

Limit Distribution Theory for f -divergences

Sreejith Sreekumar, Ziv Goldfeld, Kengo Kato

Abstract

f -divergences, which quantify discrepancy between probability distributions, are ubiquitous in information theory, machine learning, and statistics. While there are numerous methods for estimating f -divergences from data, a limit distribution theory, which quantifies fluctuations of the estimation error, is largely obscure. As limit theorems are pivotal for valid statistical inference, to close this gap, we develop a general methodology for deriving distributional limits for f -divergences based on the functional delta method and Hadamard directional differentiability. Focusing on four prominent f -divergences—Kullback-Leibler divergence, χ^2 divergence, squared Hellinger distance, and total variation distance—we identify sufficient conditions on the population distributions for the existence of distributional limits and characterize the limiting variables. These results are used to derive one- and two-sample limit theorems for Gaussian-smoothed f -divergences, both under the null and the alternative. Finally, an application of the limit distribution theory to auditing differential privacy is proposed and analyzed for significance level and power against local alternatives.

Index Terms

f -divergence, limit theorem, Hadamard differentiability, functional delta method, auditing differential privacy

I. INTRODUCTION

Statistical inference often boils down to estimation of certain functional of underlying probability measures. Discrepancy measures between probability distributions, also known as statistical divergences, such as f -divergences [3], [4], Rényi divergences [5], [6], integral probability metrics [7], [8], Wasserstein distances [9], [10], etc., form an important class of such functionals. They play a fundamental role in information theory, signal processing, and statistics, with some arising naturally as operational quantities characterizing the fundamental limits of data compression, hypothesis testing, and communication [11], [12]. Moreover, statistical divergences are potent tools for modeling, analysis, and design of machine learning algorithms, encompassing generative modeling [13]–[17], homogeneity/goodness-of-fit/independence testing [18]–[20], anomaly detection [21], [22], to name a few.

In data-driven applications, one only has samples from the population distributions, which necessitates estimating f -divergences. While there is an abundance of consistent estimators with known convergence rates (see the literature

S. Sreekumar is with the Institute for Quantum Information at RWTH Aachen University, Aachen 52074, Germany, (email: sreekumar@physik.rwth-aachen.de). He was with the School of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14850, USA, at the time of this work. Z. Goldfeld and K. Kato are with the School of Electrical and Computer Engineering and Department of Statistics and Data Science, respectively, at Cornell University (e-mail: goldfeld@cornell.edu; kk976@cornell.edu). S. Sreekumar was partially supported by the TRIPODS Center for Data Science NSF grant CCF-1740822. Z. Goldfeld is supported by NSF grants CCF-1947801, CCF-2046018, DMS-2210368, and the 2020 IBM Academic Award. K. Kato is supported by the NSF grants DMS-1952306, DMS-2014636, and DMS-2210368. Parts of this work were presented at the International Symposium on Information Theory (ISIT) in 2020 [1] and 2023 [2].

review in Section I-A), a limit distribution theory for the empirical estimation error has remained partial and premature. For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and an f -divergence $D_f(\cdot\|\cdot)$, limit theorems seek to identify the scaling rate $r_n \rightarrow \infty$ and the limiting variable G , such that the following convergence in distribution holds¹

$$r_n(D_f(\mu_n\|\nu) - D_f(\mu\|\nu)) \xrightarrow{d} G, \quad (1)$$

where μ_n is an estimate of μ from samples. As such, these results characterize the probability laws governing the random fluctuations of the error and serve as a central constituent for valid statistical inference. Indeed, distributional limits enable constructing confidence intervals, devising consistent resampling methods, proving guarantees for applications of hypothesis testing, and more.

To address the aforementioned gap, we develop a unified methodology for deriving limit distributions for f -divergences under general regularity conditions. Our approach relies on the functional delta method over normed vector spaces [23], [24] and Hadamard directional differentiability of f -divergence over a certain class of probability distributions. The Hadamard differentiability analysis captures how the f -divergence functional changes due to small perturbations of the considered distributions within the said class. However, f -divergence functionals (e.g., KL divergence) are generally non-smooth and highly sensitive to support mismatch, which may cause them to degenerate or even blow up. This irregular behaviour also carries over to their derivatives. It is therefore pivotal to identify appropriate regularity conditions under which the Hadamard directional derivatives and the corresponding distributional limits exist and can be characterized. In particular, there is a trade-off between how strict the imposed regularity is and the class of distributions that the theory accounts for. Consequently, a key technical challenge is to discern the right normed space in which the densities of the considered distributions should reside, so as to obtain a limit distribution theory that accounts for the largest possible class of distributions.

Existing approaches for deriving limit distributions for f -divergences are mostly limited to discrete distributions or compactly supported continuous distributions with smooth densities bounded away from zero, in which case the f -divergence functional as well as its derivatives become smooth. Our approach disposes of such restrictive assumptions and extends these results to the general case. We leverage the Taylor expansion of the considered f -divergences to ascertain minimal primitive regularity conditions on the population distributions that guarantee the existence of the Hadamard derivative. In particular, we identify a certain $L^2(\eta)$ space, where the measure η is defined in terms of the populations (μ, ν) , in which the Hadamard directional derivatives exist and can be characterized. Having that, the functional delta method enables lifting weak convergence of the estimates of the underlying distributions to convergence of the f -divergence between them, with the limiting variable identified in terms of this derivative.

The general framework is instantiated to obtain the one- and two-sample distributional limits, under both the null ($\mu = \nu$) and the alternative ($\mu \neq \nu$), of four popular f -divergences—Kullback-Leibler (KL) divergence, chi-squared (χ^2) divergence, squared Hellinger (H^2) distance, and total variation (TV) distance. These results hold under the high-level weak convergence assumptions on the empirical estimates of μ, ν with a given scaling law r_n . To obtain limit

¹The two-sample problem, i.e., when the first divergence term is $D_f(\mu_n\|\nu_n)$, is also of interest.

theorems under basic conditions on the population distributions with explicit rates, we consider Gaussian-smoothed f -divergences, i.e., $D_f(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)$ where $\gamma_\sigma = \mathcal{N}(0, \sigma^2 I_d)$, and estimate μ, ν by the empirical measures $\hat{\mu}_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$ and $\hat{\nu}_n = n^{-1} \sum_{i=1}^n \delta_{Y_i}$, respectively. Under this setup, we derive primitive conditions² on μ, ν that guarantees weak convergence of the smooth empirical measures $\hat{\mu}_n * \gamma_\sigma, \hat{\nu}_n * \gamma_\sigma$, utilizing the central limit theorem (CLT) in L^2 spaces [25, Proposition 2.1.11]. For KL divergence, χ^2 divergence, and H^2 distance under the null, we identify the scaling law as $r_n = n$ and the limiting variable as a weighted sum of independent and identically distributed (i.i.d.) χ^2 random variables. Under the alternative, we show that $r_n = \sqrt{n}$ and the limit is a centered Gaussian. The TV distance behaves slightly differently, with $r_n = \sqrt{n}$ in both cases and the limiting variables having a certain integral form. By virtue of our Hadamard differentiability analysis, we automatically obtain consistency of the bootstrap, which yields a computationally tractable resampling method for estimating the distributional limits.

As an application of our limit distribution theory, we consider auditing ϵ -differential privacy (DP). An audit of a black-box privacy mechanism seeks to certify whether it satisfies a promised DP guarantee. While existing auditing methods are heuristic [26], [27] or lack in formal guarantees [28], we propose a principled hypothesis testing pipeline for DP auditing with a full (asymptotic) analysis of significance level and power against local alternatives. The key idea is to relax the ϵ -DP constraint³ to a KL divergence bound, which is further relaxed to the Gaussian-smoothed KL divergence via the data-processing inequality [29, Theorem 2.15]. We then test for the smooth KL divergence value and leverage our limit theorems for the significance and power analysis. We also establish a stability lemma that bounds the gap due to smoothing, namely $|\mathcal{D}_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma) - \mathcal{D}_{\text{KL}}(\mu \| \nu)|$. This enables lifting the audit to test for the KL divergence value itself, for which we show that any non-zero significance level along with power 1 can be achieved asymptotically.

A. Related Work

Statistical analysis of divergence estimators has been an active area of research in recent years. Convergence rates for various estimators, which subsumes entropy and mutual information as special cases, have been studied in [30]–[42] (see also references therein). Literature on limit distributions for f -divergences mainly focused on analyzing specific estimators on a case-by-case basis. In [43], limit distributions for f -divergences between maximum likelihood estimates of probability distributions over a certain parametric class is established, with the limit variable shown to be either normal or χ^2 . The authors of [32] study plugin methods of kernel density estimator and show asymptotic normality subject to high Hölder smoothness and compact support of the densities. The case when the density estimates are constructed using k -nearest neighbour technique is treated in [44]. One-sample null distributional limits of Gaussian-smoothed TV distance and χ^2 divergence have been derived in [1] by invoking the CLT in $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively. Limit distributions for plug-in estimators of entropy and mutual information in the discrete setting have been considered in [45], [46]. Differing from these, the unified methodology

²The primitive conditions shown here are sharp in the one-sample null case, see e.g., Proposition 2(i).

³ ϵ -DP corresponds to an ϵ bound on the infinite order Rényi divergence between the output distribution of the mechanism applied to two neighboring databases.

developed herein enables obtaining distributional limits of f -divergences based on plugin of arbitrary estimators under general (oftentimes sufficient and necessary) conditions. In particular, our results subsume those of [1] and considerably generalize the scope of this preliminary work via markedly different proof techniques.

Apart from the f -divergence class, limit distribution theory for other divergences such as Wasserstein distances has been extensively studied [47]–[55]. There is also a surge of interest in regularized optimal transport distances driven by computational and statistical gains, encompassing techniques like smoothing, slicing, and entropic penalization. Limit theorems under these frameworks for the optimal transport cost, plan, map, and dual potentials can be found in [17], [53], [56]–[65].

B. Organization

The rest of the paper is organized as follows. Section II collects background material on Hadamard differentiability and the functional delta method. Section III formalizes the Hadamard differentiability framework for f -divergences. Section IV obtains limit distributions for f -divergences between random probability measures under general regularity conditions, with Section V focusing specifically on Gaussian-smoothed divergences. Section VI studies the application of the limit distribution theory to auditing DP. Proofs are provided in Section VII, while Section VIII provides concluding remarks and discusses future research directions.

II. BACKGROUND AND PRELIMINARIES

A. Notation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a sufficiently rich probability space on which all random variables are defined. Let $(\mathfrak{S}, \mathcal{S})$ be a separable measurable space equipped with a σ -finite measure ρ . When \mathfrak{S} is a topological space, we use $\mathcal{B}(\mathfrak{S})$ to denote the Borel σ -field on \mathfrak{S} . In the sequel, we adapt ρ on a case-by-case basis, but given ρ , all considered measures are assumed to be absolutely continuous w.r.t. it. For $\eta \ll \rho$, we write $p_\eta = d\eta/d\rho$ for the Radon-Nikodym derivative of η w.r.t. ρ . $\eta^{\otimes n}$ stands for the n -fold product measure, and δ_x represents the Dirac measure at x . $\mathbb{1}_{\mathcal{E}}$ denotes the indicator of an event \mathcal{E} .

We use $\mathcal{P}(\mathfrak{S})$ to denote the space of probability measures on $(\mathfrak{S}, \mathcal{S})$, leaving the σ -field implicit. When $\mathfrak{S} = \mathbb{R}^d$, we always take $\mathcal{S} = \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{P}(\mathbb{R}^d)$ as the set of Borel probability measures. For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, $\mu * \nu$ denotes the convolution of μ and ν ; likewise, $f * g$ represents convolution of two measurable functions $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$. We write $\gamma_\sigma = N(0, \sigma^2 I_d)$ for the centered Gaussian distribution on \mathbb{R}^d with covariance matrix $\sigma^2 I_d$, and use $\varphi_\sigma(x) = (2\pi\sigma^2)^{-d/2} e^{-\|x\|^2/(2\sigma^2)}$ ($x \in \mathbb{R}^d$) for the corresponding density. We say that $\mu \in \mathcal{P}(\mathbb{R}^d)$ is β -sub-Gaussian for $\beta \geq 0$, if $X \sim \mu$ satisfies $\mathbb{E}[e^{\alpha \cdot (X - \mathbb{E}[X])}] \leq \exp(\beta^2 \|\alpha\|^2/2)$, for all $\alpha \in \mathbb{R}^d$. Let \xrightarrow{w} and \xrightarrow{d} denote weak convergence⁴ of Borel measurable maps (or their laws) and convergence in distribution of random variables, respectively.

For $1 \leq r \leq \infty$, let $L^r(\rho) = L^r(\mathfrak{S}, \mathcal{S}, \rho)$ be the space of all real-valued measurable functions f on \mathfrak{S} such that $\|f\|_{r, \rho} := (\int_{\mathfrak{S}} |f|^r d\rho)^{1/r} < \infty$, with the usual identification of functions that are equal ρ -almost everywhere (a.e.).

⁴A sequence of Borel measurable maps X_n converges weakly to a Borel measurable map X if $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X)]$ for all bounded continuous functions f . This is denoted by $X_n \xrightarrow{w} X$.

For $1 \leq r < \infty$, the space $(L^r, \|\cdot\|_{r,\rho})$ is a separable Banach (and hence, Polish) space. When ρ is the Lebesgue measure λ on \mathbb{R}^d , we use $\|\cdot\|_r$ to denote the corresponding L^r norm. $\|\cdot\|$ designates Euclidean norm. $L^r_+(\rho)$ denotes the subset of positive functions in $L^r(\rho)$. The support of a measurable function $f : \mathfrak{S} \rightarrow \bar{\mathbb{R}}$, i.e., $\{s \in \mathfrak{S} : f(s) \neq 0\}$ is denoted by $\text{spt}(f)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ with $|\alpha| = \sum_{j=1}^d \alpha_j$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), we use D^α for the differential operator $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ with $D^0 f = f$, and employ the shorthand $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. The shorthand $a \lesssim b$ designates $a \leq cb$ for a universal constant $c > 0$. The values of constants may change from line to line of a certain derivation. Lastly, we adopt the convention $0/0 = 0$, $c/0 = \infty$ for $c > 0$, $\infty \cdot 0 = 0$, and $0 \log(c/0) = 0$ for $c \geq 0$.

B. f -Divergences

f -divergences form a broad class of discrepancy measures between probability distributions, as defined next.

Definition 1 (f -divergence) *Let $f : [0, \infty] \rightarrow (-\infty, \infty]$ be a convex function such that $f(1) = 0$ and $f(0) = f(0^+)$. For $\mu, \nu \in \mathcal{P}(\mathfrak{S})$, the f -divergence of μ from ν is*

$$D_f(\mu\|\nu) := \int_{\mathfrak{S}} f \circ \left(\frac{p_\mu}{p_\nu} \right) d\nu. \quad (2)$$

The class of f -divergences includes several popular dissimilarity measures, such as KL divergence, χ^2 divergence, H^2 distance, TV distance, and many more. Every f -divergence satisfies nonnegativity ($D_f(\mu\|\nu) \geq 0$, $\forall \mu, \nu$) and joint convexity in the pair (μ, ν) , with additional properties holding for specific instances, as mentioned below.

1) KL divergence: Setting $f(x) = f_{\text{KL}}(x) := x \log x$ in (2) yields KL divergence, $D_{\text{KL}}(\mu\|\nu) := \int_{\mathfrak{S}} \log(p_\mu/p_\nu) d\mu$ for $\mu \ll \nu$, and $D_{\text{KL}}(\mu\|\nu) = \infty$, otherwise.

2) χ^2 divergence: Setting $f(x) = f_{\chi^2}(x) := (x-1)^2$ in (2) leads to χ^2 divergence, $\chi^2(\mu\|\nu) := \int_{\mathfrak{S}} (p_\mu/p_\nu - 1)^2 d\nu$ for $\mu \ll \nu$, and $\chi^2(\mu\|\nu) = \infty$, otherwise.

3) H^2 distance: Setting $f(x) = f_{H^2}(x) := (\sqrt{x} - 1)^2$ in (2) leads to H^2 distance, $H^2(\mu, \nu) := \int_{\mathfrak{S}} (\sqrt{p_\mu} - \sqrt{p_\nu})^2 d\rho$. H^2 distance is symmetric in its arguments, and $0 \leq H^2(\mu, \nu) \leq 2$ for all $\mu, \nu \in \mathcal{P}(\mathcal{S})$.

4) TV distance: Setting $f(x) = f_{\text{TV}}(x) := |x - 1|/2$ in (2) yields the TV distance, $\|\mu - \nu\|_{\text{TV}} := \frac{1}{2} \int_{\mathfrak{S}} |p_\mu - p_\nu| d\rho = \sup_{A \in \mathcal{S}} |\mu(A) - \nu(A)|$. TV distance is symmetric in its arguments, and $0 \leq \|\mu - \nu\|_{\text{TV}} \leq 1$ for all $\mu, \nu \in \mathcal{P}(\mathcal{S})$.

C. Functional delta method

We derive limit distribution theory for f -divergences via the functional delta method, which relies on the concept of Hadamard directional differentiability. The ideas are introduced below.

Definition 2 (Hadamard directional differentiability [23]–[25]) *Let \mathfrak{D} and \mathfrak{E} be linear topological spaces, and consider a function $\Phi : \Theta \subseteq \mathfrak{D} \rightarrow \mathfrak{E}$.*

(i) Φ is first order Hadamard directionally differentiable at $\theta \in \Theta$ tangentially to Θ if there exists a map

$\Phi'_\theta : \mathfrak{T}_\Theta(\theta) \rightarrow \mathfrak{E}$ such that

$$\lim_{n \rightarrow \infty} \frac{\Phi(\theta + t_n h_n) - \Phi(\theta)}{t_n} = \Phi'_\theta(h), \quad (3)$$

for any $h \in \mathfrak{T}_\Theta(\theta)$, $t_n \downarrow 0^+$ and $h_n \rightarrow h$ in \mathfrak{D} such that $\theta + t_n h_n \in \Theta$, where

$$\mathfrak{T}_\Theta(\theta) := \left\{ h \in \mathfrak{D} : h = \lim_{n \rightarrow \infty} \frac{\theta_n - \theta}{t_n} \text{ for some } \theta_n \rightarrow \theta \text{ with } \theta_n \in \Theta \text{ and } t_n \downarrow 0^+ \right\}$$

is the tangent cone to Θ at θ .

- (ii) For convex Θ ,⁵ we say that Φ is second order Hadamard directionally differentiable at $\theta \in \Theta$ tangentially to Θ if it is first order Hadamard directionally differentiable at θ and there exists a map $\Phi'_\theta : \mathfrak{T}_\Theta(\theta) \rightarrow \mathfrak{E}$ such that

$$\Phi''_\theta(h) = \lim_{n \rightarrow \infty} \frac{\Phi(\theta + t_n h_n) - \Phi(\theta) - t_n \Phi'_\theta(h_n)}{\frac{1}{2} t_n^2}, \quad (4)$$

for any $t_n \downarrow 0^+$ and $h_n \rightarrow h$ in \mathfrak{D} (note: the convexity of Θ implies that $h_n \in \mathfrak{T}_\Theta(\theta)$, so that $\Phi'_\theta(h_n)$ is well-defined).

The functions Φ'_θ and Φ''_θ are called the first and second order Hadamard directional derivatives of Φ at θ .

For Hadamard directionally differentiable maps we have the following adaptation of the functional delta method [23]–[25]. Let $\text{int}(\mathcal{C})$ and $\partial\mathcal{C}$ denote the interior and boundary of a set \mathcal{C} , respectively.

Lemma 1 (Functional delta method) *Let \mathfrak{D} and \mathfrak{E} be metrizable linear topological spaces, $\Theta \subseteq \mathfrak{D}_\Phi \subseteq \mathfrak{D}$, and $\Phi : \mathfrak{D}_\Phi \rightarrow \mathfrak{E}$ be Hadamard directionally differentiable at $\theta \in \Theta$ tangentially to Θ with derivative $\Phi'_\theta : \mathfrak{T}_\Theta(\theta) \rightarrow \mathfrak{E}$. Let $Z_n : \Omega \rightarrow \mathfrak{D}_\Phi$, $n \in \mathbb{N}$, be measurable maps such that $r_n(Z_n - \theta) \xrightarrow{w} Z$ for some sequence $r_n \rightarrow \infty$, where Z is a random variable that takes values in $\mathfrak{T}_\Theta(\theta)$.*

- (i) *If $\mathfrak{D}_\Phi = \Theta$, then we have*

$$r_n(\Phi(Z_n) - \Phi(\theta)) \xrightarrow{d} \Phi'_\theta(Z),$$

and if Θ is convex then $r_n(\Phi(Z_n) - \Phi(\theta)) = \Phi'_\theta(r_n(Z_n - \theta)) + o_{\mathbb{P}}(1)$.

- (ii) *If $\mathfrak{D}_\Phi = \Theta$ is convex and Φ is also second order Hadamard directionally differentiable at $\theta \in \Theta$ tangentially to Θ , with derivative Φ''_θ , then*

$$r_n^2(\Phi(Z_n) - \Phi(\theta) - \Phi'_\theta(Z_n - \theta)) \xrightarrow{d} \frac{1}{2} \Phi''_\theta(Z),$$

and $r_n^2(\Phi(Z_n) - \Phi(\theta) - \Phi'_\theta(Z_n - \theta)) = \frac{1}{2} \Phi''_\theta(r_n(Z_n - \theta)) + o_{\mathbb{P}}(1)$.

- (iii) *For a convex $\Theta \subseteq \mathfrak{D}_\Phi$, suppose that Z_n and Z satisfy $\mathbb{P}^*(Z_n \notin \Theta) \rightarrow 0$ and $\mathbb{P}(Z \in \partial\mathfrak{T}_\Theta(\theta)) = 0$, where \mathbb{P}^* denotes outer probability (see [25]). Then, Part (i) and (ii) above holds.*

Part (i) and (ii) above are standard in the literature, while Part (iii) which incorporates probabilistic constraints is new and is useful for our purposes to derive limit distribution results under more general conditions. To establish this claim, we prove an adaptation of the extended continuous mapping theorem [25, Theorem 1.11.1] as stated next, which may be of independent interest.

⁵When Θ is convex, we have $\mathfrak{T}_\Theta(\theta) = \text{cl}(\{(\tilde{\theta} - \theta)/t : \tilde{\theta} \in \Theta, t > 0\})$ for all $\theta \in \text{cl}(\Theta)$; see [24].

Theorem 1 (Generalized extended continuous mapping theorem) *Let \mathfrak{D} and \mathfrak{E} be metrizable linear topological spaces, $\mathfrak{D}_n, \mathfrak{D}_g \subseteq \mathfrak{D}$, and $\mathfrak{D}_n^\rightarrow \subseteq \mathfrak{D}_n$, for all $n \in \mathbb{N}$. Suppose measurable functions $g_n : \mathfrak{D}_n \rightarrow \mathfrak{E}$ and $g : \mathfrak{D}_g \rightarrow \mathfrak{E}$ satisfy the following: if $h_n \rightarrow h$ with $h_n \in \mathfrak{D}_n^\rightarrow$ for all sufficiently large n and $h \in \mathfrak{D}_g$, then $g_n(h_n) \rightarrow g(h)$. Let $H_n : \Omega \rightarrow \mathfrak{D}_n$ and $H : \Omega \rightarrow \mathfrak{D}_g$ be measurable maps such that $H_n \xrightarrow{w} H$, $\mathbb{P}(H \in \mathfrak{D}_*) = 0$ and $\mathbb{P}(H \in \mathfrak{D}_\infty) = 1$, where $\mathfrak{D}_\infty = \{h \in \mathfrak{D} : \exists (h_n)_{n \in \mathbb{N}}, h_n \rightarrow h \text{ and } h_n \in \mathfrak{D}_n^\rightarrow \forall n \text{ sufficiently large}\}$ and $\mathfrak{D}_* = \cap_{m=1}^\infty \overline{\cup_{n=m}^\infty (\mathfrak{D} \setminus \mathfrak{D}_n^\rightarrow)}$. Then, $g_n(H_n) \xrightarrow{w} g(H)$.*

The proofs of Part (iii) of Lemma 1 and Theorem 1 are given in Appendices A-1 and A-2, respectively.

III. HADAMARD DIFFERENTIABILITY FRAMEWORK FOR f -DIVERGENCES

We specialize the Hadamard differentiability framework to treat f -divergences. The framework is first described in abstract terms, after which we instantiate it to the case of KL divergence for concreteness (see Example 1 below). The key idea is to find the right normed space over which $D_f(\cdot \|\cdot)$ can be set up as a first and second order Hadamard directionally differentiable functional w.r.t. that norm. The construction is as follows.

Let $\phi : \bar{\mathbb{R}}_{\geq 0} \times \bar{\mathbb{R}}_{\geq 0} \rightarrow \bar{\mathbb{R}}$, where $\bar{\mathbb{R}}$ is the extended reals and $\bar{\mathbb{R}}_{\geq 0}$ is its nonnegative part.

Assumption 1 $\phi(1, 1) = 0$, $\phi(0, 0) = \lim_{y \downarrow 0} \phi(0, y)$, ϕ is continuous at $(0, c)$ for $c > 0$, and all its partial derivatives of order two exists and are continuous in $(0, \infty) \times (0, \infty)$, possibly taking the values $\pm\infty$ only when at least one of its arguments is 0 or ∞ .

Let $g_1^*, g_2^* \in L_+^1(\rho)$ be such that $\|g_1^*\|_{1,\rho} \vee \|g_2^*\|_{1,\rho} \leq 1$. For a multi-index $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 2$, let $\psi_\alpha : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be measurable functions, and $\eta_1, \eta_2 \ll \rho$ be positive measures on \mathfrak{S} defined via their relative densities w.r.t. ρ as

$$\begin{aligned} p_{\eta_1} &:= 1 + |\psi_{2,0} \circ (g_1^*, g_2^*)| + |\psi_{1,1} \circ (g_1^*, g_2^*)|, \\ p_{\eta_2} &:= 1 + |\psi_{0,2} \circ (g_1^*, g_2^*)| + |\psi_{1,1} \circ (g_1^*, g_2^*)|. \end{aligned} \quad (5)$$

Given g_1^*, g_2^* , and $\psi := (\psi_{2,0}, \psi_{0,2}, \psi_{1,1})$ as above, we define the normed space

$$\mathfrak{D}_{g_1^*, g_2^*, \psi} := \left\{ (g_1 - g_1^*, g_2 - g_2^*) : g_1, g_2 \in L^1(\rho), \| (g_1 - g_1^*, g_2 - g_2^*) \|_{\mathfrak{D}_{g_1^*, g_2^*, \psi}} < \infty \right\}, \quad (6)$$

where $\|(g, \tilde{g})\|_{\mathfrak{D}_{g_1^*, g_2^*, \psi}} := \|g\|_{2, \eta_1} + \|\tilde{g}\|_{2, \eta_2}$. We henceforth use the shorthands \mathfrak{D} and $\|\cdot\|_{\mathfrak{D}}$ for the space $\mathfrak{D}_{g_1^*, g_2^*, \psi}$ and its norm.

Setting $\Theta' := \{(g_1 - g_1^*, g_2 - g_2^*) \in \mathfrak{D} : g_1 \geq 0, g_2 \geq 0, \text{spt}(g_1) \subseteq \text{spt}(g_2), \|g_1\|_{1,\rho} \vee \|g_2\|_{1,\rho} \leq 1\}$ and $\Theta'' := \{(g_1 - g_1^*, g_2 - g_2^*) \in \mathfrak{D} : g_1 \geq 0, g_2 \geq 0, \|g_1\|_{1,\rho} \vee \|g_2\|_{1,\rho} \leq 1\}$, let Θ be a convex subset of Θ' or Θ'' that contains $(0, 0)$, and consider the functional $\Phi : \Theta \rightarrow \bar{\mathbb{R}}$ given by

$$\Phi(\theta_1, \theta_2) := \int_{\mathfrak{S}} \phi(g_1^* + \theta_1, g_2^* + \theta_2) d\rho. \quad (7)$$

In addition to Assumption 1, the following assumptions are needed to state our Hadamard differentiability result.

Assumption 2 At least one of the following conditions hold:

- (i) $D^{(1,0)}\phi \circ (g_1^*, g_2^*), D^{(0,1)}\phi \circ (g_1^*, g_2^*) \in L^2(\rho)$.
- (ii) $D^{(1,0)}\phi \circ (g_1^*, g_2^*) \in L^2(\rho)$ and Θ is such that $\theta_2 = 0$ for all $(\theta_1, \theta_2) \in \Theta$.
- (iii) $D^{(0,1)}\phi \circ (g_1^*, g_2^*) \in L^2(\rho)$ and Θ is such that $\theta_1 = 0$ for all $(\theta_1, \theta_2) \in \Theta$.

Assumption 3 Φ is well-defined⁶ on Θ and $|\Phi(\theta_1, \theta_2)| < \infty$ for all $(\theta_1, \theta_2) \in \Theta$.

Assumption 4 For any $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 2$, there exists $v_\alpha \in L^1([0, 1], \lambda)$, such that for all $\theta = (\theta_1, \theta_2) \in \Theta$ and $\tau \in [0, 1]$, we have

$$(1 - \tau) \left| \theta^\alpha D^\alpha \phi \circ ((g_1^*, g_2^*) + \tau(\theta_1, \theta_2)) \right| \lesssim |\theta^\alpha \psi_\alpha \circ (g_1^*, g_2^*)| v_\alpha(\tau) \quad \rho - a.e. \quad (8)$$

Under these assumptions, we have the desired Hadamard differentiability result.

Proposition 1 (i) If Assumptions 1-4 hold for $\Theta \subseteq \Theta'$, then Φ , as defined in (7), is second order Hadamard differentiable at $\theta^* := (0, 0) \in \Theta$ tangentially to Θ with

$$\Phi'_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} (h_1 D^{(1,0)}\phi \circ (g_1^*, g_2^*) + h_2 D^{(0,1)}\phi \circ (g_1^*, g_2^*)) d\rho, \quad (9)$$

$$\Phi''_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} (h_1^2 D^{(2,0)}\phi \circ (g_1^*, g_2^*) + h_2^2 D^{(0,2)}\phi \circ (g_1^*, g_2^*) + 2h_1 h_2 D^{(1,1)}\phi \circ (g_1^*, g_2^*)) d\rho, \quad (10)$$

for all $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{\theta/t : \theta = (\theta_1, \theta_2) \in \Theta, t > 0\})$.

(ii) The above claim further extends to $\Theta \subseteq \Theta''$ provided ϕ is also continuous at $(0, 0)$.

The following example instantiates the above framework to the case of KL divergence.

Example 1 (KL divergence) Consider the KL divergence $D_{\text{KL}}(\mu \parallel \nu)$ between probability measures $\mu \ll \nu \in \mathcal{P}(\mathfrak{S})$, in which case we may take $\rho = \nu$. Assume $p_\mu > 0$. The setup above specializes the KL divergence case with $\phi(x, y) = x \log x$, $g_1^* = p_\mu$, and $g_2^* = 1$, whereby $D_{\text{KL}}(\mu \parallel \nu) = \Phi(0, 0)$ (see (7)). Further set $\psi_{2,0}(x, y) = D^{(2,0)}\phi(x, y) = 1/x$, $\psi_{1,1}(x, y) = D^{(1,1)}\phi(x, y) = 0$, and $\psi_{0,2}(x, y) = D^{(0,2)}\phi(x, y) = 0$. The measures η_1 and η_2 are defined through the densities $p_{\eta_1} = 1 + \psi_{2,0} \circ (p_\mu, 1) = 1 + 1/p_\mu$ and $p_{\eta_2} = 1$ (note that $\eta_2 = \nu$). The spaces of interest are taken as $\mathfrak{D} = \{(g_1 - p_\mu, g_2 - 1) : g_1, g_2 \in L^1(\rho), \|g_1 - p_\mu\|_{\eta_1} + \|g_2 - 1\|_\nu < \infty\}$, $\Theta = \{(g_1 - p_\mu, 0) \in \mathfrak{D} : g_1 \geq 0, \|g_1\|_{1,\nu} = 1\}$. The function v_α , $\alpha \in \mathbb{N}_0^2$, to satisfy (8) is chosen as $v_{0,2} = v_{1,1} = 0$, and $v_{2,0} = 1$. Indeed, the former null values are sufficient since $D^{(2,0)}\phi = D^{(1,1)}\phi = 0$, while for the latter, with $\theta = (\theta_1, \theta_2) = (g_1 - g_1^*, g_2 - g_2^*)$, we have

$$(1 - \tau) \left| \theta_1^2 D^{(2,0)}\phi \circ ((g_1^*, g_2^*) + \tau(\theta_1, \theta_2)) \right| = \frac{(1 - \tau)\theta_1^2}{(1 - \tau)g_1^* + \tau g_1} \leq \frac{\theta_1^2}{g_1^*} = |\theta_1^2 \psi_{2,0} \circ (g_1^*, g_2^*)|.$$

Here, the inequality holds since $g_1 \geq 0$. Note that $\mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{((g_1 - p_\mu)/t, 0) : (g_1 - p_\mu, 0) \in \Theta, t > 0\})$. Corresponding choices for other f -divergences can be found in Lemma 5 (in Section VII-B) below.

⁶When ϕ is convex and ρ is a finite measure, Jensen's inequality and Assumption 1 automatically imply that Φ is well-defined and nonnegative.

IV. LIMIT DISTRIBUTION FOR f -DIVERGENCES

We instantiate the above unified framework to derive a flexible limit distribution theory under general regularity conditions for several popular f -divergences—KL divergence, χ^2 divergence, H^2 distance, and TV distance. To maintain versatility, we model generic estimators of population probability distribution as random probability measures. These can be substituted with a specific estimator depending on the application.

Definition 3 (Random probability measure) *A random probability measure on \mathfrak{S} is a map $\zeta : \Omega \times \mathcal{S} \rightarrow [0, 1]$ satisfying*

- (i) *for every $\mathcal{C} \in \mathcal{S}$, $\omega \rightarrow \zeta(\omega, \mathcal{C})$ is measurable from (Ω, \mathcal{A}) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$;*
- (ii) *for every $\omega \in \Omega$, $\zeta(\omega, \cdot) \in \mathcal{P}(\mathfrak{S})$.*

Let $\mu, \nu \in \mathcal{P}(\mathfrak{S})$. Consider a sequence $(\mu_n, \nu_n)_{n \in \mathbb{N}}$ of random probability measures on \mathfrak{S} such that (μ_n, ν_n) converges weakly to (μ, ν) . Accordingly, (μ_n, ν_n) can be viewed as an instance of weakly convergent estimators of the population distribution (μ, ν) . Below, the one- and two-sample settings refer to when only μ or both (μ, ν) are approximated by μ_n or (μ_n, ν_n) , respectively. We also use the terms ‘null’ and ‘alternative’ for when $\mu = \nu$ or $\mu \neq \nu$, respectively. In the following results, $(r_n)_{n \in \mathbb{N}}$ denotes a diverging sequence, q , q_1 , and q_2 represent measurable functions. Also, inequalities involving relative densities, e.g. $p_\mu > 0$, are to be interpreted as holding ρ a.e., and regularity conditions involving random measures, e.g. $D_{\text{KL}}(\mu_n \| \nu) < \infty$, are required to hold only for sufficiently large n .

A. KL divergence

Theorem 2 (Limit distribution for KL divergence) *The following hold:*

- (i) *(One-sample null) Let $\mu_n \ll \mu = \rho$ be such that $D_{\text{KL}}(\mu_n \| \mu) < \infty$ almost surely (a.s.). If $r_n(p_{\mu_n} - 1) \xrightarrow{w} B$ in $L^2(\mu)$, then*

$$r_n^2 D_{\text{KL}}(\mu_n \| \mu) \xrightarrow{d} \frac{1}{2} \int_{\mathfrak{S}} B^2 d\mu. \quad (11)$$

- (ii) *(One-sample alternative) Let $\mu_n \ll \mu \ll \nu = \rho$ satisfy $\mathbb{1}_{\text{spt}(p_\mu)} \log p_\mu \in L^2(\nu)$, $D_{\text{KL}}(\mu \| \nu) < \infty$, and $D_{\text{KL}}(\mu_n \| \nu) < \infty$ a.s. If $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\eta)$, where η has relative density $p_\eta = 1 + (\mathbb{1}_{\text{spt}(p_\mu)} / p_\mu)$, then*

$$r_n(D_{\text{KL}}(\mu_n \| \nu) - D_{\text{KL}}(\mu \| \nu)) \xrightarrow{d} \int_{\text{spt}(p_\mu)} B \log p_\mu d\nu. \quad (12)$$

- (iii) *(Two-sample null) Let $\mu_n \ll \nu_n \ll \mu = \rho$ be such that $D_{\text{KL}}(\mu_n \| \nu_n) < \infty$ and $p_{\mu_n}/p_{\nu_n} \leq q$ a.s. Let $\eta_1 = \mu$ and η_2 be the measure with relative density $p_{\eta_2} = 1 + q$. If $(r_n(p_{\mu_n} - 1), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in $L^2(\eta_1) \times L^2(\eta_2)$, then*

$$r_n^2 D_{\text{KL}}(\mu_n \| \nu_n) \xrightarrow{d} \frac{1}{2} \int_{\mathfrak{S}} (B_1 - B_2)^2 d\mu. \quad (13)$$

(iv) *(Two-sample alternative)* Let $\mu_n \ll \nu_n \ll \nu = \rho$ and $\mu_n \ll \mu \ll \nu$ satisfy $p_\mu, \mathbb{1}_{\text{spt}(p_\mu)} \log p_\mu \in L^2(\nu)$, $D_{\text{KL}}(\mu \parallel \nu) < \infty$, $D_{\text{KL}}(\mu_n \parallel \nu_n) < \infty$ and $p_{\mu_n}/p_{\nu_n} \leq q$ a.s. Let η_1 and η_2 be measures with relative densities $p_{\eta_1} = 1 + (\mathbb{1}_{\text{spt}(p_\mu)}/p_\mu)$ and $p_{\eta_2} = 1 + p_\mu + q$, respectively. If $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in $L^2(\eta_1) \times L^2(\eta_2)$, then

$$r_n(D_{\text{KL}}(\mu_n \parallel \nu_n) - D_{\text{KL}}(\mu \parallel \nu)) \xrightarrow{d} \int_{\text{spt}(p_\mu)} B_1 \log p_\mu d\nu - \int_{\text{spt}(p_\mu)} B_2 d\mu. \quad (14)$$

The proof of Theorem 2 utilizes the functional delta method in Lemma 1 and Proposition 1. For the purpose of applying the latter result, we note that the KL divergence functional can be written in the form (7) with an appropriate ϕ in each of the cases above such that Assumptions 1-4 are satisfied under the conditions therein (see Example 1).

Remark 1 (Limit distribution under probabilistic constraints) Let $\mathcal{Q}_n = \{\omega \in \Omega : p_{\mu_n}(\omega, \cdot)/p_{\nu_n}(\omega, \cdot) \leq q(\cdot)\}$. It is possible to replace the constraints $p_{\mu_n}/p_{\nu_n} \leq q$ a.s. by the more relaxed constraint $\mathbb{P}^*(\Omega \setminus \mathcal{Q}_n) \rightarrow 0$ if (B_1, B_2) is continuous. Here, \mathbb{P}^* denotes outer probability which is needed since \mathcal{Q}_n may be non-measurable, in general. The continuity of (B_1, B_2) means that for every Borel set $C \in L^2(\eta_1) \times L^2(\eta_2)$, $\mathbb{P}((B_1, B_2) \in \partial C) = 0$, where ∂C denotes the boundary of C . This holds, for instance, if (B_1, B_2) is Gaussian. The proof of the claim (see Appendix A-3) follows similarly to that of Theorem 2 by using Part (iii) of Lemma 1 in place of Part (i) and (ii). Similar remarks apply to Theorem 3 and 4 which are omitted for brevity.

A few remarks about the regularity conditions in Theorem 2 are in order. In the one-sample null case, the condition $r_n(p_{\mu_n} - 1) \xrightarrow{w} B$ is the weak convergence of the (centered) density of μ_n in $L^2(\mu)$, which is a natural requirement for the existence of the KL limit. In the one-sample alternative case, we require the integrability condition $\mathbb{1}_{\text{spt}(p_\mu)} \log p_\mu \in L^2(\nu)$ along with $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\eta)$. Using $|\log x| \leq |x - 1| \vee |1 - x^{-1}|$, the former condition holds if $p_\mu, \mathbb{1}_{\text{spt}(p_\mu)}/p_\mu \in L^2(\nu)$. For the latter (weak convergence) condition to hold it is sufficient that $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\nu)$, so long as $\mathbb{1}_{\text{spt}(p_\mu)}/p_\mu \in L^\infty(\nu)$. The corresponding two-sample results hold under similar conditions plus a requirement of existence of a function q that dominates the ratio of the (random) densities of μ_n and ν_n , i.e., $p_{\mu_n}/p_{\nu_n} \leq q$ a.s., which can be relaxed to a probabilistic constraint given in Remark 1 provided the limit variables are continuous. We emphasize that this additional regularity condition necessitates that the one- and two-sample cases are stated separately, even though the one-sample limit distribution may be obtained from the two-sample result by setting $(B_1, B_2) = (B, 0)$.

To gain further insight into the applicability of Theorem 2, we next consider several important examples.

1) Finite support: Let \mathfrak{S} be a finite set, $\mu \ll \nu$, and (X_1, \dots, X_n) and (Y_1, \dots, Y_n) be i.i.d. samples from μ and ν , respectively. Set the empirical distributions $\mu_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$ and $\nu_n = (n^{-1} - n^{-2}) \sum_{i=1}^n \delta_{Y_i} + n^{-1} \mu_n$ as the random measures, where we add a vanishing regularization term $n^{-1} \mu_n$ so that $\mu_n \ll \nu_n$ and $D_{\text{KL}}(\mu_n \parallel \nu_n) < \infty$. Clearly, $\mu_n \ll \nu_n \ll \nu$ and $\mu_n \ll \mu \ll \nu$. Notice also that all the other regularity conditions in Theorem 2 are satisfied. In particular, the relevant weak convergences of random measures to Gaussian limits follow from the multivariate CLT. Moreover, by Hoeffding's inequality, there exists a constant c such that $\mathbb{P}(\|p_{\mu_n}/p_{\nu_n}\|_\infty >$

$c) \rightarrow 0$. Hence, Remark 1 applies with $q = c$ implying that (11)-(14) hold with $r_n = n^{1/2}$ and $B, (B_1, B_2)$ as Gaussian vectors (and, of course, integral replaced by summation).

- 2) Compact support with smoothed empirical measures: Consider compactly supported and continuous distributions $\mu \ll \nu$ on \mathbb{R}^d with Lebesgue densities bounded from above and away from zero on the support. In this case, the vanilla empirical distributions as defined above are not absolutely continuous w.r.t. μ and ν , respectively. To resolve this and obtain a well-posed empirical approximation setting, we convolve the empirical distributions with a mollifier⁷ (or bump function). Let μ_n and ν_n denote the smoothed empirical distributions with Lebesgue densities $p_{\mu_n} = n^{-1} \sum_{i=1}^n \delta_{X_i} * \xi$ and $p_{\nu_n} = (n^{-1} - n^{-2}) \sum_{i=1}^n \delta_{Y_i} * \xi + n^{-1} p_{\mu_n}$, respectively, where ξ denotes a non-negative mollifier with $\|\xi\|_1 = 1$. It follows by CLT in L^r spaces (see Theorem 6 below) that the weak-convergence requirements in Theorem 2 hold with $\mu * \xi, \nu * \xi$ in place of μ, ν and with $r_n = n^{1/2}$ and $B, (B_1, B_2)$ as appropriate Gaussian processes indexed by a compact subset \mathfrak{S} of \mathbb{R}^d . Moreover, using concentration bounds for separable sub-Gaussian processes (see e.g., [66, Theorem 5.29]) indexed by compact \mathfrak{S} , one can show that there exists a constant c such that $\mathbb{P}(\|p_{\mu_n}/p_{\nu_n}\|_\infty > c) \rightarrow 0$. It can be verified that the other regularity conditions in Theorem 2 also hold, and consequently the result applies via Remark 1.
- 3) Unbounded support with Gaussian smoothing: In Proposition 2 below, we further specialize Theorem 2 to Gaussian-smoothed empirical distributions on \mathbb{R}^d . This provides an instance of smooth distributions with unbounded support, for which we characterize the limit laws and derive primitive conditions in terms of μ and ν for their existence.

We mention here that while Theorem 2 provides general regularity conditions for existence of limit distributions for KL divergence, certain assumptions therein are arguably stronger than what is necessary. For instance, the requirement $\mathbb{1}_{\text{spt}(p_\mu)} \log p_\mu \in L^2(\nu)$ for (12) to hold arises from Assumption 2, which furnishes sufficient conditions for the first order Hadamard derivatives to exist. However, from the proof of Proposition 1, it is evident that alternative sufficient conditions in lieu of Assumption 2 are plausible by setting up the function space \mathfrak{D} of perturbations via the norm $\|(g, \tilde{g})\|_{\mathfrak{D}} := \|g\|_{r, \eta_1} + \|\tilde{g}\|_{r, \eta_2}$ for some $r \geq 1$. Then, the relevant condition for existence of limit distribution on account of Lemma 4 would be $\mathbb{1}_{\text{spt}(p_\mu)} \log p_\mu \in L^{r'}(\nu)$, where r' is the Hölder conjugate of r . Thus, there is some flexibility possible in the regularity conditions required for (12) to hold. That said, we do not delve into this aspect further within this paper.

B. χ^2 divergence

Theorem 3 (Limit distribution for χ^2 divergence) *The following hold:*

- (i) (One-sample null) Let $\mu_n \ll \mu = \rho$ satisfy $\chi^2(\mu_n \| \mu) < \infty$ a.s. If $r_n(p_{\mu_n} - 1) \xrightarrow{w} B$ in $L^2(\mu)$, then

$$r_n^2 \chi^2(\mu_n \| \mu) \xrightarrow{d} \int_{\mathfrak{S}} B^2 d\mu. \quad (15)$$

⁷A mollifier is a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ which is both smooth (i.e., has continuous derivatives of all orders) and compactly supported, e.g., $\xi(x) = ce^{-1/(1-\|x\|^2)}$ for $\|x\| < 1$ and zero elsewhere for some normalizing constant c .

(ii) (One-sample alternative) Let $\mu, \mu_n \ll \nu = \rho$ satisfy $\chi^2(\mu \parallel \nu) < \infty$ and $\chi^2(\mu_n \parallel \nu) < \infty$ a.s. If $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\nu)$, then

$$r_n(\chi^2(\mu_n \parallel \nu) - \chi^2(\mu \parallel \nu)) \xrightarrow{d} 2 \int_{\mathfrak{S}} B d\mu. \quad (16)$$

(iii) (Two-sample null) Let $\mu_n \ll \nu_n \ll \mu = \rho$ be such that $\chi^2(\mu_n \parallel \nu_n) < \infty$ and $p_{\mu_n}/p_{\nu_n} \leq q$ a.s. Let η_1 and η_2 be measures with relative densities $p_{\eta_1} = 1 + q$ and $p_{\eta_2} = p_{\eta_1} + q^2$, respectively. If $(r_n(p_{\mu_n} - 1), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in $L^2(\eta_1) \times L^2(\eta_2)$, then

$$r_n^2 \chi^2(\mu_n \parallel \nu_n) \xrightarrow{d} \int_{\mathfrak{S}} (B_1 - B_2)^2 d\mu. \quad (17)$$

(iv) (Two-sample alternative) Let $\mu_n \ll \nu_n \ll \nu = \rho$ and $\mu_n \ll \mu \ll \nu$ satisfy $p_\mu \in L^4(\nu)$, $\chi^2(\mu \parallel \nu) < \infty$, $\chi^2(\mu_n \parallel \nu_n) < \infty$ and $p_{\mu_n}/p_{\nu_n} \leq q$ a.s. Let η_1 and η_2 be measures with relative densities $p_{\eta_1} = 1 + p_\mu + q$ and $p_{\eta_2} = p_{\eta_1} + q^2 + p_\mu^2$, respectively. If $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in $L^2(\eta_1) \times L^2(\eta_2)$, then

$$r_n(\chi^2(\mu_n \parallel \nu_n) - \chi^2(\mu \parallel \nu)) \xrightarrow{d} 2 \int_{\mathfrak{S}} B_1 d\mu - \int_{\mathfrak{S}} B_2 p_\mu d\mu. \quad (18)$$

Theorem 3 utilizes similar proof methodology as in Theorem 2 via the machinery from Section III, except that Parts (i) and (ii) follow in a simpler manner via the continuous mapping theorem [25, Theorem 1.3.6] owing to the specific structure of the χ^2 divergence functional. Also note that the one- and two-sample null limits of the χ^2 and KL divergences are the same up to a factor of 0.5, which is in line with the fact that KL divergence is locally χ^2 (see [29, Theorem 7.18]). Via an argument similar to that in Remark 1, we observe that Theorem 3 applies to discrete and smoothed compactly supported distributions (as discussed after Theorem 2 above), with $r_n = n^{1/2}$ and Gaussian B , (B_1, B_2) . An application to Gaussian-smoothed empirical distributions will be given in Proposition 6 in Appendix E-A.

C. H^2 distance

Theorem 4 (Limit distribution for H^2 distance) *The following hold:*

(i) (One-sample null) Let $\mu_n, \mu \ll \rho$ for some finite measure ρ such that $p_\mu > 0$. If $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\eta)$, where η has relative density $p_\eta = 1 + (1/p_\mu)$, then

$$r_n^2 H^2(\mu_n, \mu) \xrightarrow{d} \frac{1}{4} \int_{\mathfrak{S}} \frac{B^2}{p_\mu} d\rho. \quad (19)$$

(ii) (One-sample alternative) Let $\mu_n, \mu, \nu \ll \rho$ for some finite measure ρ such that $p_\mu, p_\nu > 0$, and suppose that $p_\nu/p_\mu \in L^1(\rho)$. If $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\eta)$, where η has relative density $p_\eta = 1 + (p_\nu^{1/2}/p_\mu^{3/2})$, then

$$r_n(H^2(\mu_n, \nu) - H^2(\mu, \nu)) \xrightarrow{d} - \int_{\mathfrak{S}} \left(\frac{p_\nu}{p_\mu} \right)^{\frac{1}{2}} B d\rho. \quad (20)$$

(iii) (Two-sample null) Let $\mu_n, \nu_n, \mu \ll \rho$ for some finite measure ρ such that $p_\mu > 0$, $p_{\mu_n} \leq q_1$ and $p_{\nu_n} \leq q_2$ a.s. Let η_1 and η_2 be measures with relative densities $p_{\eta_1} = 1 + (1/p_\mu) + (q_2^{1/2}/p_\mu^{3/2})$ and $p_{\eta_2} = 1 + (1/p_\mu) +$

$(q_1^{1/2}/p_\mu^{3/2})$, respectively. If $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\mu)) \xrightarrow{w} (B_1, B_2)$ in $L^2(\eta_1) \times L^2(\eta_2)$, then

$$r_n^2 H^2(\mu_n, \nu_n) \xrightarrow{d} \frac{1}{4} \int_{\mathfrak{S}} \frac{(B_1 - B_2)^2}{p_\mu} d\rho. \quad (21)$$

(iv) (Two-sample alternative) Let $\mu_n, \nu_n, \mu, \nu \ll \rho$ for some finite measure ρ such that $p_\mu, p_\nu > 0$, $p_\nu/p_\mu, p_\mu/p_\nu \in L^1(\rho)$, $p_{\mu_n} \leq q_1$ and $p_{\nu_n} \leq q_2$ a.s. Let η_1 and η_2 be measures with relative densities

$$\begin{aligned} p_{\eta_1} &= 1 + \left(p_\nu^{\frac{1}{2}} + q_2^{\frac{1}{2}}\right) p_\mu^{-\frac{3}{2}} + p_\mu^{-\frac{1}{2}} p_\nu^{-\frac{1}{2}}, \\ p_{\eta_2} &= 1 + \left(p_\mu^{\frac{1}{2}} + q_1^{\frac{1}{2}}\right) p_\nu^{-\frac{3}{2}} + p_\mu^{-\frac{1}{2}} p_\nu^{-\frac{1}{2}}, \end{aligned}$$

respectively. If $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\nu)) \xrightarrow{w} (B_1, B_2)$ in $L^2(\eta_1) \times L^2(\eta_2)$, then

$$r_n(H^2(\mu_n, \nu_n) - H^2(\mu, \nu)) \xrightarrow{d} - \int_{\mathfrak{S}} \left(\frac{p_\nu}{p_\mu}\right)^{\frac{1}{2}} B_1 d\rho - \int_{\mathfrak{S}} \left(\frac{p_\mu}{p_\nu}\right)^{\frac{1}{2}} B_2 d\rho. \quad (22)$$

The proof of Theorem 4 is similar to that of Theorem 2, and is given in Section VII-B3. Note that in contrast to Theorem 2 and 3, the two-sample alternative limit distribution is symmetric in μ and ν , which is to be expected given the symmetry of $H^2(\mu, \nu)$ itself. Also, observe that the regularity conditions in Theorem 4 are satisfied for the case of discrete (when $\mu \ll \nu$) and compactly supported smoothed empirical distributions considered in Section IV-A. Consequently, (19)-(22) hold with $r_n = n^{1/2}$ and Gaussian B , (B_1, B_2) . In Proposition 7 in Appendix E-B, we will further discuss the application of Theorem 4 to Gaussian-smoothed empirical distributions.

D. TV distance

Theorem 5 (Limit distribution for TV distance) Let $\text{sgn}(x) = x/|x|$ for $x \neq 0$, and for given $\mu, \nu \ll \rho$, let $\mathcal{Q} := \{s \in \mathfrak{S} : p_\mu(s) = p_\nu(s)\}$. Then, the following hold:

(i) (One-sample null and alternative) Let $\mu_n, \mu, \nu \ll \rho$ for some measure ρ . If $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^1(\rho)$, then

$$r_n(\|\mu_n - \nu\|_{\text{TV}} - \|\mu - \nu\|_{\text{TV}}) \xrightarrow{d} \frac{1}{2} \int_{\mathcal{Q}} |B| d\rho + \frac{1}{2} \int_{\mathfrak{S} \setminus \mathcal{Q}} \text{sgn}(p_\mu - p_\nu) B d\rho. \quad (23)$$

(ii) (Two-sample null and alternative) Let $\mu_n, \nu_n, \mu, \nu \ll \rho$ for some measure ρ . If⁸ $r_n(p_{\mu_n} - p_{\nu_n} - p_\mu + p_\nu) \xrightarrow{w} B$ in $L^1(\rho)$, then

$$r_n(\|\mu_n - \nu_n\|_{\text{TV}} - \|\mu - \nu\|_{\text{TV}}) \xrightarrow{d} \frac{1}{2} \int_{\mathcal{Q}} |B| d\rho + \frac{1}{2} \int_{\mathfrak{S} \setminus \mathcal{Q}} \text{sgn}(p_\mu - p_\nu) B d\rho. \quad (24)$$

The proof for the one-sample and two-sample null cases above follow via an application of the continuous mapping theorem. The proof of the alternative requires an argument different from the divergences considered until now. To see why, recall from (2) that TV distance corresponds to an f -divergence with $f = |x - 1|/2$, which is not differentiable at $x = 1$. Hence, Proposition 1 does not apply for computing the Hadamard derivative

⁸Alternatively, we may state the weak convergence requirement in terms of the stronger condition $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\nu)) \xrightarrow{w} (B_1, B_2)$ which implies $r_n(p_{\mu_n} - p_{\nu_n} - p_\mu + p_\nu) \xrightarrow{w} B$ with $B = B_1 - B_2$.

since Assumption 1 is violated. However, utilizing the equivalence of Hadamard differentiability to Gâteaux differentiability under Lipschitzness of the functional (see [23]), we instead compute the latter simpler quantity via a direct argument. From there, the claims follow as usual via an application of the functional delta method. We note that Theorem 5 applies to the case of discrete and compactly supported smoothed empirical distributions in Section IV-A. Its application to Gaussian-smoothed case is discussed in Appendix E-C (Proposition 8).

V. LIMIT DISTRIBUTION THEORY FOR GAUSSIAN-SMOOTHED f -DIVERGENCES

We study distributional limits of Gaussian-smoothed f -divergence, i.e., the population objective is $D_f(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)$, where $\gamma_\sigma = N(0, \sigma^2 I_d)$ is the isotropic Gaussian kernel [67]. Our goal is to approximate μ (or both μ and ν) from samples, while assuming that the Gaussian kernel is known. The Gaussian smoothing alleviates mismatched support issues that f -divergences often suffer from and gives rise to a well-posed empirical approximation setting. This setup was studied in our preliminary work [1] for the χ^2 divergence and the TV distance under the one-sample null setting. The results herein significantly generalize and broaden those of [1]. We focus on the Gaussian-smoothed KL divergence in this section and relegate analogous results for χ^2 divergence, H^2 distance and TV distance to Appendix E for brevity. Throughout this section, we assume $\mathfrak{S} = \mathbb{R}^d$ and $\mathcal{S} = \mathcal{B}(\mathbb{R}^d)$. Some preliminaries are due before stating the results.

Empirical measures: In defining the empirical measures of μ and ν we allow arbitrary correlation between their samples, which is necessary for the application to auditing DP studied in Section VI. Let $(X, Y) \sim \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with X, Y marginals μ, ν , respectively. Set $\hat{\mu}_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$ as the empirical distribution of (X_1, \dots, X_n) and $\hat{\nu}_n = n^{-1} \sum_{i=1}^n \delta_{Y_i}$ be that of (Y_1, \dots, Y_n) , where $(X_i, Y_i) \sim \pi$, $1 \leq i \leq n$, are i.i.d. Recalling that φ_σ denotes the density of γ_σ , the Lebesgue densities of $\hat{\mu}_n * \gamma_\sigma$ and $\hat{\nu}_n * \gamma_\sigma$ are $\hat{\mu}_n * \varphi_\sigma$ and $\hat{\nu}_n * \varphi_\sigma$, respectively. We study distributional limits of $D_f(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)$ as well as its two-sample analogues, under the null ($\mu = \nu$) and the alternative ($\mu \neq \nu$).

Gaussian process: Our limit variables are characterized as integral forms of a certain Gaussian process, which is introduced next. Consider the 2-dimensional centered Gaussian process $(G_{\mu, \sigma}, G_{\nu, \sigma}) := (G_{\mu, \sigma}(x), G_{\nu, \sigma}(y))_{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d}$ with covariance function $\Sigma_{\mu, \nu, \sigma} : (\mathbb{R}^d \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}^{2 \times 2}$ given by

$$\begin{aligned} \Sigma_{\mu, \nu, \sigma}((x, y), (\tilde{x}, \tilde{y})) &:= \begin{bmatrix} \mathbb{E}[G_{\mu, \sigma}(x)G_{\mu, \sigma}(\tilde{x})] & \mathbb{E}[G_{\mu, \sigma}(x)G_{\nu, \sigma}(\tilde{y})] \\ \mathbb{E}[G_{\nu, \sigma}(y)G_{\mu, \sigma}(\tilde{x})] & \mathbb{E}[G_{\nu, \sigma}(y)G_{\nu, \sigma}(\tilde{y})] \end{bmatrix} \\ &= \begin{bmatrix} \text{cov}(\varphi_\sigma(x - X), \varphi_\sigma(\tilde{x} - X)) & \text{cov}(\varphi_\sigma(x - X), \varphi_\sigma(\tilde{y} - Y)) \\ \text{cov}(\varphi_\sigma(y - Y), \varphi_\sigma(\tilde{x} - X)) & \text{cov}(\varphi_\sigma(y - Y), \varphi_\sigma(\tilde{y} - Y)) \end{bmatrix}. \end{aligned} \quad (25)$$

For $i, j \in \{1, 2\}$, denote the (i, j) -th entry of $\Sigma_{\mu, \nu, \sigma}$ by $\Sigma_{\mu, \nu, \sigma}^{(i, j)}$. Note that each such entry depends only on two coordinates among $((x, y), (\tilde{x}, \tilde{y}))$. Hence, by some abuse of notation, we omit the redundant coordinates and use the remaining coordinates in the same order they appear, e.g., $\Sigma_{\mu, \nu, \sigma}^{(2, 1)}(y, \tilde{x})$ for $\Sigma_{\mu, \nu, \sigma}^{(2, 1)}((x, y), (\tilde{x}, \tilde{y}))$. Further, when $\nu = \mu$ (and consequently $X \stackrel{d}{=} Y$), we denote $G_{\nu, \sigma}$ by $\tilde{G}_{\mu, \sigma}$ to avoid confusion with $G_{\mu, \sigma}$.

Asymptotic variance: When the limiting variable is Gaussian, the asymptotic variance for different f -divergences can be expressed in a unified form, using the following definitions. For a function f corresponding to an f -divergence (see Definition 1), let

$$v_{1,f}^2(\mu, \nu, \sigma) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(1,1)}(x, y) L_{1,f}(x) L_{1,f}(y) dx dy,$$

$$v_{2,f}^2(\mu, \nu, \sigma) := \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(i,j)}(x, y) L_{i,f}(x) L_{j,f}(y) dx dy,$$

where

$$L_{1,f}(x) := L_{1,f,\mu,\nu,\sigma}(x) := f' \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right),$$

$$L_{2,f}(x) := L_{2,f,\mu,\nu,\sigma}(x) := f \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right) - \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} f' \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right).$$

Here, f' denotes the derivative of f and $v_{1,f}^2(\mu, \nu, \sigma)$, $v_{2,f}^2(\mu, \nu, \sigma)$ will be well-defined and finite in the settings we consider below. The quantities give the asymptotic variance of Gaussian-smoothed f -divergence (except TV distance) in the one-sample and two-sample alternative cases, respectively.

A. KL Divergence

The following proposition characterizes the limit distribution for Gaussian-smoothed KL divergence.

Proposition 2 (Limit distribution for Gaussian-smoothed KL divergence) *The following hold:*

(i) (One-sample null) If

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx < \infty, \quad (26)$$

then there exists a version of $G_{\mu,\sigma}$ such that $G_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ is $L^2(\mathbb{R}^d)$ -valued, and

$$nD_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma) \xrightarrow{d} \frac{1}{2} \int_{\mathbb{R}^d} \frac{G_{\mu,\sigma}^2(x)}{\mu * \varphi_\sigma(x)} dx, \quad (27)$$

where the limit can be represented as a weighted sum of i.i.d. χ^2 random variables with 1 degree of freedom (see Remark 2). In particular, (26) and (27) holds for β -sub-Gaussian μ with $\beta < \sigma$. Conversely, if (26) is violated, then we have $\liminf_{n \rightarrow \infty} n\mathbb{E}[D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] = \infty$.

(ii) (One-sample alternative) If (26) holds, $\|(\nu * \varphi_\sigma)^2 / \mu * \varphi_\sigma\|_\infty < \infty$, $\log(\mu * \varphi_\sigma / \nu * \varphi_\sigma) \in L^2(\nu * \varphi_\sigma)$ and

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx < \infty, \quad (28)$$

then

$$n^{\frac{1}{2}}(D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) - D_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)) \xrightarrow{d} N(0, v_{1,f_{\text{KL}}}^2(\mu, \nu, \sigma)), \quad (29)$$

where $v_{1,f_{\text{KL}}}^2(\mu, \nu, \sigma)$ is given in (58). In particular, (29) holds for β -sub-Gaussian μ with $\beta < \sigma$ such that $\nu \ll \mu \ll \nu$ and $\|d\mu/d\nu\|_\infty \vee \|d\nu/d\mu\|_\infty < \infty$.

(iii) (Two-sample null) If μ has compact support, then there exists a version of $G_{\mu,\sigma}$, $\tilde{G}_{\mu,\sigma}$ such that $G_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ and $\tilde{G}_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ are $L^2(\mathbb{R}^d)$ -valued, and

$$nD_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) \xrightarrow{d} \frac{1}{2} \int_{\mathbb{R}^d} \frac{(G_{\mu,\sigma}(x) - \tilde{G}_{\mu,\sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx, \quad (30)$$

where the limit can be represented as a weighted sum of i.i.d. χ^2 random variables with 1 degree of freedom.

(iv) (Two-sample alternative) If μ, ν have compact supports, then

$$n^{\frac{1}{2}} (D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - D_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)) \xrightarrow{d} N(0, v_{2,\text{fKL}}^2(\mu, \nu, \sigma)), \quad (31)$$

where $v_{2,\text{fKL}}^2(\mu, \nu, \sigma)$ is given in (69).

The proof of bulk of the claims in Proposition 2 hinges on Theorem 2 by identifying primitive conditions in terms of μ, ν , and σ that guarantee the regularity assumptions in Theorem 2. The proof of the final claim in Part (i) above, i.e., the necessity of Condition (26), relies on the following lemma, which is also applicable to other f -divergences with a twice differentiable f .

Lemma 2 (Lower bound on expected f -divergence limit) *Consider the f -divergence in Definition 1. Assume that f is continuously twice differentiable in $(0, \infty)$ with a nonnegative second derivative f'' . Then,*

$$\liminf_{n \rightarrow \infty} n\mathbb{E}[D_f(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] \geq \frac{f''(1)}{2} \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx. \quad (32)$$

The proof of Lemma 2 is given in Section VII-C2, and is based on Taylor's theorem.

Remark 2 (Simplified null limit distribution) *The RHS of (27) and (30) admit a representation as a weighted sum of i.i.d. χ^2 random variables (with 1 degree of freedom). This follows since centered $L^2(\mathbb{R}^d)$ -valued Gaussian random variables admit a representation of the form $\sum_{i \in \mathbb{N}} W_i b_i$, where $(W_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of standard Gaussian (real) random variables, and $(b_i)_{i \in \mathbb{N}}$ is an orthonormal basis of the reproducing kernel Hilbert space (see [68, Theorem 4.3]) associated with the $L^2(\mathbb{R}^d)$ -valued Gaussian variable. Hence, the RHS of (27) and (30) can be represented as $\sum_{i \in \mathbb{N}} \|b_i\|_2^2 W_i^2/2$, from which the claim follows. The same conclusion also applies to the other divergences, except for TV distance.*

Remark 3 (Unequal sample sizes) *While we consider equal number of samples from both μ and ν for simplicity, the results readily extend to the mismatched scenario. Suppose n and m denote the number of samples from μ and ν , respectively, such that $m/(n+m) \rightarrow \tau$ for some $\tau \in (0, 1)$ as $m, n \rightarrow \infty$. By minor modifications to the proof of (29) and (31), one may verify that under the same respective conditions, we have*

$$\frac{nm}{n+m} D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_m * \gamma_\sigma) \xrightarrow{d} \frac{1}{2} \int_{\mathbb{R}^d} \frac{(\sqrt{\tau} G_{\mu,\sigma}(x) - \sqrt{1-\tau} \tilde{G}_{\mu,\sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx, \quad (33)$$

and

$$\left(\frac{nm}{n+m}\right)^{\frac{1}{2}} \left(D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_m * \gamma_\sigma) - D_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma) \right) \xrightarrow{d} N(0, \bar{v}_{2, \text{fKL}}^2(\tau, \mu, \nu, \sigma)),$$

where $\bar{v}_{2, \text{fKL}}^2(\tau, \mu, \nu, \sigma) := \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c_{i,j}(\tau) \Sigma_{\mu, \nu, \sigma}^{(i,j)}(x, y) L_{i, \text{fKL}}(x) L_{j, \text{fKL}}(y) dx dy$, with $c_{1,1}(\tau) = \tau$, $c_{1,2}(\tau) = c_{2,1}(\tau) = \sqrt{\tau(1-\tau)}$, and $c_{2,2}(\tau) = 1 - \tau$. Similar claims hold for other f -divergences. Also, note that Remark 2 applies to (33) as $(\sqrt{\tau} G_{\mu, \sigma} - \sqrt{1-\tau} \tilde{G}_{\mu, \sigma}) / \sqrt{\mu * \varphi_\sigma}$ is an $L^2(\mathbb{R}^d)$ -valued Gaussian random variable.

1) Bootstrap consistency: The limit distribution from Theorem 2 are non-pivotal in the sense that they depend on the population distributions μ and ν , which are unknown in practice. To circumvent this difficulty and enable inference (e.g., construction of confidence intervals) we apply the bootstrap, which is a computationally tractable resampling method for estimating distributional limits. This section establishes consistency of the bootstrap for the Gaussian-smoothed KL divergence.

Given the data $(X_1, Y_1), \dots, (X_n, Y_n)$, let $(X_1^B, \dots, X_n^B) \sim \hat{\mu}_n^{\otimes n}$ and $(Y_1^B, \dots, Y_n^B) \sim \hat{\nu}_n^{\otimes n}$ denote bootstrap samples drawn independently from empirical distributions $\hat{\mu}_n$ and $\hat{\nu}_n$, respectively. Set $\hat{\mu}_n^B = n^{-1} \sum_{i=1}^n \delta_{X_i^B}$ and $\hat{\nu}_n^B = n^{-1} \sum_{i=1}^n \delta_{Y_i^B}$ as the bootstrap empirical distributions. Denoting by \mathbb{P}_B the conditional probability given $(X_1, Y_1), \dots, (X_n, Y_n)$, we have the following bootstrap consistency claim for the Gaussian-smoothed KL divergence limit distribution from Proposition 2.

Corollary 1 (Bootstrap consistency for KL divergence limit distribution) *Consider the setting of Proposition 2, and let $v_{1, \text{fKL}}^2(\mu, \nu, \sigma)$ and $v_{2, \text{fKL}}^2(\mu, \nu, \sigma)$ be as given in (58) and (69), respectively. Then, the following hold:*

(i) *(One-sample null and alternative) Under the conditions of Proposition 2(ii),*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}_B \left(n^{\frac{1}{2}} \left(D_{\text{KL}}(\hat{\mu}_n^B * \gamma_\sigma \| \nu * \gamma_\sigma) - D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) \right) \leq t \right) - \mathbb{P}(W_1 \leq t) \right| = o_{\mathbb{P}}(1),$$

where $W_1 \sim N(0, v_{1, \text{fKL}}^2(\mu, \nu, \sigma))$.

(ii) *(Two-sample null and alternative) If μ, ν have compact supports, then*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}_B \left(n^{\frac{1}{2}} \left(D_{\text{KL}}(\hat{\mu}_n^B * \gamma_\sigma \| \hat{\nu}_n^B * \gamma_\sigma) - D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) \right) \leq t \right) - \mathbb{P}(W_2 \leq t) \right| = o_{\mathbb{P}}(1),$$

where $W_2 \sim N(0, v_{2, \text{fKL}}^2(\mu, \nu, \sigma))$.

The proof follows from [69, Theorem 23.9] and the linearity of the first order Hadamard derivative of the KL divergence functional (see Proposition 1), once we verify that the tangent cone (domain of the Hadamard derivative) contains a non-trivial linear subspace. The details are provided in Section VII-C3. Bootstrap consistency results for other f -divergences (except TV distance) may be derived in an analogous manner, but we skip the formal statements below for brevity.

VI. APPLICATION TO AUDITING DIFFERENTIAL PRIVACY

We consider the application of our limit distribution theory to auditing DP, which was introduced in [70] as an approach for quantifying privacy leakage of privatization mechanisms. We recall some DP notions that are relevant

to our setting. Consider a set \mathcal{U} with a relation \sim such that $u \sim v$, for $u, v \in \mathcal{U}$, denotes that u and v are adjacent. In the DP context, \mathcal{U} is a set of databases and $u \sim v$ denotes that u and v are adjacent databases, differing on a single entry.

Definition 4 (DP mechanisms [70]–[72]) *Let $\epsilon, \delta \geq 0$. A randomized (measurable) mechanism $f : \mathcal{U} \rightarrow \mathbb{R}^d$ is*

- (i) *ϵ -differentially private if $\mathbb{P}(f(u) \in \mathcal{T}) \leq e^\epsilon \mathbb{P}(f(v) \in \mathcal{T})$ for every $u \sim v$ and $\mathcal{T} \in \mathcal{B}(\mathbb{R}^d)$;*
- (ii) *(ϵ, δ) -differentially private if $\mathbb{P}(f(u) \in \mathcal{T}) \leq e^\epsilon \mathbb{P}(f(v) \in \mathcal{T}) + \delta$ for every $u \sim v$ and $\mathcal{T} \in \mathcal{B}(\mathbb{R}^d)$;*
- (iii) *ϵ -KL differentially private if $D_{\text{KL}}(\mu_u \| \mu_v) \leq \epsilon$ for every $u \sim v$, where $\mu_u \in \mathcal{P}(\mathbb{R}^d)$ is the distribution of $f(u)$.*

In addition, we say that a privacy mechanism is ϵ -smoothed KL differentially private if $D_{\text{KL}}(\mu_u * \gamma_\sigma \| \mu_v * \gamma_\sigma) \leq \epsilon$ for every $u \sim v$, where $\sigma > 0$ is a pre-specified parameter.

Standard noise-injection mechanisms for DP operate as follows. Consider $g : \mathcal{U} \rightarrow \mathbb{R}^d$, a (deterministic) query function to be published about the database and define its ℓ^r -sensitivity as $\Delta_r(g) := \sup_{u, v \in \mathcal{U}: u \sim v} \|g(u) - g(v)\|_r$. A noise-injection DP mechanism is given by $f(\cdot) = g(\cdot) + W$, where W is an additive noise random variable. For instance, the Laplace mechanism [73] takes $W \sim \text{Lap}(0, b)^{\otimes d}$ (i.e., whose Lebesgue density is $\propto e^{-\|\cdot\|_1/b}$) with $b \geq \Delta_1(g)/\epsilon$, which guarantees ϵ -DP. Similarly, the Gaussian mechanism sets $W \sim \gamma_\sigma$ with $\sigma \geq \Delta_2(g)\sqrt{2\log(1.25/\delta)}/\epsilon$, which guarantees (ϵ, δ) -DP. While the above DP mechanisms add unbounded noise, a privacy mechanism that adds bounded noise in an adaptive query setting and ensures (ϵ, δ) -DP (asymptotically) for any $\delta > 0$ is proposed in [74]. As ϵ -DP is equivalent to $\sup_{u \sim v} D_\infty(\mu_u \| \mu_v) \leq \epsilon$, where D_∞ is the ∞ -order Renyi divergence, it is clear that KL DP is a relaxation of DP.⁹ By the data processing inequality, we further have that smoothed KL DP is a relaxation of KL DP.

In practice, given output samples from a privacy mechanism, one encounters the problem of ascertaining whether the mechanism is differentially private or not, referred to as auditing DP. In [28], a hypothesis test for auditing DP using regularized kernel Rényi divergence is proposed, where the null hypothesis is that the mechanism satisfies (ϵ, δ) -DP. The authors propose a decision rule achieving any non-zero significance level (type I error probability), leaving the characterization of the power (equivalently, type II error probability) open. Here, utilizing the limit distribution theory from Section V, we put forth a principled hypothesis testing pipeline for DP auditing using the Gaussian-smoothed and classic KL divergence as the privacy measures of interest. Our analysis accounts for both significance and power of the test. We start from the smoothed KL DP test.

A. Smoothed KL DP test

The main objective of a privacy audit is to identify violations. For that reason, we set up an hypothesis test where the null H_0 corresponds to when privacy holds, and consider a sequence of local alternatives $H_{1,n}$ that become harder to distinguish from H_0 as n grows. This models a situation where the alternative hypothesis is arbitrarily close to the null, and we seek a powerful test that successfully rejects the null, even under these local alternatives.

⁹In particular, ϵ -DP implies $\epsilon(1 \wedge (e^\epsilon - 1))$ -KL DP by [73, Lemma 3.18].

To define the local alternatives, we consider a sequence of privacy mechanisms that violate ϵ -smoothed KL DP by an $O(n^{-1/2})$ amount.

Fix $\sigma, \epsilon, b, C > 0$ and, for $n \in \mathbb{N}_0$, let $f_n : \mathcal{U} \rightarrow \mathcal{I}_b := [-b, b]^d$ be a sequence of privacy mechanisms. Denote a pair adjacent databases¹⁰ by $(U, V) \sim \tilde{\pi} \in \mathcal{P}(\mathcal{U} \times \mathcal{U})$. Let $\pi_n := (f_n, f_n)_{\#} \tilde{\pi}$ be the joint distribution of $(f_n(U), f_n(V))$, where $\#$ is the pushforward operation. The first and second marginals of π_n are denoted by μ_n and ν_n , respectively. We impose the following assumption on the sequence $(\pi_n)_{n \in \mathbb{N}_0}$.

Assumption 5 *The sequence $(\pi_n)_{n \in \mathbb{N}_0}$ is such that*

(i) *there exists $0 \neq h \in L^2(\pi_0)$ with $nH^2(\pi_n, \pi_0) \rightarrow \|h/2\|_{2, \pi_0}^2$, $\int_{\mathbb{R}^d \times \mathbb{R}^d} h d\pi_0 = 0$, and*

$$\begin{aligned} & \left(n^{1/2} \left(\frac{\mu_n * \varphi_\sigma - \mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} \right), n^{1/2} \left(\frac{\nu_n * \varphi_\sigma}{\nu_0 * \varphi_\sigma} - 1 \right) \right) \\ & \rightarrow \left(\frac{\mathbb{E}_{\pi_0} [h(X, Y) \varphi_\sigma(\cdot - X)]}{\nu_0 * \varphi_\sigma}, \frac{\mathbb{E}_{\pi_0} [h(X, Y) \varphi_\sigma(\cdot - Y)]}{\nu_0 * \varphi_\sigma} \right) \text{ in } L^\infty(\lambda) \times L^\infty(\lambda). \end{aligned} \quad (34)$$

(ii) $D_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) \leq \epsilon$ and $D_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) \geq \epsilon_{n,C} := \epsilon + Cn^{-1/2}$ for all n sufficiently large.

Observe that Assumption 5(ii) implies that f_0 satisfies ϵ -smoothed KL DP while f_n violates it for all n sufficiently large. On the other hand, Assumption 5(i) is a technical requirement which ensures that Gaussian-smoothed KL divergence limit theorems relevant for our purpose continue to hold in the local alternatives setting in which π_n changes with n . Proposition 4 below presents an explicit construction of $(\pi_n)_{n \in \mathbb{N}_0}$ that satisfies Assumption 5 for any $\sigma, \epsilon, b, C > 0$. For now, under this assumption, consider the following binary hypothesis test with a sequence of alternatives:

$$H_0 : D_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) \leq \epsilon, \quad (35)$$

$$H_{1,n} : D_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) \geq \epsilon_{n,C}.$$

Let $(X_1, Y_1), \dots, (X_n, Y_n) \sim \pi$ be pairwise i.i.d. samples of the privacy mechanism's output when acting on i.i.d. pairs of adjacent databases, where $\pi = \pi_0$ under H_0 and $\pi = \pi_n$ under $H_{1,n}$. Denote the empirical measures of (X_1, \dots, X_n) and (Y_1, \dots, Y_n) by $\hat{\mu}_n$ and $\hat{\nu}_n$, respectively. For a test statistic $T_n = T_n(X_1, \dots, X_n, Y_1, \dots, Y_n)$, a standard class of tests rejects H_0 if $T_n > t_n$, where t_n is a critical value chosen according to the desired level $\tau \in (0, 1)$. The operational meaning of rejecting H_0 is declaring that ϵ -smoothed KL DP is violated, and hence, also ϵ -DP itself. We say that such a sequence has *asymptotic level* τ if $\limsup_{n \rightarrow \infty} \mathbb{P}(T_n > t_n | H_0) \leq \tau$. The *power* of a test is the probability that it correctly rejects H_0 , i.e., $\mathbb{P}(T_n > t_n | H_{1,n})$, and the *asymptotic power* is $\liminf_{n \rightarrow \infty} \mathbb{P}(T_n > t_n | H_{1,n})$. Lastly, the sequence of tests is called *asymptotically consistent* if its asymptotic power is 1. The above definitions specialize to the case of a fixed alternative H_1 by taking $H_{1,n} = H_1$ and $\pi_n = \pi_1$ for all $n \in \mathbb{N}$.

¹⁰The results in this section do not depend on the database distribution $\tilde{\pi}$ per say. Also, note that the current model subsumes the case of deterministic (U, V) by taking $\tilde{\pi}$ to be a point mass on a pair of adjacent databases. In this case, the randomness in $(f_n(U), f_n(V))$ only comes from the mechanism.

For $\tau \in [0, 1]$, let $Q^{-1}(\tau) = \inf \{z \in \mathbb{R} : (2\pi)^{-1/2} \int_z^\infty e^{-u^2/2} du \leq \tau\}$ be the inverse complimentary cumulative distribution function of $N(0, 1)$. The following proposition provides a test statistic for the above hypothesis test and characterizes its asymptotic level and asymptotic power against local alternatives.

Proposition 3 (Smoothed KL DP audit) *Suppose Assumption 5 holds. For $0 < \tau, \tau' \leq 1$, let $C_{b,d,\sigma,\tau,\tau'} = c_{b,d,\sigma}(Q^{-1}(\tau) - Q^{-1}(1 - \tau'))$, where $c_{b,d,\sigma}$ is given in (79). Then the test statistic $T_n = D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)$ with critical value $t_n = \epsilon + c_{b,d,\sigma}Q^{-1}(\tau)n^{-1/2}$ achieves an asymptotic level τ and asymptotic power at least $1 - \tau'$ for the test in (35), whenever $C > C_{b,d,\sigma,\tau,\tau'} \vee 0$.*

The proof of the above claim relies on the limit distribution result for smoothed KL divergence given in (31) along with its refinement to account for the local alternatives scenario, i.e., to account for a sequence of distributions $(\mu_n * \gamma_\sigma, \nu_n * \gamma_\sigma)_{n \in \mathbb{N}}$ instead of a fixed one. This refinement (see (76)) is derived under Assumption 5 by invoking Le Cam's third lemma [25, Theorem 3.10.7]. Given these results and the fact that the relevant limit distributions are Gaussian, the claim in Proposition 3 follows by an analysis of the asymptotic level and power via the Portmanteau theorem [25, Theorem 1.3.4]. Note that the constant $C_{b,d,\sigma,\tau,\tau'}$ is positive whenever $\tau + \tau' < 1$, which is when the requirement $C > C_{b,d,\sigma,\tau,\tau'}$ is active. Operationally, $\tau + \tau' < 1$ means that the sum of type I and type II error probabilities is less than 1, which is the interesting regime for hypothesis testing; otherwise, a test based on a random coin flip is preferable.

We conclude this part by providing an explicit construction of a sequence of couplings $(\pi_n)_{n \in \mathbb{N}_0}$ that satisfies Assumption 5.

Proposition 4 (Construction satisfying Assumption 5) (i) *Let $\pi_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be such that $\mu_0 \neq \nu_0$, $\nu_0 \otimes \mu_0 \ll \pi_0$, $\|d(\nu_0 \otimes \mu_0)/d\pi_0\|_{\infty, \pi_0} < \infty$ and $\|h_{\pi_0, \bar{c}}\|_{2, \pi_0} < \infty$, where*

$$h_{\pi_0, \bar{c}} := \bar{c} \left(\frac{d(\mu_0 \otimes \nu_0)}{d\pi_0} - \frac{d(\nu_0 \otimes \mu_0)}{d\pi_0} \right), \quad (36)$$

and $\bar{c} > 0$ is an arbitrary constant. Let $\pi_n \ll \pi_0$ be the probability measure specified by the relative density

$$\frac{d\pi_n}{d\pi_0} = 1 + n^{-\frac{1}{2}} h_{\pi_0, \bar{c}}, \quad (37)$$

whenever the RHS is non-negative π_0 -a.s.; otherwise, set $\pi_n = \pi_0$. Then, π_n satisfies Assumption 5(i) with $h = h_{\pi_0, \bar{c}}$.

(ii) *Let $\pi_0 \in \mathcal{P}(\mathcal{I}_b \times \mathcal{I}_b)$ and σ be such that $\mu_0 \neq \nu_0$, $\nu_0 \otimes \mu_0 \ll \pi_0$, $\|d(\nu_0 \otimes \mu_0)/d\pi_0\|_{\infty, \pi_0} \vee \|d(\mu_0 \otimes \nu_0)/d\pi_0\|_{2, \pi_0} < \infty$ and $D_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) = \epsilon$. Then, there exists a sufficiently large \bar{c} , such that π_n as defined in (37) satisfies Assumption 5 with $h = h_{\pi_0, \bar{c}}$ for any $C > 0$.*

Proposition 4(ii), which is proven in Section VII-E, provides a method of constructing π_n for the hypothesis test in (35), given π_0 that satisfies the aforementioned regularity assumptions. This can be achieved, for instance, by choosing π_0 such that $\mu_0 \ll \nu_0 \ll \lambda$, $\|d\nu_0/d\mu_0\|_\infty \vee \|d\mu_0/d\nu_0\|_\infty < \infty$ and $D_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) = \epsilon$.

B. KL DP test

A more stringent DP audit is realized via a hypothesis test for detecting ϵ -KL DP violations, rather than its Gaussian-smoothed version. We next provide such a test against a fixed alternative, namely:

$$\begin{aligned} H_0 : D_{\text{KL}}(\mu_0 \| \nu_0) &\leq \epsilon, \\ H_1 : D_{\text{KL}}(\mu_1 \| \nu_1) &\geq \tilde{\epsilon}, \end{aligned} \tag{38}$$

where $\tilde{\epsilon} > \epsilon > 0$. For this test, we again employ the test statistic T_n from Proposition 3 with appropriately chosen σ . Doing so requires additional assumption on the output distributions of the DP mechanism, namely that μ_i, ν_i , for $i = 0, 1$, to have smooth Lebesgue densities belonging to the following class.

Definition 5 (Lipschitz class, see [75]) *For $r \in (0, \infty]$, $m \in \mathbb{N}$, and $f \in L^r(\mathbb{R}^d)$, the m -th modulus of smoothness of f is*

$$\kappa_{m,r}(f, t) := \sup_{y \in \mathbb{R}^d, \|y\| \leq t} \left\| \Delta_y^m f \right\|_r, \tag{39}$$

where $\Delta_y^m f(x) = \sum_{j=0}^m (-1)^{m-j} f(x + jy)$. For $0 < s \leq 1$ and $\mathcal{X} \subseteq \mathbb{R}^d$, the Lipschitz class with smoothness parameter s and norm parameter M is

$$\text{Lip}_{s,r}(M, \mathcal{X}) := \{f \in L^r(\mathbb{R}^d) : \|f\|_{\text{Lip}(s,r)} \leq M, \text{spt}(f) \subseteq \mathcal{X}\},$$

where $\|f\|_{\text{Lip}(s,r)} := \|f\|_r + \sup_{t>0} t^{-s} \kappa_{1,r}(f, t)$ is the Lipschitz seminorm.

Assumption 6 *For $i = 0, 1$, the Lebesgue densities $p_{\mu_i}, p_{\nu_i} \in \text{Lip}_{s,1}(M, \mathcal{I}_b)$ and $\|p_{\mu_i}/p_{\nu_i}\|_\infty \vee \|p_{\nu_i}/p_{\mu_i}\|_\infty \leq M$ for some $0 < s \leq 1$ and $M > 0$. Further, $D_{\text{KL}}(\mu_0 \| \nu_0) \leq \epsilon$ and $D_{\text{KL}}(\mu_1 \| \nu_1) \geq \tilde{\epsilon}$ for some $\tilde{\epsilon} > \epsilon > 0$.*

Assumption 6 is not very restrictive in practice. Indeed, the definition of DP itself necessitates that $\|p_{\mu_u}/p_{\mu_v}\|_\infty$ is bounded uniformly for all $u, v \in \mathcal{U}$ with $u \sim v$. Moreover, the class of Lipschitz functions grows as we shrink the smoothness parameter s , and thus $\cup_{M \geq 0} \text{Lip}_{1,1}(M, \mathcal{I}_b) \subseteq \cup_{M \geq 0} \text{Lip}_{s,1}(M, \mathcal{I}_b)$ under our assumption that $0 < s \leq 1$. As the class of functions with bounded variation (for $d = 1$) over \mathcal{I}_b is contained in $\cup_{M \geq 0} \text{Lip}_{1,1}(M, \mathcal{I}_b)$, Assumption 6 allows for most densities of practical interest.

We are now ready to state the ϵ -KL DP audit result. As it may be unrealistic to assume that the exact values of M , s , and $\tilde{\epsilon}$ are known when constructing T_n and choosing critical values, the following proposition only requires the existence of known constants \bar{M} , $\bar{\epsilon}$, \underline{s} , and \bar{s} such that $M \leq \bar{M} < \infty$, $\epsilon < \bar{\epsilon} \leq \tilde{\epsilon}$, and $0 < \underline{s} \leq s \leq \bar{s} \leq 1$.

Proposition 5 (KL DP audit) *Suppose Assumption 6 holds. Let $0 < \tau \leq 1$ and $0 < \sigma < \sigma_{\epsilon, \bar{\epsilon}, \underline{s}, \bar{s}, d, \bar{M}}$, where $\sigma_{\epsilon, \bar{\epsilon}, \underline{s}, \bar{s}, d, \bar{M}}$ is the solution of (81). Then the test statistic $T_n = D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)$ with critical value $t_n = \epsilon + c_{b,d,\sigma} Q^{-1}(\tau) n^{-1/2}$, where $c_{b,d,\sigma}$ is given in (79), is asymptotically consistent and achieves an asymptotic level τ for the test in (38).*

The key difference between the proof of this claim and Proposition 3 is that given \bar{M} , $\bar{\epsilon}$, \underline{s} , and \bar{s} , it is possible to choose $\sigma > 0$ small enough so that $D_{\text{KL}}(\mu_1 * \gamma_\sigma \| \nu_1 * \gamma_\sigma) > \epsilon$ while $D_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) \leq \epsilon$. Choosing such

a σ , the claim then follows by utilizing (31) along with the Portmanteau theorem to bound the type I and type II error probabilities associated with T_n . The aforementioned choice of σ relies on a stability lemma for smoothed KL divergence given next, which may be of independent interest.

Lemma 3 (Stability lemma for smoothed KL divergence) *Let $\mathcal{X} \subseteq \mathbb{R}^d$, and $\mu, \nu \in \mathcal{P}(\mathcal{X})$ have Lebesgue densities p_μ and p_ν , respectively. Further, assume that $p_\mu, p_\nu \in \text{Lip}_{s,1}(M, \mathcal{X})$ and $\|p_\mu/p_\nu\|_\infty \vee \|p_\nu/p_\mu\|_\infty \leq M$ for some $M \geq 1$. Then,*

$$|\text{D}_{\text{KL}}(\mu\|\nu) - \text{D}_{\text{KL}}(\mu * \gamma_\sigma\|\nu * \gamma_\sigma)| \leq c_{d,s} M (M + 1 + \log M) \sigma^s, \quad (40)$$

where $c_{d,s} := \int_{\mathbb{R}^d} \|z\|^s \varphi_1(z) dz$.

The proof of Lemma 3 upper bounds the left-hand side (LHS) of (40) using Taylor's theorem, and then exploits the boundedness and Lipschitzness of the densities to control the resulting terms.

VII. PROOFS

This section contains proofs of the results from Section III-VI.

A. Proof of Proposition 1

The derivation uses the following lemma whose proof is given in Appendix B for completeness.

Lemma 4 (Generalized Slutsky's theorem) *Let $r, r' \geq 1$ be conjugate indices, i.e., $1/r + 1/r' = 1$. The following hold:*

- (i) *If $f_n \rightarrow f$ in $L^r(\rho)$ and $g_n \rightarrow g$ in $L^{r'}(\rho)$, then $f_n g_n \rightarrow fg$ in $L^1(\rho)$.*
- (ii) *If $Y_n \xrightarrow{w} Y$ in $L^r(\rho)$ and $Z_n \xrightarrow{w} z$ in $L^{r'}(\rho)$, where z is deterministic, then $Y_n Z_n \xrightarrow{w} Yz$ in $L^1(\rho)$.*

Having that, recall that all partial derivatives of ϕ of order two exists in $(0, \infty) \times (0, \infty)$. Consequently, the multivariate second-order Taylor's expansion of $\phi(x, y)$ around $\phi(x^*, y^*)$, where $x, y, x^*, y^* > 0$, yields

$$\begin{aligned} \phi(x, y) &= \phi(x^*, y^*) + (x - x^*) D^{(1,0)} \phi(x^*, y^*) + (y - y^*) D^{(0,1)} \phi(x^*, y^*) + (x - x^*)^2 \int_0^1 D^{(2,0)} \phi(u_\tau) (1 - \tau) d\tau \\ &\quad + (y - y^*)^2 \int_0^1 D^{(0,2)} \phi(u_\tau) (1 - \tau) d\tau + 2(x - x^*)(y - y^*) \int_0^1 D^{(1,1)} \phi(u_\tau) (1 - \tau) d\tau, \end{aligned}$$

where $u_\tau := (1 - \tau)(x^*, y^*) + \tau(x, y)$. Substituting $g_1(x), g_1^*(x), g_2(y), g_2^*(y)$ for x, x^*, y, y^* , respectively, in the above equation, and setting $u_{j,\tau} = (1 - \tau)g_j^* + \tau g_j$ for $j = 1, 2$, $\theta = (\theta_1, \theta_2) = (g_1 - g_1^*, g_2 - g_2^*)$, we obtain

$$\begin{aligned} \phi \circ (g_1, g_2) &= \phi \circ (g_1^*, g_2^*) + \theta_1 D^{(1,0)} \phi \circ (g_1^*, g_2^*) + \theta_2 D^{(0,1)} \phi \circ (g_1^*, g_2^*) + \int_0^1 \theta_1^2 D^{(2,0)} \phi \circ (u_{1,\tau}, u_{2,\tau}) (1 - \tau) d\tau \\ &\quad + \int_0^1 \theta_2^2 D^{(0,2)} \phi \circ (u_{1,\tau}, u_{2,\tau}) (1 - \tau) d\tau + 2 \int_0^1 \theta_1 \theta_2 D^{(1,1)} \phi \circ (u_{1,\tau}, u_{2,\tau}) (1 - \tau) d\tau, \end{aligned}$$

for all (x, y) with $g_1(x), g_1^*(x), g_2(y), g_2^*(y) > 0$. The validity of the above equation when $g_1(x) \geq 0$, and $g_2(y), g_1^*(x), g_2^*(y) > 0$ then follows by taking limits $g_1(x) \downarrow 0$ via Assumption 1 (specifically, the continuity of ϕ at

$(0, c)$ for $c > 0$, and the continuity of second order partial derivatives) along with the dominated convergence theorem applied to the last three integrals using Assumption 4. Likewise, the above equation extends to the case $g_1(x) = 0$, $g_2(y) \geq 0$ and $g_1^*(x), g_2^*(y) > 0$ by taking limits $g_2(y) \downarrow 0$ and using $\lim_{z \downarrow 0} \phi(0, z) = \phi(0, 0)$ in Assumption 1. Note that the above scenarios correspond to the constraints $\{(g_1, g_2) : g_1 \geq 0, g_2 \geq 0, \text{spt}(g_1) \subseteq \text{spt}(g_2)\}$ in the definition of Θ' . In a similar vein, the above equation also generalizes to the case $g_1(x), g_2(y) \geq 0$, $g_1^*(x), g_2^*(y) > 0$ by taking limits $g_1(x), g_2(y) \downarrow 0$, provided that ϕ is continuous at $(0, 0)$. This corresponds to the constraints defining the set Θ'' .

Integrating w.r.t. ρ then gives

$$\begin{aligned} \Phi(\theta_1, \theta_2) &= \Phi(0, 0) + \int_{\mathfrak{S}} \theta_1 D^{(1,0)} \phi \circ (g_1^*, g_2^*) d\rho + \int_{\mathfrak{S}} \theta_2 D^{(0,1)} \phi \circ (g_1^*, g_2^*) d\rho \\ &\quad + \int_{\mathfrak{S}} \int_0^1 \theta_1^2 D^{(2,0)} \phi \circ (u_{1,\tau}, u_{2,\tau}) (1-\tau) d\tau d\rho + \int_{\mathfrak{S}} \int_0^1 \theta_2^2 D^{(0,2)} \phi \circ (u_{1,\tau}, u_{2,\tau}) (1-\tau) d\tau d\rho \\ &\quad + 2 \int_{\mathfrak{S}} \int_0^1 \theta_1 \theta_2 D^{(1,1)} \phi \circ (u_{1,\tau}, u_{2,\tau}) (1-\tau) d\tau d\rho, \end{aligned} \quad (41)$$

where we have used the definition of Φ in (7). The terms in (41) are well-defined and finite due to the following reasons. The finiteness of $\Phi(\theta_1, \theta_2)$ and $\Phi(0, 0)$ is straightforward from Assumption 3, while that of the first two integrals is a consequence of Assumption 2 and Hölder's inequality. The remaining integrals exists and are finite since for any α with $|\alpha| = 2$, we have

$$\begin{aligned} &\int_{\mathfrak{S}} \int_0^1 |\theta_1^2 D^{(2,0)} \phi \circ (u_{1,\tau}, u_{2,\tau}) + \theta_2^2 D^{(0,2)} \phi \circ (u_{1,\tau}, u_{2,\tau}) + 2\theta_1 \theta_2 D^{(1,1)} \phi \circ (u_{1,\tau}, u_{2,\tau})| (1-\tau) d\tau d\rho \\ &\lesssim \int_{\mathfrak{S}} \left(|\theta_1^2 D^{(2,0)} \psi_{2,0} \circ (g_1^*, g_2^*)| + |\theta_2^2 D^{(0,2)} \psi_{0,2} \circ (g_1^*, g_2^*)| + |\theta_1 \theta_2 D^{(1,1)} \psi_{1,1} \circ (g_1^*, g_2^*)| \right) d\rho \\ &\leq \int_{\mathfrak{S}} \left(|\theta_1^2 D^{(2,0)} \psi_{2,0} \circ (g_1^*, g_2^*)| + |\theta_2^2 D^{(0,2)} \psi_{0,2} \circ (g_1^*, g_2^*)| \right) d\rho \\ &\quad + \left(\int_{\mathfrak{S}} \theta_1^2 |D^{(1,1)} \psi_{1,1} \circ (g_1^*, g_2^*)| d\rho \right)^{\frac{1}{2}} \left(\int_{\mathfrak{S}} \theta_2^2 |D^{(1,1)} \psi_{1,1} \circ (g_1^*, g_2^*)| d\rho \right)^{\frac{1}{2}} \\ &< \infty, \end{aligned} \quad (42)$$

where the first step uses (8) and $v_\alpha \in L_+^1([0, 1], \lambda)$, the second one invokes the Cauchy-Schwarz inequality, while the finiteness follows from (5) and $(\theta_1, \theta_2) \in \Theta \subset \mathfrak{D}$.

Given the expansion from (41), setting $(\theta_{1,n}, \theta_{2,n}) = (g_{1,n} - g_1^*, g_{2,n} - g_2^*) \in \Theta$ and $h_{j,n} = \theta_{j,n}/t_n$, $t_n > 0$, $j = 1, 2$, we obtain

$$\frac{\Phi(t_n h_{1,n}, t_n h_{2,n}) - \Phi(0, 0)}{t_n} = \int_{\mathfrak{S}} \left(h_{1,n} D^{(1,0)} \phi \circ (g_1^*, g_2^*) + h_{2,n} D^{(0,1)} \phi \circ (g_1^*, g_2^*) \right) d\rho + J_n, \quad (43)$$

where $J_n := t_n \int_{\mathfrak{S}} (h_{1,n}^2 I_{1,n} + h_{2,n}^2 I_{2,n} + h_{1,n} h_{2,n} I_{3,n}) d\rho$ with

$$\begin{aligned} I_{1,n} &:= \int_0^1 D^{(2,0)} \phi \circ (g_1^* + \tau t_n h_{1,n}, g_2^* + \tau t_n h_{2,n}) (1-\tau) d\tau, \\ I_{2,n} &:= \int_0^1 D^{(0,2)} \phi \circ (g_1^* + \tau t_n h_{1,n}, g_2^* + \tau t_n h_{2,n}) (1-\tau) d\tau, \end{aligned}$$

$$I_{3,n} := 2 \int_0^1 D^{(1,1)} \phi \circ (g_1^* + \tau t_n h_{1,n}, g_2^* + \tau t_n h_{2,n}) (1 - \tau) d\tau.$$

We will show that the limit of the RHS of (43) as $n \rightarrow \infty$ evaluates to the RHS of (9) for all $t_n \downarrow 0^+$ and $(h_{1,n}, h_{2,n}) \rightarrow (h_1, h_2)$ in \mathfrak{D} for some $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*)$, which will prove (9). Here, $\mathfrak{T}_\Theta(\theta^*)$ is as given in Proposition 1 since Θ is convex and $\theta^* \in \Theta$.

Under Assumption 2, the first two terms in (43) converge to the RHS of (9) by Lemma 4(i) since $h_{1,n} \rightarrow h_1$ and $h_{2,n} \rightarrow h_2$ in $L^2(\rho)$. Thus, it remains to show that J_n converges to zero. To that end we will show that every subsequence of J_n has a further subsequence that converges to 0. Fix $(t_n, h_{1,n}, h_{2,n})_{n \in \mathbb{N}}$ and consider a subsequence $(n_k)_{k \in \mathbb{N}}$ of \mathbb{N} . Since $(h_{1,n_k}, h_{2,n_k}) \rightarrow (h_1, h_2)$ in \mathfrak{D} implies $h_{i,n_k} \rightarrow h_i$ in $L^2(\eta_i)$, for $i = 1, 2$, and L^2 convergence implies convergence in measure, there exists a further subsequence $(n_{k_l})_{l \in \mathbb{N}}$ such that $h_{i,n_{k_l}} \rightarrow h_i$ η_i -a.e., for $i = 1, 2$. Consequently, (8) with $v_\alpha \in L_+^1([0, 1], \lambda)$ and dominated convergence theorem implies that ρ -a.e.,

$$\lim_{l \rightarrow \infty} I_{1,n_{k_l}} + I_{2,n_{k_l}} + I_{3,n_{k_l}} = \frac{1}{2} \left(D^{(2,0)} \phi \circ (g_1^*, g_2^*) + D^{(0,2)} \phi \circ (g_1^*, g_2^*) + 2D^{(1,1)} \phi \circ (g_1^*, g_2^*) \right).$$

Next, we claim that $(h_{1,n_{k_l}}^2 I_{1,n_{k_l}} + h_{2,n_{k_l}}^2 I_{2,n_{k_l}} + h_{1,n_{k_l}} h_{2,n_{k_l}} I_{3,n_{k_l}})_{l \in \mathbb{N}}$ is uniformly integrable w.r.t. ρ . This along with the above equation and Vitali's convergence theorem then leads to

$$\begin{aligned} \lim_{l \rightarrow \infty} \int_{\mathfrak{S}} (h_{1,n_{k_l}}^2 I_{1,n_{k_l}} + h_{2,n_{k_l}}^2 I_{2,n_{k_l}} + h_{1,n_{k_l}} h_{2,n_{k_l}} I_{3,n_{k_l}}) d\rho \\ = \frac{1}{2} \int_{\mathfrak{S}} \left(h_1^2 D^{(2,0)} \phi \circ (g_1^*, g_2^*) + h_2^2 D^{(0,2)} \phi \circ (g_1^*, g_2^*) + 2h_1 h_2 D^{(1,1)} \phi \circ (g_1^*, g_2^*) \right) d\rho, \end{aligned} \quad (44)$$

and hence, $\lim_{l \rightarrow \infty} J_{n_{k_l}} = 0$ as $t_{n_{k_l}} \downarrow 0^+$. Thus, every subsequence $(J_{n_k})_{k \in \mathbb{N}}$ has a further subsequence $(J_{n_{k_l}})_{l \in \mathbb{N}}$ which converges to 0, which implies $\lim_{n \rightarrow \infty} J_n = 0$ and shows that

$$\lim_{n \rightarrow \infty} \frac{\Phi(t_n h_{1,n}, t_n h_{2,n}) - \Phi(0, 0)}{t_n} = \int_{\mathfrak{S}} \left(h_1 D^{1,0} \phi \circ (g_1^*, g_2^*) + h_2 D^{(0,1)} \phi \circ (g_1^*, g_2^*) \right) d\rho.$$

Since the above holds for any $t_n \downarrow 0^+$ and $(h_{1,n}, h_{2,n}) \rightarrow (h_1, h_2)$, (9) follows.

To show the uniform integrability claim mentioned above, note that for any $\mathcal{D} \subset \mathfrak{S}$

$$\begin{aligned} \int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} \left| h_{1,n_{k_l}}^2 I_{1,n_{k_l}} + h_{2,n_{k_l}}^2 I_{2,n_{k_l}} + h_{1,n_{k_l}} h_{2,n_{k_l}} I_{3,n_{k_l}} \right| d\rho \\ \leq \int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} \left| h_{1,n_{k_l}}^2 I_{1,n_{k_l}} \right| d\rho + \int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} \left| h_{2,n_{k_l}}^2 I_{2,n_{k_l}} \right| d\rho + \int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} \left| h_{1,n_{k_l}} h_{2,n_{k_l}} I_{3,n_{k_l}} \right| d\rho \\ \lesssim \int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} \left(h_{1,n_{k_l}}^2 |\psi_{2,0} \circ (g_1^*, g_2^*)| + h_{2,n_{k_l}}^2 |\psi_{0,2} \circ (g_1^*, g_2^*)| \right) d\rho \\ + \left(\int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} h_{1,n_{k_l}}^2 |\psi_{1,1} \circ (g_1^*, g_2^*)| d\rho \right)^{\frac{1}{2}} \left(\int_{\mathfrak{S}} \mathbb{1}_{\mathcal{D}} h_{2,n_{k_l}}^2 |\psi_{1,1} \circ (g_1^*, g_2^*)| d\rho \right)^{\frac{1}{2}} \\ \lesssim \left\| \mathbb{1}_{\mathcal{D}} h_{1,n_{k_l}} \right\|_{2,\eta_1}^2 + \left\| \mathbb{1}_{\mathcal{D}} h_{2,n_{k_l}} \right\|_{2,\eta_2}^2. \end{aligned}$$

where the penultimate inequality above follows via similar steps leading to the bound in (42). Also, recall that $h_{i,n_{k_l}} \rightarrow h_i$ in $L^2(\eta_i)$ implies uniform integrability of $(h_{i,n_{k_l}})_{l \in \mathbb{N}}$, for $i = 1, 2$. This along with the above equation

then shows the desired uniform integrability.

To compute the second order Hadamard derivative, from (43), we have

$$\frac{\Phi(t_n h_{1,n}, t_n h_{2,n}) - \Phi(0, 0) - t_n \Phi'_\Theta(h_{1,n}, h_{2,n})}{t_n^2} = \int_{\mathfrak{S}} (h_{1,n}^2 I_{1,n} + h_{2,n}^2 I_{2,n} + h_{1,n} h_{2,n} I_{3,n}) d\rho.$$

Using similar argument leading to (44) one readily shows that the RHS multiplied by 2 above converges to the RHS of (10). Since this holds for any $t_n \downarrow 0^+$ and $(h_{1,n}, h_{2,n}) \rightarrow (h_1, h_2)$, (10) follows, thus completing the proof of the proposition.

B. Proofs for Section IV

We first state a lemma that shows that Assumption 1 and Assumption 4 are satisfied by the functionals corresponding to f -divergences.

Lemma 5 Consider ρ , g_1^* , g_2^* and Θ as in Section III. The following hold:

(i) The functions

$$\begin{aligned}\phi_{\text{KL}}(x, y) &:= y f_{\text{KL}}(x/y) = x \log(x/y), \\ \phi_{\chi^2}(x, y) &:= y f_{\chi^2}(x/y) = (x - y)^2 / y, \\ \phi_{\text{H}^2}(x, y) &:= y f_{\text{H}^2}(x/y) = (\sqrt{x} - \sqrt{y})^2,\end{aligned}$$

are convex and satisfy Assumption 1. Moreover, ϕ_{H^2} is continuous in $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, while ϕ_{KL} and ϕ_{χ^2} are continuous in $[0, \infty) \times (0, \infty)$.

(ii) The above functions satisfy Assumption 4 under conditions enumerated below:

(a) ϕ_{KL} with $\psi_{2,0} \circ (g_1^*, g_2^*) = 1/g_1^*$, $\psi_{0,2} \circ (g_1^*, g_2^*) = g_1^*/g_2^{*2} + q/g_2^*$, $\psi_{1,1} \circ (g_1^*, g_2^*) = 1/g_2^*$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, and

$$\Theta \subseteq \bar{\Theta}(q) := \left\{ (g_1 - g_1^*, g_2 - g_2^*) \in \mathfrak{D} : g_1 \geq 0, g_2 \geq 0, \text{spt}(g_1) \subseteq \text{spt}(g_2), \|g_1\|_{1,\rho} \vee \|g_2\|_{1,\rho} \leq 1, \right. \\ \left. |g_1/g_2| \leq q, \text{ } \rho\text{-a.e.} \right\}. \quad (45)$$

(b) ϕ_{χ^2} with $\psi_{2,0} \circ (g_1^*, g_2^*) = 1/g_2^*$, $\psi_{0,2} \circ (g_1^*, g_2^*) = g_1^{*2}/g_2^{*3} + (q^2/g_2^*)$, $\psi_{1,1} \circ (g_1^*, g_2^*) = g_1^*/g_2^{*2} + q/g_2^*$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, and $\Theta \subseteq \bar{\Theta}(q)$;

(c) ϕ_{H^2} with $\psi_{2,0} \circ (g_1^*, g_2^*) = (g_2^{*1/2} + q_2^{1/2})/g_1^{*3/2}$, $\psi_{0,2} \circ (g_1^*, g_2^*) = (g_1^{*1/2} + q_1^{1/2})/g_2^{*3/2}$, $\psi_{1,1} \circ (g_1^*, g_2^*) = (g_1^* g_2^*)^{-1/2}$, $v_{2,0} = v_{0,2} = \tau^{1/2}(1 - \tau)^{-1/2}$, $v_{1,1} = 1$, and

$$\Theta \subseteq \check{\Theta}(q_1, q_2) := \left\{ (g_1 - g_1^*, g_2 - g_2^*) \in \mathfrak{D} : g_1, g_2 \geq 0, \|g_1\|_{1,\rho} \vee \|g_2\|_{1,\rho} \leq 1, |g_1| \leq q_1, |g_2| \leq q_2, \rho\text{-a.e.} \right\}.$$

The proof of Lemma 5 is given in Appendix C. We proceed with the proofs of the results in Section IV.

1) Proof of Theorem 2: We will invoke Proposition 1 to prove the claim. To that end, we specialize the Hadamard differentiability framework of Section III by identifying the relevant quantities and showing that the required

assumptions hold. For brevity, we will only prove the two-sample case and highlight its differences with the one-sample case at the end.

Part (iii): Let $\rho = \mu$, $g_1^* = g_2^* = 1$, $\psi_{2,0} \circ (1, 1) = \psi_{1,1} \circ (1, 1) = 1$, $\psi_{0,2} \circ (1, 1) = 1 + q$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 3$ and $p_{\eta_2} = 3 + q$. Note that $D_{\text{KL}}(\mu_n \| \nu_n) = \Phi(p_{\mu_n} - 1, p_{\nu_n} - 1)$ with $\phi(x, y) = \phi_{\text{KL}}(x, y) = x \log(x/y)$ in (7). Also, observe that under the assumptions of Part (i) of Theorem 2, we have $(p_{\mu_n} - 1, p_{\nu_n} - 1) \in \Theta$ a.s., where

$$\Theta = \left\{ (g_1 - 1, g_2 - 1) \in \mathfrak{D} : g_1 \geq 0, g_2 \geq 0, \text{spt}(g_1) \subseteq \text{spt}(g_2), \|g_1\|_{1,\mu} = \|g_2\|_{1,\mu} = 1, \right. \\ \left. |g_1/g_2| \leq q, \mu\text{-a.s.} \right\}, \quad (46)$$

$$\mathfrak{D} = \{(g_1, g_2) : g_1, g_2 \in L^1(\mu), \|(g_1 - 1, g_2 - 1)\|_{\mathfrak{D}} < \infty\}.$$

Note that $\Theta \subseteq \bar{\Theta}(q)$ as defined in (45). Then, Part (ii)(a) of Lemma 5 implies that Assumption 1 and 4 are satisfied, while Assumption 3 holds by hypothesis. Assumption 2 holds since from (86a), we obtain $D^{(1,0)}\phi_{\text{KL}} \circ (1, 1) = 1$ and $D^{(0,1)}\phi_{\text{KL}} \circ (1, 1) = -1$. Proposition 1(i) and (86a) then yield

$$\Phi'_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} h_1 d\mu - \int_{\mathfrak{S}} h_2 d\mu, \quad \text{and} \quad \Phi''_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} (h_1 - h_2)^2 d\mu,$$

for all $(h_1, h_2) \in \mathfrak{T}_{\Theta}(\theta^*) = \text{cl}(\{(g_1 - 1)/t, (g_2 - 1)/t : (g_1 - 1, g_2 - 1) \in \Theta, t > 0\})$ (note that Θ is convex).

Next, note that $\Phi(0, 0) = \Phi'_{\theta^*}(h_1, h_2) = 0$ for all $(h_1, h_2) \in \mathfrak{T}_{\Theta}(\theta^*)$. The latter follows from that fact that if $(h_1, h_2) \in \mathfrak{T}_{\Theta}(\theta^*)$, then $h_1, h_2 \in L^1(\mu)$ and there exists a sequence $(h_{1,n}, h_{2,n})_{n \in \mathbb{N}}$, where $h_{1,n} = (g_{1,n} - 1)/t_n$, $h_{2,n} = (g_{2,n} - 1)/t_n$, $(g_{1,n} - 1, g_{2,n} - 1) \in \Theta$, such that $\|h_{1,n} - h_1\|_{1,\mu} \vee \|h_{2,n} - h_2\|_{1,\mu} \rightarrow 0$. As $\int_{\mathfrak{S}} h_{1,n} d\mu = 0$ due to $\|g_{1,n}\|_{1,\mu} = 1$ for every n , we further have

$$\left| \int_{\mathfrak{S}} h_1 d\mu \right| = \left| \lim_{n \rightarrow \infty} \int_{\mathfrak{S}} h_{1,n} d\mu - \int_{\mathfrak{S}} h_{1,n} d\mu \right| \leq \lim_{n \rightarrow \infty} \int_{\mathfrak{S}} |h_1 - h_{1,n}| d\mu = 0. \quad (47)$$

Hence, $\int_{\mathfrak{S}} h_1 d\mu = 0$ and similarly, $\int_{\mathfrak{S}} h_2 d\mu = 0$.

To conclude, we observe that $(r_n(p_{\mu_n} - 1), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in \mathfrak{D} (w.r.t. norm of $L^2(\eta_1) \times L^2(\eta_2)$) and that $(r_n(p_{\mu_n} - 1), r_n(p_{\nu_n} - 1))$ and (B_1, B_2) take values in $\mathfrak{T}_{\Theta}(\theta^*)$. An application of Lemma 1(ii) then yields (13).

Part (iv): Taking $\rho = \nu$, first consider the case when $p_{\mu} > 0$. Let $g_1^* = p_{\mu}$, $g_2^* = 1$, $\psi_{2,0} \circ (p_{\mu}, 1) = 1/p_{\mu}$, $\psi_{1,1} \circ (p_{\mu}, 1) = 1$, $\psi_{0,2} \circ (p_{\mu}, 1) = p_{\mu} + q$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 2 + (1/p_{\mu})$ and $p_{\eta_2} = 2 + p_{\mu} + q$. We have $D_{\text{KL}}(\mu_n \| \nu_n) = \Phi(p_{\mu_n} - p_{\mu}, p_{\nu_n} - 1)$ with $\phi(x, y) = \phi_{\text{KL}}(x, y) = x \log(x/y)$ in (7). Under the hypothesis in Part (iv), $(p_{\mu_n} - p_{\mu}, p_{\nu_n} - 1) \in \Theta$ a.s., where

$$\Theta = \left\{ (g_1 - p_{\mu}, g_2 - 1) \in \mathfrak{D} : g_1 \geq 0, g_2 \geq 0, \text{spt}(g_1) \subseteq \text{spt}(g_2), \|g_1\|_{1,\nu} = \|g_2\|_{1,\nu} = 1, \right. \\ \left. |g_1/g_2| \leq q, \nu\text{-a.s.} \right\}, \quad (48)$$

$$\mathfrak{D} = \{(g_1, g_2) : g_1, g_2 \in L^1(\nu), \|(g_1, g_2)\|_{\mathfrak{D}} < \infty\}.$$

Observe that $\Theta \subseteq \bar{\Theta}(q)$ as given in (45), and that Assumptions 1, 3, and 4 are satisfied (via Part (i) and (ii)(a) of Lemma 5). Assumption 2 also holds since $D^{(1,0)}\phi_{\text{KL}} \circ (p_{\mu}, 1) = 1 + \log p_{\mu} \in L^2(\nu)$ and $D^{(0,1)}\phi_{\text{KL}} \circ (p_{\mu}, 1) =$

$-p_\mu \in L^2(\nu)$ (see (86a)) by hypothesis. It then follows from Proposition 1(i) and (86a) that

$$\Phi'_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} (1 + \log p_\mu) h_1 d\nu - \int_{\mathfrak{S}} h_2 p_\mu d\nu = \int_{\mathfrak{S}} h_1 \log p_\mu d\nu - \int_{\mathfrak{S}} h_2 d\mu, \quad (49)$$

for all $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{(g_1 - p_\mu)/t, (g_2 - 1)/t) : (g_1 - p_\mu, g_2 - 1) \in \Theta, t > 0\})$. Noting that $\Phi(0, 0) = D_{\text{KL}}(\mu \| \nu)$, we obtain (14) from Lemma 1(i) and the fact that $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in \mathfrak{D} taking values in $\mathfrak{T}_\Theta(\theta^*)$.

Finally, consider the case $p_\mu \not\asymp 0$. To handle this scenario, we restrict the probability measures to the support of p_μ , i.e., $\mathfrak{S} = \text{spt}(p_\mu)$, and apply Proposition 1(ii). Set $\rho = \tilde{\nu} := \nu|_{\text{spt}(p_\mu)}$, where $\nu|_{\text{spt}(p_\mu)}$ denotes the aforementioned restriction of ν . Then, observe that $\tilde{\mu} := \mu|_{\text{spt}(p_\mu)}$ and $\tilde{\mu}_n := \mu_n|_{\text{spt}(p_\mu)}$ are probability measures since $\mu_n \ll \mu$, while $\tilde{\nu}$ and $\tilde{\nu}_n := \nu_n|_{\text{spt}(p_\mu)}$ are possibly deficient probability measures, i.e., $0 < \tilde{\nu}(\text{spt}(p_\mu)) \leq 1$ and $0 \leq \tilde{\nu}_n(\text{spt}(p_\mu)) \leq 1$. Defining

$$\Theta = \left\{ (g_1 - p_\mu, g_2 - 1) \in \mathfrak{D} : g_1 \geq 0, g_2 \geq 0, \text{spt}(g_1) \subseteq \text{spt}(g_2), \|g_1\|_{1, \tilde{\nu}} = 1, \|g_2\|_{1, \tilde{\nu}} \leq 1, \right. \\ \left. |g_1/g_2| \leq q, \nu\text{-a.s.} \right\},$$

we have $(p_{\tilde{\mu}_n} - p_{\tilde{\mu}}, p_{\tilde{\nu}_n} - 1) \in \Theta$ a.s. It now follows that $(r_n(p_{\tilde{\mu}_n} - p_{\tilde{\mu}}), r_n(p_{\tilde{\nu}_n} - 1)) \xrightarrow{w} (\tilde{B}_1, \tilde{B}_2)$, where $(\tilde{B}_1, \tilde{B}_2)$ is the restriction of (B_1, B_2) to $\text{spt}(p_\mu)$. Having that, the same argument as above with $p_\mu, p_{\mu_n}, p_{\nu_n}, \nu$ replaced by $p_{\tilde{\mu}}, p_{\tilde{\mu}_n}, p_{\tilde{\nu}_n}, \tilde{\nu}$, respectively, yields

$$r_n(D_{\text{KL}}(\tilde{\mu}_n \| \tilde{\nu}_n) - D_{\text{KL}}(\tilde{\mu} \| \tilde{\nu})) \xrightarrow{d} \int_{\text{spt}(p_\mu)} \tilde{B}_1 \log p_\mu d\nu - \int_{\text{spt}(p_\mu)} \tilde{B}_2 d\mu = \int_{\text{spt}(p_\mu)} B_1 \log p_\mu d\nu - \int_{\text{spt}(p_\mu)} B_2 d\mu.$$

The claim then follows by noting that $D_{\text{KL}}(\tilde{\mu}_n \| \tilde{\nu}_n) = D_{\text{KL}}(\mu_n \| \nu_n)$ and $D_{\text{KL}}(\tilde{\mu} \| \tilde{\nu}) = D_{\text{KL}}(\mu \| \nu)$.

The proof for the one-sample null follows via similar arguments to the two-sample null by considering $\rho = \mu$, $g_1^* = g_2^* = 1$, $\psi_{2,0} \circ (g_1^*, g_2^*) = 1$, $\psi_{0,2} = \psi_{1,1} = 0$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, $\eta_1 = 2\mu$, $\eta_2 = \mu$, and

$$\mathfrak{D} = \{(g_1 - 1, g_2 - 1) : g_1, g_2 \in L^1(\mu), \|(g_1 - 1, g_2 - 1)\|_{\mathfrak{D}} < \infty\}, \\ \Theta = \{(g_1 - 1, 0) \in \mathfrak{D} : g_1 \geq 0, \|g_1\|_{1, \mu} = 1\}.$$

Likewise, the proof for the one-sample alternative is obtained via analogous steps to the two-sample alternative by taking $\mu \ll \nu = \rho$, $g_1^* = p_\mu$, $g_2^* = 1$, $\psi_{2,0} \circ (g_1^*, g_2^*) = 1/p_\mu$, $\psi_{0,2} = \psi_{1,1} = 0$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 1 + (1/p_\mu)$, $p_{\eta_2} = 1$, and

$$\mathfrak{D} = \{(g_1 - p_\mu, g_2 - 1) : g_1, g_2 \in L^1(\nu), \|(g_1 - p_\mu, g_2 - 1)\|_{\mathfrak{D}} < \infty\}, \\ \Theta = \{(g_1 - p_\mu, 0) \in \mathfrak{D} : g_1 \geq 0, \|g_1\|_{1, \nu} = 1\}.$$

This completes the proof of the theorem.

2) Proof of Theorem 3: For proving Part (i) and (ii), the continuous mapping theorem suffices, while for Part (iii) and (iv), we will use the functional delta method.

Part (i): The claim follows from the continuous mapping theorem [25, Theorem 1.3.6], by noting that $r_n^2 \chi^2(\mu_n \| \mu)$

$= \int_{\mathfrak{S}} (r_n(p_{\mu_n} - 1))^2 d\mu$, $r_n(p_{\mu_n} - 1) \xrightarrow{w} B$ in $L^2(\mu)$, and $f \mapsto \|f\|_{2,\mu}^2$ is a continuous functional in $L^2(\mu)$. Also, observe that $r_n(p_{\mu_n} - 1) \in L^2(\mu)$ implies that $\chi^2(\mu_n|\mu) = \|p_{\mu_n} - 1\|_{2,\mu}^2 < \infty$.

Part (ii): Note that

$$\chi^2(\mu_n|\nu) = \|p_{\mu_n} - 1\|_{2,\nu}^2 = \|p_{\mu_n} - p_\mu\|_{2,\nu}^2 + \|p_\mu - 1\|_{2,\nu}^2 + 2 \int_{\mathfrak{S}} (p_{\mu_n} - p_\mu)(p_\mu - 1) d\nu.$$

Hence,

$$\begin{aligned} r_n(\chi^2(\mu_n|\nu) - \chi^2(\mu|\nu)) &= \int_{\mathfrak{S}} \left(r_n^{\frac{1}{2}}(p_{\mu_n} - p_\mu) \right)^2 d\nu + 2 \int_{\mathfrak{S}} r_n(p_{\mu_n} - p_\mu)(p_\mu - 1) d\nu \\ &= \int_{\mathfrak{S}} \left(r_n^{\frac{1}{2}}(p_{\mu_n} - p_\mu) \right)^2 d\nu + 2 \int_{\mathfrak{S}} r_n(p_{\mu_n} - p_\mu)p_\mu d\nu. \end{aligned}$$

Since $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^2(\nu)$, Slutsky's theorem implies that $r_n^{1/2}(p_{\mu_n} - p_\mu) \xrightarrow{w} 0$ in $L^2(\nu)$. Consequently, first term in the RHS above converges weakly to zero in $L^2(\nu)$ by the continuous mapping theorem applied to the continuous functional $\|\cdot\|_{2,\nu}^2$. Next, note that $\chi^2(\mu|\nu) < \infty$ and $\chi^2(\mu_n|\nu) < \infty$ imply $p_\mu \in L^2(\nu)$ and $p_{\mu_n} \in L^2(\nu)$, respectively. Lemma 4(ii) together with the fact that $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ then imply that $r_n(p_{\mu_n} - p_\mu)p_\mu \xrightarrow{w} Bp_\mu$ in $L^1(\nu)$. Having that, the claim follows from the continuous mapping theorem since $f \mapsto \int_{\mathfrak{S}} f d\nu$ is a continuous functional in $L^1(\nu)$.

Part (iii): Let $\rho = \mu$, $g_1^* = g_2^* = 1$, $\psi_{2,0} \circ (1, 1) = 1$, $\psi_{1,1} \circ (1, 1) = 1+q$, $\psi_{0,2} \circ (1, 1) = 1+q^2$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 3+q$ and $p_{\eta_2} = 3+q+q^2$. We have $\chi^2(\mu_n|\nu_n) = \Phi(p_{\mu_n} - 1, p_{\nu_n} - 1)$ with $\phi(x, y) = \phi_{\chi^2}(x, y) = (x-y)^2/y$ in (7). Also, under the hypothesis in Part (iii), we have $p_{\nu_n} > 0$ and $(p_{\mu_n} - 1, p_{\nu_n} - 1) \in \Theta = \bar{\Theta}(q)$ a.s., where $\bar{\Theta}(q)$ and the ambient space \mathfrak{D} are given in (46). To prove (17), we next verify that Assumptions 1-4 hold in this setting and then invoke Proposition 1.

Observe that Part (ii)(b) of Lemma 5 implies that Assumption 1 and Assumption 4 are satisfied. Furthermore, Assumption 3 holds by hypothesis, while Assumption 2 is satisfied since $|D^{(1,0)}\phi_{\chi^2} \circ (1, 1)| = |D^{(0,1)}\phi_{\chi^2} \circ (1, 1)| = 0$ from (86b). Then, it follows via Proposition 1(i) and (86b) that

$$\Phi'_{\theta^*}(h_1, h_2) = 0 \quad \text{and} \quad \Phi''_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} 2(h_1 - h_2)^2 d\mu,$$

for all $(h_1, h_2) \in \mathfrak{T}_{\Theta}(\theta^*) = \text{cl}(\{(g_1 - 1)/t, (g_2 - 1)/t) : (g_1 - 1, g_2 - 1) \in \Theta, t > 0\})$. Lastly, since $\Phi(0, 0) = 0$, $(r_n(p_{\mu_n} - 1), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in \mathfrak{D} , and $(r_n(p_{\mu_n} - 1), r_n(p_{\nu_n} - 1))$ as well as (B_1, B_2) take values in $\mathfrak{T}_{\Theta}(\theta^*)$, the convergence in (17) follows from Lemma 1(ii).

Part (iv): Let $\rho = \nu$ and assume first that $p_\mu > 0$. Set $g_1^* = p_\mu$, $g_2^* = 1$, $\psi_{2,0} \circ (p_\mu, 1) = 1$, $\psi_{0,2} \circ (p_\mu, 1) = p_\mu^2 + q^2$, $\psi_{1,1} \circ (p_\mu, 1) = p_\mu + q$, $v_{2,0} = v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 2 + p_\mu + q$ and $p_{\eta_2} = 1 + p_\mu + q + p_\mu^2 + q^2$. We have $\chi^2(\mu_n|\nu_n) = \Phi(p_{\mu_n} - p_\mu, p_{\nu_n} - 1)$ with $\phi(x, y) = \phi_{\chi^2}(x, y) = (x-y)^2/y$ in (7). Under the hypothesis in Part (iii), $(p_{\mu_n} - p_\mu, p_{\nu_n} - 1) \in \Theta = \bar{\Theta}(q)$ a.s., where $\bar{\Theta}(q)$ and \mathfrak{D} are as in (48). Assumptions 1, 3, and 4 are satisfied for the same reason as in Part (iii). Assumption 2 holds since (86b) together with the hypothesis $p_\mu \in L^4(\nu)$ implies $D^{(1,0)}\phi_{\chi^2} \circ (p_\mu, 1) = 2(p_\mu - 1) \in L^2(\nu)$ and $D^{(0,1)}\phi_{\chi^2} \circ (p_\mu, 1) = 1 - p_\mu^2 \in L^2(\nu)$. Applying Proposition

1(i) while using (86b), we have

$$\Phi'_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} 2(p_\mu - 1)h_1 d\nu + \int_{\mathfrak{S}} (1 - p_\mu^2)h_2 d\nu = 2 \int_{\mathfrak{S}} h_1 d\mu - \int_{\mathfrak{S}} h_2 p_\mu d\mu,$$

for all $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{(g_1 - p_\mu)/t, (g_2 - 1)/t) : (g_1 - p_\mu, g_2 - 1) \in \Theta, t > 0\})$. Here, the last equality follows since $\int_{\mathfrak{S}} h_1 d\nu = \int_{\mathfrak{S}} h_2 d\nu = 0$ for $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*)$, similarly to (47). Finally, (18) follows from the above equation and Lemma 1(i) by noting that $\Phi(0, 0) = \chi^2(\mu\|\nu)$ and $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - 1)) \xrightarrow{w} (B_1, B_2)$ in \mathfrak{D} , taking values in $\mathfrak{T}_\Theta(\theta^*)$. Finally, the case $p_\mu \not\asymp 0$ is handled by restricting the space to $\text{spt}(p_\mu)$ as given at the end of the proof of Theorem 2, Part (iv). The claim then follows by noting that $\chi^2(\tilde{\mu}_n\|\tilde{\nu}_n) = \chi^2(\mu_n\|\nu_n)$ and $\chi^2(\tilde{\mu}\|\tilde{\nu}) = \chi^2(\mu\|\nu)$, where $\tilde{\mu}_n, \tilde{\nu}_n, \tilde{\mu}$ and $\tilde{\nu}$ are as defined therein.

3) Proof of Theorem 4: The proof uses Proposition 1 by identifying the relevant quantities in the Hadamard differentiability framework of Section III and showing that the pertinent assumptions hold. Again, we only prove the two-sample case, delineating the difference for the one-sample case at the end.

Part (iii): Let $\mu_n, \nu_n, \mu \ll \rho$ for some finite measure ρ , and set $g_1^* = g_2^* = p_\mu$, $\psi_{2,0} \circ (p_\mu, p_\mu) = (p_\mu^{1/2} + q_2^{1/2})/p_\mu^{3/2}$, $\psi_{0,2} \circ (p_\mu, p_\mu) = (p_\mu^{1/2} + q_1^{1/2})/p_\mu^{3/2}$, $\psi_{1,1} \circ (p_\mu, p_\mu) = 1/p_\mu$, $v_{2,0} = v_{0,2} = \tau^{1/2}(1 - \tau)^{-1/2}$, $v_{1,1} = 1$, $p_{\eta_1} = p_{\eta_2} = 1 + (1/p_\mu) + (q^{1/2}/p_\mu^{3/2})$, and

$$\begin{aligned} \mathfrak{D} &= \{(g_1 - p_\mu, g_2 - p_\mu) : g_1, g_2 \in L^1(\rho), \|(g_1 - p_\mu, g_2 - p_\mu)\|_{\mathfrak{D}} < \infty\}, \\ \Theta &= \{(g_1 - p_\mu, g_2 - p_\mu) \in \mathfrak{D} : g_1, g_2 \geq 0, \|g_1\|_{1,\rho} = \|g_2\|_{1,\rho} = 1, |g_1| \vee |g_2| \leq q, \rho\text{-a.e.}\}. \end{aligned}$$

The hypothesis in Part (iii) entails $(p_{\mu_n} - p_\mu, p_{\nu_n} - p_\mu) \in \Theta$ a.s. Further, $H^2(\mu_n, \nu_n) = \Phi(p_{\mu_n} - p_\mu, p_{\nu_n} - p_\mu)$ with $\phi(x, y) = \phi_{H^2}(x, y) = (\sqrt{x} - \sqrt{y})^2$ in (7). Consequently, Parts (i) and (ii)(c) of Lemma 5 shows that Assumptions 1 and 4 are satisfied since $\Theta \subseteq \check{\Theta}(q_1, q_2)$ and ϕ_{H^2} is continuous at $(0, 0)$. Assumption 3 holds by boundedness of H^2 distance. Assumption 2 follows since $D^{(1,0)}\phi_{H^2} \circ (p_\mu, p_\mu) = D^{(0,1)}\phi_{H^2} \circ (p_\mu, p_\mu) = 0$ from (86c). Proposition 1(ii) and (86c) now yield

$$\Phi'_{\theta^*}(h_1, h_2) = 0 \quad \text{and} \quad \Phi''_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} \frac{(h_1 - h_2)^2}{2p_\mu} d\rho,$$

for all $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{(g_1 - p_\mu)/t, (g_2 - p_\mu)/t) : (g_1 - p_\mu, g_2 - p_\mu) \in \Theta, t > 0\})$.

The convergence in (21) is then a consequence of Lemma 1(ii), together with $\Phi(0, 0) = 0$, $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\mu)) \xrightarrow{w} (B_1, B_2)$ in \mathfrak{D} , where $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\mu))$ and (B_1, B_2) take values in $\mathfrak{T}_\Theta(\theta^*)$.

Part (iv): Let $\mu_n, \nu_n, \mu, \nu \ll \rho$ for some finite measure ρ , and set $g_1^* = p_\mu$, $g_2^* = p_\nu$, $\psi_{2,0} \circ (p_\mu, p_\nu) = (p_\nu^{1/2} + q_2^{1/2})/p_\mu^{3/2}$, $\psi_{0,2} \circ (p_\mu, p_\nu) = (p_\mu^{1/2} + q_1^{1/2})/p_\nu^{3/2}$, $\psi_{1,1} \circ (p_\mu, p_\nu) = 1/(p_\mu p_\nu)^{1/2}$, $v_{2,0} = v_{0,2} = \tau^{1/2}(1 - \tau)^{-1/2}$, and $v_{1,1} = 1$. Further define the densities

$$p_{\eta_1} = 1 + \left(p_\nu^{\frac{1}{2}} + q_2^{\frac{1}{2}}\right)p_\mu^{-\frac{3}{2}} + p_\mu^{-\frac{1}{2}}p_\nu^{-\frac{1}{2}} \quad \text{and} \quad p_{\eta_2} = 1 + \left(p_\mu^{\frac{1}{2}} + q_1^{\frac{1}{2}}\right)p_\nu^{-\frac{3}{2}} + p_\mu^{-\frac{1}{2}}p_\nu^{-\frac{1}{2}},$$

and consider the spaces

$$\Theta = \check{\Theta}(q_1, q_2) = \left\{ (g_1 - p_\mu, g_2 - p_\nu) \in \mathfrak{D} : g_1, g_2 \geq 0, \|g_1\|_{1,\rho} = \|g_2\|_{1,\rho} = 1, |g_1| \leq q_1, |g_2| \leq q_2, \rho\text{-a.e.} \right\},$$

$$\mathfrak{D} = \left\{ (g_1 - p_\mu, g_2 - p_\nu) : g_1, g_2 \in L^1(\rho), \|(g_1 - p_\mu, g_2 - p_\nu)\|_{\mathfrak{D}} < \infty \right\}.$$

Note that under the hypothesis in Part (iv), $(p_{\mu_n} - p_\mu, p_{\nu_n} - p_\nu) \in \Theta$ a.s., and $H^2(\mu_n, \nu_n) = \Phi(p_{\mu_n} - p_\mu, p_{\nu_n} - p_\nu)$ with $\phi(x, y) = \phi_{H^2}(x, y) = (\sqrt{x} - \sqrt{y})^2$ in (7). Assumptions 1 and 4 as well as the continuity of ϕ_{H^2} at $(0, 0)$ holds via Lemma 5, Parts (i) and (ii)(c), while Assumption 3 follows by boundedness of H^2 . Assumption 2 is satisfied since from (86c), we have both $D^{(1,0)}\phi_{H^2} \circ (p_\mu, p_\nu) = 1 - p_\nu^{1/2}p_\mu^{-1/2}$ and $D^{(0,1)}\phi_{H^2} \circ (p_\mu, p_\nu) = 1 - p_\mu^{1/2}p_\nu^{-1/2}$ belonging to $L^2(\rho)$ because $p_\nu/p_\mu, p_\mu/p_\nu \in L^1(\rho)$ and ρ is a finite measure. Invoking Proposition 1(ii) and (86c), we obtain

$$\Phi'_{\theta^*}(h_1, h_2) = \int_{\mathfrak{S}} \left(1 - p_\nu^{\frac{1}{2}}p_\mu^{-\frac{1}{2}}\right) h_1 d\rho + \int_{\mathfrak{S}} \left(1 - p_\mu^{\frac{1}{2}}p_\nu^{-\frac{1}{2}}\right) h_2 d\rho = - \int_{\mathfrak{S}} p_\nu^{\frac{1}{2}}p_\mu^{-\frac{1}{2}} h_1 d\rho - \int_{\mathfrak{S}} p_\mu^{\frac{1}{2}}p_\nu^{-\frac{1}{2}} h_2 d\rho,$$

for all $(h_1, h_2) \in \mathfrak{T}_{\Theta}(\theta^*) = \text{cl}(\{(g_1 - p_\mu)/t, (g_2 - p_\nu)/t : (g_1 - p_\mu, g_2 - p_\nu) \in \Theta, t > 0\})$.

The desired result, namely (22), then follows from Lemma 1(i), along with $\Phi(0, 0) = H^2(\mu, \nu)$, $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\nu)) \xrightarrow{w} (B_1, B_2)$ in \mathfrak{D} with $(r_n(p_{\mu_n} - p_\mu), r_n(p_{\nu_n} - p_\nu))$ and (B_1, B_2) taking values in $\mathfrak{T}_{\Theta}(\theta^*)$.

The proof for the one-sample null follows via analogous arguments to the two-sample null by taking $\mu_n, \mu \ll \rho$ for some finite measure ρ , $g_1^* = g_2^* = p_\mu$, $\psi_{2,0} \circ (p_\mu, p_\mu) = 1/p_\mu$, $\psi_{0,2} = \psi_{1,1} = 0$, $v_{2,0} = (1 - \tau)^{-1/2}$, $v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 1 + (1/p_\mu)$, $\eta_2 = \rho$, and

$$\mathfrak{D} = \left\{ (g_1 - p_\mu, g_2 - p_\mu) : g_1, g_2 \in L^1(\rho), \|(g_1 - p_\mu, g_2 - p_\mu)\|_{\mathfrak{D}} < \infty \right\},$$

$$\Theta = \left\{ (g_1 - p_\mu, 0) \in \mathfrak{D} : g_1 \geq 0, \|g_1\|_{1,\rho} = 1 \right\}.$$

In a similar vein, the proof for the one-sample alternative is akin to the two-sample alternative, by considering $\mu_n, \mu, \nu \ll \rho$ for some finite measure ρ , and set $g_1^* = p_\mu$, $g_2^* = p_\nu$, $\psi_{2,0} \circ (p_\mu, p_\nu) = p_\nu^{1/2}/p_\mu^{3/2}$, $\psi_{0,2} = \psi_{1,1} = 0$, $v_{2,0} = (1 - \tau)^{-1/2}$, $v_{0,2} = v_{1,1} = 1$, $p_{\eta_1} = 1 + (p_\nu^{1/2}/p_\mu^{3/2})$, $\eta_2 = \rho$, and

$$\mathfrak{D} = \left\{ (g_1 - p_\mu, g_2 - p_\nu) : g_1, g_2 \in L^1(\rho), \|(g_1 - p_\mu, g_2 - p_\nu)\|_{\mathfrak{D}} < \infty \right\},$$

$$\Theta = \left\{ (g_1 - p_\mu, 0) \in \mathfrak{D} : g_1 \geq 0, \|g_1\|_{1,\rho} = 1 \right\}.$$

4) Proof of Theorem 5: We prove the one- and two-sample null cases together. Then, we prove both statements under the alternative.

(One-sample and two-sample null): The results in these cases follow directly by the continuous mapping theorem.

To see this, note that $f \mapsto \|f\|_{\text{TV}} = \|f\|_{1,\rho}/2$ is a continuous functional in $L^1(\rho)$. With that, the convergence of $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ and $r_n(p_{\nu_n} - p_\nu) \xrightarrow{w} B$ in $L^1(\rho)$ imply

$$r_n \|\mu_n - \mu\|_{\text{TV}} \xrightarrow{d} \frac{1}{2} \int_{\mathfrak{S}} |B| d\rho \quad \text{and} \quad r_n \|\mu_n - \nu_n\|_{\text{TV}} \xrightarrow{d} \frac{1}{2} \int_{\mathfrak{S}} |B| d\rho,$$

respectively. Note that the above limit distributions are special cases of (23) and (24) for $\mu = \nu$, since $\mathcal{Q} = \mathfrak{S}$ in

this case and the second terms in the respective equations vanish.

(One-sample and two-sample alternative): Note that $\|\mu - \nu\|_{\text{TV}} = \Phi(0, 0)$ with $\phi(x, y) = |x - y|/2$, $g_1^* = p_\mu$ and $g_2^* = p_\nu$ in (7). Hence Assumption 1 is violated (at $x = y$), and the Hadamard differentiability framework in Section III does not apply directly. To circumvent that, we provide a direct argument for computing the Hadamard directional derivative in this case.

Let $\mathfrak{D} = L^1(\rho)$, $g_1^*, g_2^* \in L^1(\rho)$ be arbitrary, set

$$\hat{\Theta} = \{g = g_1 - g_2 : g_1, g_2 \in L^1(\rho), g_1, g_2 \geq 0, \|g_1\|_{1,\rho} = \|g_2\|_{1,\rho} = 1\},$$

and consider the functional $\Upsilon : \hat{\Theta} \rightarrow \mathbb{R}_{\geq 0}$ given by $\Upsilon(g) = \int_{\mathfrak{S}} |g|/2 d\rho$. We will show that Υ is locally Lipschitz at $\theta^* = g_1^* - g_2^* \in \hat{\Theta}$, whereby its Hadamard directional derivative at θ^* coincides with the Gâteaux directional derivative [23], [24], which is defined by

$$\Upsilon'_{\theta^*}(h) = \lim_{t_n \downarrow 0^+} \frac{\Upsilon(g_1^* - g_2^* + t_n h) - \Upsilon(g_1^* - g_2^*)}{t_n}, \quad h \in \mathfrak{D}.$$

For the local Lipschitzness, observe that

$$\Upsilon(g) - \Upsilon(\tilde{g}) = \frac{1}{2} \int_{\mathfrak{S}} |g| d\rho - \frac{1}{2} \int_{\mathfrak{S}} |\tilde{g}| d\rho \leq \frac{1}{2} \int_{\mathfrak{S}} |g - \tilde{g}| d\rho = \frac{1}{2} \|g - \tilde{g}\|_{1,\rho},$$

and switch the roles of g and \tilde{g} to conclude that $|\Upsilon(g) - \Upsilon(\tilde{g})| \leq \|g - \tilde{g}\|_{1,\rho}/2$.

To compute the Gâteaux derivative of Υ , set $\mathcal{Q}^* := \{s \in \mathfrak{S} : g_1^*(s) = g_2^*(s)\}$ and write

$$\frac{\Upsilon(g_1^* - g_2^* + th) - \Upsilon(g_1^* - g_2^*)}{t} = \frac{1}{2} \int_{\mathcal{Q}^*} |h| d\rho + \frac{1}{2} \int_{\mathfrak{S} \setminus \mathcal{Q}^*} \frac{|g_1^* - g_2^* + th| - |g_1^* - g_2^*|}{t} d\rho. \quad (50)$$

Note that the integrand in the last term above is dominated by $|h| \in L^1(\rho)$ ρ -a.e. and that on $\mathfrak{S} \setminus \mathcal{Q}^*$, we have

$$\lim_{t \downarrow 0^+} \frac{|g_1^* - g_2^* + th| - |g_1^* - g_2^*|}{t} = \text{sgn}(g_1^* - g_2^*)h,$$

where $\text{sgn}(x) = x/|x|$ for $x \neq 0$ and $\text{sgn}(0) = 0$. An application of dominated convergence theorem to the last term in (50) then yields the Gâteaux derivative at θ^* given by

$$\Upsilon'_{\theta^*}(h) := \lim_{t \downarrow 0^+} \frac{\Upsilon(g_1^* - g_2^* + th) - \Upsilon(g_1^* - g_2^*)}{t} = \frac{1}{2} \int_{\mathcal{Q}^*} |h| d\rho + \frac{1}{2} \int_{\mathfrak{S} \setminus \mathcal{Q}^*} \text{sgn}(g_1^* - g_2^*) h d\rho, \quad (51)$$

which coincides with the Hadamard derivative due to the aforementioned local Lipschitzness of the functional.

We are now in place to prove the limit distribution results. For the one-sample alternative case, let $\mu_n, \mu, \nu \ll \rho$ for some measure ρ , set $\theta^* = p_\mu - p_\nu$, and consider

$$\Theta = \{g_1 - p_\nu : g_1 \in L^1(\rho), g_1 \geq 0, \|g_1\|_{1,\rho} = 1\} \subset \hat{\Theta}.$$

Note that $\mathfrak{T}_\Theta(\theta^*) = \text{cl}\left(\{(g_1 - p_\mu)/t : g_1 - p_\nu \in \Theta, t > 0\}\right)$ since Θ is convex. Setting $\mathcal{Q} := \{s \in \mathfrak{S} : p_\mu(s) =$

$p_\nu(s)\}$, the Hadamard directional derivative of Υ at θ^* is

$$\Upsilon'_{\theta^*}(h) = \frac{1}{2} \int_{\mathcal{Q}} |h| d\rho + \frac{1}{2} \int_{\mathcal{S} \setminus \mathcal{Q}} \text{sgn}(p_\mu - p_\nu) h d\rho, \quad h \in \mathfrak{T}_\Theta(\theta^*).$$

Lemma 1(i) along with the fact that $r_n(p_{\mu_n} - p_\mu) \xrightarrow{w} B$ in $L^1(\rho)$ with $r_n(p_{\mu_n} - p_\mu)$ and B taking values in $\mathfrak{T}_\Theta(\theta^*)$, leads to

$$r_n(\|p_{\mu_n} - p_\nu\|_{\text{TV}} - \|p_\mu - p_\nu\|_{\text{TV}}) = r_n(\Upsilon(p_{\mu_n} - p_\nu) - \Upsilon(p_\mu - p_\nu)) \xrightarrow{d} \Upsilon'_{\theta^*}(B),$$

which proves (23).

For two-sample case, let $\mu_n, \nu_n, \mu, \nu \ll \rho$ for some measure ρ , $\theta^* = p_\mu - p_\nu$, and $\Theta = \hat{\Theta}$. Again, $\mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{(g - (p_\mu - p_\nu))/t : g \in \Theta, t > 0\})$ since Θ is convex. The claim in (24) then follows using same arguments as above via Lemma 1(i) together with $r_n(p_{\mu_n} - p_{\nu_n} - (p_\mu - p_\nu)) \xrightarrow{w} B$ in $L^1(\rho)$ with $r_n(p_{\mu_n} - p_{\nu_n} - (p_\mu - p_\nu))$ and B supported in $\mathfrak{T}_\Theta(\theta^*)$. This completes the proof.

C. Proofs for Section V

To establish the limit distribution for Gaussian-smoothed f -divergences, we first use the CLT in L^r spaces to deduce weak convergence of the smooth empirical process, and then invoke the general limit distribution theorems from Section IV. The CLT in L^r spaces is stated next.

Theorem 6 (Proposition 2.1.11 in [25]) *Let $1 \leq r < \infty$, and Z, Z_1, \dots, Z_n be i.i.d. $L^r(S, \mathcal{S}, \rho)$ -valued random variables (recall ρ is σ -finite) with zero mean (in the sense of Bochner). The following are equivalent:*

- (i) *There exists a centered Gaussian process G in L^r with same covariance function as Z such that $n^{-1/2} \sum_{i=1}^n Z_i$ converges weakly in L^r to G .*
- (ii) $\int_{\mathcal{S}} (\mathbb{E}[|Z(s)|^2])^{r/2} d\rho(s) < \infty$ and $\mathbb{P}(\|Z\|_{r,\rho} \geq t) = o(t^{-2})$ as $t \rightarrow \infty$.

Henceforth, in this section, we apply the general results from Section 4, namely, Theorems 1-4, with $\mu \leftarrow \mu * \gamma_\sigma$, $\nu \leftarrow \nu * \gamma_\sigma$, $\mu_n = \hat{\mu}_n * \gamma_\sigma$ and $\nu_n = \hat{\nu}_n * \gamma_\sigma$. The reference measure ρ will be adapted on a case-by-case basis and specified in the proofs below.

1) Proof of Proposition 2 : Part (i): To prove (27), we apply Theorem 2(i) with $\rho = \mu * \gamma_\sigma$. To verify that the required conditions hold, first observe that $\mu * \gamma_\sigma \ll \hat{\mu}_n * \gamma_\sigma$. Furthermore, for $\hat{X} \sim \hat{\mu}_n$ and $X \sim \mu$ independent of $Y \sim \mu$, we have

$$\begin{aligned} \mathbb{E}[\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] &= \mathbb{E}\left[\text{D}_{\text{KL}}\left(\mathbb{E}_{\hat{\mu}_n}[N(\hat{X}, \sigma^2 I_d)] \middle\| \mathbb{E}_\mu[N(Y, \sigma^2 I_d)]\right)\right] \\ &\stackrel{(a)}{\leq} \mathbb{E}\left[\text{D}_{\text{KL}}(N(X, \sigma^2 I_d) \| N(Y, \sigma^2 I_d))\right] \\ &\stackrel{(b)}{=} \frac{1}{2\sigma^2} \mathbb{E}[\|X - Y\|^2] < \infty, \end{aligned}$$

where (a) is by Jensen's inequality applied to KL divergence which is jointly convex in its arguments, (b) uses the closed-form expression for KL divergence between multivariate Gaussian distributions (see [76]), and the last

inequality is because all moments of a sub-Gaussian distribution are finite (see [77]). Since KL divergence is nonnegative, this implies that $D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma) < \infty$ a.s. Setting $r_n = n^{1/2}$, (27) follows from (11) provided that

$$n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\mu * \varphi_\sigma} - 1 \right) \xrightarrow{w} \frac{G_{\mu, \sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\mu * \gamma_\sigma). \quad (52)$$

It thus remains to establish (52), for which we use Theorem 6. Note that $n^{1/2}((\hat{\mu}_n * \varphi_\sigma / \mu * \varphi_\sigma) - 1) = n^{-1/2} \sum_{i=1}^n Z_i$, where

$$Z_i = \frac{\varphi(\cdot - X_i) - \mu * \varphi_\sigma}{\mu * \varphi_\sigma}, \quad i = 1, \dots, n, \quad (53)$$

are centered i.i.d. random variables with the same distribution as $Z = (\varphi(\cdot - X) - \mu * \varphi_\sigma) / \mu * \varphi_\sigma$. Further observe that the process Z_i is jointly measurable when viewed as a map from $(\Omega \times \mathbb{R}^d, \mathcal{A} \times \mathcal{B}(\mathbb{R}^d))$ to \mathbb{R} , and has paths a.s. in $L^2(\mu * \gamma_\sigma)$; indeed,

$$\mathbb{E}[\|Z_i\|_{2, \mu * \gamma_\sigma}^2] = \int_{\mathbb{R}^d} \mathbb{E}_\mu[|Z_i(x)|^2] \mu * \varphi_\sigma(x) dx = \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx < \infty, \quad (54)$$

by Fubini's theorem and the assumption in (26). This finiteness also implies the conditions in Theorem 6(ii). The first follows directly from the equation above, while for the second, we show that if $\mathbb{P}(\|Z\|_{2, \mu * \gamma_\sigma} \geq t) = o(t^{-2})$ as $t \rightarrow \infty$ does not hold, then $\mathbb{E}[\|Z_i\|_{2, \mu * \gamma_\sigma}^2] < \infty$ is contradicted. To see this, note that the violation of the former condition implies that there exists constants $c > 0$ and $t_0 \geq 1$ such that for all $t \geq t_0$, $\mathbb{P}(\|Z\|_{2, \mu * \gamma_\sigma} \geq t) \geq ct^{-2}$. Then

$$\mathbb{E}[\|Z\|_{2, \mu * \gamma_\sigma}^2] = \int_0^\infty \mathbb{P}(\|Z\|_{2, \mu * \gamma_\sigma} \geq \sqrt{t}) dt \geq \int_{t_0}^\infty ct^{-1} dt = \infty,$$

which is the desired contradiction. Hence, the conditions in Theorem 6(ii) are satisfied under (26). Consequently, by Theorem 6(i), there exists a centered Gaussian process $G_{\mu, \sigma} / \mu * \varphi_\sigma$ such that

$$n^{-1/2} \sum_{i=1}^n Z_i \xrightarrow{w} \frac{G_{\mu, \sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\mu * \gamma_\sigma),$$

which completes the proof of (27).

The claim that (26) and (27) holds for β -sub-Gaussian μ with $\beta < \sigma$ follows from Proposition 1 in [78] by noting that the LHS of (26) equals $I_{\chi^2}(V; W)$, where $I_{\chi^2}(V; W) := \chi^2(P_{V, W} \| P_V \times P_W)$ is the χ^2 mutual information between V and $W = V + Z$, where $V \sim \mu$ and $Z \sim \gamma_\sigma$ are independent. Finally, we observe that if (26) is violated then $\liminf_{n \rightarrow \infty} n \mathbb{E}[D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] = \infty$. This is a consequence of Lemma 2, by noting that f_{KL} is continuously twice differentiable with positive second derivative.

Part (ii): To prove (29), we use Theorem 2(ii) with $\rho = \nu * \gamma_\sigma$. Note that $\mu * \gamma_\sigma, \hat{\mu}_n * \gamma_\sigma, \nu * \gamma_\sigma$ are all mutually absolutely continuous, and $p_{\mu * \gamma_\sigma} = \mu * \varphi_\sigma / \nu * \varphi_\sigma > 0$. Moreover, $\log(\mu * \varphi_\sigma / \nu * \varphi_\sigma) \in L^2(\nu * \varphi_\sigma)$ by assumption, and using similar steps as in Part (i), we have $D_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma) < \infty$ and $D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) < \infty$ a.s.

Hence, we have from (12) with $r_n = n^{1/2}$ that

$$n^{\frac{1}{2}} (\mathcal{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) - \mathcal{D}_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)) \xrightarrow{d} \int_{\mathbb{R}^d} G_{\mu, \sigma}(x) \log \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right) dx, \quad (55)$$

provided

$$n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\nu * \varphi_\sigma} - \frac{\mu * \varphi_\sigma}{\nu * \varphi_\sigma} \right) \xrightarrow{w} \frac{G_{\mu, \sigma}}{\nu * \varphi_\sigma} \quad \text{in } L^2(\eta), \quad (56)$$

where η has relative density $p_\eta = 1 + (\nu * \varphi_\sigma / \mu * \varphi_\sigma)$. To prove (29), it remains to show that (56) holds and that the RHS of (55) is $N(0, v_{1, f_{\text{KL}}}^2(\mu, \nu, \sigma))$.

We first show that (56) holds given (26), (28), and $\|(\nu * \varphi_\sigma)^2 / \mu * \varphi_\sigma\|_\infty < \infty$. The latter along with the fact that $\nu * \gamma_\sigma$ is a probability measure implies that η is σ -finite. Next, observe that $n^{1/2}(\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma) / \nu * \varphi_\sigma = n^{-1/2} \sum_{i=1}^n Z_i$, where $Z_i = (\varphi(\cdot - X_i) - \mu * \varphi_\sigma) / \nu * \varphi_\sigma$, for $i = 1, \dots, n$, are centered and i.i.d. with the same distribution as $Z = (\varphi(\cdot - X) - \mu * \varphi_\sigma) / \nu * \varphi_\sigma$. The claim then follows from Theorem 6(ii), given that the conditions therein are satisfied. Akin to the proof of Part (i) above, the process Z_i is jointly measurable with paths a.s. in $L^2(\eta)$. Indeed

$$\begin{aligned} \mathbb{E}[\|Z_i\|_{2, \eta}^2] &= \int_{\mathbb{R}^d} \mathbb{E}_\mu[|Z_i(x)|^2] \left(\nu * \varphi_\sigma(x) + \frac{(\nu * \varphi_\sigma(x))^2}{\mu * \varphi_\sigma(x)} \right) dx \\ &= \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx < \infty, \end{aligned} \quad (57)$$

where the first equality is by Fubini's theorem and last inequality is due to (26) and (28). This finiteness implies the conditions in Theorem 6(ii): the first follows trivially, while the second follows via similar arguments as in Part (i). Invoking Theorem 6, there exists a centered Gaussian process $G_{\mu, \sigma} / \nu * \varphi_\sigma$ such that (56) holds.

Next, we claim that the RHS of (55) is zero mean Gaussian random variable with variance $v_{1, f_{\text{KL}}}^2(\mu, \nu, \sigma)$. This follows from the dual characterization of a random variable B taking values in a Banach space \mathfrak{B} as Gaussian if and only if $f(B)$ is a real-valued Gaussian random variable for every f in the topological dual space of continuous linear functionals on \mathfrak{B} (see [79, Page 55]). Hence, the RHS of (55) is a real Gaussian random variable because $G_{\mu, \sigma} / \sqrt{\nu * \varphi_\sigma}$ is $L^2(\mathbb{R}^d)$ -valued Gaussian random variable and $\sqrt{\nu * \varphi_\sigma} \log(\mu * \varphi_\sigma / \nu * \varphi_\sigma) \in L^2(\mathbb{R}^d)$ (by assumption). Computing the mean and variance explicitly leads to the claim above with

$$v_{1, f_{\text{KL}}}^2(\mu, \nu, \sigma) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(1,1)}(x, y) \log \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right) \log \left(\frac{\mu * \varphi_\sigma(y)}{\nu * \varphi_\sigma(y)} \right) dx dy, \quad (58)$$

where we used $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(1,1)}(x, y) dx dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{cov}(\varphi_\sigma(x - X), \varphi_\sigma(y - X)) dx dy = 0$, which in itself follows from Fubini's theorem and $\int_{\mathbb{R}^d} \varphi_\sigma(x - \cdot) dx = 1$.

To prove the final claim in Part (ii), we note that for every $x \in \mathbb{R}^d$,

$$\left| \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right| = \left| \frac{\mathbb{E}_\mu[\varphi_\sigma(x - \cdot)]}{\mathbb{E}_\nu[\varphi_\sigma(x - \cdot)]} \right| \leq \left\| \frac{d\mu}{d\nu} \right\|_\infty < \infty, \quad (59)$$

which implies $\|\mu * \varphi_\sigma / \nu * \varphi_\sigma\|_\infty < \infty$. Similarly, $\|\nu * \varphi_\sigma / \mu * \varphi_\sigma\|_\infty < \infty$. This leads to $\|(\nu * \varphi_\sigma)^2 / \mu * \varphi_\sigma\|_\infty < \infty$.

∞ and $\log(\mu * \varphi_\sigma / \nu * \varphi_\sigma) \in L^2(\nu * \varphi_\sigma)$. On the other hand, for β -sub-Gaussian μ with $\beta < \sigma$, (26) holds using the same argument as in Part (i), while (28) follows since

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx \leq \left\| \frac{\mu * \varphi_\sigma}{\nu * \varphi_\sigma} \right\|_\infty \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma} dx < \infty. \quad (60)$$

Hence, all the conditions needed for (29) are satisfied.

Part (iii): We apply Theorem 2(iii) with $\rho = \mu * \gamma_\sigma$. We have that $\mu * \gamma_\sigma$, $\hat{\mu}_n * \gamma_\sigma$, $\hat{\nu}_n * \gamma_\sigma$ are all mutually absolutely continuous, and $p_{\hat{\nu}_n * \gamma_\sigma} = \hat{\nu}_n * \gamma_\sigma / \mu * \varphi_\sigma > 0$. Moreover, we have $D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) < \infty$ a.s. via steps similar to those in Part (i). Also, since μ has compact support and $\text{spt}(\hat{\mu}_n), \text{spt}(\hat{\nu}_n) \subset \text{spt}(\mu)$, we have

$$\left| \frac{\hat{\mu}_n * \varphi_\sigma(x)}{\hat{\nu}_n * \varphi_\sigma(x)} \right| = \frac{\sum_{i=1}^n e^{-\frac{\|x - X_i\|^2}{2\sigma^2}}}{\sum_{i=1}^n e^{-\frac{\|x - Y_i\|^2}{2\sigma^2}}} \leq \max_{1 \leq i \leq n} e^{-\frac{\|x - X_i\|^2}{2\sigma^2} + \frac{\|x - Y_i\|^2}{2\sigma^2}} \leq c' e^{c\|x\|}, \quad (61)$$

for some constants $c, c' > 0$ that depend only on σ and $\text{spt}(\mu)$. Taking $q(x) = c' e^{c\|x\|}$, we have

$$\begin{aligned} \frac{d\eta_1}{dx}(x) &= \mu * \varphi_\sigma(x), \\ \frac{d\eta_2}{dx}(x) &= \mu * \varphi_\sigma(x) + \mu * \varphi_\sigma(x) c' e^{c\|x\|}. \end{aligned}$$

Recall that $n^{1/2}((\hat{\mu}_n * \varphi_\sigma / \mu * \varphi_\sigma) - 1) = n^{-1/2} \sum_{i=1}^n Z_i$, where Z_i is given in (53). Let \tilde{Z}_i be defined similarly with X_i replaced by Y_i , so that $n^{1/2}((\hat{\nu}_n * \varphi_\sigma / \mu * \varphi_\sigma) - 1) = n^{-1/2} \sum_{i=1}^n \tilde{Z}_i$. Setting $r_n = n^{1/2}$, it follows from (13) that

$$n D_{\text{KL}}(\hat{\mu}_n * \varphi_\sigma \| \hat{\nu}_n * \varphi_\sigma) \xrightarrow{d} \frac{1}{2} \int_{\mathfrak{S}} \frac{(G_{\mu, \sigma}(x) - \tilde{G}_{\mu, \sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx$$

so long that

$$\left(n^{-1/2} \sum_{i=1}^n Z_i, n^{-1/2} \sum_{i=1}^n \tilde{Z}_i \right) \xrightarrow{w} \left(\frac{G_{\mu, \sigma}}{\mu * \varphi_\sigma}, \frac{\tilde{G}_{\mu, \sigma}}{\mu * \varphi_\sigma} \right) \quad \text{in } L^2(\eta_1) \times L^2(\eta_2),$$

where $(G_{\mu, \sigma}, \tilde{G}_{\mu, \sigma}) := (G_{\mu, \sigma}(x), \tilde{G}_{\mu, \sigma}(y))_{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d}$ is a 2-dimensional Gaussian process.

A sufficient condition for the joint weak convergence above is

$$n^{-1/2} \sum_{i=1}^n Z_i \xrightarrow{w} \frac{G_{\mu, \sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\eta_1) \quad \text{and} \quad n^{-1/2} \sum_{i=1}^n \tilde{Z}_i \xrightarrow{w} \frac{\tilde{G}_{\mu, \sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\eta_2). \quad (62)$$

To see this, first observe that both η_1 and η_2 are finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and hence $L^2(\eta_1)$ and $L^2(\eta_2)$ are Polish. Furthermore, given (62), $(n^{-1/2} \sum_{i=1}^n \tilde{Z}_i)_{n \in \mathbb{N}}$ and $(n^{-1/2} \sum_{i=1}^n \tilde{Z}_i)_{n \in \mathbb{N}}$ are both asymptotically tight and asymptotically measurable in $L^2(\eta_1)$ and $L^2(\eta_2)$, respectively [25, Lemma 1.3.8]. This implies that $(n^{-1/2} \sum_{i=1}^n \tilde{Z}_i, n^{-1/2} \sum_{i=1}^n \tilde{Z}_i)_{n \in \mathbb{N}}$ are jointly asymptotically tight and jointly measurable in $L^2(\eta_1) \times L^2(\eta_2)$ [25, Lemma 1.4.3 and 1.4.4]. Then, by Polishness of $L^2(\eta_1) \times L^2(\eta_2)$, the desired joint weak convergence holds if the finite-dimensional marginals of the joint process converge weakly (see [64, Lemma 16]), i.e., for every $x_1, \dots, x_m, \tilde{x}_1, \dots, \tilde{x}_l \in \mathbb{R}^d$,

$$\left(n^{-1/2} \sum_{i=1}^n Z_i(x_1), \dots, n^{-1/2} \sum_{i=1}^n Z_i(x_m), n^{-1/2} \sum_{i=1}^n \tilde{Z}_i(\tilde{x}_1), \dots, n^{-1/2} \sum_{i=1}^n \tilde{Z}_i(\tilde{x}_l) \right) \quad (63)$$

converges weakly. The latter follows from the multivariate CLT.

Next, we apply Theorem 6 to show that the sufficient condition given in (62) is satisfied when μ, ν have compact supports. As in Part (i), the process Z_i above is jointly measurable, has paths a.s. in $L^2(\eta_1)$, and satisfies the conditions in Theorem 6(ii). This follows by a similar argument as in Part (i), under the assumption in (26) holds. Hence, by Theorem 6, there exists a centered Gaussian process $G_{\mu, \sigma} / \mu * \varphi_\sigma$ such that the first weak convergence claim in the equation above holds. The second weak convergence claim holds via a similar argument with Z_i and η_1 replaced by \tilde{Z}_i and η_2 , respectively, provided that

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} (1 + e^{c\|x\|}) dx < \infty.$$

Thus, to prove (30), it remains to show that the above equation holds when μ has compact support. This follows via a direct computation:

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} e^{c\|x\|} dx \\ & \leq \int_{\mathbb{R}^d} \frac{\mathbb{E}_\mu[\varphi_\sigma^2(x - \cdot)]}{\mu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} \frac{\mathbb{E}_\mu[\varphi_\sigma^2(x - \cdot)]}{\mu * \varphi_\sigma(x)} e^{c\|x\|} dx \\ & \leq \int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{\sigma^2}} d\mu(y)}{\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{2\sigma^2}} d\mu(y)} dx + \int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{\sigma^2}} d\mu(y)}{\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{2\sigma^2}} d\mu(y)} e^{c\|x\|} dx \\ & \leq \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} \frac{\int_{\mathbb{R}^d} e^{\frac{2x \cdot y - \|y\|^2}{\sigma^2}} d\mu(y)}{\int_{\mathbb{R}^d} e^{\frac{2x \cdot y - \|y\|^2}{2\sigma^2}} d\mu(y)} dx + \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} \frac{\int_{\mathbb{R}^d} e^{\frac{2x \cdot y - \|y\|^2}{\sigma^2}} d\mu(y)}{\int_{\mathbb{R}^d} e^{\frac{2x \cdot y - \|y\|^2}{2\sigma^2}} d\mu(y)} e^{c\|x\|} dx \\ & \leq \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} \frac{\int_{\mathbb{R}^d} e^{\frac{2\|x\|\|y\| - \|y\|^2}{\sigma^2}} d\mu(y)}{\int_{\mathbb{R}^d} e^{\frac{-2\|x\|\|y\| - \|y\|^2}{2\sigma^2}} d\mu(y)} dx + \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} \frac{\int_{\mathbb{R}^d} e^{\frac{2\|x\|\|y\| - \|y\|^2}{\sigma^2}} d\mu(y)}{\int_{\mathbb{R}^d} e^{\frac{-2\|x\|\|y\| - \|y\|^2}{2\sigma^2}} d\mu(y)} e^{c\|x\|} dx \\ & \leq c' \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} e^{c\|x\|} dx < \infty, \end{aligned} \tag{64}$$

for some constants $c, c' > 0$ which depends on σ and $\text{spt}(\mu)$.

Part (iv): To prove (31), we utilize Theorem 2(iv) with $\rho = \nu * \gamma_\sigma$. The positivity and absolute continuity of the probability measures as well as finiteness of KL divergences required in Theorem 2(iv) follow from similar steps as above, which are thus omitted. Likewise, (61) holds (possibly with different constants c, c') since μ and ν have compact supports. Moreover, using Jensen's inequality and steps similar to those leading to (64), we have

$$\begin{aligned} \frac{\nu * \varphi_\sigma(x)^2}{\mu * \varphi_\sigma(x)} & \leq \frac{\mathbb{E}_\nu[\varphi_\sigma^2(x - \cdot)]}{\mu * \varphi_\sigma(x)} = e^{-\frac{\|x\|^2}{2\sigma^2}} \frac{\int_{\mathbb{R}^d} e^{\frac{2\|x\|\|y\| - \|y\|^2}{\sigma^2}} d\nu(y)}{\int_{\mathbb{R}^d} e^{\frac{-2\|x\|\|y\| - \|y\|^2}{2\sigma^2}} d\mu(y)} \leq c' e^{-\frac{\|x\|^2}{2\sigma^2}} e^{c\|x\|} \leq \tilde{c}, \\ \frac{\nu * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} & \leq \frac{\int_{\mathbb{R}^d} e^{\frac{2\|x\|\|y\| - \|y\|^2}{2\sigma^2}} d\nu(y)}{\int_{\mathbb{R}^d} e^{\frac{-2\|x\|\|y\| - \|y\|^2}{2\sigma^2}} d\mu(y)} \leq c' e^{c\|x\|}, \end{aligned} \tag{65}$$

for some constants $c, c', \tilde{c} > 0$ which depends only on σ and the supports of μ, ν . Hence, $\|\nu * \varphi_\sigma^2 / \mu * \varphi_\sigma\|_\infty < \infty$ and $\mu * \varphi_\sigma, \log(\mu * \varphi_\sigma / \nu * \varphi_\sigma) \in L^2(\nu * \varphi_\sigma)$ (note the above inequalities also hold with μ and ν interchanged).

Next, observe that the measures η_1 and η_2 given in Theorem 2(*iv*) with $q(x) = c'e^{c\|x\|}$ have Lebesgue densities

$$\begin{aligned}\frac{d\eta_1}{dx}(x) &= \nu * \varphi_\sigma(x) + \frac{(\nu * \varphi_\sigma(x))^2}{\mu * \varphi_\sigma(x)}, \\ \frac{d\eta_2}{dx}(x) &= \nu * \varphi_\sigma(x) + c'\nu * \varphi_\sigma(x)e^{c\|x\|} + \mu * \varphi_\sigma(x),\end{aligned}\tag{66}$$

and hence are finite measures based on the inequalities above. Then, the Polishness of $L^2(\eta_1) \times L^2(\eta_2)$ and the discussion in Part (*iii*) implies via (14) with $r_n = n^{1/2}$ that

$$\begin{aligned}n^{\frac{1}{2}} \left(D_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - D_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma) \right) \\ \xrightarrow{d} \int_{\mathbb{R}^d} G_{\mu,\sigma}(x) \log \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right) dx - \int_{\mathbb{R}^d} G_{\nu,\sigma}(x) \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} dx,\end{aligned}\tag{67}$$

provided that

$$\begin{aligned}n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\nu * \varphi_\sigma} - \frac{\mu * \varphi_\sigma}{\nu * \varphi_\sigma} \right) &\xrightarrow{w} \frac{G_{\mu,\sigma}}{\nu * \varphi_\sigma} \text{ in } L^2(\eta_1), \\ n^{1/2} \left(\frac{\hat{\nu}_n * \varphi_\sigma}{\nu * \varphi_\sigma} - 1 \right) &\xrightarrow{w} \frac{G_{\nu,\sigma}}{\nu * \varphi_\sigma} \text{ in } L^2(\eta_2).\end{aligned}\tag{68}$$

Resorting to Theorem 6 once more, we next show that the weak convergence requirements in (68) hold if

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx < \infty,$$

and

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} (1 + e^{c\|x\|}) dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot)) \mu * \varphi_\sigma(x)}{(\nu * \varphi_\sigma(x))^2} dx < \infty,$$

respectively. To see this, note that the second term in the penultimate equation and the first term in the last equation can be bounded as shown in Part (*iii*). For the remaining terms, we have from (59) via steps leading to (64)-(65) that

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot)) \mu * \varphi_\sigma(x)}{(\nu * \varphi_\sigma(x))^2} dx \leq c' \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} e^{c\|x\|} \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} dx \leq c'^2 \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} e^{2c\|x\|} dx < \infty,$$

and

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx \leq \int_{\mathbb{R}^d} \frac{\mathbb{E}_\mu[\varphi_\sigma^2(x - \cdot)]}{\nu * \varphi_\sigma(x)} dx \leq c' \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} e^{c\|x\|} dx < \infty.$$

Finally, the proof of the claim is completed by noting that the RHS of (67) is a zero mean Gaussian random variable with variance

$$\begin{aligned}v_{2,f_{\text{KL}}}^2(\mu, \nu, \sigma) &:= \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(i, j)}(x, y) L_{i, f_{\text{KL}}}(x) L_{j, f_{\text{KL}}}(y) dx dy \\ &= \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(i, j)}(x, y) \tilde{L}_{i, f_{\text{KL}}}(x) \tilde{L}_{j, f_{\text{KL}}}(y) dx dy,\end{aligned}\tag{69}$$

where $\tilde{L}_{1,f_{\text{KL}}} := \log(\mu * \varphi_\sigma / \nu * \varphi_\sigma)$ and $\tilde{L}_{2,f_{\text{KL}}} := -\mu * \varphi_\sigma / \nu * \varphi_\sigma$. The second equality above uses

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(1,1)}(x, y) dx dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{cov}(\varphi_\sigma(x - X), \varphi_\sigma(y - X)) dx dy = 0.$$

2) Proof of Lemma 2: Without loss of generality, we may assume that there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $\mathbb{E}[\mathcal{D}_f(\hat{\mu}_{n_k} * \gamma_\sigma \| \mu * \gamma_\sigma)] < \infty$ for all $k \in \mathbb{N}$, since otherwise the LHS of (32) is infinite and there is nothing to prove. Henceforth, we take n within such a subsequence. Note that since $\hat{\mu}_n * \gamma_\sigma \ll \mu * \gamma_\sigma$, $\mathcal{D}_f(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma) = \mathbb{E}_{\mu * \gamma_\sigma}[f \circ (\hat{\mu}_n * \varphi_\sigma / \mu * \varphi_\sigma)]$. Applying Taylor's expansion of $f(x)$ at $x = 1$ and observing that $f(1) = 0$, we have

$$f\left(\frac{\hat{\mu}_n * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)}\right) = f'(1)\left(\frac{\hat{\mu}_n * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} - 1\right) + \int_0^1 (1 - \tau) f''\left((1 - \tau) + \tau \frac{\hat{\mu}_n * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)}\right) \left(\frac{\hat{\mu}_n * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} - 1\right)^2 d\tau. \quad (70)$$

Let $\vartheta_n(x) := n^{1/2}(\hat{\mu}_n * \varphi_\sigma(x) - \mu * \varphi_\sigma(x))$. Taking expectation first w.r.t. to $\mu * \gamma_\sigma$ and then w.r.t. $\mu^{\otimes n}$ in the above equation, and observing that the first term on the RHS integrates to zero, we obtain

$$\begin{aligned} n\mathbb{E}[\mathcal{D}_f(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] &= \mathbb{E}\left[\int_{\mathbb{R}^d} \int_0^1 (1 - \tau) f''\left(\frac{\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} \vartheta_n(x)}{\mu * \varphi_\sigma(x)}\right) \frac{\vartheta_n^2(x)}{\mu * \varphi_\sigma(x)} d\tau dx\right] \\ &= \int_{\mathbb{R}^d} \int_0^1 (1 - \tau) \mathbb{E}\left[f''\left(\frac{\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} \vartheta_n(x)}{\mu * \varphi_\sigma(x)}\right) \frac{\vartheta_n^2(x)}{\mu * \varphi_\sigma(x)}\right] d\tau dx, \end{aligned}$$

where the final equality uses Fubini's theorem. Noting that the integrand is nonnegative (by non-negativity of f''), Fatou's lemma implies

$$\liminf_{n \rightarrow \infty} n\mathbb{E}[\mathcal{D}_f(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] \geq \int_{\mathbb{R}^d} \int_0^1 (1 - \tau) \liminf_{n \rightarrow \infty} \mathbb{E}\left[f''\left(\frac{\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} \vartheta_n(x)}{\mu * \varphi_\sigma(x)}\right) \frac{\vartheta_n^2(x)}{\mu * \varphi_\sigma(x)}\right] d\tau dx. \quad (71)$$

Next, observe that since $\vartheta_n(x) = \sum_{i=1}^n Z_i(x) / \sqrt{n}$ with $Z_i(x) = \varphi_\sigma(x - X_i) - \mu * \varphi_\sigma(x)$ and $|Z_i(x)| \leq (2\pi\sigma^2)^{-d/2}$, the CLT implies that for every $k \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[f''\left(\frac{\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} \vartheta_n(x)}{\mu * \varphi_\sigma(x)}\right) \vartheta_n^2(x) \wedge k\right] = \mathbb{E}[f''(1) G_{\mu, \sigma}^2(x) \wedge k]. \quad (72)$$

Indeed, this will follow by the definition of weak convergence applied to the bounded continuous map $y \mapsto y \wedge k$, provided that

$$f''\left(\frac{\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} \vartheta_n(x)}{\mu * \varphi_\sigma(x)}\right) \vartheta_n^2(x) \xrightarrow{d} f''(1) G_{\mu, \sigma}^2(x), \quad \forall x \in \mathbb{R}^d.$$

However, the above come from the CLT, whereby $\vartheta_n(x) \xrightarrow{d} G_{\mu, \sigma}(x)$, and the extended continuous mapping theorem (see [25, Theorem 1.11.1]) applied to the sequence of functions $g_{n,x}(y) = f''((\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} y) / \mu * \varphi_\sigma(x)) y^2$ satisfying $\lim_{n \rightarrow \infty} g_{n,x}(y_n) = g_x(y) := y^2 f''(1)$ for $y_n \rightarrow y$ (the latter follows by continuity of f'').

Then, taking $k \rightarrow \infty$ in (72) and using monotone convergence theorem yields

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[f''\left(\frac{\mu * \varphi_\sigma(x) + \tau n^{-\frac{1}{2}} \vartheta_n(x)}{\mu * \varphi_\sigma(x)}\right) \vartheta_n^2(x)\right] = f''(1) \mathbb{E}[G_{\mu, \sigma}^2(x)].$$

Substituting the above in (71) and evaluating the integral, we conclude that

$$\liminf_{n \rightarrow \infty} n \mathbb{E}[\mathcal{D}_f(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] \geq \frac{f''(1)}{2} \int_{\mathbb{R}^d} \frac{\mathbb{E}[G_{\mu,\sigma}^2(x)]}{\mu * \varphi_\sigma(x)} dx = \frac{f''(1)}{2} \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx.$$

3) Proof of Corollary 1: As already noted, the claim follows from [69, Theorem 23.9] and the linearity of the first order Hadamard derivative of the KL divergence functional, provided the tangent cone $\mathfrak{T}_\Theta(\theta^*)$ contains a non-trivial linear subspace. We show this below by considering the one-sample case. The proof for the two-sample case uses identical arguments, and hence is omitted.

Recall that $\mathcal{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) = \Phi(p_{\hat{\mu}_n * \gamma_\sigma} - p_{\mu * \gamma_\sigma}, 0)$ with $\phi(x, y) = \phi_{\text{KL},1}(x, y) = x \log x$ in (7), where $p_{\hat{\mu}_n * \gamma_\sigma} = \hat{\mu}_n * \varphi_\sigma / \nu * \varphi_\sigma$ and $p_{\mu * \gamma_\sigma} = \mu * \varphi_\sigma / \nu * \varphi_\sigma$ (see the proof of Theorem 2(ii)). The first order Hadamard derivative is given by

$$\Phi'_{\theta^*}(h_1, h_2) = \int_{\mathbb{R}^d} h_1(x) \log(p_{\mu * \gamma_\sigma}(x)) \nu * \varphi_\sigma(x) dx,$$

for all $(h_1, h_2) \in \mathfrak{T}_\Theta(\theta^*) = \text{cl}(\{(g_1 - p_{\mu * \gamma_\sigma})/t, 0) : (g_1 - p_{\mu * \gamma_\sigma}, 0) \in \Theta, t > 0\})$, where

$$\begin{aligned} \Theta &= \{(g_1 - p_{\mu * \gamma_\sigma}, 0) \in \mathfrak{D} : g_1 \geq 0, \|g_1\|_{1, \nu * \gamma_\sigma} = 1\}, \\ \mathfrak{D} &= \{(g_1 - p_{\mu * \gamma_\sigma}, g_2 - 1) : g_1, g_2 \in L^1(\nu * \gamma_\sigma), \|g_1 - p_{\mu * \gamma_\sigma}\|_{2, \eta} + \|g_2 - 1\|_{2, \nu * \gamma_\sigma} < \infty\}, \end{aligned}$$

and η that has Lebesgue density $\nu * \varphi_\sigma + (\nu * \varphi_\sigma)^2 / \mu * \varphi_\sigma$. Further, the proof of Proposition 2(ii) shows that under the conditions therein, η is a σ -finite measure and

$$n^{1/2} (p_{\hat{\mu}_n * \gamma_\sigma} - p_{\mu * \gamma_\sigma}) \xrightarrow{w} \frac{G_{\mu,\sigma}}{\nu * \varphi_\sigma} \quad \text{in } L^2(\eta). \quad (73)$$

As $(n^{1/2} (p_{\hat{\mu}_n * \gamma_\sigma} - p_{\mu * \gamma_\sigma}), 0)$ takes values in $\mathfrak{T}_\Theta(\theta^*)$, the Portmanteau theorem guarantees that $\text{spt}(G_{\mu,\sigma} / \nu * \varphi_\sigma)$ is contained in $\mathfrak{T}_\Theta(\theta^*)$. As $L^2(\eta)$ is a separable Banach space and $G_{\mu,\sigma} / \nu * \varphi_\sigma$ is a centered $L^2(\eta)$ -valued Gaussian random variable, [68, Lemma 5.1] then guarantees that $\mathfrak{T}_\Theta(\theta^*)$ contains the reproducing kernel Hilbert space corresponding to $G_{\mu,\sigma} / \nu * \varphi_\sigma$, and hence, in particular, a linear subspace.

Given the above, [69, Theorem 23.9] applies and we obtain

$$n^{\frac{1}{2}} (\mathcal{D}_{\text{KL}}(\hat{\mu}_n^B * \gamma_\sigma \| \nu * \gamma_\sigma) - \mathcal{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma)) \xrightarrow{d} \int_{\mathbb{R}^d} G_{\mu,\sigma}(x) \log\left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)}\right) dx \sim N(0, v_{1, f_{\text{KL}}}^2(\mu, \nu, \sigma)), \quad (74)$$

so long that

$$n^{1/2} \left(\frac{\hat{\mu}_n^B * \varphi_\sigma}{\nu * \varphi_\sigma} - \frac{\hat{\mu}_n * \varphi_\sigma}{\nu * \varphi_\sigma} \right) \xrightarrow{w} \frac{G_{\mu,\sigma}}{\nu * \varphi_\sigma} \quad \text{in } L^2(\eta),$$

where both convergences are conditionally in probability. However, the last weak convergence above holds via [80, Remark 2.5, Page 860] provided that (73) is satisfied.

Next, note by [25, Theorem 1.12.4] (and the discussion preceding it) that (74) is equivalent to

$$\sup_{f \in \text{BL}_1(\mathbb{R})} \mathbb{E}_B \left[\left| f \left(n^{\frac{1}{2}} (\mathcal{D}_{\text{KL}}(\hat{\mu}_n^B * \gamma_\sigma \| \nu * \gamma_\sigma) - \mathcal{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma)) \right) - f(W_1) \right| \right] = o_{\mathbb{P}}(1),$$

where \mathbb{E}_B denotes the conditional expectation given $(X_1, \dots, X_n, Y_1, \dots, Y_n)$, $\text{BL}_1(\mathbb{R})$ is the set of bounded 1-Lipschitz continuous functions, and $W_1 \sim N(0, v_{1, \text{fkl}}^2(\mu, \nu, \sigma))$. Conclude by noting that the last equation implies the claim in Corollary 1(i), which is equivalent to convergence in distribution conditionally in probability, since the so-called Dudley metric $d_{\text{BL}_1}(\mu, \nu) := \sup_{f \in \text{BL}_1(\mathbb{R})} |\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f]|$ metrizes weak convergence in \mathbb{R} .

D. Proof of Proposition 3

We will use the following reformulation of a result given in [81].

Lemma 6 (Corollary 2 in Appendix A.9 of [81]) *Suppose $(\pi_n)_{n \in \mathbb{N}_0}$ is such that there exists $h \in L^2(\pi_0)$ with $\lim_{n \rightarrow \infty} nH^2(\pi_n, \pi_0) = \|h/2\|_{2, \pi_0}^2$ and $\int_{\mathbb{R}^d \times \mathbb{R}^d} h d\pi_0 = 0$. For $(X^n, Y^n) \sim \pi_0^{\otimes n}$, define $\Lambda_{n, \pi_n, \pi_0}(X^n, Y^n) := \sum_{i=1}^n \log(d\pi_n(X_i, Y_i)/d\pi_0(X_i, Y_i))$ and $\sigma_h := \|h\|_{2, \pi_0}$. Then*

$$\begin{aligned} \Lambda_{n, \pi_n, \pi_0}(X^n, Y^n) - n^{-\frac{1}{2}} \sum_{i=1}^n h(X_i) + 0.5\sigma_h^2 &= o_{\mathbb{P}}(1), \\ \Lambda_{n, \pi_n, \pi_0}(X^n, Y^n) &\xrightarrow{d} N(-0.5\sigma_h^2, \sigma_h^2). \end{aligned} \quad (75)$$

A short proof of the above claim is given in Appendix D for completeness.

For analyzing the type II error probability of the hypothesis test in (35), we require a refined version of the limit distribution result in (31) that accounts for the dependence of π_n on n . More specifically, we will show that for $(X^n, Y^n) \sim \pi_n^{\otimes n}$,

$$\begin{aligned} n^{\frac{1}{2}} (\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma)) \\ \xrightarrow{d} \int_{\mathbb{R}^d} G_{\mu_0, \sigma}(x) \log \left(\frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} \right) dx - \int_{\mathbb{R}^d} G_{\nu_0, \sigma}(x) \frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} dx \\ \sim N(0, v_{2, \text{fkl}}^2(\mu_0, \nu_0, \sigma)), \end{aligned} \quad (76)$$

where $(G_{\mu_0, \sigma}, G_{\nu_0, \sigma})$ is a centered Gaussian process with covariance function $\Sigma_{\mu_0, \nu_0, \sigma}$ as given in (25) and $v_{2, \text{fkl}}^2(\mu_0, \nu_0, \sigma)$ is specified in (69). We shall establish the above under Assumption 5(i) using Lemma 6 and an application of Le Cam's third lemma [25, Theorem 3.10.7]. Note that $(\pi_n)_{n \in \mathbb{N}_0}$ satisfying Assumption 5 also fulfills the conditions in Lemma 6, and hence (75) holds for $(X^n, Y^n) \sim \pi_0^{\otimes n}$. Consequently, so long that (68) holds with $(\mu, \nu) = (\mu_0, \nu_0)$, we have

$$\left(n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\nu_0 * \varphi_\sigma} - \frac{\mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} \right), n^{1/2} \left(\frac{\hat{\nu}_n * \varphi_\sigma}{\nu_0 * \varphi_\sigma} - 1 \right), \Lambda_{n, \pi_n, \pi_0} \right) \xrightarrow{w} \left(\frac{G_{\mu_0, \sigma}}{\nu_0 * \varphi_\sigma}, \frac{G_{\nu_0, \sigma}}{\nu_0 * \varphi_\sigma}, W \right),$$

in $L^2(\eta_1) \times L^2(\eta_2) \times \mathbb{R}$, where η_1 and η_2 as given in (66) are finite measures for compactly supported (μ_0, ν_0) , $W \sim N(-0.5\sigma_h^2, \sigma_h^2)$, and the centered Gaussian process $(G_{\mu_0, \sigma}(x), G_{\nu_0, \sigma}(y), W)_{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d}$ has covariance function $\tilde{\Sigma}_{\mu, \nu, \sigma} : (\mathbb{R}^d \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}^{3 \times 3}$ given by

$$\tilde{\Sigma}_{\mu_0, \nu_0, \sigma}((x, y), (\tilde{x}, \tilde{y})) := \begin{bmatrix} \mathbb{E}[G_{\mu_0, \sigma}(x)G_{\mu_0, \sigma}(\tilde{x})] & \mathbb{E}[G_{\mu_0, \sigma}(x)G_{\nu_0, \sigma}(\tilde{y})] & \mathbb{E}[G_{\mu_0, \sigma}(x)h(X, Y)] \\ \mathbb{E}[G_{\nu_0, \sigma}(y)G_{\mu_0, \sigma}(\tilde{x})] & \mathbb{E}[G_{\nu_0, \sigma}(y)G_{\nu_0, \sigma}(\tilde{y})] & \mathbb{E}[G_{\nu_0, \sigma}(y)h(X, Y)] \\ \mathbb{E}[h(X, Y)G_{\mu_0, \sigma}(\tilde{x})] & \mathbb{E}[h(X, Y)G_{\nu_0, \sigma}(\tilde{y})] & \sigma_h^2 \end{bmatrix}$$

$$= \begin{bmatrix} \text{cov}(\varphi_\sigma(x-X), \varphi_\sigma(\tilde{x}-X)) & \text{cov}(\varphi_\sigma(x-X), \varphi_\sigma(\tilde{y}-Y)) & \mathbb{E}[\varphi_\sigma(x-X)h(X,Y)] \\ \text{cov}(\varphi_\sigma(y-Y), \varphi_\sigma(\tilde{x}-X)) & \text{cov}(\varphi_\sigma(y-Y), \varphi_\sigma(\tilde{y}-Y)) & \mathbb{E}[\varphi_\sigma(y-Y)h(X,Y)] \\ \mathbb{E}[h(X,Y)\varphi_\sigma(\tilde{x}-X)] & \mathbb{E}[h(X,Y)\varphi_\sigma(\tilde{y}-Y)] & \sigma_h^2 \end{bmatrix},$$

for $(X, Y) \sim \pi_0$. This follows by (75), the convergence of finite-dimensional marginals of the joint process (via the multivariate CLT), and the separability of $L^2(\eta_1) \times L^2(\eta_2) \times \mathbb{R}$; see arguments leading to (63).

Having the above, Le Cam's third lemma [25, Theorem 3.10.7 and Example 3.10.8] implies that for $(X^n, Y^n) \sim \pi_n^{\otimes n}$, we have

$$\left(n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\nu_0 * \varphi_\sigma} - \frac{\mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} \right), n^{1/2} \left(\frac{\hat{\nu}_n * \varphi_\sigma}{\nu_0 * \varphi_\sigma} - 1 \right) \right) \xrightarrow{w} \left(\frac{\bar{G}_{\mu_0, \sigma, h}}{\nu_0 * \varphi_\sigma}, \frac{\bar{G}_{\nu_0, \sigma, h}}{\nu_0 * \varphi_\sigma} \right)$$

in $L^2(\eta_1) \times L^2(\eta_2)$, where the Gaussian process $(\bar{G}_{\mu_0, \sigma, h}, \bar{G}_{\nu_0, \sigma, h})$ has mean function $(m_{1,h}, m_{2,h}) := (\mathbb{E}_{\pi_0}[h(X, Y)\varphi_\sigma(\cdot - X)], \mathbb{E}_{\pi_0}[h(X, Y)\varphi_\sigma(\cdot - Y)])$ and covariance function $\Sigma_{\mu, \nu, \sigma}$ as given in (25). Since μ_n and ν_n are compactly supported (on $[-b, b]^d$), the proof of Proposition 2(iv) applies and results in

$$\begin{aligned} n^{\frac{1}{2}} (\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma)) \\ \xrightarrow{d} \int_{\mathbb{R}^d} \bar{G}_{\mu_0, \sigma, h}(x) \log \left(\frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} \right) dx - \int_{\mathbb{R}^d} \bar{G}_{\nu_0, \sigma, h}(x) \frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} dx. \end{aligned}$$

Next, note that the finite measures η_1 and η_2 from (66) have bounded Lebesgue densities. Consequently, the convergence given in (34) in $L^\infty(\lambda) \times L^\infty(\lambda)$ implies the same in $L^\infty(\eta_1) \times L^\infty(\eta_2)$ and $L^2(\eta_1) \times L^2(\eta_2)$. It then follows via the definition of the Hadamard first derivative (see (49)), similarly to (67), that

$$\begin{aligned} n^{\frac{1}{2}} (\text{D}_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma)) \\ \xrightarrow{d} \int_{\mathbb{R}^d} m_{1,h}(x) \log \left(\frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} \right) dx - \int_{\mathbb{R}^d} m_{2,h}(x) \frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} dx. \end{aligned}$$

Subtracting the last equation from the penultimate one leads to (76).

Armed with (76), we proceed to analyze the type I and type II error probabilities, i.e., $e_{1,n}(T_n) := \mathbb{P}(T_n > t_n | H_0)$ and $e_{2,n}(T_n) := \mathbb{P}(T_n \leq t_n | H_{1,n})$, respectively. Consider the test statistic $T_n = \text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)$ with critical value $t_n = \epsilon + cn^{-1/2}$ for some constant c that will be specified later. Define the event $\mathcal{E}_{n, \epsilon, c} := \{\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) > \epsilon + cn^{-1/2}\}$. Note that $(-\infty, c)$ and $[c, \infty)$ are continuity sets for the Gaussian measure on \mathbb{R} , i.e., with boundary measure zero. Then, with $Z_{n,0} := n^{1/2}(\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma))$, we have

$$\limsup_{n \rightarrow \infty} e_{1,n}(T_n) = \limsup_{n \rightarrow \infty} \mathbb{P}(\mathcal{E}_{n, \epsilon, c} | H_0) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(Z_{n,0} > c | H_0) = Q(c/v_{2, \text{f}_{\text{KL}}}(\mu_0, \nu_0, \sigma)), \quad (77)$$

where $v_{2, \text{f}_{\text{KL}}}(\mu_0, \nu_0, \sigma)$ is as given in (69) and Q is the Q -function (or complementary error function) given by $Q(x) = (2\pi)^{-1/2} \int_x^\infty e^{-z^2/2} dz$. The inequality above is due to $\text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) \leq \epsilon$, while the final equality uses the Portmanteau theorem and (31) (the latter applies since μ_0, ν_0 have compact supports). Similarly, defining $Z_{n,1} := n^{1/2}(\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma))$ and $\bar{\mathcal{E}}_{n, \epsilon, c} := \{\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) \leq \epsilon +$

$cn^{-1/2}\}$, for the type II error probability we have

$$\begin{aligned}
\limsup_{n \rightarrow \infty} e_{2,n}(T_n) &= \limsup_{n \rightarrow \infty} \mathbb{P}(\bar{\mathcal{E}}_{n,\epsilon,c} | H_{1,n}) \\
&= \limsup_{n \rightarrow \infty} \mathbb{P}(Z_{n,1} \leq -n^{1/2}(\mathcal{D}_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) - \epsilon) + c | H_{1,n}) \\
&\leq \limsup_{n \rightarrow \infty} \mathbb{P}(Z_{n,1} \leq c - C | H_{1,n}) \\
&= 1 - Q((c - C)/v_{2,f_{\text{KL}}}(\mu_0, \nu_0, \sigma)),
\end{aligned} \tag{78}$$

where the last inequality uses $\mathcal{D}_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) \geq \epsilon + n^{-1/2}C$ and the final equality uses the Portmanteau theorem together with (76).

To arrive at the result in Proposition 3, it remains to appropriately upper bound $v_{2,f_{\text{KL}}}(\mu_0, \nu_0, \sigma)$. By using Cauchy-Schwarz inequality and the fact that second moment upper bounds variance, we have

$$\begin{aligned}
\text{cov}(\varphi_\sigma(x - X), \varphi_\sigma(y - X))^2 &\leq \mathbb{E}_{\mu_0}[\varphi_\sigma^2(x - X)] \mathbb{E}_{\mu_0}[\varphi_\sigma^2(y - X)] \\
&= (2\pi\sigma^2)^{-2d} \int_{\mathbb{R}^d} e^{-\frac{\|x-z\|^2}{\sigma^2}} d\mu_0(z) \int_{\mathbb{R}^d} e^{-\frac{\|y-z\|^2}{\sigma^2}} d\mu_0(z) \\
&\leq (2\pi\sigma^2)^{-2d} e^{\frac{2b^2d}{\sigma^2}} e^{-\frac{(\|x\|^2 + \|y\|^2)}{2\sigma^2}},
\end{aligned}$$

where the last inequality uses $\|x - z\|^2 \geq \|x\|^2/2 - \|z\|^2$. Hence

$$\Sigma_{\mu_0, \nu_0, \sigma}^{(1,1)}(x, y) = \text{cov}(\varphi_\sigma(x - X), \varphi_\sigma(y - X)) \leq (2\pi\sigma^2)^{-d} e^{\frac{b^2d}{\sigma^2}} e^{-\frac{(\|x\|^2 + \|y\|^2)}{4\sigma^2}}.$$

By following similar steps, for all $1 \leq i, j \leq 2$, we have

$$\Sigma_{\mu_0, \nu_0, \sigma}^{(i,j)}(x, y) \leq (2\pi\sigma^2)^{-d} e^{\frac{b^2d}{\sigma^2}} e^{-\frac{(\|x\|^2 + \|y\|^2)}{4\sigma^2}}.$$

Also,

$$e^{\frac{-b^2d}{2\sigma^2}} e^{\frac{-2b\sqrt{d}\|x\|}{\sigma^2}} \leq \frac{\int_{\mathbb{R}^d} e^{\frac{-2\|x\|\|z\| - \|z\|^2}{2\sigma^2}} d\mu_0(z)}{\int_{\mathbb{R}^d} e^{\frac{2\|x\|\|z\| - \|z\|^2}{2\sigma^2}} d\nu_0(z)} \leq \frac{\mu_0 * \varphi_\sigma(x)}{\nu_0 * \varphi_\sigma(x)} \leq \frac{\int_{\mathbb{R}^d} e^{\frac{2\|x\|\|z\| - \|z\|^2}{2\sigma^2}} d\mu_0(z)}{\int_{\mathbb{R}^d} e^{\frac{-2\|x\|\|z\| - \|z\|^2}{2\sigma^2}} d\nu_0(z)} \leq e^{\frac{b^2d}{2\sigma^2}} e^{\frac{2b\sqrt{d}\|x\|}{\sigma^2}}.$$

Combining the above bounds, we have from (69) that

$$\begin{aligned}
v_{2,f_{\text{KL}}}^2(\mu_0, \nu_0, \sigma) &= \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu_0, \nu_0, \sigma}^{(i,j)}(x, y) \tilde{L}_{i,f_{\text{KL}}}(x) \tilde{L}_{j,f_{\text{KL}}}(y) dx dy \\
&\leq (2\pi\sigma^2)^{-d} e^{\frac{b^2d}{\sigma^2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{(\|x\|^2 + \|y\|^2)}{4\sigma^2}} \left(\frac{(b^2d + 4b\sqrt{d}\|x\|)(b^2d + 4b\sqrt{d}\|y\|)}{4\sigma^4} \right. \\
&\quad + e^{\frac{2b^2d + 4b\sqrt{d}(\|x\| + \|y\|)}{2\sigma^2}} + e^{\frac{b^2d + 4b\sqrt{d}\|x\|}{2\sigma^2}} \frac{(b^2d + 4b\sqrt{d}\|y\|)}{2\sigma^2} \\
&\quad \left. + e^{\frac{b^2d + 4b\sqrt{d}\|y\|}{2\sigma^2}} \frac{(b^2d + 4b\sqrt{d}\|x\|)}{2\sigma^2} \right) dx dy \\
&=: c_{b,d,\sigma}.
\end{aligned} \tag{79}$$

For $0 < \tau, \tau' \leq 1$, setting $c = c_{b,d,\sigma} Q^{-1}(\tau)$, the claim of Proposition 3 follows from (77) and (78) with $C_{b,d,\sigma,\tau,\tau'} = c - c_{b,d,\sigma} Q^{-1}(1 - \tau')$.

E. Proof of Proposition 4

Part (i): Since $\|d(\nu_0 \otimes \mu_0)/d\pi_0\|_\infty < \infty$, there exists $n_0 \in \mathbb{N}$ such that the RHS of (37) is non-negative π_0 -a.s., and henceforth, we take $n \geq n_0$. Further, observe that π_n specified by (37) is a valid joint probability measure as

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} d\pi_n = \int_{\mathbb{R}^d \times \mathbb{R}^d} d\pi_0 + \bar{c} n^{-\frac{1}{2}} (d(\mu_0 \otimes \nu_0) - d(\nu_0 \otimes \mu_0)) = 1.$$

Via Taylor's theorem applied to $f_{H^2}(x) = (\sqrt{x} - 1)^2$ around $x = 1$ and using $f'(1) = f''(1) = 0$, we have

$$f_{H^2} \circ \frac{d\pi_n}{d\pi_0} = \frac{1}{2} \int_0^1 \frac{(1-\tau)}{\left((1-\tau) + \tau \frac{d\pi_n}{d\pi_0}\right)^{\frac{3}{2}}} \left(\frac{d\pi_n}{d\pi_0} - 1\right)^2 d\tau = \frac{1}{2} \int_0^1 \frac{(1-\tau)}{\left((1-\tau) + \tau \frac{d\pi_n}{d\pi_0}\right)^{\frac{3}{2}}} h_{\pi_0, \bar{c}}^2 d\tau.$$

Multiplying by n and taking expectation w.r.t. π_0 leads to

$$nH^2(\pi_n, \pi_0) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^1 \frac{(1-\tau)}{2 \left((1-\tau) + \tau \frac{d\pi_n}{d\pi_0}\right)^{\frac{3}{2}}} h_{\pi_0, \bar{c}}^2 d\tau d\pi_0.$$

Note that the integrand in the RHS above is dominated by $0.5(1-\tau)^{-1/2} h_{\pi_0, \bar{c}}^2$, which is integrable w.r.t. the product measure $\tau \otimes \pi_0$ under the assumption $\|h_{\pi_0, \bar{c}}\|_{2, \pi_0} < \infty$. Taking limit, the dominated convergence theorem and the fact that $d\pi_n/d\pi_0$ converges pointwise to 1, imply that

$$\lim_{n \rightarrow \infty} nH^2(\pi_n, \pi_0) = \frac{1}{4} \|h_{\pi_0, \bar{c}}\|_{2, \pi_0}^2.$$

Moreover, it is readily verified that $\int_{\mathbb{R}^d \times \mathbb{R}^d} h_{\pi_0, \bar{c}} d\pi_0 = 0$. Finally, the marginals of π_n satisfy $d\mu_n = d\mu_0 + n^{-1/2} \bar{c}(d\mu_0 - d\nu_0)$ and $d\nu_n = d\nu_0 + n^{-1/2} \bar{c}(d\nu_0 - d\mu_0)$, which implies

$$\begin{aligned} \mu_n * \varphi_\sigma &= \mu_0 * \varphi_\sigma + n^{-1/2} \bar{c}(\mu_0 * \varphi_\sigma - \nu_0 * \varphi_\sigma), \\ \nu_n * \varphi_\sigma &= \nu_0 * \varphi_\sigma + n^{-1/2} \bar{c}(\nu_0 * \varphi_\sigma - \mu_0 * \varphi_\sigma), \end{aligned} \tag{80}$$

and thus

$$\begin{aligned} n^{1/2} \left(\frac{\mu_n * \varphi_\sigma - \mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} \right) &= \frac{\bar{c}(\mu_0 * \varphi_\sigma - \nu_0 * \varphi_\sigma)}{\nu_0 * \varphi_\sigma} = \frac{\mathbb{E}_{\pi_0} [h_{\pi_0, \bar{c}}(X, Y) \varphi_\sigma(\cdot - X)]}{\nu_0 * \varphi_\sigma}, \\ n^{1/2} \left(\frac{\nu_n * \varphi_\sigma - \nu_0 * \varphi_\sigma}{\mu_0 * \varphi_\sigma} \right) &= \frac{\bar{c}(\nu_0 * \varphi_\sigma - \mu_0 * \varphi_\sigma)}{\mu_0 * \varphi_\sigma} = \frac{\mathbb{E}_{\pi_0} [h_{\pi_0, \bar{c}}(X, Y) \varphi_\sigma(\cdot - Y)]}{\mu_0 * \varphi_\sigma}. \end{aligned}$$

Hence, (34) is satisfied (with pointwise equality), verifying all the requirements in Assumption 5(i) and concluding the proof of Part (i).

Part (ii): Note that $\|h_{\pi_0, \bar{c}}\|_{2, \pi_0} < \infty$ as $\|d(\nu_0 \otimes \mu_0)/d\pi_0\|_{\infty, \pi_0}, \|d(\mu_0 \otimes \nu_0)/d\pi_0\|_{2, \pi_0} < \infty$. Hence all the assumptions in Part (i) are fulfilled thereby implying via the proof above that Assumption 5(i) is satisfied. So, we

only need to verify that Assumption 5(ii) holds. By Taylor's theorem applied to $\phi_{\text{KL}}(x, y) = x \log(x/y)$, we have

$$\begin{aligned}
& \mu_n * \varphi_\sigma \log \left(\frac{\mu_n * \varphi_\sigma}{\nu_n * \varphi_\sigma} \right) - \mu_0 * \varphi_\sigma \log \left(\frac{\mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} \right) \\
&= \left(1 + \log \left(\frac{\mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} \right) \right) (\mu_n * \varphi_\sigma - \mu_0 * \varphi_\sigma) - \frac{\mu_0 * \varphi_\sigma}{\nu_0 * \varphi_\sigma} (\nu_n * \varphi_\sigma - \nu_0 * \varphi_\sigma) \\
&+ (\mu_n * \varphi_\sigma - \mu_0 * \varphi_\sigma)^2 \int_0^1 \frac{(1-\tau) d\tau}{(1-\tau)\mu_0 * \varphi_\sigma + \tau\mu_n * \varphi_\sigma} \\
&+ (\nu_n * \varphi_\sigma - \nu_0 * \varphi_\sigma)^2 \int_0^1 \frac{(1-\tau)((1-\tau)\mu_0 * \varphi_\sigma + \tau\mu_n * \varphi_\sigma) d\tau}{(1-\tau)\nu_0 * \varphi_\sigma + \tau\nu_n * \varphi_\sigma} \\
&- 2(\mu_n * \varphi_\sigma - \mu_0 * \varphi_\sigma)(\nu_n * \varphi_\sigma - \nu_0 * \varphi_\sigma) \int_0^1 \frac{(1-\tau) d\tau}{(1-\tau)\nu_0 * \varphi_\sigma + \tau\nu_n * \varphi_\sigma}.
\end{aligned}$$

Taking integral w.r.t. Lebesgue measure and simplifying using (80), we obtain

$$\begin{aligned}
& \text{D}_{\text{KL}}(\mu_n * \gamma_\sigma \| \nu_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) \\
&= n^{-\frac{1}{2}} \bar{c} (\text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) + \text{D}_{\text{KL}}(\nu_0 * \gamma_\sigma \| \mu_0 * \gamma_\sigma) + \chi^2(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma)) \\
&+ \frac{\bar{c}^2}{n} \int_{\mathbb{R}^d} (\mu_0 * \varphi_\sigma - \nu_0 * \varphi_\sigma)^2 \int_0^1 \frac{(1-\tau)}{(1-\tau)\mu_0 * \varphi_\sigma + \tau\mu_n * \varphi_\sigma} d\tau dx \\
&+ \frac{\bar{c}^2}{n} \int_{\mathbb{R}^d} (\nu_0 * \varphi_\sigma - \mu_0 * \varphi_\sigma)^2 \int_0^1 \frac{(1-\tau)((1-\tau)\mu_0 * \varphi_\sigma + \tau\mu_n * \varphi_\sigma)}{(1-\tau)\nu_0 * \varphi_\sigma + \tau\nu_n * \varphi_\sigma} d\tau dx \\
&+ \frac{2\bar{c}^2}{n} \int_{\mathbb{R}^d} (\nu_0 * \varphi_\sigma - \mu_0 * \varphi_\sigma)^2 \int_0^1 \frac{(1-\tau)}{(1-\tau)\nu_0 * \varphi_\sigma + \tau\nu_n * \varphi_\sigma} d\tau dx \\
&> n^{-\frac{1}{2}} \bar{c} (\text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) + \text{D}_{\text{KL}}(\nu_0 * \gamma_\sigma \| \mu_0 * \gamma_\sigma) + \chi^2(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma)),
\end{aligned}$$

where the final inequality is due to the omitted terms being positive. The proof is concluded by noting that the RHS is larger than $Cn^{-1/2}$ for \bar{c} sufficiently large.

F. Proof of Proposition 5

We consider the test statistic $T_n = \text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)$ with critical value $t_n = \epsilon + cn^{-1/2}$ for an appropriately chosen σ (small enough) and c , and analyze the resulting asymptotic error probabilities of the hypothesis test in (38). The choice of σ relies on the stability lemma (see Lemma 3) which quantifies the deviation of KL divergence from its smoothed version as a function of σ .

Recall that $\bar{M}, \bar{\epsilon}, \underline{s}, \bar{s}$ are known constants such that $M \leq \bar{M} < \infty$, $\epsilon < \bar{\epsilon} \leq \tilde{\epsilon}$, and $0 < \underline{s} \leq s \leq \bar{s} \leq 1$. Let $\sigma_{\epsilon, \bar{\epsilon}, \underline{s}, \bar{s}, d, \bar{M}}$ equal x such that

$$c_{d, \bar{s}} \bar{M} (\bar{M} + 1 + \log \bar{M}) (x^{\underline{s}} \vee x^{\bar{s}}) = \bar{\epsilon} - \epsilon. \quad (81)$$

Choose any $\sigma < \sigma_{\epsilon, \bar{\epsilon}, \underline{s}, \bar{s}, d, \bar{M}}$, whereby $c_{d, s} M (M + 1 + \log M) \sigma^s < \bar{\epsilon} - \epsilon$. Hence, (40) and $\text{D}_{\text{KL}}(\mu_1 \| \nu_1) \geq \bar{\epsilon}$ imply $\text{D}_{\text{KL}}(\mu_1 * \gamma_\sigma \| \nu_1 * \gamma_\sigma) > \epsilon$. On the other hand, the data processing inequality along with the fact that $\text{D}_{\text{KL}}(\mu_0 \| \nu_0) \leq \epsilon$ imply $\text{D}_{\text{KL}}(\mu_0 * \gamma_\sigma \| \nu_0 * \gamma_\sigma) \leq \epsilon$. Then, for the test statistic T_n , we obtain via steps leading to

(77) that

$$\limsup_{n \rightarrow \infty} e_{1,n}(T_n) \leq Q(c/v_{2,f_{\text{KL}}}(\mu_0, \nu_0, \sigma)).$$

Setting $Z_{n,1} := n^{1/2}(\text{D}_{\text{KL}}(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \text{D}_{\text{KL}}(\mu_1 * \gamma_\sigma \| \nu_1 * \gamma_\sigma))$, we further have

$$\limsup_{n \rightarrow \infty} e_{2,n}(T_n) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(Z_{n,1} \leq -n^{1/2}(\text{D}_{\text{KL}}(\mu_1 * \gamma_\sigma \| \nu_1 * \gamma_\sigma) - \epsilon) + c | H_1) = 1 - Q(-\infty) = 0,$$

where the penultimate equality uses the Portmanteau theorem applied to $Z_{n,1}$ converging weakly to a Gaussian random variable (via (31) with $v_{2,f_{\text{KL}}}^2(\mu_1, \nu_1, \sigma) \leq c_{b,d,\sigma}$ similar to (79)), and $c - n^{1/2}(\text{D}_{\text{KL}}(\mu_1 * \gamma_\sigma \| \nu_1 * \gamma_\sigma) - \epsilon)$ diverging to $-\infty$ due to $\text{D}_{\text{KL}}(\mu_1 * \gamma_\sigma \| \nu_1 * \gamma_\sigma) > \epsilon$. Choosing $c = c_{b,d,\sigma} Q^{-1}(\tau)$ (see (79)) completes the proof.

G. Proof of Lemma 3

For $\tau \in [0, 1]$, let $z_{\mu,\tau}(x) := (1 - \tau)p_\mu(x) + \tau\mu * \varphi_\sigma(x)$ and $z_{\nu,\tau}(x) := (1 - \tau)p_\nu(x) + \tau\nu * \varphi_\sigma(x)$. By Taylor's theorem, we have

$$\begin{aligned} \mu * \varphi_\sigma(x) \log \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right) &= p_\mu(x) \log \left(\frac{p_\mu(x)}{p_\nu(x)} \right) + (\mu * \varphi_\sigma(x) - p_\mu(x)) \int_0^1 \left(1 + \log \frac{z_{\mu,\tau}(x)}{z_{\nu,\tau}(x)} \right) d\tau \\ &\quad - (\nu * \varphi_\sigma(x) - p_\nu(x)) \int_0^1 \frac{z_{\mu,\tau}(x)}{z_{\nu,\tau}(x)} d\tau. \end{aligned}$$

Note that $1/M \leq p_\mu(x)/p_\nu(x) \leq M$ by assumption, which implies $1/M \leq \mu * \varphi_\sigma(x)/\nu * \varphi_\sigma(x) \leq M$, and hence

$$\frac{1}{M} \leq \frac{p_\mu(x)}{p_\nu(x)} \wedge \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \leq \frac{z_{\mu,\tau}(x)}{z_{\nu,\tau}(x)} \leq \frac{p_\mu(x)}{p_\nu(x)} \vee \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \leq M, \quad \forall x \in \mathbb{R}^d.$$

Integrating w.r.t. Lebesgue measure in the above equation, we then obtain (note that $M \geq 1$)

$$|\text{D}_{\text{KL}}(\mu \| \nu) - \text{D}_{\text{KL}}(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)| \leq (1 + \log M) \int_{\mathbb{R}^d} |\mu * \varphi_\sigma(x) - p_\mu(x)| dx + M \int_{\mathbb{R}^d} |\nu * \varphi_\sigma(x) - p_\nu(x)| dx. \quad (82)$$

The first integral is bounded as follows:

$$\begin{aligned} \int_{\mathbb{R}^d} |p_\mu(x) - \mu * \varphi_\sigma(x)| dx &= \int_{\mathbb{R}^d} \left| p_\mu(x) - \sigma^{-d} \int_{\mathbb{R}^d} p_\mu(y) \varphi_1((x - y)\sigma^{-1}) dy \right| dx \\ &= \int_{\mathbb{R}^d} \left| p_\mu(x) - \int_{\mathbb{R}^d} p_\mu(x - \sigma z) \varphi_1(z) dz \right| dx \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (p_\mu(x) \varphi_1(z) - p_\mu(x - \sigma z) \varphi_1(z)) dz \right| dx \\ &\stackrel{(a)}{\leq} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p_\mu(x + \sigma z) - p_\mu(x)| dx \right) \varphi_1(z) dz \\ &\stackrel{(b)}{\leq} \int_{\mathbb{R}^d} \kappa_{1,1}(p_\mu, \sigma \|z\|) \varphi_1(z) dz \\ &\stackrel{(c)}{\leq} M \sigma^s \int_{\mathbb{R}^d} \|z\|^s \varphi_1(z) dz, \end{aligned}$$

where (a) uses Fubini's theorem, (b) is by the definition of the modulus of smoothness in Definition 5, while (c) is because $p_\mu \in \text{Lip}_{s,1}(M, \mathcal{X})$. Following similar steps with μ replaced by ν , the same bound holds for the second

integral term in (82), which concludes the proof.

VIII. CONCLUDING REMARKS

This paper provided a unified methodology for deriving one- and two-sample limit distributions for f -divergences, under both the null (i.e., when the population distributions are equal) and the alternative. We focused on four prominent examples, namely, KL divergence, χ^2 divergence, squared Hellinger distance, and total variation distance. The general limit theorems were stated under technical conditions on the distributions which guarantee Hadamard differentiability of the relevant f -divergence functional. The framework allows arbitrary estimators of the population measures and accounts for dependent data, which renders it rather flexible and broadly applicable. We instantiate the general limit distribution theory to the setting of Gaussian-smoothed f -divergence, showing that the limit (except for TV distance) is Gaussian under the alternative, or can be represented as a weighted sum of i.i.d. χ^2 random variables under the null. In contrast, the limit distribution for smoothed TV distance in both the above scenarios is non-Gaussian. We also proposed a hypothesis testing pipeline for auditing DP and characterized its asymptotic level and power by utilizing our distributional limits.

While this work focuses on f -divergences, a limit distribution theory for other classes of divergences, such as integral probability metrics and Bregman divergences, are largely unexplored and worth pursuing. We believe that our approach based on the functional delta method and Hadamard directional differentiability can be extended to cover those cases as well. Another appealing research avenue is to explore asymptotic distributions of more sophisticated (non plug-in) estimators of f -divergences. This includes approaches based on bias correction mechanisms (e.g., [32], [39], [40]) or variational methods (e.g., [35], [42]). While such estimators are known to achieve better minimax rates over appropriate classes of distributions, the main challenge here is to establish regularity conditions under which the Hadamard derivative of the relevant functional exists. Towards that end, recent results on Hadamard differentiability of supremum-type functionals [82] may be useful for deriving limit theorems of variational estimators. Yet another intriguing question pertains to establishing Berry-Esseen type results, which would reveal the convergence rate of the empirical f -divergence to its distributional limit. However, deriving such results is highly non-trivial and would require different tools and technical approaches than the empirical process theory-based techniques employed in this work.

APPENDIX A

PROOFS OF PART (iii) OF LEMMA 1, THEOREM 1, AND THE CLAIM IN REMARK 1

1) Proof of Part (iii) of Lemma 1: We will use similar steps as in [24, Theorem 1] and [24, Theorem 2] with the extended continuous mapping theorem replaced by Theorem 1. We highlight the relevant steps for the claim that Part (i) of Lemma 1 holds. The proof that Part (ii) holds is similar and omitted.

Recall that since Θ is convex, the tangent cone is given by (see [24])

$$\begin{aligned} \mathfrak{T}_{\Theta}(\theta) &:= \left\{ h \in \mathfrak{D} : h = \lim_{n \rightarrow \infty} \frac{\theta_n - \theta}{t_n} \text{ for some } \theta_n \rightarrow \theta \text{ with } \theta_n \in \Theta \text{ and } t_n \downarrow 0^+ \right\} \\ &= \text{cl}(\{(\tilde{\theta} - \theta)/t : \tilde{\theta} \in \Theta, t > 0\}). \end{aligned}$$

Since $\mathfrak{T}_\Theta(\theta)$ is closed, its boundary is a Borel set. Set $\mathfrak{D}_n = \{h \in \mathfrak{D} : \theta + (h/r_n) \in \mathfrak{D}_\Phi\}$, $\mathfrak{D}_n^\rightarrow = \{h \in \mathfrak{D} : \theta + (h/r_n) \in \Theta\}$, $\mathfrak{D}_g = \mathfrak{D}_\infty = \mathfrak{T}_\Theta(\theta)$, $g = \Phi'_\theta$, $g_n(h) = r_n(\Phi(\theta + (h/r_n)) - \Phi(\theta))$, $H_n = r_n(Z_n - \theta)$ and $H = Z$. Then, the claim will follow from Theorem 1, provided that $\mathbb{P}(H \in \text{int}(\mathfrak{T}_\Theta(\theta))) = 1$ and $\mathfrak{D}_* \subseteq \mathfrak{D} \setminus \text{int}(\mathfrak{T}_\Theta(\theta))$, since these imply $\mathbb{P}(H \in \mathfrak{D}_\infty) = 1$ and $\mathbb{P}(H \in \mathfrak{D}_*) = 0$.

The first condition above follows from

$$\mathbb{P}(H \in \text{int}(\mathfrak{T}_\Theta(\theta))) \stackrel{(a)}{=} 1 - \mathbb{P}(H \notin \mathfrak{T}_\Theta(\theta)) \stackrel{(b)}{=} 1 - \lim_{n \rightarrow \infty} \mathbb{P}(H_n \notin \mathfrak{T}_\Theta(\theta)) \stackrel{(c)}{\geq} 1 - \lim_{n \rightarrow \infty} \mathbb{P}^*(Z_n \notin \Theta) = 1,$$

where (a) is because $\mathbb{P}(H \in \partial \mathfrak{T}_\Theta(\theta)) = 0$, (b) is since $H_n \xrightarrow{w} H$ and Portmanteau theorem, (c) is due to $Z_n \in \Theta$ implies $H_n \in \mathfrak{T}_\Theta(\theta)$, and the final equality follows from $\mathbb{P}^*(Z_n \notin \Theta) \rightarrow 0$. To show the second condition, suppose $h \in \text{int}(\mathfrak{T}_\Theta(\theta))$. We claim that there exists $m \in \mathbb{N}$ such that $h \in \text{int}(\mathfrak{D}_n^\rightarrow)$ for all $n \geq m$ which implies that $h \notin \cup_{n \geq m} \overline{\mathfrak{D} \setminus \mathfrak{D}_n^\rightarrow} \supseteq \mathfrak{D}_*$ as desired. To see this, note that $h = (\tilde{\theta} - \theta)/t$ for some $t > 0$ and $\tilde{\theta} \in \text{int}(\Theta)$. Let $\hat{\theta}_\alpha := \alpha \tilde{\theta} + (1 - \alpha)\theta$ for $\alpha \in [0, 1]$. We have $\hat{\theta}_\alpha \in \Theta$ since Θ is convex. Then, by definition of $\mathfrak{D}_n^\rightarrow$, $\alpha_n r_n(\tilde{\theta} - \theta) = r_n(\hat{\theta}_{\alpha_n} - \theta) \in \mathfrak{D}_n^\rightarrow$. Hence, for $n \geq m = \min\{n : r_n t \geq 1\}$, we have $h \in \text{int}(\mathfrak{D}_n^\rightarrow)$ by choosing $\alpha_n = 1/(tr_n)$. Hence, all the conditions in Theorem 1 are satisfied under the assumptions in Part (iii) of Lemma 1, and consequently $r_n(\Phi(Z_n) - \Phi(\theta)) \xrightarrow{d} \Phi'_\theta(Z)$ follows. The proof of $r_n(\Phi(Z_n) - \Phi(\theta)) = \phi'_\theta(r_n(Z_n - \theta)) + o_{\mathbb{P}}(1)$ and Part (ii) follows similarly to [24] with the relevant steps adapted as above, and hence omitted.

2) Proof of Theorem 1: Since the proof follows closely to that of [25, Theorem 1.11.1], we only highlight the differences. Following similar steps as therein, we have that g restricted to $\mathfrak{D}_g \cap \mathfrak{D}_\infty$ is continuous, and hence $g(H)$ is Borel measurable.

Consider a closed set $\mathcal{C} \subseteq \mathfrak{E}$. Then, we have

$$\cap_{n=1}^\infty \overline{\cup_{m=n}^\infty g_m^{-1}(\mathcal{C})} \subseteq g^{-1}(\mathcal{C}) \cup (\mathfrak{D} \setminus (\mathfrak{D}_\infty \cap \mathfrak{D}_g)) \cup \mathfrak{D}_*, \quad (83)$$

where $\bar{\mathcal{C}}$ denotes the closure of the set \mathcal{C} . Indeed, if h is in the set on the left, then one of the following condition must hold: (i) there exists a divergent sequence $(n_k)_{k \in \mathbb{N}}$ such that $h_{n_k} \in g_{n_k}^{-1}(\mathcal{C}) \cap \mathfrak{D}_\rightarrow$ and $d(h_{n_k}, h) \rightarrow 0$, where $d(\cdot, \cdot)$ denotes the metric of \mathfrak{D} ; or (ii) there exists a divergent sequence $(n_k)_{k \in \mathbb{N}}$ such that $h_{n_k} \in g_{n_k}^{-1}(\mathcal{C}) \cap (\mathfrak{D} \setminus \mathfrak{D}_n^\rightarrow)$ and $d(h_{n_k}, h) \rightarrow 0$. If (i) holds, then by the assumption in the theorem, either $g_{n_k}(h_{n_k}) \rightarrow g(h)$ (if $h \in \mathfrak{D}_\infty \cap \mathfrak{D}_g$) or $h \in \mathfrak{D} \setminus (\mathfrak{D}_\infty \cap \mathfrak{D}_g)$. If $g_{n_k}(h_{n_k}) \rightarrow g(h)$, then $g(h) \in \mathcal{C}$ since $g_{n_k}(h_{n_k}) \in \mathcal{C}$ and \mathcal{C} is a closed set. On the other hand, if (ii) holds, then this implies that $h \in \cap_{m=1}^\infty \overline{\cup_{k=m}^\infty g_{n_k}^{-1}(\mathcal{C}) \cap (\mathfrak{D} \setminus \mathfrak{D}_{n_k}^\rightarrow)} \subseteq \cap_{m=1}^\infty \overline{\cup_{k=m}^\infty (\mathfrak{D} \setminus \mathfrak{D}_{n_k}^\rightarrow)} \subseteq \mathfrak{D}_*$. Hence, (83) holds. Then, we have for any fixed k that

$$\limsup_{n \rightarrow \infty} \mathbb{P}(g_n(H_n) \in \mathcal{C}) \leq \limsup_{n \rightarrow \infty} \mathbb{P}\left(H_n \in \overline{\cup_{m=k}^\infty g_m^{-1}(\mathcal{C})}\right) \leq \mathbb{P}\left(H \in \overline{\cup_{m=k}^\infty g_m^{-1}(\mathcal{C})}\right), \quad (84)$$

where the final inequality follows from $H_n \xrightarrow{w} H$ and Portmanteau theorem [25, Theorem 1.3.4]. Taking limit $k \rightarrow \infty$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}(g_n(H_n) \in \mathcal{C}) &\leq \mathbb{P}\left(H \in \cap_{k=1}^\infty \overline{\cup_{m=k}^\infty g_m^{-1}(\mathcal{C})}\right) \\ &\stackrel{(a)}{\leq} \mathbb{P}(H \in g^{-1}(\mathcal{C})) + \mathbb{P}(H \in \mathfrak{D} \setminus (\mathfrak{D}_\infty \cap \mathfrak{D}_g)) + \mathbb{P}(H \in \mathfrak{D}_*) \end{aligned}$$

$$\stackrel{(b)}{=} \mathbb{P}(g(H) \in \mathcal{C}), \quad (85)$$

where (a) is due to (83), and (b) is because $\mathbb{P}(H \in \mathfrak{D}_*) = 0$ by assumption and

$$\mathbb{P}(H \in \mathfrak{D} \setminus (\mathfrak{D}_\infty \cap \mathfrak{D}_g)) = 1 - \mathbb{P}(H \in \mathfrak{D}_\infty \cap \mathfrak{D}_g) = 1 - \mathbb{P}(H \in \mathfrak{D}_\infty) = 0.$$

Since \mathcal{C} is an arbitrary closed set, $g_n(H_n) \xrightarrow{w} g(H)$ again follows from (85) by Portmanteau theorem.

3) Proof of the claim in Remark 1: The proof follows by using Part (iii) of Lemma 1 in the proof of Part (iii) and (iv) of Theorem 2. Recall that $\mathcal{Q}_n := \{\omega \in \Omega : p_{\mu_n}(\omega, \cdot)/p_{\nu_n}(\omega, \cdot) \leq q(\cdot)\}$, and let $\tilde{\mathcal{Q}}_n := \{\omega \in \Omega : Z_n(\omega, \cdot) = (p_{\mu_n}(\omega, \cdot) - p_{\mu}, p_{\nu_n}(\omega, \cdot) - 1) \in \Theta\}$. Note that under the conditions in Part (iii) and (iv) of Theorem 2 sans the requirement $p_{\mu_n}/p_{\nu_n} \leq q$ a.s., $\Omega \setminus \tilde{\mathcal{Q}}_n = (\Omega \setminus \mathcal{Q}_n) \cup \mathcal{N}_0$ for some null set \mathcal{N}_0 of outer probability zero. Hence, $\mathbb{P}^*(Z_n \notin \Theta) = \mathbb{P}^*(\Omega \setminus \mathcal{Q}_n) \rightarrow 0$ under the assumption in Remark 1. Also, $Z = (B_1, B_2)$ being continuous implies $\mathbb{P}(Z \in \partial \mathfrak{T}_\Theta(\theta)) = 0$. Hence, all the conditions in Part (iii) of Lemma 1 are satisfied, and the result follows.

APPENDIX B

PROOF OF LEMMA 4

Since $g_n \rightarrow g$ in $L^{r'}(\rho)$, for every $\epsilon > 0$, there exists $n_0(\epsilon) \in \mathbb{N}$ such that $\|g_n\|_{r', \rho} \leq \|g\|_{r', \rho} + \epsilon$ for all $n \geq n_0(\epsilon)$. Thus, for every $n \geq n_0(\epsilon)$, we have by Minkowski's and Hölder's inequality that

$$\begin{aligned} \|f_n g_n - f g\|_{1, \rho} &\leq \|f_n g_n - f g_n\|_{1, \rho} + \|f g_n - f g\|_{1, \rho} \\ &\leq \|f_n - f\|_{r, \rho} \|g_n\|_{r', \rho} + \|f\|_{r, \rho} \|g_n - g\|_{r', \rho} \\ &\leq \|f_n - f\|_{r, \rho} (\|g\|_{r', \rho} + \epsilon) + \|f\|_{r, \rho} \|g_n - g\|_{r', \rho}. \end{aligned}$$

The RHS converges to zero since $f_n \rightarrow f$ in $L^r(\rho)$ and $g_n \rightarrow g$ in $L^{r'}(\rho)$, thus proving Part (i). The claim in Part (ii) follows from [25, Example 1.4.7 (Slutsky's lemma)] by the separability of $L^r(\rho)$ spaces for $1 \leq r < \infty$, and the continuous mapping theorem applied to the continuous map $(f, g) \mapsto fg$ from $L^r(\rho) \times L^{r'}(\rho) \rightarrow L^1(\rho)$ by Part (i).

APPENDIX C

PROOF OF LEMMA 5

Part (i): The convexity of ϕ_{KL} , ϕ_{χ^2} and ϕ_{H^2} follows since these are perspectives¹¹ of the convex functions f_{KL} , f_{χ^2} and f_{H^2} , respectively (see [83]). Computing the partial derivatives yields

$$\begin{aligned} D^{(1,0)}\phi_{\text{KL}}(x, y) &= 1 + \log\left(\frac{x}{y}\right), & D^{(0,1)}\phi_{\text{KL}}(x, y) &= -\frac{x}{y}, \\ D^{(1,1)}\phi_{\text{KL}}(x, y) &= -\frac{1}{y}, & D^{(2,0)}\phi_{\text{KL}}(x, y) &= \frac{1}{x}, & D^{(0,2)}\phi_{\text{KL}}(x, y) &= \frac{x}{y^2}, \end{aligned} \quad (86a)$$

¹¹The perspective of a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ is the function $f_p : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $f_p(x, t) = tf(x/t)$.

$$D^{(1,0)}\phi_{\chi^2}(x,y) = \frac{2(x-y)}{y}, \quad D^{(0,1)}\phi_{\chi^2}(x,y) = 1 - \frac{x^2}{y^2}, \quad (86b)$$

$$D^{(1,1)}\phi_{\chi^2}(x,y) = -\frac{2x}{y^2}, \quad D^{(2,0)}\phi_{\chi^2}(x,y) = \frac{2}{y}, \quad D^{(0,2)}\phi_{\chi^2}(x,y) = \frac{2x^2}{y^3},$$

$$D^{(1,0)}\phi_{\mathbf{H}^2}(x,y) = 1 - y^{\frac{1}{2}}x^{-\frac{1}{2}}, \quad D^{(0,1)}\phi_{\mathbf{H}^2}(x,y) = 1 - x^{\frac{1}{2}}y^{-\frac{1}{2}}, \quad (86c)$$

$$D^{(1,1)}\phi_{\mathbf{H}^2}(x,y) = -\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}, \quad D^{(2,0)}\phi_{\mathbf{H}^2}(x,y) = \frac{1}{2}x^{-\frac{3}{2}}y^{\frac{1}{2}}, \quad D^{(0,2)}\phi_{\mathbf{H}^2}(x,y) = \frac{1}{2}y^{-\frac{3}{2}}x^{\frac{1}{2}}.$$

These partial derivatives obviously satisfy the continuity conditions in Assumption 1 and those mentioned in the statement of Lemma 5 (i), thus completing the proof of Part (i).

Part (ii): For (ii)(a), first note that for $g_1, g_2 \geq 0$ and $g_1^*, g_2^* > 0$, we have

$$\left| D^{(2,0)}\phi_{\text{KL}} \circ ((1-\tau)(g_1^*, g_2^*) + \tau(g_1, g_2)) \right| (1-\tau) = \frac{(1-\tau)}{(1-\tau)g_1^* + \tau g_1} \leq \frac{1}{g_1^*}.$$

The case of $D^{(1,1)}\phi_{\text{KL}}$ is similar, while for $D^{(0,2)}\phi_{\text{KL}}$, we have for $(g_1, g_2) \in \bar{\Theta}(q)$ that

$$\left| D^{(0,2)}\phi_{\text{KL}} \circ ((1-\tau)(g_1^*, g_2^*) + \tau(g_1, g_2)) \right| (1-\tau) = \frac{(1-\tau)((1-\tau)g_1^* + \tau g_1)}{((1-\tau)g_2^* + \tau g_2)^2} \leq \frac{g_1^*}{g_2^{*2}} + \frac{g_1}{2g_2^*g_2} \leq \frac{g_1^*}{g_2^{*2}} + \frac{q}{g_2^*}.$$

For (ii)(b), the case of $D^{(1,1)}\phi_{\chi^2}$ and $D^{(2,0)}\phi_{\chi^2}$ again can be shown as above, while

$$\left| D^{(0,2)}\phi_{\chi^2} \circ ((1-\tau)(g_1^*, g_2^*) + \tau(g_1, g_2)) \right| (1-\tau) = \frac{2(1-\tau)((1-\tau)g_1^* + \tau g_1)^2}{((1-\tau)g_2^* + \tau g_2)^3} \lesssim \frac{g_1^{*2}}{g_2^{*3}} + \frac{g_1^2}{g_2^*g_2^2} \leq \frac{g_1^{*2}}{g_2^{*3}} + \frac{q^2}{g_2^*}.$$

Finally, for (ii)(c), let $(g_1, g_2), (g_1^*, g_2^*) \in \check{\Theta}(q_1, q_2)$. We have

$$\begin{aligned} \left| D^{(1,1)}\phi_{\mathbf{H}^2} \circ ((1-\tau)(g_1^*, g_2^*) + \tau(g_1, g_2)) \right| (1-\tau) &\lesssim \frac{(1-\tau)}{((1-\tau)g_2^* + \tau g_2)^{\frac{1}{2}}((1-\tau)g_1^* + \tau g_1)^{\frac{1}{2}}} \leq \frac{1}{(g_1^*g_2^*)^{\frac{1}{2}}}, \\ \left| D^{(2,0)}\phi_{\mathbf{H}^2} \circ ((1-\tau)(g_1^*, g_2^*) + \tau(g_1, g_2)) \right| (1-\tau) &\lesssim \frac{(1-\tau)((1-\tau)g_2^* + \tau g_2)^{\frac{1}{2}}}{((1-\tau)g_1^* + \tau g_1)^{\frac{3}{2}}} \\ &\leq \frac{g_2^{\frac{1}{2}}}{g_1^{*\frac{3}{2}}} + \frac{g_2^{\frac{1}{2}}\tau^{1/2}}{g_1^{*\frac{3}{2}}(1-\tau)^{\frac{1}{2}}} \leq \frac{g_2^{*\frac{1}{2}}}{g_1^{*\frac{3}{2}}} + \frac{q_2^{\frac{1}{2}}\tau^{\frac{1}{2}}}{g_1^{*\frac{3}{2}}(1-\tau)^{\frac{1}{2}}}, \end{aligned}$$

where $\tau^{1/2}(1-\tau)^{-\frac{1}{2}} \in L^1([0, 1], \lambda)$. The bound for $D^{(0,2)}\phi_{\mathbf{H}^2}$ can be shown similarly, thus completing the proof.

APPENDIX D

PROOF OF LEMMA 6

The claim follows from the proof of [81, Appendix A.9, Corollary 2] (see Page 500-501) by noting that the condition (a), (b) and (c) given therein which suffices for the proof to hold are satisfied under the conditions here. Specifically, condition (a) is equivalent to $\lim_{n \rightarrow \infty} n\mathbf{H}^2(\pi_n, \pi_0) = \|h/2\|_{2, \pi_0}^2$ (with $\delta = h/2$ and $\mu = \pi_0$ in the notation therein) which in turn also implies (b), while (c) is equivalent to $\int_{\mathbb{R}^d \times \mathbb{R}^d} h d\pi_0 = 0$.

APPENDIX E

LIMIT DISTRIBUTION FOR GAUSSIAN-SMOOTHED χ^2 DIVERGENCE, H^2 DISTANCE AND TV DISTANCEA. χ^2 divergence

We consider distributional limits for the Gaussian-smoothed χ^2 divergence.

Proposition 6 (Limit distribution for Gaussian-smoothed χ^2 divergence) *The following hold:*

(i) (One-sample null) *If (26) is satisfied, then there exists a version of $G_{\mu,\sigma}$ such that $G_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ is $L^2(\mathbb{R}^d)$ -valued, and*

$$n\chi^2(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma) \xrightarrow{d} \int_{\mathbb{R}^d} \frac{G_{\mu,\sigma}^2(x)}{\mu * \varphi_\sigma(x)} dx, \quad (87)$$

where the limit can be represented as a weighted sum of i.i.d. χ^2 random variables with 1 degree of freedom. In particular, (87) holds for β -sub-Gaussian μ with $\beta < \sigma$. Conversely, if (26) is violated, then $\mathbb{E}[\chi^2(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] = \infty$ for every $n \in \mathbb{N}$.

(ii) (One-sample alternative) *If (28) holds and $\chi^2(\mu * \varphi_\sigma \| \nu * \varphi_\sigma) < \infty$, then*

$$n^{\frac{1}{2}}(\chi^2(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) - \chi^2(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)) \xrightarrow{d} N(0, v_{1,f_{\chi^2}}^2(\mu, \nu, \sigma)), \quad (88)$$

where $v_{1,f_{\chi^2}}^2(\mu, \nu, \sigma)$ is given in (92). In particular, (88) holds for β -sub-Gaussian μ with $\beta < \sigma$ such that $\mu \ll \nu$ and $\|d\mu/d\nu\|_\infty < \infty$.

(iii) (Two-sample null) *If μ has compact support, then there exists a version of $G_{\mu,\sigma}$, $\tilde{G}_{\mu,\sigma}$ such that $G_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ and $\tilde{G}_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ are $L^2(\mathbb{R}^d)$ -valued, and*

$$n\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) \xrightarrow{d} \int_{\mathbb{R}^d} \frac{(G_{\mu,\sigma}(x) - \tilde{G}_{\mu,\sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx, \quad (89)$$

where the limit can be represented as a weighted sum of i.i.d. χ^2 random variables with 1 degree of freedom.

(iv) (Two-sample alternative) *If μ, ν have compact supports, then*

$$n^{\frac{1}{2}}(\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \chi^2(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)) \xrightarrow{d} N(0, v_{2,f_{\chi^2}}^2(\mu, \nu, \sigma)), \quad (90)$$

where $v_{2,f_{\chi^2}}^2(\mu, \nu, \sigma)$ is given in (94).

Proof. Since the general idea of the proof is similar to that of Proposition 2, we only provide a sketch of the proof, while highlighting the differences.

Part (i): We apply Theorem 3(i) with $\rho = \mu * \gamma_\sigma$. We have

$$\mathbb{E}[\chi^2(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma)] \stackrel{(a)}{=} \frac{1}{n^2} \int_{\mathbb{R}^d} \mathbb{E} \left[\frac{\left(\sum_{i=1}^n (\varphi_\sigma(x - X_i) - \mu * \varphi_\sigma(x)) \right)^2}{\mu * \varphi_\sigma(x)} \right] dx$$

$$\begin{aligned}
&\stackrel{(b)}{=} \frac{1}{n} \int_{\mathbb{R}^d} \mathbb{E} \left[\frac{(\varphi_\sigma(x - X) - \mu * \varphi_\sigma(x))^2}{\mu * \varphi_\sigma(x)} \right] dx \\
&= \frac{1}{n} \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx,
\end{aligned} \tag{91}$$

where (a) is by Fubini's theorem and (b) uses that X_1, \dots, X_n are i.i.d. Consequently, under (26), we obtain $\chi^2(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma) < \infty$ a.s. The rest of the proof leading to (87) via Theorem 3(i) is similar to that of Proposition 2(i) and proceeds by showing that the conditions in Theorem 6(ii) are satisfied; we omit the details to avoid repetition. The claim that (87) holds for β -sub-Gaussian μ with $\beta < \sigma$ is a consequence of [78, Proposition 1] by noting that the LHS of (26) equals $I_{\chi^2}(V; V + Z)$, with $V \sim \mu$ and $Z \sim \gamma_\sigma$ independent of each other. The final claim is obvious from (91).

Part (ii): Note that

$$\begin{aligned}
\mathbb{E}[\chi^2(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma)] &= \int_{\mathbb{R}^d} \frac{1}{n^2 \nu * \varphi_\sigma(x)} \mathbb{E} \left[\left(\sum_{i=1}^n (\varphi_\sigma(x - X_i) - \nu * \varphi_\sigma(x)) \right)^2 \right] dx \\
&\stackrel{(a)}{\leq} \int_{\mathbb{R}^d} \frac{1}{\nu * \varphi_\sigma(x)} \mathbb{E}_\mu \left[(\varphi_\sigma(x - X) - \nu * \varphi_\sigma(x))^2 \right] dx \\
&\stackrel{(b)}{=} \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx + \chi^2(\mu * \varphi_\sigma \| \nu * \varphi_\sigma),
\end{aligned}$$

where (a) uses the convexity of the map $x \mapsto x^2$ while (b) is because X_1, \dots, X_n are i.i.d. Since the RHS of the above equation is finite by assumption, we conclude that $\chi^2(\hat{\mu}_n * \gamma_\sigma \| \mu * \gamma_\sigma) < \infty$ a.s. Also, under (28), it follows that

$$n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\nu * \varphi_\sigma} - \frac{\mu * \varphi_\sigma}{\nu * \varphi_\sigma} \right) \xrightarrow{w} \frac{G_{\mu, \sigma}}{\nu * \varphi_\sigma} \quad \text{in } L^2(\nu * \varphi_\sigma),$$

via similar arguments to those in proof of Proposition 2(ii). Then, (88) is a direct consequence of Theorem 3(ii) with $\rho = \nu * \gamma_\sigma$, which implies

$$n^{\frac{1}{2}} \left(\chi^2(\hat{\mu}_n * \gamma_\sigma \| \nu * \gamma_\sigma) - \chi^2(\mu * \gamma_\sigma \| \nu * \gamma_\sigma) \right) \xrightarrow{d} 2 \int_{\mathbb{R}^d} G_{\mu, \sigma}(x) \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} dx \sim N(0, v_{1, f_{\chi^2}}^2(\mu, \nu, \sigma)),$$

where

$$v_{1, f_{\chi^2}}^2(\mu, \nu, \sigma) = 4 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(1,1)}(x, y) \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \frac{\mu * \varphi_\sigma(y)}{\nu * \varphi_\sigma(y)} dx dy. \tag{92}$$

Finally, the last claim follows from the first since (28) is satisfied for β -sub-Gaussian μ with $\beta < \sigma$ and $\|d\mu/d\nu\|_\infty < \infty$ due to (60), and by (59)

$$\chi^2(\mu * \varphi_\sigma \| \nu * \varphi_\sigma) = \int_{\mathbb{R}^d} \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} - 1 \right)^2 \nu * \varphi_\sigma(x) dx \leq 2 \left(\left\| \frac{\mu * \varphi_\sigma}{\nu * \varphi_\sigma} \right\|_\infty^2 + 1 \right) < \infty.$$

Part (iii): We specialize Theorem 3(iii) with $\rho = \mu * \gamma_\sigma$. First, note that the absolute continuity requirements on the distributions are satisfied, and $p_{\hat{\nu}_n * \gamma_\sigma} > 0$. Also, $\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) < \infty$ a.s. since with $\hat{X} \sim \hat{\mu}_n$, $\hat{Y} \sim \hat{\nu}_n$,

$X \sim \mu$ independent of $Y \sim \mu$, we have

$$\begin{aligned} \mathbb{E}[\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)] &= \mathbb{E}[\chi^2(N(\hat{X}, \sigma^2 I_d) \| N(\hat{Y}, \sigma^2 I_d))] \\ &\stackrel{(a)}{\leq} \mathbb{E}[\chi^2(N(X, \sigma^2 I_d) \| N(Y, \sigma^2 I_d))] \\ &\stackrel{(b)}{=} \mathbb{E}\left[e^{\frac{\|X-Y\|^2}{\sigma^2}}\right] \stackrel{(c)}{<} \infty. \end{aligned} \quad (93)$$

Here, (a) follows by convexity of χ^2 divergence, (b) uses the closed form expression for χ^2 divergence between multi-variate isotropic Gaussians [84], and (c) is because μ has compact support. On the other hand, taking $q(x) = c'e^{c\|x\|}$ such that $\hat{\mu}_n * \varphi_\sigma(x)/\hat{\nu}_n * \varphi_\sigma(x) \leq q(x)$ a.s. based on (61), we have

$$\begin{aligned} \frac{d\eta_1}{dx}(x) &= (1 + c'e^{c\|x\|})\mu * \varphi_\sigma(x), \\ \frac{d\eta_2}{dx}(x) &= (1 + c'e^{c\|x\|} + c'^2 e^{2c\|x\|})\mu * \varphi_\sigma(x). \end{aligned}$$

Setting $r_n = n^{1/2}$, it follows from (17) and the arguments in the proof of Part (iii) that

$$n\chi^2(\hat{\mu}_n * \varphi_\sigma \| \hat{\nu}_n * \varphi_\sigma) \xrightarrow{d} \int_{\mathfrak{S}} \frac{(G_{\mu,\sigma}(x) - \tilde{G}_{\mu,\sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx,$$

provided that (62) holds. By the same arguments as in the proof of Proposition 2(iii), this holds if

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} (1 + e^{2c\|x\|}) dx < \infty,$$

which in turn is satisfied because $\text{spt}(\mu)$ is compact (see the derivation leading to (64)).

Part (iv): We employ Theorem 3(iv) with $\rho = \nu * \gamma_\sigma$. The required positivity and absolute continuity requirements of the probability measures are readily verified. Also, using Jensen's inequality and (93), we have

$$\chi^2(\mu * \gamma_\sigma \| \nu * \gamma_\sigma) \leq \mathbb{E}[\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma)] < \infty,$$

which implies $\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) < \infty$ a.s. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{(\mu * \varphi_\sigma(x))^4}{(\nu * \varphi_\sigma(x))^3} dx &= \int_{\mathbb{R}^d} \frac{\left(\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{2\sigma^2}} d\mu(y)\right)^4}{\left(\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{2\sigma^2}} d\nu(y)\right)^3} dx \\ &= \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} \frac{\left(\int_{\mathbb{R}^d} e^{\frac{2x \cdot y - \|y\|^2}{2\sigma^2}} d\mu(y)\right)^4}{\left(\int_{\mathbb{R}^d} e^{\frac{2x \cdot y - \|y\|^2}{2\sigma^2}} d\nu(y)\right)^3} dx \\ &\leq \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2\sigma^2}} e^{c\|x\|} dx < \infty. \end{aligned}$$

Taking $q(x) = c'e^{c\|x\|}$ with $\hat{\mu}_n * \varphi_\sigma(x)/\hat{\nu}_n * \varphi_\sigma(x) \leq q(x)$ a.s. (see (61)), we have

$$\frac{d\eta_1}{dx}(x) = \nu * \varphi_\sigma(x) + \mu * \varphi_\sigma(x) + \nu * \varphi_\sigma(x) c'e^{c\|x\|},$$

$$\frac{d\eta_2}{dx}(x) = \frac{d\eta_1}{dx}(x) + \nu * \varphi_\sigma(x) c'^2 e^{2c\|x\|} + \frac{(\mu * \varphi_\sigma(x))^2}{\nu * \varphi_\sigma(x)}.$$

Similarly to the proof of Proposition 6(iv), setting $r_n = n^{1/2}$ in (18), yields

$$\begin{aligned} n^{\frac{1}{2}}(\chi^2(\hat{\mu}_n * \gamma_\sigma \| \hat{\nu}_n * \gamma_\sigma) - \chi^2(\mu * \gamma_\sigma \| \nu * \gamma_\sigma)) &\xrightarrow{d} 2 \int_{\mathbb{R}^d} G_{\mu,\sigma}(x) \frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} G_{\nu,\sigma}(x) \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right)^2 dx \\ &\sim N(0, v_{2,f_{\chi^2}}^2(\mu, \nu, \sigma)), \end{aligned}$$

provided that (68) holds. Here, the variance is given by

$$\begin{aligned} v_{2,f_{\chi^2}}^2(\mu, \nu, \sigma) &:= \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(i,j)}(x, y) L_{i,f_{\chi^2}}(x) L_{j,f_{\chi^2}}(y) dx dy \\ &= \sum_{1 \leq i, j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(i,j)}(x, y) \tilde{L}_{i,f_{\chi^2}}(x) \tilde{L}_{j,f_{\chi^2}}(y) dx dy, \end{aligned} \quad (94)$$

with $\tilde{L}_{1,f_{\chi^2}} := 2(\mu * \varphi_\sigma / \nu * \varphi_\sigma)$ and $\tilde{L}_{2,f_{\chi^2}} := (\mu * \varphi_\sigma / \nu * \varphi_\sigma)^2$. Finally, Theorem 6 implies that (68) is satisfied if

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} (1 + e^{c\|x\|}) dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot)) \mu * \varphi_\sigma(x)}{(\nu * \varphi_\sigma(x))^2} dx < \infty, \\ \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} (1 + e^{2c\|x\|}) dx + \sum_{j=1,2} \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot)) (\mu * \varphi_\sigma(x))^j}{(\nu * \varphi_\sigma(x))^{j+1}} dx < \infty, \end{aligned}$$

which holds for compactly supported μ, ν , akin to (64). This completes the proof. \square

B. H^2 distance

The following proposition obtains limit distributions for Gaussian-smoothed squared Hellinger distance.

Proposition 7 (Limit distribution for Gaussian-smoothed H^2 distance) *The following hold:*

- (i) (One-sample null) If (26) holds, then there exists a version of $G_{\mu,\sigma}$ such that $G_{\mu,\sigma} / \sqrt{\mu * \varphi_\sigma}$ is $L^2(\mathbb{R}^d)$ -valued, and

$$nH^2(\hat{\mu}_n * \gamma_\sigma, \mu * \gamma_\sigma) \xrightarrow{d} \frac{1}{4} \int_{\mathfrak{S}} \frac{G_{\mu,\sigma}^2(x)}{\mu * \varphi_\sigma(x)} dx, \quad (95)$$

where the limit can be represented as a weighted sum of i.i.d. χ^2 random variables with 1 degree of freedom. In particular, (95) holds for β -sub-Gaussian μ with $\beta < \sigma$. Conversely, if (26) is violated, then $\liminf_{n \rightarrow \infty} n\mathbb{E}[H^2(\hat{\mu}_n * \gamma_\sigma, \mu * \gamma_\sigma)] = \infty$.

- (ii) (One-sample alternative) If

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} \left(\frac{\nu * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} \right)^{\frac{1}{2}} dx < \infty, \quad (96)$$

then

$$n^{\frac{1}{2}}(H^2(\hat{\mu}_n * \gamma_\sigma, \nu * \gamma_\sigma) - H^2(\mu * \gamma_\sigma, \nu * \gamma_\sigma)) \xrightarrow{d} N(0, v_{1,f_{H^2}}^2(\mu, \nu, \sigma)), \quad (97)$$

where $v_{1,f_{H^2}}^2(\mu, \nu, \sigma)$ is given in (102). In particular, (97) holds for β -sub-Gaussian μ with $\beta < \sigma$ and $\nu \ll \mu$ such that $\|d\nu/d\mu\|_\infty < \infty$.

(iii) (Two-sample null) If

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x))^{3/2}} dx < \infty, \quad (98)$$

then there exists a version of $G_{\mu,\sigma}$, $\tilde{G}_{\mu,\sigma}$ such that $G_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ and $\tilde{G}_{\mu,\sigma}/\sqrt{\mu * \varphi_\sigma}$ are $L^2(\mathbb{R}^d)$ -valued, and

$$n H^2(\hat{\mu}_n * \gamma_\sigma, \hat{\nu}_n * \gamma_\sigma) \xrightarrow{d} \frac{1}{4} \int_{\mathbb{R}^d} \frac{(G_{\mu,\sigma}(x) - \tilde{G}_{\mu,\sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx, \quad (99)$$

where the limit can be represented as a weighted sum of i.i.d. χ^2 random variables with 1 degree of freedom. In particular, (98) and (99) holds for β -sub-Gaussian μ with $\beta < \sigma/\sqrt{6}$.

(iv) (Two-sample alternative) If either of the following conditions hold

a) μ, ν have compact supports;

b) $\|d\mu/d\nu\|_\infty < \infty$, and

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x))^{3/2}} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{(\nu * \varphi_\sigma(x))^{3/2}} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{(\nu * \varphi_\sigma(x))^{3/2}} dx < \infty; \quad (100)$$

c) The conditions in (b) hold with the role of μ and ν interchanged;

then

$$n^{\frac{1}{2}}(H^2(\hat{\mu}_n * \gamma_\sigma, \hat{\nu}_n * \gamma_\sigma) - H^2(\mu * \gamma_\sigma, \nu * \gamma_\sigma)) \xrightarrow{d} N(0, v_{2,f_{H^2}}^2(\mu, \nu, \sigma)), \quad (101)$$

where $v_{2,f_{H^2}}^2(\mu, \nu, \sigma)$ is specified in (105). In particular, (101) holds for β -sub-Gaussian μ, ν with $\beta < \sigma/\sqrt{6}$ such that $\mu \ll \nu$ and $\|d\mu/d\nu\|_\infty \vee \|d\nu/d\mu\|_\infty < \infty$.

Proof. The proof relies on Theorem 4 by ensuring that the relevant conditions therein are fulfilled.

Part (i): We apply Theorem 4(i) with $\rho = \mu * \gamma_\sigma$. Observe that $p_\eta = 2$ and that the positivity and absolute continuity assumptions are satisfied. Then, (95) will follow from (19), provided (52) holds, which in turn is fulfilled under (26) (see (54) and the following discussion). The penultimate claim is because β -sub-Gaussian μ with $\beta < \sigma$ satisfies (26), as mentioned in the proof of Proposition 2(i). The final claim follows by instantiating Lemma 2 to H^2 distance and noting that f_{H^2} satisfy the regularity conditions therein.

Part (ii): We apply Theorem 4(ii) with $\rho = \mu * \gamma_\sigma$. Note that $\|\nu * \varphi_\sigma / \mu * \varphi_\sigma\|_{1, \mu * \varphi_\sigma} = 1$ and $d\eta/dx = \mu * \varphi_\sigma + (\mu * \varphi_\sigma)^{1/2}(\nu * \varphi_\sigma)^{1/2}$, which implies that η is a finite measure. Thus, provided that

$$n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\mu * \varphi_\sigma} - 1 \right) \xrightarrow{w} \frac{G_{\mu,\sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\eta),$$

we obtain from (20) the limit distribution

$$n^{\frac{1}{2}} \left(H^2(\hat{\mu}_n * \gamma_\sigma, \nu * \gamma_\sigma) - H^2(\mu * \gamma_\sigma, \nu * \gamma_\sigma) \right) \xrightarrow{d} \int_{\mathbb{R}^d} \left(\frac{\nu * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} \right)^{\frac{1}{2}} G_{\mu, \sigma}(x) dx \sim N(0, v_{1, f_{H^2}}^2(\mu, \nu, \sigma)),$$

where

$$v_{1, f_{H^2}}^2(\mu, \nu, \sigma) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu, \nu, \sigma}^{(1,1)}(x, y) \left(\frac{\nu * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} \right)^{\frac{1}{2}} \left(\frac{\nu * \varphi_\sigma(y)}{\mu * \varphi_\sigma(y)} \right)^{\frac{1}{2}} dx dy. \quad (102)$$

To establish the weak convergence condition we again verify Condition (ii) of Theorem 6, which uses the assumption from (96), and then invoke Theorem 6(i) (see the proof of Proposition 2). The final claim follows as the LHS of (96) is finite for β -sub-Gaussian μ with $\beta < \sigma$ and $\|d\nu/d\mu\|_\infty < \infty$, based on (59) and the arguments at the end of proof of Proposition 2(i).

Part (iii): We specialize Theorem 4(iii) with $\rho = \mu * \gamma_\sigma$. Setting $q_1 = q_2 = c_{d, \sigma} / \mu * \gamma_\sigma$ where $c_{d, \sigma} = (2\pi\sigma^2)^{-d/2}$, we have that $\eta_1 = \eta_2 = \eta$ is a σ -finite measure specified by the Lebesgue density $d\eta/dx = 2\mu * \varphi_\sigma + (c_{d, \sigma} \mu * \varphi_\sigma)^{1/2}$ (note that both terms are bounded on \mathbb{R}^d). Then, from (21), we obtain

$$nH^2(\hat{\mu}_n * \varphi_\sigma, \hat{\nu}_n * \varphi_\sigma) \xrightarrow{d} \frac{1}{4} \int_{\mathfrak{S}} \frac{(G_{\mu, \sigma}(x) - \tilde{G}_{\mu, \sigma}(x))^2}{\mu * \varphi_\sigma(x)} dx,$$

provided that

$$n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\mu * \varphi_\sigma} - 1 \right) \xrightarrow{w} \frac{G_{\mu, \sigma}}{\mu * \varphi_\sigma} \quad \text{and} \quad n^{1/2} \left(\frac{\hat{\nu}_n * \varphi_\sigma}{\mu * \varphi_\sigma} - 1 \right) \xrightarrow{w} \frac{\tilde{G}_{\mu, \sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\eta).$$

Applying Theorem 6(ii), the above weak convergence holds once (98) is satisfied.

To prove the final claim in Part (iii), we first note that

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x))^{\frac{3}{2}}} dx &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\varphi_\sigma^2(x - y)}{(\mu * \varphi_\sigma(x))^{\frac{3}{2}}} dx d\mu(y) \\ &= \int_{\mathbb{R}^d} e^{-\|y\|^2/\sigma^2} \frac{1}{(2\pi\sigma^2)^d} \int_{\mathbb{R}^d} \frac{e^{2y \cdot x/\sigma^2 - \|x\|^2/\sigma^2}}{(\mu * \varphi_\sigma(x))^{\frac{3}{2}}} dx d\mu(y). \end{aligned}$$

Since $\|x - y\|^2 \leq (1 + \tau)\|x\|^2 + (1 + \tau^{-1})\|y\|^2$, $\forall \tau \in (0, 1)$, Jensen's inequality implies

$$(\mu * \varphi_\sigma(x))^{\frac{3}{2}} \geq \mathbb{E}_\mu \left[\varphi_\sigma^{\frac{3}{2}}(x - \cdot) \right] \geq \frac{e^{-3(1+\tau)\|x\|^2/(4\sigma^2)}}{(2\pi\sigma^2)^{\frac{3d}{4}}} \underbrace{\int_{\mathbb{R}^d} e^{-3(1+\tau^{-1})\|y\|^2/(4\sigma^2)} d\mu(y)}_{=: c_{\mu, \tau}}.$$

Then

$$\frac{1}{(2\pi\sigma^2)^d} \int_{\mathbb{R}^d} \frac{e^{2y \cdot x/\sigma^2 - \|x\|^2/\sigma^2}}{(\mu * \varphi_\sigma(x))^{\frac{3}{2}}} dx \leq \frac{2^{\frac{3d}{4}} \sigma^{\frac{d}{2}} \pi^{\frac{d}{4}}}{(1 - 3\tau)^{\frac{d}{2}} c_{\mu, \tau}} \int_{\mathbb{R}^d} e^{2y \cdot x/\sigma^2} \varphi_{\sqrt{2}\sigma/\sqrt{1-3\tau}}(x) dx = \frac{2^{\frac{3d}{4}} \sigma^{\frac{d}{2}} \pi^{\frac{d}{4}}}{(1 - 3\tau)^{\frac{d}{2}} c_{\mu, \tau}} e^{\frac{4\|y\|^2}{(1-3\tau)\sigma^2}},$$

where in the last equation we used $\mathbb{E}[e^{\alpha \cdot W}] = e^{\|\alpha\|^2 \sigma^2/2}$ for $W \sim N(0, \sigma^2 I_d)$. Conclude that

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x))^{\frac{3}{2}}} dx \leq \frac{2^{\frac{3d}{4}} \sigma^{\frac{d}{2}} \pi^{\frac{d}{4}}}{(1 - 3\tau)^{\frac{d}{2}} c_{\mu, \tau}} \int_{\mathbb{R}^d} e^{\frac{3(1+\tau)\|y\|^2}{(1-3\tau)\sigma^2}} d\mu(y) \leq \frac{2^{\frac{3d}{4}} \sigma^{\frac{d}{2}} \pi^{\frac{d}{4}}}{(1 - 3\tau)^{\frac{d}{2}} c_{\mu, \tau}} \left(1 - \frac{6(1+\tau)\beta^2}{(1-3\tau)\sigma^2} \right)^{-\frac{d}{2}},$$

provided that $3(1+\tau)/((1-3\tau)\sigma^2) < 1/(2\beta^2)$, i.e., $\beta < \sigma\sqrt{1-3\tau}/\sqrt{6(1+\tau)}$. Here, the final inequality uses that for any β -sub-Gaussian X with mean zero and $0 \leq \tau' < 1/(2\beta^2)$, we have $\mathbb{E}[e^{\tau'\|X\|^2}] \leq (1-2\beta^2\tau')^{-d/2}$ (see [1, Equation 7]). Since $\tau \in (0, 1)$ is arbitrary and $c_{\mu,\tau} > 0$ for $\tau > 0$, the desired result follows.

Part (iv): We use Theorem 4(iv) with $\rho = \mu * \gamma_\sigma$. Observe that $\|\nu * \varphi_\sigma / \mu * \varphi_\sigma\|_{1, \mu * \varphi_\sigma} = 1$, and since $\|d\mu/d\nu\|_\infty < \infty$ by assumption, we further have $\mu * \varphi_\sigma / \nu * \varphi_\sigma \in L^1(\mu * \gamma_\sigma)$. With $q_1 = q_2 = c_{d,\sigma}/\mu * \gamma_\sigma$, the measures η_1 and η_2 in Theorem 4(iv) are specified by the Lebesgue densities

$$\frac{d\eta_1}{dx} = \mu * \varphi_\sigma + (\mu * \varphi_\sigma \nu * \varphi_\sigma)^{\frac{1}{2}} + (c_{d,\sigma} \mu * \varphi_\sigma)^{\frac{1}{2}} + (\mu * \varphi_\sigma)^{\frac{3}{2}} (\nu * \varphi_\sigma)^{-\frac{1}{2}}, \quad (103)$$

$$\frac{d\eta_2}{dx} = \mu * \varphi_\sigma + (\mu * \varphi_\sigma)^{\frac{5}{2}} (\nu * \varphi_\sigma)^{-\frac{3}{2}} + c_{d,\sigma}^{\frac{1}{2}} (\mu * \varphi_\sigma)^2 (\nu * \varphi_\sigma)^{-\frac{3}{2}} + (\mu * \varphi_\sigma)^{\frac{3}{2}} (\nu * \varphi_\sigma)^{-\frac{1}{2}}. \quad (104)$$

Thus, if the weak convergence conditions

$$\begin{aligned} n^{1/2} \left(\frac{\hat{\mu}_n * \varphi_\sigma}{\mu * \varphi_\sigma} - 1 \right) &\xrightarrow{w} \frac{G_{\mu,\sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\eta_1), \\ n^{1/2} \left(\frac{\hat{\nu}_n * \varphi_\sigma}{\mu * \varphi_\sigma} - \frac{\nu * \varphi_\sigma}{\mu * \varphi_\sigma} \right) &\xrightarrow{w} \frac{G_{\nu,\sigma}}{\mu * \varphi_\sigma} \quad \text{in } L^2(\eta_2). \end{aligned}$$

hold, then (22) yields

$$\begin{aligned} &n^{\frac{1}{2}} \left(H^2(\hat{\mu}_n * \gamma_\sigma, \hat{\nu}_n * \gamma_\sigma) - H^2(\mu * \gamma_\sigma, \nu * \gamma_\sigma) \right) \\ &\xrightarrow{d} \int_{\mathbb{R}^d} \left(\frac{\nu * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} \right)^{\frac{1}{2}} G_{\mu,\sigma}(x) dx + \int_{\mathbb{R}^d} \left(\frac{\mu * \varphi_\sigma(x)}{\nu * \varphi_\sigma(x)} \right)^{\frac{1}{2}} G_{\nu,\sigma}(x) dx \\ &\sim N(0, v_{2,f_{H^2}}^2(\mu, \nu, \sigma)), \end{aligned}$$

where

$$\begin{aligned} v_{2,f_{H^2}}^2(\mu, \nu, \sigma) &:= \sum_{1 \leq i,j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu,\nu,\sigma}^{(i,j)}(x,y) L_{i,f_{H^2}}(x) L_{j,f_{H^2}}(y) dx dy \\ &= \sum_{1 \leq i,j \leq 2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Sigma_{\mu,\nu,\sigma}^{(i,j)}(x,y) \tilde{L}_{i,f_{H^2}}(x) \tilde{L}_{j,f_{H^2}}(y) dx dy, \end{aligned} \quad (105)$$

with $\tilde{L}_{1,f_{H^2}} := (\nu * \varphi_\sigma / \mu * \varphi_\sigma)^{1/2}$ and $\tilde{L}_{2,f_{H^2}} := (\mu * \varphi_\sigma / \nu * \varphi_\sigma)^{1/2}$. Since $\|d\mu/d\nu\|_\infty < \infty$ by assumption, η_1 and η_2 are σ -finite measures because the terms in the RHS of (103) and (104) either have finite Lebesgue integrals or are bounded on \mathbb{R}^d . Hence, the above weak convergences hold if the conditions in Theorem 6(ii) are satisfied. This in turn happens if

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\mu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x))^{3/2}} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x) \nu * \varphi_\sigma(x))^{1/2}} dx < \infty,$$

and

$$\int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{(\nu * \varphi_\sigma(x))^{3/2}} dx + \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{(\mu * \varphi_\sigma(x) \nu * \varphi_\sigma(x))^{1/2}} dx < \infty.$$

The above conditions simplify to that in (100) when $\|d\mu/d\nu\|_\infty < \infty$, thus proving the desired claim under the conditions in (b). The validity of (101) under conditions in (c) is a consequence of the symmetry of H^2 distance in its

arguments. For Condition (a), we need to show that (100) holds when μ, ν have compact supports. Using steps similar to those in (65), one readily verifies (100), as well as that η_1 and η_2 are σ -finite, and $\mu * \varphi_\sigma / \nu * \varphi_\sigma \in L^1(\mu * \gamma_\sigma)$. Hence, all the sufficient conditions required for (101) to hold are verified.

The final claim in Part (iv) follows from the proof of the final claim in Part (iii) which shows that the first and last terms in the LHS of (100) are finite for β -sub-Gaussian μ, ν with $\beta < \sigma/\sqrt{6}$, and

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\mu * \varphi_\sigma(x)} dx &= \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} \frac{\nu * \varphi_\sigma(x)}{\mu * \varphi_\sigma(x)} dx \\ &\leq \left\| \frac{d\nu}{d\mu} \right\|_\infty \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{\nu * \varphi_\sigma(x)} dx \\ &\leq (2\pi\sigma^2)^{-d/4} \left\| \frac{d\nu}{d\mu} \right\|_\infty \int_{\mathbb{R}^d} \frac{\text{Var}_\nu(\varphi_\sigma(x - \cdot))}{(\nu * \varphi_\sigma(x))^{3/2}} dx < \infty. \end{aligned}$$

This completes the proof. \square

C. TV distance

Before stating limit distributions for Gaussian-smoothed TV distance, we recall the definition of tightness or stochastic boundedness of a sequence of random variables. A sequence of real-valued random variables $Z_n, n \in \mathbb{N}$, is tight if for any $\epsilon > 0$, there exists a constant c_ϵ such that $\mathbb{P}(|Z_n| > c_\epsilon) \leq \epsilon$ for all n [25].

Proposition 8 (Gaussian-smoothed TV distance limit distribution) *Let $\rho = \lambda$, $\mathcal{Q} = \{s \in \mathbb{R}^d : p_{\mu * \gamma_\sigma}(s) = p_{\nu * \gamma_\sigma}(s)\}$ and $\text{sgn}(x) = x/|x|$ for $x \neq 0$. Then, the following hold:*

(i) (One-sample null and alternative) *If*

$$\int_{\mathbb{R}^d} \sqrt{\text{Var}_\mu(\varphi_\sigma(x - \cdot))} dx < \infty, \quad (106)$$

then there exists a version of $G_{\mu,\sigma}$ which is $L^1(\mathbb{R}^d)$ valued, and

$$\begin{aligned} n^{\frac{1}{2}} (\|\hat{\mu}_n * \gamma_\sigma - \nu * \gamma_\sigma\|_{\text{TV}} - \|\mu * \gamma_\sigma - \nu * \gamma_\sigma\|_{\text{TV}}) \\ \xrightarrow{d} \frac{1}{2} \int_{\mathcal{Q}} |G_{\mu,\sigma}| dx + \frac{1}{2} \int_{\mathbb{S} \setminus \mathcal{Q}} \text{sgn}(p_{\mu * \gamma_\sigma} - p_{\nu * \gamma_\sigma}) G_{\mu,\sigma} dx. \end{aligned} \quad (107)$$

*In particular, (106) and (107) holds if μ has finite $(2d + \epsilon)$ moment for some $\epsilon > 0$, i.e., $\mathbb{E}[\|X\|^{2d+\epsilon}] < \infty$ for $X \sim \mu$. Conversely, if (106) is violated, then the sequence $n^{1/2} \|\hat{\mu}_n * \gamma_\sigma - \mu * \gamma_\sigma\|_{\text{TV}}$ is not tight.*

(ii) (Two-sample null and alternative) *If*

$$\int_{\mathbb{R}^d} \sqrt{\text{Var}_\mu(\varphi_\sigma(x - \cdot))} dx + \int_{\mathbb{R}^d} \sqrt{\text{Var}_\nu(\varphi_\sigma(x - \cdot))} dx < \infty, \quad (108)$$

then there exists a version of $G_{\mu,\sigma}$ and $G_{\nu,\sigma}$ which are $L^1(\mathbb{R}^d)$ valued such that

$$\begin{aligned} n^{\frac{1}{2}} (\|\hat{\mu}_n * \gamma_\sigma - \hat{\nu}_n * \gamma_\sigma\|_{\text{TV}} - \|\mu * \gamma_\sigma - \nu * \gamma_\sigma\|_{\text{TV}}) \\ \xrightarrow{d} \frac{1}{2} \int_{\mathcal{Q}} |G_{\mu,\sigma} - G_{\nu,\sigma}| dx + \frac{1}{2} \int_{\mathbb{S} \setminus \mathcal{Q}} \text{sgn}(p_{\mu * \gamma_\sigma} - p_{\nu * \gamma_\sigma}) (G_{\mu,\sigma} - G_{\nu,\sigma}) dx. \end{aligned} \quad (109)$$

In particular, (108) and (109) holds if μ and ν have finite $(2d + \epsilon)$ moments for some $\epsilon > 0$. Conversely, if $\mu = \nu$ and (108) does not hold, then $n^{1/2} \|\hat{\mu}_n * \gamma_\sigma - \hat{\nu}_n * \gamma_\sigma\|_{TV}$ is not tight.

We note here that in contrast to Proposition 2, 6, and 7, the limit distribution for smoothed TV distance in the one- and two-sample alternative is not Gaussian. This is a consequence of the non-linearity of the first-order Hadamard derivative (see (51)).

Remark 4 (Upper bound for (106)) *To establish that (106) holds if μ has finite $(2d + \epsilon)$ moments for some $\epsilon > 0$, we show in the proof that for $X \sim \mu$,*

$$\int_{\mathbb{R}^d} \sqrt{\text{Var}_\mu(\varphi_\sigma(x - \cdot))} dx \leq 8^{d/2} + \frac{2^{d/2+1}}{\sigma^d \Gamma(d/2)} \int_0^\infty t^{d-1} \sqrt{\mathbb{P}(\|X\| > t)} dt, \quad (110)$$

where Γ denotes the Gamma function. The RHS is then finite by Markov's inequality provided $\mathbb{E}[\|X\|^{2d+\epsilon}] < \infty$.

Proof. We apply Theorem 5 with $\rho = \lambda$.

Part (i): Observe that (107) is a direct consequence of (23), provided that

$$n^{1/2} (\hat{\mu}_n * \gamma_\sigma - \mu * \gamma_\sigma) \xrightarrow{w} G_{\mu, \sigma} \quad \text{in } L^1(\mathbb{R}^d).$$

Let $Z_i(x) = \varphi_\sigma(x - X_i) - \mu * \varphi_\sigma(x)$ and $Z(x) = \varphi_\sigma(x - X) - \mu * \varphi_\sigma(x)$, where $X \sim \mu$. Note that $\|Z\|_1 \leq 2$ a.s. and $\sqrt{n}(\hat{\mu}_n * \gamma_\sigma - \mu * \gamma_\sigma) = (1/\sqrt{n}) \sum_{i=1}^n Z_i$. Since $\mathbb{P}(\|Z\|_1 \geq t) = 0$ for $t > 2$, Theorem 6(ii) implies that the weak convergence above holds if

$$\int_{\mathbb{R}^d} \left(\mathbb{E}[|Z(x)|^2] \right)^{\frac{1}{2}} dx = \int_{\mathbb{R}^d} \left(\text{Var}_\mu(\varphi_\sigma(x - \cdot)) \right)^{\frac{1}{2}} dx < \infty.$$

The penultimate and final claim in Proposition 8(i) come from [1, Lemma 1] and [1, Proposition 1], respectively, whose proofs we repeat here for completeness. To prove the former, note that for $X \sim \mu$, one has

$$\text{Var}_\mu(\varphi_\sigma(x - \cdot)) \leq \mathbb{E}[\varphi_\sigma^2(x - X)] = \frac{1}{(2\pi\sigma^2)^d} \int_{\mathbb{R}^d} e^{-\|x-y\|^2/\sigma^2} d\mu(y). \quad (111)$$

Splitting the integral over \mathbb{R}^d into $\|y\| \leq \|x\|/2$ and $\|y\| > \|x\|/2$, we further obtain

$$\int_{\mathbb{R}^d} e^{-\|x-y\|^2/\sigma^2} d\mu(y) \leq \int_{\|y\| \leq \|x\|/2} e^{-\|x-y\|^2/\sigma^2} d\mu(y) + \mathbb{P}(\|X\| > \|x\|/2). \quad (112)$$

Changing to polar coordinates leads to

$$\int_{\mathbb{R}^d} \sqrt{\mathbb{P}(\|X\| > \|x\|/2)} dx = \frac{2^{d+1}\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty t^{d-1} \sqrt{\mathbb{P}(\|X\| > t)} dt. \quad (113)$$

Next, using $\|x - y\|^2 \geq \|x\|^2/2 - \|y\|^2$, we have

$$\int_{\|y\| \leq \|x\|/2} e^{-\|x-y\|^2/\sigma^2} d\mu(y) \leq e^{-\|x\|^2/(4\sigma^2)} \int_{\|y\| \leq \|x\|/2} d\mu(y) \leq e^{-\|x\|^2/(4\sigma^2)}, \quad (114)$$

and the square root of the RHS integrates to $(16\pi\sigma^2)^{d/2}$. Combining (111)–(114), we obtain inequality (110).

Finally, if μ has finite $(2d + \epsilon)$ moments, then by Markov's inequality

$$t^{d-1} \sqrt{\mathbb{P}(\|X\| > t)} \leq t^{d-1} \wedge \sqrt{\mathbb{E}[\|X\|^{2d+\epsilon}] t^{-1-(\epsilon/2)}}.$$

The RHS is integrable on $[0, \infty)$, thus showing that (106) (and consequently (107)) holds under the assumption that μ has a finite $(2d + \epsilon)$ -th moment, for some $\epsilon > 0$.

The proof of the final claim is divided into two steps. We first show that if $(\sqrt{n} \|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}})_{n \in \mathbb{N}}$ is tight, then its first moment is uniformly bounded for all n . Then we prove that under this uniform boundedness, Condition (106) holds.

For the first step, define $S_i = \sum_{j=1}^i (\varphi_\sigma(x - X_j) - \mu * \varphi_\sigma(x))$ for $1 \leq i \leq n$, and note that $\sqrt{n} \|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}} = \|S_n / \sqrt{n}\|_1 / 2$. We want to show that if $\|S_n / \sqrt{n}\|_1 / 2$ is tight, then

$$\sup_n \mathbb{E}[\|S_n / \sqrt{n}\|_1] = \sup_n \sqrt{n} \mathbb{E}[\|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}}] < \infty.$$

By Hoffmann-Jørgensen's inequality (see [79, Proposition 6.8]), we have

$$\mathbb{E}[\|S_n\|_1] \lesssim \mathbb{E} \left[\max_{1 \leq i \leq n} \|Z_i\|_1 \right] + t_{n,0},$$

where $t_{n,0} = \inf \{t > 0 : \mathbb{P}(\max_{1 \leq i \leq n} \|S_i\|_1 > t)\} \leq 1/8$. The first term on the RHS is bounded by 2. In addition, by Montgomery-Smith's inequality [85, Corollary 4], there exists a universal constant c such that

$$t_{n,0} \leq \inf \{t > 0 : \mathbb{P}(\|S_n\|_1 > ct) \leq c\}.$$

Thus, if $\|S_n / \sqrt{n}\|_1 / 2$ is tight (uniformly for all n), then $\sup_n t_{n,0} / \sqrt{n} < \infty$, which implies $\sup_n \mathbb{E}[\|S_n / \sqrt{n}\|_1] < \infty$, as desired.

Next, we prove that the uniform boundedness of $\sqrt{n} \mathbb{E}[\|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}}]$ implies Condition (106) holds. Let k be any positive integer. With $Z_i(x) = \varphi_\sigma(x - X_i) - \mu * \varphi_\sigma(x)$ and $\bar{Z}_n = (1/n) \sum_{i=1}^n Z_i$, Fubini's theorem yields

$$\sqrt{n} \mathbb{E}[\|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}}] \geq \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}[(\sqrt{n} |\bar{Z}_n(x) - \mu * \varphi_\sigma(x)| \wedge k)] dx.$$

Since $|Z_i(x)| \leq (2\pi\sigma^2)^{-d/2}$, the CLT implies that for any $x \in \mathbb{R}^d$,

$$\lim_{n \rightarrow \infty} \mathbb{E}[(\sqrt{n} |\bar{Z}_n(x) - \mu * \varphi_\sigma(x)|) \wedge k] = \mathbb{E}[|G_{\mu,\sigma}(x)| \wedge k].$$

Indeed, this follows from the CLT, i.e., $\sqrt{n}(\bar{Z}_n(x) - \mu * \varphi_\sigma(x)) \xrightarrow{d} N(0, \sigma_x^2)$ with $\sigma_x^2 = \text{Var}_\mu(\varphi_\sigma(x - \cdot))$, and the definition of weak convergence since $y \mapsto |y| \wedge k$ is bounded (by k) and (Lipschitz) continuous. Together with Fatou's lemma, we have

$$\liminf_{n \rightarrow \infty} \sqrt{n} \mathbb{E}[\|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}}] \geq \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}[|G_{\mu,\sigma}(x)| \wedge k] dx.$$

Taking $k \rightarrow \infty$, we conclude by monotone convergence theorem that

$$\liminf_{n \rightarrow \infty} \sqrt{n} \mathbb{E} \left[\|\hat{\mu}_n * \varphi_\sigma - \mu * \varphi_\sigma\|_{\text{TV}} \right] \geq \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}[|G_{\mu,\sigma}(x)|] dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \left(\text{Var}_\mu(\varphi_\sigma(x - \cdot)) \right)^{\frac{1}{2}} dx,$$

where the second equality is because $\mathbb{E}[|W|] = \sqrt{2\mathbb{E}[W^2]/\pi}$ for a centered Gaussian variable W . This completes the proof of Part (i).

Part (ii): The claim follows from (24), provided that

$$n^{1/2} (\hat{\mu}_n * \varphi_\sigma - \hat{\nu}_n * \varphi_\sigma - \mu * \gamma_\sigma + \nu * \gamma_\sigma) \xrightarrow{w} G_{\mu,\sigma} - G_{\nu,\sigma} \quad \text{in } L^1(\mathbb{R}^d),$$

where $G_{\mu,\sigma}$ and $G_{\nu,\sigma}$ are $L^1(\mathbb{R}^d)$ -valued Gaussian random variables. Since $L^1(\mathbb{R}^d)$ is Polish, arguments similar to those in the proof of Proposition 2(iii), imply that the above weak convergence holds if

$$n^{1/2} (\hat{\mu}_n * \varphi_\sigma - \mu * \gamma_\sigma) \xrightarrow{w} G_{\mu,\sigma} \quad \text{and} \quad (\hat{\nu}_n * \varphi_\sigma - \nu * \gamma_\sigma) \xrightarrow{w} G_{\nu,\sigma} \quad \text{in } L^1(\mathbb{R}^d).$$

The latter holds under (108) by Theorem 6(ii) via similar arguments to those presented in the proof of Part (i) above. The proof of the converse claim is again analogous to that in Part (i) with $S_n = \sum_{i=1}^n \varphi_\sigma(\cdot - X_i) - \varphi_\sigma(\cdot - Y_i)$; details are omitted. This concludes the proof. \square

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