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Ten Compatible Poisson Brackets on $\mathbb{P}^5$

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Abstract. We give explicit formulas for ten compatible Poisson brackets on $\mathbb{P}^5$ found in arXiv:2007.12351.

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1 Introduction

The goal of this paper is to present explicit formulas for certain algebraic Poisson brackets on $\mathbb{P}^5$.

Recall that two Poisson brackets $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ are called *compatible* if any linear combination $\{\cdot,\cdot\}_1+\lambda\cdot\{\cdot,\cdot\}_2$ is still a Poisson bracket (i.e., satisfies the Jacobi identity). Pairs of compatible Poisson brackets play an important role in the theory of integrable systems.

With every normal elliptic curve C in $\mathbb{P}^n$ one can associate naturally a Poisson bracket on $\mathbb{P}^n$, called a *Feigin–Odesskii bracket of type* $q_{n+1,1}$. The corresponding quadratic Poisson brackets on $\mathbb{A}^{n+1}$ arise as quasi-classical limits of Feigin–Odesskii elliptic algebras. On the other hand, they can be constructed using the geometry of vector bundles on C (see [2, 8]).

It was discovered by Odesskii–Wolf [6] that for every n there exists a family of 9 linearly independent mutually compatible Poisson brackets on $\mathbb{P}^n$, such that their generic linear combinations are Feigin–Odesskii brackets of type $q_{n+1,1}$. In [3], this construction was explained and extended in terms of anticanonical line bundles on del Pezzo surfaces. It was observed in [3, Example 4.6] that in this framework one also obtains 10 linearly independent mutually compatible Poisson brackets on $\mathbb{P}^5$. In this paper, we will produce explicit formulas for these 10 brackets (see Theorem 3.2).

2 Homological perturbation for $\mathbb{P}^n$

2.1 Formula for the homotopy

Let

$$H=\bigoplus_{p\geq 0,\,q\in\mathbb{Z}} H^p(\mathbb{P}^n,\mathcal{O}(q))$$

be the cohomology algebra of line bundles on $\mathbb{P}^n$, and

$$A=\Big(\bigoplus_{p\geq 0,\,q\in\mathbb{Z}} C^p(\mathbb{P}^n,\mathcal{O}(q)),d\Big)$$

the Čech complex with respect to the standard open covering $U_i=(x_i\neq 0)$ of $\mathbb{P}^n$. There is a natural dg-algebra structure on A, such that the corresponding cohomology algebra is H. The multiplication on A is defined as follows. For $\alpha \in C^p(\mathbb{P}^n, \mathcal{O}(q))$ and $\beta \in C^{p'}(\mathbb{P}^n, \mathcal{O}(q'))$, we define $\alpha\beta \in C^{p+p'}(\mathbb{P}^n, \mathcal{O}(q+q'))$ by

$$(\alpha\beta)_{i_0 i_1 \dots i_{p+p'}} := \alpha_{i_0 \dots i_p}|_{U_{i_0 \dots i_{p+p'}}} \cdot \beta_{i_p \dots i_{p+p'}}|_{U_{i_0 \dots i_{p+p'}}},$$

where on the right hand side we use the multiplication map $\mathcal{O}(q) \otimes \mathcal{O}(q') \to \mathcal{O}(q+q')$.

The homological perturbation lemma equips H with a minimal A_∞-structure (m_n), where m_2 is the usual product on H. We will use the form of this lemma due to Kontsevich–Soibelman [5], which gives formulas for m_n as sums over trees. To apply homological perturbation, we need the following data:

- a projection $\pi\colon A \to H$,
- an inclusion $\iota\colon H \to A$, and
- a homotopy Q such that $\pi\iota = \mathrm{id}_H$ and $\mathrm{id}_A - \iota\pi = \mathrm{d}Q + Q\mathrm{d}$.

Recall that $H^0 = \mathbb{C}[x_0, \dots, x_n]$,

$$H^n \simeq \bigoplus_{e_0, \dots, e_n < 0} \mathbf{k} \cdot x_0^{e_0} x_1^{e_1} \cdots x_n^{e_n} \subset A^n,$$

and $H^i = 0$ for $i \neq 0, n$. We define ι in degree zero by $\iota(f)_k = f$ for $k = 0, 1, \dots, n$. We define ι in degree n by $\iota(g)_{0\dots n} = g$. We define the projection in degree zero to be

$$\pi(\gamma) = \begin{cases} \gamma_n & \text{if } \gamma_n \in \mathbb{C}[x_0, \dots, x_n], \\ 0 & \text{else.} \end{cases}$$

To define π in degree n, we observe that

$$A^n = \bigoplus_{e_0, \dots, e_n \in \mathbb{Z}} \mathbf{k} \cdot x_0^{e_0} x_1^{e_1} \cdots x_n^{e_n},$$

and we let π be the natural projection to H^n.

To define the homotopy, we use that A decomposes as a direct sum of chain complexes

$$A = \oplus_{\vec{e} \in \mathbb{Z}^{n+1}} A(\vec{e}),$$

where $A(\vec{e})$ consists of all elements in A whose components are scalar multiples of $x^{\vec{e}} := x_0^{e_0} x_1^{e_1} \cdots x_n^{e_n}$. In other words, $A(\vec{e})$ is the $\vec{e}$-isotypical summand with respect to the action of the group $\mathbb{G}_m^{n+1}$.

Let us set for $\vec{e} \in \mathbb{Z}^{n+1}$,

$$k(\vec{e}) := \max\{i \mid e_i \geq 0\}$$

(which is equal to $-\infty$ if all e_i are negative). There is then a standard homotopy Q defined on an element $\gamma \in A(\vec{e})^p$ by $Q(\gamma)_{i_0 i_1 \dots i_{p-1}} = \gamma_{k(\vec{e}) i_0 \dots i_{p-1}}$ if $k(\vec{e}) > -\infty$ and $Q(\gamma)_{i_0 i_1 \dots i_{p-1}} = 0$ otherwise (i.e., if all e_i are negative).

For a Laurent monomial $x^{\vec{e}}$ and a subset $I = \{i_0, \dots, i_p\} \subset \{0, 1, \dots, n\}$ such that $I \supset \{0 \leq i \leq n | e_i < 0\}$, let us denote by $x_I^{\vec{e}}$ the element of A^p given by

$$(x_I^{\vec{e}})_{j_0 \dots j_p} = \begin{cases} x^{\vec{e}} & \text{if } \{j_0, \dots, j_p\} = I, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the condition $I \supset \{0 \le i \le n \,|\, e_i < 0\}$ guarantees that $x^{\vec{e}}$ is a regular section of the appropriate line bundle over $U_{i_0 \ldots i_p}$. Clearly, these elements form a basis for A and our homotopy operator Q is given by

$$Q\big(x_I^{\vec{e}}\big) = \begin{cases} (-1)^j x^{\vec{e}}_{I \setminus k(\vec{e})} & \text{if } k(\vec{e}) = i_j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

With these data one can in principle calculate all the higher products on the cohomology algebra H. Below, we will get explicit formulas in the case we need.

2.2 Calculation of m_4 for $\mathbb{P}^2$

We now specialize to the case of the projective plane $\mathbb{P}^2$. We have no higher products of odd degree because H and $H^{\otimes n}$ only live in even degrees. Also, for degree reasons the product m_4 will only be non-zero on elements $e \otimes f \otimes g \otimes h \in H^{\otimes 4}$ where one or two of the arguments lie in H^2 and the rest in H^0. Below, we will explicitly compute the product m_4 involving one argument in H^2. Thus, the following special case of the multiplication in A will be relevant: for a monomial $x^{\vec{e}}$ and a Laurent monomial $x^{\vec{e}'}$, we have

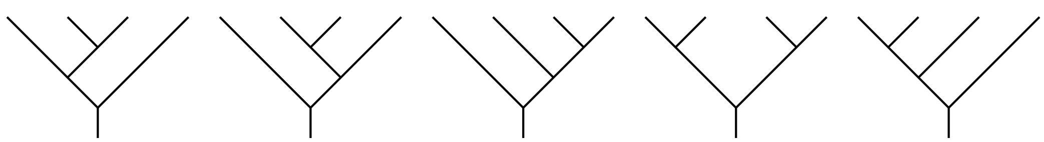
$$\iota_0\big(x^{\vec{e}}\big) \cdot x_I^{\vec{e}'} = x_I^{\vec{e}'} \cdot \iota_0\big(x^{\vec{e}}\big) = x_I^{\vec{e}+\vec{e}'}.$$

We use the formula

$$m_4(e,f,g,h) = -\sum_T \epsilon(T) m_T(e,f,g,h),$$

where the sum runs over all rooted binary trees with 4 leaves labeled e, f, g and h (from left to right). For each such tree T the expression $m_T(e,f,g,h)$ is computed by moving the inputs through that tree, applying ι at the leaves, applying the homotopy Q on each interior edge, multiplying elements of A at each inner vertex and finally applying the projection π at the bottom.

We have to sum over the following five trees, which we denote $T_1, \ldots, T_5$, respectively,



Let us first consider the case $e \in H^2$ and $f, g, h \in H^0$ and let's take them all to be basis elements of H^2 and H^0:

$$e = \big(x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2}\big)_{\{0,1,2\}}, \qquad f = x_0^{a_0} x_1^{a_1} x_2^{a_2}, \qquad g = x_0^{b_0} x_1^{b_1} x_2^{b_2}, \qquad h = x_0^{c_0} x_1^{c_1} x_2^{c_2},$$

where $\alpha_0, \alpha_1, \alpha_2 < 0$ and $a_i, b_i, c_i \ge 0$ for $i = 0, 1, 2$. In this case only one of the trees above can be non-zero in the expression for $m_4(e,f,g,h)$, namely T_5, because in all other trees at some point the homotopy Q will be applied to an element of A^0. Below is a picture of the different summands in $A^\bullet$ and the possible ways the homotopy Q can map a monomial element in each summand:

$$
\begin{tikzcd}
 & & H^2 \arrow[d, "\iota_2"] & \\
A^2: & & \bullet_{0,1,2} \arrow[dl] \arrow[d, "(1)"] \arrow[dr, "(3)"] & \\
A^1: & \bullet_{0,1} \arrow[d] \arrow[dr] & \bullet_{0,2} \arrow[dl] \arrow[dr, "(2)"] & \bullet_{1,2} \arrow[dl] \arrow[d, "(4)"] \\
A^0: & \bullet_0 & \bullet_1 & \bullet_2 \arrow[d, "\pi_0"] \\
 & & & H^0
\end{tikzcd}
$$

When computing $m_{T_5}(e,f,g,h)$ we should move e through this diagram; at every node it gets multiplied by one of the other arguments and then it moves downwards along one of the arrows. We see that we get non-zero result if we move either along (1) followed by (2) or along (3) followed by (4) (so that we land in $\bullet_2$). We claim that only the second route is possible. The reason is that at each node we multiply e by a monomial, so the exponents of x_0, x_1, x_2 will not decrease at any time. By the definition of Q, if e gets moved along (1) then after the multiplication at $\bullet_{0,1,2}$ the exponent of x_1 is non-negative while the exponent of x_2 is negative. Hence, after performing the multiplication at $\bullet_{0,2}$ the exponent of x_1 is still non-negative. It follows then from the definition of Q that e cannot move along (2) after moving along (1).

Now comes the computation of $m_{T_5}(e,f,g,h)$. Below, we denote by μ the multiplication in A. Then

$$
\begin{aligned}
m_{T_5}(e,f,g,h) &= \pi\mu(Q\mu(Q\mu(e,f),g),h) \\
&= \pi\mu\big(Q\mu\big(Q\big(x_0^{\alpha_0+a_0}x_1^{\alpha_1+a_1}x_2^{\alpha_2+a_2}\big)_{\{0,1,2\}},g\big),h\big) \\
&\overset{(*)}{=} \pi\mu\big(Q\mu\big(\big(x_0^{\alpha_0+a_0}x_1^{\alpha_1+a_1}x_2^{\alpha_2+a_2}\big)_{\{1,2\}},g\big),h\big) \\
&= \pi\mu\big(Q\big(x_0^{\alpha_0+a_0+b_0}x_1^{\alpha_1+a_1+b_1}x_2^{\alpha_2+a_2+b_2}\big)_{\{1,2\}},h\big) \\
&\overset{(**)}{=} \pi\big(\mu\big(\big(x_0^{\alpha_0+a_0+b_0}x_1^{\alpha_1+a_1+b_1}x_2^{\alpha_2+a_2+b_2}\big)_{\{2\}},h\big)\big) \\
&= \pi\big(\big(x_0^{\alpha_0+a_0+b_0+c_0}x_1^{\alpha_1+a_1+b_1+c_1}x_2^{\alpha_2+a_2+b_2+c_2}\big)_{\{2\}}\big) \\
&\overset{(***)}{=} x_0^{\alpha_0+a_0+b_0+c_0}x_1^{\alpha_1+a_1+b_1+c_1}x_2^{\alpha_2+a_2+b_2+c_2},
\end{aligned}
$$

where the symbols $(*)$, $(**)$ are $(***)$ mean that we get zero unless the following conditions hold:

$$
(*)\ \begin{cases} \alpha_0+a_0\ge 0, \\ \alpha_1+a_1<0, \\ \alpha_2+a_2<0, \end{cases} \qquad (**)\ \begin{cases} \alpha_1+a_1+b_1\ge 0, \\ \alpha_2+a_2+b_2<0, \end{cases} \qquad (***)\ \begin{cases} \alpha_0+a_0+b_0+c_0\ge 0, \\ \alpha_1+a_1+b_1+c_1\ge 0, \\ \alpha_2+a_2+b_2+c_2\ge 0. \end{cases}
$$

In the end, we have

$$
m_4(e,f,g,h) = -m_{T_5}(e,f,g,h) = -\rho\big(\vec{\alpha};\vec{a},\vec{b},\vec{c}\big)\cdot x^{\vec{\alpha}+\vec{a}+\vec{b}+\vec{c}},
$$

where

$$
\rho\big(\vec{\alpha};\vec{a},\vec{b},\vec{c}\big) := \begin{cases} 1 & \text{if } \alpha_0+a_0\ge 0,\ \alpha_1+a_1<0,\ \alpha_1+a_1+b_1\ge 0, \\ & \quad \alpha_2+a_2+b_2<0,\ \alpha_2+a_2+b_2+c_2\ge 0, \\ 0 & \text{else.} \end{cases}
$$

Similarly, we compute m_4 applied to e, f, g, h in any given order. We have

$$m_4(e,f,g,h) = -\rho\big(\vec{\alpha};\vec{a},\vec{b},\vec{c}\big)\cdot x^{\vec{\alpha}+\vec{a}+\vec{b}+\vec{c}},$$
$$m_4(f,e,g,h) = \big[-\rho\big(\vec{\alpha};\vec{a},\vec{b},\vec{c}\big)+\rho\big(\vec{\alpha};\vec{b},\vec{a},\vec{c}\big)-\rho\big(\vec{\alpha};\vec{b},\vec{c},\vec{a}\big)\big]\cdot x^{\vec{\alpha}+\vec{a}+\vec{b}+\vec{c}},$$
$$m_4(f,g,e,h) = \big[\rho\big(\vec{\alpha};\vec{b},\vec{a},\vec{c}\big)-\rho\big(\vec{\alpha};\vec{b},\vec{c},\vec{a}\big)+\rho\big(\vec{\alpha};\vec{c},\vec{b},\vec{a}\big)\big]\cdot x^{\vec{\alpha}+\vec{a}+\vec{b}+\vec{c}},$$
$$m_4(f,g,h,e) = \rho\big(\vec{\alpha};\vec{c},\vec{b},\vec{a}\big)\cdot x^{\vec{\alpha}+\vec{a}+\vec{b}+\vec{c}}.$$

3 Feigin–Odesskii brackets

3.1 Bivectors on projective spaces

It is well known that every $\mathbb{G}_m$-invariant bivector on a vector space V leads to a bivector on the projective space $\mathbb{P}V$. A bivector on V can be thought of as a skew-symmetric bracket $\{\cdot,\cdot\}$ on the polynomial algebra $S(V^*)$, which is a biderivation. Such a bracket is $\mathbb{G}_m$-invariant if and only if the bracket of two linear forms is a quadratic form. In other words, such a bracket can be viewed as a skew-symmetric pairing

$$b\colon\ V^*\times V^*\to S^2(V^*).$$

The corresponding bivector Π on the projective space $\mathbb{P}V$ is determined by the skew-symmetric forms Π_v on $T^*_v\mathbb{P}V$ for each point $\langle v\rangle\in\mathbb{P}V$. We have an identification

$$T^*_v\mathbb{P}V = \langle v\rangle^\vee\subset V^*.$$

It is easy to see that under this identification we have

$$\Pi_v(s_1\wedge s_2) = b(s_1,s_2)(v), \tag{3.1}$$

where $s_1,s_2\in\langle v\rangle^\vee$. Here we take the value of the quadratic form $b(s_1\wedge s_2)$ at v.

We can use the above formula in reverse. Namely, suppose for some bivector Π on $\mathbb{P}V$ we found a skew-symmetric pairing b such that (3.1) holds. Then the $\mathbb{G}_m$-invariant bracket $\{\cdot,\cdot\}$ on $S(V)$ given by b induces the bivector Π on $\mathbb{P}V$. Note that if Π is a Poisson bivector on $\mathbb{P}V$, it is not guaranteed that the $\mathbb{G}_m$-invariant bracket $\{\cdot,\cdot\}$ on $S(V)$ is also Poisson, i.e., satisfies the Jacobi identity (but it is known that $\{\cdot,\cdot\}$ can be chosen to be Poisson, see [1, 7]).

3.2 Recollections from [3]

Below, we will denote simply by L_1L_2 the tensor product of line bundles L_1 and L_2.

Let ξ be a line bundle of degree n on an elliptic curve C. We fix a trivialization $\omega_C\simeq\mathcal{O}_C$. Then the associated Feigin–Odesskii Poisson structure Π (to which we will refer as *FO bracket*) on $\mathbb{P}H^1\big(\xi^{-1}\big)\simeq\mathbb{P}H^0(\xi)^*$ is given by the formula (see [3, Lemma 2.1])

$$\Pi_\phi(s_1\wedge s_2) = \langle\phi,\mathrm{MP}(s_1,\phi,s_2)\rangle, \tag{3.2}$$

where $\langle\phi\rangle\in\mathbb{P}\,\mathrm{Ext}^1(\xi,\mathcal{O})$, and $s_1,s_2\in\langle\phi\rangle^\perp$. Here we use the Serre duality pairing $\langle\cdot,\cdot\rangle$ between $H^0(\xi)$ and $H^1\big(\xi^{-1}\big)$ and the triple Massey product

$$\mathrm{MP}\colon\ H^0(\xi)\otimes H^1\big(\xi^{-1}\big)\otimes H^0(\xi)\to H^0(\xi)$$

that also agrees with the triple product m_3 obtained by homological perturbation from the natural dg enhancement of the derived category of coherent sheaves on C. There is some ambiguity in a choice of m_3 but for $s_1, s_2 \in \langle\phi\rangle^\perp$, the expression in the right-hand side of (3.2) is well defined.

Next, assume that S is a smooth projective surface, L is a line bundle on S such that $H^*(S, LK_S) = 0$, and let $C \subset S$ be a smooth connected anticanonical divisor (which is an elliptic curve), so we have an exact sequence of coherent sheaves on S,

$$0 \to K_S \xrightarrow{F} \mathcal{O}_S \to \mathcal{O}_C \to 0. \tag{3.3}$$

We have a natural restriction map

$$H^0(S, L) \to H^0(C, L|_C).$$

The exact sequence

$$0 \to LK_S \xrightarrow{F} L \to L_C \to 0 \tag{3.4}$$

shows that under our assumptions this restriction map is an isomorphism.

Thus, the FO bracket on $\mathbb{P}H^0(L|_C)^*$ associated with $(C, L|_C)$ (defined up to rescaling) can be viewed as a Poisson structure on a fixed projective space $\mathbb{P}V^*$, where

$$V := H^0(S, L).$$

By [3, Theorem 4.4], the Poisson brackets on $\mathbb{P}V^*$ associated with different anticanonical divisors are compatible. More precisely, we get a linear map from $H^0\big(S, K_S^{-1}\big)$ to the space of bivectors on $\mathbb{P}V^*$, whose image lies in the space of Poisson brackets.

3.3 Feigin–Odesskii bracket for an anticanonical divisor

We keep the data (S, L) of the previous subsection. Let $i\colon C \hookrightarrow S$ be an anticanonical divisor in S, with the equation $F \in H^0\big(S, K_S^{-1}\big)$. We want to write a formula for the FO bracket $\Pi = \Pi_F$ on $\mathbb{P}V^*$ in terms of higher products on the surface S and the equation F. For this we rewrite the right-hand side of formula (3.2). Let us write the triple product in this formula as MP^C to remember that it is defined for the derived category of C.

Proposition 3.1.

(*i*) *In the above situation, given $e \in V^*$ and $s_1, s_2 \in \langle e\rangle^\perp$, one has*

$$\big\langle e, \mathrm{MP}^C(s_1|_C, e, s_2|_C)\big\rangle = \langle m_4(F, s_1, e, s_2) - m_4(s_1, F, e, s_2), e\rangle,$$

where we use the identification $V^* \simeq H^2\big(S, L^{-1}K_S\big)$ given by Serre duality and consider the A_∞-products on S,

$$\begin{aligned} m_4\colon\ & H^0\big(K_S^{-1}\big)H^0(L)H^2\big(L^{-1}K_S\big)H^0(L) \to H^0(L),\\ & H^0(L)H^0\big(K_S^{-1}\big)H^2\big(L^{-1}\big)H^0(L) \to H^0(L), \end{aligned}$$

obtained by the homological perturbation.

(*ii*) *Assume that a generic anticanonical divisor is smooth (and connected). Then*

$$\Pi_F|_e(s_1 \wedge s_2) := \langle m_4(F, s_1, e, s_2) - m_4(s_1, F, e, s_2), e\rangle$$

gives a collection of compatible Poisson brackets on $\mathbb{P}V$ depending linearly on F.

Proof. (i) By Serre duality, $H^*\big(S, L^{-1}\big) = 0$, so the map

$$H^1\big(C, L^{-1}\big|_C\big) \to H^2\big(S, L^{-1}K_S\big),$$

induced by the exact sequence

$$0 \to L^{-1}K_S \to L^{-1} \to L^{-1}\big|_C \to 0,$$

is an isomorphism. It is a standard fact that this isomorphism is the dual to the isomorphism $H^0(S,L) \to H^0(C, L|_C)$ given by the restriction, via Serre dualities on S and C. Let us denote by $e_C \in H^1\big(C, L^{-1}\big|_C\big)$ the element corresponding to $e \in H^2\big(S, L^{-1}K_S\big)$ under the above isomorphism.

We claim that the triple Massey product $\mathrm{MP}^C(s_1|_C, e_C, s_2|_C) = m_3(s_1|_C, e_C, s_2|_C)$ corresponding to the arrows

$$\mathcal{O}_C \xrightarrow{s_2|_C} L|_C \xrightarrow{e_C} \mathcal{O}_C \xrightarrow{s_1|_C} L|_C$$

(where the middle arrow has degree 1) agrees with the corresponding triple Massey product on S,

$$\mathcal{O}_S \xrightarrow{s_2} L \xrightarrow{e_C} \mathcal{O}_C \xrightarrow{s_1|_C} L|_C.$$

Indeed, the relevant spaces are identified via the restriction maps. Let

$$r\colon\ \mathcal{O}_S \to \mathcal{O}_C, \qquad r_L\colon\ L \to L|_C$$

be the natural maps. Then we have to check that for $s_1, s_2 \in \langle e\rangle^\perp \subset H^0(S,L)$, one has

$$m_3(s_1|_C, e_C, s_2|_C)r \equiv m_3(s_1|_C, e_C r_L, s_2) \mod \langle s_1|_C r, s_2|_C r\rangle,$$

where we view this as equality of cosets in $\mathrm{Hom}(\mathcal{O}_S, L|_C)$. The A_∞-identities imply that

$$m_3(s_1|_C, e_C, s_2|_C)r = m_3(s_1|_C, e_C, s_2|_C r) \pm s_1|_C m_3(e_C, s_2|_C, r),$$

where $s_2|_C r = r_L s_2$, and

$$m_3(s_1|_C, e_C, r_L s_2) = m_3(s_1|_C, e_C r_L, s_2) \pm s_1|_C m_3(e_C, r_L, s_2) \pm m_3(s_1|_C, e_C, r_L)s_2.$$

Combining these two identities, we deduce our claim.

Thus, it is enough to calculate the Massey product $\mathrm{MP}(s_1|_C, e_C r_L, s_2)$. Using the exact sequences (3.3) and (3.4), we can represent $\mathcal{O}_C$ (resp. L_C) by the twisted complex $[K_S[1] \to \mathcal{O}_S]$ (resp. $[LK_S[1] \to L]$).

In terms of these resolutions, the elements of $\mathrm{Ext}^1(L, \mathcal{O}_C)$ get represented by $\mathrm{Ext}^2(L, K_S) \subset \mathrm{hom}^\bullet(L, [K_S[1] \to \mathcal{O}_S])$, while the element of $\mathrm{Hom}(\mathcal{O}_C, L|_C)$ corresponding to $s \in H^0(S,L) \simeq H^0(C, L|_C)$ is given by the natural map of twisted complexes induced by the multiplication by s. The elements of $\mathrm{Hom}(\mathcal{O}_S, L|_C)$ are identified with $\mathrm{Hom}(\mathcal{O}_S, L) \simeq \mathrm{hom}^0(\mathcal{O}_S, [LK_S[1] \to L])$. Thus, the m_3 product we are interested is given by the following triple product in the category of twisted complexes over S:

$$\begin{tikzcd} \mathcal{O}_S \arrow[d, "s_2"'] & \\ L \arrow[d, "e"'] & \\ K_S[1] \arrow[r, "F"] \arrow[d, "s_1"'] & \mathcal{O}_S \arrow[d, "s_1"'] \\ LK_S[1] \arrow[r, "F"] & L, \end{tikzcd}$$

where we view e as a morphism of degree 1 from L to $K_S[1]$. Now the formula for m_3 on twisted complexes (see [4, Section 7.6]) gives

$$m_4(F, s_1, e, s_2) - m_4(s_1, F, e, s_2)$$

(here the insertions of F correspond to insertions of the differentials in the twisted complexes).

(ii) It is clear that Π_F gives a linear map from $H^0\big(S, \omega_S^{-1}\big)$ to the space of bivectors on $\mathbb{P}V$. By (i), for generic F we get a Poisson bracket. Hence, this is true for all F. ■

3.4 The case leading to 10 compatible brackets on $\mathbb{P}^5$

We can apply Proposition 3.1 to the case $S = \mathbb{P}^2$ and $L = \mathcal{O}(2)$. Note that the assumptions are satisfied in this case since $LK_S = \mathcal{O}(-1)$ has vanishing cohomology. Thus, for each $F \in H^0\big(\mathbb{P}^2, \mathcal{O}(3)\big)$ giving a smooth cubic, we get a formula for the FO-bracket Π_F on $\mathbb{P}H^0\big(\mathbb{P}^2, \mathcal{O}(2)\big)^* = \mathbb{P}^5$. Hence, we get a family of 10 (the dimension of $H^0\big(\mathbb{P}^2, \mathcal{O}(3)\big)$ compatible brackets on $\mathbb{P}^5$ (we also know this from [3, Proposition 4.7]). The fact that these 10 brackets are linearly independent follows from the compatibility of this construction with the GL_3-action and is explained in [3, Proposition 4.7].

Now we will derive formulas for the brackets $\{\ ,\ \}_F$ on the algebra of polynomials in 6 variables which induce the above Poisson brackets on $\mathbb{P}V \simeq \mathbb{P}^5$, where

$$V = H^0\big(\mathbb{P}^2, \mathcal{O}(2)\big)^*.$$

They depend linearly on F, so we will just give formulas for $\{\ ,\ \}_{x^{\vec{c}}}$, where $x^{\vec{c}}$ runs through all 10 monomials of degree 3 in (x_0, x_1, x_2).

Let us set

$$\Delta(n) := \begin{cases} \big\{(a_0, a_1, a_2) \in \mathbb{Z}^3 \mid a_0 + a_1 + a_2 = n,\ a_i \geq 0 \text{ for } i = 0, 1, 2\big\} & \text{if } n \geq 0, \\ \big\{(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3 \mid \alpha_0 + \alpha_1 + \alpha_2 = n,\ \alpha_i < 0 \text{ for } i = 0, 1, 2\big\} & \text{if } n < 0. \end{cases}$$

Note that $\{x^{\vec{e}} \mid \vec{e} \in \Delta(n)\}$ forms a basis for $H^0\big(\mathbb{P}^2, \mathcal{O}(n)\big)$ when $n \geq 0$, while $\big\{x^{\vec{e}}_{\{0,1,2\}} \mid \vec{e} \in \Delta(n)\big\}$ is a basis for $H^2\big(\mathbb{P}^2, \mathcal{O}(n)\big)$ when $n < 0$. In particular, we use $\big\{x^{\vec{a}} \mid \vec{a} \in \Delta(2)\big\}$ as a basis in $V^* = H^0\big(\mathbb{P}^2, \mathcal{O}(2)\big)$. Our brackets should associate to a pair of elements of this basis a quadratic form in the same variables.

Theorem 3.2. *One has for $\vec{a}, \vec{b} \in \Delta(2)$, $\vec{c} \in \Delta(3)$,*

$$\big\{x^{\vec{a}}, x^{\vec{b}}\big\}_{x^{\vec{c}}} := \sum_{\vec{a}\,', \vec{b}\,' \in \Delta(2)} \Big[\sum_{\sigma} -\operatorname{sgn}(\sigma)\tilde{\rho}\big(\sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}, \vec{a}\,', \vec{b}\,'\big)\Big] x^{\vec{a}\,'} x^{\vec{b}\,'}, \tag{3.5}$$

where the second sum is over the symmetric group on the letters $\{a, b, c\}$ and

$$\tilde{\rho}(\vec{a}, \vec{b}, \vec{c}, \vec{a}\,', \vec{b}\,') := \begin{cases} 1 & \text{if } a_0' \leq a_0 - 1,\ a_1' > a_1 - 1,\ a_1' \leq a_1 + b_1 - 1, \\ & \quad a_2 + b_2 < a_2' + 1,\ c_2 + a_2 + b_2 \geq a_2' + 1, \\ & \quad a_0' + b_0' = a_0 + b_0 + c_0 - 1,\ a_1' + b_1' = a_1 + b_1 + c_1 - 1, \\ 0 & \text{else.} \end{cases}$$

Proof. By Serre duality, we can identify $V = H^0\big(\mathbb{P}^2, \mathcal{O}(2)\big)^*$ with $H^2\big(\mathbb{P}^2, \mathcal{O}(-5)\big)$. By Proposition 3.1, the bracket $\{x^{\vec{a}}, x^{\vec{b}}\}_{x^{\vec{c}}}$ is the quadratic form on $V \simeq H^2\big(\mathbb{P}^2, \mathcal{O}(-5)\big)$ given by

$$Q(e) := \big\langle e, m_4\big(x^{\vec{c}}, x^{\vec{a}}, e, x^{\vec{b}}\big) - m_4\big(x^{\vec{a}}, x^{\vec{c}}, e, x^{\vec{b}}\big)\big\rangle.$$

We can write

$$e = \sum_{\vec{\alpha} \in \Delta(-5)} c_{\vec{\alpha}} x^{\vec{\alpha}}_{\{0,1,2\}} \in H^2\big(\mathbb{P}^2, \mathcal{O}(-5)\big).$$

Using the formulas for m_4 from the end of Section 2.2, we get

$$Q(e) = \sum_{\vec{\alpha}, \vec{\beta} \in \Delta(-5)} \Big[\sum_{\sigma} -\operatorname{sgn}(\sigma)\rho\big(\vec{\alpha}; \sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}\big)\Big]\delta\big(\vec{\alpha}, \vec{\beta}, \vec{a}, \vec{b}, \vec{c}\big) c_{\vec{\alpha}} c_{\vec{\beta}},$$

where the second sum runs over the symmetric group on the letters $\{a, b, c\}$ and

$$\delta(\vec{\alpha}, \vec{\beta}, \vec{a}, \vec{b}, \vec{c}) = \begin{cases} 1 & \text{if } \vec{\alpha} + \vec{\beta} + \vec{a} + \vec{b} + \vec{c} = (-1, -1, -1), \\ 0 & \text{else.} \end{cases}$$

We have to show that the element in $S^2\big(H^0\big(\mathbb{P}^2, \mathcal{O}(2)\big)\big)$ given by the right-hand side of (3.5) defines the same quadratic form Q on $H^2\big(\mathbb{P}^2, \mathcal{O}(-5)\big)$. To see this, we apply it to the element $e = \sum_{\vec{\alpha} \in \Delta(-5)} c_{\vec{\alpha}} x^{\vec{\alpha}}_{\{0,1,2\}} \in H^2\big(\mathbb{P}^2, \mathcal{O}(-5)\big)$. For $\vec{\alpha} \in \Delta(-5)$, we set $\vec{\alpha}^* := (-1, -1, -1) - \vec{\alpha}$ and then we compute

$$\begin{aligned} &\Bigg(\sum_{\vec{a}\,', \vec{b}\,' \in \Delta(2)} \Big[\sum_{\sigma} -\operatorname{sgn}(\sigma)\tilde{\rho}\big(\sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}, \vec{a}\,', \vec{b}\,'\big)\Big] x^{\vec{a}\,'} x^{\vec{b}\,'}\Bigg)(e) \\ &\quad = \sum_{\vec{\alpha}, \vec{\beta} \in \Delta(-5)} \sum_{\vec{a}\,', \vec{b}\,' \in \Delta(2)} \Big[\sum_{\sigma} -\operatorname{sgn}(\sigma)\tilde{\rho}\big(\sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}, \vec{a}\,', \vec{b}\,'\big)\Big] \langle x^{\vec{a}\,'}, x^{\vec{\alpha}}_{\{0,1,2\}}\rangle \langle x^{\vec{b}\,'}, x^{\vec{\beta}}_{\{0,1,2\}}\rangle c_{\vec{\alpha}} c_{\vec{\beta}} \\ &\quad = \sum_{\vec{\alpha}, \vec{\beta} \in \Delta(-5)} \Big[\sum_{\sigma} -\operatorname{sgn}(\sigma)\tilde{\rho}\big(\sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}, \vec{\alpha}^*, \vec{\beta}^*\big)\Big] c_{\vec{\alpha}} c_{\vec{\beta}}. \end{aligned}$$

Now it only remains to note that for any permutation σ, one has

$$\tilde{\rho}\big(\sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}, \vec{\alpha}^*, \vec{\beta}^*\big) = \rho\big(\vec{\alpha}; \sigma\vec{a}, \sigma\vec{b}, \sigma\vec{c}\big)\delta\big(\vec{\alpha}, \vec{\beta}, \vec{a}, \vec{b}, \vec{c}\big),$$

for $\tilde{\rho}$ given in the formulation of the theorem. ■

Remarks 3.3.

1. Note that when we take $\vec{c} = (0, 0, 3)$ only two permutations σ, namely, $\sigma = 1$ and $\sigma = (a\ b)$, can give non-zero terms in the formula of Theorem 3.2. When $\vec{c} = (1, 2, 0)$ all permutations except $\sigma = 1$ and $\sigma = (a\ b)$ may give non-zero terms. When $\vec{c} = (1, 1, 1)$ all permutations can give non-zero terms.
2. It is not true that formulas (3.5) define compatible Poisson brackets on the algebra of polynomials in 6 variables: this is true only for the induced brackets on $\mathbb{P}^5$ (in other words, the relevant identities hold only for the ratios of coordinates x_i/x_j).

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