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ABSTRACT

Let $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$, with $q_j \in \mathbb{Z}^+$ for each $j \in \{1, \ldots, d\}$, and denote by Δ the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$. Using Macaulay2, we first numerically find complex-valued Γ-periodic potentials $V : \mathbb{Z}^d \to \mathbb{C}$ such that the operators $\Delta + V$ and Δ are Floquet isospectral. We then use combinatorial methods to validate these numerical solutions.

I. INTRODUCTION AND MAIN RESULTS

Let Δ denote the discrete Laplacian on $\mathbb{Z}^d$:

$$(\Delta u)(n) = \sum_{|n'-n|_1=1} u(n'),$$

where for $n = (n_1, n_2, \ldots, n_d)$, $n' = (n_1', n_2', \ldots, n_d') \in \mathbb{Z}^d$ we write

$$|n' - n|_1 := \sum_{i=1}^{d} |n_i - n_i'|.$$

In this paper, we study finite difference equations of the type

$$(\Delta u)(n) + V(n)u(n) = \lambda u(n), \quad \text{for all } n \in \mathbb{Z}^d, \tag{1}$$

subject to the Floquet boundary condition

$$u(n + q_j e_j) = e^{2\pi i k_j} u(n), \quad \text{for all } n \in \mathbb{Z}^d, k_j \in [0,1] \quad \text{and } j \in [d] := \{1, \ldots, d\} \tag{2}$$

where $\{e_j\}_{j=1}^d$ is the standard basis in $\mathbb{R}^d$ and $\{q_j\}_{j=1}^d$ are positive integers. The potential $V : \mathbb{Z}^d \to \mathbb{C}$ is assumed to be Γ-periodic. Namely, letting $\Gamma = q_1\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$, we impose that

$$V(n+m) = V(n), \quad \text{for all } n \in \mathbb{Z}^d \text{ and } m \in \Gamma.$$

Writing $Q = \prod_{i=1}^d q_i$, Eq. (1) with the boundary condition (2) can be realized as the eigen-equation of a finite $Q \times Q$ matrix $D_V(k)$, where $k = (k_1, \ldots, k_d)$. Denote by $\sigma_V(k) = \sigma(D_V(k))$ the set of eigenvalues of $D_V(k)$, including algebraic multiplicity.

Definition 1. Assume that V and V' are Γ-periodic potentials. Two operators $\Delta + V$ and $\Delta + V'$ are called Floquet isospectral if $\sigma_V(k) = \sigma_{V'}(k)$ for all $k \in \mathbb{R}^d$. In this case it is also common to say that the Γ-periodic potentials V and V' are Floquet isospectral.

Understanding when two potentials V and V' are Floquet isospectral is an interesting problem with a rich history of study.[12–14,20] In this paper, we focus on the discrete case; however, there is also a deep body of work for the continuous case.[4–6,8,10,23,27]

Floquet isospectrality belongs to the class of inverse spectral problems, where one aims at recovering information about the original operator from certain spectral data. Among such problems, we also highlight Fermi isospectrality,[1,22] Borg's Theorem[2,3,21,25] and the recovery of the potential from its integrated density of states.[7] For more background and history of inverse spectral problems for periodic potentials, we refer the reader to the following recent surveys.[15,16,19]

In this paper, we focus on the Ambarzumyan-type inverse problem of finding potentials isospectral to the zero potential, denoted henceforth by **0**. It is a well-known and classical result that there are no real non-zero potentials Floquet isospectral to **0** in both the continuous and discrete cases. For more general work on the well-studied class of Ambarzumyan-type problems, we refer the reader to Ref. 18 (Chap. 14). For the continuous case, it has been shown that there are many complex-valued potentials that are Floquet isospectral to **0** [e.g., Refs. 11 and 17 (Theorem 11)]. It is a folklore result that there exist complex-valued Γ-periodic potentials V which are Floquet isospectral to **0** in the discrete case. Our goal of this paper is multi-fold. Firstly, we present explicit potentials isospectral to **0**. Secondly, our method of proof is, to the best of our knowledge, new. More concretely, we introduce combinatorial language and techniques which might be of interest to other problems in the realm of the spectral theory of discrete Schrödinger operators.

Theorem I.1. *Let $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$. Assume that at least one of q_j, $j = 1, 2, \ldots, d$ is even. Then there exists a nonzero Γ-periodic function V Floquet isospectral to* **0**.

Remark 1. We prove Theorem I.1 *by constructing explicit potentials isospectral to* **0** *(see Theorem* III.2).

In order to construct these explicit examples, we began by experimentally finding solutions using Macaulay2.[9] Although we do not use all the numerical solutions found, our experimental data enabled us to notice a particular pattern for when potentials are Floquet isospectral to **0**. In this paper, we begin with this pattern and focus on proving it. The reader interested in how this pattern was discovered by us is invited to consult our annotated Macaulay2 code at the github repository.[24] By Floquet theory, the discrete Schödinger operator can be represented as the direct sum of a family of finite square matrices. By modeling this matrix as a finite directed graph, we are able to use graph theory and algebra to verify the solutions suggested by the observed numerical pattern.

The remainder of this note is organized as follows. In Sec. II we give a brief background on the combinatorial constructions we will employ. In Sec. III we reduce Theorem I.1 to Theorem III.2. In Sec. IV we prove Theorem III.2.

II. COMBINATORIAL BACKGROUND AND NOTATION

In this section we introduce basic terminology and also make a few remarks which will be relevant to the proof of the main results of this note. Let us start by introducing some notation.

(i) Fix a $m \times m$ matrix $M = (M_{ij})_{m\times m}$ with entries in a ring R. The digraph of M is a weighted directed graph $G := (\mathcal{V}, \mathcal{E}, w)$ with vertex set $\mathcal{V} = [m]$ and edges given by $(i,j) \in \mathcal{E}$ if $M_{i,j} \neq 0$. The weight function $w : \mathcal{E} \to R$ is naturally inherited from M via $w(i,j) = M_{i,j}$ for all $(i,j) \in \mathcal{E}$.
(ii) A vertex cycle cover is a union of cycles which are subgraphs of G and contains all vertices of G.
(iii) A disjoint cycle cover of a digraph is a vertex cycle cover for which two different cycles have no vertices in common.
(iv) We denote by $\mathcal{S}_m$ the symmetric group of m elements. The permutations in $\mathcal{S}_m$ will be denoted by σ.
(v) We also let $M_\sigma = \prod_{i=1}^m M_{i,\sigma(i)}$.
(vi) Since any permutation $\sigma \in \mathcal{S}_m$ is a product of disjoint cycles, it is easy to see that if $M_\sigma \neq 0$ then σ induces a disjoint cycle cover η of G such that

$$w(\eta) := \prod_{e\in\eta} w(e) = M_\sigma$$

We then denote $\operatorname{sgn}(\eta) := \operatorname{sgn}(\sigma)$.

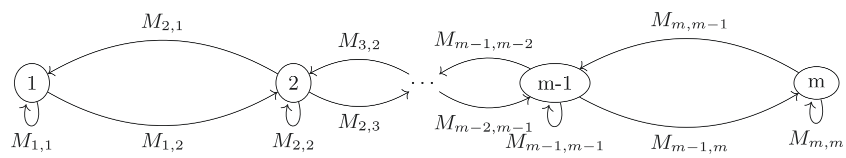

FIG. 1. The weighted digraph J_m of an $m \times m$ Jacobi matrix M.

(vii) We let

$$\mathcal{P} = \{\sigma \in \mathcal{S}_m : \ M_\sigma \neq 0\}.$$

Note that $\mathcal{P}$ is in bijection with the collection of disjoint cycle covers of G, denoted henceforth by $\mathcal{U}_G$.

We now make the following remarks:

(R1) Notice that the digraph of M is a directed version of the adjacency matrix.

(R2) With the given definitions, one has that

$$\det(M) = \sum_{\sigma \in P} \operatorname{sgn}(\sigma) M_\sigma = \sum_{\eta \in U} \operatorname{sgn}(\eta) w(\eta). \tag{3}$$

(viii) Let $[t]f$ denote the coefficient of the monomial term t of a polynomial f. For example if $f(z) = 4z^2 + 9z - 2$ then $[z^2]f(z) = 4$, $[z]f(z) = 9$, and $[z^0]f(z) = -2$.

(ix) We say that $M = (M_{ij})_{m\times m}$ is a Jacobi matrix if it has the following property: $M_{ij} \neq 0$ for some $i, j \in [m]$ if and only if $|i - j| \leq 1$. In this case, we denote by J_m the digraph of M, as shown in Fig. 1. It is easy to see that if η is a disjoint cycle cover of J_m, then η must be composed of only cycles of length 1 and 2.

(x) Let $S(m, p)$ denote the number of disjoint cycle covers of J_m with exactly p 2-cycles. This quantity will be used in the Proof of Theorem III.1. Although we do not need the explicit value of $S(m, p)$, by basic combinatorial identities one finds that $S(m, p) = \binom{m-p}{p}$.

III. REDUCTION OF MAIN THEOREM

To prove Theorem I.1, we provide an explicit potential which is Floquet isospectral to $\mathbf{0}$.

Suppose that $m \in \mathbb{Z}^+$. Let us define a vector $v = (v_1, \ldots, v_{2m}) \in \mathbb{C}^{2m}$ via

$$v \in \mathbb{C}^{2m}, v_1 = 1 + i, v_2 = 1 - i, v_{m+1} = -1 + i, v_{m+2} = -1 - i \ \text{ and } \ v_l = 0 \ \text{ otherwise.} \tag{4}$$

As a $2m\mathbb{Z}$-periodic potential r is determined by the values $r_1 := r(1), \ldots, r_{2m} := r(2m)$, we often abuse notation and identify r with the vector $r = (r_1, \ldots, r_{2m})$.

Theorem III.1. *Let $m \in \mathbb{Z}^+$ and $\Gamma = 2m\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$, and define $V : \mathbb{Z}^d \to \mathbb{C}$ to be the separable Γ-periodic potential given by $V(n_1, \ldots, n_d) = v_{(n_1 \bmod 2m)}$ with v given by (4). Then V is Floquet isospectral to $\mathbf{0}$.*

Theorem III.1 clearly implies Theorem I.1.

To prove Theorem III.1, we need only show that the following theorem is true.

Theorem III.2. *Let v be given by (4), then the following two matrices have the same eigenvalues, including multiplicity*

$$D_v = \begin{pmatrix} v_1 & 1 & 0 & \ldots & 0 & 1 \\ 1 & v_2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 1 & v_{2m-1} & 1 \\ 1 & 0 & \ldots & 0 & 1 & v_{2m} \end{pmatrix}, \quad D_{\mathbf{0}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 1 & 0 & 1 \\ 1 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}. \tag{5}$$

Definition 2. A function $V : \mathbb{Z}^d \to \mathbb{C}$ is called completely separable if there exists $V_j : \mathbb{Z} \to \mathbb{C}$ such that $V(n) = V(n_1, \ldots, n_d) = V_1(n_1) + \cdots + V_d(n_d)$ for all $n \in \mathbb{Z}^d$. If V is completely separable we write $V = V_1 \oplus \cdots \oplus V_d$.

Lemma III.3. Let $\Gamma = q_1\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$. Assume $V = V_1 \oplus \cdots \oplus V_d$ and $W = W_1 \oplus \cdots \oplus W_d$ are completely separable Γ-periodic potentials. If W_j and V_j are Floquet isospectral for each $j \in [d]$, then V and W are Floquet isospectral.

Proof. This follows from a well-known result e.g. Ref. 26 (Theorem 4.17). □

Proof of Reduction of Theorem 3.1 to Theorem 3.2. Notice that V and $\mathbf{0}$ are both completely separable potentials. By Lemma III.3, to prove Theorem III.1 it suffices to show that the $2m\mathbb{Z}$-periodic potential v is isospectral to the $2m\mathbb{Z}$-periodic $\mathbf{0}$. Furthermore, for the one dimensional Schrödinger operator, it is well-known that the Floquet isospectrality of two potentials is determined by their values at a single quasi-momenta [e.g., Ref. 13 (page 1)]. Let us fix the quasi-momtenta $k = 0$. Then V and $\mathbf{0}$ are isospectral if D_v and $D_{\mathbf{0}}$ have the same eigenvalues, each with the same multiplicity for each respective matrix. □

IV. COMBINATORIAL FORMULATION AND PROOF OF THEOREM III.2

Let $P_v(\lambda)$ and $P_{\mathbf{0}}(\lambda)$ denote the characteristic polynomials of D_v and $D_{\mathbf{0}}$, respectively. In order to prove Theorem III.2, it suffices to check that

$$P_v(\lambda) = P_{\mathbf{0}}(\lambda) \quad \text{for all } \lambda \in \mathbb{C}. \tag{6}$$

Before taking v to have the specific form (4), we investigate the difference $P_v(\lambda) - P_{\mathbf{0}}(\lambda)$ when

$$v = (v_1, v_2, 0, \ldots, 0, v_3, v_4, 0, \ldots, 0). \tag{7}$$

for arbitrary values of $v_1, \ldots, v_4$. Note that in this case the polynomial $P_v(\lambda) - P_{\mathbf{0}}(\lambda)$ has degree at most λ^{2m-1}, and each of its coefficients must depend on at least one of the variables $v_1, \ldots, v_4$. More can be said and, in fact, we have the following complete description of the above difference. Define

$$F_i(v) := [\lambda^i](P_v(\lambda) - P_{\mathbf{0}}(\lambda)) = 0 \quad \text{for } i \in \{0, \ldots, 2m-1\}.$$

Theorem IV.1. *The following formulas for $F_{2m-k}(v)$ hold:*

$$\begin{aligned} F_{2m-k}(v) = v_1v_2v_3v_4(-1)^{\ell}\left(\sum_{i=0}^{\ell-2} S(m-2,i)S(m-2,\ell-2-i)\right) + (v_1v_2 + v_3v_4)(-1)^{\ell-1}S(2m-2,\ell-1) \\ +(v_1v_3 + v_2v_4)(-1)^{\ell-1}\left(\sum_{i=0}^{\ell-1} S(m-1,i)S(m-1,\ell-1-i)\right) \\ +(v_1v_4 + v_2v_3)(-1)^{\ell-1}\left(\sum_{i=0}^{\ell-1} S(m,i)S(m-2,\ell-1-i)\right), \end{aligned}$$

if $k = 2\ell$ for $\ell \in [m-1]$.

$$\begin{aligned} F_{2m-k}(v) = (v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_2v_3v_4)(-1)^{\ell}\left(\sum_{i=0}^{\ell-1} S(m-2,i)S(m-1,\ell-1-i)\right) \\ +(v_1 + v_2 + v_3 + v_4)(-1)^{\ell+1}S(2m-1,\ell), \end{aligned}$$

if $k = 2\ell + 1$ for $\ell \in \{0, \ldots, m-1\}$.

Remark 2. It readily follows that, for the above range of values for ℓ, the parity of k must agree with the parity of the degree of any of the monomials appearing in $F_{2m-k}(v)$.

Remark 3. While there are previous works regarding explicit constructions in the case of real-valued potentials,[12,14] it is an interesting question to see whether a construction similar to ours can produce examples of complex potentials Floquet isospectral to an arbitrary real and nonconstant potential V_0. This is necessarily rare, as it is well known, by Ref. 12 (Proposition 2.2), that for a real potential v with pairwise distinct coordinates, for all t sufficiently large the potential tv has only real Floquet isospectral potentials.

We chose to focus on the case of a constant real potential (in this case $V_0 = \mathbf{0}$), since there are no non-trivial real potentials Floquet isospectral to it. In fact, if V_0 is a non-constant real-valued potential, there must exist a real potential $V \neq V_0$ Floquet isospectral to V_0. Thus, not only is it is interesting to ask whether complex Floquet isospectral potentials exist, but, in the case of a constant real potential, these are the only possible non-trivial Floquet isospectral potentials.

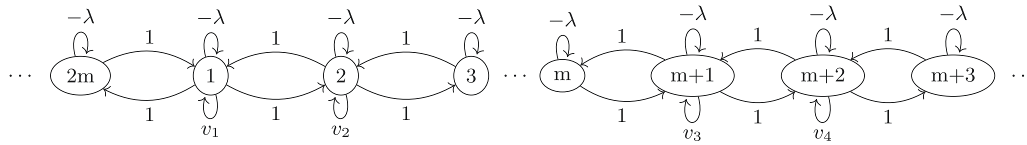

FIG. 2. The refined digraph G'. Notice we do not include the weight 0 self loops after specializing.

Remark 4. While it is natural to start by choosing v taking many zero values, its specific form (7) turned out to be the most convenient for us. The recursive patterns in Theorem IV.1 *were indicated through computations carried out with Maculay2.*[9]

Proof.

Here, we spell out the details when k is even. The proof for odd values of k is completely analogous. Let $k = 2\ell$ for some $\ell \in [m-1]$. Let G be the digraph of $D_v(z) - \lambda I$ and let $\mathcal{U}_G$ be the collection of disjoint cycle covers of G. By (3), we have that

$$P_v(\lambda) = \sum_{\eta \in \mathcal{U}_G} \operatorname{sgn}(\eta) w(\eta).$$

Define $G' = (\mathcal{V}, \mathcal{E}', w')$ to be a refined version of G, where edges of the form (i,i) are replaced by two new edges $(i,i)_1$ and $(i,i)_2$ with weights $w'((i,i)_1) = v_i$, $w'((i,i)_2) = -\lambda$. All other edges and weight assignments are kept fixed (see Fig. 2). Let $\mathcal{U}_{G'}$ be the collection of disjoint cycle covers of G'. Notice that if $\eta \in \mathcal{U}_G$ does not contain (i,i), then $\eta \in \mathcal{U}_{G'}$ and $w(\eta) = w'(\eta)$. Furthermore, given an $\eta \in \mathcal{U}_G$ with a single self loop edge (i,i) there exist disjoint cycle covers α and β in $\mathcal{U}_{G'}$ such that $w(\eta) = w'(\alpha) + w'(\beta)$. In general, $\eta \in \mathcal{U}_G$ splits into 2^l disjoint cycle covers $\alpha_1, \ldots, \alpha_{2^l}$ of G' where l is the number of self loops in η, such that $w(\eta) = \sum_{i=1}^{2^l} w(\alpha_i)$. Thus, we have that

$$P_v(\lambda) = \sum_{\eta \in \mathcal{U}_{G'}} \operatorname{sgn}(\eta) w'(\eta).$$

To solve for $[v_1 v_2 v_3 v_4 \lambda^{2m-k}] P_v(\lambda)$, consider the collection $\mathcal{C}$ of $\eta \in \mathcal{U}_{G'}$ with the following properties

- $w'(\eta) \neq 0$.
- η contains the edges $(1,1)_1$, $(2,2)_1$, $(m+1,m+1)_1$, $(m+2,m+2)_1$.
- η has $2m-k$ other edges of the form $(i,i)_2$.

Given an $\eta \in \mathcal{C}$, after fixing the edges $(1,1)_1$, $(2,2)_1$, $(m+1,m+1)_1$, and $(m+2,m+2)_1$, the remaining digraph $G' \setminus \{1,2,m+1,m+2\}$ is a disjoint union of two $J_{(m-2)}$ digraphs, denoted G_1' and G_2'. It follows that the remaining cycles of η are the $2m-k$ 1-cycles given by $(i,i)_2$ and $\ell - 2$ 2-cycles. Therefore,

$$[v_1 v_2 v_3 v_4 \lambda^{2m-k}] \operatorname{sgn}(\eta) \eta_{w'} = (-1)^{\ell}. \tag{8}$$

In particular,

$$[v_1 v_2 v_3 v_4 \lambda^{2m-k}] P_v(\lambda) = (-1)^{\ell} |\mathcal{C}|, \tag{9}$$

and thus we are left to count the elements of T. Notice that after fixing the edges $(1,1)_1$, $(2,2)_1$, $(m+1,m+1)_1$, and $(m+2,m+2)_1$, the remaining 1-cycles of η in $\mathcal{C}$ are determined by the 2-cycles of η. To count T, we only need to count the number of ways we can choose $\ell - 2$ 2-cycles from the disjoint subdigraphs G_1' and G_2'. As we must choose $\ell - 2$ 2-cycles, if we choose i 2-cycles from G_1', then we must choose $\ell - 2 - i$ 2-cycles from G_2'. It follows that

$$|\mathcal{C}| = \sum_{i=1}^{\ell-2} S(m-2,i) S(m-2,\ell-2-i). \tag{10}$$

Combining Eqs. (8)–(10), we conclude that

$$[v_1 v_2 v_3 v_4] F_{2m-k} = (-1)^{\ell} \sum_{i=1}^{\ell-2} S(m-2,i) S(m-2,m-2-i).$$

The other cases follow through similar arguments. See Fig. 3 for an illustration of why the coefficients of $v_1 v_2$ and $v_3 v_4$ agree. □.

Remark 5. The reason why Remark 2 holds is because if t is a monomial of F_{2m-k}, then $\deg(t) + 2m - k$ is the number of 1-cycles in an $\eta \in \mathcal{U}_{G'}$ such that $[t\lambda^{2m-k}] w'(\eta) \neq 0$. Thus, the remaining cycles must be 2-cycles, and so $k - \deg(t)$ must be even; that is, the parities of k and $\deg(t)$ must agree.

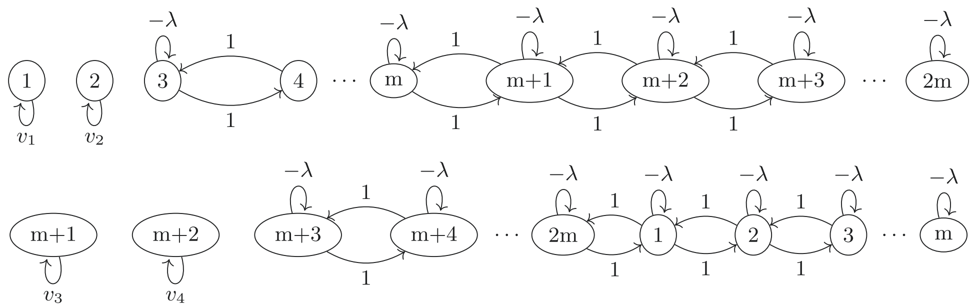

FIG. 3. (Above) G' after fixing the cycles v_1 and v_2 and removing cycles v_3 and v_4. (Below) G' after fixing the cycles v_3 and v_4 and removing cycles v_1 and v_2. Notice how these are the same digraphs, just relabeled; thus, the coefficients of v_1v_2 and v_3v_4 are indeed the same.

Now we can prove Theorem III.2.

A. Proof of Theorem III.2

Lemma IV.2. Let

$$v = (v_1, v_2, 0, \ldots, 0, v_3, v_4, 0, \ldots, 0)$$

with $v_1 = 1 + i$, $v_2 = 1 - i$, $v_3 = -1 + i$, *and* $v_4 = -1 - i$. *Then* $F_{2m-k}(v) = 0$ *for all* $k \in [2m]$.

We split the proof into two cases.

(i) *k is odd.*

Notice that $v_1 = -v_4$ and $v_2 = -v_3$, thus we have that

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 0. \\ v_1v_2v_3 + v_2v_3v_4 + v_1v_3v_4 + v_1v_2v_4 = 0. \end{cases}$$

Combined with Theorem IV.1, this completes the proof in the case that k is odd.

(ii) *k is even* ($k = 2\ell$).

With the given choice of v, we have

$$\begin{cases} v_1v_2v_3v_4 = 4 \\ v_1v_2 + v_3v_4 = 4 \\ v_1v_3 + v_2v_4 = -4 \\ v_1v_4 + v_2v_3 = 0. \end{cases}$$

Thus, according to Theorem IV.1, we are left to show that

$$S(2m-2, \ell-1) = \sum_{i=0}^{\ell-2} S(m-2, i)S(m-2, \ell-2-i) + \sum_{i=0}^{\ell-1} S(m-1, i)S(m-1, \ell-1-i). \tag{11}$$

To show that (11) holds, we will prove that both sides count the same collection of objects. Indeed, according to the definition of $S(2m-2, \ell-1)$, the left-hand side counts the number of disjoint cycle covers of the digraph $J_{(2m-2)}$ that have $\ell - 1$ 2-cycles. We now claim that the right-hand side enumerates the same collection. Indeed, suppose we have a $J_{(2m-2)}$ digraph and we wish to count the number of disjoint cycle covers with $\ell - 1$ 2-cycles. We may partition our collection of disjoint cycle covers into two sets: those that contain the cycle $((m-1, m), (m, m-1))$, denoted by $\mathcal{C}_1$, and those that do not, which we call $\mathcal{C}_2$.

To count $\mathcal{C}_1$, we can start by removing the vertices $m-1$ and m from $J_{(2m-2)}$. We are left with the disjoint union of two $J_{(m-2)}$ digraphs, which we denote G_1' and G_2', respectively. In particular, the number of elements in $\mathcal{C}_1$ is the number of disjoint cycle covers of the disjoint union of G_1' and G_2' with $\ell - 2$ 2-cycles. It readily follows that

$$|\mathcal{C}_1| = \sum_{i=0}^{\ell-2} S(m-2, i)S(m-2, \ell-2-i).$$

We now proceed to count the elements of $\mathcal{C}_2$. Note that the edges $(m-1,m)$ and $(m,m-1)$ cannot appear in any element of $\mathcal{C}_2$. Removing these edges from $J_{(2m-2)}$, it follows that $\eta \in \mathcal{C}_2$ if and only if η is a disjoint cycle cover of $G_1' \dot\sqcup G_2'$ with $\ell-12$-cycles, where G_1' and G_2' are two $J_{(m-1)}$ digraphs and $\dot\sqcup$ denotes a disjoint union. It readily follows that

$$|\mathcal{C}_2| = \sum_{i=0}^{\ell-1} S(m-1,i)S(m-1,\ell-1-i).$$

As $S(2m-2,\ell-1) = |\mathcal{C}_1| + |\mathcal{C}_2|$, we have established (11), finishing the proof.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Matthew Faust: Data curation (equal); Investigation (equal); Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). **Wencai Liu**: Conceptualization (equal); Investigation (equal); Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). **Rodrigo Matos**: Investigation (equal); Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). **Jenna Plute**: Data curation (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Jonah Robinson**: Data curation (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Yichen Tao**: Data curation (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Ethan Tran**: Data curation (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Cindy Zhuang**: Data curation (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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