SCHMIDT RANK OF QUARTICS OVER PERFECT FIELDS

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Abstract. Let $\mathbf{k}$ be a perfect field of characteristic $\neq 2$. We prove that the Schmidt rank (also known as strength) of a quartic polynomial f over $\mathbf{k}$ is bounded above in terms of only the Schmidt rank of f over $\overline{\mathbf{k}}$, an algebraic closure of $\mathbf{k}$.

1. Introduction

Recall that the *Schmidt rank* (also known as *strength*) of a homogeneous polynomial $f \in \mathbf{k}[x_1, \ldots, x_n]$ (see [1], [2] and references therein) is defined as the minimal number r such that f admits a decomposition $f = g_1 h_1 + \ldots + g_r h_r$, with $\deg(g_i)$ and $\deg(h_i)$ smaller than $\deg(f)$. We denote the Schmidt rank of f as $\mathrm{rk}^S_{\mathbf{k}}(f)$.

It is conjectured in [1] that for a homogeneous polynomial f of degree d over a non-closed field $\mathbf{k}$ one has

$$\mathrm{rk}^S_{\mathbf{k}}(f) \le \kappa_d \cdot \mathrm{rk}^S_{\overline{\mathbf{k}}}(f),$$

where $\overline{\mathbf{k}}$ is an algebraic closure of $\mathbf{k}$. This is known to be true for $d \le 3$ with $\kappa_d = d$ since in this case the Schmidt rank is equal to the *slice rank* defined as the minimal r such that $f \in (l_1, \ldots, l_r)$, where $\deg(l_i) = 1$ (see [5, Thm. A] for the case of cubics).

One can also ask a weaker question whether there exists a function $c(r, d)$ such that

$$\mathrm{rk}^S_{\mathbf{k}}(f) \le c(\mathrm{rk}^S_{\overline{\mathbf{k}}}(f), d).$$

Our main result is that this weaker question has a positive answer in the case of quartic polynomials.

Theorem A. *Assume that the ground field $\mathbf{k}$ is perfect of characteristic $\neq 2$. Then there exists a function $r \mapsto c(r)$ such that for any homogeneous quartic polynomial $f(x_1, \ldots, x_n)$, such that $\mathrm{rk}^S_{\overline{\mathbf{k}}}(f) = r$, one has $\mathrm{rk}^S_{\mathbf{k}}(f) \le c(r)$.*

We find it convenient to use the following refined version of Schmidt rank.

Definition 1.1. For a collection of nonnegative integers $(r_k, \ldots, r_1)$ and a homogeneous polynomial f of degree d, we say that the *refined Schmidt rank of f is at most* $(r_k, \ldots, r_1)$, and write $\mathrm{rk}^S_{\mathbf{k}}(f) \le (r_k, \ldots, r_1)$, if there exists a decomposition over $\mathbf{k}$,

$$f = \sum_{j=1}^{k} \sum_{i=1}^{r_j} f_{ij} g_{ij},$$

where $\deg(f_{ij}) = d - \deg(g_{ij}) = j$ for $j = 1, \ldots, k$.

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Thus, for a quartic polynomial f, we have $\mathrm{rk}^S_{\mathbf{k}}(f) \le (r_2, r_1)$ if there exist quadrics $q_1, \dots, q_{r_2}$ and linear forms $l_1, \dots, l_{r_1}$, such that $f \in (q_1, \dots, q_{r_2}, l_1, \dots, l_{r_1})$. In our proof of Theorem A we show the existence of functions $(c_2(r_2, r_1), c_1(r_2, r_1))$, such that if $\mathrm{rk}^S_{\overline{\mathbf{k}}}(f) \le (r_2, r_1)$ then $\mathrm{rk}^S_{\mathbf{k}}(f) \le (c_2(r_2, r_1), c_1(r_2, r_1))$. Then one can set

$$c(r) = \max_{r_1+r_2=r}(c_2(r_2, r_1) + c_1(r_2, r_1)).$$

One can work through our proof of Theorem A and get explicit formulas for $(c_2(r_2, r_1), c_1(r_2, r_1))$. As an illustration of this, we give formulas for $(c_2(1, r_1), c_1(1, r_1))$.

Theorem B. *Let f be a homogeneous quartic polynomial defined over a perfect field $\mathbf{k}$ with* $\mathrm{char}(\mathbf{k}) \ne 2$. *Assume that $\mathrm{rk}^S_{\overline{\mathbf{k}}}(f) \le (1, r)$. Then $\mathrm{rk}^S_{\mathbf{k}}(f) \le (2, C(r))$, where*

$$C(r) = 8r(41 + 20 \cdot (10r + 1)^{10r+1}).$$

The main idea of the proof of Theorem A is to study decompositions of a quartic polynomial f of the form

$$f = \sum_{i=1}^{r} q_i q_i' \mod (P),$$

where q_i, q_i' are of degree 2 and P is a subspace of linear forms. The main result about such decompositions is that if the rank of any linear combination of $(q_\bullet, q_\bullet')$ is sufficiently large then the above decomposition is essentially unique (possibly after enlarging P): the only way to get a new decomposition is by making an orthogonal change of basis in the linear space with the basis $(q_\bullet, q_\bullet')$. We then apply this result to the decompositions obtained from a given one over $\overline{\mathbf{k}}$ by applying the Galois group action. If the rank of $(q_\bullet, q_\bullet')$ is sufficiently large, then we obtain a 1-cocycle of the Galois group with values in the orthogonal group measuring how the decomposition transforms under the Galois action. We can assume that this 1-cocycle is trivial (after passing to an extension of $\mathbf{k}$ of small degree). We then use a certain linear algebra result from [4] to prove the existence of a decomposition with $(q_\bullet, q_\bullet')$ defined over $\mathbf{k}$. Furthermore, we have a bound on the slice rank of $f - \sum_{i=1}^r q_i q_i'$ over $\overline{\mathbf{k}}$, and hence over $\mathbf{k}$ by [5, Thm. A]. This gives the required bound on the Schmidt rank of f over $\mathbf{k}$.

2. Preliminaries

2.1. **Criterion for an ideal generated by quadrics and linear forms to be prime.** In this subsection we fix a ground field $\mathbf{k}$ (and omit it from the notation). By a *quadric* we mean an element of $\mathbf{k}[V]_2$, i.e., a quadratic form.

Definition 2.1. For a subspace of quadrics Q, and a collection of quadrics $q_1, \dots, q_r$, we define $\mathrm{srk}(q_1, \dots, q_r, Q)$ as the minimum of $\mathrm{srk}(\sum_i c_i q_i + q)$, where $q \in Q$ and c_i are constants, such that either $q \ne 0$ or $(c_\bullet) \ne 0$. In particular, $\mathrm{srk}(Q)$ is the minimal slice rank of a nonzero element of Q.

We denote by $\mathrm{rk}(q)$ the usual rank of a quadric q. It is easy to see that the rank and the slice rank of a quadric q are related by

$$2\mathrm{srk}(q) - 1 \le \mathrm{rk}(q) \le 2\mathrm{srk}(q).$$

In other words, we have

$$\mathrm{srk}(q) = \lceil \frac{\mathrm{rk}(q)}{2} \rceil.$$

Lemma 2.2. *Let $q_1, \dots, q_r$ be quadratic forms such that*

$$R = \min_{(c_1,\dots,c_r)\neq 0} \mathrm{rk}(c_1 q_1 + \dots c_r q_r) \ge 2r + 1.$$

Then the subscheme $q_1 = \dots = q_r = 0$ is normal connected of codimension r.

Proof. Consider the Jacobian matrix $J(q_1, \dots, q_r)$. The locus $S(q_1, \dots, q_r)$ where $J(q_1, \dots, q_r)$ has rank $< r$ coincides with the union of kernels of $\sum c_j q_j$ over $(c_1, \dots, c_r) \neq 0$. Hence,

$$\dim S(q_1, \dots, q_r) \le (n - R) + (r - 1),$$

where $n - R$ is the maximal dimension of the kernels and $r - 1$ is the dimension of the base $\mathbb{P}^{r-1}$ of the family of quadrics. Thus,

$$\mathrm{codim}_V S(q_1, \dots, q_r) \ge R - (r - 1) \ge r + 2.$$

Since the codimension of $X := (q_1 = \dots = q_r = 0) \subset \mathbb{A}^n$ is $\le r$, on a nonempty Zariski open subset of X, the rank of the Jacobian equals r. This implies that X has codimension r, and so $S(q_1, \dots, q_r) \cap X$ has codimension ≥ 2 in X. Thus, X is Cohen-Macaulay (as a complete intersection), nonsingular in codimension 1. Therefore, by Serre's R_1+S_2 criterion (see [3, Thm. 8.22A], X is normal. Finally, X is connected as a complete intersection. □

Proposition 2.3. *(i) Let Q be a subspace of quadratic forms such that* $\mathrm{srk}(Q) \ge \dim Q + 1$. *Then the ideal (Q) is prime.*
(ii) Let L be a subspace of linear forms, Q a subspace of quadratic forms. Assume that $\mathrm{srk}(Q) \ge \dim Q + \dim L + 1$. *Then the ideal (Q, L) is prime.*

Proof. (i) For any $q \in Q$ we have $\mathrm{rk}(q) \ge 2\mathrm{srk}(q) - 1 \ge 2\dim Q + 1$. Hence, by Lemma 2.2, the subscheme defined by (Q) is normal connected, so integral. Therefore, the ideal (Q) is prime.
(ii) Consider the quotient $\overline{S} = S/(L)$ of the algebra of polynomials S by the ideal (L). For any $q \in Q$ we have $\mathrm{srk}(\overline{q}) \ge \mathrm{srk}(q) - \dim L$, where $\overline{q}$ is the image of q in $\overline{S}$. Hence, the image $\overline{Q}$ of Q in $\overline{S}$ satisfies the assumptions of (i), so the ideal $(\overline{Q})$ in $\overline{S}$ is prime. Therefore, its preimage in S, namely (Q, L), is also prime. □

2.2. **Almost invariant quadratic forms.** We will use the following result from [4].

Theorem 2.4. *Let $E/\mathbf{k}$ be a finite Galois extension with the Galois group G, and let V_0, V_0' be finite dimensional $\mathbf{k}$-vector spaces. Let us set $V = V_0 \otimes_{\mathbf{k}} E$, $V' = V_0' \otimes_{\mathbf{k}} E$. Suppose $T : V \to V'$ is an E-linear operator such that for any $\sigma \in G$, one has*

$$\mathrm{rk}_E(\sigma(T) - T) \le r,$$

for some $r \geq 0$. *Then there exists a* $\mathbf{k}$*-linear operator* $T_0 : V_0 \to V_0'$*, such that*

$$\mathrm{rk}_E(T - T_0) \leq r(2 + (r+1)^{r+1}),$$

where we view T_0 *as an operator* $V \to V'$ *by extension of scalars.*

We need the following consequence of this theorem for quadratic forms.

Corollary 2.5. *Let* $E/\mathbf{k}$ *be a finite Galois extension with the Galois group* G*, where* $\mathrm{char}(\mathbf{k}) \neq 2$*, and let* V_0 *be a finite dimensional* $\mathbf{k}$*-vector space,* $V = V_0 \otimes_{\mathbf{k}} E$*. Assume that* q *is a quadratic form on* V *such that for any* $\sigma \in G$*, one has* $\mathrm{rk}_E(\sigma(q) - q) \leq r$ *for some* $r \geq 0$*, where* rk_E *is the usual rank of the quadratic form. Then there exists a quadratic form* q_0 *on* V_0 *such that* $\mathrm{rk}_E(q - q_0) \leq 2r(2 + (r+1)^{r+1})$*.*

Proof. Let $T : V \to V^*$ be the symmetric linear map associated with q. Our assumption implies that $\mathrm{rk}_E(\sigma(T) - T) \leq r$ for any $\sigma \in G$. By Theorem 2.4, there exists an operator $T_0 : V_0 \to V_0^*$ such that $\mathrm{rk}_E(T - T_0) \leq r(2 + (r+1)^{r+1})$. Let $T_0^* : V_0 \to V_0^*$ be the dual operator. Then

$$\mathrm{rk}_E(T - \frac{1}{2}(T_0 + T_0^*)) \leq 2r(2 + (r+1)^{r+1}),$$

so we can let q_0 be the quadratic form corresponding to $\frac{1}{2}(T_0 + T_0^*)$. ☐

3. Schmidt rank for quartics

From now on we assume that the ground field $\mathbf{k}$ is perfect and has characteristic $\neq 2$.

3.1. **Case** $r_2 = 1$**.** We start with a proof of Theorem B dealing with the case $r_2 = 1$, since it is simpler but still shows the main idea.

Lemma 3.1. *Let* $\mathbf{k}'/\mathbf{k}$ *be a quadratic extension, and let* f *be a homogeneous polynomial over* $\mathbf{k}$ *such that* $\mathrm{rk}^S_{\mathbf{k}'}(f) \leq (r_2, r_1)$*. Then* $\mathrm{rk}^S_{\mathbf{k}}(f) \leq (2r_2, 2r_1)$*.*

Proof. By assumption $f \in (Q, P)$, where Q is a subspace of quadrics and P is a subspace of linear forms, both defined over $\mathbf{k}'$. Hence, $f \in (Q + \sigma(Q), P + \sigma(P))$, where σ is the generator of the Galois group of $\mathbf{k}'/\mathbf{k}$. Since the subspaces $Q + \sigma(Q)$ and $P + \sigma(P)$ are defined over $\mathbf{k}$, this implies the assertion. ☐

Proof of Theorem B. We have to check that if $E/\mathbf{k}$ is a finite Galois extension, and

$$f \equiv qq' \mod (P),$$

where q, q' are quadratic forms over E and P is an r-dimensional subspace of linear forms defined over E, then $\mathrm{rk}^S_{\mathbf{k}}(f) \leq (2, C(r))$.

If $\mathrm{srk}_E(q) \leq 9r$ or $\mathrm{srk}_E(q') \leq 9r$ then $\mathrm{srk}_E(f) \leq 10r$, and so $\mathrm{srk}_{\mathbf{k}}(f) \leq 40r \leq C(r)$. Thus, we can assume that $\mathrm{srk}_E(q) > 9r$ and $\mathrm{srk}_E(q') > 9r$. Let G be the Galois group of $E/\mathbf{k}$. For any $\sigma \in G$ we have

$$f \equiv \sigma(q)\sigma(q') \mod (\sigma(P)).$$

By assumption, the slice rank of $\overline{q} = q \mod (P + \sigma(P))$ is ≥ 2, hence the quadric $\overline{q}$ is irreducible. In other words, the ideal $(q, P + \sigma(P))$ is prime. Since $qq' \in (\sigma(q), P + \sigma(P))$, we have either $q \in (\sigma(q), P + \sigma(P))$ or $q' \in (\sigma(q), P + \sigma(P))$. Since the slice ranks of q and q' are $> 2r$, this means that either

$$\sigma(q) \equiv c(\sigma) \cdot q \mod (P + \sigma(P)),\ \sigma(q') \equiv c(\sigma)^{-1} \cdot q' \mod (P + \sigma(P)),\ \text{or}$$
$$\sigma(q) \equiv c(\sigma) \cdot q' \mod (P + \sigma(P)),\ \sigma(q') \equiv c(\sigma)^{-1} \cdot q \mod (P + \sigma(P)),$$

for some $c(\sigma) \in E^*$.

Let $H \subset G$ be the set of $\sigma \in G$ for which the first possibility holds. Let us consider separately two cases.

Case $\mathrm{srk}_E(q, q') > 3r$. Assume first that $\sigma_1, \sigma_2 \in H$. Then we have

$$\sigma_1\sigma_2(q) \equiv \sigma_1(c(\sigma_2)) \cdot \sigma_1(q) \equiv c(\sigma_1)\sigma_1(c(\sigma_2)) \cdot q \mod (P + \sigma_1(P) + \sigma_1\sigma_2(P)).$$

If $\sigma_1\sigma_2 \notin H$, we would get that a nontrivial linear combination of q and q' is in $(P + \sigma_1(P) + \sigma_1\sigma_2(P))$, contradicting the assumption $\mathrm{srk}_E(q, q') > 3r$. Hence, $\sigma_1\sigma_2 \in H$, so we have

$$\sigma_1\sigma_2(q) \equiv c(\sigma_1\sigma_2) \cdot q \mod (P + \sigma_1\sigma_2(P)).$$

Comparing this with the previous congruence, we get

$$[c(\sigma_1)\sigma_1(c(\sigma_2)) - c(\sigma_1\sigma_2)] \cdot q \equiv 0 \mod (P + \sigma_1(P) + \sigma_1\sigma_2(P)).$$

Since $\mathrm{srk}_E(q) > 9r \geq 3r$, a nonzero multiple of q cannot be contained in $(P + \sigma_1(P) + \sigma_1\sigma_2(P))$, we obtain

$$c(\sigma_1)\sigma_1(c(\sigma_2)) - c(\sigma_1\sigma_2) = 0,$$

i.e., $c(\sigma)$ is a 1-cocycle of H. A similar argument shows that if exactly one of σ_1, σ_2 belongs to H then $\sigma_1\sigma_2 \notin H$, and if $\sigma_1, \sigma_2 \notin H$ then $\sigma_1\sigma_2 \in H$, proving that H is a subgroup of index ≤ 2 in G.

Case $\mathrm{srk}_E(q, q') \leq 3r$. In this case we have $q' \equiv cq \mod (P_0)$ for some subspace of linear forms P_0 of dimension $\leq 3r$ (defined over E) and some $c \in E^*$. This implies that for $\sigma \notin H$ we have

$$\sigma(q) \equiv c(\sigma) \cdot q' \equiv c(\sigma)c \cdot q \mod (P + \sigma(P) + P_0).$$

Redefining $c(\sigma)$ for $\sigma \notin H$ we obtain that

$$\sigma(q) \equiv c(\sigma) \cdot q \mod (P + \sigma(P) + P_0), \quad \sigma(q') \equiv c(\sigma)^{-1} \cdot q' \mod (P + \sigma(P) + P_0) \tag{3.1}$$

for all $\sigma \in G$. Now for $\sigma_1, \sigma_2 \in G$ we have

$$\sigma_1\sigma_2(q) \equiv \sigma_1(c(\sigma_2)) \cdot \sigma_1(q) \equiv c(\sigma_1)\sigma_1(c(\sigma_2)) \cdot q \mod (P + \sigma_1(P) + \sigma_1\sigma_2(P) + P_0 + \sigma_1(P_0)).$$

On the other hand,

$$\sigma_1\sigma_2(q) \equiv c(\sigma_1\sigma_2) \cdot q \mod (P + \sigma_1\sigma_2(P) + P_0).$$

Thus, we get

$$[c(\sigma_1)\sigma_1(c(\sigma_2)) - c(\sigma_1\sigma_2)] \cdot q \equiv 0 \mod (P + \sigma_1(P) + \sigma_1\sigma_2(P) + P_0 + \sigma_1(P_0)).$$

Note that $\dim(P + \sigma_1(P) + \sigma_1\sigma_2(P) + P_0 + \sigma_1(P_0)) \leq 9r$. Since $\mathrm{srk}_E(q) > 9r$, this is possible only if $c(\sigma_1)\sigma_1(c(\sigma_2)) - c(\sigma_1\sigma_2) = 0$, i.e., $c(\sigma)$ is a 1-cocycle of H.

In either case we obtain that for a subgroup $H \subset G$ of index ≤ 2 and a subspace of linear forms P_0 of dimension $\leq 3r$, the congruences (3.1) hold for some 1-cocycle $c : H \to E^*$.

Let $\mathbf{k}'/\mathbf{k}$ be the subextension of E corresponding to the subgroup $H \subset G$, so that the extension $E/\mathbf{k}'$ is Galois with the Galois group H. By Hilbert's Theorem 90, the cocycle $c(\sigma)$ of H is trivial, so rescaling q and q', we can assume that

$$\sigma(q) \equiv q \mod (P + \sigma(P) + P_0), \quad \sigma(q') \equiv q' \mod (P + \sigma(P) + P_0)$$

for all $\sigma \in H$. But this implies that $\mathrm{srk}_E(\sigma(q) - q) \le 5r$ and $\mathrm{srk}_E(\sigma(q') - q') \le 5r$. Hence, we obtain

$$\mathrm{rk}_E(\sigma(q) - q) \le 10r, \quad \mathrm{rk}_E(\sigma(q') - q') \le 10r$$

for all $\sigma \in H$. By Corollary 2.5, there exist quadrics q_0 and q_0' defined over $\mathbf{k}'$ such that

$$\max(\mathrm{rk}_E(q - q_0), \mathrm{rk}_E(q' - q_0')) \le 20r(2 + (10r + 1)^{10r+1}).$$

Hence,

$$\max(\mathrm{srk}_E(q - q_0), \mathrm{srk}_E(q' - q_0')) \le 10r(2 + (10r + 1)^{10r+1}).$$

It follows that

$$\mathrm{srk}_E(f - q_0q_0') \le \mathrm{srk}_E(f - qq') + \mathrm{srk}_E(qq' - q_0q_0') \le r + 20r(2 + (10r+1)^{10r+1}) = r(41 + 20\cdot(10r+1)^{10r+1}).$$

By [5, Thm. A], this implies that

$$\mathrm{srk}_{\mathbf{k}'}(f - q_0q_0') \le 4r(41 + 20 \cdot (10r + 1)^{10r+1}),$$

hence, $\mathrm{rk}^S_{\mathbf{k}'}(f) \le (1, 4r(41 + 20 \cdot (10r + 1)^{10r+1}))$. Since the extension $\mathbf{k}'/\mathbf{k}$ is either trivial or quadratic, applying Lemma 3.1 we get the result. □

3.2. **Quadratic decompositions of quartics.**

Lemma 3.2. *Assume we have a collection of quadrics*

$$q_1, \dots, q_r, q_1', \dots, q_r', p_1, \dots, p_s, p_1', \dots, p_s',$$

where $r > s$, and a subspace of quadrics Q, such that

$$\sum_{i=1}^{r} q_iq_i' \equiv \sum_{i=1}^{s} p_ip_i' \mod (Q).$$

Then for some constants $a_1, \dots, a_r, a_1', \dots, a_r'$ such that $\sum_i a_ia_i' = 0$, we have

$$\mathrm{srk}(\sum_i (a_iq_i + a_i'q_i'), Q) \le c(r, s, \dim Q) := 2^s(r + \dim Q) + 2^{s-1}(s - 2).$$

Proof. We use the induction on s. In the case $s = 0$ we have to prove that

$$\mathrm{srk}(\sum_i (a_iq_i + a_i'q_i'), Q) \le c(r, 0, \dim Q) = r + \dim Q - 1$$

for some isotropic $(a_\bullet, a_\bullet')$. Indeed, assume this is not true. Then $\mathrm{srk}(q_1, \dots, q_{r-1}, Q) \ge r + \dim Q$, so by Proposition 2.3, the ideal $(q_1, \dots, q_{r-1}, Q)$ is prime. But we have

$$q_rq_r' \in (q_1, \dots, q_{r-1}, Q).$$

Hence, swapping q_r with q_r' if necessary, we deduce that $q_r \in (q_1, \dots, q_{r-1}, Q)$. But this implies that $\mathrm{srk}(q_r + c_1q_1 + \dots + c_{r-1}q_{r-1} + q) = 0$ for some constants c_i and $q \in Q$, which contradicts the assumption $\mathrm{srk}(q_\bullet, Q) \ge r + \dim Q \ge r > 0$.

Assume the assertion holds for $s-1$. We have

$$\sum_{i=1}^{r} q_i q_i' \equiv \sum_{i=2}^{s} p_i p_i' \mod (p_1, Q).$$

Hence, by the induction assumption,

$$\operatorname{srk}(\sum_i (a_i q_i + a_i' q_i'), p_1, Q) \le c(r, s-1, \dim Q + 1)$$

for some isotropic $(a_\bullet, a_\bullet')$. Changing the basis in $(q_\bullet)$, we can assume that either $\operatorname{srk}(q_1, Q) \le c(r, s-1, \dim Q + 1)$, or there exists a subspace L of linear forms of dimension $\le c(r, s-1, \dim Q + 1)$ such that $p_1 \in (q_1, Q, L)$. In the former case we are done since $c(r, s, \dim Q) \ge c(r, s-1, \dim Q + 1)$. In the latter case, we have

$$\sum_{i=2}^{r} q_i q_i' \equiv \sum_{i=2}^{s} p_i p_i' \mod (q_1, Q, L).$$

Applying the induction assumption we obtain that there exists an isotropic vector $(a_{>1}, a_{>1}')$ such that

$$\operatorname{srk}_L(\sum_{i=2}^{r} (a_i q_i + a_i' q_i'), q_1, Q) \le c(r-1, s-1, \dim Q + 1),$$

hence

$$\operatorname{srk}(\sum_{i=2}^{r} (a_i q_i + a_i' q_i'), q_1, Q) \le c(r-1, s-1, \dim Q + 1) + \dim L \le$$
$$c(r-1, s-1, \dim Q + 1) + c(r, s-1, \dim Q + 1) = c(r, s, \dim Q).$$

Since any linear combination of $\sum_{i=2}^{r} (a_i q_i + a_i' q_i')$ with q_1 will correspond to an isotropic vector, the assertion follows. □

Proposition 3.3. *Let Q be a subspace of quadrics,*

$$q_1, \dots, q_r, q_1', \dots, q_r', p_1, \dots, p_r, p_1', \dots, p_r'$$

quadratic forms, such that

$$\sum_{i=1}^{r} q_i q_i' \equiv \sum_{i=1}^{r} p_i p_i' \mod (Q). \tag{3.2}$$

Assume that for any constants $a_1, \dots, a_r, a_1', \dots, a_r'$ such that $\sum_i a_i a_i' = 0$, we have

$$\operatorname{srk}(\sum_i (a_i q_i + a_i' q_i'), Q) \ge C(r, \dim Q) := 2^r (r + \dim Q) + 2^{r-1}(r-2) + 1.$$

Then there exists a subspace of linear forms L of dimension at most

$$D(r, \dim Q) := (2^r - 1)(r + \dim Q - 1) + r \cdot 2^{r-1}$$

and a linear transformation $A : \mathbf{k}^{2r} \to \mathbf{k}^{2r}$ preserving the quadratic form $\sum_{i=1}^{r} x_i y_i$, such that for the linear operator ϕ from $\mathbf{k}^{2r}$ to the space of quadrics sending the standard basis $(e_\bullet, f_\bullet)$ to $(q_\bullet, q_\bullet')$, we have

$$p_i \equiv \phi(A e_i), \quad p_i' \equiv \phi(A f_i) \mod (Q, L).$$

Proof. We use induction on r. In the case $r = 0$ we can take $L = 0$, $D(0, \dim Q) = 0$.

Assume the assertion holds for $r - 1$. We have

$$\sum_{i=1}^{r} q_i q_i' \equiv \sum_{i=2}^{r} p_i p_i' \mod (p_1, Q).$$

Hence, by Lemma 3.2, changing $(q_\bullet, q_\bullet')$ by an orthogonal transformation, we can achieve that

$$\mathrm{srk}(q_1, p_1, Q) \le c(r, r-1, \dim Q + 1).$$

Since $\mathrm{srk}(q_1, Q) \ge C(r, \dim Q) \ge c(r, r-1, \dim Q + 1) + 1$, this implies that there exists a subspace of linear forms L of dimension $\le c(r, r-1, \dim Q + 1)$, such that

$$p_1 \in (q_1, Q, L).$$

Note that if $p_1 \in (Q, L)$ then we get

$$\sum_{i=1}^{r} q_i q_i' \equiv \sum_{i=2}^{r} p_i p_i' \mod (Q, L).$$

Hence, by Lemma 3.2, we would get

$$\mathrm{srk}(\sum_i (a_i q_i + a_i' q_i'), Q) \le \mathrm{srk}_L(\sum_i (a_i q_i + a_i' q_i'), Q) + \dim L \le$$
$$c(r, r-1, \dim Q) + c(r, r-1, \dim Q + 1) \le C(r, \dim Q) - 1,$$

which is a contradiction. Hence, rescaling q_1 and q_1', we can assume that

$$p_1 \equiv q_1 \mod (Q, L).$$

Also, from $p_1 \in (q_1, Q, L)$ we deduce that

$$\sum_{i=2}^{r} q_i q_i' \equiv \sum_{i=2}^{r} p_i p_i' \mod (q_1, Q, L).$$

Since for any isotropic vector $(a_{>1}, a_{>1}')$ one has

$$\mathrm{srk}_L(\sum_{i=2}^{r} (a_i q_i + a_i' q_i'), q_1, Q) \ge \mathrm{srk}(\sum_{i=2}^{r} (a_i q_i + a_i' q_i'), q_1, Q) - \dim L \ge C(r, \dim Q) - \dim L \ge$$
$$C(r, \dim Q) - c(r, r-1, \dim Q + 1) \ge C(r-1, \dim Q + 1),$$

we can apply the induction hypothesis and deduce that for some subspace of linear forms $L' \supset L$ of dimension

$$D(r-1, \dim Q + 1) + \dim L \le D(r-1, \dim Q + 1) + c(r, r-1, \dim Q + 1) = D(r, \dim Q),$$

after changing the basis $(q_2, \dots, q_r, q_2', \dots, q_r')$ by an orthogonal transformation, we have

$$p_i \equiv q_i, \quad p_i' \equiv q_i' \mod (q_1, Q, L'),$$

for $i \ge 2$. This means that we have

$$p_i \equiv q_i + c_i q_1, \quad p_i' \equiv q_i' + c_i' q_1 \mod (Q, L'),$$

for $i \ge 2$. Substituting this into (3.2) and recalling that $p_1 \equiv q_1 \mod (Q, L)$, we get

$$q_1 q_1' \equiv q_1 \cdot [p_1' + \sum_{i=2}^{r} (c_i q_i' + c_i' q_i) + (\sum_{i=2}^{r} c_i c_i') q_1] \mod (Q, L').$$

Since

$$\operatorname{srk}(Q) \ge C(r, \dim Q) \ge \dim Q + D(r, \dim Q) + 1 \ge \dim Q + \dim L' + 1,$$

by Proposition 2.3, the ideal (Q, L') is prime, so we get

$$p_1' \equiv q_1' - \sum_{i=2}^{r} (c_i q_i' + c_i' q_i) - (\sum_{i=2}^{r} c_i c_i') q_1 \mod (Q, L').$$

It remains to observe that the linear transformation

$$Ae_1 = e_1, \quad Af_1 = f_1 - \sum_{i=2}^{r} (c_i f_i + c_i' e_i) - (\sum_{i=2}^{r} c_i c_i') e_1,$$
$$Ae_i = e_i + c_i e_1, \quad Af_i = f_i + c_i' e_1, \text{ for } i \ge 2,$$

preserves the quadratic form $\sum x_i y_i$. □

We are mainly interested in the case $Q = 0$ in the above proposition (the case of general Q was introduced in order for the inductive argument to work).

Corollary 3.4. *(i) Assume that*

$$\sum_{i=1}^{r} q_i q_i' = \sum_{i=1}^{r} p_i p_i',$$

where q_i, q_i', p_i, p_i' are quadrics and

$$\operatorname{srk}(\sum_i (a_i q_i + a_i' q_i')) \ge C(r, 0) = (r-1) \cdot 2^r + r \cdot 2^{r-1} + 1$$

for any isotropic $(a_\bullet, a_\bullet')$. Then there exists a subspace of linear forms L of dimension at most

$$D(r, 0) = (r-1) \cdot (2^r - 1) + r \cdot 2^{r-1} \le C(r, 0)$$

and a linear transformation $A : \mathbf{k}^{2r} \to \mathbf{k}^{2r}$ preserving the quadratic form $\sum_{i=1}^{r} x_i y_i$, such that for the linear operator ϕ from $\mathbf{k}^{2r}$ to the space of quadrics sending the standard basis $(e_\bullet, f_\bullet)$ to $(q_\bullet, q_\bullet')$, we have

$$p_i \equiv \phi(Ae_i), \quad p_i' \equiv \phi(Af_i) \mod (L).$$

(ii) Assume that

$$q_0^2 + \sum_{i=1}^{r} q_i q_i' = p_0^2 + \sum_{i=1}^{r} p_i p_i',$$

where q_i, q_i', p_i, p_i' are quadrics and for any constants $a_0, \dots, a_r, a_1', \dots, a_r'$ such that $a_0^2 + 4 \sum_i a_i a_i' = 0$, one has $\operatorname{srk}(a_0 q_0 + \sum_{i=1}^{r} (a_i q_i + a_i' q_i')) \ge c(r+1, r, 0) + C(r, 0) + 1$. Then there exists a subspace of linear forms L of dimension at most $D(r, 0) + c(r+1, r, 0)$, such that after making a linear change in $(q_0, \dots, q_r, q_1', \dots, q_r')$ preserving the quadratic form $a_0^2 + 4 \sum_{i=1}^{r} a_i a_i'$, one has

$$q_0 \equiv p_0, \quad q_i \equiv p_i, \quad q_i' \equiv p_i' \mod (L).$$

Proof. (i) This is the case $Q=0$ of Proposition 3.3.
(ii) We have

$$(q_0-p_0)(q_0+p_0)+\sum_{i=1}^r q_iq_i'=\sum_{i=1}^r p_ip_i'.$$

Hence, by Lemma 3.2, there exists a nonzero vector $(a_0,\ldots,a_r,a_1',\ldots,a_r',b)$ such that

$$(a_0-b)(a_0+b)+\sum_{i=1}^r a_ia_i'=0$$

and

$$\mathrm{srk}((a_0-b)(q_0+p_0)+(a_0+b)(q_0-p_0)+\sum_{i=1}^r(a_iq_i+a_i'q_i'))\le c(r+1,r,0).$$

We can rewrite these conditions as

$$a_0^2+\sum_{i=1}^r a_ia_i'=b^2$$

and $\mathrm{srk}(a_0q_0+\frac{1}{2}\sum_{i=1}^r(a_iq_i+a_i'q_i')-bp_0,Q)\le c(r+1,r,0)$. Note that if $b=0$ we would get a contradiction with the assumption that $\mathrm{srk}(a_0q_0+\frac{1}{2}\sum_{i=1}^r(a_iq_i+a_i'q_i')-bp_0,Q)>c(r+1,r,0)+C(r,0)\ge c(r+1,r,0)$. Hence, we necessarily have $b\ne 0$. Thus, after making an orthogonal transformation of $(q_0,\ldots,q_r,q_1',\ldots,q_r')$, we can assume that $\mathrm{srk}(q_0-p_0)\le c(r+1,r,0)$. Thus, we have $q_0\equiv p_0 \mod (L_0)$,

$$\sum_{i=1}^r q_iq_i'\equiv\sum_{i=1}^r p_ip_i' \mod (L_0),$$

for some subspace of linear forms L_0 of dimension $\le c(r+1,r,0)$.

Now applying part (i), we find a subspace of linear forms $L\supset L_0$ of dimension $\le D(r,0)+c(r+1,r,0)$, such that after an orthogonal change of $(q_1,\ldots,q_r,q_1',\ldots,q_r')$, we have

$$p_i\equiv q_i \mod (L),\quad p_i'\equiv q_i' \mod (L).$$

□

3.3. **Proof of Theorem A.** We will prove the existence of functions $c_1(r_2,r_1)$ and $c_2(r_2,r_1)$ such that if a quartic f satisfies $\mathrm{rk}^S_{\mathbf{k}}(f)\le(r_2,r_1)$ then $\mathrm{rk}^S_{\mathbf{k}}(f)\le(c_2(r_2,r_1),c_1(r_2,r_1))$.

We use the induction on r_2. In the case $r_2=0$ we just have the slice rank, so by [5, Thm. A], we can set

$$c_1(0,r_1)=4r_1,\quad c_2(0,r_1)=0.$$

Now assume that the functions $c_1(r_1,r_2)$ and $c_2(r_1,r_2)$ are already constructed for $r_2<r$. Let f be a quartic over $\mathbf{k}$, and $E/\mathbf{k}$ is a finite Galois extension such that $\mathrm{rk}^S_E(f)\le(r,p)$, i.e., over E we have a decomposition

$$f\equiv\sum_{i=1}^r q_iq_i' \mod (P), \tag{3.3}$$

where (q_i,q_i') are quadrics, and P is a subspace of linear forms of dimension p.

Below we will use constants $C(r,0)$, $D(r,0)$ and $c(r,s,0)$ introduced in Sec. 3.2, and we set

$$N := 2p + C(r,0).$$

Case 1. Assume first that $\mathrm{srk}(q_\bullet, q'_\bullet) > 6p + 3C(r,0)$.

Note that for any element σ of the Galois group $\mathrm{Gal}(E/\mathbf{k})$ we have

$$\sum_{i=1}^{r} q_i q'_i \equiv \sum_{i=1}^{r} \sigma(q_i)\sigma(q'_i) \mod (P + \sigma(P)).$$

Since $\mathrm{srk}_{P+\sigma(P)}(\sum_i (a_i q_i + a'_i q'_i)) > 4p + 3C(r,0) \ge C(r,0)$ for any nonzero $(a_\bullet, a'_\bullet)$, by Corollary 3.4(i), there exists a subspace of linear forms $L_\sigma \supset P + \sigma(P)$ of dimension $\le N = 2p + C(r,0)$ and an E-linear orthogonal transformation A_σ of the $2r$ dimensional space with the basis $(q_\bullet, q'_\bullet)$ such that

$$\sigma(q_i) \equiv A_\sigma(q_i), \quad \sigma(q'_i) \equiv A_\sigma(q'_i) \mod (L_\sigma)$$

(note that here $A_\sigma(q_i)$ and $A_\sigma(q'_i)$ are linear combinations of $(q_\bullet, q'_\bullet)$).

We claim that $\sigma \mapsto A_\sigma$ defines a cocycle with values in the orthogonal group. Indeed, applying an element σ_1 of the Galois group to the congruence $\sigma_2(q_i) \equiv A_{\sigma_2}(q_i) \mod (L_{\sigma_2})$, we get

$$\sigma_1\sigma_2(q_i) \equiv \sigma_1(A_{\sigma_2})(\sigma_1(q_i)) \mod (\sigma_1(L_{\sigma_2})),$$

where $\sigma_1(A_{\sigma_2})$ is the orthogonal matrix obtained by applying σ_1 to the matrix A_{σ_2}. Hence, we have

$$\sigma_1(A_{\sigma_2})A_{\sigma_1}(q_i) \equiv \sigma_1(A_{\sigma_2})(\sigma_1(q_i)) \equiv \sigma_1\sigma_2(q_i) \mod (\sigma_1(L_{\sigma_2}) + L_{\sigma_1}),$$

hence,

$$q_i \equiv A^{-1}_{\sigma_1\sigma_2}\sigma_1(A_{\sigma_2})A_{\sigma_1}(q_i) \mod (\sigma_1(L_{\sigma_2}) + L_{\sigma_1} + L_{\sigma_1\sigma_2}).$$

Similarly,

$$q'_i \equiv A^{-1}_{\sigma_1\sigma_2}\sigma_1(A_{\sigma_2})A_{\sigma_1}(q'_i) \mod (\sigma_1(L_{\sigma_2}) + L_{\sigma_1} + L_{\sigma_1\sigma_2}).$$

Since $\mathrm{srk}(q_\bullet, q'_\bullet) > 3N \ge \dim(\sigma_1(L_{\sigma_2}) + L_{\sigma_1} + L_{\sigma_1\sigma_2})$, this implies that $A_{\sigma_1\sigma_2} = \sigma_1(A_{\sigma_2})A_{\sigma_1}$.

Recall that the nonabelian H^1 of the Galois group of $E/\mathbf{k}$ with values in the group of E-linear transformations preserving the standard quadratic form $Q_0 = \sum_{i=1}^r x_i y_i$ classifies equivalence classes of nondegenerate quadratic forms on $\mathbf{k}^{2r}$, which become equivalent to Q_0 over E (see [6, III.1.2]), so that the trivial class in H^1 corresponds to forms equivalent to Q_0 over $\mathbf{k}$.

Let Q be the quadratic form over $\mathbf{k}$ corresponding to our cocycle $\sigma \mapsto A_\sigma$. We can find a basis such that

$$Q = \sum_{i=1}^{r} (\lambda_i x_i^2 + \mu_i y_i^2),$$

for some $\lambda_i, \mu_i \in \mathbf{k}^*$. Let $\mathbf{k}' \supset \mathbf{k}$ denote the field extension obtained by adjoining r square roots $(\sqrt{-\mu_i/\lambda_i})$ to $\mathbf{k}$. Then over $\mathbf{k}'$ we can write

$$Q = \sum_{i=1}^{r} \lambda_i (x_i + \sqrt{-\mu_i/\lambda_i}\, y_i)(x_i - \sqrt{-\mu_i/\lambda_i}\, y_i),$$

so Q is equivalent to Q_0 over $\mathbf{k}'$. Note that $[\mathbf{k}':\mathbf{k}]\le 2^r$.

Without loss of generality we can assume that $\mathbf{k}'\subset E$. The fact that Q becomes equivalent to Q_0 over $\mathbf{k}'$ means that the cocycle $\sigma\mapsto A_\sigma$ becomes a coboundary when restricted to $\mathrm{Gal}(E/\mathbf{k}')$. Thus, we can make an orthogonal change of basis in $(q_\bullet,q'_\bullet)$ such that

$$\sigma(q_i)\equiv q_i,\quad \sigma(q'_i)\equiv q'_i \mod (L_\sigma)$$

for any σ in the Galois group $\mathrm{Gal}(E/\mathbf{k}')$. By Corollary 2.5, there exist quadratic forms $\overline{q}_\bullet$, $\overline{q}'_\bullet$ defined over $\mathbf{k}'$, such that

$$\mathrm{srk}(q_i-\overline{q}_i)\le N',\quad \mathrm{srk}(q'_i-\overline{q}'_i)\le N',$$

with $N'=N(2+(2N+1)^{2N+1})$.

Thus, we have

$$f-\sum_{i=1}^r \overline{q}_i\overline{q}'_i\in(P')$$

for some subspace of linear forms P' over E of dimension $\le N''=p+2rN'$. In other words, the slice rank of $\widetilde{f}=f-\sum_{i=1}^r\overline{q}_i\overline{q}'_i$ over E is $\le N''$. By [5, Thm. A], this implies that the slice rank of $\widetilde{f}$ over $\mathbf{k}'$ is $\le 4N''$. This means that

$$\mathrm{rk}^S_{\mathbf{k}'}(f)\le(r,4N'').$$

By Lemma 3.1, it follows that

$$\mathrm{rk}^S_{\mathbf{k}'}(f)\le(2^r\cdot r,2^r\cdot 4N'').$$

Case 2. Next, assume that $\mathrm{srk}(q_\bullet,q'_\bullet)\le 3N$, so there exists a nontrivial linear combination $\sum_i a_iq_i+\sum_i a'_iq'_i$ which has slice rank $\le 3N$. The restriction of the quadratic form $\sum_{i=1}^r x_iy_i$ to the linear subspace $\sum_i a_ix_i+\sum_i a'_iy_i=0$ has rank $2r-1$ or $2r-2$. So enlarging E if necessary we can find linear combinations $\overline{q}_0,(\overline{q}_i,\overline{q}'_i)_{i=1}^{r-1}$ of $(q_\bullet,q'_\bullet)$ (where possibly $\overline{q}=0$) such that

$$\sum_{i=1}^r q_iq'_i\equiv\overline{q}_0^2+\sum_{i=1}^{r-1}\overline{q}_i\overline{q}'_i \mod (\sum_i a_iq_i+\sum_i a'_iq'_i).$$

Hence, renaming $\overline{q}_i$, $\overline{q}'_i$ by q_i, q'_i, we obtain

$$f\equiv q_0^2+\sum_{i=1}^{r-1}q_iq'_i \mod (P'),$$

where P' is a space of linear forms of dimension $\le p'=\dim P'\le p+3N$.

Furthermore, we claim that we can assume that $\mathrm{srk}(q_0,q_\bullet,q'_\bullet)>3N'$, where

$$N'=D(r-1,0)+c(r,r-1,0)+2p'.$$

Indeed, otherwise arguing as above we obtain a decomposition (3.3) with r replaced by $r-1$ and p replaced by $p''=p'+3N'$. In other words, we would have $\mathrm{rk}^S_E(f)\le(r-1,p'')$, so by the induction assumption we would deduce that

$$\mathrm{rk}^S_{\mathbf{k}}(f)\le(c_2(r-1,p''),c_1(r-1,p'')).$$

Since for every $\sigma \in \mathrm{Gal}(E/\mathbf{k})$, one has

$$q_0^2 + \sum_{i=1}^{r-1} q_i q_i' \equiv \sigma(q_0)^2 + \sum_{i=1}^{r-1} \sigma(q_i)\sigma(q_i') \mod (P' + \sigma P'),$$

by Corollary 3.4(ii) with r replaced by $r-1$, we get that

$$\sigma(q_0) \equiv A_\sigma q_0, \quad \sigma(q_i) \equiv A_\sigma q_i, \quad \sigma(q_i') \equiv A_\sigma q_i' \mod (L_\sigma),$$

for some orthogonal transformation A_σ and a subspace of linear forms $L_\sigma \supset P' + \sigma P'$ of dimension $\le N'$ (here to verify the assumptions of Corollary 3.4(ii) we use the inequality

$$3N' \ge 3D(r-1,0) + 3c(r,r-1,0) \ge C(r-1,0) + c(r-1,r-2,0)$$

which is easy to check). Since $\mathrm{srk}(q_0, q_\bullet, q_\bullet') > 3N'$, as in Case 1, this implies that $\sigma \mapsto A_\sigma$ is a 1-cocycle.

Arguing as in Case 1, we find a subextension $\mathbf{k}' \subset E$ obtained by adjoining at most $r-1$ square roots to $\mathbf{k}$, such that after making a change of basis in $(q_\bullet, q_\bullet')$, we get

$$\sigma(q_i) \equiv q_i, \ \sigma(q_i') \equiv q_i' \mod (L_\sigma)$$

for any $\sigma \in \mathrm{Gal}(E/\mathbf{k}')$. Hence, by Corollary 2.5, there exist quadratic forms $\overline{q}_\bullet$, $\overline{q}_\bullet'$ defined over $\mathbf{k}'$, such that

$$\mathrm{srk}(q_i - \overline{q}_i) \le N'' \text{ for } i = 0, \dots, r-1, \quad \mathrm{srk}(q_i' - \overline{q}_i') \le N'' \text{ for } i = 1, \dots, r-1,$$

with $N'' = N'(2 + (2N'+1)^{2N'+1})$, and we get

$$\mathrm{srk}_E(f - \overline{q}_0^2 - \sum_{i=1}^{r-1} \overline{q}_i \overline{q}_i') \le M = p' + (2r-1)N''.$$

By [5, Thm. A], this implies that the slice rank of $\widetilde{f}$ over $\mathbf{k}'$ is $\le 4M$, so we get

$$\mathrm{rk}^S_{\mathbf{k}'}(f) \le (r, 4M),$$

and so by Lemma 3.1,

$$\mathrm{rk}^S_{\mathbf{k}'}(f) \le (2^r \cdot r, 2^r \cdot 4M).$$

References

[1] K. Adiprasito, D. Kazhdan, T. Ziegler, *On the Schmidt and analytic ranks for trilinear forms*, arXiv:2102.03659.

[2] E. Ballico, A. Bik, A. Oneto, E. Ventura, *Strength and slice rank of forms are genericaly equal*, arXiv:2102.11549.

[3] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.

[4] D. Kazhdan, A. Polishchuk, *Almost invariant subspaces and operators*, arXiv:2107.08085.

[5] D. Kazhdan, A. Polishchuk, *Linear subspaces of minimal codimension in hypersurfaces*, arXiv:2107.08080.

[6] J.-P. Serre, *Cohomologie Galoisienne*, Springer-Verlag, Berlin, 1994.

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