

Multiplicity scaling of fragmentation function

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The single-particle inclusive fragmentation function and the particle multiplicity are observables of fundamental importance in studying properties of quantum chromodynamics at colliders. It is well known that at high energies the multiplicity distribution satisfies Koba-Nielsen-Olesen (KNO) scaling in which all moments are proportional to powers of the mean multiplicity. We prove that, under weak assumptions, the leading dependence of the fragmentation function on multiplicity is itself a kind of KNO scaling in which all moments are inversely proportional to powers of the mean multiplicity. This scaling with multiplicity additionally accounts for the dominant dependence on collision energy in the fragmentation function. The proof relies crucially on properties of the fragmentation function conditioned on the total multiplicity and application of the Stieltjes moment problem. In the process, we construct a novel basis of the fragmentation function expressed as an overall exponential suppression times a series of Laguerre polynomials. We study this scaling of the fragmentation function in experimental electron-positron collision data and observe that residual scale violations are significantly reduced.

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A quantity of central importance in understanding the properties and behavior of the theory of the strong force, quantum chromodynamics (QCD), is the distribution of a particle's energy fraction that was produced in a collision event. A whole chapter is devoted to this single-particle inclusive fragmentation function in the Particle Data Group's review of particle physics [1] and its features in threshold limits, as a function of collision energy, and sensitivity to hadron species reveals a rich and varied structure and provides validation of the whole theoretical framework. In the collisions of electrons and positrons, $e^+e^- \rightarrow \gamma/Z \rightarrow h + X$, as we study in this article, particle h 's energy fraction z_h is typically defined as [2,3]

$$z_h = \frac{2E_h}{\sqrt{s}}, \quad (1)$$

where \sqrt{s} is the center-of-mass collision energy of the events. This definition satisfies the energy conservation sum rule

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$$\sum_{h \in \text{event}} z_h = 2, \quad (2)$$

while momentum conservation restricts any individual energy fraction in the range $z_h \in [0, 1]$.

With unit normalization, the so-called fragmentation structure function in e^+e^- collisions,

$$F(z) = \frac{1}{\sigma} \frac{d\sigma^h}{dz}, \quad (3)$$

can be further factorized as a convolution of the standard fragmentation functions $D_{h/i}(z)$ with perturbative coefficient functions [1]. Note that the fragmentation function $D_{h/i}(z)$ represents the probability that a particle h carries energy fraction z of the fragmenting parton i [4]. In this paper, we will concentrate on studying the scaling behavior of the fragmentation structure function $F(z)$. Because of its close connection to the standard fragmentation function, we will abuse the terminology a bit below to often call $F(z)$ just as the inclusive fragmentation function. Clearly, $F(z)$ is related to the particle multiplicity N of the event as a larger multiplicity would correspond to less energy per particle on average. This can be made precise by introducing the fragmentation function conditioned on the multiplicity N , $F(z|N)$ [5]. The inclusive fragmentation function is defined as

$$F(z) = \int dN F(z|N) p(N), \quad (4)$$

where $p(N)$ is the probability distribution of multiplicity. The mean of the conditional fragmentation function is fixed by the multiplicity,

$$\langle z \rangle_N \equiv \int dz z F(z|N) = \frac{2}{N}. \quad (5)$$

We will always use the subscript N to denote expectation values of the conditional fragmentation function.

In this article, we establish general, robust properties of $F(z|N)$ that hold in the limit of large multiplicity, $N \rightarrow \infty$. With these results, we will show that the leading dependence of the inclusive fragmentation function on multiplicity is reminiscent of the Koba-Nielsen-Olesen (KNO) scaling of particle multiplicity distribution [6,7], where

$$F(z) = \langle N \rangle \Phi(\langle N \rangle z), \quad (6)$$

for a function $\Phi(x)$ that has no residual dependence on the mean multiplicity. Despite the extensive precise theoretical analysis, formulation, and predictions of the fragmentation function over its greater than 50 year history, including its perturbative scale evolution through the celebrated Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations [8–12], we are unaware of any study that provided a quantitative, model-independent relationship between the multiplicity and the energy fraction of a single particle produced in a collision event. Along the way to this goal, we construct a novel basis to represent the functional form of the fragmentation function that can provide a foundation for future theoretical studies, on which we can only briefly comment here.

In the limit of large multiplicity, the probability distribution of multiplicity $p(N)$ obeys KNO scaling, where

$$p(N) = \frac{1}{\langle N \rangle} \psi\left(\frac{N}{\langle N \rangle}\right), \quad (7)$$

where $\psi(x)$ is a universal function. KNO scaling, and its violation, is well established experimentally, e.g., at LEP [13–20] and the LHC [21–25], and specific functional forms of $\psi(x)$ have been proposed, e.g., Refs. [26–29]. This result of Eq. (5) then implies that the mean of the fragmentation function is

$$\langle z \rangle = \int dz z F(z) = 2\langle N^{-1} \rangle. \quad (8)$$

With KNO scaling, the mean of the fragmentation function scales inversely proportional to the mean multiplicity, $\langle z \rangle \propto \langle N \rangle^{-1}$. Without scaling properties of the conditional fragmentation function $F(z|N)$, we cannot establish general relationships between higher moments of the fragmentation function and the multiplicity distribution.

While one could establish properties of $F(z|N)$ through its perturbative scale evolution for example, here we will

instead start from a set of weak assumptions and employ consistency relations and enforce qualities required of all probability distributions. This procedure will result in a useful framework and functional form for calculations of the fragmentation function, which we leave to future work. The assumptions that we employ here are as follows.

- (1) The number of particles in the event N is large, $N \rightarrow \infty$.
- (2) The distribution of particles is sufficiently smooth away from the boundaries of phase space.
- (3) Only particle energies are measured so all particles are treated as identical and indistinguishable.

The conditional fragmentation function $F(z|N)$ can then be calculated from

$$F(z|N) = \lim_{N \rightarrow \infty} \int_0^1 \prod_{i=1}^N [dz_i] f(z_1, \dots, z_N) \delta\left(2 - \sum_{i=1}^N z_i\right) \times \delta(z - z_1), \quad (9)$$

where $f(z_1, z_2, \dots, z_N)$ is a positive semidefinite, integrable and appropriately normalized function that is permutation symmetric in the particle energy fractions z_i . The lower bound on the integration symbol means that all energy fractions are non-negative, $z_i \geq 0$, but are not constrained from above except by total energy conservation. Exploiting permutation symmetry, we select particle 1 to measure the energy fraction z . We also slightly abuse notation and employ $\lim_{N \rightarrow \infty}$ to mean taking the leading functional dependence on N as $N \rightarrow \infty$. That is, as $N \rightarrow \infty$, the ratio of the left- and right-hand sides of this equation approaches unity. We will take limits of the right-hand side in parts, and so we hope that the notation $\lim_{N \rightarrow \infty}$ reminds the reader where limits still need to be taken.

Performing the integral over z_1 and introducing the variables $z_i \equiv x_i(2 - z)$, we then have

$$F(z|N) = \lim_{N \rightarrow \infty} (2 - z)^{N-2} \int_0^1 \prod_{i=2}^N [dx_i] \times f(z, x_2(2 - z), \dots, x_N(2 - z)) \delta\left(1 - \sum_{i=2}^N x_i\right). \quad (10)$$

We note that there are $N - 1$ factors of $2 - z$ that come from the Jacobians dz_i/dx_i and then one inverse factor of $2 - z$ from pulling it out of the argument of the energy conservation δ function. Taking the $N \rightarrow \infty$ limit of the factor outside the integrals produces an exponential, where

$$F(z|N) = e^{-\frac{N}{2}z} \lim_{N \rightarrow \infty} \int_0^1 \prod_{i=2}^N [dx_i] \times f(z, x_2(2 - z), \dots, x_N(2 - z)) \delta\left(1 - \sum_{i=2}^N x_i\right). \quad (11)$$

In this and expressions that follow, we ignore overall z -independent factors as they will always be fixed by normalization anyway. The overall exponential factor restricts the region where there is significant support to $z \sim 2/N \ll 1$, so, to leading order in $1/N$, we set $2 - z \rightarrow 2$ in the argument of the function f :

$$F(z|N) = e^{-\frac{N}{2}z} \lim_{N \rightarrow \infty} \int_0^1 \prod_{i=2}^N [dx_i] f(z, 2x_2, \dots, 2x_N) \times \delta\left(1 - \sum_{i=2}^N x_i\right). \quad (12)$$

Note that the remaining δ function restricts the x_i variables to lie in an $N - 2$ simplex.

If the function $f(z, 2x_2, \dots, 2x_N)$ were analytic in its arguments about the origin, we could Taylor expand and ignore terms at linear order and beyond because they would be suppressed by powers of $z \sim 2/N$. That is, in the large- N limit, the conditional fragmentation function would reduce to

$$F(z|N) \propto e^{-\frac{N}{2}z}, \quad (13)$$

the maximally entropic distribution with a fixed mean $\langle z \rangle_N = 2/N$. We should expect that the function $f(z, 2x_2, \dots, 2x_N)$ is nonanalytic at the boundaries of phase space so this Taylor expansion about the origin will in general not be valid. However, we can instead expand around the mean value of $\langle z \rangle_N = 2/N$, which, while small, is displaced from the boundary of phase space. Generically, distributions should be smooth on the interior of phase space unless there were resonances or other new states, which here we do not consider.

The exponential form of the conditional fragmentation function has effectively been considered in related contexts previously, e.g., Ref. [30]. However, this was studied within a particular model for multiparticle production, with explicit choices for the distributions of particle energies and multiplicity. This and other studies therefore lacked a general framework or clear separation between fundamental hypotheses and constraints imposed by the form of model.

To perform the expansion away from the boundary, we introduce the variables

$$x_i \equiv \frac{1}{N} + \epsilon_i, \quad (14)$$

and so the conditioned fragmentation function becomes

$$F(z|N) = e^{-\frac{N}{2}z} \lim_{N \rightarrow \infty} \int_{-1/N}^1 \prod_{i=2}^N [d\epsilon_i] f \times \left(z, \frac{2}{N} + 2\epsilon_2, \dots, \frac{2}{N} + 2\epsilon_N\right) \times \delta\left(\frac{1}{N} - \sum_{i=2}^N \epsilon_i\right). \quad (15)$$

Now, by permutation symmetry and the remaining δ function, the expectation value of any of the ϵ_i variables is $\langle \epsilon_i \rangle_N \rightarrow 1/N^2$, in the large- N limit. Thus again, to leading order in $1/N$, we can set $\frac{1}{N} + \epsilon_i \rightarrow \frac{1}{N}$. Therefore, to leading order in $1/N$, the conditioned fragmentation function can be expressed as

$$F(z|N) = e^{-\frac{N}{2}z} f\left(z, \frac{2}{N}, \dots, \frac{2}{N}\right) \equiv e^{-\frac{N}{2}z} f\left(\frac{N}{2}z\right). \quad (16)$$

At right, we have concatenated the function into a function of a single argument, $f(Nz/2)$. The argument must be of the form $Nz \sim 1$; otherwise, we could expand the function about $z \sim 2/N$ and ignore terms that depend on the displacement from $2/N$ to leading order in $1/N$. However, there may be additional dependence on the multiplicity N in the function $f(Nz/2)$, which we address as follows.

The conditioned fragmentation function can be expanded in terms of Laguerre polynomials as

$$\tilde{F}(x|N) \equiv \frac{2}{N} F(z|N) = e^{-x} \left[1 + \sum_{n=2}^{\infty} c_n L_n(x)\right], \quad (17)$$

where $x = Nz/2$, is unit normalized and has unit mean, $\langle x \rangle_N = 1$. The Laguerre polynomials are familiar as the radial energy eigenstate wave functions of the hydrogen atom, are defined to be

$$L_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{k!} x^k, \quad (18)$$

and are orthonormal with respect to the exponential kernel

$$\delta_{mn} = \int_0^{\infty} dx e^{-x} L_m(x) L_n(x). \quad (19)$$

In general, it might seem that the coefficients c_n in Eq. (17) can have dependence on multiplicity N , but we will show that any dependence in the $N \rightarrow \infty$ limit leads to a pathological expansion. By the orthogonality of the Laguerre polynomials, the coefficients c_n can be isolated by taking moments of $\tilde{F}(x|N)$. These moments can be evaluated to find

$$\langle x^l \rangle_N = l! \left(1 + \sum_{n=2}^l (-1)^n \binom{l}{n} c_n\right). \quad (20)$$

To ensure that $\tilde{F}(x|N)$ is a probability distribution, namely, positive and integrable on $x \in [0, \infty)$, there are strong constraints imposed on the moments, as established by the Stieltjes moment problem [31]. This was also recently

exploited to establish positivity constraints on general effective Lagrangians [32].

The solution to the Stieltjes moment problem is that the Hankel matrix of moments must be completely positive. For brevity here, we will only explicitly consider the lowest dimension nontrivial Hankel matrices and just comment on generalization to higher dimensions that include more moments. To ensure positivity, the 2×2 Hankel matrices

$$\Delta_1 \equiv \begin{pmatrix} \langle x^0 \rangle_N & \langle x^1 \rangle_N \\ \langle x^1 \rangle_N & \langle x^2 \rangle_N \end{pmatrix} \quad (21)$$

and

$$\Delta_1^{(1)} \equiv \begin{pmatrix} \langle x^1 \rangle_N & \langle x^2 \rangle_N \\ \langle x^2 \rangle_N & \langle x^3 \rangle_N \end{pmatrix} \quad (22)$$

must have positive determinants $\det \Delta_1 > 0$ and $\det \Delta_1^{(1)} > 0$.

The determinants of the Hankel matrices and the Stieltjes constraints, using Eq. (20), are then

$$\det \Delta_1 = 2c_2 + 1 > 0 \quad (23)$$

and

$$\det \Delta_1^{(1)} = 6 + 18c_2 - 6c_3 - 4(1 + c_2)^2 > 0. \quad (24)$$

If the coefficient c_2 has dependence on N as any function that diverges as $N \rightarrow \infty$, for example, powers such as N^a or logarithms $\log^a N$ for $a > 0$, then the constraints imposed by the Stieltjes moment problem cannot be satisfied while maintaining sufficient smoothness of the fragmentation function. Specifically, if we assume that $c_2 = g(N)$ such that $g(N \rightarrow \infty) \rightarrow \infty$, then the constraint on Δ_1 can easily be satisfied. However, with this assumption, the constraint on $\Delta_1^{(1)}$ then requires, in the large- N limit,

$$\det \Delta_1^{(1)} \rightarrow -6c_3 - 4c_2^2 > 0 \quad (25)$$

or that

$$c_3 \lesssim -g(N)^2. \quad (26)$$

Therefore, to ensure that $\tilde{F}(x|N)$ is a probability distribution, the expansion coefficient c_3 must scale with multiplicity N parametrically larger than c_2 as $N \rightarrow \infty$. One finds that c_4 must correspondingly scale parametrically larger with N than c_3 , and this pattern continues to higher coefficients. Such an expansion is terribly nonconvergent, which violates our assumption that distributions on the interior of phase space are sufficiently smooth. Therefore, there can be no explicit dependence on the multiplicity N in the conditioned fragmentation function of Eq. (17); in the large- N multiplicity limit, the coefficients c_n are

independent of N . These coefficients may still depend on other, suppressed, parameters, like the collision energy or particle masses, but we leave that study for future work.

This result implies that this conditional fragmentation function satisfies a KNO-like scaling, where

$$F(z|N) = \frac{1}{\langle z \rangle_N} \mathcal{F}\left(\frac{z}{\langle z \rangle_N}\right) = \frac{N}{2} \mathcal{F}\left(\frac{N}{2} z\right) \quad (27)$$

and the function $\mathcal{F}(x)$ is independent of multiplicity N in the limit $N \rightarrow \infty$. This result enables direct calculation of the leading dependence of the fragmentation function on the multiplicity distribution. In this limit, we have established the scaling properties of the conditional fragmentation function $F(z|N)$, and the multiplicity distribution $p(N)$ satisfies KNO scaling. We have

$$\begin{aligned} F(z) &\rightarrow \int dN \frac{N}{2} \mathcal{F}\left(\frac{N}{2} z\right) \frac{1}{\langle N \rangle} \psi\left(\frac{N}{\langle N \rangle}\right) \\ &= \frac{\langle N \rangle}{2} \int dy y \mathcal{F}\left(\langle N \rangle z \frac{y}{2}\right) \psi(y) \\ &\equiv \langle N \rangle \Phi(\langle N \rangle z), \end{aligned} \quad (28)$$

for a function $\Phi(x)$ that has no residual dependence on mean multiplicity $\langle N \rangle$. $\Phi(x)$ may still depend on the collision energy but only in a way that is independent of $\langle N \rangle$. Additionally, KNO scaling is violated beyond leading-logarithmic accuracy, and so we expect that there are corrections to this rescaling of the inclusive fragmentation function at higher perturbative orders. However, the assumptions we employ here are sufficiently general that the scaling of the conditional fragmentation function in the large multiplicity limit should hold to any perturbative order. We leave further study of the consequences of this scaling on the perturbative aspects of the fragmentation function to future work.

We can observe this scaling of the fragmentation function in simulated data. We generated $e^+e^- \rightarrow$ hadrons collision events in Pythia 8.309 [33] at center-of-mass collision energies of $\sqrt{s} = 22, 55, 91, 206$ GeV. We start by establishing KNO scaling within this simulated data in Fig. 1. As the energy increases, the particle multiplicity correspondingly increases, but when the distributions are rescaled by their respective means, they collapse to a nearly identical, universal, distribution.

We focus on validation of the main result of the article, that the conditioned fragmentation function $F(z|N)$ is purely a function of zN in the large multiplicity N limit

$$\lim_{N \rightarrow \infty} F(z|N) = \frac{N}{2} \mathcal{F}\left(\frac{N}{2} z\right), \quad (29)$$

where $\mathcal{F}(x)$ with $x = Nz/2$ is independent of N . This relationship implies that all moments in z of $F(z|N)$ are inversely proportional to powers of multiplicity N :

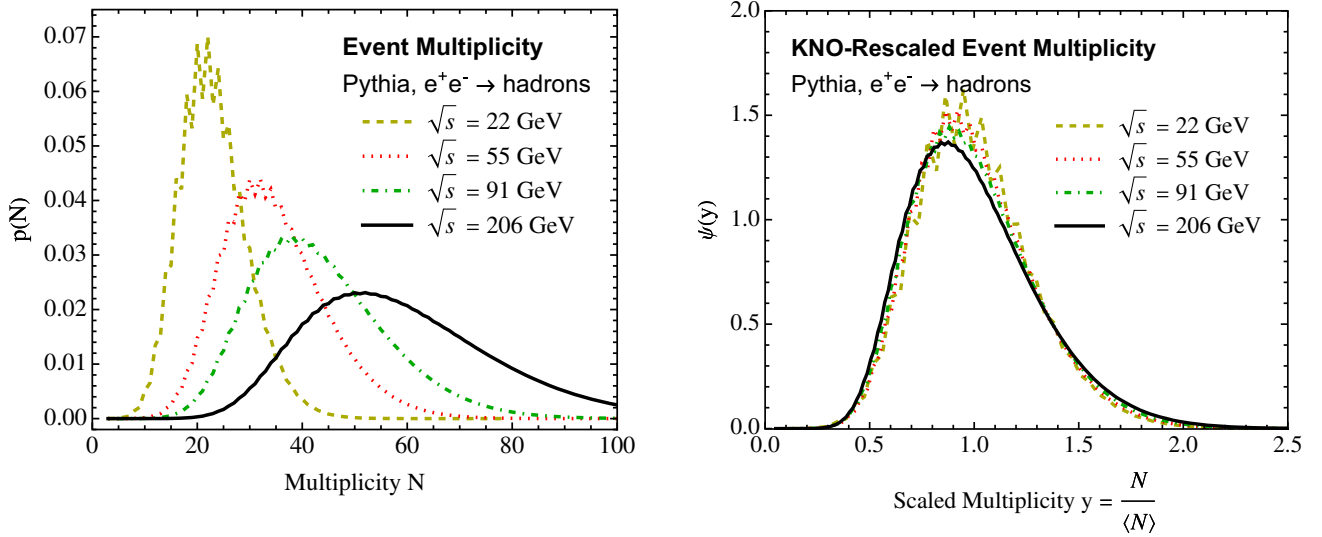


FIG. 1. The particle multiplicity distributions (left) in simulated $e^+e^- \rightarrow \text{hadron}$ events and the corresponding KNO-rescaled distributions (right) at four different center-of-mass energies.

$$\langle z^n \rangle_N = \int dz z^n F(z|N) \propto N^{-n}, \quad (30)$$

for any n . To test this, we will study the scaled moments $\langle x^n \rangle_N$, which should be independent of N at sufficiently high multiplicity.

In the simulated $e^+e^- \rightarrow \text{hadron}$ event samples from Pythia, we generate 50 million events at each center-of-mass collision energy $\sqrt{s} = 22, 55, 91, 206$ GeV. On these events, we measure the moments $\langle x^n \rangle_N$ for $n = 1, 2, 3, 4, 5, 6, 7$ and bin them in multiplicity N . The result of this analysis is presented in Fig. 2. At all energies, there is a steep rise in the moments at small multiplicity, but then the moments plateau, demonstrating to very good approximation that they do indeed lose dependence on multiplicity N at sufficiently large multiplicity. Some residual N dependence is observed; at the very highest multiplicities, the value of the moments decrease slightly, but this is a small effect over a wide range of multiplicities. Therefore, these simulated data exhibit behavior with multiplicity N that are consistent with the scaling of the conditioned fragmentation function $F(z|N)$ established above.

In Fig. 3, we plot the single-particle inclusive fragmentation function multiplied by the energy fraction $zF(z)$ at left. As anticipated, at higher energies and therefore higher average multiplicities, the fragmentation function becomes more and more strongly peaked near $z = 0$. However, by instead plotting the rescaled fragmentation function of Eq. (28), at right in Fig. 3, the broad spread between the distributions at different energies is dramatically reduced, especially at small values of the parameter $x = \langle N \rangle z$, denoted as “scaled energy fraction” in the figure. Sample dependence is observed at the upper limits, where the rescaled fragmentation function is most sensitive to the deviations from the asymptotic $N \rightarrow \infty$ limit.

With data from e^+e^- collider experiments, we can test this leading scaling behavior of the fragmentation function directly. To do this comparison, we use four datasets at collision energies comparable to the energies used in simulated data. The datasets we use here are, with their measured average charged particle multiplicities:

- (i) TASSO at $\sqrt{s} = 22$ GeV [34,35], $\langle N_{\text{ch}} \rangle = 11.22 \pm 0.07$,
- (ii) AMY at about $\sqrt{s} = 55$ GeV [36,37], $\langle N_{\text{ch}} \rangle = 16.82 \pm 0.22$,
- (iii) DELPHI at $\sqrt{s} = 91$ GeV [38], $\langle N_{\text{ch}} \rangle = 21.21 \pm 0.2$, and
- (iv) OPAL at $\sqrt{s} = 206$ GeV [39], $\langle N_{\text{ch}} \rangle = 27.75 \pm 0.67$.

Assuming that isospin (or flavor symmetry) is approximately conserved, the mean charged particle multiplicity just differs by a constant factor from the mean total multiplicity, independent of energy scale. The fragmentation functions measured in these data are plotted in Fig. 4. At left, are the raw fragmentation functions that exhibit a structure that peaks more sharply near $z = 0$ as collision energy or multiplicity increases. At right, we rescale the fragmentation function by the mean charged particle multiplicity and observe that the scale dependence is reduced as multiplicity or collision energy increases, experimentally demonstrating this scaling relation of the fragmentation function.

We have established the leading scaling law for the single-particle inclusive fragmentation function with the mean particle multiplicity. The proof of this scaling required construction of a systematic expansion of the fragmentation function conditioned on multiplicity in terms of Laguerre polynomials and constraints imposed on its coefficients by the Stieltjes moment problem. Residual

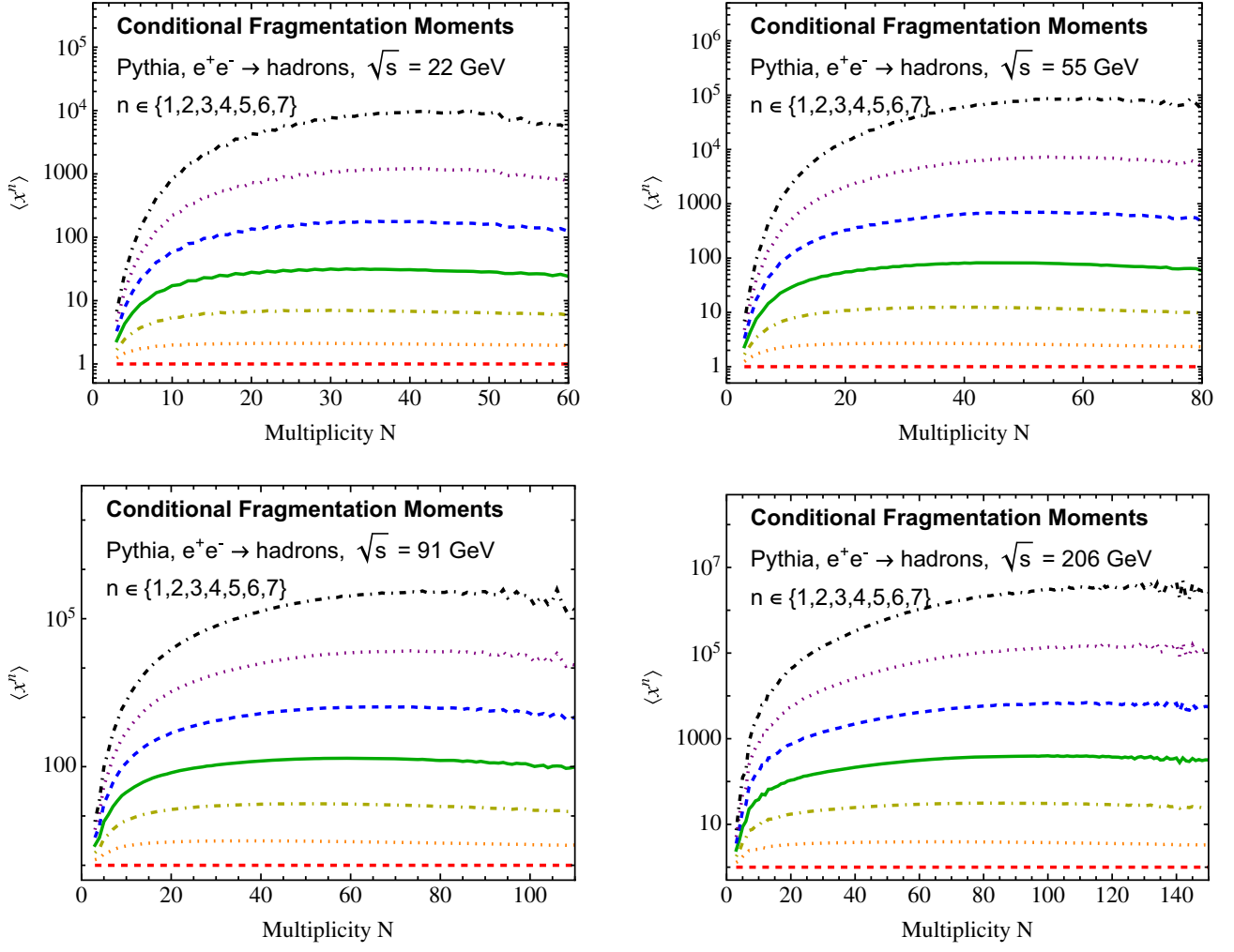


FIG. 2. Plots of the scaled moments $\langle x^n \rangle_N$ of the fragmentation function conditioned on multiplicity, where $x = Nz/2$, for $n = 1, 2, 3, 4, 5, 6, 7$ ordered in increasing value in the plots and colored red, orange, yellow, green, blue, purple, and black, respectively. These data are measured in $e^+e^- \rightarrow \text{hadron}$ events generated in Pythia at center-of-mass energies of $\sqrt{s} = 22, 55, 91, 206$ GeV.

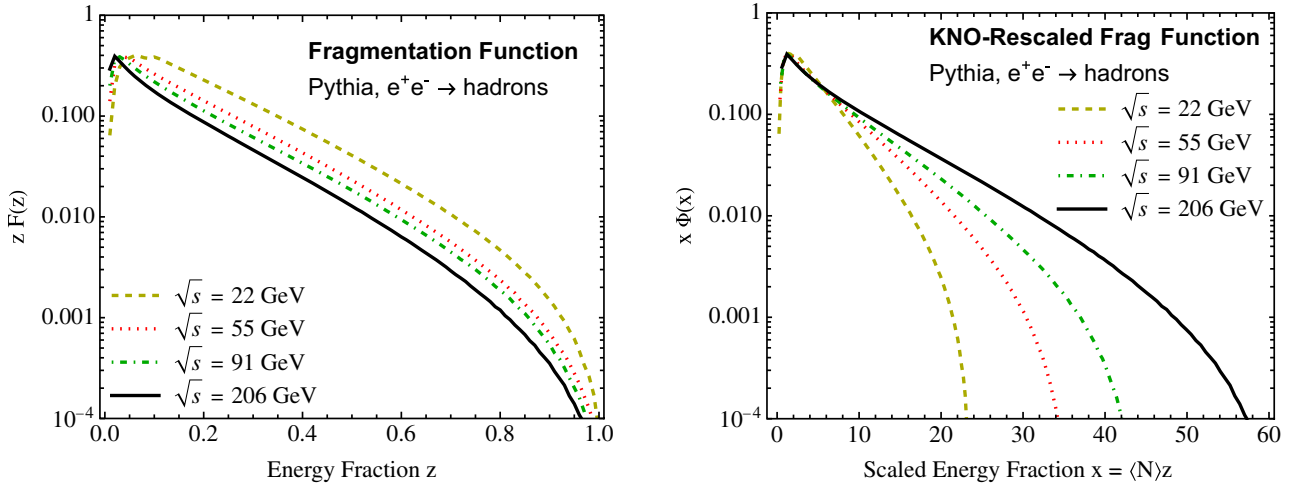


FIG. 3. The single-particle inclusive fragmentation function (left) in simulated $e^+e^- \rightarrow \text{hadron}$ events and the corresponding KNO-rescaled fragmentation functions (right) at four different center-of-mass energies.

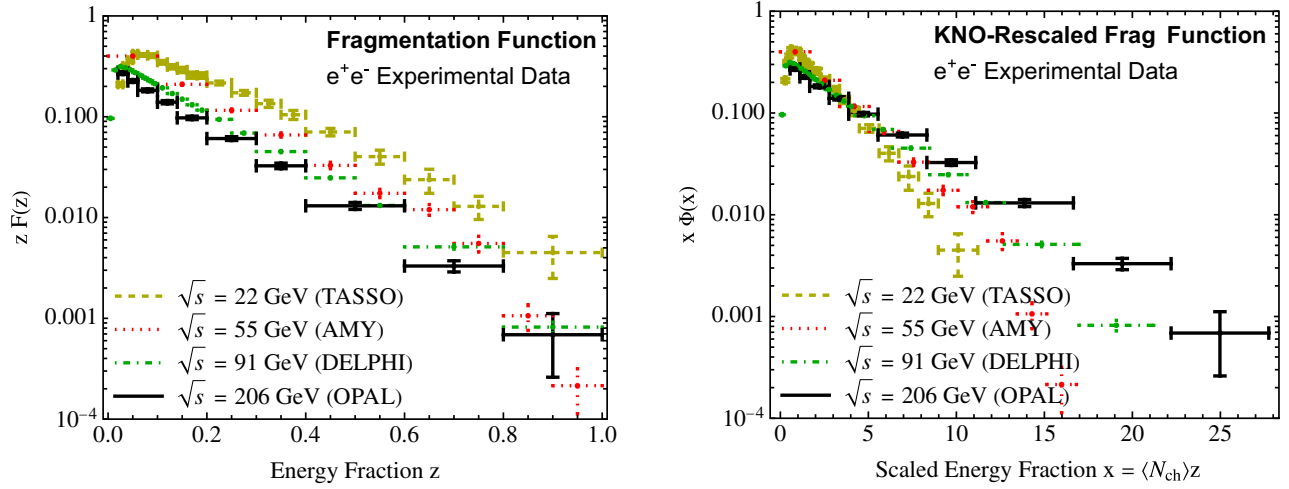


FIG. 4. The single-particle inclusive fragmentation function (left) in $e^+e^- \rightarrow \text{hadron}$ experimental data and the corresponding KNO-rescaled fragmentation functions (right) at four different center-of-mass energies.

dependence on collision energy is then largely uncorrelated with the mean multiplicity and enables isolation of other physical effects that control the features of the fragmentation function. This general analysis and dependence on multiplicity may provide insight into features of the fragmentation function, from the Gaussian peak's dependence on energy in the distribution of $\log 1/z$ [1,3,40] to the observed transition between recombination and fragmentation of partons in high-multiplicity heavy ion collisions [41,42]. We look forward to explorations and refinements of this analysis in the future.

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- [1] R. L. Workman *et al.* (Particle Data Group), Review of particle physics, *Prog. Theor. Exp. Phys.* **2022**, 083C01 (2022).
 - [2] Guido Altarelli, Partons in quantum chromodynamics, *Phys. Rep.* **81**, 1 (1982).
 - [3] R. Keith Ellis, W. James Stirling, and B. R. Webber, *QCD and Collider Physics* (Cambridge University Press, Cambridge, England, 2011), Vol. 8.
 - [4] John C. Collins and Davison E. Soper, Parton distribution and decay functions, *Nucl. Phys.* **B194**, 445 (1982).
 - [5] Zhong-Bo Kang, Andrew J. Larkoski, and Jinghong Yang, Towards a nonperturbative formulation of the jet charge, *Phys. Rev. Lett.* **130**, 151901 (2023).
 - [6] A. M. Polyakov, A Similarity hypothesis in the strong interactions. I. Multiple hadron production in e^+e^- annihilation, *Zh. Eksp. Teor. Fiz.* **59**, 542 (1970).
 - [7] Z. Koba, Holger Bech Nielsen, and P. Olesen, Scaling of multiplicity distributions in high-energy hadron collisions, *Nucl. Phys.* **B40**, 317 (1972).
 - [8] V. N. Gribov and L. N. Lipatov, Deep inelastic e p scattering in perturbation theory, *Sov. J. Nucl. Phys.* **15**, 438 (1972).
 - [9] V. N. Gribov and L. N. Lipatov, e^+e^- pair annihilation and deep inelastic e p scattering in perturbation theory, *Sov. J. Nucl. Phys.* **15**, 675 (1972).
 - [10] L. N. Lipatov, The parton model and perturbation theory, *Yad. Fiz.* **20**, 181 (1974).
 - [11] Yuri L. Dokshitzer, Calculation of the structure functions for deep inelastic scattering and e^+e^- annihilation by perturbation theory in quantum chromodynamics, *Sov. Phys. JETP* **46**, 641 (1977).
 - [12] Guido Altarelli and G. Parisi, Asymptotic freedom in parton language, *Nucl. Phys.* **B126**, 298 (1977).
 - [13] P. Abreu *et al.* (DELPHI Collaboration), Charged particle multiplicity distributions in Z^0 hadronic decays, *Z. Phys. C* **50**, 185 (1991).
 - [14] D. Decamp *et al.* (ALEPH Collaboration), Measurement of the charged particle multiplicity distribution in hadronic Z decays, *Phys. Lett. B* **273**, 181 (1991).
 - [15] P. Abreu *et al.* (DELPHI Collaboration), Charged particle multiplicity distributions in restricted rapidity intervals in Z^0 hadronic decays, *Z. Phys. C* **52**, 271 (1991).

- [16] P. D. Acton *et al.* (OPAL Collaboration), A study of charged particle multiplicities in hadronic decays of the Z^0 , *Z. Phys. C* **53**, 539 (1992).
- [17] P. Abreu *et al.* (DELPHI Collaboration), Charged particle multiplicity in e^+e^- interactions at $s = 130$ GeV, *Phys. Lett. B* **372**, 172 (1996).
- [18] R. Barate *et al.* (ALEPH Collaboration), Studies of quantum chromodynamics with the ALEPH detector, *Phys. Rep.* **294**, 1 (1998).
- [19] P. Abreu *et al.* (DELPHI Collaboration), Charged particle multiplicity in $e^+e^- \rightarrow q\bar{q}$ events at 161-GeV and 172-GeV and from the decay of the W boson, *Phys. Lett. B* **416**, 233 (1998).
- [20] P. Abreu *et al.* (DELPHI Collaboration), Charged and identified particles in the hadronic decay of W bosons and in $e^+e^- \rightarrow q\bar{q}$ from 130-GeV to 200-GeV, *Eur. Phys. J. C* **18**, 203 (2000); **25**, 493(E) (2002).
- [21] K. Aamodt *et al.* (ALICE Collaboration), Charged-particle multiplicity measurement in proton-proton collisions at $\sqrt{s} = 0.9$ and 2.36 TeV with ALICE at LHC, *Eur. Phys. J. C* **68**, 89 (2010).
- [22] Vardan Khachatryan *et al.* (CMS Collaboration), Charged particle multiplicities in pp interactions at $\sqrt{s} = 0.9$, 2.36, and 7 TeV, *J. High Energy Phys.* **01** (2011) 079.
- [23] R. Aaij *et al.* (LHCb Collaboration), Measurement of charged particle multiplicities in pp collisions at $\sqrt{s} = 7$ TeV in the forward region, *Eur. Phys. J. C* **72**, 1947 (2012).
- [24] Jaroslav Adam *et al.* (ALICE Collaboration), Charged-particle multiplicities in proton-proton collisions at $\sqrt{s} = 0.9$ to 8 TeV, *Eur. Phys. J. C* **77**, 33 (2017).
- [25] S. Acharya *et al.* (ALICE Collaboration), Charged-particle multiplicity distributions over a wide pseudorapidity range in proton-proton collisions at $\sqrt{s} = 0.9$, 7, and 8 TeV, *Eur. Phys. J. C* **77**, 852 (2017).
- [26] Fernand Hayot and George F. Sterman, The shape of the KNO scaling function in QCD, *Phys. Lett.* **121B**, 419 (1983).
- [27] A. Bassetto, KNO scaling in QCD jets and the negative binomial distribution, *Nucl. Phys.* **B303**, 703 (1988).
- [28] Yizhuang Liu, Maciej A. Nowak, and Ismail Zahed, Mueller's dipole wave function in QCD: Emergent KNO scaling in the double logarithm limit, *Phys. Rev. D* **108**, 034017 (2023).
- [29] Yizhuang Liu, Maciej A. Nowak, and Ismail Zahed, Universality of Koba-Nielsen-Olesen scaling in QCD at high energy and entanglement, [arXiv:2302.01380](https://arxiv.org/abs/2302.01380).
- [30] Grzegorz Wilk and Zbigniew Włodarczyk, Some intriguing aspects of multiparticle production processes, *Int. J. Mod. Phys. A* **33**, 1830008 (2018).
- [31] T-J Stieltjes, Recherches sur les fractions continues, in *Annales de la Faculté des sciences de Toulouse: Mathématiques*, Vol. **8** (1894), pp. J1–J122.
- [32] Nima Arkani-Hamed, Tzu-Chen Huang, and Yu-tin Huang, The EFT-hedron, *J. High Energy Phys.* **05** (2021) 259.
- [33] Christian Bierlich *et al.*, A comprehensive guide to the physics and usage of Pythia 8.3, *SciPost Phys. Codeb.* **2022**, 8 (2022).
- [34] W. Braunschweig *et al.* (TASSO Collaboration), Global jet properties at 14-GeV to 44-GeV center-of-mass energy in e^+e^- annihilation, *Z. Phys. C* **47**, 187 (1990).
- [35] M. Althoff *et al.* (TASSO Collaboration), Jet production and fragmentation in e^+e^- annihilation at 12-GeV to 43-GeV, *Z. Phys. C* **22**, 307 (1984).
- [36] Y. K. Li *et al.* (AMY Collaboration), Multi-hadron event properties in e^+e^- annihilation at $\sqrt{s} = 52$ GeV to 57-GeV, *Phys. Rev. D* **41**, 2675 (1990).
- [37] H. W. Zheng *et al.* (AMY Collaboration), Charged hadron multiplicities in e^+e^- annihilations at $\sqrt{s} = 50$ GeV—61.4 GeV, *Phys. Rev. D* **42**, 737 (1990).
- [38] P. Abreu *et al.* (DELPHI Collaboration), Measurement of the quark and gluon fragmentation functions in Z^0 hadronic decays, *Eur. Phys. J. C* **6**, 19 (1999).
- [39] G. Abbiendi *et al.* (OPAL Collaboration), Scaling violations of quark and gluon jet fragmentation functions in e^+e^- annihilations at $\sqrt{s} = 91.2$ -GeV and 183-GeV to 209-GeV, *Eur. Phys. J. C* **37**, 25 (2004).
- [40] Yuri L. Dokshitzer, Victor S. Fadin, and Valery A. Khoze, Double logs of perturbative QCD for parton jets and soft hadron spectra, *Z. Phys. C* **15**, 325 (1982).
- [41] R. J. Fries, Berndt Muller, C. Nonaka, and S. A. Bass, Hadronization in heavy ion collisions: Recombination and fragmentation of partons, *Phys. Rev. Lett.* **90**, 202303 (2003).
- [42] V. Greco, C. M. Ko, and P. Levai, Parton coalescence and anti-proton/pion anomaly at RHIC, *Phys. Rev. Lett.* **90**, 202302 (2003).