacian. In this case, the equation for the velocity potential becomes

$$\left[\left(N^2 - \mu d_2 \nabla^2\right)\right] \nabla^2 \nabla_h^2 \Phi - \frac{d_2}{\mu}\left[\mu^2 \nabla^4 \Phi_{zz} + (2\Omega)^2 \Phi_{nn}\right] = \nabla^2 S_z. \tag{24}$$

The equations for the other variables follow in much the same manner, and we do not record them here as they don't necessarily provide any more insights into the solutions. However, we note that in the case of diffusive damping, we must solve elliptic equations for all the variables, whereas for linear damping we need only solve a single elliptic equation for Φ.

4. The Traditional and Non-Traditional Coriolis Terms

We now look at the two cases where either only the NCTs or the only the traditional Coriolis terms (TCTs) are retained in equations (3a)-(3e). The former case occurs at the equator, and is obtained by setting $\lambda = 0$, and $\partial/\partial n = \partial/\partial y$. The latter case occurs at the north pole, and is obtained by setting $\lambda = \pi/2$, and $\partial/\partial n = \partial/\partial z$. For the purposes of this discussion, instead of using the equation for the velocity potential (19), we will recast it in terms of the vertical velocity by substituting (5).

Specifically, at the equator, only the non-traditional Coriolis terms are active, and the elliptic equation for the vertical velocity becomes

$$\left[N^2 + d_1 d_2\right] \nabla_h^2 w + d_1 d_2 \left[w_{zz} + \left(\frac{2\Omega}{d_1}\right)^2 w_{yy}\right] = \nabla_h^2 S, \tag{25}$$

while the kinematic equation for the stream function in terms of the velocity potential becomes

$$d_1 \frac{\partial \psi}{\partial z} = 2\Omega \frac{\partial \Phi}{\partial y}. \tag{26}$$

There are two cases of note that occur at the equator. In the case of $d_2 = 0$, the equation for the vertical velocity is independent of latitude, and simplifies to $\nabla_h^2 w = N^{-2} \nabla_h^2 S$, whose solution is $w = N^{-2} S$. Thus, in the absence of radiative damping, we obtain the WTG approximation (Hittmeir and Klein 2018), the direct diagnosis of vertical velocity from heating.

The second case occurs if both d_1 and d_2 are non-zero, but their product is small enough to neglect $d_1 d_2$, corresponding to $d_1 d_2 << N^2$. In this case, the equation for the vertical velocity at the equator becomes

$$\nabla_h^2 w + \frac{4\Omega^2}{N^2} \frac{d_2}{d_1} w_{yy} = \frac{1}{N^2} \nabla_h^2 S, \tag{27}$$

which is an equation that would allow for the vertical velocity to be diagnosed directly if not for the term proportional to d_2/d_1. So in the case of non-zero radiation, we have an equation for the vertical velocity similar to WTG, but with a modification induced by the presence of radiation and the non-traditional Coriolis force terms that requires the inversion of an elliptic operator. Thus, radiation makes the velocity a non-local function of the heating, particularly in the meridional direction.

We point out that there are cases that we have not considered where vertical non-locality induced by the presence of the w_{zz} term in equation (25) is important (Kuo and Neelin 2022). Our focus, however, is on the impact of the non-traditional Coriolis terms, which manifest themselves through the w_{yy} term in equation (25). By considering the case where $d_1 d_2 << N^2$, we can isolate the impact of the NCTs alone.

Irrespective of the momentum damping coefficient, at the equator the secondary horizontal circulation described by ψ is proportional to the meridional derivative of Φ - i.e. the horizontal circulation induced by the NCT at the equator is proportional to the meridional component of the poloidal circulation. Thus we expect poloidal flows which are symmetric about the equator to induce secondary circulations which are antisymmetric about the equator. This symmetry breaking has important implications for upscale momentum fluxes which we will pursue in future work.

At the north pole, the vertical velocity satisfies

$$\left[N^2 + d_1 d_2\right] \nabla_h^2 w + d_1 d_2 \left[1 + \left(\frac{2\Omega}{d_1}\right)^2\right] w_{zz} = \nabla_h^2 S \tag{28}$$

and the stream function is proportional to the velocity potential

$$d_1 \psi = 2\Omega\Phi. \tag{29}$$

The relationship of the stream function to the velocity potential in equation (29) describes the well known behavior of geostrophically balanced flows: areas of horizontal convergence of the poloidal flow will drive cyclonic rotation. Usually this occurs in the lower troposphere where the flow is convergent, while the compensating, divergent, anticyclonic circulation occurs in the upper troposphere.

In contrast to the NCT equation in (25), where the non-WTG terms (those proportional to d_2) manifest as both horizontal and vertical derivatives of w in the elliptic operator, in the case of TCT, given in equation (28), the additional term is only proportional to vertical derivatives of w. This w_{zz} term generates a vertically non-local response to localized diabatic heating, and it is the effect of damped gravity waves generated by a convective heating source. The coefficient multiplying the vertical derivatives in equation (28) is a complicated combination of the rotation rate of the Earth, the momentum damping, and the ratio of thermal to momentum damping; that is to say that their effects combine in a manner to be indistinguishable from one another in the solution to the vertical velocity.

In the companion paper, we will present an extensive study of solutions to the balanced framework. But in order to provide a preliminary illustration of the phenomena that the balanced framework describes, we compute approximate solutions for the velocity potential and stream function at the equator (NCT) and the north pole (TCT), for a horizontally localized heating profile, which maximizes in mid troposphere, thereby resembling the latent heat released by a convective cloud,

$$S = \begin{cases} \dfrac{S_0}{N^2}\sin(\pi z/H) & \text{if } \sqrt{x^2+y^2} < L \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

where $S_0 = 10^{-4}\,\mathrm{m\,s^{-3}} = 0.36\,\mathrm{m\,s^{-2}\,hr^{-1}}$, $H = 3$ km, $L = 3$ km, and $d_1 = 10^{-4}\mathrm{s}^{-1}$.

Figure 1 (a) shows a horizontal cross section of the secondary circulation, and the vector field $(-\psi_y, \psi_x)$, at the north pole at the bottom of the troposphere, where only the TCT are present. In this case, heating drives a cyclonic secondary circulation whose maximum strength occurs at the bottom and top of the troposphere. Figure 1 (b) shows the secondary circulation at the equator in the middle of the troposphere, when only the non-traditional Coriolis terms are present. In this case, the secondary circulation is antisymmetric about the equator. In figure 1 (c), the velocity potential and vector field $(-\Phi_x, -\Phi_y)$ are plotted at the bottom of the troposphere at the north pole, where only the traditional Coriolis terms are present. Figure 1 (d) shows the velocity potential and vector field $(-\Phi_x, -\Phi_y)$ at the bottom of the troposphere at the equator. In both cases the flow is convergent at the bottom of the troposphere, and divergent at the top.

5. Summary

In this paper, we discuss a framework for studying convective dynamics under the influence of heating, the full Coriolis force, thermal, and momentum damping. The circulation strengths and length scales we consider allow for the study of steady, linear equilibrated convective flows, and constitutes the first step in studying momentum and buoyancy fluxes from the convective scales to the mesoscales. We use the Helmholtz decomposition of the velocity field as a tool to disentangle the effects of heating, Coriolis force, and damping on the convective circulation,

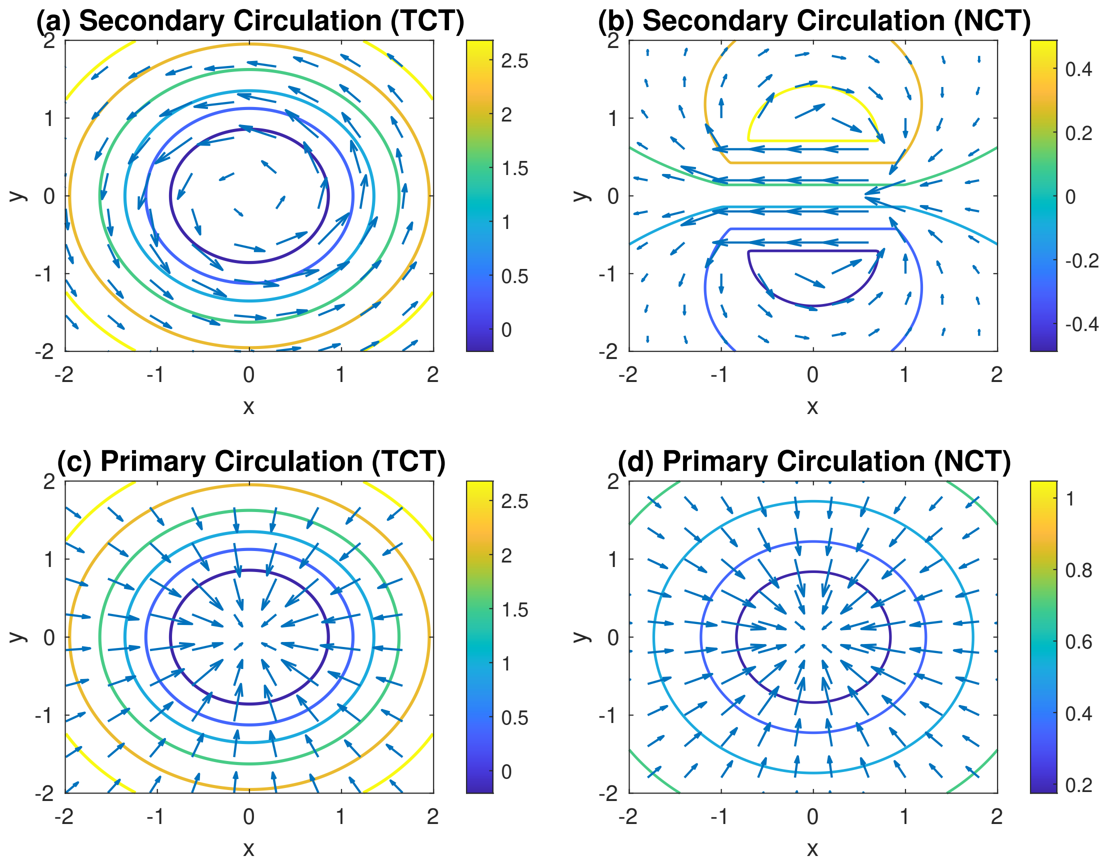

FIG. 1: Figures (a) and (b) show contours of the secondary circulation, ψ, and the vector field $(-\psi_y, \psi_x)$ at the bottom of the troposphere at the north pole, and in the middle of the troposphere at the equator, respectively. Figures (c) and (d) show contours of the velocity potential and the vector field $(-\Phi_x, -\Phi_y)$ at the bottom of the troposphere at the north pole, and at the bottom of the troposphere at the equator, respectively. The axes and variables are scaled to the horizontal length scale, L, and the colorbar is in m/s.

$(-\Phi_x, -\Phi_y, w)$, and the secondary horizontal velocity, $(-\psi_y, \psi_x, 0)$, that arises in response to it. The schematic panels in figure 2 depict (left) the primary convective circulation in the absence of radiative damping and Coriolis force, (center) the symmetric primary circulation and the rotational secondary circulation in the presence of radiative damping and the traditional Coriolis force terms, and (right) the primary and secondary circulation in the presence of radiative damping and the non-traditional Coriolis force terms.

The framework is encapsulated by two equations. The first equation arises from torque balance, described in (18), which determines the secondary horizontal circulation, ψ, given the velocity potential. The response of the secondary circulation depends on the latitude of the convection,

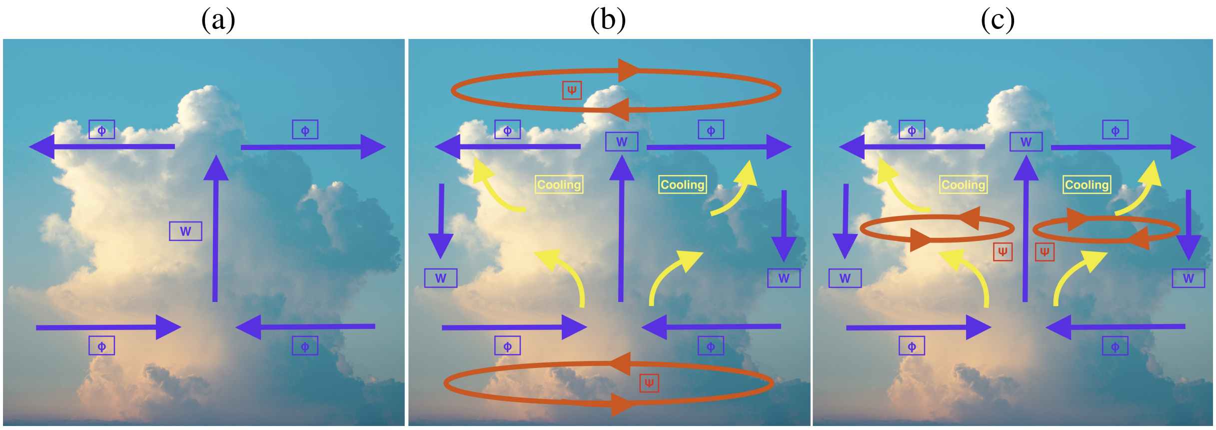

FIG. 2: Schematic representations of the solutions of the convective WTG framework. In purple is the primary poloidal circulation, both the vertical velocity, w, and the horizontal velocity due to the potential, Φ. In yellow is an indication of radiative cooling. Red depicts the secondary circulation ψ, due to the Coriolis force and damping. (a) Shows the WTG without damping or Coriolis force. (b) Shows WTG with radiative cooling and traditional Coriolis terms. (c) Shows WTG with radiative cooling and non-traditional Coriolis terms.

in the absence of the Coriolis terms and radiation, there would only be a poloidal circulation, figure 2 (a). The TCT (poles, figure 2 (b)) drives a cyclonic circulation in response to horizontal convergence, while the NCT (equator, figure 2 (c)) drives an antisymmetric response proportional to the meridional component of the convective velocity field. The red curves in figures 2 (b) and (c) are placed at the heights where the maximum secondary circulation occurs for each case. For the TCT (panel b), equation (29) shows that the the secondary circulation is largest at heights where the horizontal convergence of the convection is largest. From equation (5) we see that this occurs where the vertical derivative of the vertical velocity (and thus the vertical derivative of the heating) is maximum; at the top and bottom of the troposphere. For the NCT (panel c), equation (26) shows the secondary circulation is largest at the height where the meridional velocity of the convection vanishes. At such elevations, the vertical component of the velocity is maximal, therefore the secondary circulation due to the NCT is maximal at the height of the maximum upward velocity in the convection.

The second equation in this framework is an elliptic operator (equation (19)) whose solution yields the velocity potential, Φ, given the diabatic heating, S; if dissipation is used instead of drag, then the theory is described by equation (24). In the absence of radiative damping, the operator is exactly the weak temperature Gradient approximation, but on convective length scales. Radiative damping generates a response in the vertical component of the velocity field away from the diabatic heating source, and thus we describe this as a non-local response. In the case of the NCT (equation (28)) the non-locality is in the vertical direction (figure 2 (c)) while in the case of the TCT (equation (25)) the non-locality is both in the vertical and meridional directions (figure 2 (b)).

In a companion paper we study the solutions of this Convective-Coriolis balanced framework. Future work will describe the convective momentum and temperature fluxes which arise from diabatic heat sources, and the implications of these fluxes for the parameterization of convection in meso- and synoptic scale dynamics, especially in the tropics.

Acknowledgments. This work was partially supported by the NSF under award AGS-2224293.

Data availability statement. No datasets were generated or analyzed during the current study.

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