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RESEARCH ARTICLE

The eleventh cohomology group of $\overline{\mathcal{M}}_{g,n}$

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Abstract

We prove that the rational cohomology group $H^{11}(\overline{\mathcal{M}}_{g,n})$ vanishes unless $g = 1$ and $n \geq 11$. We show furthermore that $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge–Tate for all even $k \leq 12$ and deduce that $\#\overline{\mathcal{M}}_{g,n}(\mathbb{F}_q)$ is surprisingly well approximated by a polynomial in q. In addition, we use $H^{11}(\overline{\mathcal{M}}_{1,11})$ and its image under Gysin push-forward for tautological maps to produce many new examples of moduli spaces of stable curves with nonvanishing odd cohomology and nontautological algebraic cycle classes in Chow cohomology.

1. Introduction

The Langlands program makes a number of striking predictions about the Hodge structures and Galois representations that appear in the cohomology of moduli spaces of stable curves; see [8, Section 1.2] and [4, 3]. While the conjectured correspondence with algebraic cuspidal automorphic representations of conductor 1 remains out of reach, these representations have been classified up to weight 22 [7], and some of the resulting predictions can now be verified unconditionally. Bergström, Faber and the third author recently proved that $H^k(\overline{\mathcal{M}}_{g,n})$ vanishes for all odd $k \leq 9$ and all g and n [4]. For $k = 11$, the conjectural correspondence predicts that $H^{11}(\overline{\mathcal{M}}_{g,n})$ is isomorphic to a direct sum of copies of $H^{11}(\overline{\mathcal{M}}_{1,11})$ and hence should vanish in all cases where $\overline{\mathcal{M}}_{g,n}$ is unirational. We confirm this prediction unconditionally and show that $H^{11}(\overline{\mathcal{M}}_{g,n})$ vanishes in an even wider range of cases.

Theorem 1.1. *The cohomology group $H^{11}(\overline{\mathcal{M}}_{g,n})$ is nonzero if and only if $g = 1$ and $n \geq 11$.*

For $g = 1$ and $n \geq 11$, $H^{11}(\overline{\mathcal{M}}_{1,n})$ is isomorphic to a direct sum of $\binom{n-1}{10}$ copies of $H^{11}(\overline{\mathcal{M}}_{1,11})$ [14]; in particular, it decomposes as $H^{11,0}(\overline{\mathcal{M}}_{1,n}) \oplus H^{0,11}(\overline{\mathcal{M}}_{1,n})$. We show that $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ is generated by the pullbacks of the distinguished generator of $H^{11,0}(\overline{\mathcal{M}}_{1,11})$, which corresponds to the weight 12 cusp form Δ for $\mathrm{SL}_2(\mathbb{Z})$, under the $\binom{n}{11}$ forgetful maps and describe the relations among these generators. In this way, we show that $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ is an irreducible $\mathbb{S}_n$-representation isomorphic to the Specht module $V_{n-10,1^{10}}$.

Next, we address the Hodge structures and Galois representations that appear in other low degrees. The Langlands program predicts that the cohomology of $\overline{\mathcal{M}}_{g,n}$ should be pure Hodge–Tate in all even degrees less than or equal to 20. This prediction was previously confirmed only in the cases where these cohomology groups are known to be generated by tautological classes, for example, for $g \leq 2$ [18, 14, 23], for $k \leq 2$ [1], and for $k = 4$ and $g \geq 8$ [25]. Our second result extends the confirmation of this prediction to a much wider range of cases. The proof is a double induction on g and n. The base cases

are given by recent results of the first two authors, who showed that $H^*(\overline{\mathcal{M}}_{g,n})$ is tautological for $g \geq 3$ and $2g + n \leq 14$ [5].

Theorem 1.2. *For any even $k \leq 12$, the cohomology group $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge–Tate.*

It remains an open problem whether $H^k(\overline{\mathcal{M}}_{g,n})$ is generated by algebraic cycle classes for even $4 \leq k \leq 12$.

As an application of these two theorems, we show that the point count $\#\overline{\mathcal{M}}_{g,n}(\mathbb{F}_q)$ is surprisingly well approximated by a polynomial in q.

Corollary 1.3. *Assume $g \geq 2$, and let $d = \dim \overline{\mathcal{M}}_{g,n}$. Then*

$$\#\overline{\mathcal{M}}_{g,n}(\mathbb{F}_q) = \sum_{i=0}^{6} h^{2i}(\overline{\mathcal{M}}_{g,n}) q^{d-i} + O(q^{d-13/2}).$$

Remark 1.4. The point count $\#\overline{\mathcal{M}}_{0,n}(\mathbb{F}_q)$ is a polynomial in q, for all n, as is $\#\overline{\mathcal{M}}_{1,n}(\mathbb{F}_q)$ for $n \leq 10$. For $n \geq 11$, $\#\overline{\mathcal{M}}_{1,n}(\mathbb{F}_q)$ was determined by Getzler [14]; it has an approximation to order $O(q^{n-13/2})$ by a polynomial in q minus the correction term $\binom{n-1}{10}\tau(q)$, where $\tau(q)$ denotes the coefficient of q in the Fourier expansion of the weight 12 cusp form Δ for $\mathrm{SL}_2(\mathbb{Z})$.

Unlike the cohomology groups in smaller odd degrees, $H^{13}(\overline{\mathcal{M}}_{g,n})$ is nonvanishing in a wide range of cases, including for large g, as are all higher degree odd cohomology groups. Indeed, Pikaart showed that $H^{13}(\overline{\mathcal{M}}_{g,n})$ is nonvanishing for $n \geq 10$ and g sufficiently large, as is $H^{33}(\overline{\mathcal{M}}_g)$ [24]. Similar nonvanishing statements in higher degrees follow immediately, by Hard Lefschetz. The bounds on g that come from Pikaart's method are large, typically in the thousands. For instance, van Zelm computes that Pikaart's method yields $H^{33}(\overline{\mathcal{M}}_{g,1}) \neq 0$ for $g \geq 8069$. The bounds for $H^{13}(\overline{\mathcal{M}}_{g,10})$ and $H^{33}(\overline{\mathcal{M}}_g)$ are not explicitly stated in the literature, but there is substantial evidence that such bounds should be far from optimal. While Pikaart's constructions prove the existence of nontautological algebraic cycle classes on $\overline{\mathcal{M}}_g$ for $g \geq 16192$, van Zelm proved that this holds for $g \geq 12$ [28]. Also, Bergström and Faber have recently shown that $H^{13}(\overline{\mathcal{M}}_{2,n}) \neq 0$ for $n \geq 10$. They also prove that the nonvanishing of $H^{13}(\overline{\mathcal{M}}_{3,n})$ for $n \geq 10$ follows from conjectural parts of the Langlands correspondence. Here, we prove the latter statement unconditionally, extend it to all higher genera, and also improve Pikaart's bound for the nonvanishing of $H^{33}(\overline{\mathcal{M}}_g)$.

Theorem 1.5. *Assume $k \leq 11$. Let $g_1, \ldots, g_k$ be distinct positive integers, and set $g = 1 + g_1 + \cdots + g_k$. Then*

$$H^{11+2k}(\overline{\mathcal{M}}_{g,n}) \neq 0 \text{ for } n \geq 11 - k.$$

In particular, $H^{13}(\overline{\mathcal{M}}_{g,n}) \neq 0$ for $g \geq 2$ and $n \geq 10$, and $H^{33}(\overline{\mathcal{M}}_g) \neq 0$ for $g \geq 67$.

As a further application, we prove the existence of nontautological classes in the Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ in a number of cases where this was not previously known.

Corollary 1.6. *Consider $\overline{\mathcal{M}}_{g,n}$ as a stack over $\mathbb{C}$. For any (g,n) as in Theorem 1.5, the quotient $A^*(\overline{\mathcal{M}}_{g,n})/R^*(\overline{\mathcal{M}}_{g,n})$ is uncountable, as is the subgroup of $A^*(\overline{\mathcal{M}}_{g,n})$ generated by cycles algebraically equivalent to zero.*

This provides many new examples of (g,n) for which $A^*(\overline{\mathcal{M}}_{g,n})$ is not tautological. In particular, the existence of nontautological Chow classes is new for $(4,n)$ with $9 \leq n \leq 16$; for $(5,n)$ with $9 \leq n \leq 13$; for $(6,n)$ with $9 \leq n \leq 11$ and for $(7,8)$ and $(7,9)$.

Existence results for nontautological classes come in two flavors. There are cases where one can write down *explicit* examples of nontautological Chow classes. Graber and Pandharipande gave the first such example when $(g,n) = (2,20)$ [17]. Van Zelm generalized their example to show the existence

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of explicit nontautological Chow classes on $\overline{\mathcal{M}}_{g,n}$ for $2g+n \geq 24$ and $g \geq 2$ [28]. A nice feature of these examples is that they are nontautological in both Chow and cohomology. There are also the *inexplicit* nontautological Chow classes, which arise from the existence of odd cohomology. The first such examples are for $g = 1, n \geq 11$, where the existence of a holomorphic 11-form implies that $A_0(\overline{\mathcal{M}}_{1,n})$ is infinite dimensional [26]. Bergström and Faber showed that there is odd cohomology on $\overline{\mathcal{M}}_{2,n}$ for $n \geq 10$ [3], implying that there are nontautological Chow classes as well, by results of Kimura and Totaro [19, 27]; see Theorem 7.1. The examples provided by Corollary 1.6 are also of this inexplicit form. We do not know whether all of the nontautological classes in these inexplicit cases are homologically equivalent to zero.

1.1. Methods

Arbarello and Cornalba introduced an inductive method for studying cohomology groups of $\overline{\mathcal{M}}_{g,n}$ and applied this to prove the vanishing of $H^k(\overline{\mathcal{M}}_{g,n})$ for $k \in \{1,3,5\}$ [1]. The same method was used to prove vanishing for $k \in \{7,9\}$ after establishing the additional base cases needed to run the induction, via point counting over finite fields [4]. Our proof of Theorems 1.1 and 1.2 start from the observation that the same induction can be used to control the Hodge structures and Galois representations that appear in $H^k(\overline{\mathcal{M}}_{g,n})$ even when these groups do not vanish. The first two authors recently established the base cases needed for $k \leq 12$ [5]. Running this induction when k is even leads directly to Theorem 1.2. Doing so for $k = 11$ shows that $H^{11}(\overline{\mathcal{M}}_{g,n})$ injects into a direct sum of copies of $H^{11}(\overline{\mathcal{M}}_{1,11})$ (Proposition 3.5). This is enough to confirm the prediction that $H^{11}(\overline{\mathcal{M}}_{g,n})$ vanishes whenever $\overline{\mathcal{M}}_{g,n}$ is unirational, but a different argument is needed to prove that it vanishes whenever $g \geq 2$.

The Arbarello–Cornalba induction uses the excision sequence for the pair of $\overline{\mathcal{M}}_{g,n}$ with its boundary $\partial\mathcal{M}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$, along with the map

$$H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\widetilde{\partial\mathcal{M}}_{g,n}) \tag{1.1}$$

given by pullback to the normalization of the boundary. Our proof of Theorem 1.1 uses the observation that the map (1.1) is the first arrow in a natural chain complex whose jth term may be identified with the cohomology of the normalization of the closure of the codimension j boundary strata with coefficients in a natural local system, the determinant of the permutation representation on the branches of the boundary divisor. This complex has several natural interpretations: It is the kth weight-graded piece of the Feynman transform of the modular operad that takes the value $\mathbb{Q}$ for every (g,n) [16]. It is also the weight k row in the E_1-page of a natural spectral sequence obtained via Poincaré duality from Deligne's weight spectral sequence for the pair $(\overline{\mathcal{M}}_{g,n}, \partial\mathcal{M}_{g,n})$ and has a natural interpretation as a decorated graph complex [20, Section 2.3].

For $k = 11$, we examine the first two maps in this complex. Assuming that $H^{11}(\overline{\mathcal{M}}_{2,n})$ vanishes for all n, a double induction on g and n shows that $H^{11}(\overline{\mathcal{M}}_{g,n})$ vanishes whenever $g > 2$. In Section 4.3, we prove the needed base cases, that is, the vanishing of $H^{11}(\overline{\mathcal{M}}_{2,n})$ for all n, by explicit calculations using the generators and relations for $H^{11}(\overline{\mathcal{M}}_{1,n})$. It should also be possible to deduce these base cases from results of Petersen [23, 22]; see Remark 4.3.

1.2. Structure of the paper

In Section 2, we recall how to extract $H^{11}(\overline{\mathcal{M}}_{1,n})$ with its Hodge structure or Galois representation from the work of Getzler [14]. We then describe generators and relations for this group and describe the $\mathbb{S}_n$-action and the pullback under tautological morphisms in terms of these generators. In Section 3, we recall the inductive method of Arbarello and Cornalba and use it to prove Theorem 1.2. In Section 4, we present the inductive argument for vanishing of $H^{11}(\overline{\mathcal{M}}_{g,n})$ for $g > 2$, using the weight spectral sequence, assuming the vanishing for $g = 2$. We then prove the vanishing in the necessary base cases,

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for $g = 2$, using the explicit generators and relations for $g = 1$. In Sections 5, 6 and 7, we prove Corollary 1.3, Theorem 1.5 and Corollary 1.6, respectively.

1.3. Notations and conventions

We denote by $H^k(\overline{\mathcal{M}}_{g,n})$ either the $\mathbb{Q}$-Hodge structure $H^k((\overline{\mathcal{M}}_{g,n})_{\mathbb{C}}, \mathbb{Q})$ or the absolute Galois representation $H^k_{\text{ét}}((\overline{\mathcal{M}}_{g,n})_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. We write L for the Tate motive. We say that $H^k(\overline{\mathcal{M}}_{g,n})$ is *pure Hodge–Tate* if it is isomorphic to a direct sum of the Betti or ℓ-adic realizations of powers of L. We write S_{12} for the motive associated to the weight 12 modular form Δ, whose Betti and ℓ-adic realizations are $H^{11}(\overline{\mathcal{M}}_{1,11})$. We denote by $A^*(X)$ the Chow ring with rational coefficients of a variety or Deligne–Mumford stack X.

2. Genus 1

In this section, we give explicit generators and relations for $H^{11}(\overline{\mathcal{M}}_{1,n})$ and describe the $\mathbb{S}_n$-action and the pullback to boundary divisors in terms of these generators. These formulas will be used in Section 4.3.2, in our proof that $H^{11}(\overline{\mathcal{M}}_{2,n}) = 0$.

2.1. Dimension and Hodge structure

We start by explaining how to extract $H^{11}(\overline{\mathcal{M}}_{1,n})$ with its Hodge structure or Galois representation from [14]. Getzler gives generating functions that determine the cohomology groups $H^*(\overline{\mathcal{M}}_{1,n})$ with their $\mathbb{S}_n$-actions. These formulas simplify substantially when forgetting the $\mathbb{S}_n$-action, so we begin by using Getzler's formula to extract $H^{11}(\overline{\mathcal{M}}_{1,n})$ nonequivariantly. Below, as in [14], we write L for the Tate motive and S_{2k+2} for the Hodge structure associated to the space of cusp forms of weight $2k+2$ (see [14, p. 489] for definition). We note that $\mathsf{S}_{2k+2} = 0$ for $k \leq 4$.

Lemma 2.1. *The cohomology group $H^{11}(\overline{\mathcal{M}}_{1,n})$ is a direct sum of $\binom{n-1}{10}$ copies of S_{12}.*

Proof. Let $\mathsf{e}^{\mathbb{S}_n}(\mathcal{M})$ denote the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{M}$ in the Grothendieck ring of equivariant mixed Hodge structures. Getzler defines two families of generating functions $\mathbf{a}_i = \sum \mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{i,n})$ and $\mathbf{b}_i = \sum \mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{i,n})$. For $i = 0$ or 1, these generating functions are power series in the Hodge structures L and S_{2k+2} whose coefficients are symmetric functions.

To get to the ordinary Euler characteristic generating function $\mathsf{e}(\mathcal{M})$ from $\mathsf{e}^{\mathbb{S}_n}(\mathcal{M})$, we apply Getzler's rk functor, which is defined by setting the power sums to $p_1 = x$ and $p_n = 0$ for $n > 0$. It sends $\mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{i,n})$ to $\mathsf{e}(\overline{\mathcal{M}}_{i,n})\frac{x^n}{n!}$ [14, p. 484]. We shall write $a_i = \mathrm{rk}(\mathbf{a}_i)$ and $b_i = \mathrm{rk}(\mathbf{b}_i)$, which are power series in L and S_{2k+2} when $i = 0$ or 1. For example,

$$b_0 = \mathrm{rk}(\mathbf{b}_0) = \sum_{n \geq 3} \mathsf{e}(\overline{\mathcal{M}}_{0,n}) \frac{x^n}{n!}.$$

Important for us is that

$$1 + x + b_0' = e^x + \text{terms divisible by } \mathsf{L}.$$

For b_1, we are interested in the coefficient of $\frac{x^n}{n!}\mathsf{S}_{12}$, which is equal to the negative of the multiplicity of S_{12} in $H^{11}(\overline{\mathcal{M}}_{1,n})$ by construction. Applying rk to [14, Theorem 2.5] relates b_1 to a_1. Note that the symbol $\circ$ in [14, Theorem 2.5] denotes plethysm of symmetric functions; applying rk turns this plethysm into composition of functions, as can be seen from the properties characterizing plethysm in [13, Section 5.2]. In particular, we find

$$b_1 = a_1(x + b_0') + \text{terms built from } a_0. \tag{2.1}$$

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The terms built from a_0 cannot contribute to the coefficient of S_{12}. (We note that there is a small error in [14, Theorem 2.5], which is corrected in [9, p. 306], but it occurs in these terms built from a_0, and thus will not affect the outcome of our calculation.)

To expand the right-hand side of equation (2.1), apply rk to the equation for $\mathbf{a}_1$ in [14, p. 489], and then plug in $x + b_0'$ for x (so substitute $p_1 = x + b_0'$ and $p_n = 0$ for $n > 1$):

$$a_1(x + b_0') = \operatorname{res}_0\left[\frac{(1 + x + b_0')^{1-\omega-\mathsf{L}/\omega+\mathsf{L}} - 1}{1 - \omega - \mathsf{L}/\omega + \mathsf{L}} \times \left(\sum_{k=1}^{\infty}\left(\frac{\mathsf{S}_{2k+2} + 1}{\mathsf{L}^{2k+1}}\right)\omega^{2k} - 1\right)(\omega - \mathsf{L}/\omega)d\omega\right].$$

In the middle parenthesized term, S_{12} is multiplied by $\omega^{10}/\mathsf{L}^{11}$. Since we need to take the residue at 0 with respect to ω, the coefficient of S_{12} appears when the other terms combine to give $\mathsf{L}^{11}/\omega^{11}$. To get $\mathsf{L}^{11}/\omega^{11}$, we must use the $-\mathsf{L}/\omega$ piece of the $(\omega - \mathsf{L}/\omega)$ term. Similarly, when we expand the first term, only the powers of L/ω are relevant. From this, we see that the coefficient of S_{12} in the above display is the negative of the coefficient of y^{10} in

$$\frac{(1 + x + b_0')^{1+y}}{1 + y} = \frac{e^{x(1+y)}}{1 + y} + \langle \mathsf{L} \rangle = \sum_{n=1}^{\infty} \frac{x^n}{n!}(y + 1)^{n-1} + \langle \mathsf{L} \rangle.$$

In conclusion, $H^{11}(\overline{\mathcal{M}}_{1,n})$ consists of $\binom{n-1}{10}$ copies of S_{12}. □

Getzler's formulas also encode the $\mathbb{S}_n$-action on $H^{11}(\overline{\mathcal{M}}_{1,n})$. We recover this information in a different way, by describing generators on which the $\mathbb{S}_n$-action is evident, as follows.

2.2. Generators and their pullbacks

To begin, in the case $n = 11$, Lemma 2.1 tells us

$$H^{11}(\overline{\mathcal{M}}_{1,11}) \otimes \mathbb{C} = H^{11,0}(\overline{\mathcal{M}}_{1,11}) \oplus H^{0,11}(\overline{\mathcal{M}}_{1,11}).$$

The weight 12 cusp form of $\mathrm{SL}_2(\mathbb{Z})$ gives rise to a distinguished generator $\omega \in H^{11,0}(\overline{\mathcal{M}}_{1,11})$; see [12, p. 14] for an explicit geometric construction. It is evident from this construction (or from [14]) that $\mathbb{S}_{11}$ acts by the sign representation.

We now describe a natural collection of forms in $H^{11,0}(\overline{\mathcal{M}}_{1,n})$, which we will soon see are generators. These forms come from pulling back the distinguished generator of $H^{11,0}(\overline{\mathcal{M}}_{1,11})$ under the various forgetful maps $\overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,11}$. Precisely, given an ordered subset $A \subset \{1, \dots, n\}$ with $|A| = 11$, write $f_A \colon \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,A} \cong \overline{\mathcal{M}}_{1,11}$ for the projection map and define $\omega_A := f_A^*\omega \in H^{11,0}(\overline{\mathcal{M}}_{1,n})$.

The pullbacks of these forms to boundary divisors follow a simple rule. By the Künneth formula, the only boundary divisors of $\overline{\mathcal{M}}_{1,n}$ with nonzero H^{11} are those of the form

$$D_B = \overline{\mathcal{M}}_{1,B\cup p} \times \overline{\mathcal{M}}_{0,B^c\cup q}, \tag{2.2}$$

where $|B| \geq 10$ and $\iota_B \colon D_B \to \overline{\mathcal{M}}_{1,n}$ is the map that glues p to q. In this case, projection onto the first factor $\mathrm{pr}_1 \colon D_B \to \overline{\mathcal{M}}_{1,B\cup p}$ induces an isomorphism

$$\mathrm{pr}_1^* \colon H^{11}(\overline{\mathcal{M}}_{1,B\cup p}) \xrightarrow{\sim} H^{11}(D_B). \tag{2.3}$$

Given ordered subsets A and B of $\{1, \dots, n\}$ with $|A| = 11$ and $|A \cap B| = 10$, there is a unique element $i \in A$ such that $i \notin B$. Let $\epsilon(A)$ denote the ordered set obtained from A by replacing i with p, so $\epsilon(A)$ is a subset of B. If $|A \cap B| = 11$ so that A is already contained in B, then we set $\epsilon(A) = A$.

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Lemma 2.2. *Given a boundary divisor $\iota_B \colon D_B \to \overline{\mathcal{M}}_{1,n}$, and $A \subset \{1, \dots, n\}$ with $|A| = 11$, we have*

$$\iota_B^* \omega_A = \begin{cases} \mathrm{pr}_1^* \omega_{\epsilon(A)} & \text{if } |A \cap B| \geq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, suppose $|A \cap B| \leq 9$, so $|A \cap B^c| \geq 2$. Then there is a commutative diagram

$$\begin{array}{ccc} D_B & \xrightarrow{\iota_B} & \overline{\mathcal{M}}_{1,n} \\ \downarrow & & \downarrow f_A \\ \overline{\mathcal{M}}_{1,(A\cap B)\cup p} \times \overline{\mathcal{M}}_{0,(A\cap B^c)\cup q} & \longrightarrow & \overline{\mathcal{M}}_{1,A}, \end{array} \tag{2.4}$$

where the horizontal maps glue p to q and the vertical maps forget markings not in A. In this case, the image of $f_A \circ \iota_B$ is a proper boundary divisor in $\overline{\mathcal{M}}_{1,A}$, which has no holomorphic 11-forms. Hence, the pullback of the generator of $H^{11,0}(\overline{\mathcal{M}}_{1,A})$ to D_B vanishes.

Now, suppose $|A \cap B| \geq 10$, so $|A \cap B^c| \leq 1$. Then, the lower left hand term in diagram (2.4) must be replaced by $\overline{\mathcal{M}}_{(A\cap B)\cup p}$. Thus, there is another commutative diagram

$$\begin{array}{ccc} D_B & \xrightarrow{\iota_B} & \overline{\mathcal{M}}_{1,n} \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow f_A \\ \overline{\mathcal{M}}_{1,B\cup p} & \xrightarrow[f_A]{} & \overline{\mathcal{M}}_{1,A}, \end{array} \tag{2.5}$$

where if $|A \cap B| = 10$, we identify p with the unique symbol of A not contained in B. □

2.3. *Relations and the $\mathbb{S}_n$-action*

The group $\mathbb{S}_n$ acts on the subsets $A \subset \{1, \dots, n\}$ and correspondingly on the subspace of $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ generated by the ω_A. Note that, for any permutation σ in the subgroup of $\mathbb{S}_n$ fixing A, we have $\omega_{\sigma(A)} = \mathrm{sign}(\sigma)\omega_A$.

To identify our representation, we briefly recall some of the combinatorial objects that arise in the representation theory of $\mathbb{S}_n$.

A *tabloid* is an equivalence class of tableaux, which identifies tableaux up to reordering rows. Given a tableau T, we write $\{T\}$ for the corresponding tabloid. Given a partition λ of n, we denote by M_λ the vector space with basis given by tabloids of shape λ. The *Specht module generator* associated to a tableau T is the vector

$$\sum_{\sigma \in C_T} \mathrm{sign}(\sigma)\{\sigma(T)\} \in M_\lambda, \tag{2.6}$$

where $C_T \subset \mathbb{S}_n$ is the subgroup that preserves the columns of T setwise. The subspace of M_λ generated by the vectors (2.6) as T runs over all tableaux is an irreducible representation called the *Specht module* V_λ. The Specht module generators associated to the standard tableaux on λ form a basis for V_λ.

To each ordered subset $A \subset \{1, \dots, n\}$ of size 11, we associate a tableau T_A of shape $(n-10, 1^{10})$ which has the symbols of A in order down the first column and the rest of the first row filled in increasing order.

Proposition 2.3. *There is an $\mathbb{S}_n$-equivariant isomorphism $H^{11,0}(\overline{\mathcal{M}}_{1,n}) \xrightarrow{\sim} V_{n-10,1^{10}}$ taking ω_A to the Specht module generator associated to T_A.*

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Proof. The dimension of the Specht module $V_{n-10,1^{10}}$ is the number of standard tableaux of shape $(n-10,1^{10})$, which is $\binom{n-1}{10}$. Therefore, by Lemma 2.1, it will suffice to give a $\mathbb{S}_n$-equivariant map from the subspace of $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ generated by $\{\omega_A\}$ to $V_{n-10,1^{10}}$ that takes ω_A to the Specht module generator associated to A.

We are going to study the image of subspace generated by $\{\omega_A\}$ under the pullback map

$$H^{11,0}(\overline{\mathcal{M}}_{1,n}) \to \bigoplus_{|B|=10} H^{11,0}(D_B), \tag{2.7}$$

where B is an ordered subset of $\{1,\dots,n\}$ and D_B is as in equation (2.2). Let W_B be the element in the target of the map (2.7) which has component $\mathrm{pr}_1^*\omega \in H^{11,0}(D_B)$ and 0 in all other components. The collection of ordered subsets B of size 10 is in bijection with tabloids on $(n-10,1^{10})$, where B^c fills the row and B (in order) fills the column. Thus, the right-hand side is identified with the vector space of tabloids $M_{n-10,1^{10}}$; given a tabloid $\{T\}$ corresponding to B we write $W_{\{T\}} = W_B$.

Fix $A \subset \{1,\dots,n\}$ with $|A| = 11$. If $B \subset A$, with $|B| = 10$, then A is a permutation of $i \cup B$ for some $i \notin B$. Let $\sigma_{A\to B} : i \cup B \to A$ denote the corresponding permutation. By Lemma 2.2, the image of ω_A under the map (2.7) is

$$\omega_A \mapsto \sum_{B\subset A} \mathrm{sign}(\sigma_{B\to A}) W_B = \sum_{\sigma\in S_A} \mathrm{sign}(\sigma) W_{\{\sigma(T_A)\}},$$

which is the Specht module generator defined in the expression (2.6), and the proposition follows. □

Corollary 2.4. *The forms $\{\omega_A : 1 \in A, A \text{ increasing}\}$ form a basis for $H^{11}(\overline{\mathcal{M}}_{1,n})$.*

Corollary 2.5. *The pullback map $H^{11,0}(\overline{\mathcal{M}}_{1,n}) \to \bigoplus_{|B|=10} H^{11,0}(D_B)$ is injective.*

Remark 2.6. Dan Petersen suggested an alternate method to obtain several of the results in this section, which avoids the manipulations with generating functions in Lemma 2.1. We sketch his argument here. Let $f^1 : \mathcal{E} \to \mathcal{M}_{1,1}$ denote the universal elliptic curve and $\mathcal{E}^n$ denote the n-fold fiber product of $\mathcal{E}$ with itself over $\mathcal{M}_{1,1}$. Note that $\mathcal{M}_{1,n}$ is an open substack of $\mathcal{E}^{n-1}$. By the long exact sequences for the pairs $(\overline{\mathcal{M}}_{1,n}, \mathcal{M}_{1,n})$ and $(\mathcal{E}^{n-1}, \mathcal{M}_{1,n})$, we see that there are natural isomorphisms

$$W_{11}H^{11}(\mathcal{E}^{n-1}) \cong W_{11}H^{11}(\mathcal{M}_{1,n}) \cong H^{11}(\overline{\mathcal{M}}_{1,n}).$$

One can then study the Leray spectral sequence for the smooth morphism $f^{n-1} : \mathcal{E}^{n-1} \to \mathcal{M}_{1,1}$. Let $\mathbb{V}$ denote the local system $R^1 f^1_* \mathbb{Q}$. Then by the Künneth formula,

$$Rf^{n-1}_*\mathbb{Q} \cong (Rf^1_*\mathbb{Q})^{\otimes n-1} \cong (\mathbb{Q} \oplus \mathbb{V}[-1] \oplus \mathbb{Q}(-1)[-2])^{\otimes n-1}.$$

The pure cohomology $W_{11}H^{11}(\mathcal{E}^{n-1})$ arises from the $\binom{n-1}{10}$ summands $\mathbb{V}[-1]^{\otimes 10}$, each of which gives a copy of S_{12}. This gives Lemma 2.1. To identify the $\mathbb{S}_n$ representation as in Proposition 2.3, one first notes that as an $\mathbb{S}_{n-1}$ representation, we have

$$W_{11}H^{11}(\mathcal{E}_{n-1}) \cong \mathsf{S}_{12} \otimes \mathrm{Ind}_{\mathbb{S}_{10}\times\mathbb{S}_{n-11}}^{\mathbb{S}_{n-1}} (\mathrm{sgn} \boxtimes \mathrm{triv}).$$

By the Pieri formula and the branching rule for the symmetric group, it follows that as an $\mathbb{S}_n$ representation,

$$W_{11}H^{11}(\mathcal{E}_{n-1}) \cong \mathsf{S}_{12} \otimes V_{n-10,1^{10}}.$$

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3. Applying the Arbarello–Cornalba induction

We start by recalling the inductive method of Arbarello and Cornalba [1], by excision of the boundary and pullback to its normalization. We then apply this method to prove Theorem 1.2 and a preliminary proposition about the degree 11 cohomology of $\overline{\mathcal{M}}_{g,n}$.

3.1. *Restricting to boundary divisors*

Consider the excision long exact sequence associated to the boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$:

$$\cdots \to H^k_c(\mathcal{M}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n}) \to H^{k+1}_c(\mathcal{M}_{g,n}) \to \cdots .$$

Note that this sequence is in fact a long exact sequence of mixed Hodge structures or ℓ-adic Galois representations. In particular, when $H^k_c(\mathcal{M}_{g,n}) = 0$, there is an injective morphism

$$H^k(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^k(\partial\mathcal{M}_{g,n}). \tag{3.1}$$

Let $\widetilde{\partial\mathcal{M}}_{g,n}$ denote the normalization of $\partial\mathcal{M}_{g,n}$. Arbarello and Cornalba improve on the injectivity of the map (3.1), as follows.

Lemma 3.1 (Lemma 2.6 of [1]). *Suppose $H^k_c(\mathcal{M}_{g,n}) = 0$. Then the pullback*

$$H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\widetilde{\partial\mathcal{M}}_{g,n})$$

is injective.

For fixed k, the following proposition gives vanishing of compactly supported cohomology in all but an explicit finite collection of cases.

Proposition 3.2 (Proposition 2.1 of [4]). *Assume $g \geq 1$.*

$$H^k_c(\mathcal{M}_{g,n}) = 0 \text{ for } \begin{cases} k < 2g \text{ and } n = 0, 1 \\ k < 2g - 2 + n \text{ and } n \geq 2. \end{cases}$$

3.2. *The case of even degrees $k \leq 12$*

Let $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ be the tautological cohomology ring. Tautological classes are algebraic and defined over $\mathbb{Z}$, so if $H^k(\overline{\mathcal{M}}_{g,n}) = RH^k(\overline{\mathcal{M}}_{g,n})$, then $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge–Tate. The next lemma provides the necessary base cases for the inductive argument.

Lemma 3.3. *If $2g - 2 + n \leq 12$ and $k \leq 12$ is even, then $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge–Tate.*

Proof. For $g = 0$ and any n, all cohomology is tautological [18], as is all *even* cohomology for $g = 1$ [21], and for $g = 2$ and $n < 20$ [23]. Finally, for $g \geq 3$, all cohomology is tautological for $2g - 2 + n \leq 12$ [5, Theorem 1.4]. □

Proof of Theorem 1.2. We induct on g and n. By Lemma 3.3, we can assume $2g - 2 + n > k$. By Lemma 3.2, we have an injection

$$H^k(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^k(\widetilde{\partial\mathcal{M}}_{g,n}). \tag{3.2}$$

Each component of $\widetilde{\partial\mathcal{M}}_{g,n}$ is a quotient by a finite group of $\overline{\mathcal{M}}_{g-1,n+2}$ or $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$, where $g_1 + g_2 = g$, $n_1 + n_2 = n$, and $2g_i - 2 + n_i > 0$. By induction on g, we know $H^k(\overline{\mathcal{M}}_{g-1,n+2})$ is pure

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Hodge–Tate. Meanwhile, note that $H^j(\overline{\mathcal{M}}_{g',n'}) = 0$ for all (g', n') and odd $j \leq k/2 \leq 6$ by [1]. Hence, the Künneth formula shows that

$$H^k(\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}) = \bigoplus_{i=0}^{k/2} H^{2i}(\overline{\mathcal{M}}_{g_1,n_1+1}) \otimes H^{k-2i}(\overline{\mathcal{M}}_{g_2,n_2+1}). \tag{3.3}$$

Inductively, we know that the right-hand side of equation (3.3) is pure Hodge–Tate. Thus, the map (3.2) is an injection of $H^k(\overline{\mathcal{M}}_{g,n})$ into a Hodge structure or Galois representation that is pure Hodge–Tate, and it follows that $H^k(\overline{\mathcal{M}}_{g,n})$ is pure Hodge–Tate as well. □

3.3. The case of degree $k = 11$

The base cases required to run an analogous induction for $k = 11$ are those where $H^{11}(\overline{\mathcal{M}}_{g,n})$ does not inject into $H^{11}(\widetilde{\partial\mathcal{M}}_{1,11})$. Recent results of the first two authors rule out any such bases cases with $g \geq 2$.

Lemma 3.4. *If $g \geq 2$, then $H^{11}(\overline{\mathcal{M}}_{g,n}) \to H^{11}(\widetilde{\partial\mathcal{M}}_{g,n})$ is injective.*

Proof. If $2g - 2 + n > 11$, this follows by combining Lemma 3.1 and Proposition 3.2. If $2g - 2 + n \leq 11$ and $g \geq 2$, then [5, Theorem 1.4] shows $H^{11}(\overline{\mathcal{M}}_{g,n}) = 0$. □

Proposition 3.5. *For any g, there is an injection*

$$H^{11}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow \bigoplus H^{11}(\overline{\mathcal{M}}_{1,11}).$$

Proof. The result is known for $g \leq 1$, so we may assume $g \geq 2$. By Lemma 3.4, we have an injection

$$H^{11}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^{11}(\widetilde{\partial\mathcal{M}}_{g,n}).$$

Each component of $\widetilde{\partial\mathcal{M}}_{g,n}$ is a quotient by a finite group of $\overline{\mathcal{M}}_{g-1,n+2}$ or $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$, where $k_1 + k_2 = n$, $2g_i - 2 + n_i > 0$. Because $H^k(\overline{\mathcal{M}}_{g',n'}) = 0$ for all (g', n') and $k = 1, 3, 5, 7, 9$ [1, 4], the Künneth formula shows that

$$H^{11}(\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}) = H^{11}(\overline{\mathcal{M}}_{g_1,n_1+1}) \oplus H^{11}(\overline{\mathcal{M}}_{g_2,n_2+1}). \tag{3.4}$$

Note that either $g_1 < g$ or $g_1 = g$ and $n_1 + 1 < n$, and analogously for (g_2, n_2). Therefore, there is an injective morphism of Hodge structures from $H^{11}(\overline{\mathcal{M}}_{g,n})$ into a direct sum of Hodge structures of the form $H^{11}(\overline{\mathcal{M}}_{\gamma,\nu})$, where $\gamma < g$ or $\gamma = g$ and $\nu < n$. By double induction on g and n and using that $H^{11}(\overline{\mathcal{M}}_{0,n}) = 0$ and $H^{11}(\overline{\mathcal{M}}_{1,n}) = 0$ for $n \leq 10$, we conclude that there is an injective morphism of Hodge structures or ℓ-adic Galois representations

$$H^{11}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow \bigoplus H^{11}(\overline{\mathcal{M}}_{1,11}).$$ □

4. An induction via the weight spectral sequence

In this section, we prove the vanishing of $H^{11}(\overline{\mathcal{M}}_{g,n})$ for $g \geq 2$. We do so by identifying the injection $H^{11}(\overline{\mathcal{M}}_{g,n}) \hookrightarrow H^{11}(\widetilde{\partial\mathcal{M}}_{g,n})$ (Lemma 3.4) as the first map in a complex and showing that the next map in the complex is injective. The complex we use is obtained from the E_1-page in the weight spectral sequence associated to the compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ [10], via Poincaré duality. It is also the weight 11 summand of the Feynman transform of the modular operad $H^*(\overline{\mathcal{M}}_{g,n})$ [16], and therefore has a natural graph complex interpretation [20, Section 2.3].

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4.1. The first two maps in the weight k complex

Let Γ denote a stable n-marked graph of genus g; the underlying graph is connected, each vertex v is labeled by an integer g_v and the valence of the vertex v is denoted n_v. The stability condition is that $2g_v + n_v - 2 > 0$. Set $\overline{\mathcal{M}}_\Gamma = \prod_v \overline{\mathcal{M}}_{g_v,n_v}$. There is the natural gluing map

$$\xi_\Gamma \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}.$$

The normalization of $\xi_\Gamma(\overline{\mathcal{M}}_\Gamma)$ is isomorphic to $\overline{\mathcal{M}}_\Gamma/\mathrm{Aut}(\Gamma)$. Thus, the first map in the weight k complex, the pullback of H^k to the normalization of the boundary, can be rewritten as:

$$H^k(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\alpha} \bigoplus_{|E(\Gamma)|=1} H^k(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}\,\Gamma}. \tag{4.1}$$

The target of the jth map in the weight k complex is $\bigoplus_{|E(\Gamma)|=j}(H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\mathrm{Aut}\,\Gamma}$. Here, $\det E(\Gamma)$ denotes the determinant of the permutation representation of $\mathrm{Aut}(\Gamma)$ acting on the set of edges. We will only need the first and second maps.

Let us describe the second map. We define

$$\bigoplus_{|E(\Gamma)=1|} H^k(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)} \xrightarrow{\beta} \bigoplus_{|E(\Gamma)=2|} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\mathrm{Aut}(\Gamma)} \tag{4.2}$$

as follows. For each graph Γ with two edges, choose an ordering of the edges $E(\Gamma) = \{e_1, e_2\}$ and say e_i corresponds to the node obtained by gluing the marked points p_i and q_i. Let $\phi_i(\Gamma)$ denote the graph with one edge obtained by contracting e_i. Gluing p_i and q_i induces a map $\xi_i \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\phi_i(\Gamma)}$, which in turn gives a map $\xi_i^* \colon H^k(\overline{\mathcal{M}}_{\phi_i(\Gamma)}) \to H^k(\overline{\mathcal{M}}_\Gamma)$. This induces

$$H^{11}(\overline{\mathcal{M}}_{\phi_i(\Gamma)}) \xrightarrow{\xi_i^*} H^{11}(\overline{\mathcal{M}}_\Gamma). \tag{4.3}$$

If $E(\Gamma)$ is a nontrivial representation of $\mathrm{Aut}(\Gamma)$, then $\xi_2 = \xi_1 \circ \sigma$, where $\sigma \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ is the automorphism that simultaneously swaps p_1 with p_2 and q_1 with q_2 (corresponding to the automorphism of Γ that swaps the two edges). In particular, it follows that the image of $\xi_1^* - \xi_2^*$ lies in the subspace $(H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\mathrm{Aut}(\Gamma)} \subset H^k(\overline{\mathcal{M}}_\Gamma)$. Then β is defined by taking the sum over all two-edge graphs. Then $\beta \circ \alpha = 0$ because each component is the difference of the pullbacks under two copies of the same gluing maps.

4.2. Proof of Theorem 1.1, assuming it holds for $g = 2$

We first give a short inductive proof of Theorem 1.1, assuming that $H^{11}(\overline{\mathcal{M}}_{2,n}) = 0$ for all n. Fix $k = 11$ and $g \geq 3$. By Lemma 3.4, the map α in (4.1) is injective. We claim that the map β in (4.2) is also injective. The theorem follows from this claim since $\beta \circ \alpha = 0$.

Consider the domain of β, which is $\bigoplus_{|E(\Gamma)=1|} H^k(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)}$. If Γ has a single vertex with a loop edge, then $H^{11}(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)} = H^{11}(\overline{\mathcal{M}}_{g-1,n+2})^{\mathbb{S}_2}$, which vanishes by induction on g.

Suppose Γ has two vertices joined by an edge. Then $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{a,A\cup p} \times \overline{\mathcal{M}}_{b,A^c\cup q}$. By the Künneth formula and the vanishing of lower degree odd cohomology,

$$H^{11}(\overline{\mathcal{M}}_{a,A\cup p} \times \overline{\mathcal{M}}_{b,A^c\cup q}) = H^{11}(\overline{\mathcal{M}}_{a,A\cup p}) \oplus H^{11}(\overline{\mathcal{M}}_{b,A^c\cup q}). \tag{4.4}$$

Assume $a \geq b$. By induction on g and n, $H^{11}(\overline{\mathcal{M}}_\Gamma)$ vanishes unless $b = 1$ and $|A^c| \geq 10$, in which case it is $H^{11}(\overline{\mathcal{M}}_{1,A^c\cup q})$. Note that, in this case, $\mathrm{Aut}(\Gamma)$ is trivial.

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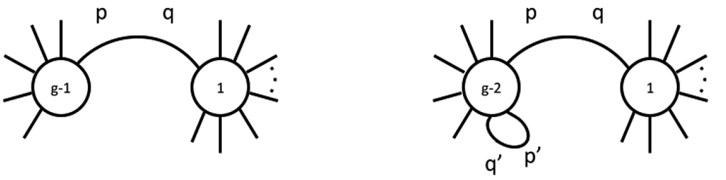

Figure 1. *The graph* Γ *on the left and* Γ' *on the right.*

Let Γ' be the graph obtained by attaching a loop to the vertex of genus $g-1$ (and decreasing its genus accordingly), as shown in Figure 1. As in equation (4.4), we know that $H^{11}(\overline{\mathcal{M}}_{\Gamma'})$ contains $H^{11}(\overline{\mathcal{M}}_{1,A^c\cup q})$ as a summand. Then β maps $H^{11}(\overline{\mathcal{M}}_{\Gamma})$ injectively into this summand of $H^{11}(\overline{\mathcal{M}}_{\Gamma'})$, and it follows that β is injective, as claimed.

4.3. Proof of Theorem 1.1 for $g = 2$

The proof that $H^{11}(\overline{\mathcal{M}}_{g,n}) = 0$ for all n follows a similar strategy to the inductive argument for $g \geq 3$. By Lemma 3.4, we know that α is injective, and we claim that β is also injective. The theorem follows from this claim. However, the proof that β is injective is more involved in this case.

We begin by describing some of the components of β as concretely as possible.

Example 4.1. Suppose Γ is the leftmost graph in Figure 3. In this case, $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ are not isomorphic, so each component of $\beta|_{H^{11}(\overline{\mathcal{M}}_\Gamma)}$ is one of the usual restriction maps, up to sign.

Example 4.2. Suppose Γ is the rightmost graph in Figure 3. In this case, $\phi_1(\Gamma) \cong \phi_2(\Gamma)$. Let $\sigma\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$ be the automorphism that simultaneously swaps p_1 with p_2 and q_1 with q_2 (corresponding to the automorphism of Γ swapping the two edges). Then $\xi_2 = \xi_1 \circ \sigma$. By Proposition 2.3, we know $\sigma^*\colon H^{11}(\overline{\mathcal{M}}_\Gamma) \to H^{11}(\overline{\mathcal{M}}_\Gamma)$ is multiplication by -1. It follows that $\beta|_{H^{11}(\overline{\mathcal{M}}_\Gamma)}$ is again one of the usual restriction maps, up to rescaling.

We will show that, for each 1-edge graph Γ there is a collection of 2-edge graphs $\{\Gamma_i\}$ such that β maps $H^{11}(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)}$ injectively into $\bigoplus_i (H^{11}(\overline{\mathcal{M}}_{\Gamma_i}) \otimes \det(E_{\Gamma_i}))^{\mathrm{Aut}(\Gamma_i)}$. Moreover, we order the one-edge graphs in such a way that, at each step, none of the two-edge graphs $\{\Gamma_i\}$ admit edge contractions to any of the one-edge graphs that came earlier in the order. In this way, we see that β can be represented by a block upper diagonal matrix with injective blocks, and hence is injective, as required.

For $n \leq 9$, the cohomology $H^*(\overline{\mathcal{M}}_{2,n})$ is tautological [5], and hence $H^{11}(\overline{\mathcal{M}}_{2,n}) = 0$, as required. The cases $n = 10$ and 11 can be handled by an argument similar to that used for the cases $n \geq 12$, below, but the details are more involved. Instead, we note that $\overline{\mathcal{M}}_{2,n}$ is rational in these cases [6], and hence, $H^{11,0}(\overline{\mathcal{M}}_{2,n}) = 0$. By Proposition 3.5, it follows that $H^{11}(\overline{\mathcal{M}}_{2,n}) = 0$. For the remainder of this subsection, we therefore assume $n \geq 12$.

There are three types of 1-edge graphs to consider: those with a vertex of genus 2, those with a unique vertex of genus 1 and a self-edge and those with two vertices of genus 1.

4.3.1. Graphs with a genus 2 vertex

Let Γ be a graph with one edge and a genus 2 vertex. Then $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{2,A\cup p} \times \overline{\mathcal{M}}_{0,A^c\cup q}$. Because $|A \cup p| < n$, we see that $H^{11}(\overline{\mathcal{M}}_{2,A\cup p}) = 0$ by induction. By the Künneth formula and the vanishing of lower degree odd cohomology, $H^{11}(\overline{\mathcal{M}}_\Gamma) = 0$.

4.3.2. Graphs with a single genus 1 vertex

Let Γ be the graph with one vertex of genus 1, n legs and one self-edge. Consider $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{1,n+2}$ labeled $\{1,\dots,n,p_1,q_1\}$, where p_1, q_1 are glued together to get curves with dual graph Γ. As in Section 2.2, given a subset $B \subset \{1,\dots,n,p_1,q_1\}$ of cardinality 10, we let $D_B = \overline{\mathcal{M}}_{1,B\cup p_2} \times \overline{\mathcal{M}}_{0,B^c\cup q_2}$. (This is

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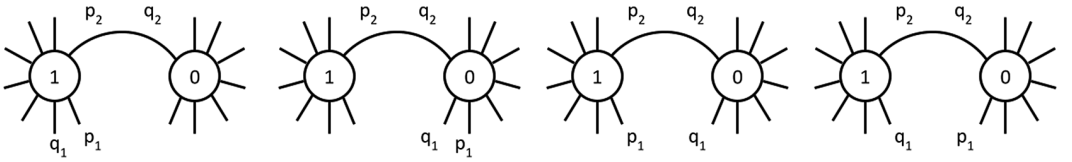

Figure 2. *The four flavors of D_B.*

nonempty because $n \geq 12$.) There are four flavors of subsets B: both of p_1, q_1 are in B; neither p_1 nor q_1 is in B; only p_1 is in B; or only q_1 is in B. These correspond to 4 types of codimension 1 strata under the map $D_B \to \overline{\mathcal{M}}_{1,n+2}$ that glues p_2 and q_2 (see Figure 2).

For each subset B, let Γ_B be the two-edge graph obtained by further gluing p_1 and q_1 (see Figure 3). Note that $\overline{\mathcal{M}}_{\Gamma_B} = D_B$. We want to show that

$$H^{11}(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)} \to \bigoplus_{|B|=10} H^{11}(\overline{\mathcal{M}}_{\Gamma_B} \otimes \det E(\Gamma_B))^{\mathrm{Aut}(\Gamma_B)}$$

is injective.

When we glue p_1 and q_1, the last two types of subsets B give the same type of two-edge graph Γ_B:

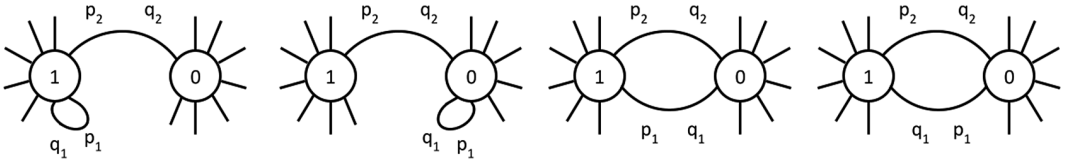

Figure 3. *Gluing p_1 and q_1.*

Swapping p_1 and q_1 preserves the first two maps $D_B \to \overline{\mathcal{M}}_{1,n+2}$ but exchanges the other two. Correspondingly, the edge representation is trivial in the first two cases and nontrivial in cases three and four.

By Corollary 2.5, we have an injection

$$H^{11}(\overline{\mathcal{M}}_{1,n+2}) \hookrightarrow \bigoplus_{|B|=10} H^{11}(D_B).$$

Now, $\mathbb{S}_2$ acts on both sides by swapping p_1 and q_1. Taking $\mathbb{S}_2$-invariants of both sides, we have an injection from $H^{11}(\overline{\mathcal{M}}_{1,n+2})^{\mathbb{S}_2}$ into

$$\bigoplus_{\substack{\{p_1,q_1\}\subset B \text{ or } \{p_1,q_1\}\subset B^c \\ |B|=10}} H^{11}(D_B)^{\mathbb{S}_2} \oplus \bigoplus_{\substack{p_1\in B \text{ and } q_1\notin B \\ |B|=10}} [H^{11}(D_B) \oplus H^{11}(D_{(B\setminus p_1)\cup q_1})]^{\mathbb{S}_2}.$$

The first collection of terms is $(H^{11}(\overline{\mathcal{M}}_{\Gamma_B}) \otimes \det E(\Gamma_B))^{\mathbb{S}_2}$ for the graphs Γ_B of the first and second flavor in Figure 3, which have trivial edge representation. For the terms in square brackets, we have $H^{11}(D_B) \cong H^{11}(D_{(B\setminus p_1)\cup q_1})$ and the $\mathbb{S}_2$-action switches the two factors. Hence, the space of $\mathbb{S}_2$-invariants is $H^{11}(\overline{\mathcal{M}}_{\Gamma_B})$. When $p_1 \in B$ and $q_1 \notin B$, note that Γ_B is the type of graph considered in Example 4.2. In particular $\mathrm{Aut}(\Gamma_B)$ acts by the sign representation, so $(H^{11}(\overline{\mathcal{M}}_{\Gamma_B}) \otimes \det E(\Gamma_B))^{\mathrm{Aut}(\Gamma_B)} = H^{11}(\overline{\mathcal{M}}_{\Gamma_B})$. In summary, we have given an injection

$$H^{11}(\overline{\mathcal{M}}_\Gamma)^{\mathrm{Aut}(\Gamma)} = H^{11}(\overline{\mathcal{M}}_{1,n+2})^{\mathbb{S}_2} \hookrightarrow \bigoplus_{|B|=10} (H^{11}(\overline{\mathcal{M}}_{\Gamma_B}) \otimes \det E(\Gamma_B))^{\mathrm{Aut}(\Gamma_B)}. \tag{4.5}$$

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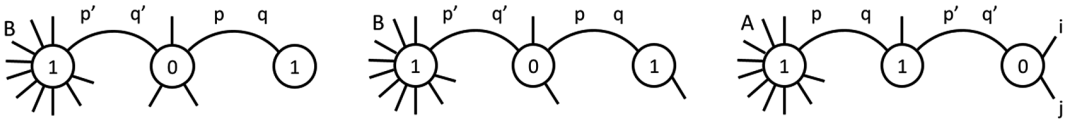

Figure 4. *The graphs in Cases 1, 2 and 3.*

4.3.3. Graphs with two genus 1 vertices

Suppose Γ has two genus 1 vertices, so

$$\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_{1,A\cup p} \times \overline{\mathcal{M}}_{1,A^c\cup q}.$$

We may assume $|A^c| \le |A|$. Figure 4 shows three types of graphs that will be used in the three cases of the argument. Note that none of them have a loop, so none admit edge contractions to the one-edge graphs of Section 4.3.2.

Case 1: $|A^c| = 0$. By Corollary 2.5, there is an injection

$$H^{11}(\overline{\mathcal{M}}_\Gamma) = H^{11}(\overline{\mathcal{M}}_{1,A\cup p}) \hookrightarrow \bigoplus_{\substack{B\subset A\\ |B|=10}} H^{11}(\overline{\mathcal{M}}_{1,B\cup p'}) = H^{11}(\overline{\mathcal{M}}_{\Psi_B}),$$

where Ψ_B is the first graph in Figure 4, which has

$$\overline{\mathcal{M}}_{\Psi_B} = \overline{\mathcal{M}}_{1,B\cup p'} \times \overline{\mathcal{M}}_{0,(A\setminus B)\cup\{q',p\}} \times \overline{\mathcal{M}}_{1,\{q\}}.$$

Case 2: $|A^c| = 1$. Again, by Corollary 2.5, there is an injection

$$H^{11}(\overline{\mathcal{M}}_\Gamma) = H^{11}(\overline{\mathcal{M}}_{1,A\cup p}) \hookrightarrow \bigoplus_{\substack{B\subset A\\ |B|=10}} H^{11}(\overline{\mathcal{M}}_{1,B\cup p'}) = H^{11}(\overline{\mathcal{M}}_{\Psi'_B}),$$

where this time Ψ'_B is the second graph in Figure 4, which has

$$\overline{\mathcal{M}}_{\Psi'_B} = \overline{\mathcal{M}}_{1,B\cup p'} \times \overline{\mathcal{M}}_{0,(A\setminus B)\cup\{q',p\}} \times \overline{\mathcal{M}}_{1,A^c\cup q}.$$

Note that our assumptions $n \ge 12$ and $|A^c| = 1$ ensure $|A \setminus B| \ge 1$, so the middle genus 0 vertex has at least 3 markings. Note that no edge contraction of Ψ'_B gives a one-edge graph Γ of the type in Case 1 or a different Γ of the type in Case 2.

Case 3: $|A^c| \ge 2$. Choose any $i, j \in A^c$ and $k, \ell \in A$ and define

$$X_{A^c\cup q} = \overline{\mathcal{M}}_{1,(A^c\setminus\{i,j\})\cup\{q,p'\}} \times \overline{\mathcal{M}}_{0,\{q',i,j\}}$$

and

$$X_{A\cup p} = \overline{\mathcal{M}}_{1,(A\setminus\{k,\ell\})\cup\{p,p'\}} \times \overline{\mathcal{M}}_{0,\{q',k,\ell\}}.$$

Associated to these are two-edge graphs Φ_A and Φ_{A^c} so that

$$\overline{\mathcal{M}}_{\Phi_A} = \overline{\mathcal{M}}_{1,A\cup p} \times X_{A^c\cup q} \qquad \text{and} \qquad \overline{\mathcal{M}}_{\Phi_{A^c}} = X_{A\cup p} \times \overline{\mathcal{M}}_{1,A^c\cup q}$$

and contracting the edge connecting p' and q' gives Γ. See the last graph in Figure 4 for a picture of Φ_A. Notice that no edge contraction of Φ_A or Φ_{A^c} gives a one-edge graph appearing in Cases 1 or 2 above or a different one-edge graph of the type in Case 3.

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By the Künneth formula,

$$H^{11}(\overline{\mathcal{M}}_\Gamma) = H^{11}(\overline{\mathcal{M}}_{1,A\cup p}) \oplus H^{11}(\overline{\mathcal{M}}_{1,A^c\cup q}),$$
$$H^{11}(\overline{\mathcal{M}}_{\Phi_A}) = H^{11}(\overline{\mathcal{M}}_{1,A\cup p}) \oplus H^{11}(\overline{\mathcal{M}}_{1,(A^c\setminus\{i,j\})\cup\{q,p'\}}),$$

and

$$H^{11}(\overline{\mathcal{M}}_{\Phi_{A^c}}) = H^{11}(\overline{\mathcal{M}}_{1,A^c\cup q}) \oplus H^{11}(\overline{\mathcal{M}}_{1,(A\setminus\{k,\ell\})\cup\{p,p'\}}).$$

It follows that $H^{11}(\overline{\mathcal{M}}_\Gamma)$ injects into the sum

$$H^{11}(\overline{\mathcal{M}}_{\Phi_A}) \oplus H^{11}(\overline{\mathcal{M}}_{\Phi_{A^c}}).$$

Remark 4.3. It should also be possible to deduce the vanishing of $H^{11}(\overline{\mathcal{M}}_{2,n})$ from the work of Dan Petersen as follows. There are exact sequences

$$H^{k-2}(\overline{\mathcal{M}}_{1,n+2})(-1) \to H^k(\overline{\mathcal{M}}_{2,n}) \to W_k H^k(\mathcal{M}^{\mathrm{ct}}_{2,n}) \to 0.$$

Specializing to $k = 11$, we see that

$$H^{11}(\overline{\mathcal{M}}_{2,n}) \cong W_{11} H^{11}(\mathcal{M}^{\mathrm{ct}}_{2,n}).$$

By [23, Theorem 2.1 and Remark 2.2], there is an isomorphism

$$H^{11}(\mathcal{M}^{\mathrm{ct}}_{2,n}) \cong \bigoplus_{p+q=11} H^p(\mathcal{M}^{\mathrm{ct}}_2, A^q) \oplus H^p(\mathrm{Sym}^2 \mathcal{M}_{1,1}, B^q),$$

where A^q and B^q are certain direct sums of Tate twists of symplectic local systems. The cohomology of these local systems has been determined by Petersen in [22]. Petersen's work makes significant use of high-powered machinery, including mixed Hodge modules, perverse sheaves, the decomposition theorem for the map $\mathcal{M}^{\mathrm{ct}}_{2,n} \to \mathcal{M}^{\mathrm{ct}}_2$ and the Eichler–Shimura isomorphism concerning modular forms. The proof we present here is relatively elementary and highlights a combinatorial perspective on the vanishing of $H^{11}(\overline{\mathcal{M}}_{2,n})$.

5. Application to point counting

The weighted count of $\mathbb{F}_q$ points on a Deligne–Mumford stack X is

$$\#X(\mathbb{F}_q) := \sum_{x\in|X(\mathbb{F}_q)|} \frac{1}{\#\operatorname{Aut}(x)},$$

where $|X(\mathbb{F}_q)|$ denotes the set of isomorphism classes of the groupoid $X(\mathbb{F}_q)$. This point count is related to the trace of the Frobenius map Φ_q on cohomology by Behrend's Grothendieck–Lefschetz trace formula [2, Theorem 3.1.2]:

$$\#X(\mathbb{F}_q) = q^{\dim X} \sum_{k\geq 0} (-1)^k \operatorname{tr} \Phi_q | H^k_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell).$$

Using this formula in the case $X = \overline{\mathcal{M}}_{g,n}$ leads to the proof of Corollary 1.3.

Proof of Corollary 1.3. Let $d = \dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$. The eigenvalues of Frobenius acting on $H^k_{\text{ét}}((\overline{\mathcal{M}}_{g,n})_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ are Weil numbers of weight k, meaning that under any embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ they have absolute value $q^{k/2}$ [11]. Thus, the point count $\#\overline{\mathcal{M}}_{g,n}(\mathbb{F}_q)$ is determined up to $O(q^{d-13/2})$ by the

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eigenvalues of Frobenius on $H^{2d-k}(\overline{\mathcal{M}}_{g,n})$ for $k \leq 12$. Theorem 1.2 and Poincaré duality tells us that Frobenius acts on the even cohomology $H^{2d-k}(\overline{\mathcal{M}}_{g,n})$ for $k \leq 12$ by $q^{d-k/2}$. Moreover, the groups $H^{2d-k}(\overline{\mathcal{M}}_{g,n})$ vanish for odd $k \leq 9$ by [1, 4]. Thus, up to $O(q^{d-13/2})$, the only other contribution to $\#\overline{\mathcal{M}}_{g,n}(\mathbb{F}_q)$ is from $H^{2d-11}(\overline{\mathcal{M}}_{g,n})$, which vanishes when $g \geq 2$ by Theorem 1.1 and Poincaré duality. □

6. Higher odd cohomology groups

In this section, we prove Theorem 1.5. The main tool is the push-pull formula. This formula is proven for manifolds in [29, Corollary 2.2]. The proof for orbifolds or smooth Deligne–Mumford stacks goes through analogously.

Lemma 6.1. *Suppose $i\colon X \hookrightarrow Y$ is a closed embedding of codimension d between smooth Deligne–Mumford stacks. Let $N_{X/Y}$ denote the normal bundle. Then for any cohomology class $\alpha \in H^*(X)$*

$$i^* i_*(\alpha) = c_d(N_{X/Y}) \cup \alpha.$$

Proof of Theorem 1.5. Set $g = 1 + \sum_{i=1}^{k} g_i$. Assume $\{g_i\}$ is ordered so that $g_1 < \cdots < g_k$. It suffices to prove the result for $\overline{\mathcal{M}}_{g,11-k}$, as the pull back maps

$$H^i(\overline{\mathcal{M}}_{g,n}) \to H^i(\overline{\mathcal{M}}_{g,n+1})$$

are injective for all n. Consider the gluing morphism

$$\xi\colon \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{g_1,1} \times \cdots \times \overline{\mathcal{M}}_{g_k,1} \to \overline{\mathcal{M}}_{g,11-k}$$

attaching the marked point on the $\overline{\mathcal{M}}_{g_i,1}$ component to the ith marked point on $\overline{\mathcal{M}}_{1,11}$. Let $\alpha \in H^{11,0}(\overline{\mathcal{M}}_{1,11})$ be a nonzero holomorphic 11-form. We will show that

$$\xi_*(\alpha \otimes 1 \otimes \cdots \otimes 1) \neq 0 \in H^{11+2k}(\overline{\mathcal{M}}_{g,11-k}).$$

It suffices to show that $\xi^*\xi_*(\alpha \otimes 1 \otimes \cdots \otimes 1)$ is nonzero. If $g_1 \neq 1$, set

$$U := \mathcal{M}_{1,11} \times \mathcal{M}_{g_1,1} \times \cdots \times \mathcal{M}_{g_k,1}.$$

If $g_1 = 1$, set

$$U := \mathcal{M}_{1,11} \times \overline{\mathcal{M}}_{1,1} \times \mathcal{M}_{g_2,1} \times \cdots \mathcal{M}_{g_k,1}.$$

We will show that

$$\xi^*\xi_*(\alpha \otimes 1 \otimes \cdots \otimes 1)|_U \neq 0.$$

Let $\Gamma_{g,11-k}$ denote the stable graph with $k+1$ vertices as follows. One vertex is of genus 1 with $11-k$ half-edges, and the rest of the vertices are of genus g_i. There is exactly one edge between the first vertex and each of the latter vertices, and no other edges, unless $g_1 = 1$ in which case we allow ourselves to replace the g_1 vertex with a genus 0 vertex with a self-edge (see Figure 5).

Let $V_{g,11-k}$ denote the open substack of $\overline{\mathcal{M}}_{g,11-k}$ parametrizing curves whose dual graphs are obtained from $\Gamma_{g,11-k}$ by edge contraction. Then

$$\xi|_U : U \to \overline{\mathcal{M}}_{g,11-k}$$

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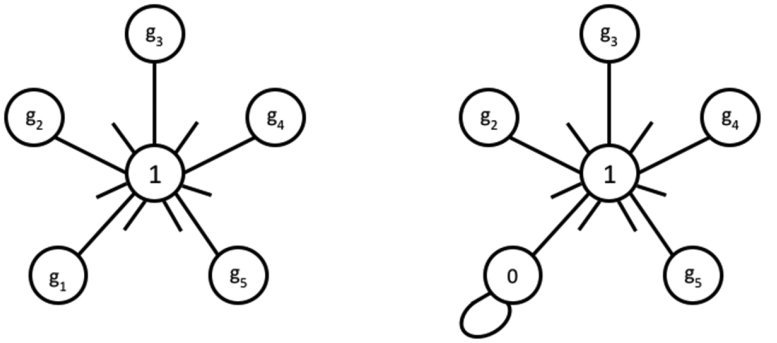

Figure 5. *The graphs* $\Gamma_{g,11-k}$.

factors through the open substack $V_{g,11-k}$. The induced map

$$i\colon U \hookrightarrow V_{g,11-k}$$

is a closed embedding between smooth stacks of codimension k. Here, we are using that the genera $g_1, \ldots, g_k$ are distinct. Let β denote the restriction of α to $\mathcal{M}_{1,11}$, which is nonzero (for example, by [15]). By Lemma 6.1

$$i^* i_*(\beta \otimes 1 \otimes \cdots \otimes 1) = c_k(N_i) \cup (\beta \otimes 1 \otimes \cdots \otimes 1).$$

Let ψ_{g_j} denote the (unique) ψ class on the $(j+1)$st component of U. Let ψ_j denote the jth ψ class on $\mathcal{M}_{1,11}$. It is well-known that

$$c_k(N_i) = \prod_{j=1}^{k} (-\psi_j \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes \psi_{g_j} \otimes \cdots \otimes 1).$$

See, for example, the discussion in [17, Section A.4]. Because $H^{13}(\mathcal{M}_{1,11}) = 0$ [15], all products of the form $\beta\psi_j$ vanish. Therefore,

$$c_k(N_i) \cup (\beta \otimes 1 \otimes \cdots \otimes 1) = (-1)^k (\beta \otimes \psi_{g_1} \otimes \cdots \otimes \psi_{g_k}) \neq 0. \qquad \square$$

7. Application to Chow rings

We denote by $A^*(X)$ the Chow ring of X with $\mathbb{Q}$-coefficients.

Theorem 7.1 (Kimura [19], Totaro [27]). *Suppose that X is a smooth, proper Deligne–Mumford stack over $\mathbb{C}$. If $A^*(X)$ is a countable $\mathbb{Q}$-vector space, then the cycle class map*

$$A^*(X) \to H^*(X)$$

is an isomorphism.

Proof. First, note that X is defined over a subfield $E \subset \mathbb{C}$ that is finitely generated over $\mathbb{Q}$ and hence countable. Suppose $A^*(X)$ is countable. Then there is a countable extension F of E such that $A^*(X_F) \to A^*(X)$ is surjective. Let G be a finitely generated extension of F. Then G can be embedded in $\mathbb{C}$, so we have a morphism

$$A^*(X_F) \to A^*(X_G) \to A^*(X).$$

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Each of the maps above is injective by [19, Proposition 3.2]. Because the composite is surjective, it follows that the first morphism is also surjective. By [27, Theorem 4.1], it follows that the motive of X_F is pure Hodge–Tate, and thus the motive of X is as well. Note that [27, Theorem 4.1] is stated for schemes, but the same proof goes through for Deligne–Mumford stacks. In particular, the cycle class map is an isomorphism. □

Proof of Corollary 1.6. The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ is a finite-dimensional $\mathbb{Q}$-vector space by [17, Corollary 1]. By Theorem 7.1, we know that $A^*(\overline{\mathcal{M}}_{g,n})$ is uncountable whenever there is odd cohomology. Therefore, the quotient $A^*(\overline{\mathcal{M}}_{g,n})/R^*(\overline{\mathcal{M}}_{g,n})$ is uncountable for the values of (g,n) in Theorem 1.5. It follows that the subgroup of $A^*(\overline{\mathcal{M}}_{g,n})$ generated by cycles algebraically equivalent to zero is also uncountable since the Hilbert scheme has only countably many connected components. □

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